TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY NOVEMBER 13: MATRIX MULTIPLICATION: ROWS, COLUMNS, ORTHOGONALITY, AND OTHER MISCELLANEA

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

25 people took this 13-question quiz. The score distribution was as follows:

- Score of 4: 2 people
- Score of 5: 1 person
- Score of 6: 1 person
- Score of 7: 3 people
- Score of 8: 3 people
- Score of 9: 5 people
- Score of 10: 4 people
- Score of 11: 4 people
- Score of 12: 2 people

The mean score was 8.68.

The question-wise answers and performance review are as follows:

(1) Option (D): 24 people
(2) Option (C): 22 people
(3) Option (D): 22 people
(4) Option (E): 19 people
(5) Option (A): 17 people
(6) Option (B): 23 people
(7) Option (E): 14 people
(8) Option (E): 20 people
(9) Option (C): 7 people
(10) Option (A): 12 people
(11) Option (B): 7 people
(12) Option (E): 14 people
(13) Option (B): 16 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The purpose of this quiz is two-fold. First, many of the ideas related to matrix multiplication are at the stage where a bit of review will help prevent their fading out. Drawing from the best research on *spaced repetition* (see for instance `http://en.wikipedia.org/wiki/Spaced_repetition`) we will try to recall some of the stuff. But with a twist, because we consider it from a somewhat different angle.

Second, the new angle will also turn out to be useful for later material.

For Questions 1-5: Given a n-dimensional vector $\langle a_1, a_2, \dots, a_n \rangle \in \mathbb{R}^n$, the vector can be interpreted as a $n \times 1$ matrix (a column vector). This is the default interpretation. But there are also two other interpretations: as a $1 \times n$ matrix (a row vector) and as a diagonal $n \times n$ matrix.

Also note that for Questions 1-5, all the three ways of representing vectors coincide with each other for $n = 1$, so the questions are uninteresting for $n = 1$ because all answer options are equivalent. You may therefore assume that $n > 1$ for these questions, though obviously the correct answers are correct for $n = 1$ as well.

(1) Suppose I want to add two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ to obtain the output vector $\langle a_1 + b_1, a_2 + b_2, \dots, a_n + b_n \rangle$ using matrix addition. What format (row vector, column vector, or diagonal matrix) should I use? Please see Option (D) before answering and select the option that best describes your view.
(A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors and interpret the sum as a row vector.
(B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors and interpret the sum as a column vector.
(C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices and interpret the sum as a diagonal matrix.
(D) We can use any of the above.
Answer: Option (D)
Explanation: Regardless of whether we represent the vectors as row vectors, column vectors, or diagonal matrices, matrix addition is executed coordinate-wise, as desired.
Performance review: 24 out of 25 people got this. 1 chose (B).

(2) Suppose I want to perform coordinate-wise multiplication on two vectors. Explicitly, I have two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ and I want to obtain the output vector $\langle a_1 b_1, a_2 b_2, \dots, a_n b_n \rangle$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left and the matrix for $\vec{b}$ written on the right). What format (row vector, column vector, or diagonal matrix) should I use? Please see Option (D) before answering and select the option that best describes your view.
(A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors and interpret the matrix product as a row vector.
(B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors and interpret the matrix product as a column vector.
(C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices and interpret the matrix product as a diagonal matrix.
(D) We can use any of the above.
Answer: Option (C)
Explanation: The multiplication of diagonal matrices is coordinate-wise along the diagonal. For instance, when $n = 2$, we get:

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{bmatrix}$$

In general:

$$\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & a_n \end{bmatrix} \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & a_n b_n \end{bmatrix}$$

Note that for Options (A) and (B), it does not even make sense to try computing the product.
Option (A): Here, $\vec{a}$ and $\vec{b}$ are both represented by n-dimensional row vectors, i.e., $1 \times n$ matrices, so we cannot multiply them because the number of columns of the first matrix does not equal the number of rows of the second matrix.
Option (B): Here, $\vec{a}$ and $\vec{b}$ are both represented by n-dimensional column vectors, i.e., $n \times 1$ matrices, so we cannot multiply them because the number of columns of the first matrix does not equal the number of rows of the second matrix.
Performance review: 22 out of 25 got this. 3 chose (D).

