TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY NOVEMBER 6: GEOMETRY OF LINEAR TRANSFORMATIONS (ABSTRACT)

MATH 196, SECTION 57 (VIPUL NAIK)

1. PERFORMANCE REVIEW

27 people took this 14-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 4: 1 person
- Score of 6: 3 people
- Score of 7: 2 people
- Score of 8: 4 people
- Score of 9: 4 people
- Score of 10: 5 people
- Score of 11: 4 people
- Score of 12: 2 people
- Score of 14: 1 person

The mean score was 8.8.

The question-wise answers and performance review are as follows:

 (1) Option (B): 18 people
 (2) Option (C): 22 people
 (3) Option (C): 17 people
 (4) Option (C): 18 people
 (5) Option (A): 10 people
 (6) Option (E): 22 people
 (7) Option (C): 25 people
 (8) Option (E): 16 people
 (9) Option (C): 13 people
(10) Option (B): 17 people
(11) Option (B): 12 people
(12) Option (A): 20 people
(13) Option (C): 15 people
(14) Option (C): 13 people

2. SOLUTIONS

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz tests for a deep abstract understanding of linear transformations and their geometry. It is related to Section 2.2 of the book and also to the `Geometry of linear transformations` lecture notes.

For the questions here, please use the following terminology.

Suppose n is a fixed natural number greater than 1. For ease of geometric visualization, you can take $n = 2$ for the discussion.

- A *linear automorphism* of $\mathbb{R}^n$ is defined as a bijective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- An *affine linear automorphism* of $\mathbb{R}^n$ is defined as a bijective function from $\mathbb{R}^n$ to itself that preserves collinearity, i.e., it sends lines to lines. In addition, it preserves the ratios of lengths within each line. This can be included as part of the definition or deduced from the fact that collinearity is preserved for $n > 1$.

- A *self-isometry* of $\mathbb{R}^n$ is defined as a bijective function from $\mathbb{R}^n$ to itself that preserves Euclidean distance: for all pairs of points $\vec{x}, \vec{y} \in \mathbb{R}^n$, the Euclidean distance between $\vec{x}$ and $\vec{y}$ equals the Euclidean distance between $T(\vec{x})$ and $T(\vec{y})$.
- A *self-homothety* (or *similitude transformation* or *similarity transformation*) of $\mathbb{R}^n$ is defined as a bijective function from $\mathbb{R}^n$ to itself that multiplies all distances by a fixed number called the *factor of similitude* (dependent on the transformation): if the factor of similitude is λ, then for all pairs of points $\vec{x}, \vec{y} \in \mathbb{R}^n$, the distance between $T(\vec{x})$ and $T(\vec{y})$ equals λ times the distance between $\vec{x}$ and $\vec{y}$.

