TAKE-HOME CLASS QUIZ SOLUTIONS: DUE OCTOBER 12: SEQUENCES AND MISCELLANEA

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

41 people took this quiz. The score distribution was as follows:

- Score of 5: 2 people.
- Score of 6: 1 person.
- Score of 8: 2 people.
- Score of 9: 2 people.
- Score of 10: 6 people.
- Score of 11: 3 people.
- Score of 12: 25 people.

The mean score was 10.8. The question wise answer choices and performance were as follows:

(1) Option (E): 38 people.
(2) Option (E): 35 people.
(3) Option (D): 40 people.
(4) Option (D): 41 people (everybody).
(5) Option (C): 40 people.
(6) Option (C): 32 people.
(7) Option (E): 33 people.
(8) Option (B): 38 people.
(9) Option (A): 36 people.
(10) Option (B): 36 people.
(11) Option (C): 36 people.
(12) Option (C): 38 people.

2. Solutions

(1) Consider the sequence $a_n = 2a_{n-1} - \alpha$, with $a_1 = \beta$, for α, β real numbers. What can we say about this sequence for sure?
   (A) (a_n) is eventually increasing for all values of α, β.
   (B) (a_n) is eventually decreasing for all values of α, β.
   (C) (a_n) is eventually constant for all values of α, β.
   (D) (a_n) is either increasing or decreasing, and which case occurs depends on the values of α and β.
   (E) (a_n) is increasing, decreasing, or constant, and which case occurs depends on the values of α and β.

   Answer: Option (E)

   Explanation: See the answer explanation for the next question. Note that the *constant* case could arise when $\alpha = \beta = 1$.

   Performance review: 38 out of 41 got this. 3 chose (D).

   Historical note (last year): 7 out of 11 got this correct. 2 chose (A), 2 chose (D).

(2) *This is a generalization of the preceding question.* Suppose f is a continuous increasing function on $\mathbb{R}$. Define a sequence recursively by $a_n = f(a_{n-1})$, with a_1 chosen separately. What can we say about this sequence for sure?
   (A) (a_n) is eventually increasing regardless of the choice of a_1.
   (B) (a_n) is eventually decreasing regardless of the choice of a_1.

(C) (a_n) is eventually constant regardless of the choice of a_1.
(D) (a_n) is either increasing or decreasing, and which case occurs depends on the value of a_1 and the nature of f.
(E) (a_n) is increasing, decreasing, or constant, and which case occurs depends on the value of a_1 and the nature of f.

Answer: Option (E)

Explanation: Since f is increasing, if $f(a_1) < a_1$, then $f(f(a_1)) < f(a_1)$. we can inductively show that if $f(a_1) < a_1$, then (a_n) is decreasing. If $f(a_1) = a_1$, then (a_n) is constant. If $f(a_1) > a_1$, then (a_n) is increasing. Any of these cases may occur (as we can see using specific example from the previous problem). So, (a_n) is either increasing, decreasing or constant, but which case occurs depends on the value of a_1 and the nature of f.

For a function $f : \mathbb{R} \to \mathbb{R}$ and a particular element $a \in \mathbb{R}$, define $g : \mathbb{N} \to \mathbb{R}$ by $g(n) = f(f(\dots(f(a))\dots))$ with the f occurring $n-1$ times. Thus, $g(1) = a$, $g(2) = f(a)$, and so on. Choose the right expression for g for each of these choices of f.

Performance review: 35 out of 41 got this. 4 chose (A), 2 chose (D).

Historical note (last year): 7 out of 11 got this correct. 2 chose (A), 1 each chose (C) and (D).

(3) $f(x) := x + \pi$.
(A) $g(n) := a + n\pi$.
(B) $g(n) := a + n\pi - 1$.
(C) $g(n) := a + n(\pi - 1)$.
(D) $g(n) := a + \pi(n - 1)$.
(E) $g(n) := \pi + n(a - 1)$.

Answer: Option (D)

Explanation: Straightforward summation/induction/observation. If you aren't able to arrive at the expression, just plug in and check the values $n = 1$ and $n = 2$.

