Modeling and Control of a Two Wheeled Auto-Balancing Robot: a didactic platform for control engineering education

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1 Introduction

The Two Wheeled Automatic Balancing Robot (TWABR) is an unstable and non-linear electromechanical system consisting of two side wheels that have contact with the floor surface. The wheels are independently driven to balance in the gravity center above the axis of the wheel's rotation. The wheels are driven by two motors coupled to each of them. The motors can be of a DC nature and are controlled by electrical signals by means of a control system based on the inclination reading and the velocity of their gravity center. Its operation is similar to the classic inverted pendulum system. The control objective is to stabilize the TWABR by keeping it in a vertical equilibrium position.The complex dynamics inherent in this platform finds its application in the design and development of control systems for cars, spacecraft, domestic transportation, military transport, among others.

This experiment aims to demonstrate the design of a controllers for a TWABR by means of a linear control tools. In implementing such a control system the following topics will be covered.

1. Modelling the dynamics of an TWABR using the Euler-Lagrange equation
2. Obtaining a linear state-space representation of the system
3. Creating a Simulink model to simulate the non-linear behaviour of the pendulum.
4. Designing a state-feedback control law that improves damping for the pendulum in the downward position and balances it at its vertical upward position
5. Implementing the designed control law on the Simulink model and verifying its performance

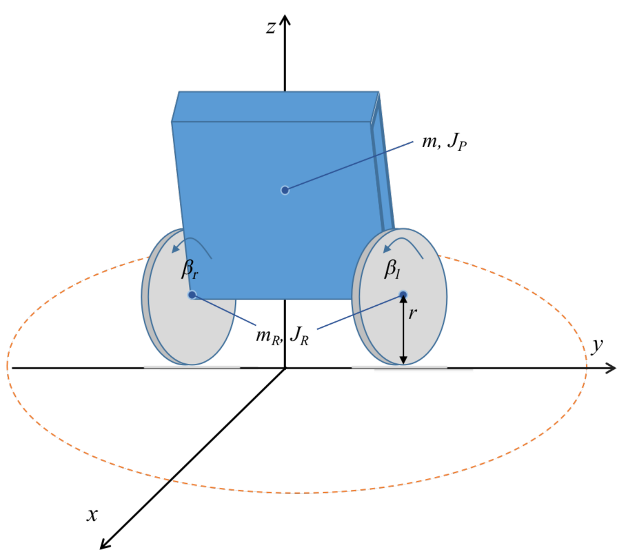

Figure 1: Balanced Free-body diagram of the TWABR system.

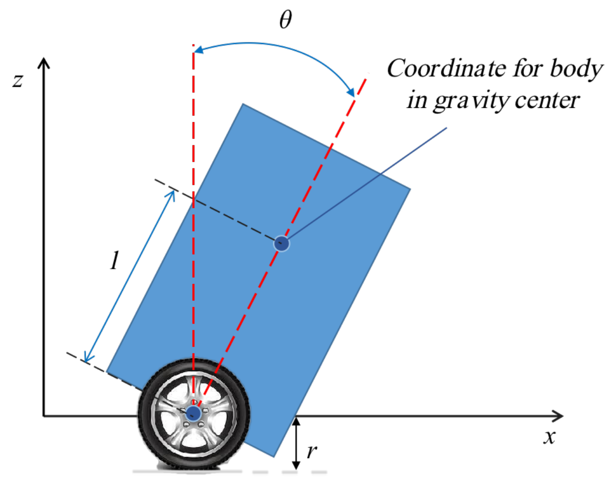

Figure 2: Inclined Free-body diagram of the TWABR system.

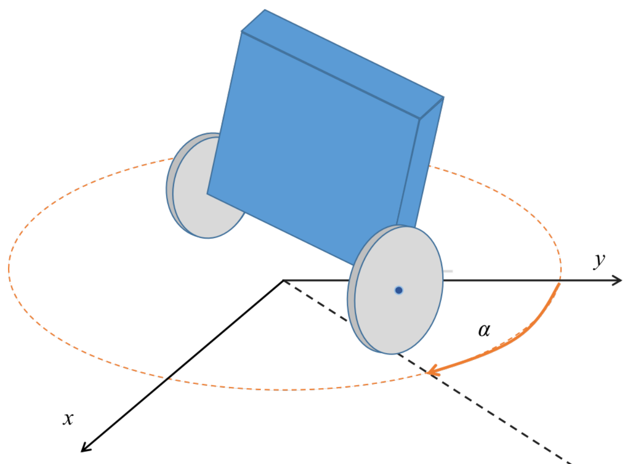

Figure 3: Rotated Free-body diagram of the TWABR system.

