CLASS QUIZ SOLUTIONS: FEBRUARY 29: SERIES

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 10-question quiz. The score distribution was as follows:

- Score of 3: 2 people
- Score of 4: 1 person
- Score of 5: 4 people
- Score of 6: 2 people
- Score of 7: 2 people

Below are the question wise answers and performance review:

(1) Option (C): 9 people
(2) Option (B): 3 people
(3) Option (E): 5 people
(4) Option (B): 10 people
(5) Option (A): 7 people
(6) Option (D): 5 people
(7) Option (E): 3 people
(8) Option (D): 3 people
(9) Option (E): 5 people
(10) Option (C): 6 people

2. Solutions

(1) Suppose p is a polynomial that take positive values on all nonnegative integers. Consider the summation $\sum_{k=1}^{\infty} \frac{(k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.
(A) The summation converges if and only if the degree of p is *at least* one
(B) The summation converges if and only if the degree of p is *at least* two
(C) The summation converges if and only if the degree of p is *at least* three
(D) The summation converges if and only if the degree of p is *at most* two
(E) The summation converges if and only if the degree of p is *at most* one

Answer: Option (C)

Explanation: This is a straightforward application of the degree difference rule in its more general form. The numerator has degre $4/3$, so the degree difference is $\deg(p) - (4/3)$. This difference needs to be greater than 1 for the series to converge, so we need that the degree of the denominator is strictly greater than $7/3$. The smallest integer greater than $7/3$ is 3, so that is the answer.

Performance review: 9 out of 11 got this. 2 chose (B).

(2) Suppose p is a polynomial that take positive values on all nonnegative integers. Consider the summation $\sum_{k=1}^{\infty} \frac{(-1)^k(k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.
(A) The summation converges if and only if the degree of p is *at least* one
(B) The summation converges if and only if the degree of p is *at least* two
(C) The summation converges if and only if the degree of p is *at least* three
(D) The summation converges if and only if the degree of p is *at most* two
(E) The summation converges if and only if the degree of p is *at most* one

Answer: Option (B)

Explanation: The previous question tells us that if the degree of p is at least three, then the series of absolutely convergent.

If the degree of p is two, then the series is conditionally convergent. To see this, note that $p(k)$ is always positive, so the summation is an alternating series summation. Also, the terms are eventually decreasing in magnitude, and they go to zero. Thus, by the alternating series theorem, the series converges. On the other hand, the degree difference rule tells us that it does not converge absolutely.

If the degree of p is one or zero, then the terms of the series do not approach zero, so the series does not converge.

Performance review: 3 out of 11 got this. 7 chose (C), 1 chose (E).

(3) Which of the following series converges? Assume for all series that the starting point of summation is large enough that the terms are well defined.
(A) $\sum 1/(k\ln(\ln k))$
(B) $\sum 1/(k\ln k)$
(C) $\sum 1/(k(\ln(\ln k))^2)$
(D) $\sum 1/(k(\ln k)(\ln(\ln k)))$
(E) $\sum 1/(k(\ln k)(\ln(\ln k))^2)$

Answer: Option (E)

Explanation: Options (B) and (D) diverge by the integral test. As for options (A) and (C), these have smaller denominators, hence larger terms, than option (B), hence, by basic comparison, these diverge too. This leaves option (E), which converges by the integral test.

Performance review: 5 out of 11 got this. 4 chose (B), 2 chose (D).

Historical note (last year): 11 out of 25 people got this correct. 7 chose (C), 3 each chose (A) and (B), and 1 left the question blank.

The main attraction of (C) seems to have been its superficial resemblance to $1/(k(\ln k)^2)$ which does converge.

(4) Which of the following series converges?
(A) $\sum \frac{k+\sin k}{k^2+1}$
(B) $\sum \frac{k+\cos k}{k^3+1}$
(C) $\sum \frac{k^2-\sin k}{k+1}$
(D) $\sum \frac{k^3+\cos k}{k^2+1}$
(E) $\sum \frac{k}{\sin(k^3+1)}$

Answer: Option (B)

Explanation: We can use a comparison test, either rigorously or in the form of a heuristic of looking at degree of denominator minus degree of numerator. Note that for (A), the degree difference is 1, so it diverges. For (C) and (D), the numerator actually has larger degree than the denominator, so it diverges. For (E), the denominator is bounded in $[-1,1]$, and the numerator goes to ∞, so it diverges. This leaves (B), which converges because the degree of denominator minus degree of numerator equals 2.

