REVIEW SHEET FOR FINAL

MATH 153, SECTION 55 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet AND the previous review sheet to the review session.

This review sheet does *not* repeat review of material in the midterm 1 or midterm 2 syllabus, although there is some coverage of ideas introduced earlier that have been expanded upon more recently. So, please go through the midterm 1 and midterm 2 review sheets as well.

New feature: This review sheet has "Cautionary notes" and "Error-spotting exercises." The error-spotting exercises include lists of erroneous statements that may be found in student answers. Your task is to identify all errors in these statements.

1. Series and convergence

Words ...

(1) *Not for discussion*: The $\sum$ summation notation is used to compactly express a sum of many (finitely or infinitely many) terms. The terms of the summation are called *summands* and the variable that changes value across summands is termed the *variable of summation* or *index of summation*, and is a *dummy variable*. Some variants include: (i) writing the start of summation at the bottom and the end of summation at the top, (ii) writing the set constraint at the bottom, (iii) doing (i) or (ii) but omitting the index of summation.

(2) The infinite sum $\sum_{k=1}^{\infty} f(k)$ is defined as the limit $\lim_{n\to\infty} \sum_{k=1}^{n} f(k)$. The sums $\sum_{k=1}^{n} f(k)$ are termed the *partial sums*. We use the term *series* for a sequence to be summed up. The sum of a series is the limit of the sequence of partial sums. The summands are called the *terms* of the series.

(3) For a series of nonnegative terms, the sum is independent of the ordering of the terms. It can also be determined by grouping together the terms in any manner whatsoever. Thus, sums of nonnegative terms are commutative and associative in a strong sense.

(4) Summation is linear: it is additive and scalar multiples can be pulled out. In other words, $\sum(f(k)+g(k)) = \sum f(k) + \sum g(k)$. On the other hand, summation is *not* multiplicative. In other words, $\sum f(k)g(k)$ is not the same thing as $(\sum f(k))(\sum g(k))$.

(5) If $g = \Delta f$ where Δ is the forward difference operator, then $\sum_{k=a}^{b} g(k) = f(b+1) - f(a)$. This has a more general version called telescoping. Telescoping can be thought of as the discrete analogue of the fundamental theorem of calculus.

(6) There are four kinds of things we can do concretely for a series of nonnegative terms: (i) show that the series diverges, (ii) show that the series converges, and find its sum, (iii) show that the series converges, and find bounds on its sum, without finding an explicit summation-free expression for the sum, (iv) show that the series converges, without any explicit bounds on its sum.

(7) A series of nonnegative terms converges to the least upper bound of its sequence of partial sums. Note that the sequence of partial sums is a non-decreasing sequence precisely because the terms of the series (i.e., the summands) are nonnegative.

(8) If a series of (possibly mixed sign) terms converges, the magnitudes of the terms must go to 0. The contrapositive is that if the terms of a series do not go to 0, the series does not converge. This establishes a *necessary but not sufficient condition* for the convergence of a series. *Note that we are talking about the terms here, not the partial sums. Also, the terms going to zero does not directly say anything about the value of the sum of the series.*

(9) Left shifts and/or changing finitely many terms does not change the convergence of a series though it may change the value of the sum of the series. This result holds for a series with mixed sign terms.

(10) A geometric series is a series where the quotient of successive terms is constant. The constant (successor term over current term) is termed the *common ratio*. A geometric series of possibly mixed sign terms converges if and only if the common ratio has absolute value strictly less than 1. If the first term is a and the common ratio is r, the geometric series converges to $a/(1-r)$.
(11) The sum of a *finite* segment of a geometric series with n terms, first term a, and common ratio r, is $a(1-r^n)/(1-r)$ if $r \neq 1$, and na if $r = 1$.
(12) The integral test gives both a computational estimate for the sum of a series and a conditional test for whether the series converges. In particular, it states that for a (eventually) nonnegative, (eventually) continuous, (eventually) decreasing function, the integral is finite if and only if the sum is. (See the class notes for the details on the computational estimate; the book concentrates on the conditionality aspect).
(13) The basic comparison test states that if $0 \leq a_k \leq b_k$ for all sufficiently large k, and $\sum b_k$ converges, so does $\sum a_k$. Similarly, if $\sum a_k$ diverges, so does $\sum b_k$.
(14) The limit comparison test states that if $\lim_{k\to\infty} a_k/b_k$ is finite and positive, then both series have the same convergence/divergence behavior.
(15) For $p > 0$, consider the p-series $\sum_{k=1}^{\infty} k^{-p}$. This series diverges for $p \leq 1$ and converges for $p > 1$. We define the zeta function $\zeta(p)$ as the sum of the series. We note that $\max\{1, 1/(p-1)\} \leq \zeta(p) \leq p/(p-1)$ for all p, by the integral test. [Review how this is derived]. ζ is a continuous decreasing function on $(1,\infty)$ with $\lim_{p\to 1^+} \zeta(p) = \infty$ and $\lim_{p\to\infty} \zeta(p) = 1$. Also, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. [Note: We have *not* seen how to derive these values.]

