TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 2: SERIES CONVERGENCE

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

42 people took this quiz. The score distribution was as follows:

- Score of 5: 1 person
- Score of 7: 2 people
- Score of 8: 1 person
- Score of 9: 5 people
- Score of 10: 1 person
- Score of 11: 5 people
- Score of 12: 2 people
- Score of 13: 4 people
- Score of 14: 7 people
- Score of 15: 2 people
- Score of 16: 12 people

The mean score was about 12.7. The median score was 13.5 and the modal score was 16.
The question-wise answers and performance are as follows:

(1) Option (C): 37 people
(2) Option (B): 29 people
(3) Option (E): 34 people
(4) Option (D): 36 people
(5) Option (B): 41 people
(6) Option (A): 34 people
(7) Option (D): 33 people
(8) Option (C): 39 people
(9) Option (D): 33 people
(10) Option (C): 30 people
(11) Option (E): 33 people
(12) Option (A): 34 people
(13) Option (D): 37 people
(14) Option (E): 21 people
(15) Option (D): 27 people
(16) Option (E): 35 people

2. Solutions

(1) Suppose p is a polynomial that takes positive values on all positive integers. Consider the summation $\sum_{k=1}^\infty \frac{(k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.
(A) The summation converges if and only if the degree of p is *at least* one
(B) The summation converges if and only if the degree of p is *at least* two
(C) The summation converges if and only if the degree of p is *at least* three
(D) The summation converges if and only if the degree of p is *at most* two
(E) The summation converges if and only if the degree of p is *at most* one

Answer: Option (C)

Explanation: This is a straightforward application of the degree difference rule in its more general form. The numerator has degre $4/3$, so the degree difference is $\deg(p) - (4/3)$. This difference needs to be greater than 1 for the series to converge, so we need that the degree of the denominator is strictly greater than $7/3$. The smallest integer greater than $7/3$ is 3, so that is the answer.

Performance review: 37 out of 42 got this correct. 4 chose (B), 1 chose (E).

Historical note (last year): 9 out of 11 got this. 2 chose (B).

(2) Suppose p is a polynomial that takes positive values on all positive integers. Consider the summation $\sum_{k=1}^{\infty} \frac{(-1)^k (k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.

(A) The summation converges if and only if the degree of p is *at least* one
(B) The summation converges if and only if the degree of p is *at least* two
(C) The summation converges if and only if the degree of p is *at least* three
(D) The summation converges if and only if the degree of p is *at most* two
(E) The summation converges if and only if the degree of p is *at most* one

Answer: Option (B)

Explanation: The previous question tells us that if the degree of p is at least three, then the series of absolutely convergent.

If the degree of p is two, then the series is conditionally convergent. To see this, note that $p(k)$ is always positive, so the summation is an alternating series summation. Also, the terms are eventually decreasing in magnitude, and they go to zero. Thus, by the alternating series theorem, the series converges. On the other hand, the degree difference rule tells us that it does not converge absolutely.

If the degree of p is one or zero, then the terms of the series do not approach zero, so the series does not converge.

Performance review: 29 out of 42 got this correct. 10 chose (C), 2 chose (A), 1 chose (D).

Historical note (last year): 3 out of 11 got this. 7 chose (C), 1 chose (E).

(3) Which of the following series converges? Assume for all series that the starting point of summation is large enough that the terms are well defined. *Two years ago:* 11/25 *correct*

(A) $\sum 1/(k \ln(\ln k))$
(B) $\sum 1/(k \ln k)$
(C) $\sum 1/(k(\ln(\ln k))^2)$
(D) $\sum 1/(k(\ln k)(\ln(\ln k)))$
(E) $\sum 1/(k(\ln k)(\ln(\ln k))^2)$

Answer: Option (E)

Explanation: Options (B) and (D) diverge by the integral test. As for options (A) and (C), these have smaller denominators, hence larger terms, than option (B), hence, by basic comparison, these diverge too. This leaves option (E), which converges by the integral test.

Performance review: 34 out of 42 got this correct. 5 chose (C), 2 chose (D), 1 chose (B).

Historical note (last year): 5 out of 11 got this. 4 chose (B), 2 chose (D).

