TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY FEBRUARY 20: INTEGRATION TECHNIQUES (ONE VARIABLE)

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this quiz. The score distribution was as follows:

- Score of 3: 1 person.
- Score of 5: 1 person.
- Score of 6: 1 person.
- Score of 9: 1 person.
- Score of 10: 1 person.
- Score of 12: 1 person.
- Score of 15: 1 person.
- Score of 18: 7 people.
- Score of 19: 2 people.
- Score of 20: 4 people.
- Score of 21: 5 people.

Here are the question wise answers and performance review:

(1) Option (A): 20 people.
(2) Option (B): 22 people.
(3) Option (A): 17 people.
(4) Option (C): 19 people.
(5) Option (B): 19 people.
(6) Option (D): 21 people.
(7) Option (B): 20 people.
(8) Option (D): 22 people.
(9) Option (B): 21 people.
(10) Option (C): 21 people.
(11) Option (C): 20 people.
(12) Option (D): 20 people.
(13) Option (A): 22 people.
(14) Option (E): 21 people.
(15) Option (D): 22 people.
(16) Option (B): 15 people.
(17) Option (D): 21 people.
(18) Option (B): 17 people.
(19) Option (E): 9 people.
(20) Option (D): 23 people.
(21) Option (D): 17 people.

2. Solutions

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

(1) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? Please see Option (E) before answering.
(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.
Answer: Option (A).
Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.
Performance review: 20 out of 25 people got this. 3 chose (E), 1 each chose (B) and (D).
Historical note (earlier appearance this quarter): 15 out of 24 got this. 6 chose (E), 3 chose (B).

(2) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
(A) $k - 2$
(B) $k - 1$
(C) k
(D) $k + 1$
(E) There is no bound in terms of k.
Answer: Option (B)
Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F - G$ have k^{th} derivative equal to zero. For $k = 1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k - 1$.
Performance review: 22 out of 25 people got this. 2 chose (E), 1 chose (C).
Historical note (earlier appearance this quarter): 21 out of 24 got this. 2 chose (A), 1 chose (E).

(3) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).
(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.
Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 17 out of 25 people got this. 5 chose (D), 2 chose (E), 1 chose (B).

Historical note (earlier appeance this quarter): 18 out of 24 got this. 3 chose (C), 2 chose (E), 1 chose (B).

(4) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see options (D) and (E) before answering.
(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 19 out of 25 people got this. 6 chose (D).

Historical note (earlier appearance this quarter): 16 out of 24 got this. 8 chose (D).

(5) Suppose F, G are continuously differentiable functions defined on all of $\mathbb{R}$. Suppose a, b are real numbers with $a < b$. Suppose, further, that $G(x)$ is identically zero everywhere except on the open interval (a, b). Then, what can we say about the relationship between the numbers $P = \int_a^b F(x)G'(x)\,dx$ and $Q = \int_a^b F'(x)G(x)\,dx$?
(A) $P = Q$
(B) $P = -Q$
(C) $PQ = 0$
(D) $P = 1 - Q$
(E) $PQ = 1$

Answer: Option (B)

Explanation: Integration by parts gives us that:

$$\int_a^b F(x)G'(x)\,dx = [F(x)G(x)]_a^b - \int_a^b F'(x)G(x)\,dx$$

Since $G(x) = 0$ outside (a, b), we get that $G(a) = G(b) = 0$, so that the evaluation of $[F(x)G(x)]_a^b$ gives 0. We are thus left with:

$$P = -Q$$

Performance review: 19 out of 25 people got this. 3 chose (D), 2 chose (A), 1 chose (C).

Historical note (Math 153): 32 out of 41 got this. 5 chose (A), 2 chose (C), 1 chose (D), 1 wrote multiple options.

(6) Consider the integration $\int p(x)q''(x)\,dx$. Apply integration by parts twice, first taking p as the part to differentiate, and q as the part to integrate, and then again apply integration by parts to avoid a circular trap. What can we conclude?
(A) $\int p(x)q''(x)\,dx = \int p''(x)q(x)\,dx$

(B) $\int p(x)q''(x)\,dx = \int p'(x)q'(x)\,dx - \int p''(x)q(x)\,dx$
(C) $\int p(x)q''(x)\,dx = p'(x)q'(x) - \int p''(x)q(x)\,dx$
(D) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) + \int p''(x)q(x)\,dx$
(E) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) - \int p''(x)q(x)\,dx$

Answer: Option (D)

Explanation: Just write it out.

Performance review: 21 out of 25 people got this. 2 chose (C), 1 each chose (B) and (E).

Historical note (Math 153): 35 out of 41 got this. 4 chose (E) (sign error, didn't notice double negative), 2 chose (A).

