TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY OCTOBER 7: LINEAR FUNCTIONS AND EQUATION-SOLVING (PART 2)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

28 people took this 5-question quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 3 people
- Score of 2: 9 people
- Score of 3: 5 people
- Score of 4: 6 people
- Score of 5: 2 people

The question-wise answers and performance summary are given below.

(1) Option (C): 25 people
(2) Option (D): 12 people
(3) Option (E): 12 people
(4) Option (C): 15 people
(5) Option (D): 6 people

On comparison with last time: Last year, I had set the due date for the take-home quiz as Wednesday rather than Monday, so that might partly explain the better performance of students on the quiz last time. The main question where performance diverged considerable from last time was Q5, and this is the question where the sophistication of one extra class can go a long way.

2. Solutions

This quiz covers some basics involving linear functions and equation-solving (notes at `Linear functions: a primer` and `Equation-solving with a special focus on the linear case`). The quiz tests for the following:

- The distinction between behavior relative to the variables (the inputs) and behavior relative to the parameters.
- Counting the number of parameters by creating the explicit general functional form from a verbal description (with a special focus on polynomial functional forms).
- Figuring out how to "ask the right questions" with respect to input choices, so that the answers provide meaningful information. This builds towards the ideas of hypothesis testing, rank, and overdetermination that we will see in the future.

(1) Suppose f is a polynomial function of x of degree at most a *known number* n. What is the minimum number of (input,output) pairs that we need in order to determine f uniquely? *Extra information: Somewhat surprisingly, in this case, we do not need to be judicious about our input choices. Any set of distinct inputs of the required number will do. This has something to do with the "Vandermonde matrix" and "Vandermonde determinant" and is also related to the Lagrange interpolation formula.*
(A) $n-1$
(B) n
(C) $n+1$
(D) $2n$
(E) n^2

Answer: Option (C)

Explanation: There are $n+1$ unknown coefficients, namely, the coefficients of x^i for $0 \le i \le n$. In general, a polynomial of degree n is of the form:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Every input-output pair gives rise to a linear equation in terms of the coefficients. With $n+1$ input-output pairs, we get $n+1$ equations in $n+1$ variables where the "variables" here are the parameters). Due to linearity in the parameters, we actually get a system of $n+1$ *linear* equations in the $n+1$ variables. We would expect the system to have a unique solution if the inputs are chosen judiciously.

In fact, it turns out that the system always has a unique solution. The existence of a solution is given by the Lagrange interpolation formula, and its uniqueness follows from the fact that any polynomial of degree $\le n$ that has $n+1$ distinct roots must be the zero polynomial. This is also related to the idea of the Vandermonde determinant, but that is beyond the scope of the current discussion.

Update for after understanding coefficient matrices and ranks: Choosing the inputs judiciously basically amounts to choosing the inputs in a manner that the coefficient matrix (a $(n+1)\times(n+1)$ square matrix) has full rank $n+1$. The statement above, that distinct inputs always work, is the statement that the coefficient matrix always has full rank $n+1$ as long as the inputs are distinct. This type of matrix is called a Vandermonde matrix and there is a considerable theory related to these in algebra.

Performance review: 25 out of 28 got this. 3 chose (B).

Historical note (last time): 27 out of 28 got this. 1 chose (B).

(2) f is a polynomial function of two variables x and y of total degree at most 2. In other words, for each monomial occurring in f, the total of the degrees of x and y in that monomial is at most 2. No other information is given about f. What is the minimum number of judiciously chosen (input,output) pairs we need in order to determine f uniquely?

(A) 2
(B) 3
(C) 4
(D) 6
(E) 7

Answer: Option (D)

Explanation: First, a little bit to clarify the question. The total degree of a monomial $x^i y^j$ is $i+j$. For instance, the degree of x^3y^2 is $3+2=5$.

The total degree of a polynomial that involves addition and scalar multiplication of monomials is the maximum of the degrees of the individual monomials. This is very similar to the situation with polynomials of one variable: the degree of the polynomial is the maximum of the degrees of the constituent monomials. Thus, for instance, $x - y + 17xy - x^2y$ has total degree 3 because the constituent monomials have degree 1, 1, 2, and 3, and the maximum of these is 3.

In order to determine how many input-output pairs we would need, we need to count the number of parameters in the generic functional form. So, the first step is to figure out a *general* expression for all the possibilities we can have such polynomials of total degree at most 2. Since all such polynomials are obtained by (linearly) combining monomials of the sort, our first step is to list the monomials that are allowed.

The possible monomials involved in f are 1, x, y, x^2, xy, and y^2. There are six of them, hence there are 6 coefficients to be determined by setting up and solving the linear system. The generic form is:

$$a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

Each (input,output) pair gives a *linear* equation in terms of $a_1, a_2, \dots, a_6$. Six suitably chosen inputs will give six linear equations that we can then solve to uniquely determine the inputs. Note that unlike the functions of one variable case, it is possible to choose the inputs "badly", i.e., in a way that does not reveal information.

