TAKE-HOME CLASS QUIZ SOLUTIONS: DECEMBER 7: DIFFERENTIAL EQUATIONS

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

35 people took this quiz. The score distribution was as follows:

- Score of 1: 3 people
- Score of 2: 2 people
- Score of 3: 2 people
- Score of 4: 1 person
- Score of 5: 6 people
- Score of 6: 5 people
- Score of 7: 6 people
- Score of 8: 3 people
- Score of 9: 7 people

The question wise answers and performance review were as follows:

(1) Option (D): 28 people
(2) Option (A): 26 people
(3) Option (B): 22 people
(4) Option (B): 23 people
(5) Option (A): 22 people
(6) Option (C): 13 people
(7) Option (A): 4 people
(8) Option (E): 14 people
(9) Option (E): 20 people
(10) Option (C): 25 people
(11) Option (E): 9 people

2. Solutions

(1) It takes time T for $1/10$ of a radioactive substance to decay. How much does it take for $3/10$ of the same substance to decay?
   (A) Between T and $2T$
   (B) Between $2T$ and $3T$
   (C) Exactly $3T$
   (D) Between $3T$ and $4T$
   (E) Between $4T$ and $5T$

   Answer: Option (D)

   Explanation: In time T, the material reduces to 0.9 of its original value. In time $3T$, it reduces to $0.9^3 = 0.729$ of its original value. In time $4T$, it reduces to $0.9^4 = 0.6561$ of its original value. The time taken for $3/10$ to decay, which means that it must reduce to 0.7 of its original value, is thus between $3T$ and $4T$.

   Intuitive rationale: The time taken for the first $1/10$ to decay is less than the time taken for the next $1/10$ to decay, because the next $1/10$ is, as a fraction, $1/9$ of what is left. Thus, the total time taken for $2/10$ to decay is slightly more than twice the time taken for $1/10$ to decay. Reasoning similarly, we see that the total time taken for $3/10$ to decay is slightly more than thrice the time taken for $1/10$ to decay.

Performance review: 28 out of 35 got this. 4 chose (A), 1 each chose (B), (C), and (E).

Historical note (last year): 5 out of 10 got this. 4 chose (E), 1 chose (C).

Historical note (two years ago): 22 out of 26 people got this correct. 2 people chose (A) and 1 person each chose (B) and (C). Many people did lengthy calculations involving ln.

Action point: Please make sure you understand the *intuitive rationale* presented above, so that you can answer this question faster.

(2) Suppose a function f satisfies the differential equation $f''(x) = 0$ for all $x \in \mathbb{R}$. Which of the following is true about $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$?

(A) If either limit is finite, then both are finite and they are equal. Otherwise, both the limits are infinities of opposite signs.

(B) If either limit is finite, then both are finite and they are equal. Otherwise, both the limits are infinities of the same sign.

(C) One of the limits is finite and the other is infinite.

(D) Both the limits are finite and unequal.

(E) Both the limits are infinite but they may be of the same or of opposite signs.

Answer: Option (A)

Explanation: Solving, we see that $f(x)$ is a function of the form $ax + b$, where a and b are constants. There are three cases: $a = 0$, in which case f is a constant function, $a > 0$, in which case f approaches $+\infty$ as $x \to \infty$ and approaches $-\infty$ as x approaches $-\infty$, and $a < 0$, in which case f approaches $-\infty$ as x approaches $+\infty$ and approaches $-\infty$ as x approaches $+\infty$.

Performance review: 26 out of 35 got this. 5 chose (E), 2 chose (D), 1 each chose (B) and (C).

Historical note (last year): All 10 got this.

Historical note (two years ago): 13 out of 26 people got this correct. 6 people chose (B) and 7 people chose (E). Of the people who chose (B), some seem to have mistakenly considered the general solution to be quadratic rather than linear.

(3) For y a function of x, consider the differential equation $(y')^2 - 3yy' + 2y^2 = 0$. What is the description of the **general solution** to this differential equation?

(A) $y = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are arbitrary real numbers.

(B) $y = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are real numbers satisfying $C_1C_2 = 0$ (i.e., at least one of them is zero)

(C) $y = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are real numbers satisfying $C_1 + C_2 = 0$.

(D) $y = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are real numbers satisfying $C_1C_2 = 1$.

(E) $y = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are real numbers satisfying $C_1 + C_2 = 1$.

