CLASS QUIZ SOLUTIONS: JANUARY 31: PARTIAL FRACTIONS AND RADICALS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

27 people took this 8-question quiz. The score distribution was as follows:

- Score of 1: 2 people.
- Score of 2: 5 people.
- Score of 3: 4 people.
- Score of 4: 9 people.
- Score of 5: 4 people.
- Score of 6: 3 people.

The mean score was 3.63 and the median and modal scores were 4.
Here is the performance summary by question:

(1) Option (D): 22 people.
(2) Option (B): 4 people. *Please review this solution.* This is similar to part of Question 9 on your recent midterm, in the sense that it defines a function of one variable using a definite integral on another variable, and then seeks to find the intervals where the value of the integral increases and decreases.
(3) Option (D): 21 people.
(4) Option (B): 14 people. *Please review this solution.*
(5) Option (D): 12 people.
(6) Option (E): 3 people. *This will come up again later in the course, so you might benefit from internalizing the answer right now.*
(7) Option (D): 15 people. *Please review this solution if you got it wrong.*
(8) Option (D): 7 people.

2. Solutions

(1) Which of these functions of x is *not* elementarily integrable?
(A) $x\sqrt{1+x^2}$
(B) $x^2\sqrt{1+x^2}$
(C) $x(1+x^2)^{1/3}$
(D) $x\sqrt{1+x^3}$
(E) $x^2\sqrt{1+x^3}$
Answer: Option (D)
—em Explanation: For options (A) and (C), the substitution $u = 1 + x^2$ works fine. For option (E), the substitution $u = 1 + x^3$ works fine. For option (B), we can solve the problem using a trigonometric substitution. This leaves option (D) (which, incidentally, requires the use of elliptic integrals).
Post-performance review: 22 out of 27 people got this correct. 4 people chose (B) and 1 person chose (C).

(2) Consider the function $f(k) := \int_1^2 \frac{dx}{\sqrt{x^2+k}}$. f is defined for $k \in (-1, \infty)$. What can we say about the nature of f within this interval?
(A) f is increasing on the interval $(-1, \infty)$.
(B) f is decreasing on the interval $(-1, \infty)$.
(C) f is increasing on $(-1, 0)$ and decreasing on $(0, \infty)$.
(D) f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

(E) f is increasing on $(-1,0)$, decreasing on $(0,2)$, and increasing again on $(2,\infty)$.

Answer: Option (B)

Explanation: For any fixed value of $x \in [1,2]$, the integrand $1/\sqrt{x^2+k}$ is a *decreasing* function of k for $k \in (-1,\infty)$. Hence, the value we get upon integrating it for $x \in [1,2]$ should also be a decreasing function of k.

Post-performance review: 4 out of 27 people got this correct. 11 people chose (A), 7 people chose (C), 3 people chose (D), 1 person chose (E), and 1 person left the question blank.

Mainly, people confused the roles of the dummy variable x (which gets integrated away) and the variable k.

Action point: This is qualitatively similar to Question 9 on the midterm and a corresponding question on the mock midterm. The solution approach is also the same. After reviewing this solution, you may wish to take another look at that question.

(3) For which of these functions of x does the antiderivative necessarily involve *both* arctan *and* ln?

(A) $1/(x+1)$
(B) $1/(x^2+1)$
(C) $x/(x^2+1)$
(D) $x/(x^3+1)$
(E) $x^2/(x^3+1)$

Answer: Option (D)

Explanation: Option (A) integrates to $\ln|x|$, option (B) integrates to $\arctan x$, option (C) integrates to $(1/2)\ln(x^2+1)$, and option (E) integrates to $(1/3)\ln|x^3+1|$. For option (D), we need to use partial fractions with denominators $x+1$ and x^2-x+1, and we end up getting nonzero coefficients on terms that integrate to ln and to arctan.

Post-performance review: 21 out of 27 people got this correct. 2 people chose (E), and 1 person each chose (A), (B), and (C). 1 person left the question blank.

(4) Suppose F is a (not known) function defined on $\mathbb{R}\setminus\{-1,0,1\}$, differentiable everywhere on its domain, such that $F'(x) = 1/(x^3-x)$ everywhere on $\mathbb{R}\setminus\{-1,0,1\}$. For which of the following sets of points is it true that knowing the value of F at these points **uniquely** determines F?

(A) $\{-\pi,-e,1/e,1/\pi\}$
(B) $\{-\pi/2,-\sqrt{3}/2,11/17,\pi^2/6\}$
(C) $\{-\pi^3/7,-\pi^2/6,\sqrt{13},11/2\}$
(D) Knowing F at any of the above determines the value of F uniquely.
(E) None of the above works to uniquely determine the value of F.

Answer: Option (B)

Explanation: The domain of F has four connected components: the open intervals $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$. We need to know the value of F at one point in each of these intervals. By computing values, we see that the set of points in option (B) has the property that it contains one point in each of these intervals, and those in options (A) and (C) do not.

Post-performance review: 14 out of 27 people got this correct. 6 people chose (D), 4 people chose (C), and 3 people chose (A).

