CLASS QUIZ SOLUTIONS: NOVEMBER 2: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

This quiz actually happened on November 7. 11 people took this quiz. The score distribution was as follows:

- Score of 0: 2 people
- Score of 1: 2 people
- Score of 2: 4 people
- Score of 3: 2 people
- Score of 4: 1 person

The mean score was 1.8. The problem wise solutions and scores:

(1) Option (B): 7 people
(2) Option (A): 7 people
(3) Option (C): 1 person
(4) Option (B): 5 people
(5) Option (E): Nobody

Overall, performance on Questions 1 and 3 was not as good as it had been last year, with Question 3 the most prominent of the lot. I think this may have been because I explained something very similar to the questions in class last year.

2. Solutions

(1) Suppose f and g are both functions on $\mathbb{R}$ with the property that f'' and g'' are both everywhere the zero function. For which of the following functions is the second derivative *not necessarily* the zero function everywhere?
   (A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
   (B) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
   (C) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
   (D) All of the above, i.e., the second derivative need not be identically zero for any of these functions.
   (E) None of the above, i.e., for all these functions, the second derivative is the zero function.

   Answer: Option (B)

   Explanation: f and g are both polynomial functions of degree at most 1, i.e., they are both constant or linear. A sum of two such functions is again of the same type (i.e., constant or linear). A composite of two such functions is also of the same type (i.e., constant or linear). On the other hand, a product of two such functions need not be of that type, e.g., x times x is x^2.

   Note that the question can also be solved without explicitly using the actual form of f and g, i.e., by just computing $(f+g)''$, $(f \cdot g)''$, and $(f \circ g)''$. However, this is somewhat more time-consuming.

   Performance review: 7 out of 11 got this correct. 3 chose (E) and 1 chose (D).

   Historical note (last year): 14 out of 15 people got this correct. 1 person chose (D).

(2) Suppose f and g are both functions on $\mathbb{R}$ with the property that f''' and g''' are both everywhere the zero function. For which of the following functions is the third derivative *necessarily* the zero function everywhere?
   (A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
   (B) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
   (C) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
   (D) All of the above, i.e., the third derivative is identically zero for all of these functions.

(E) None of the above, i.e., the third derivative is not guaranteed to be the zero function for any of these.

Answer: Option (A)

Explanation: Clearly, $(f+g)''' = f''' + g'''$, so if $f''' = g''' = 0$, then $(f+g)''' = 0$.

Counterexamples for the others: for products, take $f(x) = g(x) = x^2$. For composites, the same counterexample works. What's different from the previous problem is that while a composite of linear polynomials is linear, a composite of quadratic polynomials has degree 4.

Performance review: 7 out of 11 got this correct. 2 chose (B) and 2 chose (D).

Historical note (last year): 12 out of 15 people got this correct. 3 people chose (D).

(3) Suppose f is a function on an interval $[a,b]$, that is continuous except at finitely many interior points $c_1 < c_2 < \cdots < c_n$ $(n \geq 1)$, where it has jump discontinuities (hence, both the left-hand limit and the right-hand limit exist but are not equal). Define $F(x) := \int_a^x f(t)\,dt$. Which of the following **is true**?

(A) F is continuously differentiable on (a,b) and the derivative equals f wherever f is continuous.

(B) F is differentiable on (a,b) but the derivative is not continuous, and $F' = f$ on the entire interval.

(C) F has one-sided derivatives on (a,b) and the left-hand derivative of F at any point equals the left-hand limit of f at that point, while the right-hand derivative of F at any point equals the right-hand limit of f at that point.

(D) F has one-sided derivatives on all points of (a,b) except at the points $c_1, c_2, \dots, c_n$; it is continuous at all these points but does not have one-sided derivatives.

(E) F is continuous at all points of (a,b) except at the points $c_1, c_2, \dots, c_n$.

Answer: Option (C)

Explanation: On each interval $[c_i, c_{i+1}]$, the function is differentiable on the interior and has one-sided derivatives at the endpoints, which equal the corresponding one-sided limits of f. Piecing together the intervals, we get the desired result.

Performance review: 1 out of 11 got this correct. 4 each chose (B) and (E), 2 chose (D).

Historical note (last year): 8 out of 15 people got this correct. 5 people chose (B) and 2 people chose (D).

Action point: We'll review this in class next time.

(4) (**) For a continuous function f on $\mathbb{R}$ and a real number a, define $F_{f,a}(x) = \int_a^x f(t)\,dt$. Which of the following is **true**?

(A) For every continuous function f and every real number a, $F_{f,a}$ is an antiderivative for f, and every antiderivative of f can be obtained in this way by choosing a suitably.

(B) For every continuous function f and every real number a, $F_{f,a}$ is an antiderivative for f, but it is not necessary that every antiderivative of f can be obtained in this way by choosing a suitably. (i.e., there are continuous functions f where not every antiderivative can be obtained in this way).

(C) For every continuous function f, every antiderivative of f can be written as $F_{f,a}$ for some suitable a, but there may be some choices of f and a for which $F_{f,a}$ is not an antiderivative of f.

(D) There may be some choices for f and a for which $F_{f,a}$ is not an antiderivative for f, and there may be some choices of f for which there exist antiderivatives that cannot be written in the form $F_{f,a}$.

(E) None of the above.

Answer: Option (B)

Explanation: The first clause: for every continuous function f and every real number a, $F_{f,a}$ is an antiderivative of f is just a restatement of Theorem 5.3.5, which we covered. This already whittles our options down to (A) and (B). To see why (B) is true, imagine a situation where $F_{f,0}$ does not take all real values, e.g., it is an increasing function with horizontal asymptotes at -1 and 1. We have $F_{f,0} - F_{f,a} = F_{f,0}(a)$ (by properties of integrals). Thus, there is no way we can choose a value of a for which $F_{f,a} = F_{f,0} + 5$.

In more intuitive terms, the problem is that whereas for getting all antiderivatives, we should be able to add arbitrary constants, there could be cases where the definite integral between two points cannot be made to include the set of all constants.

An explicit functional example (unfamiliar to you at this stage) is where $f(x) = 1/(x^2+1)$. Then $F_{f,0} = \arctan$ is bounded between $-\pi/2$ and $\pi/2$. Thus, say the function $20 + \arctan x$ cannot be realized as $F_{f,a}$ for any a.

Performance review: 5 out of 11 got this correct. 5 chose (C), 1 chose (A).

Historical note (last year): 5 out of 15 people got this correct. 5 people chose (A), 4 people chose (C), and 1 person chose (D).

Action point: We will return to this in Math 153 when we study improper integrals.

(5) (**) Suppose F is a differentiable function on an open interval (a, b) and F' is not a continuous function. Which of these discontinuities can F' have?

(A) A removable discontinuity (the limit exists and is finite but is not equal to the value of the function)
(B) An infinite discontinuity (one or both the one-sided limits is infinite)
(C) A jump discontinuity (both one-sided limits exist and are finite, but not equal)
(D) All of the above
(E) None of the above

Answer: Option (E)

Explanation: This is hard – perhaps part of a future challenge problem, so won't say more more. Briefly, the only kinds of discontinuities allowed are oscillatory discontinuities, of the kind seen with the derivative of $x^2 \sin(1/x)$ at 0.

Performance review: Nobody got this correct. 3 each chose (B), (C), (D), and 1 chose (A).

Historical note (last year): Nobody got this correct.