Quickest Search Over Multiple Sequences

Group Meeting

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Outline

Motivation

Model

Solutions

- Finite Horizon
- Infinite Horizon

Motivation

Motivation

It has many applicaitons:

- Locate attack point in sensor networks
- Find free frequency channel in cognitive radio systems
- Quality monitoring of manufacturing machines
- Sequentially search for certain data over multiple databases

Model

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Consider N sequences

- N sequences: $\{Y_k^i,\ k = 1, 2, \ldots\},\ i = 1, \ldots, N$
- For each i: $\{Y_k^i,\ k = 1, 2, \ldots\}$ are i.i.d. observations
- Two hypotheses:

$$\begin{array}{rl} 1 - \pi_0 \qquad & H_0 :\ Y_k^i \sim Q_0,\ k = 1, 2, \ldots \\ \pi_0 \qquad & H_1 :\ Y_k^i \sim Q_1,\ k = 1, 2, \ldots \end{array}$$

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prior prob.

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Intuitively, we want to quickly search the seq. generated by Q_1 with **minimal delay** and **maximal accuracy**.

Model (cont.)

- Each time select a sequence, say seq. j
- Draw a smaple Y from selected seq.
- Make one of the actions:
  - **Decision**: stop and claim
  - **Observation**: continue sampling from the same seq.
  - **Exploration**: observe next seq.
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- Consider $N = \infty$
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- A1: All seq.s are independent
- A2: A seq.'s distribution is independent of all others'

Notations

- Time $k = 1, 2, \ldots$
- s_k: seq. index observed at time k
- $Y_k^{s_k}$: the observation at time k
- Filtration $\mathcal{F}_k = \sigma(Y_1^{s_1}, Y_2^{s_2}, \ldots, Y_k^{s_k})$: knowledge up to time k
- Stopping time τ: stop at seq. s_τ and claim
- $\mathcal{T}$: the set of all stopping times w.r.t. $\mathcal{F}_k$
- Switching function

$$\phi_k(\mathcal{F}_k) = \begin{cases} 1, & \text{exploration}, s_{k+1} = s_k + 1 \\ 0, & \text{observation}, s_{k+1} = s_k \end{cases}$$

Problem

Two performance metrics

- **False alarm:** $P(H^{s_\tau} = H_0)$
- **Avg. # of samples:** $\mathbb{E}(\tau)$

This leads to the core problem

Problem

$$\inf_{\tau \in \mathcal{T}, \phi} \left[P(H^{s_\tau} = H_0) + c\mathbb{E}(\tau) \right] \tag{1}$$

Solutions

Finite Horizon

First, we consider a finite-horizon ver. of Problem (1) by restricting $\tau \in [0, T]$. This means we must stop at time T.

Posterior Prob.

Define $\pi_k = P(H^{s_k} = H_1 \mid \mathcal{F}_k)$ as the posterior prob. that seq. s_k is generated by Q_1 after observing $\mathcal{F}_k$,

$$
\begin{aligned}
\pi_1 =& \frac{\pi_0 q_1(Y_1^1)}{\pi_0 q_1(Y_1^1) + (1-\pi_0) q_0(Y_1^1)} \\
&\vdots \\
\pi_{k+1} =& \frac{\pi_k q_1(Y_{k+1}^{s_{k+1}})}{\pi_k q_1(Y_{k+1}^{s_{k+1}}) + (1-\pi_k) q_0(Y_{k+1}^{s_{k+1}})} \mathbf{1}(\phi_k = 0) \\
&+ \frac{\pi_0 q_1(Y_{k+1}^{s_{k+1}})}{\pi_0 q_1(Y_{k+1}^{s_{k+1}}) + (1-\pi_0) q_0(Y_{k+1}^{s_{k+1}})} \mathbf{1}(\phi_k = 1)
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$$\boxed{P(H^1 = H_1)}$$

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This calls the concept of function $\tilde{J}_k^T(\mathcal{F}_k)$, standing for the minimal cost-to-go at time k.

