# CLASS QUIZ SOLUTIONS: FRIDAY FEBRUARY 1: MULTIVARIABLE FUNCTION BASICS

MATH 195, SECTION 59 (VIPUL NAIK)

## 1. Performance review

27 people took this 5-question quiz. The score distribution was as follows:

- Score of 1: 4 people (this was mostly people who missed class!)
- Score of 2: 7 people
- Score of 3: 12 people
- Score of 4: 2 people
- Score of 5: 2 people

The question wise answers and performance summary:

(1) Option (B): 21 people.
(2) Option (D): 8 people. *Please review this solution!*
(3) Option (A): 12 people. *Please review this solution!*
(4) Option (C): 20 people.
(5) Option (D): 11 people. *Please review this solution!*

## 2. Solutions

(1) Suppose $f$ is a function of two variables, defined on all of $\mathbb{R}^2$, with the property that $f(x, y) = f(y, x)$ for all real numbers $x$ and $y$. What does this say about the symmetry of the graph $z = f(x, y)$ of $f$?
  (A) It has mirror symmetry about the plane $z = x + y$.
  (B) It has mirror symmetry about the plane $x = y$.
  (C) It has mirror symmetry about the plane $z = x - y$.
  (D) It has half turn symmetry about the line $x = y = z$.
  (E) It has half turn symmetry about the origin.

  *Answer*: Option (B)

  *Explanation*: The condition $f(x, y) = f(y, x)$ implies that if the point $(x, y, z)$ lies in the graph, so does the point $(y, x, z)$. These two points are mirror images of each other with respect to the plane $x = y$.

  *Performance review*: 21 out of 27 got this. 4 chose (D), 1 each chose (A) and (E).

  *Historical note (last time)*: 16 out of 21 people got this correct. 2 each chose (A) and (D) and 1 chose (E).

(2) Consider the function $f(x, y) := ax + by$ where $a$ and $b$ are fixed nonzero reals. The level curves for this function are a bunch of parallel lines. What vector are they all parallel to?
  (A) $\langle a, b\rangle$
  (B) $\langle a, -b\rangle$.
  (C) $\langle b, a\rangle$
  (D) $\langle b, -a\rangle$
  (E) $\langle a - b, a + b\rangle$

  *Answer*: Option (D)

  *Explanation*: This can be seen by noting that the slope of the line $ax + by = c$ is $-a/b$. It can also be seen using dot products. The expression $ax + by$ is the dot product of the vector $\langle a, b\rangle$ and the vector $\langle x, y\rangle$. To keep this dot product constant (i.e., move along a level curve) one must move along a vector orthogonal to $\langle a, b\rangle$. Of the given vectors, $\langle b, -a\rangle$ is orthogonal to $\langle a, b\rangle$.

  Note that it is true that all the lines are *perpendicular* to $\langle a, b\rangle$, but they are not parallel to $\langle a, b\rangle$.

*Performance review*: 8 out of 27 got this. 12 chose (B), 6 chose (A), 1 chose (E).

*Historical note (last time)*: 5 out of 21 got the question correct. 14 chose (A), 2 chose (B).

(3) Suppose $f$ is a function of one variable and $g$ is a function of two variables. What is the relationship between the level curves of $f \circ g$ and the level curves of $g$?

(A) Each level curve of $f \circ g$ is a union of level curves of $g$ corresponding to the pre-images of the point under $f$.

(B) Each level curve of $f \circ g$ is an intersection of level curves of $g$ corresponding to the pre-images of the point under $f$.

(C) The level curves of $f \circ g$ are precisely the same as the level curves of $g$.

(D) Each level curve of $g$ is a union of level curves of $f \circ g$.

(E) Each level curve of $g$ is an intersection of level curves of $f \circ g$.

*Answer*: Option (A)

*Explanation*: If $(f \circ g)(x, y) = c$, this means that $f(g(x, y)) = c$, so $g(x, y)$ is one of the pre-images of $c$ under $f$. The set of possibilities for $(x, y)$ is thus the union of the set of level curves for each of the pre-images of $c$ under $f$.

Basically, the application of $f$ can unite level curves, but it cannot separate them again, because once the $g$-values already agree, the $f \circ g$-values must also agree.

*Performance review*: 12 out of 27 got this. 6 chose (E), 5 chose (B), 3 chose (D), 1 chose (C).

*Historical note (last time)*: 7 out of 21 people got this correct. 8 chose (B), 3 chose (C), 3 chose (E).

(4) Consider the following function $f$ from $\mathbb{R}^2$ to $\mathbb{R}^2$: the function that sends $\langle x, y \rangle$ to $\langle \frac{x+y}{2}, \frac{x-y}{2} \rangle$. What is the image of $\langle x, y \rangle$ under $f \circ f$?

(A) $\langle x, y \rangle$

(B) $\langle 2x, 2y \rangle$

(C) $\langle x/2, y/2 \rangle$

(D) $\langle x + (y/2), y + (x/2) \rangle$

(E) $\langle 2x + y, 2x - y \rangle$

*Answer*: Option (C)

*Explanation*: We apply $f$ to $\langle (x + y)/2, (x - y)/2 \rangle$ and get the first coordinate as $((x + y)/2 + (x - y)/2)/2 = x/2$ and the second coordinate as $((x + y)/2 - (x - y)/2)/2 = y/2$.

*Performance review*: 20 out of 27 got this. 3 chose (A), 2 chose (D), 1 each chose (B) and (E).

*Historical note (last time)*: 12 out of 21 people got this correct. 4 chose (A), 3 chose (D), 2 chose (E).

(5) Consider the following functions defined on the subset $x > 0$ of the $xy$-plane: $f(x, y) = x^y$. Consider the surface $z = f(x, y)$. What do the intersections of this surface with planes parallel to the $xz$-plane and $yz$-plane look like (ignore the following two special intersections: intersection with the plane $x = 1$ and intersection with the plane $y = 0$, also ignore intersections that turn out to be empty).

(A) Intersections with any plane parallel to the $xz$ or $yz$ plane look like graphs of exponential functions.

(B) Intersections with any plane parallel to the $xz$ or $yz$ plane look like graphs of power functions (only positive inputs allowed).

(C) Intersections with any plane parallel to the $xz$-plane look like graphs of exponential functions, and intersections with any plane parallel to the $yz$-plane look like graphs of power functions (only positive inputs allowed).

(D) Intersections with any plane parallel to the $yz$-plane look like graphs of exponential functions, and intersections with any plane parallel to the $xz$-plane look like graphs of power functions (only positive inputs allowed).

(E) All the intersections are straight lines.

*Answer*: Option (D)

*Explanation*: A plane parallel to the $yz$-plane corresponds to fixing a value of $x$. The intersection with such a plane is the graph of the function $y \mapsto x^y$ with $x$ a constant. By assumption, $x \neq 1$ and

$x > 0$, so if we set $k = \ln x$, this becomes $y \mapsto \exp(ky)$. This is an exponential function (increasing if $k > 0$, decreasing if $k < 0$).

A plane parallel to the $xz$-plane corresponds to a fixed value of $x$. The intersection with such a plane is the graph of the function $x \mapsto x^y$ with $y$ a constant. By assumption $y \neq 0$. We thus get a power function, and $x$ is restricted to being positive.

*Performance review*: 11 out of 27 got this. 8 chose (C), 6 chose (E), 2 chose (A).

*Historical note (last time)*: 5 out of 21 people got this correct. 8 chose (C), 3 each chose (B) and (E), and 2 chose (A).