Performance review: 40 out of 41 got this. 1 chose (E).

Historical note (last year): 9 out of 11 got this correct. 1 each chose (B) and (C).

(4) $f(x) := mx$, $m \neq 0$.
(a) $g(n) := mna$.
(b) $g(n) := m^n a$.
(c) $g(n) := n^m a$.
(d) $g(n) := m^{n-1} a$.
(e) $g(n) := n^{m-1} a$.

Answer: Option (D)

Explanation: Straightforward summation/induction/observation. If you aren't able to arrive at the expression, just plug in and check the values $n = 1$ and $n = 2$.

Performance review: All 41 got this.

Historical note (last year): 9 out of 11 got this. 2 chose (B).

(5) $f(x) := x^2$.
(A) $g(n) := a^{2^n} - 1$.
(B) $g(n) := a^{2^n - 1}$.
(C) $g(n) := a^{2^{n-1}}$.
(D) $g(n) := a^{2^{n^{-1}}}$.
(E) $g(n) := (a^{2^n})^{-1}$.

Answer: Option (C)

Explanation: Each time we square, the exponent gets multiplied by 2. Thus, the exponent itself is growing like 2^{n-1} (it starts out at 1).

Performance review: 40 out of 41 got this. 1 chose (D).

Historical note (last year): Everybody got this correct.

(6) One of these sequences can *not* be obtained using the procedure described in the previous questions (i.e., iterated application of a function). Identify this sequence. Only the first five terms of the sequence are presented:

(A) $1, 2, 3, 3, 3$
(B) $1, 2, 3, 2, 3$
(C) $1, 2, 3, 2, 1$
(D) $1, 2, 3, 4, 5$
(E) $1, 2, 3, 4, 3$

Answer: Option (C).

Explanation: For a sequence obtained by function iteration, it must be true that the successor of an element is uniquely determined by that element. For the sequence with first five terms $1, 2, 3, 2, 1$, we note that at one place in the sequence, 2 is followed by 3, but at another place, 2 is followed by one. This is not possible, because $f(2)$ cannot be both 3 and 1.

Performance review: 32 out of 41 got this. 9 chose (A).

Historical note (last year): Nobody got this correct! 10 chose (A), 1 chose (E).

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Identify which of these definitions is *not* correct for $\lim_{x\to c} f(x) = L$, where c and L are both finite real numbers.
(A) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.
(B) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.
(C) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.
(D) For every $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.
(E) None of these, i.e., all definitions are correct.

Answer: Option (E)

Explanation: Although the usual $\varepsilon - \delta$ definition uses centered intervals, i.e., intervals centered at the points c and L, this is not a necessary aspect of the definition. So, instead of taking centered intervals $(c-\delta, c+\delta)$ or $(L-\varepsilon, L+\varepsilon)$, we could consider open intervals that have different amounts on the left and on the right. Thus, all four definitions are correct.

Performance review: 33 out of 41 got this. 3 chose (C). 2 each chose (B) and (D), 1 left the question blank.

Historical note (last year): 1 out of 11 got this correct. 4 chose (B), 3 chose (A), 2 chose (D), 1 chose (C).

(8) In the usual $\varepsilon - \delta$ definition of limit for a given limit $\lim_{x\to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\varepsilon > 0$, then which of the following is true?
(A) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
(B) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
(C) Every larger value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
(D) Every larger value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
(E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ε also works for larger εs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 38 out of 41 got this. 2 chose (D), 1 chose (C).

Historical note (last year): 9 out of 11 got this correct. 2 chose (C).