(7) Suppose p is a polynomial function. In order to find the indefinite integral for a function of the form $x \mapsto p(x)\exp(x)$, the general strategy, which always works, is to take $p(x)$ as the part to differentiate and $\exp(x)$ as the part to integrate, and keep repeating the process. Which of the following is the best explanation for why this strategy works?

(A) exp can be repeatedly differentiated (staying exp) and polynomials can be repeatedly integrated (giving polynomials all the way).
(B) exp can be repeatedly integrated (staying exp) and polynomials can be repeatedly differentiated, eventually becoming zero.
(C) exp and polynomials can both be repeatedly differentiated.
(D) exp and polynomials can both be repeatedly integrated.
(E) We need to use the recursive version of integration by parts whereby the original integrand reappears after a certain number of applications of integration by parts (i.e., the polynomial equals one of its higher derivatives, up to sign and scaling).

Answer: Option (B)

Explanation: This follows because the polynomial is the part that we are choosing to differentiate.

Performance review: 20 out of 25 people got this. 2 each chose (C) and (E). 1 left the question blank.

Historical note (Math 153): All 41 got this.

(8) Consider the function $x \mapsto \exp(x)\sin x$. This function can be integrated using integration by parts. What can we say about how integration by parts works?

(A) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once to get the answer directly.
(B) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(C) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice to get the answer directly.
(D) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(E) We choose exp as the part to integrate and sin as the part to differentiate, and we apply integration by parts four times to get the answer directly.

Answer: Option (D)

Explanation: sin is the negative of its second derivative, exp equals its second antiderivative.

Performance review: 22 out of 25 people got this. 3 chose (C).

Historical note (Math 153): 38 out of 41 got this. 3 chose (C).

(9) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is *guaranteed to be* **elementarily integrable**?

(A) 1
(B) 2
(C) 3

(D) 4
(E) 5

Answer: Option (B)

Explanation: Via integration by parts, integrating f m times is equivalent to finding antiderivatives for $f(x)$, $xf(x)$, and so on till $x^{m-1}f(x)$. In our case, f can be integrated 3 times, so the largest k is $3-1=2$.

Performance review: 21 out of 25 people got this. 3 chose (C), 1 chose (D).

Historical note (last time): 5 out of 18 people got this correct. 6 chose (D), 5 chose (C), and 2 chose (E).

(10) Suppose f is a continuous function on $(0,\infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0,\infty)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (C)

Explanation: Via the u-substitution $u = x^{1/k}$, we get $\int ku^{k-1}f(u)\,du$. Now using the previous question, the maximum value of $k-1$ possible is 2, so the maximum possible value is 3.

We can also do a direct integration by parts taking 1 as the second part.

Performance review: 21 out of 25 people got this. 2 each chose (A) and (B).

Historical note (last time): 8 out of 18 people got this correct. 5 chose (A), 3 chose (D), and 2 chose (B).

(11) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.
(A) $x \mapsto x\sin x$
(B) $x \mapsto x\cos x$
(C) $x \mapsto x\tan x$
(D) $x \mapsto x\sin^2 x$
(E) $x \mapsto x\tan^2 x$

Answer: Option (C)

Explanation: If f is elementarily integrable, then $xf(x)$ is elementarily integrable iff f is twice elementarily integrable; this is easily seen using integration by parts. Of the function options given here, tan is the only function that is not twice elementarily integrable, because the first integration gives $-\ln|\cos x|$ which cannot be integrated. Of the others, note that sin, cos, and $\sin^2$ can be integrated using elementary functions infinitely many times. $\tan^2$ is twice elementarily integrable but no further: integrates the first time to $\tan x - x$, which integrates one more time to $-\ln|cosx| - x^2/2$, which cannot be integrated further.

Performance review: 20 out of 25 people got this. 2 each chose (D) and (E), 1 chose (B).

Historical note (last time): 8 out of 18 people got this correct. 8 people chose (E) and 2 people chose (D).

(12) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin\sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0,1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**?
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.
(D) f_2 and f_3.
(E) f_2 and f_4.

Answer: Option (D)

Explanation: Integration of f_3 is a standard procedure, so we say nothing about that. As for f_2, recall that integrating $f(x^{1/k})$ is equivalent to integrating $u^{k-1}f(u)$ where $u = x^{1/k}$, which in turn is equivalent to integrating f k times. Since sin can be integrated as many times as we wish, f_2 can be integrated.

The reason why f_1 and f_4 are not elementarily integrable is subtler but it's clear that none of the obvious methods work.

Performance review: 20 out of 25 people got this. 2 chose (B), 1 each chose (A), (C), and (E).

