DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 25: MATRIX MULTIPLICATION (BASIC)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

26 people took this 5-quesion quiz. The score distribution was as follows:

- Score of 0: 4 people
- Score of 1: 1 person
- Score of 2: 4 people
- Score of 3: 9 people
- Score of 4: 3 people
- Score of 5: 5 people

The mean score was 2.8.

The question-wise answers and performance review were as follows:

(1) Option (D): 17 people
(2) Option (B): 8 people
(3) Option (A): 14 people
(4) Option (D): 14 people
(5) Option (B): 20 people

2. Solutions

PLEASE DO *NOT* DISCUSS ANY QUESTIONS

This quiz tests for basic comprehension of the setup for matrix multiplication. It corresponds to the material from Sections 1-6 (excluding Section 4) of the `Matrix multiplication and inversion` notes, and also to Section 2.3 of the book.

(1) *Do not discuss this!*: Suppose A and B are (not necessarily square) matrices. Then, which of the following describes correctly the relationship between the existence and value of the (alleged) matrix product AB and the existence and value of the (alleged) matrix product BA?
(A) AB is defined if and only if BA is defined, and if so, they are equal.
(B) AB is defined if and only if BA is defined, but they need not be equal.
(C) If AB and BA are both defined, then $AB = BA$. However, it is possible for one of AB and BA to be defined and the other to not be defined.
(D) It is possible for only one of AB and BA to be defined. It is also possible for both AB and BA to be defined, but to not be equal to each other.

Answer: Option (D)

Explanation: Suppose A is a $m \times n$ matrix (i.e., it has m rows and n columns) and B is a $p \times q$ matrix (i.e., it has p rows and q columns), where m, n, p, and q are positive integers. AB is defined if and only if $n = p$, i.e., the number of columns of A equals the number of rows of B. BA is defined if and only if $m = q$, i.e., the number of columns of B equals the number of rows of A. The conditions are independent of one another, so it is possible for only one of AB and BA to be defined.

Suppose now that AB and BA are both defined. Then, $n = p$ and $m = q$, so B is a $n \times m$ matrix. Thus, AB is a $m \times m$ matrix and BA is a $n \times n$ matrix, with m and n possibly different. Therefore, AB and BA do not even necessarily have the same dimensions, and therefore they definitely are not required to be equal.

Not clear to you?: Try picking actual numerical values of m, n, p, and q, write down actual example matrices, and see how the multiplication works.

Performance review: 17 out of 26 got this. 4 chose (B), 3 chose (C), 2 chose (A).

(2) *Do not discuss this!*: Suppose A and B are matrices such that both AB and BA are defined. Which of the following correctly describes what we know about AB and BA?
   (A) Both AB and BA are square matrices and have the same dimensions, i.e., in both AB and BA, the number of rows equals the number of columns, and further, the number of rows of AB equals the number of rows of BA.
   (B) Both AB and BA are square matrices (the number of rows equals the number of columns) but they may not have the same dimensions: the number of rows in AB need not equal the number of rows in BA.
   (C) AB and BA need not be square matrices but both must have the same dimensions: the number of rows in AB equals the number of rows in BA, and the number of columns in AB equals the number of columns in BA.
   (D) AB and BA need not be square matrices and they need not have the same row count or the same column count, i.e., the number of rows in AB need not equal the number of rows in BA, and the number of columns in AB need not equal the number of columns in BA.

   Answer: Option (B)

   Explanation: See the second paragraph of the explanation for the preceding question. The conclusion there was that if AB and BA are both defined, then for A a $m \times n$ matrix, B is a $n \times m$ matrix, and thus AB is $m \times m$ and BA is $n \times n$. m need not be equal to n.

   Not clear to you?: Just try picking positive integer values of m and n that are not equal, and try constructing a $m \times n$ matrix A and a $n \times m$ matrix B. Then, compute AB and BA and verify that they are square matrices of different dimensions.

   Performance review: 8 out of 26 got this. 8 chose (D), 7 chose (A), 3 chose (C).

(3) *Do not discuss this!*: Suppose A and B are matrices such that both AB and $A + B$ are defined. Which of the following correctly describes what we know about A and B?
   (A) Both A and B are square matrices and have the same dimensions, i.e., in both A and B, the number of rows equals the number of columns, and further, the number of rows of A equals the number of rows of B.
   (B) Both A and B are square matrices (the number of rows equals the number of columns) but they may not have the same dimensions: the number of rows in A need not equal the number of rows in B.
   (C) A and B need not be square matrices but both must have the same dimensions: the number of rows in A equals the number of rows in B, and the number of columns in A equals the number of columns in B.
   (D) A and B need not be square matrices and they need not have the same row count or the same column count, i.e., the number of rows in A need not equal the number of rows in B, and the number of columns in A need not equal the number of columns in B.