Performance review: 10 out of 11 got this. 1 chose (A).

Historical note (last year): 23 out of 25 people got this correct. 1 person chose (C) and 1 person chose (D).

(5) Consider the series $\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}$. What can we say about the sum of this series?
(A) It is finite and strictly between 0 and 1.
(B) It is finite and equal to 1.
(C) It is finite and strictly between 1 and 2.
(D) It is finite and equal to 2.
(E) It is infinite.

Answer: Option (A)

Explanation: The summation goes like:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \dots$$

Notice that the series being summed is a subseries of the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

In particular, the sum of the former is less than the sum of the latter. The latter sums up to 1, so the sum of the former is less than 1. Also, since the first term is 1/2, it must be greater than 1/2. Thus, the series sum is between 1/2 and 1. Option (A) is the best fit.

Performance review: 7 out of 11 got this. 3 chose (C), 1 chose (B).

Historical note (last year): 14 out of 26 people got this correct. 9 chose (C) (possibly because of getting the first term wrong?), 2 chose (E), 1 chose (B).

(6) For one of the following functions f on $(0, \infty)$, the integral $\int_0^\infty f(x)\,dx$ converges but $\int_0^\infty |f(x)|\,dx$ does not converge. What is that function f? (Note that this is similar to, but not quite the same as, the absolute versus conditional convergence notion for series).

(A) $f(x) = \sin x$
(B) $f(x) = \sin(\sin x)$
(C) $f(x) = (\sin\sqrt{x})/\sqrt{x}$
(D) $f(x) = (\sin x)/x$
(E) $f(x) = (\sin^3 x)/x^3$

Answer: Option (D)

Explanation: For options (A) and (B), the integral $\int_0^\infty f(x)\,dx$ does not converge. The reason is simple: in neither case is it true that $\lim_{x\to\infty} f(x) = 0$. The function itself going to zero is a necessary (but not sufficient) condition for the integral to converge.

For option (C), the antiderivative for f is $-2\cos\sqrt{x}$, evaluated between limits 0 and ∞. However, the limit for the antiderivative at ∞ does not exist, hence the integral does not converge.

This leaves options (D) and (E). For option (D), it is a well known (?) fact that:

$$\int_0^\infty \frac{\sin x\,dx}{x} = \frac{\pi}{2}$$

Hence, the integral does converge. However, if we consider the integral:

$$\int_0^\infty \frac{|\sin x|\,dx}{|x|}$$

This integral does not converge, a fact that we can prove by bounding it in terms of the summation $\sum_{n=1}^\infty 1/n$, which diverges.

Finally, for option (E), both $\int_0^\infty f(x)\,dx$ and $\int_0^\infty |f(x)|\,dx$ converge. To see this, first split the integral as $\int_0^1 + \int_1^\infty$. The former integral is finite because it is integrating a bounded function over a bounded interval (note that the limit of the function at 0 is 1). The latter integral is finite because we can compare it to $1/x^3$ which has a finite integral. The reasoning works for both f and $|f|$, so we are done.

Performance review: 5 out of 11 got this. 4 chose (C), 2 chose (D).

(7) Consider the function $F(x,p) := \sum_{n=1}^\infty \frac{x^n}{n^p}$ with x and p both real numbers. For what values of x and what values of p does this summation converge?

(A) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| \geq 1$, it does not converge for any p.
(B) For $|x| \leq 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p.
(C) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $|x| = 1$, it converges if and only if $p > 1$.
(D) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $x = 1$, it converges if and only if $p > 0$. For $x = -1$, it converges if and only if $p > 1$.
(E) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any $p \in \mathbb{R}$. For $x = 1$, it converges if and only if $p > 1$. For $x = -1$, it converges if and only if $p > 0$.

Answer: Option (E)

Explanation: When $|x| < 1$, then the series is absolutely convergent and when $|x| > 1$ it diverges, as we can see by the root test or ratio test. What's happening is that $(1/n^p)^{1/n} \to 1$ (regardless of p), so the radius of convergence is 1.

This leaves the case $|x| = 1$. If $x = 1$, then we get the usual p-series, which we know converges iff $p > 1$. If $x = -1$, then the terms have alternating signs. Obviously, the series cannot converge for $p \leq 0$ because the terms do not tend to 0. For $p > 0$, on the other hand, the terms are alternating in sign and decrease monotonically, tending to 0. Thus, by the alternating series theorem, it converges for $p > 0$.

