DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE WEDNESDAY NOVEMBER 6: IMAGE AND KERNEL (BASIC)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

28 people took this 5-question quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 7 people
- Score of 4: 9 people
- Score of 5: 10 people

The mean score was slightly under 4.

The question-wise answers and performance review are below:

(1) Option (B): 26 people
(2) Option (A): 22 people
(3) Option (C): 17 people
(4) Option (B): 19 people
(5) Option (C): 27 people

2. Solutions

PLEASE DO NOT DISCUSS *ANY* QUESTIONS.

The questions here test for a very rudimentary understanding of the ideas covered in the lectures notes titled `Image and kernel of a linear transformation`. The corresponding section of the book is Section 3.1.

(1) *Do not discuss this!*: For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the kernel of T is defined as the set of vectors $\vec{x} \in \mathbb{R}^m$ satisfying the condition that $T(\vec{x}) = \vec{0}$. Which of the following correctly describes the type of subset of $\mathbb{R}^m$ that the kernel must be? Note that, as usual, we identify a set of vectors with the set of corresponding points.
(A) The kernel is a line segment in $\mathbb{R}^m$.
(B) The kernel is a linear subspace of $\mathbb{R}^m$, i.e., it passes through the origin and, for any two points in the kernel, the line joining them is completely inside the kernel.
(C) The kernel is an affine linear subspace of $\mathbb{R}^m$ but it need not be linear, i.e., it is non-empty and the line joining any two points in it is also in it, but it need not contain the origin.
(D) The kernel is a curve in $\mathbb{R}^m$ with a parametric description.

Answer: Option (B)

Explanation: See Section 4.3 of the lecture notes titled `Image and kernel of a linear transformation`.

Briefly: we can readily verify that $T(\vec{0}) = \vec{0}$, and we can verify that the kernel is a linear subspace based on our earlier definition (closed under addition and scalar multiplication). It's easy to see that this also coincides with our new definition of linear subspace.

Performance review: 26 out of 28 got this. 2 chose (C).

(2) *Do not discuss this!*: For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the kernel of T is defined as the set of vectors $\vec{x} \in \mathbb{R}^m$ satisfying the condition that $T(\vec{x}) = \vec{0}$. Given a vector $\vec{y} \in \mathbb{R}^n$, the set of solutions to $T(\vec{x}) = \vec{y}$ is either empty, or it bears some relation with the kernel. What relation does it bear to the kernel if it is nonempty?

(A) The solution set is an affine linear subspace of $\mathbb{R}^m$ (see definition in Option (C) of Q1) that is a translate of the kernel, i.e., there is a vector $\vec{v}$ such that the vectors in the solution set are precisely the vectors expressible as ($\vec{v}$ plus a vector in the kernel).
(B) The solution set coincides precisely with the kernel.
(C) The solution set comprises a single point (i.e., a single vector) that is not in the kernel.

Answer: Option (A)

Explanation: See Section 5 of the lecture notes titled `Image and kernel of a linear transformation`.

Performance review: 22 out of 28 got this. 5 chose (B), 1 chose (C).

(3) *Do not discuss this!*: Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, recall that we say that T is *injective* if for every $\vec{y} \in \mathbb{R}^n$, there exists *at most* one $\vec{x} \in \mathbb{R}^m$ satisfying $T(\vec{x}) = \vec{y}$. Another way of formulating this is that if A is the $n \times m$ matrix for T, then the linear system $A\vec{x} = \vec{y}$ has at most one solution for $\vec{x}$ for each fixed value of $\vec{y}$. We had earlier worked out that this condition is equivalent to full column rank (recall: all the variables need to be leading variables), which in this case means rank m.

What is the relationship between the injectivity of T and the kernel of T?
(A) T is injective if and only if the kernel of T is empty.
(B) If T is injective, then the kernel of T is empty. However, the converse is not in general true.
(C) T is injective if and only if the kernel of T comprises only the zero vector.
(D) If T is injective, then the kernel of T comprises only the zero vector. However, the converse is not in general true.
(E) If the kernel of T comprises only the zero vector, then T is injective. However, the converse is not in general true.

Answer: Option (C)

Explanation: See Sections 5 and 6 of the lecture notes titled `Image and kernel of a linear transformation`.

Performance review: 17 out of 28 got this. 8 chose (D), 3 chose (E).

(4) *Do not discuss this!*: For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the image of T is defined as the set of vectors $\vec{y} \in \mathbb{R}^n$ satisfying the condition that there exists a vector $\vec{x} \in \mathbb{R}^m$ satisfying $T(\vec{x}) = \vec{y}$. In other words, the image of T equals the range of T as a function. Which of the following correctly describes the type of subset of $\mathbb{R}^n$ that the image must be? Note that, as usual, we identify a set of vectors with the set of corresponding points.
(A) The image is a line segment in $\mathbb{R}^n$.
(B) The image is a linear subspace of $\mathbb{R}^n$, i.e., it passes through the origin and, for any two points in the image, the line joining them is completely inside the image.
(C) The image is an affine linear subspace of $\mathbb{R}^n$ but it need not be linear, i.e., it is non-empty and the line joining any two points in it is also in it, but it need not contain the origin.
(D) The image is a curve in $\mathbb{R}^n$ with a parametric description.

Answer: Option (B)

Explanation: See Section 4.1 of the `Image and kernel of a linear transformation` lecture notes.

Performance review: 19 out of 28 got this. 7 chose (C), 2 chose (A).

(5) *Do not discuss this!*: Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, recall that we say that T is *surjective* if for every $\vec{y} \in \mathbb{R}^n$, there exists *at least* one $\vec{x} \in \mathbb{R}^m$ satisfying $T(\vec{x}) = \vec{y}$. Another way of formulating this is that if A is the $n \times m$ matrix for T, then the linear system $A\vec{x} = \vec{y}$ has at least one solution for $\vec{x}$ for each fixed value of $\vec{y}$. We had earlier worked out that this condition is equivalent to full row rank (recall: we need all rows in the rref to be nonzero in order to avoid the potential for inconsistency), which in this case means rank n.

What is the relationship between the surjectivity of T and the image of T?
(A) T is surjective if and only if the image of T is empty.
(B) T is surjective if and only if the image of T comprises only the zero vector.
(C) T is surjective if and only if the image of T is all of $\mathbb{R}^n$.

Answer: Option (C)

Explanation: This is obvious from the definition.

Performance review: 27 out of 28 got this. 1 chose (B).