CLASS QUIZ SOLUTIONS: JANUARY 10: HYPERBOLIC FUNCTIONS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

28 students took this 5-question quiz. The score distribution was as follows:

- Score of 1: 2 people.
- Score of 2: 5 people.
- Score of 3: 4 people.
- Score of 4: 6 people.
- Score of 5: 11 people.

The mean score was 3.68, standard deviation was 1.36, median score was 4, and modal score was 5. The correlation with doing Math 152 under me last quarter was -0.53 (indicating that students who did Math 152 under me last quarter did substantially worse on average) and the correlation with doing Math 152 under some other instructor last quarter was $+0.26$ (indicating the students who did Math 152 under another instructor did better on average). A partial explanation may be that many students coming from other sections of Math 152 had already seen hyperbolic functions in their previous 152 section. It may also be the case that students new to the section are more likely to put in effort into the quizzes prior to class and are hence performing better.

The answers by question number are:

(1) Option (B): 21 people got this correct.
(2) Option (E): 24 people got this correct.
(3) Option (D): 19 people got this correct. *Please look at the solution!*
(4) Option (C): 24 people got this correct.
(5) Option (B): 15 people got this correct. *Please look at the solution!*

2. Solutions

(1) What is the limit $\lim_{x \to \infty} (\cosh x)/e^x$?
   (A) 0
   (B) 1/2
   (C) 1
   (D) 2
   (E) The limit does not exist.

   Answer: Option (B)

   Explanation: We have:

$$\lim_{x \to \infty} \frac{\cosh x}{e^x} = \lim_{x \to \infty} \frac{e^x + e^{-x}}{2e^x} = \frac{1}{2} + \lim_{x \to \infty} \frac{e^{-2x}}{2} = \frac{1}{2}$$

   The key thing to note is that as $x \to \infty$, e^{-2x} tends to 0.

   Post-performance review: 21 out of 28 students got this correct. 3 people chose (C), 3 people chose (E), and 1 person chose (A).

(2) What is the limit $\lim_{x \to -\infty} (\cosh x)/e^x$?
   (A) 0
   (B) 1/2
   (C) 1
   (D) 2
   (E) The limit does not exist.

Answer: Option (E)

Explanation: We have

$$\lim_{x\to-\infty}\frac{\cosh x}{e^x}=\lim_{x\to-\infty}\frac{e^x+e^{-x}}{2e^x}=\frac{1}{2}+\lim_{x\to-\infty}\frac{e^{-2x}}{2}$$

As $x\to-\infty$, $e^{-2x}\to\infty$, so the overall limit is undefined.

Alternatively, we can note that as $x\to-\infty$, $\cosh x\to+\infty$ and $e^x\to 0$, so the quotient goes off to infinity.

Post-performance review: 24 out of 28 people got this correct. 3 people chose (C) and 1 person chose (B).

(3) Consider the function $y=f(x)$ where $f(x):=\arctan(\sinh x)$. Which of the following does $\cosh x$ necessarily equal?
(A) $\sin y$
(B) $\cos y$
(C) $\cot y$
(D) $\sec y$
(E) $\csc y$

Answer: Option (D)

Explanation: We have the relationship $\tan y=\sinh x$. Squaring and adding 1 to both sides, we get:

$$\sec^2 y=\cosh^2 x$$

Now, we note that cosh is always positive, and for $y\in(-\pi/2,\pi/2)$, which it must be to be arctan of something, $\sec y$ is also positive. Thus, taking square roots on both sides yields:

$$\sec y=\cosh x$$

Post-performance review: 19 out of 28 people got this correct. 8 people chose (C) and 1 person chose (A).

(4) Consider the function $y=f(x)$ where $f(x):=\arctan(\sinh x)$ (same as in the previous question). The function is a one-to-one increasing function on its domain. What are its domain and range?
(A) The domain and range are both equal to $\mathbb{R}$
(B) The domain and range are both equal to the open interval $(-\pi/2,\pi/2)$
(C) The domain equals $\mathbb{R}$ and the range equals the open interval $(-\pi/2,\pi/2)$
(D) The domain equals the open interval $(-\pi/2,\pi/2)$ and the range equals $\mathbb{R}$
(E) The domain equals the open interval $(-\pi/2,\pi/2)$ and the range equals the closed interval $[-\pi/2,\pi/2]$

Answer: Option (C)

Explanation: The sinh function has domain and range both $\mathbb{R}$. The arctan function has domain $\mathbb{R}$ and range $(-\pi/2,\pi/2)$. Thus, the composite has domain $\mathbb{R}$ (any real input is permisible) and range equal to $(-\pi/2,\pi/2)$.

Post-performance review: 24 out of 28 people got this correct. 3 people chose (D) and 1 person chose (B).

(5) sinh is a one-to-one function with domain and range both equal to $\mathbb{R}$. Hence, it must have an inverse function with domain and range both equal to $\mathbb{R}$. What is this inverse function?
(A) $x\mapsto(\ln(x)-\ln(-x))/2$
(B) $x\mapsto(1/2)\ln(x^2+1)$
(C) $x\mapsto\ln[x+\sqrt{x^2+1}]$
(D) $x\mapsto\ln[x-\sqrt{x^2+1}]$
(E) $x\mapsto\ln[\sqrt{x^2+1}-x]$

Answer: Option (C)

Explanation: If $x = \sinh t$, then $\cosh^2 t = x^2 + 1$. Taking square roots, and using that cosh is always positive, we get $\cosh t = \sqrt{x^2+1}$. Thus, $\exp(t) = \sinh(t) + \cosh(t) = x + \sqrt{x^2+1}$. Taking ln both sides, we get $t = \ln[x + \sqrt{x^2+1}]$.

Post-performance review: 15 out of 28 people got this correct. 1 person appears to have missed the question because it was printed on the back side page. 6 people chose (B), 3 people chose (A), 2 people chose (E), and 1 person chose (D).