(1) What is the relationship between linear automorphisms and affine linear automorphisms of $\mathbb{R}^n$?
  (A) Being a linear automorphism is precisely the same as being an affine linear automorphism.
  (B) Every linear automorphism is an affine linear automorphism, but not every affine linear automorphism is a linear automorphism.
  (C) Every affine linear automorphism is a linear automorphism, but not every linear automorphism is an affine linear automorphism.
  (D) A linear automorphism need not be affine linear, and an affine linear automorphism need not be linear.
  Answer: Option (B)
  Explanation: Linear automorphisms are precisely those affine linear automorphisms that send the origin to itself. In general, an affine linear automorphism can be expressed as a composite of a translation and a linear automorphism.
  Performance review: 18 out of 27 got this. 8 chose (C), 1 chose (A).
  Historical note (last time): 16 out of 27 got this. 10 chose (C), 1 chose (D).
(2) What is the relationship between self-homotheties and self-isometries of $\mathbb{R}^n$?
  (A) Being a self-homothety is precisely the same as being a self-isometry.
  (B) Every self-homothety is a self-isometry, but not every self-isometry is a self-homothety.
  (C) Every self-isometry is a self-homothety, but not every self-homothety is a self-isometry.
  (D) A self-homothety need not be a self-isometry, and a self-isometry need not be a self-homothety.
  Answer: Option (C)
  Explanation: We can define a self-isometry as a self-homothety where the factor of homothety (the factor of similitude) is 1.
  Performance review: 22 out of 27 got this. 4 chose (B), 1 chose (D).
  Historical note (last time): 21 out of 27 got this. 4 chose (D), 1 each chose (A) and (B).
(3) What is the relationship between self-homotheties and affine linear automorphisms of $\mathbb{R}^n$?
  (A) Being an affine linear automorphism is precisely the same as being a self-homothety.
  (B) Every affine linear automorphism is a self-homothety, but not every self-homothety is an affine linear automorphism.
  (C) Every self-homothety is an affine linear automorphism, but not every affine linear automorphism is a self-homothety.
  (D) An affine linear automorphism need not be a self-homothety, and a self-homothety need not be affine linear.
  Answer: Option (C)
  Explanation: A self-homothety must preserve ratios of distances. In particular, thanks to the triangle inequality, it must preserve collinearity with betweennness. In other words, if T is a self-homothety of $\mathbb{R}^n$ and A, B, C are points in $\mathbb{R}^n$ with B between A and C, then $T(A)$, $T(B)$, and $T(C)$ are also points in $\mathbb{R}^n$ with $T(B)$ between $T(A)$ and $T(C)$, because of the triangle inequality.
  There are affine linear automorphisms, such as shear automorphisms, that are not self-homotheties. Alternatively, we could also consider a dilation along one axis keeping the other axis fixed.
  Performance review: 17 out of 27 got this. 6 chose (B), 4 chose (A).
  Historical note (last time): 18 out of 27 got this. 6 chose (B), 2 chose (D), 1 chose (A).
(4) What is the relationship between affine linear automorphisms and self-isometries of $\mathbb{R}^n$?
  (A) Being an affine linear automorphism is precisely the same as being a self-isometry.
  (B) Every affine linear automorphism is a self-isometry, but not every self-isometry is an affine linear automorphism.

(C) Every self-isometry is an affine linear automorphism, but not every affine linear automorphism is a self-isometry.

(D) An affine linear automorphism need not be a self-isometry, and a self-isometry need not be affine linear.

Answer: Option (C)

Explanation: This can be deduced from the two preceding questions, and can also be seem directly.

Performance review: 18 out of 27 got this. 3 each chose (B) and (D), 2 chose (A), 1 left the question blank.

Historical note (last time): 22 out of 27 got this. 4 chose (D), 1 chose (A).

(5) There is a special kind of bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$ called a *translation*. A translation with translation vector $\vec{v}$ is defined as the bijection $\vec{x} \mapsto \vec{x} + \vec{v}$. A *nontrivial* translation is a translation whose translation vector is not the zero vector. Which of the following is an automorphism type that nontrivial translations are *not*? Please see Option (E) before answering.

(A) Linear automorphism

(B) Affine linear automorphism

(C) Self-isometry

(D) Self-homothety

(E) None of the above, i.e., nontrivial translations are of all these types

Answer: Option (A)

Explanation: Translations are self-isometries. Hence, they are also self-homotheties and affine linear automorphisms. They are not linear automorphisms because they do not preserve the origin.

Performance review: 10 out of 27 got this. 7 chose (E), 5 chose (C), 4 chose (D), 1 chose (B).

Historical note (last time): 12 out of 27 got this. 11 chose (E), 3 chose (C), and 1 chose (D).

(6) A collection of bijections from $\mathbb{R}^n$ to itself is said to form a *group* if it satisfies all these three conditions:

- The composite of any two (possibly equal, possibly distinct) bijections in the collection is also in the collection.
- The identity bijection (i.e., the map sending every vector to itself) is in the collection.
- For every bijection in the collection, the inverse bijection is also in the collection.

For fixed n, which of the following collections of bijections from $\mathbb{R}^n$ to itself does *not* form a group? Please see Option (E) before answering.

(A) The collection of all linear automorphisms of $\mathbb{R}^n$

(B) The collection of all affine linear automorphisms of $\mathbb{R}^n$

(C) The collection of all self-isometries of $\mathbb{R}^n$

(D) The collection of all self-homotheties of $\mathbb{R}^n$

(E) None of the above, i.e., each of them is a group

Answer: Option (E)

Explanation: This is obvious from the definition and concept of symmetry.