Historical note (last time): 5 out of 18 people got this correct. 5 people each chose (B) and (E), 2 chose (C), 1 chose (A).

(13) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?

(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2e^{-x^2}$
(E) $x \mapsto x^4e^{-x^2}$

Answer: Option (A)

Explanation: We show the equivalence with the others.

Option (D): We use integration by parts, writing $x^2e^{-x^2}$ as $x \cdot (xe^{-x^2})$ and taking xe^{-x^2} as the part to integrate, so that x is the part to differentiate. An antiderivative for xe^{-x^2} is $(-1/2)e^{-x^2}$, so we get:

$$\frac{-x}{2}e^{-x^2} - \int \frac{-1}{2}e^{-x^2}\,dx$$

We thus see that it reduces to $\int e^{-x^2}\,dx$.

Option (E), via reduction to option (D): We use integration by parts, taking x^3 as the part to differentiate and xe^{-x^2} as the part to integrate. One application of integration by parts reduces this to $\int x^2e^{-x^2}$, which is option (D).

Option (B), via reduction to option (D): Start with $\int e^{-x^{2/3}}\,dx$. Put $u = x^{1/3}$. The substitution gives (up to scalars) $\int u^2e^{-u^2}\,du$, which is option (D).

Option (C), via reduction to option (D): Start with $\int e^{-x^{2/5}}\,dx$. Put $u = x^{1/5}$. The substitution gives (up to scalars) $\int u^4e^{-u^2}\,du$, which is option (E).

Performance review: 22 out of 25 people got this. 1 each chose (B), (D), and (E).

Historical note (last time): 6 out of 18 people got this correct. 6 chose (C), 2 each chose (B), (D), (E).

(14) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?

(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\dots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Answer: Option (E)

Explanation: Review the material on rational function integration.

Performance review: 21 out of 25 got this. 2 chose (D), 1 each chose (B) and (C).

Historical note (last time): 3 out of 18 people got this correct. 7 chose (C), 6 chose (D), 1 each chose (A) and (B).

(15) Which of these functions of x is *not* elementarily integrable?

(A) $x\sqrt{1+x^2}$
(B) $x^2\sqrt{1+x^2}$
(C) $x(1+x^2)^{1/3}$
(D) $x\sqrt{1+x^3}$
(E) $x^2\sqrt{1+x^3}$

Answer: Option (D)

—em Explanation: For options (A) and (C), the substitution $u = 1 + x^2$ works fine. For option (E), the substitution $u = 1 + x^3$ works fine. For option (B), we can solve the problem using a trigonometric substitution. This leaves option (D) (which, incidentally, requires the use of elliptic integrals).

Performance review: 22 out of 25 got this. 3 chose (C).

Historical note (last time): 9 out of 18 people got this correct. 4 chose (C), 4 chose (B), 1 chose (A).

(16) Consider the function $f(k) := \int_1^2 \frac{dx}{\sqrt{x^2+k}}$. f is defined for $k \in (-1, \infty)$. What can we say about the nature of f within this interval?

(A) f is increasing on the interval $(-1, \infty)$.
(B) f is decreasing on the interval $(-1, \infty)$.
(C) f is increasing on $(-1, 0)$ and decreasing on $(0, \infty)$.
(D) f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.
(E) f is increasing on $(-1, 0)$, decreasing on $(0, 2)$, and increasing again on $(2, \infty)$.

Answer: Option (B)

Explanation: For any fixed value of $x \in [1, 2]$, the integrand $1/\sqrt{x^2+k}$ is a *decreasing* function of k for $k \in (-1, \infty)$. Hence, the value we get upon integrating it for $x \in [1, 2]$ should also be a decreasing function of k.

Performance review: 15 out of 25 got this. 6 chose (D), 3 chose (C), 1 chose (A).

Historical note (last time): 6 out of 18 people got this correct. 3 chose (A), 5 chose (C), 2 chose (E), 1 chose (D), 1 left the question blank.

(17) For which of these functions of x does the antiderivative necessarily involve *both* arctan *and* ln?

(A) $1/(x+1)$
(B) $1/(x^2+1)$
(C) $x/(x^2+1)$
(D) $x/(x^3+1)$
(E) $x^2/(x^3+1)$

Answer: Option (D)

Explanation: Option (A) integrates to $\ln|x+1|$, option (B) integrates to $\arctan x$, option (C) integrates to $(1/2)\ln(x^2+1)$, and option (E) integrates to $(1/3)\ln|x^3+1|$. For option (D), we need to use partial fractions with denominators $x+1$ and x^2-x+1, and we end up getting nonzero coefficients on terms that integrate to ln and to arctan.