2 MATHEMATIC MODEL OF THE TWABR SYSTEM

the following considerations are assumed: 1) the wheels follow a non-slip movement, 2) the wheels of the rotation system are rigid and similar (same radius and masses) and 3) that the weight in the main body is evenly distributed. Frictions in movements are initially considered, but are later omitted.

The position and velocity of the main body are given as:

$$P = \begin{bmatrix} x + l\sin\theta\cos\alpha \\ y + l\sin\theta\sin\alpha \\ r + l\cos\alpha \end{bmatrix}, \quad V = \begin{bmatrix} \dot{x} + l(\dot{\theta}\cos\theta\cos\alpha - \dot{\alpha}\sin\theta\sin\alpha) \\ \dot{y} + l(\dot{\theta}\cos\theta\sin\alpha + \dot{\alpha}\sin\theta\cos\alpha) \\ -l\dot{\alpha}\sin\alpha \end{bmatrix} \tag{1}$$

and the angular velocities of the mobile base and the main body are given as:

$$\omega_m = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}, \quad \omega_p = \begin{bmatrix} -\dot{\alpha}\sin\theta \\ \dot{\theta} \\ \dot{\alpha}\cos\theta \end{bmatrix} \tag{2}$$

The motion equations that define the behavior of the TWABR system can be obtained using the Lagrangian dynamics .

The kinetic energy of the main body due to angular displacement can be represented as:

$$KE_p = \frac{1}{2}MV^TV + \frac{1}{2}\omega_p^T J_p \omega_p \tag{3}$$

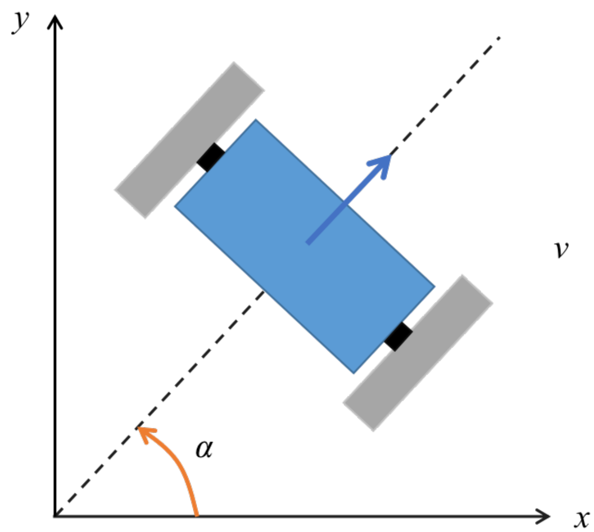

Figure 4: Translational and rotational velocities of a wheeled mobile robot.

The kinetic energy of the mobile base can be represented as:

$$KE_m = \frac{1}{2}mr^2(\dot{\beta_l}^2 + \dot{\beta_r}^2) + \frac{1}{2}(J_{wa} + J_{ra}\gamma^2)(\dot{\beta_l}^2 + \dot{\beta_r}^2) + (J_{wd} + J_{wr})\dot{\alpha}^2 \tag{4}$$

The kinetic energy of the total system is:

$$KE_s = T_p + T_m \tag{5}$$

The potential energy of the system can be represented as:

$$PE_s = Mgl\cos\theta \tag{6}$$

The damping energy of the system can be represented as:

$$D = \frac{1}{2}c_r(\dot{\beta}_r - \dot{\theta})^2 + \frac{1}{2}c_l(\dot{\beta}_l - \dot{\theta})^2 \tag{7}$$