(3) Suppose I am given two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ and I want to obtain a 1×1 matrix with entry $\sum_{i=1}^{n} a_i b_i$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left and the matrix for $\vec{b}$ written on the right). What format (row vector, column vector, or diagonal matrix) should I use?
(A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors.
(B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors.
(C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices.

(D) Represent $\vec{a}$ as a row vector and $\vec{b}$ as a column vector.
(E) Represent $\vec{a}$ as a column vector and $\vec{b}$ as a row vector.

Answer: Option (D)

Explanation: The dimensions match: $\vec{a}$ is represented by a $1 \times n$ matrix and $\vec{b}$ is represented by a $n \times 1$ matrix. The product definition also matches. Note that this particular form of product is the *dot product* of the vectors.

Note that Option (C) would give a diagonal matrix with the products $a_i b_i$ along the diagonal, but would not add them up. Options (A) and (B) do not make sense, for the same reason as discussed in the answer to Question 2. Option (E) would give a product that is a $n \times n$ matrix whose $(ij)^{th}$ entry is the product $a_i b_j$. The matrix described by Option (E) is termed the *Hadamard product*.

Performance review: 22 out of 25 got this. 2 chose (E), 1 left the question blank.

(4) Suppose I am given three vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$, $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$, and $\vec{c} = \langle c_1, c_2, \dots, c_n \rangle$. I want to obtain a 1×1 matrix with entry $\sum_{i=1}^{n} (a_i b_i c_i)$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left, the matrix for $\vec{b}$ written in the middle, and the matrix for $\vec{c}$ written on the right). What format should I use?
(A) $\vec{a}$ as a row vector, $\vec{b}$ as a column vector, $\vec{c}$ as a diagonal matrix.
(B) $\vec{a}$ as a column vector, $\vec{b}$ as a row vector, $\vec{c}$ as a diagonal matrix.
(C) $\vec{a}$ as a diagonal matrix, $\vec{b}$ as a row vector, $\vec{c}$ as a column vector.
(D) $\vec{a}$ as a column vector, $\vec{b}$ as a diagonal matrix, $\vec{c}$ as a row vector.
(E) $\vec{a}$ as a row vector, $\vec{b}$ as a diagonal matrix, $\vec{c}$ as a column vector.

Answer: Option (E)

Explanation: First, note that the dimensions match. $\vec{a}$ is represented by a $1 \times n$ matrix, $\vec{b}$ by a diagonal $n \times n$ matrix, and $\vec{c}$ by a $n \times 1$ matrix. The multiplication makes sense, and the product is a 1×1 matrix.

Second, note that the matrix we get is the correct one. The product of the diagonal matrix for $\vec{b}$ and the column vector for $\vec{c}$ gives a column vector whose i^{th} coordinate is $b_i c_i$. The product of the row vector for $\vec{a}$ with this column vector gives $\sum_{i=1}^{n} (a_i b_i c_i)$ as desired.

Here is how it looks in the case $n = 2$:

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

If we begin our simplification by multiplying the second and third matrix, we obtain:

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 c_1 \\ b_2 c_2 \end{bmatrix}$$

We now do the remaining multiplication and obtain the desired 1×1 matrix:

$$\begin{bmatrix} a_1 b_1 c_1 + a_2 b_2 c_2 \end{bmatrix}$$

Alternatively, we could simplify the original product by multiplying the first two matrices to begin with, and obtain:

$$\begin{bmatrix} a_1 b_1 & a_2 b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Multiplying them out gives the desired result. *Note*: It's not surprising that both ways of simplifying the product of three matrices give the same result. That follows from associativity of matrix multiplication. We did it both ways for a sanity check.

Performance review: 19 out of 25 got this. 2 each chose (A) and (C). 1 each chose (B) and (D).

The next few questions rely on the concept of orthogonality (*orthogonal* is a synonym for *perpendicular* or *at right angles*). We say that two vectors (of the same dimension) are orthogonal if their dot product is zero. By this definition, the zero vector of a given dimension is orthogonal to every vector of that dimension. Note that it does not make sense to talk of orthogonality for vectors with different dimensions, i.e., with different numbers of coordinates.

(5) Suppose A is a $n \times m$ matrix. We can think of solving the system $A\vec{x} = \vec{0}$ (where $\vec{x}$ is a $m \times 1$ column vector of unknowns) as trying to find all the vectors orthogonal to all the vectors in a given set of vectors. What set of vectors is that?

(A) The set of row vectors of A, i.e., the rows of A, viewed as m-dimensional vectors.
(B) The set of column vectors of A, i.e., the columns of A, viewed as n-dimensional vectors.