Actions ...

(1) Telescoping is a powerful tool. It allows us to use partial fractions to sum up various kinds of sums with quadratic denominatorics. In particular, if $g(k) = f(k) - f(k+m)$, with $f(k) \to 0$ as $n \to \infty$, them $\sum g(k) = f(1) + f(2) + \cdots + f(m)$.
(2) We can also use the known formulas for summing up $\sum 1$, $\sum k$, $\sum k^2$ and $\sum k^3$ along with linearity to calculate summations where the summands are polynomials of degree at most 3 in k.
(3) We can sum up an *eventually* geometric series by summing up the eventual geometric part of the series and separately handling the first few anomalous terms. *It often happens in real world series that the series is eventually geometric but the first few terms are anomalous. This is due to boundary effects/startup issues.* [For instance, the bouncing ball distance traveled problem.]
(4) For a series with alternating common ratios, we can sum up by splitting into two geometric series. (see example from class notes).
(5) The geometric series can be interpreted as an expansion for $1/(1-x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots, |x| < 1$$

As we see later, this is the Taylor series for the function $1/(1-x)$.
(6) As a corollary of the convergence results on p-series and the basic comparison test, we have the following rule: for a rational function f, the series $\sum f(k)$ converges if the degree of the denominator exceeds the degree of the numerator by at least 2, and diverges otherwise.
(7) More generally, for series involving polynomial-like things, if the degree of the denominator (possibly fractional) exceeds the degree of the numerator by something strictly greater than 1, the series converges, otherwise it diverges.
(8) Things get a little trickier when $\ln k$ appears in the terms. ln can be thought of as being a polynomial in k of degree 0^+ – slightly greater than 0 but less than any positive number. If, in the given setup, the degree of the denominator minus the degree of the numerator is strictly less than 1, 1^- or 1, the series diverges. If it is strictly greater than 1, the series converges. If it is 1^+, then the situation is indeterminate, and we may need to use the integral test. For instance, $\sum(\ln k)/k$ clearly diverges, because the difference is 1^-. On the other hand $\sum 1/(k\ln(k+1))$ and $1/(k[(\ln k)^2+1])$ are ambigious, because we get 1^+ in both cases. In fact, the series diverges in the former case and converges in the latter case, as we can see using the integral test.
(9) sin and cos are bounded between -1 and 1, and this, along with the basic comparison test, can often be used to ascertain behavior about these functions. For instance, consider $\sum(2+\sin k)/k^2$.

Cautionary notes ...

(1) *Where you start – it matters*: In some cases, we index the terms of a series with labels starting from 0 rather than 1. This is most customarily done when dealing with power series, but is also something done in other contexts. *Please be careful when writing expressions for the general term whether the term labels begin at* 0 *or* 1.
(2) *Start late to determine convergence*: If you are asked to judge whether a series converges, but are not told a starting point, then always start with a large enough starting point so that the terms beyond that are well defined. For instance, if dealing with $1/(k \ln(k-1))$, you need to start from $k = 3$. Note that even if you start later than the absolute earliest possible starting point, this will not affect the conclusion on convergence.
(3) *Use the correct letter*: When writing the general expression for the term of a series (or sequence) please *use the same letter* as the subscript for the general term. If the general term is written as a_k, then you should write it as a function of k, not of some other variable.
(4) *Dummy variable cannot appear outside*: The sum of a series whose terms are indexed by k *cannot* have a k in it, because k, the *index of summation*, is a *dummy variable*.
(5) *Keep distinction between terms and partial sums clear*: The sum of a series is the limit of the partial sums. When talking of the *terms* of a series $\sum a_k$, simply write a_k, do *not* write $\sum a_k$. The partial sum $s_n = \sum_{k=1}^n a_k$ is not the same thing as a term of the series.