Historical note (two years ago): 11 out of 25 people got this correct. 7 chose (C), 3 each chose (A) and (B), and 1 left the question blank.

The main attraction of (C) seems to have been its superficial resemblance to $1/(k(\ln k)^2)$ which does converge.

(4) Which of the following series converges? If all converge, select option (E). Assume for all series that the starting point of summation is large enough that the terms are well defined.

(A) $\sum 1/(k \ln(k^2 + 1))$
(B) $\sum 1/(k(\ln(k^2) + 1))$
(C) $\sum 1/((k \ln(k^2)) + 1)$
(D) $\sum 1/(k((\ln k)^2 + 1))$
(E) All of the above converge

Answer: Option (D)

Explanation: All the other options are, by limit comparison, equivalent to the summation of $1/(k \ln k)$, which diverges. Option (D) is equivalent to $1/(k(\ln k)^2)$, which converges.

Performance review: 36 got this correct. 6 chose (E).

This question did not appear in previous years.

(5) Which of the following series converges? *Two years ago:* 23/25 *correct*

(A) $\sum \frac{k+\sin k}{k^2+1}$
(B) $\sum \frac{k+\cos k}{k^3+1}$
(C) $\sum \frac{k^2-\sin k}{k+1}$
(D) $\sum \frac{k^3+\cos k}{k^2+1}$
(E) $\sum \frac{k}{\sin(k^3+1)}$

Answer: Option (B)

Explanation: We can use a comparison test, either rigorously or in the form of a heuristic of looking at degree of denominator minus degree of numerator. Note that for (A), the degree difference is 1, so it diverges. For (C) and (D), the numerator actually has larger degree than the denominator, so it diverges. For (E), the denominator is bounded in $[-1, 1]$, and the numerator goes to ∞, so it diverges. This leaves (B), which converges because the degree of denominator minus degree of numerator equals 2.

Performance review: 41 got this correct. 1 left the question blank.

Historical note (last year): 10 out of 11 got this. 1 chose (A).

Historical note (two years ago): 23 out of 25 people got this correct. 1 person chose (C) and 1 person chose (D).

(6) Consider the series $\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}$. What can we say about the sum of this series? *Two years ago:* 14/26 *correct*

(A) It is finite and strictly between 0 and 1.
(B) It is finite and equal to 1.
(C) It is finite and strictly between 1 and 2.
(D) It is finite and equal to 2.
(E) It is infinite.

Answer: Option (A)

Explanation: The summation goes like:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \dots$$

Notice that the series being summed is a subseries of the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

In particular, the sum of the former is less than the sum of the latter. The latter sums up to 1, so the sum of the former is less than 1. Also, since the first term is 1/2, it must be greater than 1/2. Thus, the series sum is strictly between 1/2 and 1. Option (A) is the best fit.

Performance review: 34 out of 42 got this correct. 7 chose (C), 1 chose (B).

Historical note (last year): 7 out of 11 got this. 3 chose (C), 1 chose (B).

Historical note (two years ago): 14 out of 26 people got this correct. 9 chose (C) (possibly because of getting the first term wrong?), 2 chose (E), 1 chose (B).

(7) For one of the following functions f on $(0, \infty)$, the integral $\int_0^\infty f(x)\,dx$ converges but $\int_0^\infty |f(x)|\,dx$ does not converge. What is that function f? (Note that this is similar to, but not quite the same as, the absolute versus conditional convergence notion for series).

(A) $f(x) = \sin x$
(B) $f(x) = \sin(\sin x)$
(C) $f(x) = (\sin \sqrt{x})/\sqrt{x}$
(D) $f(x) = (\sin x)/x$
(E) $f(x) = (\sin^3 x)/x^3$

Answer: Option (D)

Explanation: For options (A) and (B), the integral $\int_0^\infty f(x)\,dx$ does not converge. The reason is simple: in neither case is it true that $\lim_{x\to\infty} f(x) = 0$. The function itself going to zero is a necessary (but not sufficient) condition for the integral to converge.

