CLASS QUIZ SOLUTIONS: MARCH 4: SERIES CONVERGENCE

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

26 people took this 8-question quiz. The mean score was 3.46. The score distribution was as follows:

- Score of 1: 4 people.
- Score of 2: 3 people.
- Score of 3: 6 people.
- Score of 4: 5 people.
- Score of 5: 6 people.
- score of 6: 2 people.

Here are the answers and performance summary by question. A (*) in front of a question indicates that there was a single incorrect option that was chosen by more people than the correct option for that question:

(1) Option (A): 14 people.
(2) (*) Option (E): 7 people. *Master this!*
(3) (*) Option (D): 8 people. *Master this!*
(4) Option (E): 14 people.
(5) Option (C): 14 people.
(6) Option (B): 16 people. *Master this!*
(7) Option (D): 10 people.
(8) Option (A): 7 people.

2. Solutions

(1) Consider the series $\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}$. What can we say about the sum of this series?
(A) It is finite and strictly between 0 and 1.
(B) It is finite and equal to 1.
(C) It is finite and strictly between 1 and 2.
(D) It is finite and equal to 2.
(E) It is infinite.

Answer: Option (A)

Explanation: The summation goes like:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \dots$$

Notice that the series being summed is a subseries of the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

In particular, the sum of the former is less than the sum of the latter. The latter sums up to 1, so the sum of the former is less than 1. Also, since the first term is 1/2, it must be greater than 1/2. Thus, the series sum is between 1/2 and 1. Option (A) is the best fit.

Performance review: 14 out of 26 people got this correct. 9 chose (C) (possibly because of getting the first term wrong?), 2 chose (E), 1 chose (B).

(2) Consider the function $F(x,p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. For what values of x and what values of p does this summation converge?
(A) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| \geq 1$, it does not converge for any p.

(B) For $|x| \leq 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p.
(C) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $|x| = 1$, it converges if and only if $p > 1$.
(D) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $x = 1$, it converges if and only if $p > 0$. For $x = -1$, it converges if and only if $p > 1$.
(E) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any $p \in \mathbb{R}$. For $x = 1$, it converges if and only if $p > 1$. For $x = -1$, it converges if and only if $p > 0$.

Answer: Option (E)

Explanation: When $|x| < 1$, then the series is absolutely convergent and when $|x| > 1$ it diverges, as we can see by the root test or ratio test. What's happening is that $(1/n^p)^{1/n} \to 1$ (regardless of p), so the radius of convergence is 1.

This leaves the case $|x| = 1$. If $x = 1$, then we get the usual p-series, which we know converges iff $p > 1$. If $x = -1$, then the terms have alternating signs. Obviously, the series cannot converge for $p \leq 0$ because the terms do not tend to 0. For $p > 0$, on the other hand, the terms are alternating in sign and decrease monotonically, tending to 0. Thus, by the alternating series theorem, it converges for $p > 0$.

There is a result of calculus which states that, under suitable conditions, if $f_1, f_2, \dots, f_n, \dots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$. In other words, under suitable assumptions, we can differentiate a sum of countably many functions by differentiating each of them and adding up the derivatives.

We will not be going into what those assumptions are, but will consider some applications where you are explicitly told that these assumptions are satisfied.

Performance review: 7 out of 26 people got this correct. 10 chose (C), 4 chose (D), 3 chose (A), 2 chose (B). The large vote for (C) indicates that many people did not notice the special application of the alternating series theorem to the case of $x = -1$.

Action point: Please review what happens in the case $x = -1$. This will be covered in more detail in class when we study the notion of interval of convergence of a power series.

(3) Consider the summation $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. Assume that the required assumptions are valid for this summation, so that $\zeta'(p)$ is the sum of the derivatives of each of the terms (summands) with respect to p. What is the correct expression for $\zeta'(p)$?
(A) $\sum_{n=1}^{\infty} \frac{-p}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{-1}{(p+1)n^{p+1}}$
(C) $\sum_{n=1}^{\infty} \frac{p}{n^{p-1}}$
(D) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^p}$
(E) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^{p+1}}$

Answer: Option (D)

Explanation: We need to differentiate $(1/n)^p$ with respect to p. This is the same as differentiating a^x with respect to x, which gives $a^x \ln a$. In our case, we get $(1/n)^p \ln(1/n)$ which is $(-\ln n)/n^p$.

Note that Option (A) arises if we try to differentiate formally with respect to n, which is not the correct operation at all. n is a dummy variable and the expression should be differentiated with respect to p.