Consequently, the Lagrangian of the system is $L = KE_s - PE_s$. The Lagrangian equation is expressed as:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad i = 1, 2, \dots, n \tag{8}$$

where Q, q and n denote a generalized force, coordinate, and coordinate number, respectively. We define a generalized coordinate as $q = [x, y, \alpha, \theta, \beta_r, \beta_l]^T$, the six Lagrangian equations are:

$$\begin{aligned}
&\mathbf{Eq1}: \ M\,\ddot{x} - M\,l\,\cos\alpha\,\sin\theta\,\dot{\alpha}^2 - M\,l\,\cos\alpha\,\sin\theta\,\dot{\theta}^2 + M\,l\,\cos\alpha\,\cos\theta\,\ddot{\theta} - M\,l\,\sin\alpha\,\sin\theta\,\ddot{\alpha} \\
&-2\,M\,l\,\sin\alpha\,\cos\theta\,\dot{\theta}\,\dot{\alpha} = 0 \\
&\mathbf{Eq2}: \ M\,\ddot{y} - M\,l\,\sin\alpha\,\sin\theta\,\dot{\alpha}^2 - M\,l\,\sin\alpha\,\sin\theta\,\dot{\theta}^2 + M\,l\,\cos\alpha\,\sin\theta\,\ddot{\alpha} + M\,l\,\sin\alpha\,\cos\theta\,\ddot{\theta} \\
&+2\,M\,l\,\cos\alpha\,\cos\theta\,\dot{\theta}\,\dot{\alpha} = 0 \\
&\mathbf{Eq3}: \ (2\,J_{\mathrm{rd}} + 2\,J_{\mathrm{wd}} + J_{\mathrm{xx}} - J_{\mathrm{xx}}\,\cos^2\theta + J_{\mathrm{zz}}\,\cos^2\theta + M\,l^2 - M\,l^2\,\cos^2\theta)\,\ddot{\alpha} + M\,l\,\cos\alpha\,\sin\theta\,\ddot{y} \\
&+(2\,J_{\mathrm{xx}}\,\cos\theta\,\sin\theta - 2\,J_{\mathrm{zz}}\,\cos\theta\,\sin\theta + 2\,M\,l^2\,\cos\theta\,\sin\theta)\,\dot{\theta}\,\dot{\alpha} - M\,l\,\sin\alpha\,\sin\theta\,\ddot{x} = 0 \\
&\mathbf{Eq4}: \ (c_l + c_r)\,\dot{\theta} - c_l\,\dot{\beta}_l - c_r\,\dot{\beta}_r + (J_{\mathrm{yy}} + M\,l^2)\,\ddot{\theta} + (J_{\mathrm{zz}}\,\cos\theta\,\sin\theta - J_{\mathrm{xx}}\,\cos\theta\,\sin\theta - M\,l^2\,\cos\theta\,\sin\theta)\,\dot{\alpha}^2 \\
&-m\,g\,l\,\sin\theta + M\,l\,\cos\alpha\,\cos\theta\,\ddot{x} + M\,l\,\sin\alpha\,\cos\theta\,\ddot{y} = -T_l - T_r \\
&\mathbf{Eq5}: \ (J_{\mathrm{ra}}\,\gamma^2 + m\,r^2 + J_{\mathrm{wa}})\,\ddot{\beta}_r + c_r\,\dot{\beta}_r - c_r\,\dot{\theta} = T_r \\
&\mathbf{Eq6}: \ (J_{\mathrm{ra}}\,\gamma^2 + m\,r^2 + J_{\mathrm{wa}})\,\ddot{\beta}_l + c_l\,\dot{\beta}_l - c_l\,\dot{\theta} = T_l
\end{aligned} \tag{9}$$