Performance review: 22 out of 27 got this. 2 each chose (B) and (D), 1 chose (C).

Historical note (last time): 24 out of 27 got this. 2 chose (C), 1 chose (D).

For the remaining questions, we deal with the case $n = 2$.

We consider two special types of bijections from $\mathbb{R}^2$ to $\mathbb{R}^2$: *rotations* (a rotation is specified by the center of rotation and the angle of rotation) and *reflections* (a reflection is specified by the line of reflection).

The identity map (i.e., the map sending every point to itself) is considered both a translation and a rotation. It is the translation by the zero vector. It can be viewed as a rotation about any point by the zero angle.

Note that for a rotation, the angle of rotation is determined uniquely up to additive multiples of 2π. The center of rotation is determined uniquely for all nontrivial rotations.

(7) What is the composite of two rotations centered at the same point in $\mathbb{R}^2$? Assume for simplicity that the composite is not the identity, i.e., the two rotations do not cancel each other. Note that the rotations must commute, so the order of operation does not matter.

(A) It must be a reflection about a line passing through that center point.

(B) It must be a reflection about a line *not* passing through that center point.

(C) It must be a rotation centered at the same point

(D) It must be a rotation but it need not be centered at the same point.

(E) It must be a translation.

Answer: Option (C)

Explanation: This is physically obvious.

Performance review: 25 out of 27 got this. 1 chose (D), 1 left the question blank.

Historical note (last time): 18 out of 27 got this. 5 chose (D), 3 chose (E), 1 chose (A).

(8) What is the composite of two reflections about lines in $\mathbb{R}^2$, if the two lines of reflection are known to be parallel but distinct? Although the two reflections do not commute, the *type* of their composite does not depend upon the order in which we compose them.

(A) It must be a reflection about a third line which is parallel to both the lines and is equidistant from them.

(B) It must be a reflection about a third line which is perpendicular to both the lines.

(C) It must be a rotation about a point that is equidistant from both lines.

(D) It must be a translation by a vector parallel to the lines about which we are reflecting.

(E) It must be a translation by a vector perpendicular to the lines about which we are reflecting.

Answer: Option (E)

Explanation: The vector will in fact be twice the perpendicular difference vector between the lines.

You can think about it in terms of double mirrors.

Performance review: 16 out of 27 got this. 6 chose (A), 3 chose (D), 1 chose (B).

Historical note (last time): 20 out of 27 got this. 3 chose (D), 2 each chose (A) and (C).

(9) What is the composite of two reflections about lines in $\mathbb{R}^2$, if the two lines of reflection are distinct and intersect? Once again, the reflections do not in general commute, but the *type* of the composite does not depend on the order of composition.

(A) It must be a reflection about a third line which passes through the point of intersection of the two lines of reflection.

(B) It must be a reflection about a third line which does not pass through the point of intersection of the two lines of reflection.

(C) It must be a rotation about the point of intersection.

(D) It must be a translation by a vector that makes equal angles with both the lines.

(E) It need not be a translation, rotation, or reflection.

Answer: Option (C)

Explanation: In fact, it will be a rotation with the angle of rotation equal to *twice* the angle between the lines. We can easily see this using basic Euclidean geometry. The simplest case to think of is the case of perpendicular lines of reflection. In this case, the composite is a rotation by π, also known as a half-turn.

Performance review: 13 out of 27 people got this. 8 chose (A), 2 each chose (D) and (E), 1 chose (B).

Historical note (last time): 18 out of 27 got this. 5 chose (A), 2 each chose (B) and (E).

(10) What is the composite of a nontrivial rotation in $\mathbb{R}^2$ (i.e., the angle of rotation is not a multiple of 2π) and a nontrivial translation?

(A) It must be a rotation with the same center of rotation but with a different angle of rotation.

(B) It must be a rotation with the same angle of rotation but with a different center of rotation.

(C) It must be a reflection about a line passing through the center of rotation.

(D) It must be a reflection about a line *not* passing through the center of rotation.

(E) It must be a translation.

Answer: Option (B)

Explanation: The angle of rotation must remain the same. We can see this by imagining some kind of physical figure that is made to undergo the rotation and then the translation. The rotation changes the direction of the figure by the angle of rotation. The translation then preserves the direction of the figure. The composite should also change the direction of the figure by the same

angle, hence must be a rotation by the same angle. The center of rotation may well be different, and in fact, must be different if the translation is nontrivial.

