DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 18: LINEAR TRANSFORMATIONS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

27 people took this 5-question quiz. The score distribution was as follows:

- Score of 1: 2 people
- Score of 2: 4 people
- Score of 3: 5 people
- Score of 4: 8 people
- Score of 5: 8 people

The mean score was 3.59 out of 5.

The question-wise answers and performance review were as follows:

(1) Option (B): 24 people
(2) Option (B): 22 people
(3) Option (D): 19 people
(4) Option (E): 19 people
(5) Option (D): 13 people

2. Solutions

PLEASE DO *NOT* DISCUSS ANY QUESTIONS.

The quiz covers basics related to linear transformations (notes titled `Linear transformations`, corresponding section in the book Section 2.1). Explicitly, the quiz covers:

- Representation of a linear transformation using a matrix, and identifying the domain and co-domain in terms of the row and column counts of the matrix.
- Relationship between injectivity, surjectivity, rank, row count, and column count.
- Relationship between the entries of the matrix and the images of the standard basis vectors under the corresponding linear transformation.

The question are fairly easy if you understand the material. But it's important that you be able to answer them, otherwise what we study later will not make much sense.

(1) *Do not discuss this!*: Which of the following correctly describes a $m \times n$ matrix?
  (A) There are m rows, and each row gives a vector with m coordinates. There are n columns, and each column gives a vector with n coordinates.
  (B) There are m rows, and each row gives a vector with n coordinates. There are n columns, and each column gives a vector with m coordinates.
  (C) There are n rows, and each row gives a vector with m coordinates. There are m columns, and each column gives a vector with n coordinates.
  (D) There are n rows, and each row gives a vector with n coordinates. There are m columns, and each column gives a vector with m coordinates.

  Answer: Option (B)

  Explanation: This should be obvious by looking at the matrix. For instance, a 2×3 matrix is of the form:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Performance review: 24 out of 27 got this. 3 chose (A).

(2) *Do not discuss this!*: For a $p \times q$ matrix A, we can define a linear transformation T_A by $T_A(\vec{x}) := A\vec{x}$. What type of linear transformation is T_A?
(A) T_A is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^q$
(B) T_A is a linear trnasformation from $\mathbb{R}^q$ to $\mathbb{R}^p$
(C) T_A is a linear transformation from $\mathbb{R}^{\max\{p,q\}}$ to $\mathbb{R}^{\min\{p,q\}}$
(D) T_A is a linear transformation from $\mathbb{R}^{\min\{p,q\}}$ to $\mathbb{R}^{\max\{p,q\}}$

Answer: Option (B)

Explanation: The way matrix-vector multiplication works, a $p \times q$ matrix multiplies with a $q \times 1$ vector to give a $p \times 1$ vector. Thus the linear transformation takes as input a q-dimensional vector and gives as output a p-dimensional vector, and is hence a transformation from $\mathbb{R}^q$ to $\mathbb{R}^p$.

Performance review: 22 out of 27 got this. 3 chose (A), 2 chose (C).

(3) *Do not discuss this!*: With the same notation as for the preceding question, which of the following is true?
(A) If $p < q$, T_A must be injective
(B) If $p > q$, T_A must be injective
(C) If $p = q$, T_A must be injective
(D) If $p < q$, T_A cannot be injective
(E) If $p > q$, T_A cannot be injective

Answer: Option (D)

Explanation: For the linear transformation T_A to be injective, the matrix A needs to have full column rank q, because that is what it means for there to be no non-leading variables and for the solution to therefore be unique if it exists (see the lecture notes for more). In other words, we require that the matrix have rank q. However, we know that the rank of a matrix is at most equal to the minimum of the number of rows and the number of columns. Thus, if $p < q$, the matrix cannot have full column rank, and the linear transformation cannot be injective.

Intuitively, the linear transformation goes from $\mathbb{R}^q$ to $\mathbb{R}^p$, so in order for it to be injective, the target space should be at least as big as the domain. Thus, $p < q$ is incompatible with injectivity.

Performance review: 19 out of 27 got this. 4 chose (E), 3 chose (C), 1 left the question blank.

(4) *Do not discuss this!*: With the same notation as for the previous two questions, which of the following is true?
(A) If $p < q$, T_A must be surjective
(B) If $p > q$, T_A must be surjective
(C) If $p = q$, T_A must be surjective
(D) If $p < q$, T_A cannot be surjective
(E) If $p > q$, T_A cannot be surjective

Answer: Option (E)

Explanation: For the linear transformation T_A to be surjective, the matrix A needs to have full row rank p, because we want the system to always be consistent and therefore we want that there should be no zero rows in the rref. We also know that the rank of the matrix is at most equal to the minimum of the number of rows and number of columns. Therefore, if $p > q$, A cannot have full row rank and T_A cannot be surjective.

Performance review: 19 out of 27 got this. 5 chose (D), 2 chose (C).

(5) *Do not discuss this!*: With the same notation as for the last three questions, which of the following is true?
(A) The rows of A are the images under T_A of the standard basis vectors of $\mathbb{R}^p$.
(B) The columns of A are the images under T_A of the standard basis vectors of $\mathbb{R}^p$.
(C) The rows of A are the images under T_A of the standard basis vectors of $\mathbb{R}^q$.
(D) The columns of A are the images under T_A of the standard basis vectors of $\mathbb{R}^q$.

Answer: Option (D)

Explanation: See the lecture notes for details. Note, however, that dimension considerations can get the answer here immediately. T_A is a map from $\mathbb{R}^q$ to $\mathbb{R}^p$, so only Options (C) and (D) make sense from the domain perspective. Further, the rows of A are q-dimensional whereas the columns are p-dimensional, so only the columns have the right dimension, thus making (D) the only legitimate option.

Performance review: 13 out of 27 got this. 11 chose (D), 2 chose (C), 1 chose (A).