$$\tilde{J}_k^T(\mathcal{F}_k) = \min \left\{ \underbrace{1 - \pi_k}_{\text{stop cost}},\ \underbrace{c + \inf_{\phi_k} \mathbb{E}\left\{ \tilde{J}_{k+1}^T(\mathcal{F}_{k+1}) \mid \mathcal{F}_k, \phi_k \right\}}_{\text{cont. cost}} \right\} \quad (3)$$

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More precisely, at time k, based on $\mathcal{F}_k$, take action (stop or continue?), pay a cost $\tilde{J}_k^T$.

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- e.g., $\tilde{J}_T^T(\mathcal{F}_T) = 1 - \pi_T$

Finite Horizon

Lemma 1

For each k, function $\tilde{J}_k^T(\mathcal{F}_k)$ depends only on π_k, say $J_k^T(\pi_k)$, so as the optimal switching rules ϕ_k.

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- Assume $\tilde{J}_{k+1}^T(\mathcal{F}_{k+1})$ depends on $\pi_{k+1} \implies \tilde{J}_k^T(\mathcal{F}_k)$ depends only on π_k

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- So as ϕ_k

Proof of Lemma 1

Recall (3),

$$\tilde{J}_k^T(\mathcal{F}_k) = \min\left\{1 - \pi_k,\ c + \inf_{\phi_k} \mathbb{E}\left\{\tilde{J}_{k+1}^T(\mathcal{F}_{k+1}) \mid \mathcal{F}_k, \phi_k\right\}\right\}.$$

The second item can be rewritten

$$c + \min\{A_{k,c}^T(\pi_k), A_{k,s}^T\},$$

so $\tilde{J}_k^T(\mathcal{F}_k)$ depends on π_k, and ϕ_k is decided by

$$\phi_k = \begin{cases} 1, & \text{if } A_{k,c}^T(\pi_k) > A_{k,s}^T \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

which also depends only on π_k.

Proof of Lemma 1 (cont.)

$$\tilde{J}_k^T(\mathcal{F}_k) = \min\left\{1-\pi_k,\ c + \boxed{\inf_{\phi_k} \mathbb{E}\left\{\tilde{J}_{k+1}^T(\mathcal{F}_{k+1}) \mid \mathcal{F}_k, \phi_k\right\}}\right\}$$

$$\Longrightarrow\ \min\left\{\mathbb{E}_{\phi_k=0}(\dots),\ \mathbb{E}_{\phi_k=1}(\dots)\right\}$$

$$\Longrightarrow\ \min\left\{\int J_{k+1}^T(\pi_{k+1}) f_c(y|\mathcal{F}_k)dy,\ \int J_{k+1}^T(\pi_{k+1}) f_s(y|\mathcal{F}_k)dy\right\}$$

Proof of Lemma 1 (cont.)

$$\tilde{J}_k^T(\mathcal{F}_k) = \min\left\{1-\pi_k,\ c + \boxed{\inf_{\phi_k} \mathbb{E}\left\{\tilde{J}_{k+1}^T(\mathcal{F}_{k+1}) \mid \mathcal{F}_k, \phi_k\right\}}\right\}$$

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Two conditional PDF:

- **stay:** $f_c(y|\mathcal{F}_k) = \pi_k q_1(y) + (1-\pi_k) q_0(y)$
- **switch:** $f_s(y|\mathcal{F}_k) = \pi_0 q_1(y) + (1-\pi_0) q_0(y)$

Additionally, π_{k+1} depends on π_k or π_0 by (2).

Proof of Lemma 1 (cont.)

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$A_{k,c}^T(\pi_k)$

$A_{k,s}^T$

Lemma 2

The functions $J_k^T(\pi_k)$ and $A_{k,c}^T(\pi_k)$ are nonnegative concave w.r.t. $\pi_k \in [0,1]$. And $J_k^T(1) = A_{k,c}^T(1) = 0$.

