TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY JANUARY 23: VECTORS, 3D, AND PARAMETRIC STUFF – MISCELLANEA

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

23 people took this 12-question quiz. The score distribution was as follows:

- Score of 7: 2 people.
- Score of 8: 4 people.
- Score of 9: 5 people.
- Score of 10: 9 people.
- Score of 11: 3 people.

The question-wise answers and performance review were as follows:

(1) Option (E): 20 people.
(2) Option (B): 23 people.
(3) Option (E): 22 people.
(4) Option (E): 20 people.
(5) Option (D): 21 people.
(6) Option (C): 23 people.
(7) Option (E): 12 people.
(8) Option (E): 20 people.
(9) Option (A): 10 people.
(10) Option (D): 22 people.
(11) Option (D): 7 people.
(12) Option (A): 14 people.

2. Solutions

(1) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x, y, z) = 0$, Γ_2 is the set of solutions to $F_2(x, y, z) = 0$, and Γ_3 is the set of solutions to $F_3(x, y, z) = 0$. Which of the following equations gives precisely the set of points that lie in *at least two* of the subsets Γ_1, Γ_2, and Γ_3?
(A) $F_1(x, y, z)F_2(x, y, z)F_3(x, y, z) = 0$
(B) $(F_1(x, y, z))^2 + (F_2(x, y, z))^2 + (F_3(x, y, z))^2 = 0$
(C) $(F_1(x, y, z) + F_2(x, y, z) + F_3(x, y, z))^2 = 0$
(D) $(F_1(x, y, z)F_2(x, y, z)) + (F_2(x, y, z)F_3(x, y, z)) + (F_3(x, y, z)F_1(x, y, z)) = 0$
(E) $(F_1(x, y, z)F_2(x, y, z))^2 + (F_2(x, y, z)F_3(x, y, z))^2 + (F_3(x, y, z)F_1(x, y, z))^2 = 0$

Answer: Option (E)

Explanation: Option (E) means that all the statements $F_1(x, y, z)F_2(x, y, z) = 0$, $F_2(x, y, z)F_3(x, y, z) = 0$, and $F_3(x, y, z)F_1(x, y, z) = 0$ must simultaneously be true. This forces at least two of the values $F_1(x, y, z)$, $F_2(x, y, z)$, and $F_3(x, y, z)$ to equal zero. Conversely, if two or more of these are zero, so are all the pair products. Thus, a point satisfies this equation if and only if it lies in at least two of the three subsets Γ_1, Γ_2, Γ_3.

Performance review: 20 out of 23 got this. 2 chose (B), 1 chose (D).

(2) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x, y, z) = 0$, Γ_2 is the set of solutions to $F_2(x, y, z) = 0$, and Γ_3 is the set of solutions to $F_3(x, y, z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$?
(A) $F_1(x, y, z) + F_2(x, y, z)F_3(x, y, z) = 0$

(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Answer: Option (B)

Explanation: Option (B) involves a sum of two squares being zero, so both the things being squared must equal zero. Thus, $F_1(x,y,z) = 0$ and $F_2(x,y,z)F_3(x,y,z) = 0$. The solution set to the latter is the union of the solution sets for F_2 and F_3, so is $\Gamma_2 \cup \Gamma_3$. So the overall solution set is $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$.

Performance review: All 23 got this correct.

(3) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x,y,z) = 0$, Γ_2 is the set of solutions to $F_2(x,y,z) = 0$, and Γ_3 is the set of solutions to $F_3(x,y,z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cup (\Gamma_2 \cap \Gamma_3)$?
(A) $F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z) = 0$
(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Answer: Option (E)

Explanation: The solution set to Option (E) is the union of the solution sets for F_1 and for $F_2^2 + F_3^2$. The latter is precisely the set of points for which $F_2 = F_3 = 0$, so is $\Gamma_2 \cap \Gamma_3$. The overall solution is thus $\Gamma_1 \cup (\Gamma_2 \cap \Gamma_3)$.

Performance review: 22 out of 23 got this. 1 chose (D).

(4) Start with two vectors a and b in $\mathbb{R}^3$ such that $a \times b \neq 0$. Consider a sequence of vectors $c_1, c_2, \dots, c_n, \dots$ in $\mathbb{R}^3$ (note: each c_n is a three-dimensional vector) defined as follows: $c_1 = a \times b$ and $c_{n+1} = a \times c_n$ for $n \geq 1$. Which *one* of the following statements is **false** about the c_ns? (5 points)
(A) All the vectors c_n are nonzero vectors.
(B) c_n and c_{n+1} are orthogonal for every n.
(C) c_n and c_{n+2} are parallel for every n.
(D) c_n and a are orthogonal for every n.
(E) c_n and b are orthogonal for every n.

Answer: Option (E)

Explanation: Note first that since $a \times b \neq 0$, both a and b are nonzero vectors.

In fact, although c_1 is orthogonal to both a and b, c_2, being orthogonal to c_1 and a, is in the plane of a and b and is orthogonal to a. Since a and b are not parallel, c_2 is not orthogonal to b.