For option (C), the antiderivative for f is $-2\cos\sqrt{x}$, evaluated between limits 0 and ∞. However, the limit for the antiderivative at ∞ does not exist, hence the integral does not converge.

This leaves options (D) and (E). For option (D), it is a well known (?) fact that:

$$\int_0^\infty \frac{\sin x\,dx}{x} = \frac{\pi}{2}$$

Hence, the integral does converge. However, if we consider the integral:

$$\int_0^\infty \frac{|\sin x|\,dx}{|x|}$$

This integral does not converge, a fact that we can prove by bounding it in terms of the summation $\sum_{n=1}^\infty 1/n$, which diverges.

Finally, for option (E), both $\int_0^\infty f(x)\,dx$ and $\int_0^\infty |f(x)|\,dx$ converge. To see this, first split the integral as $\int_0^1 + \int_1^\infty$. The former integral is finite because it is integrating a bounded function over a bounded interval (note that the limit of the function at 0 is 1). The latter integral is finite because we can compare it to $1/x^3$ which has a finite integral. The reasoning works for both f and $|f|$, so we are done.

Performance review: 33 out of 42. 6 chose (C), 2 chose (A), 1 chose (E).

Historical note (last year): 5 out of 11 got this. 4 chose (B), 2 chose (C).

(8) The series $\sum_{n=1}^\infty \frac{1}{n}$ diverges. Since it is a series of positive terms, this means that the partial sums get arbitrarily large. What is the approximate smallest value of N such that $\sum_{n=1}^N \frac{1}{n} > 100$? *Two years ago:* 14/26 *correct*

(A) Between 90 and 110
(B) Between 2000 and 3000
(C) Between 10^{40} and 10^{50}
(D) Between 10^{90} and 10^{110}
(E) Between 10^{220} and 10^{250}

Answer: Option (C)

Explanation: We can see that $\sum_{n=1}^N 1/n$ is approximately $\ln N$. More precisely, we can use the standard methods for comparising integrals and summations and obtain that the finite sum is between $\ln N$ and $1 + \ln N$. In particular, the N that works must have $\ln N$ between 99 and 100. Thus, $\log_{10} N$ is between $99/(\ln 10)$ and $100/(\ln 10)$. Both these numbers are between 40 and 50, so Option (C) is the correct choice.

Performance review: 39 out of 42 got this. 3 chose (D).

Historical note (last year): 6 out of 11 got this correct. 2 each chose (D) and (E), 1 chose (A).

Historical note (two years ago): 14 out of 26 got this correct. 6 chose (D), 2 chose (B), 2 chose (E), 1 chose (A), and 1 left the question blank.

(9) Consider the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Which of the following is **true** about the series?

(A) Every rearrangement of the series converges to $\pi^2/6$.
(B) Every rearrangement of the series converges to $\ln 2$.
(C) The series converges to $\pi^2/6$ and there is a rearrangement of the series that converges to $\ln 2$.
(D) The series converges to $\ln 2$ and there is a rearrangement of the series that converges to $\pi^2/6$.
(E) The series does not converge.

Answer: Option (D)

Explanation: We know that the series converges by the alternating series theorem, but does not converge absolutely by the p-series rule. Thus, it is conditionally convergent. Further, by examining the alternating series theorem, we see that all the partial sums are between $1/2$ and 1, so the sum is between $1/2$ and 1. Of the two possible sums given, $\ln 2$ is the only one that fits the bill.

Thus, by option elimination, the series converges to $\ln 2$. Because the convergence is only conditional, there is a rearrangement of the series that converges to $\pi^2/6$ (or for that matter, a rearrangement for any desired new sum).

Performance review: 33 out of 42 got this correct. 3 each chose (B) and (C). 1 each chose (A) and (E). 1 left the question blank.

