CLASS QUIZ SOLUTIONS: SEPTEMBER 28; TOPIC: FUNCTIONS

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1. Performance review

12 students submitted solutions. Here is the score distribution:

(1) Score of 0: Nobody.
(2) Score of 1: Nobody.
(3) Score of 2: 2 persons.
(4) Score of 3: 2 persons.
(5) Score of 4: 5 persons.
(6) Score of 5: 3 persons.

The mean score was 3.75 and the median was 4.

If you scored poorly on this quiz, there is no need to get discouraged, since these question types are probably new for you even if you're familiar with the material. Please take the time to go through the solutions so that you are able to get similar questions correct if you see them in the future.

Note: You are allowed and encouraged to discuss/collaborate with others on some questions (the star-marked ones) but you should ultimately put the solution that *best fits your conscience* and not simply go with the crowd against what you believe to be the correct answer.

Here is the summary of problem wise scores:

(1) Option (E): 9/12 correct.
(2) (*) Option (C): 11/12 correct. *Magic of collaboration?*
(3) Option (D): 12/12 correct.
(4) (*) Option (A): 8/12 correct.
(5) (*) Option (E): 5/12 correct. *Needs review!*

More details in the next section.

2. Solutions

(1) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose both f and g are even, i.e., $f(x) = f(-x)$ for all $x \in \mathbb{R}$ and $g(x) = g(-x)$ for all $x \in \mathbb{R}$. Which of the following is *not* guaranteed to be an even function from the given information?

Note: For this question, it is possible to solve the question by taking a few simple examples. You're free to do this, but the recommended method for tackling the question is to handle it **abstractly**, *i.e., try to prove or disprove in general for each function whether it is even.*

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be even functions.

Answer: Option (E)

Explanation: We show that $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all even. Below are given short versions of "proofs" of these facts.

$f + g$ is even because

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$$

Hence $(f + g)(-x) = (f + g)(x)$ for all x.

$f - g$ is even because

$$(f-g)(-x) = f(-x) - g(-x) = f(x) - g(x) = (f-g)(x)$$

$f \cdot g$ is even because

$$(f \cdot g)(-x) = f(-x)g(-x) = f(x)g(x) = (f \cdot g)(x)$$

$f \circ g$ is even because:

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$$

Note that $f \circ g$ is somewhat special, because for this case, we only use that g is even – we do not use or require f to be even.

Performance review: 9 out of 12 people got this correct. 3 chose (D).

Historical note (last year): 11 out of 15 people got this correct. 3 people chose (D) and 1 person chose (B).

(2) (*) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose both f and g are odd, i.e., $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Which of the following is *not* guaranteed to be an odd function from the given information?

Note: For this question, it is possible to solve the question by taking a few simple examples. You're free to do this, but the recommended method for tackling the question is to handle it **abstractly**, *i.e., try to prove or disprove in general for each function whether it is odd.*

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be odd functions.

Answer: Option (C)

Explanation: An example is when $f(x) := x$ and $g(x) := x$. Both f and g are odd functions. But the product $f \cdot g$ the function $x \mapsto x^2$, which is an even function.

The other choices:

Option (A): If f and g are both odd, then $f + g$ has to be odd. Here's why:

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) + (-g(x)) = -[f(x) + g(x)] = -(f+g)(x)$$

Option (B): If f and g are both odd, then $f - g$ has to be odd. Here's why:

$$(f-g)(-x) = f(-x) - g(-x) = -f(x) - (-g(x)) = -[f(x) - g(x)] = -(f-g)(x)$$

Option (D): If f and g are both odd, then $f \circ g$ has to be odd. Here's why:

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$$

Option (E) is ruled out because option (C) works.

Performance review: 11 out of 12 got this correct. 1 chose (D). *Performance this year was much better than last year, possibly due to the benefits of collaboration?*

Historical note (last year): 7 out of 15 people got this correct. 5 people chose (E) and 3 people chose (D).

(3) For which of the following pairs of polynomial functions f and g is it true that $f \circ g \neq g \circ f$?

(A) $f(x) := x^2$ and $g(x) := x^3$
(B) $f(x) := x + 1$ and $g(x) := x + 2$
(C) $f(x) := x^2 + 1$ and $g(x) := x^2 + 1$
(D) $f(x) := -x$ and $g(x) := x^2$
(E) $f(x) := -x$ and $g(x) := x^3$

Answer: Option (D)

Explanation: For option (D), $f(g(x)) = -x^2$ and $g(f(x)) = (-x)^2 = x^2$. The two polynomials take distinct values for all nonzero x, hence they are not equal as functions.

The other choices:

Option (A): $f(g(x)) = (x^3)^2 = x^{3 \cdot 2} = x^6$. Similarly $g(f(x)) = (x^2)^3 = x^{2 \cdot 3} = x^6$. So in this case $f \circ g = g \circ f$.