- These two functions are both somewhat nonnegative costs
- Concavity reserved
- Since $J_k^T(1) = \min\{0, \text{something}\} \geq 0 \implies J_k^T(1) = 0$, and

$$A_{k,c}^T(1) = \mathbb{E}\left\{J_{k+1}^T(1) \mid \mathcal{F}_k, \phi_k = 0\right\} = 0$$

Finite Horizon

$J_k^T(\pi_k)$ satisfy:

- $\leq 1 - \pi_k$
- $J_k^T(1) = 0$
- concave w.r.t. π_k

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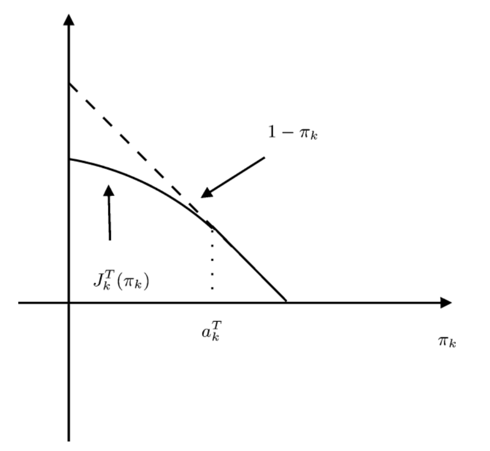

src: L Lai, et al. "Quickest search"

Figure 1: An illustration of $J_k^T(\pi_k)$

Finite Horizon (solution)

Theorem 3

For the finite-horizon version of Problem (1), the optimal stopping time is $\tau_{opt} = \inf\{k : \pi_k > a_k^T\}$*, in which* a_k^T *is given by the following equation:*

$$1 - a_k^T = c + \min\{A_{k,c}^T(a_k^T),\ A_{k,s}^T\}.$$

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Recall that

$$J_k^T(\pi_k) = \min\left\{1 - \pi_k,\ c + \min\{A_{k,c}^T(\pi_k), A_{k,s}^T\}\right\}.$$

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Once $\pi_k > a_k^T$, we have

$$1 - \pi_k < 1 - a_k^T = c + \min\{..\}$$

which means stop cost $<$ cont. const. $\implies$ **stop!**

Infinite Horizon

Now let $T \to \infty$, we get the infinite horizon ver. of Problem (1).

Infinite Horizon

Observing that

- **Constraint relaxed**: $J_k^{T+1}(\pi) \leq J_k^T(\pi)$
- **Lower bounded**: $\lim_{T\to\infty} J_k^T(\pi) = \inf_{T>k} J_k^T(\pi) = J_k^{\infty}(\pi)$
- **Concavity**: $J_k^T(\pi)$ is concave for each $T,\ k$
- **I.I.D. observation**: $J_k^{\infty}(\pi) = J_{k+1}^{\infty}(\pi)$?

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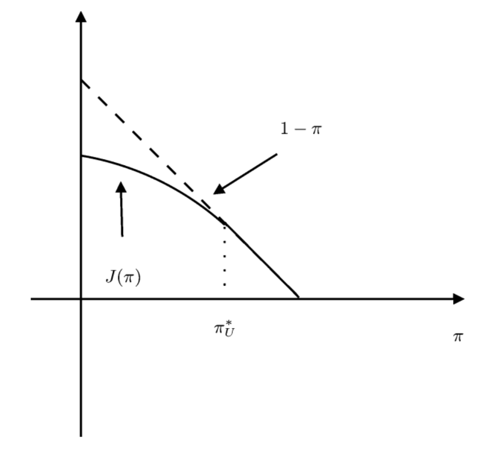
$\implies J(\pi) := J_k^\infty(\pi)$ concave w.r.t. π

Infinite Horizon



src: L Lai, et al. "Quickest search"

