HOMEWORK 3 CHECKLIST: DUE WEDNESDAY OCTOBER 23

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 2.1.4 (Page 53): Find the matrix of the linear transformation:

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3 \end{aligned}$$

You can literally read off the matrix as the coefficients.

(2) Exercise 2.1.5 (Page 53): Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}$$

Find the matrix A of T.

The matrix is a 2×3 matrix. The first column is the image of the first standard basis vector. The second column is the image of the second standard basis vector. The third column is the image of the third standard basis vector.

(3) Exercise 2.1.32 (Page 54): Find an $n \times n$ matrix A such that $A\vec{x} = 3\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

Recall that the linear transformation that sends everything to itself has matrix the identity matrix. To get something that sends everything to thrice of itself, we need three times the identity matrix. This is a scalar matrix with scalar value 3, or equivalently, a diagonal matrix with all diagonal entries 3.

(4) Exercise 2.1.35 (Page 55): In the example about the French coast guard in Section 2.1, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

- When the actual position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$, they radio $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$.
- When the actual position is $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$, they radio $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?
Suppose the coding matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, we have two matrix-vector product equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$$

We get four equations:

$$\begin{aligned} 5a + 42b &= 89 \\ 5c + 42d &= 52 \\ 6a + 41b &= 88 \\ 6c + 41d &= 53 \end{aligned}$$

Interchange the second and third equation:

$$\begin{aligned} 5a + 42b &= 89 \\ 6a + 41b &= 88 \\ 5c + 42d &= 52 \\ 6c + 41d &= 53 \end{aligned}$$

We can think of this in many ways. The most direct is to think of a linear system with variables a, b, c, and d. The system has augmented matrix:

$$\begin{bmatrix} 5 & 42 & 0 & 0 \\ 6 & 41 & 0 & 0 \\ 0 & 0 & 5 & 42 \\ 0 & 0 & 6 & 41 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 89 \\ 88 \\ 52 \\ 53 \end{bmatrix}$$

The augmented matrix is:

$$\begin{bmatrix} 5 & 42 & 0 & 0 & | & 89 \\ 6 & 41 & 0 & 0 & | & 88 \\ 0 & 0 & 5 & 42 & | & 52 \\ 0 & 0 & 6 & 41 & | & 53 \end{bmatrix}$$

We can solve this directly. However, it is more clever to solve the system by noting that it is a block diagonal system, and that both blocks have the same coefficient matrix. We can thus use a 2×2 coefficient matrix with *two* augmenting columns, one for the values a and b, one for the values c and d. We get:

$$\begin{bmatrix} 5 & 42 & | & 89 & 52 \\ 6 & 41 & | & 88 & 53 \end{bmatrix}$$

We now row reduce the coefficient matrix and copy the operations onto the augmenting columns. Once the coefficient matrix is reduced to the identity matrix, the first augmenting column gives the values a and b, and the second augmenting column gives the values c and d.

(5) Exercise 2.1.58 (was 2.1.50 in the 4th Edition) (Page 57): A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

(a) You can skip this.
(b) Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix}$$

for any piece of jewelry the goldsmith makes.
The total mass is just the sum of the masses, so we need to multiply the masses by 1 and add. The volume is the sum of the volumes of the platinum and silver alloys. *However, please be careful here: to get the volume of each alloy, we need to divide, rather than multiply, the mass by the density.*
If you are not comfortable thinking directly in terms of linear transformations, first write down the expressions for the total mass and total volume in terms of the masses of the two alloys. Then, read off the coefficient matrix of the system.
(c) Is the matrix A in part (b) invertible? If so, find the inverse. You can use the result from Question 1 of the advanced homework for this. (You can skip the rest of part (c)).
A square matrix is invertible if and only if it has full rank. For a 2×2 matrix, we can also verify invertibility, and compute the inverse, using the result of the advanced Question 1. If you did not have access to the final answer of the advanced Question 1, you could still mimic the process used there to answer this question.

(6) Exercise 2.1.59 (was 2.1.51 in the 4th Edition) (Page 57): The conversion matrix $C = \frac{5}{9}(F - 32)$ from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.
(a) Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$. (The second row of A will be $\begin{bmatrix} 0 & 1 \end{bmatrix}$.)
This question essentially relates affine linear functions (i.e., those with nonzero intercepts) to linear algebra. The idea is to treat the intercept as a scalar times 1. This should be straightforward.
(b) Is the matrix A in part (a) invertible? If so, find the inverse. You can use the result from Question 1 of the advanced homework for this. Use the result to write a formula expressing F in terms of C.
You can cross-check the formula you get by algebraically obtaining the formula for F in terms of C. It should be the same as the one you get through the linear algebra process described.