(10) Consider a series $\sum a_n$ whose terms have alternating signs, with the first term (i.e., a_1) positive in sign, such that $|a_n|$ form a decreasing sequence and $\lim_{n\to\infty} |a_n| = 1$. Let $b_k = \sum_{n=1}^{2k-1} a_n$ (so these are the sums of odd numbers of initial terms), with $b = \lim_{k\to\infty} b_k$, and $c_k = \sum_{n=1}^{2k} a_n$ (so these are sums of even numbers of initial terms), with $c = \lim_{k\to\infty} c_k$. Which of the following is **true** about the sequences b_k and c_k and the limits b and c? *Hint: This is similar to the alternating series theorem. Make a picture of the number line and hop on it.*

(A) b_k form an increasing sequence, c_k form a decreasing sequence, and $b = c$.
(B) b_k form an increasing sequence, c_k form a decreasing sequence, and $b - c = 1$
(C) b_k form a decreasing sequence, c_k form an increasing sequence, and $b - c = 1$
(D) b_k form an increasing sequence, c_k form a decreasing sequence, and $c - b = 1$
(E) b_k form a decreasing sequence, c_k form an increasing sequence, and $c - b = 1$

Answer: Option (C)

Explanation: This is similar to the proof of the alternating series theorem. Please review the videos on the alternating series theorem if this is unclear.

Performance review: 30 out of 42 got this. 5 chose (B), 4 chose (A), 2 chose (D), 1 left the question blank.

For the next few questions, let $T : \mathbb{R} \to \mathbb{R}$ be a function defined as follows. For $a \in \mathbb{R}$, define:

$$T(a) := \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

Note that T is well defined because the summation on the right converges for every a.

(11) Which of the following function types does T have?

(A) Constant
(B) Linear
(C) Periodic
(D) Odd
(E) Even

Answer: Option (E)

Explanation: By inspection, we see that $T(a) = T(-a)$ for any $a \in \mathbb{R}$.

Note that this question should *not* be done by using the integral test because the summation is not equal to the integral, and the even or odd nature of the integral is not a good guide to the nature of the summation. But in this case, the integral also turns out to be an even functino.

Performance review: 33 out of 42 got this. 5 chose (D), 4 chose (C).

(12) Which of the following is true about T?

(A) T attains its absolute maximum at 0 and has no absolute minimum
(B) T attains its absolute maximum at 0 and its absolute minimum at 1 and -1
(C) T attains its absolute minimum at 0 and has no absolute maximum
(D) T attains its absolute minimum at 0 and its absolute maximum at 1 and -1
(E) T has no absolute maximum or absolute minimum

Answer: Option (A)

Explanation: Each of the summands of T is largest at $a = 0$ and gets smaller as $|a| \to \infty$. So the sum has an absolute maximum at $a = 0$ (value $\pi^2/6$) and has horizontal asymptotes to a value that isn't attained as $|a| \to \infty$. The value of the limit as $|a| \to \infty$ of $T(a)$ is 0, but you are not asked here to show that.

Performance review: 34 out of 42 got this. 3 chose (C), 1 each chose (B), (D), and (E). 2 left the question blank.

(13) How can we express the summation:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 3n^2 + 2}$$

in terms of T?
(A) $T(1) - T(2)$
(B) $T(2) - T(1)$
(C) $T(1) + T(2)$
(D) $T(1) - T(\sqrt{2})$
(E) $T(\sqrt{2}) - T(1)$

Answer: Option (D)

Explanation: Rewrite:

$$\frac{1}{n^4 + 3n^2 + 2} = \frac{1}{(n^2+1)(n^2+2)} = \frac{(n^2+2) - (n^2+1)}{(n^2+1)(n^2+2)} = \frac{1}{n^2+1} - \frac{1}{n^2+2}$$

The summation of $\frac{1}{n^2+1}$ is $T(1)$ and the summation of $\frac{1}{n^2+2}$ is $T(\sqrt{2})$.

Because everything being added here has an absolutely convergent summation, we can rearrange and re-group and get the sum as $T(1) - T(\sqrt{2})$.

Performance review: 37 out of 42 got this correct. 2 each chose (C) and (E), 1 left the question blank.

(14) Consider the function $F(x, p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. For what values of x and what values of p does this summation converge? *Two years ago:* 7/26 *correct*
(A) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| \geq 1$, it does not converge for any p.
(B) For $|x| \leq 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p.
(C) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $|x| = 1$, it converges if and only if $p > 1$.
(D) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $x = 1$, it converges if and only if $p > 0$. For $x = -1$, it converges if and only if $p > 1$.
(E) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any $p \in \mathbb{R}$. For $x = 1$, it converges if and only if $p > 1$. For $x = -1$, it converges if and only if $p > 0$.