Many people spent time trying to determine the coefficients of the partial fraction decomposition. This is not relevant to what we need to do in the question.

Action point: The idea here (that you need to know the value at one point in each connected component) is a crucial one that you should understand.

(5) Consider a rational function $f(x) := p(x)/q(x)$ where p and q are nonzero polynomials and the degree of p is strictly less than the degree of q. Suppose $q(x)$ is monic of degree n and has n distinct real roots $a_1, a_2, \ldots, a_n$, so $q(x) = \prod_{i=1}^n (x-a_i)$. Then, we can write:

$$f(x) = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \cdots + \frac{c_n}{x-a_n}$$

for suitable constants $c_i \in \mathbb{R}$. What can we say about the sum $\sum_{i=1}^n c_i$?

(A) The sum is always 0.
(B) The sum equals the leading coefficient of p.

(C) The sum is 0 if p has degree $n-1$. If the degree of p is smaller, the sum equals the leading coefficient of p.
(D) The sum is 0 if p has degree smaller than $n-1$. If p has degree equal to $n-1$, the sum is the leading coefficient of p.
(E) The sum is 0 if p is a constant polynomial. Otherwise, it equals the leading coefficient of p.

Answer: Option (D)

Explanation: If we take a common denominator and simplify the right side, we see that the coefficient of x^{n-1} on the numerator of the right side is $\sum_{i=1}^n c_i$. Equating coefficients, we obtain that the coefficient of x^{n-1} on the left side is also $\sum_{i=1}^n c_i$. If p has degree less than $n-1$, the coefficient on the left side is 0, so $\sum_{i=1}^n c_i = 0$. If p has degree equal to $n-1$, the coefficient on the left side is the leading coefficient of p.

Post-performance review: 12 out of 27 people got this correct. 7 people chose (E), 3 people each chose (B) and (C), 1 person chose (A).

When the denominator is quadratic, then options (D) and (E) are euqivalent. This is what might have led to many people choosing option (E).

(6) *Hard right now, will become easier later*: Suppose F is a continuously differentiable function whose domain contains (a,∞) for some $a \in \mathbb{R}$, and $F'(x)$ is a rational function $p(x)/q(x)$ on the domain of F. Further, suppose that p and q are nonzero polynomials. Denote by d_p the degree of p and by d_q the degree of q. Which of the following is a **necessary and sufficient condition** to ensure that $\lim_{x\to\infty} F(x)$ is finite?
(A) $d_p - d_q \geq 2$
(B) $d_p - d_q \geq 1$
(C) $d_p = d_q$
(D) $d_q - d_p \geq 1$
(E) $d_q - d_p \geq 2$

Answer: Option (E)

Explanation: This can be justified in terms of partial fractions. Th ecase where q is a product of linear factors can be justified using the previous question. But that is not the most elegant justification. When we cover sequences and series, we will see some comparison tests that make it clear why this holds. The basic example you can keep in mind is that the antiderivative of $1/x^2$ is $-1/x$, which has a finite limit as $x \to \infty$.

Post-performance review: 3 out of 27 people got this correct. 10 people chose (D), 7 people chose (C), 6 people chose (B), and 1 person chose (A).

Those who chose (D) had the right idea but failedto account for the extra margin that needs to be maintained because an integration is being performed.

For the remaining questions, we build on the observation: For any nonconstant monic polynomial $q(x)$, there exists a finite collection of transcendental functions $f_1, f_2, \dots, f_r$ such that the antiderivative of any rational function $p(x)/q(x)$, on an open interval where it is defined and continuous, can be expressed as $g_0 + f_1 g_1 + f_2 g_2 + \dots + f_r g_r$ where $g_0, g_1, \dots, g_r$ are rational functions.

(7) For the polynomial $q(x) = 1 + x^2$, what collection of f_is works (all are written as functions of x)?
(A) $\arctan x$ and $\ln|x|$
(B) $\arctan x$ and $\arctan(1+x^2)$
(C) $\ln|x|$ and $\ln(1+x^2)$
(D) $\arctan x$ and $\ln(1+x^2)$
(E) $\ln|x|$ and $\arctan(1+x^2)$

Answer: Option (D)

Explanation: Follows from the standard partial fraction decomposition. $2x/(1+x^2)$ gives the ln integration and $1/(1+x^2)$ gives the arctan integration.

Post-performance review: 15 out of 27 people got this correct. 7 people chose (A), 2 people chose (C), 1 person each chose (A) and (E), and 1 person left the question blank.

(8) For the polynomial $q(x) := 1 + x^2 + x^4$, what is the size of the smallest collection of f_is that works?
(A) 1
(B) 2

(C) 3
(D) 4
(E) 5

Answer: Option (D)

Explanation: The denominator factors into $x^2 - x + 1$ and $x^2 + x + 1$. Each of these contributes one arctan possibility and one ln possibility. A total of 4 possibilities is achieved.

In general, if there are no repeated factors, the smallest number of pieces equals the degree of the polynomial.

Post-performance review: 7 out of 27 people got this correct. 9 people chose (C), 6 people chose (B), 4 people chose (A), and 1 person left the question blank.