(7) Exercise 2.1.62 (was 2.1.54 in the 4th Edition) (Page 57-58): Consider an arbitrary currency exchange matrix A (see Exercises 60 and 61 from the book, which were 52 and 53 in the 4th edition).
Note: The printed version of the assignment stated an incorrect number for the exercises. It's been updated in the online version, and the update was also sent out over email.
(a) What are the diagonal entries a_{ii} of A?
This is the exchange rate between a currency and itself. Therefore, it should be ...
(b) What is the relationship between a_{ij} and a_{ji}?
This is asking for how the exchange rate of one currency relative to another relates to the exchange rate of the same currencies flipped around. The rates should be reverses of each other, which mathematically means that ...
(c) What is the relationship between a_{ik}, a_{kj}, and a_{ij}?
This is comparing a direct conversion from, say, dollars to yen versus a conversion from dollars to euros and then from euros to yens.
(d) What is the rank of A? What is the relationship between A and $\text{rref}(A)$?

All the later rows are related to the first row in a particular manner. What manner is that? The only reason for writing all the rows is that it means people don't have to do too much mental math; as such, later rows are redundant.

2. Problems for your own review

(1) Exercise 2.1.40 (Page 55): Describe all linear transformations from $\mathbb{R}$ ($= \mathbb{R}^1$) to $\mathbb{R}$. What do their graphs look like?
(2) Exercise 2.1.41 (Page 55): Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. What do their graphs look like?
(3) Exercise 2.1.44 (Page 56): The cross product of two vectors in $\mathbb{R}^3$ is given by:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

3. Advanced problems

(1) Exercise 2.1.13 (Page 54): Prove the following facts:
   (a) The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   is invertible if and only if $ad - bc \neq 0$. *Hint from the book*: Consider the cases $a \neq 0$ and $a = 0$ separately.
   For the matrix to be invertible, we need its rref to be the identity matrix, which happens if and only if $ad - bc \neq 0$ (this will become clear when you work it out).
   (b) If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   The formula here is worth memorizing.
   Augment this with a generic vector $\begin{bmatrix} p \\ q \end{bmatrix}$. Then, row reduce to the identity matrix to get the vector that would map to the generic vector under this matrix. Explicitly:

$$\begin{bmatrix} a & b & | & p \\ c & d & | & q \end{bmatrix}$$

   when converted to rref has the form:

$$\begin{bmatrix} 1 & 0 & | & \text{Some linear expression in } p, q \\ 0 & 1 & | & \text{Some linear expression in } p, q \end{bmatrix}$$

   Now, construct the matrix of the linear transformation that would send $\begin{bmatrix} p \\ q \end{bmatrix}$ to the final augmented column.
   Alternative method: Augment the original matrix with the identity matrix, row reduce the original matrix completely, and then whatever we get on the augmented side is the inverse. Explicitly, start with:

$$\begin{bmatrix} a & b & | & 1 & 0 \\ c & d & | & 0 & 1 \end{bmatrix}$$

At the end, get

$$\begin{bmatrix} 1 & 0 & | & ? & ? \\ 0 & 1 & | & ? & ? \end{bmatrix}$$

(2) Exercise 2.1.39 (Page 55): Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = x_1 T(\vec{e_1}) + x_2 T(\vec{e_2}) + \cdots + x_m T(\vec{e_m})$$

where $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_m}$ are the standard vectors in $\mathbb{R}^m$.

First, justify the statement *without* the T. In other words, verify that:

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = x_1 \vec{e_1} + x_2 \vec{e_2} + \cdots + x_m \vec{e_m}$$

Now, apply T to both sides, and use the linearity of T.

(3) Exercise 2.1.45 (Page 56): Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$ where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$, and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L(T(\vec{x}))$ linear as well? [The transformation $\vec{z} = L(T(\vec{x}))$ is called the *composite* of T and L.]

Use the additive + scalar multiples definition of linearity. Use that L and T individually are linear to do the splitting and scalar pulling-out, first on the inside, then on the outside.