Answer: Option (E)

Explanation: When $|x| < 1$, then the series is absolutely convergent and when $|x| > 1$ it diverges, as we can see by the root test or ratio test. What's happening is that $(1/n^p)^{1/n} \to 1$ (regardless of p), so the radius of convergence is 1.

This leaves the case $|x| = 1$. If $x = 1$, then we get the usual p-series, which we know converges iff $p > 1$. If $x = -1$, then the terms have alternating signs. Obviously, the series cannot converge for $p \leq 0$ because the terms do not tend to 0. For $p > 0$, on the other hand, the terms are alternating in sign and decrease monotonically, tending to 0. Thus, by the alternating series theorem, it converges for $p > 0$.

Performance review: 21 out of 42 got this correct. 15 chose (C), 4 chose (A), 2 chose (B).

Historical note (last year): 3 out of 11 got this. 4 chose (C), 2 each chose (B) and (D).

Historical note (two years ago): 7 out of 26 people got this correct. 10 chose (C), 4 chose (D), 3 chose (A), 2 chose (B). The large vote for (C) indicates that many people did not notice the special application of the alternating series theorem to the case of $x = -1$.

There is a result of calculus which states that, under suitable conditions, if $f_1, f_2, \dots, f_n, \dots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$. In other words, under suitable assumptions, we can differentiate a sum of countably many functions by differentiating each of them and adding up the derivatives.

We will not be going into what those assumptions are, but will consider some applications where you are explicitly told that these assumptions are satisfied.

(15) Consider the summation $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. Assume that the required assumptions are valid for this summation, so that $\zeta'(p)$ is the sum of the derivatives of each of the terms (summands) with respect to p. What is the correct expression for $\zeta'(p)$? *Two years ago:* 8/26 *correct*

(A) $\sum_{n=1}^{\infty} \frac{-p}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{-1}{(p+1)n^{p+1}}$
(C) $\sum_{n=1}^{\infty} \frac{p}{n^{p-1}}$
(D) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^p}$
(E) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^{p+1}}$

Answer: Option (D)

Explanation: We need to differentiate $(1/n)^p$ with respect to p. This is the same as differentiating a^x with respect to x, which gives $a^x \ln a$. In our case, we get $(1/n)^p \ln(1/n)$ which is $(-\ln n)/n^p$.

Note that Option (A) arises if we try to differentiate formally with respect to n, which is not the correct operation at all. n is a dummy variable and the expression should be differentiated with respect to p.

Performance review: 27 out of 42 got this correct. 12 chose (A), 2 chose (E), 1 chose (B).

Historical note (last year): 3 out of 11 got this. 8 chose (A). This indicates that many people differentiated with respect to the wrong variable.

Historical note (two years ago): 8 out of 26 people got this correct. 13 chose (A), 3 chose (B), 1 chose (C), and 1 left the question blank. The most commonly chosen wrong option, (A), indicates that many people differentiated with respect to the wrong variable.

(16) Recall that we defined $F(x, p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. Assume that, for a particular fixed value of p, the summation satisfies the conditions as a function of x for $|x| < 1$. What is its derivative with respect to x, keeping p constant? *Last year:* 14/26 *correct*

(A) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p-1}}$
(C) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p+1}}$
(D) $\sum_{n=1}^{\infty} \frac{x^{n-1} \ln n}{n^{p+1}}$
(E) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p-1}}$

Answer: Option (E)

Explanation: We need to differentiate each term with respect to x. Differentiating x^n/n^p with respect to x gives nx^{n-1}/n^p, which, upon rearrangement, gives x^{n-1}/n^{p-1}.

Performance review: 35 out of 42 got this correct. 3 chose (C), 2 each chose (B) and (D).

Historical note (last year): 5 out of 11 got this correct. 3 each chose (C) and (D).

Historical note (two years ago): 14 out of 26 got this correct. 4 each chose (B), (C), (D), possibly indicating minor computational errors.