Answer: Option (A)

Explanation: $A\vec{x}$ is a $n \times 1$ matrix (i.e., a column vector with n coordinates) and its i^{th} entry is the dot product of the i^{th} row of A and the vector $\vec{x}$, both of which are m-dimensional vectors. This is zero if and only if the i^{th} row of A and the vector $\vec{x}$ are orthogonal to each other. In order to have $A\vec{x} = \vec{0}$, we need $\vec{x}$ to be orthogonal to all the rows of A.

Performance review: 17 out of 25 got this. 8 chose (B).

(6) Suppose A is a $p \times q$ matrix and B is a $q \times r$ matrix where p, q, and r are positive integers. The matrix product AB is a $p \times r$ matrix. What orthogonality condition corresponds to the condition that the matrix product AB is a zero matrix (i.e., all its entries are zero)?

(A) Every row of A is orthogonal to every row of B.
(B) Every row of A is orthogonal to every column of B.
(C) Every column of A is orthogonal to every row of B.
(D) Every column of A is orthogonal to every column of B.

Answer: Option (B)

Explanation: The ik^{th} entry of AB can be viewed as the dot product of the i^{th} row of A and the k^{th} column of B. In particular, this entry is zero if and only if the i^{th} row of A is orthogonal to the k^{th} column of B. In order for AB to be the zero matrix, we need this to hold for *every* row of A and *every* column of B, giving the answer option.

Performance review: 23 out of 25 got this. 2 chose (C).

(7) Suppose A is an invertible $n \times n$ square matrix. Which of the following correctly characterizes the $n \times n$ matrix A^{-1} using orthogonality? Recall that AA^{-1} and $A^{-1}A$ are both equal to the $n \times n$ identity matrix.

(A) For every i in $\{1, 2, \dots, n\}$, the i^{th} row of A is orthogonal to the i^{th} row of A^{-1}. The dot product of the i^{th} row of A and the j^{th} row of A^{-1} for distinct i, j in $\{1, 2, \dots, n\}$ equals 1.
(B) For every i in $\{1, 2, \dots, n\}$, the i^{th} column of A is orthogonal to the i^{th} column of A^{-1}. The dot product of the i^{th} row of A and the j^{th} column of A^{-1} for distinct i, j in $\{1, 2, \dots, n\}$ equals 1.
(C) For every distinct i, j in $\{1, 2, \dots, n\}$, the i^{th} row of A is orthogonal to the j^{th} row of A^{-1}. The dot product of the i^{th} row of A with the i^{th} row of A^{-1} equals 1.
(D) For every i in $\{1, 2, \dots, n\}$, the i^{th} row of A is orthogonal to the i^{th} column of A^{-1}. The dot product of the i^{th} row of A and the j^{th} column of A^{-1} for distinct i, j in $\{1, 2, \dots, n\}$ equals 1.
(E) For every distinct i, j in $\{1, 2, \dots, n\}$, the i^{th} row of A is orthogonal to the j^{th} column of A^{-1}. The dot product of the i^{th} row of A and the i^{th} column of A^{-1} equals 1.

Answer: Option (E)

Explanation: The product AA^{-1} is the identity matrix. The entries of this matrix can be described as follows:

- For distinct i, j in $\{1, 2, \dots, n\}$, the $(ij)^{th}$ entry of AA^{-1} is 0. This translates to saying that the i^{th} row of A is orthogonal to the j^{th} column of A^{-1}.
- For any i in $\{1, 2, \dots, n\}$, the $(ii)^{th}$ entry of AA^{-1} is 1. This translates to saying that the dot product of the i^{th} row of A and the i^{th} column of A^{-1} is equal to 1.

These correspond to Option (E).

Note that it is *also* true that $A^{-1}A$ is the identity matrix. We can use this to obtain an alternative characterization of A^{-1}. This condition will use the rows of A^{-1} and the columns of A. Explicitly:

- For distinct i, j in $\{1, 2, \dots, n\}$, the $(ij)^{th}$ entry of $A^{-1}A$ is 0. This translates to saying that the i^{th} row of A^{-1} is orthogonal to the j^{th} column of A.

- For any i in $\{1, 2, \dots, n\}$, the $(ii)^{th}$ entry of $A^{-1}A$ is 1. This translates to saying that the dot product of the i^{th} row of A^{-1} and the i^{th} column of A is equal to 1.