Performance review: 3 out of 11 got this. 4 chose (C), 2 each chose (B) and (D).

Historical note (last year): 7 out of 26 people got this correct. 10 chose (C), 4 chose (D), 3 chose (A), 2 chose (B). The large vote for (C) indicates that many people did not notice the special application of the alternating series theorem to the case of $x = -1$.

Action point: Please review what happens in the case $x = -1$. This will be covered in more detail in class when we study the notion of interval of convergence of a power series.

There is a result of calculus which states that, under suitable conditions, if $f_1, f_2, \dots, f_n, \dots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f'(x) = \sum_{n=1}^{\infty} f'_n(x)$. In other words, under suitable assumptions, we can differentiate a sum of countably many functions by differentiating each of them and adding up the derivatives.

We will not be going into what those assumptions are, but will consider some applications where you are explicitly told that these assumptions are satisfied.

(8) Consider the summation $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. Assume that the required assumptions are valid for this summation, so that $\zeta'(p)$ is the sum of the derivatives of each of the terms (summands) with respect to p. What is the correct expression for $\zeta'(p)$?
(A) $\sum_{n=1}^{\infty} \frac{-p}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{-1}{(p+1)n^{p+1}}$
(C) $\sum_{n=1}^{\infty} \frac{p}{n^{p-1}}$
(D) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^p}$
(E) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^{p+1}}$

Answer: Option (D)

Explanation: We need to differentiate $(1/n)^p$ with respect to p. This is the same as differentiating a^x with respect to x, which gives $a^x \ln a$. In our case, we get $(1/n)^p \ln(1/n)$ which is $(-\ln n)/n^p$.

Note that Option (A) arises if we try to differentiate formally with respect to n, which is not the correct operation at all. n is a dummy variable and the expression should be differentiated with respect to p.

Performance review: 3 out of 11 got this. 8 chose (A). This indicates that many people differentiated with respect to the wrong variable.

Historical note (last year): 8 out of 26 people got this correct. 13 chose (A), 3 chose (B), 1 chose (C), and 1 left the question blank. The most commonly chosen wrong option, (A), indicates that many people differentiated with respect to the wrong variable.

(9) Going back to question 2, recall that we defined $F(x, p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. Assume that, for a particular fixed value of p, the summation satisfies the conditions as a function of x for $|x| < 1$. What is its derivative with respect to x, keeping p constant?
(A) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p-1}}$
(C) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p+1}}$
(D) $\sum_{n=1}^{\infty} \frac{x^{n-1} \ln n}{n^{p+1}}$
(E) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p-1}}$

Answer: Option (E)

Explanation: We need to differentiate each term with respect to x. Differentiating x^n/n^p with respect to x gives nx^{n-1}/n^p, which, upon rearrangement, gives x^{n-1}/n^{p-1}.

Performance review: 5 out of 11 got this correct. 3 each chose (C) and (D).

Historical note (last year): 14 out of 26 got this correct. 4 each chose (B), (C), (D), possibly indicating minor computational errors.

(10) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Since it is a series of positive terms, this means that the partial sums get arbitrarily large. What is the approximate smallest value of N such that $\sum_{n=1}^{N} \frac{1}{n} > 100$?

(A) Between 90 and 110
(B) Between 2000 and 3000
(C) Between 10^{40} and 10^{50}
(D) Between 10^{90} and 10^{110}
(E) Between 10^{220} and 10^{250}

Answer: Option (C)

Explanation: We can see that $\sum_{n=1}^{N} 1/n$ is approximately $\ln N$. More precisely, we can use the standard methods for comparising integrals and summations and obtain that the finite sum is between $\ln N$ and $1 + \ln N$. In particular, the N that works must have $\ln N$ between 99 and 100. Thus, $\log_{10} N$ is between $99/(\ln 10)$ and $100/(\ln 10)$. Both these numbers are between 40 and 50, so Option (C) is the correct choice.

Performance review: 6 out of 11 got this correct. 2 each chose (D) and (E), 1 chose (A).

Historical note (last year): 14 out of 26 got this correct. 6 chose (D), 2 chose (B), 2 chose (E), 1 chose (A), and 1 left the question blank.