DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE WEDNESDAY OCTOBER 2: VECTORS (BASIC STUFF)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

30 people took this 3-question quiz. The score distribution was as follows:

- Score of 1: 2 people
- Score of 2: 10 people
- Score of 3: 18 people

The mean score was about 2.5.
The question-wise answers and performance review were as follows:

(1) Option (A): 22 people
(2) Option (E): 24 people
(3) Option (C): 30 people

2. Solutions

Many of you are familiar with vectors, either from Math 195 or some exposure to vectors in high school (or perhaps both). This quiz is to help gauge your level of understanding coming in. We will not get to start using the ideas in their full depth until a few weeks later.

For the benefit of those who haven't seen vectors at all, the definitions are briefly provided.

There are many ways of describing the vector in $\mathbb{R}^n$ with coordinates $a_1, a_2, \dots, a_n$. You may have seen the vector described using angled braces as $\langle a_1, a_2, \dots, a_n \rangle$. In this linear algebra course, we will customarily write the vector as a *column* vector, i.e., the coordinates will be written in a vertical column. For instance, the vector $\langle 2, 3, 7 \rangle$ will be written as the column vector $\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$.

Two vectors in $\mathbb{R}^n$ can be added with each other (note that both vectors need to be in the *same* $\mathbb{R}^n$ in order to be added). The addition is coordinate-wise:

$$\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n + w_n \end{bmatrix}$$

Also, given any real number λ (called a *scalar* to distinguish from a vector) and a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$, we can define:

$$\lambda\vec{v} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} := \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda v_n \end{bmatrix}$$

We can identify the set of n-dimensional vectors with the set of points in $\mathbb{R}^n$. The vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$ in this case corresponds to the point with coordinates $(v_1, v_2, \ldots, v_n)$.

(1) *Do not discuss this!*: For a n-dimensional vector $\vec{v}$, the *set of scalar multiples* of $\vec{v}$ is the set of vectors that can be expressed in the form $\lambda\vec{v}$, $\lambda \in \mathbb{R}$. Assume that $\vec{v}$ is a nonzero vector. What can we say geometrically about the set of points in $\mathbb{R}^n$ that correspond to the scalar multiples of $\vec{v}$?
(A) It is a straight line in $\mathbb{R}^n$ that passes through the origin.
(B) It is a straight line in $\mathbb{R}^n$. However, it need not pass through the origin.
(C) It is a straight half-line in $\mathbb{R}^n$ with the endpoint at the origin.
(D) It is a straight half-line in $\mathbb{R}^n$, but the endpoint need not be at the origin.
(E) It is a line segment in $\mathbb{R}^n$.

Answer: Option (A)

Explanation: This is clear from the way scalar multiplication of vectors is described visually. Note that if we restrict λ to the nonnegative reals (i.e., $\lambda \in (0, \infty)$), then we get a half-line (excluding the endpoint). However, the definition of scalar multiple allows for negative values of λ (which give the opposite half-line) and for $\lambda = 0$ (which gives the origin). We therefore get the whole line.

Performance review: 22 out of 30 people got this. 2 each chose (B), (C), (D), and (E).

(2) *Do not discuss this!*: Given two n-dimensional vectors $\vec{v}$ and $\vec{w}$, the *set of linear combinations* of $\vec{v}$ and $\vec{w}$ is the set of all vectors that can be written in the form $\lambda\vec{v} + \mu\vec{w}$ where $\lambda, \mu \in \mathbb{R}$ (note that λ and μ can take arbitrary real values, and are allowed to be equal to each other). In other words, you can take scalar multiples, and you can then add these scalar multiples.

The set of linear combinations of $\vec{v}$ and $\vec{w}$ is sometimes also called the *span* of $\vec{v}$ and $\vec{w}$.

What is the span of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$?
(A) The zero vector only, because that is the only vector that can be expressed both as a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and as a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
(B) The set of vectors that can be expressed as a scalar multiple either of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
(C) The set of vectors that can be expressed as a scalar multiple of at least one of these three vectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
(D) All vectors in the first quadrant of $\mathbb{R}^2$, including the bounding half-lines. In other words, the set of vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x \geq 0$ and $y \geq 0$.
(E) All vectors in $\mathbb{R}^2$.

Answer: Option (E)

Explanation: For a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we can write:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, we have expressed our arbitrary vector of $\mathbb{R}^2$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In the notation given, $\lambda = x$ and $\mu = y$.

Note the key point that the λ and μ values can vary arbitrarily. They are not restricted to be nonnegative (Option (D)) and they are not required to have at least one of them zero, or to be equal to each other (Options (A)-(C)).

Performance review: 24 out of 30 people got this. 3 chose (C), 2 chose (D), 1 chose (A).

(3) *Do not discuss this!*: Consider the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that interchanges the coordinates of a vector. Explicitly, the transformation is given as:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$$

Which of the following describes the transformation geometrically, with $\mathbb{R}^2$ viewed as the xy-plane?

(A) It is a reflection about the x-axis in $\mathbb{R}^2$, i.e., the axis for the first coordinate.
(B) It is a reflection about the y-axis in $\mathbb{R}^2$, i.e., the axis for the second coordinate.
(C) It is a reflection about the line $y = x$ in $\mathbb{R}^2$, i.e., the line of vectors where both coordinates are equal.
(D) It is a reflection about the line $y = -x$ in $\mathbb{R}^2$, i.e., the line of vectors where the coordinates are negatives of each other.

Answer: Option (C)

Explanation: You need to visualize this geometrically. The midpoint of (x_0, y_0) and (y_0, x_0) is $((x_0 + y_0)/2, (x_0 + y_0)/2)$, which lies on the $y = x$ line. Further, the line joining the points (x_0, y_0) and (y_0, x_0) has slope -1, hence is perpendicular to the $y = x$ line. Therefore, the line $y = x$ is the perpendicular bisector of the line segment joining (x_0, y_0) and (y_0, x_0). In other words, reflecting about the line $y = x$ sends (x_0, y_0) to the point (y_0, x_0).

You've probably seen this in calculus already: remember that to obtain the graph of f^{-1} from the graph of f, you reflected about the $y = x$ line? That's because reflection about the $y = x$ line reverses the roles of x and y.

Performance review: All 30 people got this.