   Answer: Option (A)

   Explanation: Suppose A is a $m \times n$ matrix (i.e., it has m rows and n columns) and B is a $p \times q$ matrix (i.e., it has p rows and q columns). The condition that $A + B$ is defined tells us that $m = p$ (i.e., A and B have the same number of rows as each other) and that $n = q$ (i.e., A and B have the same number of columns as each other). Thus, both A and B are $m \times n$ matrices. In order for AB to make sense, we need n (the number of columns of A) to equal m (the number of rows of B). Thus, $m = n$, so that both A and B are $m \times m$ matrices.

   Not clear to you?: Try writing an example that violates the conditions of Option (A) and see for yourself that you'll run into trouble either with computing $A + B$ or with computing AB.

   Performance review: 14 out of 26 got this. 12 chose (C).

(4) *Do not discuss this!*: Suppose A is a $p \times q$ matrix and B is a $q \times r$ matrix. The product matrix AB is a $p \times r$ matrix. Using the convention of matrices as linear transformations via their action by multiplication on column vectors, what is the appropriate interpretation of the matrix product in terms of composing linear transformations?

(A) A corresponds to a linear transformation T_A from $\mathbb{R}^p$ to $\mathbb{R}^q$, and B corresponds to a linear transformation T_B from $\mathbb{R}^q$ to $\mathbb{R}^r$. The product AB therefore corresponds to a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^r$ that is the composite of the two linear transformations, with T_A applied first (to the domain) and then T_B (T_B being applied to the intermediate space obtained after applying T_A).
(B) A corresponds to a linear transformation T_A from $\mathbb{R}^p$ to $\mathbb{R}^q$, and B corresponds to a linear transformation T_B from $\mathbb{R}^q$ to $\mathbb{R}^r$. The product AB therefore corresponds to a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^r$ that is the composite of the two linear transformations, with T_B applied first (to the domain) and then T_A (T_A being applied to the intermediate space obtained after applying T_B).
(C) A corresponds to a linear transformation T_A from $\mathbb{R}^q$ to $\mathbb{R}^p$, and B corresponds to a linear transformation T_B from $\mathbb{R}^r$ to $\mathbb{R}^q$. The product AB therefore corresponds to a linear transformation from $\mathbb{R}^r$ to $\mathbb{R}^p$ that is the composite of the two linear transformations, with T_A applied first (to the domain) and then T_B (T_B being applied to the intermediate space obtained after applying T_A).
(D) A corresponds to a linear transformation T_A from $\mathbb{R}^q$ to $\mathbb{R}^p$, and B corresponds to a linear transformation T_B from $\mathbb{R}^r$ to $\mathbb{R}^q$. The product AB therefore corresponds to a linear transformation from $\mathbb{R}^r$ to $\mathbb{R}^p$ that is the composite of the two linear transformations, with T_B applied first (to the domain) and then T_A (T_A being applied to the intermediate space obtained after applying T_B).

Answer: Option (D)

Explanation: Review the lecture notes regarding the interpretation of matrix multiplication as composition.

Performance review: 14 out of 26 got this. 9 chose (C), 2 chose (B), 1 chose (A).

(5) *Do not discuss this!*: Suppose A, B, and C are matrices. Which of the following is true?
(A) If ABC is defined, then so are BCA and CAB.
(B) If ABC and BCA are both defined, then so is CAB. However, it is possible to have a situation where ABC is defined but BCA and CAB are not defined.
(C) It is possible to have a situation where ABC and BCA are both defined but CAB is not defined.

Answer: Option (B)

Explanation: Suppose A is a $m \times n$ matrix (i.e., it has m rows and n columns), B is a $p \times q$ matrix (i.e., it has p rows and q columns), and C is a $r \times s$ matrix (i.e., it has r rows and s columns). In order for ABC to be defined, we need $n = p$ (for AB to be defined) and $q = r$ (for BC to be defined). However, we do not need $s = m$, the condition that would allow us to multiply C with A. Therefore, we have no guarantee that the products BCA and CAB are defined.

If both ABC and BCA are defined, then we get the additional condition that $s = m$, and this allows us to define CAB.

Performance review: 20 out of 26 got this. 4 chose (A), 2 chose (C).