HOMEWORK 4 CHECKLIST: DUE MONDAY OCTOBER 28

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

The computational problem numbers have moved a bit since the 4th Edition; if using the 4th Edition, please check the correct problem number before comparing your answer with that at the back of the book (if you're doing so) for odd-numbered problems.

(1) Exercise 2.3.3 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Something goes wrong ...

(2) Exercise 2.3.6 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

You can do the multiplication by hand. All the products being summed up are 0, so ... The final answer is a 2×2 matrix with entries ...

Matrices of the sort seen here are discussed in Section 10 of the "Matrix multiplication and inversion" notes.

(3) Exercise 2.3.7 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The product is a 3×3 matrix. Note that computing each entry involves computing a dot product of three-dimensional vectors, so it involves computing three products and doing two additions.

(4) Exercise 2.3.8 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(5) Exercise 2.3.10 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Just do it! The answer is a 3×3 matrix, and it will be a rank one matrix (i.e., all rows are multiples of each other). This special case of matrix multiplication that involves a column vector times a row vector is also called the *Hadamard product* or *outer product*. The term "outer product" signifies the contrast with *inner product*, which is an alternative term for the dot product.

(6) Exercise 2.3.11 (Page 85): Compte the matrix product by hand:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Just do it! This is a dot product, and the answer is a 1×1 matrix.

(7) Exercise 2.3.13 (Page 85): Compute the matrix product by hand:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Matrix multiplication is associative, so you could multiply in either of these ways:

$$\left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

Let's look at the left-associated way of computing it (the one written earlier). In this case, the first product gives a 1×3 matrix that essentially picks out one of the rows of the 3×3 matrix (which row?) and the product that we then have to do is a dot product that picks a particular entry from that row. In other words, the upshot is that we are picking a single entry from the 3×3 matrix.

For the next few questions, you can simultaneously address the question of invertibility and the question of finding the inverse. Namely, you take the matrix, augment it with the identity matrix, then perform the row reductions. If the rref of the matrix is the identity matrix, then the matrix is invertible and the augmented side describes the inverse. If the rref is not the identity matrix, then the matrix is non-invertible (i.e., one or more of the rows is redundant, so it does not have full rank).

Note that if at any stage we get a zero row, then we do not need to complete all the steps towards computing the rref, because the matrix does not have full rank.

(8) Exercise 2.4.2 (Page 97): Determine whether the matrix is invertible. If it is, find the inverse. Do the computations with paper and pencil.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

In this case, it will turn out the the matrix does not have full rank. The rank is 1, because the second row is redundant given the first. So, the matrix is non-invertible.

(9) Exercise 2.4.8 (Page 97) (was 2.4.10 in the 4th Edition): Determine whether the matrix is invertible. If it is, find the inverse. Do the computations with paper and pencil.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Apply the procedure. In this case, we only need to use the shear operations (subtracting a multiple of one row from another) and we do not need to use other operations such as permuting two rows or multiplying a row by a scalar. The matrix is invertible (its rref is the identity matrix) and its inverse matrix can be obtained through the "augmentation with the identity matrix" procedure.

(10) Exercise 2.4.10 (Page 97): Determine whether the matrix is invertible. If it is, find the inverse. Do the computations with paper and pencil.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrix can be converted to rref *purely by means of row interchange operations.* The inverse matrix can be obtained by doing those same operations on the identity matrix.

Both the original matrix and its inverse matrix are permutation matrices (we'll see the meaning of the term in more detail soon if we haven't by the time you're reading this). Essentially, both matrices send standard basis vectors to each other.

The matrix given to us sends $\vec{e}_1$ to $\vec{e}_3$, sends $\vec{e}_2$ to itself, and sends $\vec{e}_3$ to $\vec{e}_1$. Its inverse should do the opposite (which in this case turns out to be not all that different).

(11) Exercise 2.4.12 (Page 97) (was 2.4.14 in the 4th Edition) : Determine whether the matrix is invertible. If it is, find the inverse. Do the computations with paper and pencil.

$$\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

We can do this directly for the 4×4 matrix, but it's also worth noting, conceptually, that each 2×2 matrix can be inverted in place.

Explicitly, recall that the inverse of a diagonal matrix can be obtained by inverting each entry in place:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix}$$

The same principle extends to *block* diagonal matrices. Suppose A and D are square matrices (of possibly different sizes). Then:

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$$

Note that the 0 on the top right and bottom left are rectangular matrices of appropriate sizes (if A and D are square of the same size, then both of those are also square of the same size).