For the other options:

Options (A) and (D): c_1 is orthogonal to a because it is a cross product involving a and a nonzero vector. At each stage, we are taking a cross product of a nonzero vector orthogonal to a with the nonzero vector a, so we get a nonzero vector orthogonal to a.

Option (B): c_{n+1} is a cross product of a and c_n, and a and c_n are both nonzero, so c_{n+1} is orthogonal to c_n.

Option (C): All the c_ns are orthogonal to a, so they are all in the plane orthogonal to a. Within this plane, each is perpendicular to its predecessor. Thus, c_n and c_{n+2} must be collinear. In fact, they points in opposite directions to each other, but are in the same line.

Performance review: 20 out of 23 got this. 2 chose (B), 1 chose (D).

Historical note (last time): 9 out of 23 people got this correct. 5 chose (C), 4 chose (D), 3 chose (A), 2 chose (B).

(5) As a general rule, what would you expect should be the dimensionality of the set of solutions to m independent and consistent equations in n variables? By solution, we mean here that the solution should be the n-tuples with coordinates in $\mathbb{R}$ (or elements of $\mathbb{R}^n$) that satisfy all the m equations. Assume $n \geq m \geq 1$.
(A) n

(B) m
(C) $n-1$
(D) $n-m$
(E) 1

Answer: Option (D)

Explanation: As a general rule, we start out with the whole space, and each new constraint, if independent of prior constraints, whittles down the dimension by 1. Thus, introducing m constraints in n-dimensional space gives a dimension of $n-m$.

Note that this is not a hard-and-fast rule, because we can use tricks like the *sum of squares* trick to combine multiple equations into a single equation. However, it is a good rule of thumb for generic equations.

Performance review: 21 out of 23 got this. 2 chose (B).

(6) As a general rule, what would you expect should be the dimensionality of the set of points in $\mathbb{R}^n$ that satisfy at least one of m independent and consistent equations in n variables? Assume $n \geq m \geq 1$.
(A) n
(B) m
(C) $n-1$
(D) $n-m$
(E) 1

Answer: Option (C)

Explanation: We get a union of solution sets for equations, and each of these solution sets is of dimension $n-1$ (since it is obtained by imposing a single constraint on n-dimensional space). The union should thus also have dimension $n-1$.

Performance review: All 23 got this correct.

(7) Measuring time t in seconds since the beginning of the year 2013, and stock prices on a 24×7 stock exchange in predetermined units, the stock prices of companies A, B, and C were found to be given by $30 + t/5000000 - \sin(t/10000)$, $16 + 7t/3000000$, and $40 + t/1000000 - 5\sin(t/10000)$. To what extent can we deduce the stock prices of the companies from each other at a given point in time, without knowing what the time is?
(A) The stock price of any of the three companies can be used to deduce the other stock prices.
(B) The stock price of company A can be used to deduce the stock prices of companies B and C, but no other deductions are possible.
(C) The stock price of company A can be used to deduce the stock prices of companies B and C, and the stock price of company C can be used to deduce the stock prices of companies A and B.
(D) The stock price of company B can be used to determine the stock prices of companies A and C, and no other deductions are possible.
(E) The stock price of company B can be used to determine the stock prices of companies A and C, and the stock prices of companies A and C can be used to deduce each other but cannot be used to uniquely deduce the stock price of company B.

Answer: Option (E)

Explanation: Since the stock price of company B is linear in t, we can determine t from it, and use this to determine the stock prices of A and C. This means that the only options we need to consider are (D) and (E).

The stock prices of A and C are related by a linear relation $5(A-30) = C-40$, or equivalently, $5A = C + 110$. Thus, either is expressible in terms of the other.

Performance review: 12 out of 23 got this correct. 10 chose (D), 1 chose (C).

(8) Lushanna is coaching 30 young athletes for a 100 meter sprint. Every day, at the beginning of the day, she asks the athlete to run 100 meters as fast as they can and notes the time taken. She thus gets a vector with 30 coordinates (measuring the time taken by all the athletes) everyday. Lushanna then plots a graph in thirty-dimensional space that includes all the points for her daily measurements. Each of the following is a sign that Lushanna's young athletes are improving. Which of these signs is **strongest**, in the sense that it would imply all the others?

(A) The norm (length) of the vector every day (after the first) is less than the norm of the vector the previous day.
(B) The sum of the coordinates of the vector every day (after the first) is less than the sum of the coordinates of the vector the previous day.
(C) The minimum of the coordinates of the vector every day (after the first) is less than the minimum of the coordinates of the vector the previous day.
(D) The maximum of the coordinates of the vector every day (after the first) is less than the maximum of the coordinates of the vector the previous day.
(E) Each of the coordinates of the vector every day (after the first) is less than the corresponding coordinate of the vector the previous day.

Answer: Option (E)

Explanation: If each of the coordinates goes down, then all the measures (the maximum, minimum and various average measures) go down. However, it is possible for any one of these measures to go down while the others don't.

