CLASS QUIZ SOLUTIONS (TAKE-HOME): MARCH 2: LOOSE ENDS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

25 people took this quiz. The mean score was about 7. The score distribution was as follows:

- Score of 2: 2 people.
- Score of 4: 2 people.
- Score of 5: 4 people.
- Score of 6: 1 person.
- Score of 7: 3 people.
- Score of 8: 6 people.
- Score of 9: 1 person.
- Score of 10: 5 people.
- Score of 11: 1 person.

Here is the performance by question:

(1) Option (B): 21 people.
(2) Option (B): 8 people. *Master this!*
(3) Option (B): 11 people. *Master this!*
(4) Option (C): 8 people.
(5) Option (B): 7 people. *Master this!*
(6) Option (D): 13 people.
(7) Option (E): 10 people. *Master this!*
(8) Option (E): 11 people.
(9) Option (B): 23 people.
(10) Option (C): 17 people.
(11) Option (A): 17 people.
(12) Option (C): 15 people.
(13) Option (D): 12 people.
(14) Option (A): 4 people. *Master this!*

2. Solutions

(1) Which of the following is the correct definition of $\lim_{x \to \infty} f(x) = L$ for L a finite number?
(A) For every $\epsilon > 0$ there exists $a \in \mathbb{R}$ such that if $0 < |x - L| < \epsilon$ then $f(x) > a$.
(B) For every $\epsilon > 0$ there exists $a \in \mathbb{R}$ such that if $x > a$ then $|f(x) - L| < \epsilon$.
(C) For every $a \in \mathbb{R}$ there exists $\epsilon > 0$ such that if $x > a$ then $|f(x) - L| < \epsilon$.
(D) For every $a \in \mathbb{R}$ there exists $\epsilon > 0$ such that if $0 < |x - L| < \epsilon$ then $f(x) > a$.
(E) There exists $a \in \mathbb{R}$ and $\epsilon > 0$ such that if $x > a$ then $|f(x) - L| < \epsilon$.
Answer: Option (B)
Explanation: Straightforward unraveling of the definition.
Performance review: 21 out of 25 people got this correct. 1 person each chose options (A), (C), (D), and (E).

(2) Suppose $\lim_{x \to \infty} f'(x)$ is finite. Which of the following is true (be careful about f versus f' when reading the choices)?
(A) If $\lim_{x \to \infty} f'(x)$ is zero, then $\lim_{x \to \infty} f(x)$ is finite.
(B) If $\lim_{x \to \infty} f(x)$ is finite, then $\lim_{x \to \infty} f'(x)$ is zero.
(C) If $\lim_{x \to \infty} f(x)$ is finite, then $\lim_{x \to \infty} f(x)$ is zero.

(D) All of the above.
(E) None of the above.

Answer: Option (B)

Explanation: If f is going to a finite value, its derivative cannot have a nonzero limit, because that would mean a roughly linear behavior. Since the derivative has a limit, it must go to zero.

Note that *if we were not explicitly told that the derivative has a limit*, then option (E) would be the correct option.

Note that option (A) is wrong, because we can construct counterexamples such as ln and the square root function, where the derivative of the function goes to zero but the function does not have a finite limit at the point.

Performance review: 8 out of 25 people got this correct. 10 people chose (A), 5 people chose (E), and 2 people chose (C).

Action point: Make sure you understand this and get it right in the future!

(3) Suppose a function y of time t satisfies the differential equation $y' = f(y)$ for all time t, where f is a continuous function on $\mathbb{R}$. Further, suppose we know that $\lim_{t \to \infty} y = L$ for some finite L. What can we conclude is true about L?

(A) $f(L) = L$
(B) $f(L) = 0$
(C) $f'(L) = L$
(D) $f'(L) = 0$
(E) $f''(L) = 0$

Answer: Option (B)

Explanation: We in particular have that $\lim_{t \to \infty} y' = \lim_{t \to \infty} f(y) = f(L)$, hence $\lim_{t \to \infty} y'$ is finite. Applying the previous question, we obtain that this limit must be 0, so we get $f(L) = 0$.

Performance review: 11 out of 25 people got this correct. 6 people chose (D), 5 people chose (C), and 3 people chose (A).

Those who chose option (A) probably used the discrete analogy. While the analogy works qualitatively, the conclusion was wrongly applied. Here, f is not the function being iterated (which is the discrete setup) but rather, it is the derivative of a changing value. When the value becomes constant, f must become zero.

Action point: Make sure you understand this and get it right in the future.

(4) A sequence a_n is found to satisfy the recurrence $a_{n+1} = 2a_n(1 - a_n)$. Assume that a_1 is strictly between 0 and 1. What can we say about the sequence (a_n)?

(a) It is monotonic non-increasing, and its limit is 0.
(b) It is monotonic non-decreasing, and its limit is 1.
(c) From a_2 onward, it is monotonic non-decreasing, and its limit is 1/2.
(d) From a_2 onward, it is monotonic non-increasing, and its limit is 1/2.
(e) It is either monotonic non-decreasing or monotonic non-increasing everywhere, and its limit is 1/2.