Performance review: 8 out of 26 people got this correct. 13 chose (A), 3 chose (B), 1 chose (C), and 1 left the question blank. The most commonly chosen wrong option, (A), indicates that many people differentiated with respect to the wrong variable.

(4) Going back to question 2, recall that we defined $F(x,p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. Assume that, for a particular fixed value of p, the summation satisfies the conditions as a function of x for $|x| < 1$. What is its derivative with respect to x, keeping p constant?
(A) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p-1}}$
(C) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p+1}}$
(D) $\sum_{n=1}^{\infty} \frac{x^{n-1} \ln n}{n^{p+1}}$

(E) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p-1}}$

Answer: Option (E)

Explanation: We need to differentiate each term with respect to x. Differentiating x^n/n^p with respect to x gives nx^{n-1}/n^p, which, upon rearrangement, gives x^{n-1}/n^{p-1}.

Performance review: 14 out of 26 got this correct. 4 each chose (B), (C), (D), possibly indicating minor computational errors.

(5) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Since it is a series of positive terms, this means that the partial sums get arbitrarily large. What is the approximate smallest value of N such that $\sum_{n=1}^{N} \frac{1}{n} > 100$?

(A) Between 90 and 110
(B) Between 2000 and 3000
(C) Between 10^{40} and 10^{50}
(D) Between 10^{90} and 10^{110}
(E) Between 10^{220} and 10^{250}

Answer: Option (C)

Explanation: We can see that $\sum_{n=1}^{N} 1/n$ is approximately $\ln N$. More precisely, we can use the standard methods for comparising integrals and summations and obtain that the finite sum is between $\ln N$ and $1 + \ln N$. In particular, the N that works must have $\ln N$ between 99 and 100. Thus, $\log_{10} N$ is between $99/(\ln 10)$ and $100/(\ln 10)$. Both these numbers are between 40 and 50, so Option (C) is the correct choice.

Performance review: 14 out of 26 got this correct. 6 chose (D), 2 chose (B), 2 chose (E), 1 chose (A), and 1 left the question blank.

(6) Consider the function $f(x) := \sum_{n=1}^{\infty} \frac{x^n}{n(n+2)}$ defined on the closed interval $[-1, 1]$. What are the values of $f(1)$ and $f(-1)$?

(A) $f(1) = 3/4$ and $f(-1) = 1/4$
(B) $f(1) = 3/4$ and $f(-1) = -1/4$
(C) $f(1) = 3/4$ and $f(-1) = -3/4$
(D) $f(1) = 1/4$ and $f(-1) = 3/4$
(E) $f(1) = 1/4$ and $f(-1) = -1/4$

Answer: Option (B)

Explanation: $f(1) = \sum_{n=1}^{\infty} \frac{1}{n(n+2)}$. Rewrite each summand as:

$$\frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

The terms cancel (telescoping) and we are left with:

$$\frac{1}{2}\left(1 + \frac{1}{2}\right)$$

Simplifying, we get $3/4$.

Similarly, for $f(-1)$, the summands are:

$$\frac{(-1)^n}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

Because the term repetition is every two steps, the n^{th} summand and $(n+2)^{th}$ summand have the same outer sign, so cancellation still proceeds as before. We are left with:

$$\frac{1}{2}\left[\frac{(-1)^1}{1} + \frac{(-1)^2}{2}\right]$$

Simplifying, we get $-1/4$.

"Alternative" approach: Even if you're unable to do these summations, you can estimate the sums quickly using the first few terms and rule out all the other possibilities. For $f(1)$, we have:

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots$$

The sum of the first three terms is $21/40$, which is slightly more than $1/2$, so $3/4$ is the only viable option.

For $f(-1)$, we have:

$$-\frac{1}{3}+\frac{1}{8}-\frac{1}{15}+\dots$$

The sum of the first three terms is $-11/30$, which bounds the alternating sum from *below*. On the other hand, the sum of the first two terms, $-5/24$, bounds the alternating sum from *above*. Among the given options, the only possibility is $-1/4$.

Performance review: 16 out of 26 got this correct. 5 chose (C), 3 chose (A), and 1 each chose (D) and (E).

(7) Given that we have the following: $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$ for all $-1 < x < 1$ and the series converges absolutely in the interval, what is an explicit expression for the summation $\sum_{n=1}^{\infty} x^n/(n(n+1))$ for $x \in (-1,1) \setminus \{0\}$?