Option (B): $f(g(x)) = (x+2)+1 = x+3$ and $g(f(x)) = (x+1)+2 = x+3$. So in this case $f \circ g = g \circ g$.

Option (C): Here, $f = g$ so both $f \circ g$ and $g \circ f$ are equal to $f \circ f$. Note that we do not need to explicitly compute $f \circ g$ in this case.

Option (E): Here, $f(g(x)) = -x^3$ and $g(f(x)) = (-x)^3 = (-1)^3 x^3 = -x^3$.

Additional remark: If $f \circ g = g \circ f$, we say that the functions f and g *commute*. You may have heard about the commutativity law for addition and multiplication. In the case of function composition, commutativity is *not* a law. It holds for some pairs of functions (such as options (A), (B), (D), (E) here) and not for others (such as option (C)).

Note that for the function $f(x) := -x$, a function h commutes with f if and only if h is an odd function. Thus, in option (E), the cube map is an odd function. And option (D) fails because the square map is *not* an odd function.

Performance review: Everybody(12 out of 12) got this correct.

Historical note (last year): 14 out of 15 people got this correct. 1 person chose (B).

Action point: Even if you got this correct, it may be helpful to try to understand how to more quickly see that for all the other pairs, $f \circ g = g \circ f$.

(4) (*) Which of the following functions is *not* periodic?

(A) $\sin(x^2)$
(B) $\sin^2 x$
(C) $\sin(\sin x)$
(D) $\sin(x + 13)$
(E) $(\sin x) + 13$

Answer: Option (A)

Explanation: It's somewhat hard to show that $\sin(x^2)$ is not a periodic function. (This will be an advanced problem in a subsequent homework).

On the other hand, it is easy to solve this problem by elimination, since the other four options are periodic, as explained below.

The other choices:

Option (B): $\sin^2$ is periodic and has period π. This follows from the fact that $\sin(\pi + x) = -\sin x$. Even if you don't notice that the period is π, you can still deduce 2π-periodicity from the fact that sin is 2π-periodic.

Option (C): $\sin \circ \sin$ is 2π-periodic.

Note that options (B) and (C) both from a more general fact: if $f, g : \mathbb{R} \to \mathbb{R}$ are functions, and g is periodic, so is $f \circ g$, and any choice of $h > 0$ that works for g also works for $f \circ g$ (though the period for $f \circ g$ may be smaller than that for g)

Option (D): This is 2π-periodic. In graphical terms, it is obtained by shifting the graph of the sin function 13 units to the left, and the periodicity pattern is unaffected.

Option (E): This is 2π-periodic. Adding a constant function (in this case, 13) to a periodic function (in this case, sin) gives a periodic function with the same period.

Performance review: 8 out of 12 got this correct. 2 each chose (B) and (C).

Historical note (last year): 8 out of 15 people got this correct. 3 people chose (B), 3 people chose (C), and 1 person chose (D).

Action point: Please make sure you understand the reasoning for why the other functions are periodic. You may also experiment with plotting the graphs of these functions using Mathematica or a graphing calculator.

(5) (*) What is the domain of the function $\sqrt{1-x} + \sqrt{x-2}$? Here, domain refers to the *largest set* on which the function can be defined.

(A) $(1, 2)$
(B) $[1, 2]$
(C) $(-\infty, 1) \cup (2, \infty)$
(D) $(-\infty, 1] \cup [2, \infty)$
(E) None of the above

Hint: Think clearly, first about what the domain of each of the two functions being added is, and then about whether you need to take the union or the intersection of the domains of the individual functions.

Answer: Option (E)

Explanation: For $\sqrt{1-x}$ to be defined, we need $1 - x \geq 0$, so $x \leq 1$. For $\sqrt{x-2}$ to be defined, we need $2 - x \geq 0$, so $x \geq 2$. Thus, we require that $x \leq 1$ and $x \geq 2$ hold *simultaneously*. In set theory terms, we need to take the *intersection* of the solution sets to $x \leq 1$ (which is $(-\infty, 1]$) and to $x \geq 2$ (which is $[2, \infty)$).

The two conditions cannot hold together, i.e., the intersection of the solutions for the two constraints is empty. Hence, the domain of the function is in fact empty, i.e., the function is defined *nowhere*.

The other choices: Options (C) and (D) are the most sophisticated distractors. Option (D) is the union of the domains of $\sqrt{1-x}$ and $\sqrt{x-2}$. However, what we need here is the *intersection* of the domains, not the union.

Performance review: 5 out of 12 got this correct. 7 chose (D), which is the *union* rather than the *intersection*. So, people fell for the sophisticated distractors, not the silly ones.

Historical note (last year): 8 out of 15 people got this correct. 5 people chose (D), 1 person chose (B), and 1 person chose (C).

Action point: Please make sure you understand why we need to *intersect* the domains of f and g rather than take the union. This goes back to the fact discussed in class that the domain of $f + g$ is the intersection of the domains of f and g.