Answer: Option (B)

Explanation: Factorize to obtain:

$$(y' - y)(y' - 2y) = 0$$

Thus, either $y' = y$ or $y' = 2y$. Note that for both these solutions to hold together, we must have $y = 0$ at some point, in which case it is identically zero. Thus, it cannot shift from one solution to the other. So, either $y' = y$ identically or $y' = 2y$ identically.

In case $y' = y$ identically, we get $y = C_1e^x$ and in case $y' = 2y$ identically, we get $y = C_2e^{2x}$. The general solution can be written as $C_1e^x + C_2e^{2x}$, with the proviso that at least one among C_1 and C_2 is zero.

Performance review: 22 out of 35 got this. 6 chose (D), 4 chose (C), and 3 chose (A).

Historical note (last year): 7 out of 10 got this. 1 each chose (A), (C), and (E).

Historical note (two years ago): 12 out of 26 people got this correct. 8 people chose (C), 4 people chose (A), 1 person chose (E), and 1 person left the question blank.

(4) Suppose $F(t)$ represents the number of gigabytes of disk space that can be purchased with one dollar at time t in commercially available disk drive formats (not adjusted for inflation). Empirical observation shows that $F(1980) \approx 5 * 10^{-6}$, $F(1990) \approx 10^{-4}$, $F(2000) \approx 10^{-1}$, and $F(2010) \approx 10$. From these data, what is a good estimate for the "doubling time" of F, i.e., the time it takes for the number of gigabytes purchaseable with a dollar to double?

(A) Between 6 months and 1 year.

(B) Between 1 year and 2 years.
(C) Between 2 years and 4 years.
(D) Between 4 years and 5 years.
(E) Between 5 years and 6 years.

Answer: Option (B)

Explanation: From the given data, the amount by which F multiplies in ten years is roughly 100. Note that the first doubling time is about 20, the next one is about 1000, and the next one is 100. That's just the way real-world data is messy!

Overall, it seems to be greater than 30 (although the 1980-1990 period comes slightly less than that) and less than 1000.

The first period (1980-1990) could be a little misleading in this sense. To get the best estimate, it makes sense to look at a longer time period, so looking at the overall time period of 30 years from 1980 to 2010 gives a total multiplication of $2 * 10^6$ over 30 years, which is closest to multiplication by 10^2 every 10 years.

If the doubling time is 1 year, then in ten years, F would multiply by $2^{10} = 1024$, which is too lrage. If the doubling time is 2 years, then in ten years, F would multiply by $2^5 = 32$, which is too small. The right doubling time is likely to therefore be between 1 and 2 years.

Real world thinking: Do you remember what USB drives, external disk drives, etc. used to cost two years ago per GB? Compare those costs with today. Do you remember how much disk space was there in a typical iPod, iPhone, or other smartphone? Compare that disk space with today. Do you see the doubling?

By the way, this is related to (but not the same as)"Kryder's law" which in turn is analogous to Moore's law.

Performance review: 23 out of 35 got this. 6 chose (C), 3 chose (A), 2 chose (D), 1 chose (E).

Historical note (last year): 9 out of 10 got this. 1 chose (C).

Historical note (two years ago): 10 out of 26 people got this correct. 7 people chose (A) (mild optimism!), 6 people chose (C) (mild pessimism!), 2 people chose (E) (superstrong pessimism!), and 1 person chose (D) (strong pessimism!).

Action point: This is a real-life question with real-world data! These are the kinds of questions for which you should have an intuitive feel.

(5) The size S of an online social network satisfies the differential equation $S'(t) = kS(t)(1-(S(t))/(W(t)))$ where $W(t)$ is the world population at time t. Suppose $W(t)$ itself satisfies the differential equation $W'(t) = k_0 W(t)$ where k_0 is positive but much smaller than k. How would we expect S to behave, assuming that initially, $S(t)$ is positive but much smaller than $W(t)$?

(A) It initially appears like an exponential function with exponential growth rate k, but over time, it slows down to (roughly) an exponential function with exponential growth rate k_0.
(B) It initially appears like an exponential function with exponential growth rate k_0, but over time, it speeds up to (roughly) an exponential function with exponential growth rate k.
(C) It behaves roughly like an exponential function with growth rate k_0 for all time.
(D) It behaves roughly like an exponential function with growth rate k for all time.
(E) It initially behaves like an exponential function with exponential growth rate k but then it starts declining.