In the particular case, the matrices A and D are as follows:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

(12) Exercise 2.4.15 (Page 97): Determine whether the matrix is invertible. If it is, find the inverse. Do the computations with paper and pencil.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}$$

Same procedure as previous questions. This is *not* block diagonal, so we cannot use the shortcut approach used in the previous question.

(13) Exercise 2.4.25 (Page 97): Is this nonlinear transformation invertible? If so, find the inverse.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix}$$

The goal is to write x_1 in terms of both y_1 and y_2, and to write x_2 in terms of both y_1 and y_2, and then express that as a vector:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{expression for } x_1 \text{ in terms of } y_1 \text{ and } y_2 \\ \text{expression for } x_2 \text{ in terms of } y_1 \text{ and } y_2 \end{bmatrix}$$

Note that *invertibility* is equivalent to bijectivity. We can break this down into two aspects:

- *Surjectivity* means that we can always find values of x_1 and x_2 for any given values of y_1 and y_2.
- *Injectivity* (or being *one-one*) means that knowing y_1 and y_2, if they did arise from actual values of x_1 and x_2, *uniquely* determines the values x_1 and x_2 from which they arose.

For this particular case, the function is of a *diagonal* nature, i.e., each y_i is functionally dependent on the corresponding x_i. Thus, the invertibility of the function as a whole is equivalent to the invertibility of each of the coordinate functions. Further, the inverse to the function as a whole is also of a diagonal nature with the coordinate functions each being inverses of the original coordinate functions.

(14) Exercise 2.4.26 (Page 97): Is this nonlinear transformation invertible? If so, find the inverse.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^3 + x_2 \end{bmatrix}$$

The generic part of the hint is the same as the preceding question, but the function is no longer diagonal in nature. Rather, it is *triangular* in nature: one of the outputs depends on only one of the inputs, but the other output depends on both the inputs.

(15) Exercise 2.4.27 (Page 97): Is this nonlinear transformation invertible? If so, find the inverse.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix}$$

This is not invertible, because interchanging the roles of x_1 and x_2 gives the same answer. In other words, the output is symmetric in x_1 and x_2. For instance, $(2, 3)$ and $(3, 2)$ map to the same output. So, the map is not injective.

The map is also not surjective. To see this, note that x_1 and x_2 are the solutions to the quadratic equation:

$$x^2 - y_1 x + y_2 = 0$$

But we could choose values of y_1 and y_2 so that the quadratic has no solution. For instance, $y_1 = 0$, $y_2 = 1$.

(16) Exercise 2.4.29 (Page 97): For which values of the constant k is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

Do the general row reduction, and determine the condition needed to be able to get to the identity matrix. Conditions may emerge at steps where you are dividing by something and need to assume it is nonzero.

(17) Exercise 2.4.30 (Page 98): For which values of the constants b and c is the following matrix invertible?

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Again, do the general row reduction, similar to the preceding problem.

2. Problems for your own review, not for submission

(1) Exercise 2.4.34 (Page 98): Consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

(a) For which values of a, b, and c is A invertible? If it is invertible, what is A^{-1}?
(b) For which values of the diagonal elements is a diagonal matrix (of arbitrary size) invertible?

Checklist hint begins: This has to do with whether the diagonal entries are zero or nonzero.

(2) Exercise 2.4.35 (Page 98): Consider the upper triangular 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

(a) For which values of a, b, c, d, e, and f is A invertible?
(b) More generally, when is an upper triangular matrix (of arbitrary size) invertible?
(c) If an upper triangular matrix is invertible, is its inverse an upper triangular matrix as well?
(d) When is a lower triangular matrix invertible?

Checklist hint begins: This has to do with whether the diagonal entries are zero.

(3) Exercise 2.4.67 (Page 100): For two invertible $n \times n$ matrices A and B, determine whether the formula is true:

$$(A + B)^2 = A^2 + 2AB + B^2$$

Checklist hint begins: We can expand $(A + B)(A + B)$ using distributivity. But we do not in general have commutativity, so we cannot rewrite BA as AB.

If you're able to come up with examples of $n \times n$ matrices that do not commute, those would give you examples of matrices where the above identity fails.

(4) Exercise 2.4.70 (Page 100): For an invertible $n \times n$ matrix A, determine whether A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$.

Yes ...

(5) Exercise 2.4.68 (Page 100): For two invertible $n \times n$ matrices A and B, determine whether the formula is true:

$$(A - B)(A + B) = A^2 - B^2$$

Checklist hint begins: We can expand by distributivity, but AB and BA need not in general be equal.

If you're able to come up with examples of $n \times n$ matrices that do not commute, those would give you examples of matrices where the above identity fails.