Update for after understanding coefficient matrices and ranks: Choosing the inputs judiciously basically amounts to choosing the inputs in a manner that the coefficient matrix (a 6×6 square matrix) has full rank 6. In this case, a bad choice of inputs could lead to a coefficient matrix of lower rank, leading to redundant equations. Unfortunately, it is beyond the current scope to describe the geometry of sets of inputs for which the system does not have full rank (though one example would be where all inputs have the same y-value), but anyway, "most" random choices of inputs will give systems with full rank.

Performance review: 12 out of 28 got this. 6 chose (B), 5 chose (C), 4 chose (A), 1 chose (E).

Historical note (last time): 15 out of 28 got this. 7 chose (B), 3 chose (A), 2 chose (C), 1 chose (E).

(3) f is a polynomial function of two variables x and y of total degree at most 3. In other words, for each monomial occurring in f, the total of the degrees of x and y in that monomial is at most 3. No other information is given about f. What is the minimum number of judiciously chosen (input,output) pairs we need in order to determine f uniquely?
(A) 3
(B) 6
(C) 8
(D) 9
(E) 10

Answer: Option (E)

Explanation: In addition to the 6 monomials for the preceding question, there are 4 monomials of degree three, namely x^3, x^2y, xy^2, and y^3. Thus, there is a total of 10 monomials with unknown coefficients, so we need 10 data points to pin down the values of the coefficients. Explicitly, the general form looks like:

$$a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3$$

Each (input,output) pair gives a *linear* equation in terms of $a_1, a_2, \dots, a_{10}$. We want 10 such equations, so we want 10 judiciously chosen input-output pairs.

Update for after understanding coefficient matrices and ranks: Choosing the inputs judiciously basically amounts to choosing the inputs in a manner that the coefficient matrix (a 10×10 square matrix) has full rank 10. In this case, a bad choice of inputs could lead to a coefficient matrix of lower rank, leading to redundant equations. Unfortunately, it is beyond the current scope to describe the geometry of sets of inputs for which the system does not have full rank (though one example would be where all inputs have the same y-value), but anyway, "most" random choices of inputs will give systems with full rank.

Performance review: 12 out of 28 got this. 7 chose (A), 5 chose (B), 2 each chose (C) and (D).

Historical note (last time): 14 out of 28 got this. 8 chose (A), 2 each chose (B), (C), and (D).

(4) (*) *The perils of overfitting; see also Occam's Razor*: Suppose we are trying to model a function that we expect to behave in a polynomial-like manner, though we don't really have a good reason to believe this. Additionally, there is a possibility for measurement error in our observations. Our goal is to find the parameters so that we can both predict unmeasured values and do a qualitative analysis of the nature of the function and its derivatives and integrals.

We have a large number of observations (say, several thousands). We could attempt to "fit" the function using a polynomial of degree n for some fixed n using all those data points, and we will get a certain "best fit" that minimizes the deviation between the curve used for fitting and the function being fit. For instance, for $n = 1$, we are trying to find the best fit by a straight line function. For $n = 2$, we are trying to find the best fit by a polynomial of degree at most 2. We could try fitting using different values of n. Which of the following is true?

If you are interested in more on this, look up "overfitting". A revealing quote is by mathematician and computer scientist John von Neumann: "With four parameters I can fit an elephant. And with five I can make him wiggle his trunk." Another is by prediction guru Nate Silver: "The wide array of *statistical methods available to researchers enables them to be no less fanciful and no more scientific than a child finding animal patterns in clouds."*

(A) Larger values of n give better fits, therefore the larger the value of n we use, the better.
(B) Smaller values of n give better fits, therefore the smaller the value of n we use, the better.
(C) Larger values of n give better fits, therefore the larger the value of n we use, the less impressive a good fit (i.e., low deviation between the polynomial and the actual set of observations) should be.
(D) Smaller values of n give better fits, therefore the smaller the value of n we use, the less impressive a good fit (i.e., low deviation between the polynomial and the actual set of observations) should be.
(E) The value of n we use for trying to get a good fit is irrelevant. A good fit is a good fit, regardless of the type of function used.

Answer: Option (C)

Explanation: Larger values of n mean more parameters, and we can use more parameters to get a better fit, generally speaking. However, that better fit may well be *fitting the noise in the measurements rather than the signal*. Even without measurement error, if we do not have *a priori* theoretical reasons to be sure of the model, it may just be fitting idiosyncracies of the observed values that do not extend to other values. For both these reasons, using too many parameters gives a misleading sense of complacency based on what appears like a good fit, but is a result of sheer chance.

For instance, *any* collection of n data points can be fitted (without need for error tolerance!) on a polynomial of degree at most $n-1$. But what if there were actually measurement error? Then that polynomial of degree $n-1$ would be a fake good fit. Imagine that the actual function is $f(x) = x$, but we are measuring values near 0 and the best fit for the measured values turns out to be $f(x) = x + x^3$. This may very well be a much better fit for values close to the origin because of biased measurement errors, but extrapolating it to a larger domain could go really awry.

Performance review: 15 out of 28 got this. 9 chose (E), 2 chose (B), 1 chose (A), and 1 did not attempt the question.