Answer: Option (A)

Explanation: Here is a conceptual explanation. Initially, the growth of the social network is not directly or visibly constrained by the size of the world population. The factor $1-(S(t))/(W(t))$ is very close to 1 because $S(t)$ is much smaller than $W(t)$. Thus, the differential equation is approximately $S'(t) \approx kS(t)$, which is exponential with exponential growth rate k.

When S starts becoming comparable to W, then $1 - S(t)/W(t)$ becomes notably smaller than 1. The asymptotic steady state would occur when $1 - S(t)/W(t) = k_0/k$, i.e., $S(t) = W(t)(1 - (k_0/k))$. If this state is achieved, then we would get $S'(t) = k_0 S(t)$, and also $W'(t) = k_0 W(t)$. Thus, the size of the social network and the world population are growing at the same exponential rate, which means that the social network is used by a constant fraction of the world's population.

This equilibrium steady state will not in practice be achieved in finite time, but the asymptotic tendency will be to approach this. Note that the smaller k_0 is compared to k, the larger the equilibrium fraction S/W. If $k_0 = 0$ (so world population is static), then $S/W \to 1$.

Real world thinking: For instance, think of Facebook, which opened at Harvard in February 2004. Here, the k of Facebook is much higher than the k_0 for world population, so Facebook's initial growth was viral, reaching about 150,000 in about three months. Then the (exponential) growth rate sort of slowed down, so Facebook reached a million users by about November 2004. If the same (or even a slightly lower) *exponential* growth rate had continued, Facebook would have already saturated the human population well before the end of 2009, and would have had to start looking at non-human "people" to maintain exponential growth.

Things have been (by and large) getting slower when measured in terms of *exponential* growth rates, though faster in terms of *linear* growth rates. In other words, Facebook's number of new users per month is much higher today than it was in 2004, but its *proportion* of new users per month is much lower (less than 5%). Now that Facebook has about 12% of the world's population, it may soon be getting to the stage where the rate of Facebook growth is limited by the rate of population growth.

Performance review: 22 out of 35 got this. 5 chose (B), 4 chose (E), 2 each chose (C) and (D).

Historical note (last year): 7 out of 10 got this. 1 each chose (B), (D), and (E).

Historical note (two years ago): 11 out of 26 people got this correct. 8 people chose (B), 3 people chose (E), 2 people chose (D), 1 person chose (C), and 1 person left the question blank.

(6) Suppose the growth of a population P with time is described by the equation $dP/dt = aP^{1-\beta}$ with $a > 0$ and $0 < \beta < 1$. What can we say about the nature of the population as a function of t, assuming that the population at time 0 is positive?

(A) The population grows as a sub-linear power function of t, i.e., roughly like t^γ where $0 < \gamma < 1$.
(B) The population grows as a linear power function of t, i.e., roughly like t.
(C) The population grows as a superlinear power function of t, i.e., roughly like t^γ where $\gamma > 1$.
(D) The population grows like an exponential function of t, i.e., roughly like e^{kt} for some $k > 0$.
(E) The population grows super-exponentially, i.e., it eventually surpasses any exponential function.

Answer: Option (C)

Explanation: Rearranging, we get:

$$P^{\beta-1} dP = dt$$

Integrating both sides, we get:

$$P^\beta/\beta = t + C$$

Rearranging, we get:

$$P = (\beta(t + C))^{1/\beta}$$

Since $0 < \beta < 1$, $1/\beta > 1$. This form most closely matches (C), with $\gamma = 1/\beta$.

Performance review: 13 out of 35 got this. 16 chose (A), 3 each chose (B) and (D).

Historical note (last year): 9 out of 10 got this. 1 chose (A).

Historical note (two years ago): 8 out of 26 students got this correct. 9 people chose (A), which is probably because they got to $P^\beta/\beta = t + C$ but failed to rearrange to express P in terms of t. 4 people each chose (B) and (D) and 1 person left the question blank.

Action point: Please consider re-attempting this problem prior to reviewing course material for the final.

(7) Suppose the growth of a population P with time is described by the equation $dP/dt = aP^{1+\theta}$ with $0 < \theta$ and $a > 0$. What can we say about the nature of the population as a function of t, assuming that the population at time 0 is positive?

(A) The population approaches infinity in finite time, and the differential equation makes no sense beyond that.