Performance review: 20 out of 23 got this. 3 chose (B).

(9) In a closed system (no mass exchanged with the surroundings) a reversible chemical reaction $A+B \to C + D$, and its reverse, are proceeding. There are no other chemicals in the system, and no other reactions are proceeding in the system. A chemist studying the reaction decides to track the masses of A, B, C, and D in the system as a function of time, and plots a parametric curve in four-dimensional space. What can we say about the nature of the curve, ignoring the parametrization (i.e., just looking at the set of points covered)?
(A) It is a part of a straight line.
(B) It is a part of a circle.
(C) It is a part of a parabola.
(D) It is a part of an astroid.
(E) It is a part of a cissoid.

Answer: Option (A)

Explanation: This follows from the law of constant proportions in chemistry! Basically, the amounts of gain/loss in each coordinate are fixed in proportion based on the stoichiometry of the reaction.

Performance review: 10 out of 23 got this correct. 10 chose (B), 2 chose (D), 1 chose (C).

Lobbying special: Casa is a lobbyist for a special interest group. There are three politicians P_1, P_2, P_3 competing for a general election. Casa has computed that the probabilities of the politicians winning are p_1 for P_1, p_2 for P_2, and p_3 for P_3, with $p_1, p_2, p_3 \in [0, 1]$ and $p_1 + p_2 + p_3 = 1$. Casa estimates a payoff of m_1 money units to her special interest group if P_1 wins, m_2 money units if P_2 wins, and m_3 money units if P_3 wins. (These payoffs may be in terms of passage of favorable laws, repeal of unfavorable laws, or enforcement of laws unfavorable to competitors).

(10) What is the expected payoff to the special interest group that Casa represents?
(A) $m_1 + m_2 + m_3$
(B) $(m_1 + m_2 + m_3)/3$
(C) $(p_1 + p_2 + p_3)(m_1 + m_2 + m_3)$
(D) $p_1m_1 + p_2m_2 + p_3m_3$
(E) $\sqrt{m_1^2 + m_2^2 + m_3^2}$

Answer: Option (D)

Explanation: The expected payoff contributions for each victory are p_1m_1, p_2m_2, and p_3m_3 respectively.

Performance review: 22 out of 23 got this. 1 chose (C).

(11) Casa has discovered that the bribe multipliers of the politicians are the positive reals b_1, b_2, and b_3 respectively. In other words, if Casa donates u_i money units to P_i, then the expected payoff from politician P_i winning is now $m_i + b_iu_i$. Consider the vectors $p = \langle p_1, p_2, p_3\rangle$, $m = \langle m_1, m_2, m_3\rangle$, $c = \langle p_1b_1, p_2b_2, p_3b_3\rangle$, and $f = \langle p_1/b_1, p_2/b_2, p_3/b_3\rangle$ and let $u = \langle u_1, u_2, u_3\rangle$ be the vector of the bribe quantities Casa gives to the politicians respectively. Assume that bribing politicians does not affect the relative probabilities of winning the election. Which of the following describes Casa's expected

payoff from the election, once the bribe is made (if you want to include the cost of bribes, you'd need to subtract $u_1 + u_2 + u_3$ from this answer, but we're not doing that. *Note: Some of the answer options may not make sense from a dimensions/units viewpoint, but the correct answer does make sense.*

(A) $p \cdot (m + u)$
(B) $p \cdot (m + (b \cdot u))$
(C) $p \cdot (m + (f \cdot u))$
(D) $(p \cdot m) + (c \cdot u)$
(E) $p \cdot (f \cdot m + u)$

Answer: Option (D)

Explanation: The expected payoff for each politician is $m_i + b_i u_i$, so the expected payoff accounting for probability is $p_i m_i + p_i b_i u_i = p_i m_i + c_i u_i$. The sum thus becomes $(p \cdot m) + (c \cdot u)$.

Performance review: 7 out of 23 got this. 14 chose (B), 1 chose (E), 1 left the question blank.

(12) Continuing with the full setup of the preceing question, what is Casa's optimal bribing strategy on a fixed budget of money to be used for bribes?

(A) Donate all the money to the politician with the maximum value of $p_i b_i$, i.e., to the politician corresponding to the largest coordinate of the vector c.
(B) Donate all the money to the politician with the minimum value of $p_i b_i$, i.e., to the politician corresponding to the smallest coordinate of the vector c.
(C) Donate all the money to the politician with the maximum value of p_i/b_i, i.e., to the politician corresponding to the largest coordinate of the vector f.
(D) Donate all the money to the politician with the minimum value of p_i/b_i, i.e., to the politician corresponding to the smallest coordinate of the vector f.
(E) Split the bribery budget between the politicians in the ratio $p_1 b_1 : p_2 b_2 : p_3 b_3$.

Answer: Option (A)

Explanation: $p_i b_i$ is the overall return on investment multiplier for bribing any politician. It makes sense to spend scarce bribery resources on the politician with the highest return on investment.

Performance review: 14 out of 23 got this. 6 chose (E), 3 chose (C).