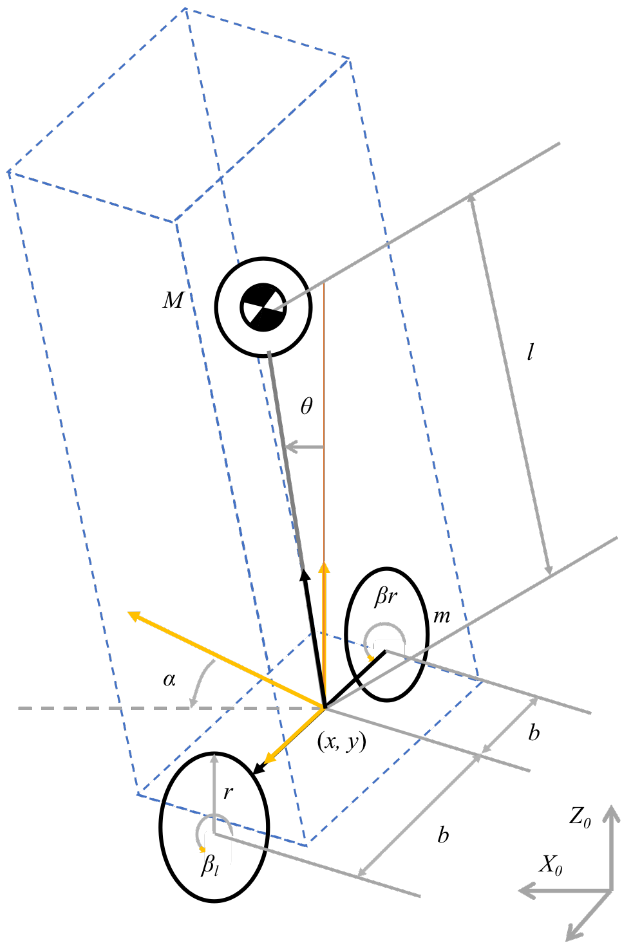

Figure 5: Model of 3DOF wheeled inverted pendulum robot.

which can be rearranged in the matrix form:

$$M(q)\ddot{q} + V\dot{q} + H(q, \dot{q}) = E\tau \tag{10}$$

where

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_r \\ \tau_l \end{bmatrix} \tag{11}$$

The structure of the fourth row of E arises because the motors are mounted on the main body. Notice that there are three constraints on the mobile base by assuming that the wheels do not slip:

$$\dot{x}\sin\alpha - \dot{y}\cos\alpha = 0 \tag{12}$$

$$\dot{x}\cos\alpha + \dot{y}\sin\alpha = \frac{r}{2}(\dot{\beta}_r + \dot{\beta}_l) \tag{13}$$

$$\dot{\alpha} = \frac{r}{2b}(\dot{\beta}_r - \dot{\beta}_l) \tag{14}$$

The motion equation with constraints is expressed as:

$$M(q)\ddot{q} + V\dot{q} + H(q, \dot{q}) = E\tau + A_q^T\lambda \tag{15}$$

where λ is a Lagrangian multiplier and A_q is written as:

$$A_q = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 & 0 & 0 & 0 \\ \cos\alpha & \sin\alpha & b & 0 & -r & 0 \\ \cos\alpha & \sin\alpha & -b & 0 & 0 & -r \end{bmatrix} \tag{16}$$

Because it is difficult to find λ, we define a matrix S_q composing linear independent vector in the null-space of A_q:

$$S_q = \begin{bmatrix} \cos\alpha & 0 & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/r & b/r & 0 \\ 1/r & -b/r & 0 \end{bmatrix} \tag{17}$$

which shows $A_q S_q = 0$. Next, we follow the standard procedure for the elimination of Lagrange multiplier λ by premultiplication (15) with S_q^T:

$$S_q^T M(q)\ddot{q} + S_q^T V\dot{q} + S_q^T H(q,\dot{q}) = S_q^T E\tau \tag{18}$$

Define $\nu = [v, \dot{\alpha}, \dot{\theta}]^T$, we see $\dot{q} = S_q\nu$ and $\ddot{q} = \dot{S}_q\nu + S_q\dot{\nu}$. Then (18) can be rewritten as:

$$\hat{M}\dot{\nu} + \hat{V}\nu + \hat{H}(\nu,\dot{\nu}) = \hat{E}\tau \tag{19}$$

where $\hat{M} = S_q^T M(q)S_q$, $\hat{H} = S_q^T[M(q)\dot{S}_q\nu + H(q,\dot{q})]$, $\hat{V} = S_q^T V(q)S_q$, $\hat{E} = S_q^T E$.

3 Linear System Analysis

3.1 Linearization

Q1) Define the control variable $x = [x, v, \alpha, \dot{\alpha}, \theta, \dot{\theta}]^T$ and linearize the system around the equilibrium point $[0,0,0,0,0,0]^T$ and obtain the expression of matrix A and B:

$$\dot{x} = Ax + Bu \tag{20}$$