Answer: Option (C)

Explanation: Whatever the value of a_1, $0 < a_2 \leq 1/2$. Once we are in this interval, we see that $f(x) \geq x$ for all x in the interval, and $f(x)$ is also in the interval. Thus, the sequence is monotonic non-decreasing from a_2 onward, and is bounded from above by 1/2. It converges to its greatest lower bound, which we know must be fixed under f. Hence, it must converge to 1/2, which is the only positive number fixed under f.

(More details can be worked out using algebra/calculus).

Performance review: 8 out of 25 people got this correct. 8 people chose (D), 4 chose (A), 3 chose (E), 2 chose (B).

(5) Suppose f is a continuous function on $\mathbb{R}$ and (a_n) is a sequence satisfying the recurrence $f(a_n) = a_{n+1}$ for all n. Further, suppose the limit of the a_ns for odd n is L and the limit of the a_ns for even n is M. What can we say about L and M?

(A) $f(L) = L$ and $f(M) = M$
(B) $f(L) = M$ and $f(M) = L$

(C) $f(L) = f(M) = 0$
(D) $f'(L) = f'(M) = 0$
(E) $f'(L) = M$ and $f'(M) = L$

Answer: Option (B)

Explanation: Each even indexed term is obtained by applying f to the preceding odd indexed term, and each odd indexed term is obtained by applying f to the preceding even indexed term. Taking appropriate limits, we get the desired conclusion.

Performance review: 7 out of 25 people got this correct. 8 chose (A), 4 each chose (C) and (D), 1 chose (E), and 1 left the question blank.

Action point: After understanding the solution, you should not forget the idea!

(6) Consider a function f on the natural numbers defined as follows: $f(m) = m/2$ if m is even, and $f(m) = 3m + 1$ if m is odd. Consider a sequence where a_1 is a natural number and we define $a_n := f(a_{n-1})$. It is conjectured (see *Collatz conjecture*) that (a_n) is eventually periodic, regardless of the starting point, and that there is only one possibility for the eventual periodic fragment. Which of the following can be the eventual periodic fragment?
(A) $(1, 2, 3)$
(B) $(1, 3, 2)$
(C) $(1, 2, 4)$
(D) $(1, 4, 2)$
(E) $(1, 3, 4)$

Answer: Option (D)

Explanation: Can be seen by applying the definition.

Performance review: 13 out of 25 people got this correct. 5 chose (C), 3 chose (B), 2 chose (A), 1 chose (E), and 1 left the question blank.

Those who chose (C) probably overlooked the issue of the cyclic ordering of elements within the periodic sequence.

Extra credit: Prove the Collatz conjecture.

(7) For which of the following properties p of sequences of real numbers does p equal *eventually* p?
(A) Monotonicity
(B) Periodicity
(C) Being a polynomial sequence (i.e., given by a polynomial function)
(D) Being a constant sequence
(E) Boundedness

Answer: Option (E)

Explanation: If a sequence is eventually bounded, then that means that excluding the first few terms gives a bounded sequence. But throwing back these finitely many terms, which have a fixed maximum and minimum, will still give a bounded sequence.

Performance review: 10 out of 25 people got this correct. 5 each chose (A) and (D), 2 each chose (B) and (C), and 1 left the question blank.

Action point: You should definitely understand, appreciate, and remember this one!

(8) Which of the following series converges? Assume for all series that the startin point of summation is large enough that the terms are well defined.
(A) $\sum 1/(k \ln(\ln k))$
(B) $\sum 1/(k \ln k)$
(C) $\sum 1/(k(\ln(\ln k))^2)$
(D) $\sum 1/(k(\ln k)(\ln(\ln k)))$
(E) $\sum 1/(k(\ln k)(\ln(\ln k))^2)$

Answer: Option (E)

Explanation: Options (B) and (D) diverge by the integral test. As for options (A) and (C), these have smaller denominators, hence larger terms, than option (B), hence, by basic comparison, these diverge too. This leaves option (E), which converges by the integral test.

Perofmrance review: 11 out of 25 people got this correct. 7 chose (C), 3 each chose (A) and (B), and 1 left the question blank.

The main attraction of (C) seems to have been its superficial resemblance to $1/(k(\ln k)^2)$ which does converge.

(9) Which of the following series converges?
(A) $\sum \frac{k+\sin k}{k^2+1}$
(B) $\sum \frac{k+\cos k}{k^3+1}$
(C) $\sum \frac{k^2-\sin k}{k+1}$
(D) $\sum \frac{k^3+\cos k}{k^2+1}$
(E) $\sum \frac{k}{\sin(k^3+1)}$

Answer: Option (B)

Explanation: We can use a comparison test, either rigorously or in the form of a heuristic of looking at degree of denominator minus degree of numerator. Note that for (A), the degree difference is 1, so it diverges. For (C) and (D), the numerator actually has larger degree than the denominator, so it diverges. For (E), the denominator is bounded in $[-1, 1]$, and the numerator goes to ∞, so it diverges. This leaves (B), which converges because the degree of denominator minus degree of numerator equals 2.