(9) In the usual $\varepsilon - \delta$ definition of limit, we find that the value $\delta = 0.2$ for $\varepsilon = 0.7$ for a function f at 0, and the value $\delta = 0.5$ works for $\varepsilon = 1.6$ for a function g at 0. What value of δ *definitely* works for $\varepsilon = 2.3$ for the function $f + g$ at 0?
(A) 0.2
(B) 0.3
(C) 0.5
(D) 0.7
(E) 0.9
Answer: Option (A)
Explanation: We choose the *smaller* of the δs to guarantee that *both* f and g are within their respective ε-distances of the targets – 0.7 in the case of f and 1.6 in the case of g. Now, the triangle inequality guarantees that $f + g$ is within 2.3 of its proposed limit.
Performance review: 36 out of 41 got this. 3 chose (D), 1 each chose (C) and (E).
Historical note (last year): 7 out of 11 got this correct. 2 chose (E), 1 each chose (C) and (D).

(10) The sum of limits theorem states that $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ *if* the right side is defined. One of the choices below gives an example where the left side of the equality is defined and finite but the right side makes no sense. Identify the correct choice.
(A) $f(x) := 1/x$, $g(x) := -1/(x + 1)$, $c = 0$.
(B) $f(x) := 1/x$, $g(x) := (x - 1)/x$, $c = 0$.
(C) $f(x) := \arcsin x$, $g(x) := \arccos x$, $c = 1/2$.
(D) $f(x) := 1/x$, $g(x) = x$, $c = 0$.
(E) $f(x) := \tan x$, $g(x) := \cot x$, $c = 0$.
Answer: Option (B)
Explanation: $f + g$ is the constant function 1, so it has a limit. On the other hand, both f and g have one-sided limits of $\pm\infty$.
For options (A), (D), and (E), one of the function f and g has a finite limit, and the other has an infinite or undefined limit, and the sum has an infinite or undefined limit. Option (C) is a case where f, g, and $f + g$ all have finite limits.
Performance review: 36 out of 41 got this. 2 each chose (A) and (E), 1 chose (C).
Historical note (last year): 9 out of 11 got this correct. 1 each chose (D) and (E).

(11) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x$?
(A) 0
(B) A
(C) A^2
(D) $g(A)$
(E) $g(A)/A$
Answer: Option (C)
Explanation: We have $\lim_{x \to 0} g(x) = \lim_{x \to 0}(g(x)/x) \lim_{x \to 0} x = A \cdot 0 = 0$.
Also, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x} = \lim_{x \to 0} \frac{g(g(x))}{g(x)} \lim_{x \to 0} \frac{g(x)}{x}$$

The second limit is A. For the first limit, note that as $x \to 0$, we also have $g(x) \to 0$, so the first limit can be rewritten as $\lim_{y \to 0} g(y)/y$, which is also equal to A. Hence, the overall limit is the product A^2.
Performance review: 36 out of 41 got this. 2 chose (D), 1 each chose (A) and (E). 1 left the question blank.
Historical note (last year): 6 out of 11 got this. 3 chose (B) and 2 chose (D).

(12) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x^2 = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x^4$?

(A) A
(B) A^2
(C) A^3
(D) $A^2 g(A)$
(E) $g(A)/A^2$

Answer: Option (C)

Explanation: First, note that since $g(x)/x^2 \to A$ as $x \to 0$, we must have $g(x) \to 0$ as $x \to 0$. In particular, $g(0) = 0$.

Now, consider:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \cdot \frac{(g(x))^2}{x^4}$$

Splitting the limit, we get:

$$\lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \left(\frac{g(x)}{x^2} \right)^2$$

Setting $u = g(x)$ for the first limit, and using the fact that as $x \to 0$, $u \to 0$ we see that the first limit is A. For the second limit, pulling the square out yields that the second limit is A^2. The overall limit is thus $A \cdot A^2 = A^3$.

We can also use an actual example to solve this problem. For instance, consider the extreme case where $g(x) = Ax^2$ (identically). In this case, $g(g(x)) = A(Ax^2)^2 = A^3x^4$. Thus, $g(g(x))/x^4 = A^3$, and the limit is thus A^3.

Even more generally, if $\lim_{x \to 0} g(x)/x^n = A$, then $\lim_{x \to 0} g(g(x))/x^{n^2} = A^{n+1}$.

Performance review: 38 out of 41 got this. 3 chose (E).

Historical note (last year): 5 out of 11 got this. 3 chose (D), 2 chose (E) 1 chose (B).