Performance review: 14 out of 25 got this. 6 chose (C), 3 chose (D), 1 each chose (A) and (B).

The remaining questions review your skills at abstract behavior prediction.

(8) Suppose n is a positive integer greater than 1. Which of the following is always true for two invertible $n \times n$ matrices A and B?
(A) $A + B$ is invertible, and $(A + B)^{-1} = A^{-1} + B^{-1}$
(B) $A + B$ is invertible, and $(A + B)^{-1} = B^{-1} + A^{-1}$
(C) $A + B$ is invertible, though neither of the formulas of the preceding two options is correct
(D) AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$
(E) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

Answer: Option (E)

Explanation: A and B being invertible does not imply that $A + B$ is invertible. For instance, A may be the identity matrix and B may be its negative.

On the other hand, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. This is because when we invert (i.e., go backward) we must do so in reverse.

Performance review: 20 out of 25 got this. 3 chose (D), 1 each chose (A) and (B).

Historical note (last time, appeared in a midterm): 21 out of 30 people got this. 5 chose (C), 4 chose (D).

(9) Suppose n is a positive integer greater than 1. For a nilpotent $n \times n$ matrix C, define the *nilpotency* of C as the smallest positive integer r such that $C^r = 0$. Note that the nilpotency is not defined for a non-nilpotent matrix. Given two $n \times n$ matrices A and B, what is the relation between the nilpotencies of AB and BA?
(A) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must be equal.
(B) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must differ by 1.
(C) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must either be equal or differ by 1.
(D) It is possible for AB to be nilpotent and BA to be non-nilpotent; however, *if* both are nilpotent, then their nilpotencies must be equal.
(E) It is possible for AB to be nilpotent and BA to be non-nilpotent, however *if* both are nilpotent, then their nilpotencies must differ by 1.

Answer: Option (C)

Explanation: We have seen examples where the nilpotencies are not equal. For instance:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = 0$ but BA is not zero.

On the other hand, we also have examples where the nilpotencies are equal. For instance:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note, however, that $(AB)^r = 0$ implies $(BA)^{r+1} = 0$ and $(BA)^s = 0$ implies $(AB)^{s+1} = 0$. Thus, the nilpotencies can differ by at most one, since each nilpotency is bounded by 1 more than the other.

Performance review: 7 out of 25 got this. 9 chose (D), 4 chose (E), 2 chose (A), 1 chose (B), and 2 left the question blank.

Historical note (last time, appeared in a midterm): 5 out of 30 got this. 12 chose (E), 11 chose (D), and 2 chose (B).

(10) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$?
(A) 1

(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Answer: Option (A)

Explananation: We can take $A = [-1]$ and $B = [1]$.

Performance review: 12 out of 25 got this. 10 chose (B), 2 chose (E), 1 chose (C).

(11) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^3 = B^3$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Answer: Option (B)

Explanation: Note first that $n = 1$ does not work, because if two real numbers have the same cube, they must be equal (this is because cubing is a one-one function).

However, $n = 2$ works. Explicitly, we can take A as a rotation matrix by $2\pi/3$ and B as the identity matrix. Both A^3 and B^3 equal the identity matrix. Explicitly:

$$A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If you prefer dealing only with matrices with integer entries, consider the following matrix. This is harder to think of, however:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Performance review: 7 out of 25 got this. 13 chose (C), 4 chose (E), 1 chose (D).

(12) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$ and $A^3 = B^3$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Answer: Option (E)

Explanation: Suppose $A^2 = B^2$ with A and B both invertible. Then, $A^{-2} = B^{-2}$ as well. We are also given $A^3 = B^3$. Multiplying the two equations, we get $A = B$. Thus, the specification required is not possible.

Performance review: 14 out of 25 got this. 7 chose (B), 3 chose (C), 1 chose (D).

(13) What is the smallest n for which there exist examples of (not necessarily invertible) $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$ and $A^3 = B^3$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Answer: Option (B)

Explanation: Note that the condition $A^3 = B^3$ would imply $A = B$ if $n = 1$. Therefore, the smallest possible case is $n = 2$. We will furnish an example in this case. In our example, we take:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note how and why this example works: $A^2 = B^2 = 0$, so all higher powers of A and of B are equal to 0.

Performance review: 16 out of 25 got this. 5 chose (E), 3 chose (C), 1 chose (A).