(B) The population increases at a decreasing rate and approaches a horizontal asymptote, i.e., it proceeds to a finite limit as time approaches infinity.
(C) The population grows linearly.
(D) The population grows super-linearly but sub-exponentially.
(E) The population grows exponentially.

Answer: Option (A)

Explanation: Rearranging, we get:

$$\int P^{-\theta-1}\,dP = \int dt$$

Integrating from time 0, we get:

$$\frac{P(0)^{-\theta} - P(t)^{-\theta}}{\theta} = t$$

Thus, we get:

$$P(t)^{-\theta} = P(0)^{-\theta} - t\theta$$

Thus, we get:

$$P(t) = [P(0)^{-\theta} - t\theta]^{-1/\theta}$$

In particular, as $t \to P(0)^{-\theta}/\theta$, $P(t) \to \infty$.

Using the specific value $\theta = 1$ may make the preceding discussion easier to follow.

Performance review: 4 out of 35 got this. 20 chose (B), 6 chose (E), 5 chose (D).

Historical note (last year): 7 out of 10 got this. 2 chose (B), 1 chose (D).

Historical note (two years ago): 3 out of 26 students got this correct. 10 people chose (B) (which would be sort of correct, if it weren't the case that the population had already gone off to infinity), 7 people chose (E), 5 people chose (D), and 1 person chose (C).

Action point: Please consider re-attempting this problem during review for the next midterm and final.

(8) Let $r(t)$ denote the fractional growth rate per annum in per capita income, which we denote by $I(t)$. In other words, $r(t) = I'(t)/I(t)$, measured in units of (per year). It is observed that, over a certain time period, $r'(t) = kr(t)$ for a positive constant k. Assuming that the initial values of $I(t)$ and $r(t)$ are positive, what best describes the nature of the function $I(t)$?
(A) $I(t)$ is a linear function of t, i.e., per capita income is getting incremented by a constant *amount* (rather than a constant proportion).
(B) $I(t)$ is a super-linear but sub-exponential function of t, i.e., per capita income is rising, but less than exponentially.
(C) $I(t)$ is an exponential function of t, i.e., per capita income is rising by a constant proportion per year.
(D) $I(t)$ is a super-exponential function of t but slower than a doubly exponential function of t.
(E) $I(t)$ is a doubly exponential function of t.

Answer: Option (E)

Explanation: Intuitively, the exponential rate of growth is itself growing exponentially, so the overall growth is doubly exponential.

Formally, $r(t) = r(0)e^{kt}$. Then, we have:

$$\frac{dI}{Idt} = r(t)$$

Thus, we get:

$$\frac{dI}{Idt} = r(0)e^{kt}$$

Rearranging, we get:

$$\frac{dI}{I} = r(0)e^{kt}\,dt$$

Integrating from 0 to t, we get:

$$\ln(I(t)/I(0)) = \frac{r(0)(e^{kt}-1)}{k}$$

Exponentiating, we get:

$$I(t) = I(0)\exp[\frac{r(0)(e^{kt}-1)}{k}]$$

This is doubly exponential in t.

Real world thinking: This is a very important question since the answer to it reflects the way you think about growth. When you see a *constant* growth rate for income measured in percentage terms, that means that income is growing exponentially. When the growth rate *itself* is growing exponentially, then income is growing doubly exponentially.

Does doubly exponential growth occur in the real world? Possibly, but one reason why it goes unnoticed is that the exponential rate of growth of the exponential rate of growth is too slow and is hidden by seasonal fluctuations. Taking the long view, we see that rates of growth *have* increased. Annual per capita income growth before 1800 was less than 1% almost everywhere in the world, now it is 2% or higher in developed countries and often more than 5% in developing countries. Similarly, the "doubling time" of a number of technologies (Moore's law-related) has been falling, albeit very slowly. It may not be the case, however, that this double exponentiality will be a continuing feature.

Performance review: 14 out of 35 got this. 8 each chose (C) and (D), 4 chose (B), 1 chose (A).

Historical note (last year): 3 out of 10 got this. 5 chose (B), 2 chose (D).

Historical note (two years ago): 2 people got this correct, 5 people chose option (D), 10 people chose option (C) (pessimists!), 6 people chose option (B) (strong pessimists)!, 2 people chose option (A) (super-strong pessimists!), and 1 person left the question blank.

Action point: Get some real world intuition! Please re-attempt this problem some time and review the solution.