(A) $1+\ln(1-x)$
(B) $1-\ln(1-x)$
(C) $1+\frac{(1+x)\ln(1-x)}{x}$
(D) $1+\frac{(1-x)\ln(1-x)}{x}$
(E) $1+\frac{(x-1)\ln(1-x)}{x}$

Answer: Option (D)

Explanation: We telescope and rewrite the summands as:

$$x^n\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

Due to absolute convergence, we can split the summation across the $-$ sign and get:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}-\sum_{n=1}^{\infty}\frac{x^n}{n+1}$$

The first sum is $-\ln(1-x)$. We now calculate the second sum. If $x \neq 0$, we can multiply and divide by x to obtain:

$$\sum_{n=1}^{\infty}\frac{x^n}{n+1}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}$$

Choose $m = n+1$ to rewrite the right side as $\frac{1}{x}\sum_{m=2}^{\infty}\frac{x^m}{m}$. Note that the sum starts from 2, so adding and subtracting the case $m = 1$ gives:

$$\frac{1}{x}\left[\frac{-x^1}{1}+\sum_{m=1}^{\infty}\frac{x^m}{m}\right]$$

This becomes:

$$\frac{1}{x}\left[-\ln(1-x)-x\right]$$

which becomes:

$$\frac{-\ln(1-x)}{x}-1$$

Plugging this back into the original, we get:

$$-\ln(1-x)-\left[\frac{-\ln(1-x)}{x}-1\right]$$

This simplifies to $1+\ln(1-x)(-1+(1/x))$ which simplifies to option (D).

Reality check: There are two reality checks we can perform on the expression obtained: taking the limit as $x \to 1^-$, and $x \to 0$. We first consider the limit as x approaches 1 from the left.

In this case, the terms approach $1/(n(n+1))$, which is $(1/n)-(1/(n+1))$, which upon telescoping cancellation gives 1.

On the other hand, the limit:

$$\lim_{x \to 1^-} 1 + \frac{(1-x)\ln(1-x)}{x}$$

is also 1.

Let's now consider the case that x approaches 0.

In this case, the summation approaches 0 because all its terms approach zero.

For the expression, we have:

$$\lim_{x \to 0} 1 + \frac{(1-x)\ln(1-x)}{x}$$

Take out the 1+ and we get:

$$1 + \lim_{x \to 0}(1-x) \lim_{x \to 0} \frac{\ln(1-x)}{x}$$

Simple stripping or LH rule gives that the remaining limit is -1, so the overall answer is $1+(-1) = 0$, as desired.

Performance review: 10 out of 26 got this correct. 6 chose (C), 4 chose (A), 4 chose (E), 2 chose (B).

(8) Given that we have the following: $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$ for all $-1 < x < 1$ and the series converges absolutely in the interval, what is an explicit expression for the summation $\sum_{n=1}^{\infty} x^n/(n(n+2))$ for $x \in (-1,1) \setminus \{0\}$?

(A) $\frac{1}{4} + \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}$
(B) $\frac{1}{4} + \frac{1}{2x} + \frac{(x^2-1)\ln(1-x)}{2x^2}$
(C) $\frac{1}{4} - \frac{1}{2x} + \frac{(x^2-1)\ln(1-x)}{2x^2}$
(D) $\frac{1}{4} - \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}$
(E) $\frac{1}{4} + \frac{1}{2x}$

Answer: Option (A)

Explanation: This is similar to the previous question – work it out yourself. It is character building.

Reality check: We now have three interesting limit cases to check: 1^-, -1^+, and 0. We know from Question 6 that the limits for 1^- and -1^+ should be $3/4$ and $-1/4$ respectively. Plugging in the values gives the same answer.

This leaves the case $x = 0$. Here, the limit of the summation should be 0, because all the terms tend to 0. Let's see if this is indeed the case:

$$\lim_{x \to 0} \left[\frac{1}{4} + \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}\right]$$

We can take the $1/4$ out and take $2x^2$ as a common denominator on the rest:

$$\frac{1}{4} + \lim_{x \to 0} \frac{x + (1-x^2)\ln(1-x)}{2x^2}$$

The limit is now a $\to 0/ \to 0$ form, so we can use LH rule:

$$\frac{1}{4} + \lim_{x \to 0} \frac{1 - 2x\ln(1-x) - (1-x^2)/(1-x)}{4x}$$

Simplifying:

$$\frac{1}{4} + \lim_{x \to 0} \frac{1 - 2x\ln(1-x) - (1+x)}{4x}$$

Simplify further:

$$\frac{1}{4} + \lim_{x \to 0} \frac{x(-1 - 2\ln(1-x))}{4x}$$

Cancel the x and evaluate. We get $-1/4$ for the limit, and adding to the outer $+1/4$, we get 0, as expected.

Performance review: 7 out of 26 got this correct. 6 each chose (B), (C), (D), and 1 person left the question blank.