Figure 2: An illustration of $J(\pi)$

As $T \to \infty$,

- $A_{k,c}^{T}(\pi) \to A_c(\pi)$
- $A_{k,s}^{T} \to A_s$

Lemma 4

$$A_c(\pi) \begin{cases} > A_s, & \textit{if } \pi < \pi_0 \\ = A_s, & \textit{if } \pi = \pi_0 \\ < A_s, & \textit{if } \pi > \pi_0 \end{cases} \tag{5}$$

Infinite Horizon

- $A_c(0) > A_s$
- $A_c(1) = 0 < A_s$
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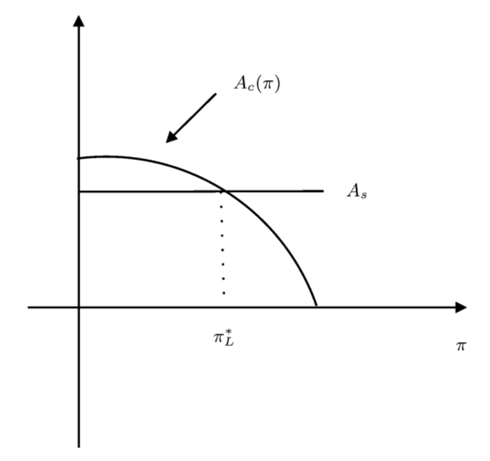

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Figure 3: An illustration of $A_c(\pi)$ and A_s

Infinite Horizon (solution)

Theorem 5

The optimal stopping time for Problem (1) is given by $\tau_{opt} = \inf\{k : \pi_k > \pi_U^*\}$ *in which*

$$1 - \pi_U^* = c + \min\{A_c(\pi_U^*),\ A_s\}.$$

Infinite Horizon (solution)

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Recall that

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Infinite Horizon (solution)

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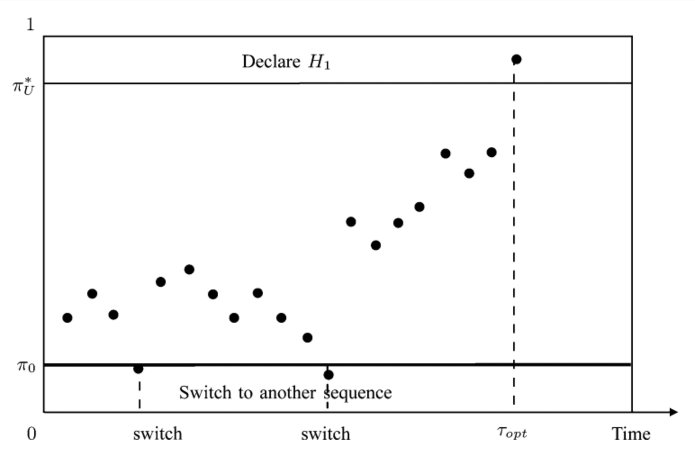
Process Tracking



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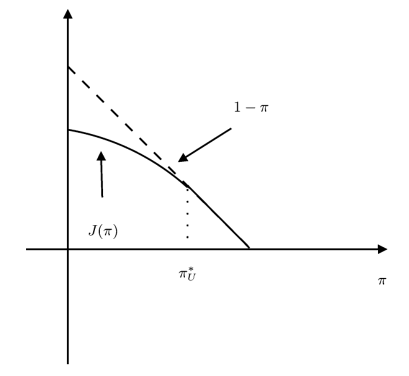
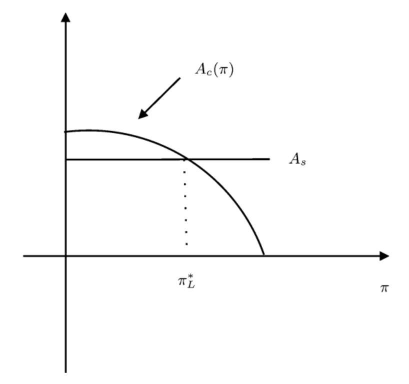
For Convenience





Thank you!