Historical note (last time): 14 out of 28 people got this. 9 chose (D), 4 chose (E), and 1 chose (A).

(5) (*) F is an affine linear function of two variables x and y, i.e., it has the form $F(x, y) := ax + by + c$ with a, b, and c real numbers. We want to determine the values of the parameters a, b, and c by using input-output pairs. It is, however, costly to find input-output pairs. We have already found $F(1, 3)$ and $F(3, 7)$. We want to find F for one other pair of inputs to determine a, b, and c. Which of these will *not* be a good choice?

(A) $F(2, 2)$, i.e., the input $x = 2$, $y = 2$
(B) $F(2, 3)$, i.e., the input $x = 2$, $y = 3$
(C) $F(2, 4)$, i.e., the input $x = 2$, $y = 4$
(D) $F(2, 5)$, i.e., the input $x = 2$, $y = 5$
(E) $F(2, 6)$, i.e., the input $x = 2$, $y = 6$

Answer: Option (D)

Explanation: The point $(2, 5)$ is collinear with the points $(1, 3)$ and $(3, 7)$ (in fact, it is their midpoint) so the value at that point can be predicted based on the values at $(1, 3)$ and $(3, 7)$ as being the arithmetic mean between these values. Thus, it does not provide new information. Mathematically, if we use this point to get the third equation, that equation will be redundant with the existing equations. In equational form:

$$F(2, 5) = \frac{F(1, 3) + F(3, 7)}{2}$$

Purely arithmetic version of observation: We can obtain this observation computationally even without explicitly noting the observation about the midpoint. Explicitly, we have:

$$\begin{aligned} F(1,3) &= a + 3b + c \\ F(3,7) &= 3a + 7b + c \end{aligned}$$

Adding, we get:

$$F(1,3) + F(3,7) = 4a + 10b + 2c = 2(2a + 5b + c) = 2F(2,5)$$

Thus, we get:

$$F(2,5) = \frac{F(1,3) + F(3,7)}{2}$$

Alternative geometric explanation: Think of the inputs as living in the xy-plane, and the output axis as the z-axis. The graph $z = F(x,y) = ax + by + c$ gives a plane in the three-dimensional space with coordinates x, y, z. We know that a plane is determined by knowing three non-collinear points on it. The points are of the form $(x, y, F(x,y))$ where x and y vary freely. The graph is a plane *because* F is a linear function. In general, the graph would be a surface.

The inputs $(1,3)$, $(2,5)$, and $(3,7)$ being collinear, along with the fact that F is affine linear, tells us that the triples $(1,3,F(1,3))$, $(2,5,F(2,5))$, and $(3,7,F(3,7))$ are collinear in three-dimensional space. In fact, they lie on the line obtained by intersecting the plane that is the graph of F with the plane parallel to the z-axis whose intersection with the xy-plane passes through the points $(1,3)$ and $(3,7)$ (explicitly, this is the plane with equation $y = 2x + 1$). In this particular case, since $(2,5)$ is the midpoint between the points $(1,3)$ and $(3,7)$, the point $(2,5,F(2,5))$ is the midpoint between the points $(1,3,F(1,3))$ and $F(3,7,F(3,7))$.

Thus, if we use these three input-output pairs, then we get three *collinear* points in the plane we are trying to find, and we cannot determine the plane uniquely (any plane through the line joining the points works). If we chose an input (x_0, y_0) that was not collinear with the points $(1,3)$ and $(3,7)$, we would get a point $(x_0, y_0, F(x_0, y_0))$ that was not collinear with the points $(1,3,F(1,3))$ and $(3,7,F(3,7))$, and therefore, the plane would be determined uniquely.

Update for after understanding coefficient matrices and ranks: When trying to find the parameters a, b, and c, we need to set up a system of simultaneous linear equations. Each input-output pair gives one equation, and therefore, one row of the augmenting matrix. The augmenting column corresponds to the outputs, and the coefficient matrix part corresponds to the inputs. In other words, each input determines a row of the coefficient matrix.

For an input (x_i, y_i), the corresponding equation is:

$$ax_i + by_i + c = F(x_i, y_i)$$

We are now viewing this as an equation in the variables a, b, and c. The row of the coefficient matrix when we set up the linear system in terms of the parameters is:

$$\begin{bmatrix} x_i & y_i & 1 \end{bmatrix}$$

For our system with three input-output pairs, the coefficient matrix becomes:

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & 7 & 1 \\ x & y & 1 \end{bmatrix}$$

where (x,y) is the third input that we choose. The input that would be bad to choose is the input for which the coefficient matrix has rank two rather than the expected rank of three. We can see that $x = 2$, $y = 5$ is the only such input. In other words, the matrix:

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & 7 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$

has rank two. We can verify this by carrying out Gauss-Jordan elimination (the row reduction process) and obtaining that the last column is all zeros. On the other hand, the coefficient matrices for all the other choices of the third input have full rank three.

Performance review: 6 out of 28 got this. 10 chose (B), 7 chose (E), 3 chose (A), 2 chose (C).

Historical note (last time): 16 out of 28 got this. 9 chose (E), 2 chose (B), 1 chose (C).