Performance review: 23 out of 25 people got this correct. 1 person chose (C) and 1 person chose (D).

Suppose F is a function of two real variables, say x and t, so $F(x, t)$ is a real number for x and t restricted to suitable open intervals in the real number. Suppose, further, that F is jointly continuous (whatever that means) in x and t.

Define $f(t) := \int_0^\infty F(x, t)\, dx$. Here, while doing the integration, t is treated as a constant. x, the variable of integration, is being integrated on $[0, \infty)$.

Suppose further that f is defined and continuous for t in $(0, \infty)$. *Note that similar computations we did in the midterm review session involved integration from $-\infty$ to ∞.*

In the next few questions, you are asked to compute the function f explicitly given the function F, for $t \in (0, \infty)$.

(10) $F(x, t) := e^{-tx}$. Find f.
(A) $f(t) = e^{-t}/t$
(B) $f(t) = e^t/t$
(C) $f(t) = 1/t$
(D) $f(t) = -1/t$
(E) $f(t) = -t$

Answer: Option (C)

Explanation: The integral becomes $[-e^{-tx}/t]_0^\infty$. Plugging in at ∞ gives 0 and plugging in at 0 gives $-1/t$. Since the value at 0 is being subtracted, we eventually get $1/t$.

Note that the answer must be positive for the simple reason that we are integrating a positive function from left to right across an interval.

Performance review: 17 out of 25 people got this correct. 4 chose (A), 3 chose (D), and 1 chose (E).

(11) $F(x, t) := 1/(t^2 + x^2)$. Find f.
(A) $f(t) = \pi/(2t)$
(B) $f(t) = \pi/t$
(C) $f(t) = 2\pi/t$
(D) $f(t) = \pi t$
(E) $f(t) = 2\pi t$

Answer: Option (A)

Explanation: We get $[(1/t)\arctan(x/t)]_0^\infty$. The evaluation at ∞ gives $\pi/(2t)$ and the evaluation at 0 gives 0. Subtracting, we get $\pi/(2t)$.

Performance review: 17 out of 25 got this correct. 5 chose (B), 2 chose (D), 1 chose (C).

(12) $F(x, t) := 1/(t^2 + x^2)^2$. Find f.
(A) $f(t) = \pi/t^3$
(B) $f(t) = \pi/(2t^3)$

(C) $f(t) = \pi/(4t^3)$
(D) $f(t) = \pi/(8t^3)$
(E) $f(t) = 3\pi/(8t^3)$

Answer: Option (C)

Explanation: Put in $\theta = \arctan(x/t)$. Substitute, and we get $(1/t^3)\int_0^{\pi/2}\cos^2\theta\,d\theta$. Integrating, we get $[\theta/2t^3 + \sin(2\theta)/4t^3]_0^{\pi/2}$. The trigonometric part vanishes between limits, and we are left with $\pi/(4t^3)$

Performance review: 15 out of 25 people got this correct. 5 chose (B), 2 chose (A), 1 each chose (D) and (E), 1 left the question blank.

(13) $F(x,t) = \exp(-(tx)^2)$. Use that $\int_0^\infty \exp(-x^2)\,dx = \sqrt{\pi}/2$. Find f.
(A) $f(t) = t^2\sqrt{\pi}/2$
(B) $f(t) = t\sqrt{\pi}/2$
(C) $f(t) = \sqrt{\pi}/2$
(D) $f(t) = \sqrt{\pi}/(2t)$
(E) $f(t) = \sqrt{\pi}/(2t^2)$

Answer: Option (D)

Explanation: Put $u = tx$, get a $1/t$ on the outside, giving $(1/t)\int_0^\infty \exp(-u^2)\,du$.

Performance review: 12 out of 25 people got the question correct. 6 chose (E), 3 chose (C), 2 each chose (A) and (B).

(14) In the same general setup as above (but with none of these specific Fs), which of the following is a *sufficient* condition for f to be an increasing function of t?
(A) $t \mapsto F(x_0, t)$ is an increasing function of t for every choice of $x_0 \geq 0$.
(B) $x \mapsto F(x, t_0)$ is an increasing function of x for every choice of $t_0 \in (0, \infty)$.
(C) $t \mapsto F(x_0, t)$ is a decreasing function of t for every choice of $x_0 \geq 0$.
(D) $x \mapsto F(x, t_0)$ is a decreasing function of x for every choice of $t_0 \in (0, \infty)$.
(E) None of the above.

Answer: Option (A)

Explanation: If F is increasing in t for every value of x_0, then that means that as t gets bigger, the function F being integrated gets bigger everywhere in x, i.e., if $t_1 < t_2$, then $F(t_1, x_0) < F(t_2, x_0)$ for every $x_0 \geq 0$. The integral for the larger value t_2 must therefore also be bigger. (We looked at this stuff in Section 5.8 of the book).

Performance review: 4 out of 25 got the question correct. 10 chose (B), 5 chose (E), 3 chose (C), 2 chose (D), and 1 left the question blank.

(A) was the "obvious" choice – people may have tried to seek more subtletly in the question than it had.

Action point: This should not trip *anybody* in the future.