CLASS QUIZ SOLUTIONS: OCTOBER 14: DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this quiz. Everybody got all questions correct!
The problem wise answers and performance review are below:

(1) Option (E): Everybody
(2) Option (C): Everybody
(3) Option (D): Everybody
(4) Option (D): Everybody

Good job!

2. Solutions

(1) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$ that are everywhere differentiable. Which of the following functions is/are guaranteed to be everywhere differentiable?
(A) $f + g$
(B) $f - g$
(C) $f \cdot g$
(D) $f \circ g$
(E) All of the above

Answer: Option (E)

Explanation: In fact, we have explicit formulas for the derivatives of all of these in terms of the derivatives of f and g. We have $(f + g)' = f' + g'$ and $(f - g)' = f' - g'$. We also have the product rule and chain rule for options (C) and (D).

Note that for the composition, we are using something more: since these are functions on the whole real line $\mathbb{R}$, the value $g(x)$ also lies in the domain of f, hence it makes sense to compose.

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose a multitude of options. *Note: The answer should always be exactly one option.*

(2) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \cdot g)''$?
(A) $f'' \cdot g + f \cdot g''$
(B) $f'' \cdot g + f' \cdot g' + f \cdot g''$
(C) $f'' \cdot g + 2f' \cdot g' + f \cdot g''$
(D) $f'' \cdot g - f' \cdot g' + f \cdot g''$
(E) $f'' \cdot g - 2f' \cdot g' + f \cdot g''$

Answer: Option (C)

Explanation: We differentiate once to get:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Now we differentiate both sides. The left side becomes $(f \cdot g)''$. The right side is a sum of two terms, so we get:

$$(f \cdot g)'' = (f' \cdot g)' + (f \cdot g')'$$

We now apply the product rule to each piece on the right side to get:

$$(f \cdot g)'' = [f'' \cdot g + f' \cdot g'] + [f' \cdot g' + f \cdot g'']$$

Combining terms, we get option (C).

Remark: In general, there is a binomial theorem-like formula for the n^{th} derivative of $f \cdot g$. I've given the formula below, but it will make sense only to people who have seen summation notation and the binomial coefficients, which we have not yet done:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

This is a lot like the binomial theorem expansion for $(a+b)^n$. It can be proved purely formally using induction from the product rule.[1]

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose option (E), though that person's rough work gave option (C).

(3) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \circ g)''$?
(A) $(f'' \circ g) \cdot g''$
(B) $(f'' \circ g) \cdot (f' \circ g') \cdot g''$
(C) $(f'' \circ g) \cdot (f' \circ g') \cdot (f \circ g'')$
(D) $(f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$
(E) $(f' \circ g') \cdot (f \circ g) + (f'' \circ g'')$

Answer: Option (D)

Explanation: This question is tricky because it requires the application of both the product rule and the chain rule, with the latter being used twice. We first note that:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Now, we differentiate both sides:

$$(f \circ g)'' = [(f' \circ g) \cdot g']'$$

The expression on the right side that needs to be differentiated is a product, so we use the product rule:

$$(f \circ g)'' = [(f' \circ g)' \cdot g'] + [(f' \circ g) \cdot g'']$$

Now, the inner composition $f' \circ g$ needs to be differentiated. We use the chain rule and obtain that $(f' \circ g)' = (f'' \circ g) \cdot g'$. Plugging this back in, we get:

$$(f \circ g)'' = (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$$

Remark: What's worth noting here is that in order to differentiate composites of functions, you need to use both composites *and* products (that's the chain rule). And in order to differentiate products, you need to use both products *and* sums (that's the product rule). Thus, in order to differentiate a composite twice, we need to use composites, products, *and* sums.

Performance review: Everybody got this correct

Historical note (last year): 14 out of 14 people got this correct. This is great! I had expected that many of you would be put off by the messy computation, but apparently you were unfazed.

(4) Suppose f is an everywhere differentiable function on $\mathbb{R}$ and $g(x) := f(x^3)$. What is $g'(x)$?
(A) $3x^2 f(x)$
(B) $3x^2 f'(x)$
(C) $3x^2 f(x^3)$
(D) $3x^2 f'(x^3)$

[1] One of the things I'm doing research on has to do with the fact above, albeit with a completely different notion of differentiation.

(E) $f'(3x^2)$

Answer: Option (D)

Explanation: Put $h(x) := x^3$. Then $g = f \circ h$. Thus, $g'(x) = f'(h(x))h'(x) = f'(x^3) \cdot (3x^2)$, giving option (D).

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose option (B), which is a close distractor if you're not paying attention.