HOMEWORK 6 CHECKLIST: DUE MONDAY NOVEMBER 11

MATH 196, SECTION 57 (VIPUL NAIK)

Note: I've kept the homework short because you have a number of lengthy quizzes due in the immediate neighborhood of the homework, and it's more worthwhile for you to concentrate on the quizzes. However, if you feel that the homework provides you with insufficient computational practice, please do additional practice problems from the relevant section of the book.

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 3.1.4 (Page 119) (was 3.1.2 in the 4th Edition): Find vectors that span the kernel of A, where A is the matrix

$$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Standard procedure based on rref. Note that the rank is 1, and the number of columns is 2, so the dimension of the kernel is expected to be 2 - 1 = 1. Here, *dimension* means the minimum possible value of the size of a spanning set (a spanning set of minimum size is called a *basis*).

Alternatively, try to think of a nonzero vector that is orthogonal to both rows of this matrix. That is a spanning vector for the one-dimensional kernel. Recall that $[a\ b]$ and $[b\ -a]$ are orthogonal vectors.

(2) Exercise 3.1.7 (Page 119): Find vectors that span the kernel of A, where A is the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Standard procedure based on rref. Alternatively, we can think of trying to find a vector that is orthogonal to all the row vectors.

Note that if the matrix has full column rank, then the kernel is zero-dimensional, and can be spanned by the empty set (or alternatively, by the zero vector). If the rank is one less than the number of columns, the kernel can be spanned by a single vector. If the rank is two less than the number of columns, the kernel can be spanned by two vectors.

(3) Exercise 3.1.8 (Page 119): Find vectors that span the kernel of A, where A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Standard procedure based on rref. Similar remarks apply as in the preceding question.

(4) Exercise 3.1.11 (Page 119): Find vectors that span the kernel of A, where A is the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

Standard procedure based on rref. Similar remarks apply as in the preceding question.

(5) Exercise 3.1.31 (Page 120): Give an example of a matrix A such that $\text{im}(A)$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

Find a spanning set (note: I originally wrote *basis*, which is a term for a minimal spanning set) for the space of vectors orthogonal to this vector (take the *row matrix* for this vector and solve that matrix times $\vec{x}$ equals zero for $\vec{x}$). Now, write a matrix whose columns are that spanning set. The matrix would be a 3×2 matrix (you could use more columns if you wished, but minimalism is valuable).

(6) Exercise 3.1.32 (Page 120): Give an example of a linear transformation whose image is the line spanned by

$$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

The linear transformation whose matrix is quite literally that matrix will do.

(7) Exercise 3.1.33 (Page 120): Give an example of a linear transformation whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

The function $x + 2y + 3z$ will do. This can be thought of as a dot product, i.e., it can be thought of as having the form:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

In other words, the matrix in question is the matrix written on the left in the expression above.

(8) Exercise 3.1.34 (Page 120): Give an example of a linear transformation whose kernel is the line spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^3$.

The procedure is similar to that for 3.1.31 (why?).

2. Problems for your own review, not for submission

(1) Exercise 3.1.51 (Page 121): Consider a $n \times p$ matrix A and a $p \times m$ matrix B such that $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$. Find $\ker(AB)$.

This is very similar to various questions in the November 15 take-home class quiz on image and kernel. Take a look at this question in conjunction with those quiz questions and understand both better.

3. Advanced problems

(1) Exercise 3.1.35 (Page 120): Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Arguing geometrically, describe the image and kernel of the linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}$ given by the dot product

$$T(\vec{x}) = \vec{v} \cdot \vec{x}$$

Checklist hint begins: You can use that $T(\vec{x}) = |\vec{v}||\vec{x}| \cos \theta$ where θ is the angle between the vectors $\vec{v}$ and $\vec{x}$. $\vec{v}$ is nonzer, so the only way that $\vec{x}$ is in the kernel is if $\cos \theta = 0$, so ...

The image is all of $\mathbb{R}$, i.e., every real number arises as a posisble output of the operation.

Interpretation in matrix terms: the matrix for the linear transformation coincides with the *row* vector for $\vec{v}$.

(2) Exercise 3.1.36 (Page 120): Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Using a geometric argument, describe the kernel of the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$T(\vec{x}) = \vec{v} \times \vec{x}$$

Checklist hint begins: The rule for the cross product explicitly tells us that the cross product is perpendicular to both the vectors being multiplied (see if you can connect this with the explicit definition, using dot product to confirm perpendicularity). Thus, the image is the plane ... The kernel is the line ...

(3) Exercise 3.1.37 (Page 120): For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

describe the images and kernels of the matrices A, A^2, and A^3 geometrically.

The linear transformation corresponding to A is one of the finite state automata types, so you should be able to use the ideas for composing those types of transformations to cross-check your reasoning.