Performance review: 17 out of 27 got this. 3 each chose (A) and (E), 2 chose (D), 1 chose (C), 1 left the question blank.

Historical note (last time): 16 out of 27 got this. 7 chose (A), 2 chose (C), 1 each chose (D) and (E).

(11) An affine linear automorphism of $\mathbb{R}^2$ is termed *area-preserving* if it preserves areas, i.e., the area of the image of any triangle under the automorphism is the same as the area of the original triangle. What is the relation between being a self-isometry and being an area-preserving affine linear automorphism of $\mathbb{R}^2$?

(A) Being a self-isometry is precisely the same as being an area-preserving affine linear automorphism.

(B) Every self-isometry is area-preserving, but not every area-preserving affine linear automorphism is a self-isometry.

(C) Every area-preserving affine linear automorphism is a self-isometry, but not every self-isometry is area-preserving.

(D) A self-isometry need not be an area-preserving affine linear automorphism, and an area-preserving affine linear automorphism need not be a self-isometry.

Answer: Option (B)

Explanation: If something preserves lengths, it sends triangles to congruent triangles, hence it also preserves areas.

There can be area-preserving affine linear automorphisms that are not self-isometries. Shear operations, such as the one with this matrix:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are examples. The point is that areas still get preserved. For instance, a triangle based on the x-axis gets bent out of shape, but its base and height remain the same, hence its area remains the same.

Performance review: 12 out of 27 got this. 9 chose (A), 4 chose (C), 1 chose (D), 1 left the question blank.

Historical note (last time): 15 out of 27 got this. 5 chose (A), 3 each chose (C) and (D), 1 left the question blank.

(12) An affine linear automorphism of $\mathbb{R}^2$ is termed *orientation-preserving* if it preserves orientation, i.e., it does not interchange left with right. An affine linear automorphism of $\mathbb{R}^2$ is termed *orientation-reversing* if it reverses orientation, i.e., it interchanges the roles of left and right. Obviously, the composite of two orientation-preserving affine linear automorphisms is orientation-preserving. What can we say about the composite of two orientation-reversing affine linear automorphisms?

(A) It must be orientation-preserving

(B) It must be orientation-reversing

(C) It may be orientation-preserving or orientation-reversing

Answer: Option (A)

Explanation: Reversing the orientation means switching the roles of left and right. Reversing the orientation a second time means switching the roles back. Thus, the composite of orientation-preserving.

Performance review: 20 out of 27 got this. 6 chose (C), 1 left the question blank.

Historical note (last time): 24 out of 27 got this. 2 chose (B), 1 chose (C).

(13) The linear automorphism of $\mathbb{R}^2$ with matrix:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is an example of a *shear automorphism*. Which of the following is this automorphism *not*? Please see options (D) and (E) before answering.

(A) Area-preserving

(B) Orientation-preserving

(C) Self-isometry

(D) None of the above, i.e., it is area-preserving, orientation-preserving, and a self-isometry

(E) All of the above, i.e., it is not area-preserving, not orientation-preserving, and not a self-isometry of $\mathbb{R}^2$.

Answer: Option (C)

Explanation: The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The original vector had length 1, but the new vector has length $\sqrt{2}$, so the map does not preserve length. It does preserve area and orientation.

Performance review: 15 out of 27 got this. 6 chose (E), 3 chose (D), 2 chose (A).

Historical note (last time): 16 out of 27 got this. 4 each chose (A) and (D), 3 chose (B).

(14) Which of the following is guaranteed to send any triangle in $\mathbb{R}^2$ to a similar triangle? Please see Options (D) and (E) before answering.

(A) Linear automorphism

(B) Affine linear automorphism

(C) Self-homothety

(D) All of the above

(E) None of the above

Answer: Option (C)

Explanation: Follows from the definition.

Performance review: 13 out of 27 got this. 10 chose (D), 2 chose (E), 1 chose (B), 1 left the question blank.

Historical note (last time): 20 out of 27 got this. 5 chose (D), 1 each chose (A) and (E).