Performance review: 21 out of 25 people got this. 2 chose (E), 1 each chose (B) and (C).

Historical note (last time): 13 out of 18 people got this correct. 3 chose (C), 1 each chose (B) and (E).

(18) Suppose F is a (not known) function defined on $\mathbb{R} \setminus \{-1, 0, 1\}$, differentiable everywhere on its domain, such that $F'(x) = 1/(x^3 - x)$ everywhere on $\mathbb{R} \setminus \{-1, 0, 1\}$. For which of the following sets of points is it true that knowing the value of F at these points **uniquely** determines F?
(A) $\{-\pi, -e, 1/e, 1/\pi\}$
(B) $\{-\pi/2, -\sqrt{3}/2, 11/17, \pi^2/6\}$
(C) $\{-\pi^3/7, -\pi^2/6, \sqrt{13}, 11/2\}$
(D) Knowing F at any of the above determines the value of F uniquely.
(E) None of the above works to uniquely determine the value of F.
Answer: Option (B)
Explanation: The domain of F has four connected components: the open intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We need to know the value of F at one point in each of these intervals. By computing values, we see that the set of points in option (B) has the property that it contains one point in each of these intervals, and those in options (A) and (C) do not.
Performance review: 17 out of 25 people got this. 4 chose (A), 2 chose (E), 1 each chose (C) and (D).
Historical note (last time): 8 out of 18 people got this correct. 5 chose (D), 3 chose (A), 2 left the question blank.

(19) Suppose F is a continuously differentiable function whose domain contains (a, ∞) for some $a \in \mathbb{R}$, and $F'(x)$ is a rational function $p(x)/q(x)$ on the domain of F. Further, suppose that p and q are nonzero polynomials. Denote by d_p the degree of p and by d_q the degree of q. Which of the following is a **necessary and sufficient condition** to ensure that $\lim_{x\to\infty} F(x)$ is finite?
(A) $d_p - d_q \geq 2$
(B) $d_p - d_q \geq 1$
(C) $d_p = d_q$
(D) $d_q - d_p \geq 1$
(E) $d_q - d_p \geq 2$
Answer: Option (E)
Explanation: This can be justified in terms of partial fractions. Th ecase where q is a product of linear factors can be justified using the previous question. But that is not the most elegant justification. When we cover sequences and series, we will see some comparison tests that make it clear why this holds. The basic example you can keep in mind is that the antiderivative of $1/x^2$ is $-1/x$, which has a finite limit as $x \to \infty$.
Performance review: 9 out of 25 people got tihs. 13 chose (D), 2 chose (C), 1 chose (B).
Historical note (last time): 3 out of 18 people got this correct. 7 chose (D), 5 chose (B), 2 chose (C), 1 chose (A).
Those who chose (D) had the right idea but failed to account for the extra margin that needs to be maintained because an integration is being performed.
For the next two questions, build on the observation: For any nonconstant monic polynomial $q(x)$, there exists a finite collection of transcendental functions $f_1, f_2, \ldots, f_r$ such that the antiderivative of any rational function $p(x)/q(x)$, on an open interval where it is defined and continuous, can be expressed as $g_0 + f_1 g_1 + f_2 g_2 + \cdots + f_r g_r$ where $g_0, g_1, \ldots, g_r$ are rational functions.

(20) For the polynomial $q(x) = 1 + x^2$, what collection of f_is works (all are written as functions of x)?
(A) $\arctan x$ and $\ln|x|$
(B) $\arctan x$ and $\arctan(1 + x^2)$
(C) $\ln|x|$ and $\ln(1 + x^2)$
(D) $\arctan x$ and $\ln(1 + x^2)$
(E) $\ln|x|$ and $\arctan(1 + x^2)$
Answer: Option (D)
Explanation: Follows from the standard partial fraction decomposition. $2x/(1 + x^2)$ gives the ln integration and $1/(1 + x^2)$ gives the arctan integration.
Performance review: 23 out of 25 people got this. 1 each chose (A) and (C).
Historical note (last time): 5 out of 18 people got this correct. 4 each chose (C) and (E), 3 chose (A), 2 chose (B).

(21) For the polynomial $q(x) := 1 + x^2 + x^4$, what is the size of the smallest collection of f_is that works?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (D)

Explanation: The denominator factors into $x^2 - x + 1$ and $x^2 + x + 1$. Each of these contributes one arctan possibility and one ln possibility. A total of 4 possibilities is achieved.

In general, if there are no repeated factors, the smallest number of pieces equals the degree of the polynomial.

Performance review: 17 out of 25 people got this. 8 chose (B).

Historical note (Math 153): 30 out of 44 got this. 8 chose (B), 5 chose (C), 1 chose (E).