CLASS QUIZ SOLUTIONS: OCTOBER 15: ORDER OF ZERO, L'HOPITAL'S RULE

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

42 people took the quiz. The score distribution was as follows:

- Score of 2: 3 people
- Score of 3: 13 people
- Score of 4: 26 people

The mean score was 3.54. The question wise answers and performance review are below:

(1) Option (B): 37 people
(2) Option (D): 42 people (everybody)
(3) Option (E): 41 people
(4) Option (A): 29 people

2. Solutions

(1) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the pointwise sum $f + g$ at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3
(D) 5
(E) 6

Answer: Option (B)

Explanation: The general rule is that when f and g both have zeros of *different* orders at a point, the order of zero of their sum is the minimum of the orders of zeros for the individual functions. We can interpret this result in terms of the limit definitions, or in terms of what's the first iterated derivative to take a nonzero value.

Performance review: 37 out of 42 got this. 4 chose (C), 1 chose (D).

Historical note (last year): 6 out of 11 got this. 4 chose (C), 1 chose (A).

(2) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the pointwise product fg at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3
(D) 5
(E) 6

Answer: Option (D)

Explanation: The order of zero for a product of two function is the sum of the orders of zeros for the two functions. This can be seen by thinking of the limit definition of order of zero.

Performance review: All 42 people got this.

Historical note (last year): 8 out of 11 got this. 2 chose (E), 1 chose (C).

(3) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the composite function $f \circ g$ at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3

(D) 5
(E) 6

Answer: Option (E)

Explanation: Roughly speaking, the order of zero of the composite function is the product of the orders of zeros. This is valid when $c = 0$, i.e., we are taking the order of zero at zero. Otherwise, the statement needs to be modified somewhat.

Performance review: 41 out of 42 got this. 1 chose (C).

Historical note (last year): 9 out of 11 got this. 1 each chose (C) and (D).

(4) The L'Hopital rule can be related with order of zero in the following manner: Every time the rule is applied to a $(\to 0)/(\to 0)$ form, the order of zero of the numerator and denominator go *down by one*. Repeated application hopefully yields a situation where either the numerator or the denominator has a nonzero limiting value.

Assume that we start with a limit $\lim_{x \to c} f(x)/g(x)$ where both f and g are infinitely differentiable at c, and further, that $f(c) = g(c) = 0$. If the order of zero of f is d_f and the order of zero of g is d_g, which of the following is true?

(A) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a nonzero numerator and zero denominator, so the limit is undefined. If $d_g < d_f$, then we apply the LH rule d_g times to get a zero numerator and nonzero denominator, so the limit is zero.
(B) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a zero numerator and nonzero denominator, so the limit is undefined. If $d_g < d_f$, then we apply the LH rule d_g times to get a nonzero numerator and zero denominator, so the limit is zero.
(C) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a nonzero numerator and zero denominator, so the limit is zero. If $d_g < d_f$, then we apply the LH rule d_g times to get a zero numerator and nonzero denominator, so the limit is undefined.
(D) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a zero numerator and nonzero denominator, so the limit is zero. If $d_g < d_f$, then we apply the LH rule d_g times to get a nonzero numerator and zero denominator, so the limit is undefined.
(E) In all cases, we perform the LH rule $\min\{d_f, d_g\}$ times and obtain a nonzero numerator and nonzero denominator.

Answer: Option (A)

Explanation: Each time the LH rule is applied, the order of zero on the numerator goes down by one and the order of zero on the denominator goes down by one. Thus, we need to perform the LH rule $\min\{d_f, d_g\}$ times to reach a situation where either the numerator or the denominator gets a zero order of zero, which means (from the information we have) that it evaluates to something nonzero.

If $d_f = d_g$, then applying the LH rule d_f times yields a situation where both the numerator and denominator become nonzero.

If $d_f < d_g$ then we need to apply the LH rule d_f times. The denominator in this case still has a zero of order $d_g - d_f$, hence evaluates to zero. The numerator has a zero of order zero, i.e., it evaluates to something nonzero. The (nonzero)/(zero) form means that the limit is undefined.

If $d_g < d_f$, then we need to apply the LH rule d_g times. The numerator in this case still has a zero of order $d_f - d_g$, so is zero, whereas the denominator is nonzero. The (zero)/(nonzero) form means that the limit is zero.

Performance review: 29 out of 42 got this. 11 chose (D), 1 each chose (B) and (E).

Historical note (last year): 7 out of 11 got this. 3 chose (D), 1 chose (B).