(9) Suppose a function P of time t has the property that $P(0) > 1$, and $dP/dt = P \ln P$ for all $t \geq 0$. Which of the following best describes P as a function of t, for $t \geq 0$?
(A) P grows logarithmically in t.
(B) P grows linearly in t.
(C) P grows super-linearly but sub-exponentially in t.
(D) P grows exponentially in t.
(E) P grows super-exponentially in t.

Answer: Option (E)

Explanation: We have:

$$\int \frac{dP}{P \ln P} = \int dt$$

Integrating both sides, and noting that $P(0) > 1$ so $\ln P > 0$, we get:

$$\ln(\ln P) = t + C$$

Exponentiating twice and setting $k = e^C$, we get:

$$P = \exp(ke^t)$$

This is doubly exponential, and hence, super-exponential.

Performance review: 20 out of 35 got this. 12 chose (C), 1 each chose (A), (B), and (D).

Historical note (last year): 6 out of 11 got this correct. 3 chose (D), 2 chose (C).

(10) An irreversible chemical reaction with reactants A and B and product C begins at time $t = 0$ with the quantity of C being 0 and with finite positive masses of A and B. The rate of reaction at any time t is proportional to the product of quantities (masses) of A and B at that time. By the law of conservation of mass, the total mass of the system is constant. What can we say about the **quantity (mass)** of C as a **function of time** t?

(A) It is increasing, concave up, and has a vertical asymptote.
(B) It is increasing and concave up till a finite time after which it becomes constant.
(C) It is increasing, concave down, and has a horizontal asymptote.
(D) It is concave down, initially increasing and later decreasing after reaching a local maximum.
(E) It is increasing, initially concave up and later concave down.

Answer: Option (C).

Explanation: While this answer can be obtained by solving the differential equation, it can also be obtained directly by general reasoning. Since the reaction proceeds in the forward direction, the quantity of C is increasing. Since the quantities of A and B decrease with time, and the rate of reaction is proportional to the product of these, the rate of reaction decreases – hence concave down. Finally, the quantity is bounded by the law of conservation of mass. Since the function is increasing, it must therefore have a horizontal asymptote.

Performance review: 25 out of 35 got this. 3 each chose (B), (D), and (E). 1 chose (A).

Historical note (last year): 6 out of 11 got this. 3 chose (B), 2 chose (E).

(11) Consider the differential equation $(y' - 2x)(y' - 3x^2) = 0$ with independent variable x and dependent variable y. We are interested in the *global* solutions to this differential equation, i.e., the solutions to this differential equation for y as a continuously differentiable function of x defined on all $\mathbb{R}$ and satisfying this condition globally. What can we say about such solutions?

(A) Any function that is a global solution to this differential equation is infinitely differentiable on all of $\mathbb{R}$.
(B) Any function that is a global solution to this differential equation is twice differentiable on all of $\mathbb{R}$, but there exist solutions that are not thrice differentiable *anywhere* on $\mathbb{R}$.
(C) Any function that is a global solution to this differential equation is twice differentiable on all of $\mathbb{R}$, but there exist solutions that are not thrice differentiable at some isolated points in $\mathbb{R}$ (but are thrice differentiable elsewhere).
(D) Any function that is a global solution to this differential equation is once differentiable on all of $\mathbb{R}$, but there exist solutions that are not twice differentiable *anywhere* on $\mathbb{R}$.
(E) Any function that is a global solution to this differential equation is once differentiable on all of $\mathbb{R}$, but there exist solutions that are not twice differentiable at some isolated points in $\mathbb{R}$ (but are twice differentiable elsewhere).

Answer: Option (E)

Explanation: Just by the nature of the equation, any global solution must be differentiable everywhere. There do, however exist some "mixed" solutions with piecewise definitions, such as:

$$y = \begin{cases} x^2 + C, & x \le 0 \\ x^3 + C, & x > 0 \end{cases}$$

This "mixed" solution function is infinitely differentiable everywhere except at the point 0, where its second derivative does not exist (the second derivative from the left is 2 and the second derivative from the right is 0).

For more on how to solve this differential equation, see the "Factorization method for solving differential equations: mixed solutions" video here: `http://www.youtube.com/watch?v=468iN7oIF58`.

Performance review: 9 out of 35 got this. 9 chose (B), 8 chose (A), 5 chose (D), 4 chose (C).