Error-spotting exercises (be more nitpicky than ever before – find all the errors)...

(1) *Horribly wrong*: The sum $\sum_{k=0}^\infty 1/k^2$ converges. One way of seeing this is that when $k = \infty$, $1/k^2 = 1/\infty^2 = 0$. Hence, we know that the terms approach 0. We know that for a series to converge, the terms must go to zero. Hence, the terms go to zero. Hence, the series converges.
(2) *Somewhat wrong*: The sum $\sum_{k=1}^\infty 1/x^2$ converges. One way of seeing this is to use the integral test. We know that $1/x^2$ is a nonnegative continuous decreasing function of x. So we can apply the integral test to it, and we get:

$$\sum_{k=1}^{\infty} 1/x^2 = \int_1^\infty dx/x^2 = -1/\infty - (-1/1) = 1$$

So the sum is 1, which is a finite number.
(3) *Horribly wrong*: The sum $\sum_{k=0}^\infty 1/(k^3 + k^2)$ converges, because as we all know:

$$\sum_{k=0}^{\infty} \frac{1}{k^3 + k^2} = \sum_{k=0}^{\infty} \frac{1}{k^3} + \sum_{k=0}^{\infty} \frac{1}{k^2}$$

The sum on the right side converges.

2. Root and ratio tests

Words ...

(1) *Not for discussion*: A geometric series with common ratio less than 1 is a discrete analogue of exponential decay. A geometric series with common ratio greater than 1 is a discrete analogue of exponential growth. The common ratio of the geometric series is a parameter controlling growth, just as the constant k controls growth in e^{kx}.
(2) If a series of nonnegative terms is eventually bounded from above by a geometric series with common ratio less than 1, then the series converges. This is the idea behind both the root test and the ratio test.
(3) The root test for a nonnegative series $\sum a_k$ looks at the limit $\lim_{k\to\infty} a_k^{1/k}$. If this limit is less than 1, the root test tells us that the series converges. If the limit is greater than 1, the series diverges. If the limit equals 1, the root test is indecisive (i.e., the series may converge or it may diverge).
(4) The ratio test for a nonnegative series $\sum a_k$ looks at the limit $\lim_{k\to\infty} a_{k+1}/a_k$. If this limit is less than 1, the series converges. If the limit is greater than 1, the series diverges. If the limit equals 1, the ratio test is indecisive (i.e., the series may converge or it may diverge).

(5) Here is a slight modification of the root test: if $a_k^{1/k}$ is greater than 1 for infinitely many k, the series diverges. This is for the simple reason that the terms cannot go to 0. On the other hand, if the sequence $a_k^{1/k}$ is *eventually bounded away from and below* 1 (i.e., bounded from above by a number strictly less than 1) then the series converges. The inconclusive case is thus where the sequence does eventually get below 1 but cannot be bounded away from 1 (i.e., it has terms arbitrarily close to 1).
(6) Here is a slight modification of the ratio test: if a_{k+1}/a_k approaches 1 from the right, or, more generally, if it is ≥ 1 for all sufficiently large k, the series diverges. This is because the terms do not go to 0. On the other hand, if a_{k+1}/a_k is bounded away from and bleow 1, the series converges. The inconclusive case is where the series comes really close to or overshoots 1 infinitely often.
(7) The root test is stronger than the ratio test. The reason is that the ratio test is highly sensitive to the precise orderings of the terms, while the root test can handle small permutations. [An example of this is in one of the advanced homework problems. Please look it up to refresh your memory.]

Actions ...

(1) The root test is more useful for power functions.
(2) The ratio test is more useful for factorials.
(3) For rational functions and for p-series, both tests are indecisive (the limit becomes 1), and we fall back on the rule covered earlier about the difference of degrees of numerator and denominator.
(4) In some cases, it is somewhat more convenient to massage the series a little before applying the root and ratio tests. As long as this massaging does not change the property of whether or not the series converges, that is perfectly fine.

Error-spotting exercises ...

(1) We can use the root test to show that $\sum_{k=1}^{\infty} 1/k^2$ is convergent as follows. The k^{th} root of the k^{th} term is $(1/k)^{2/k}$. As $k \to \infty$, $1/k \to 0^+$, so $(1/k)^{2/k} \to 0$. Hence, the root test applies and the series converges.
(2) By the ratio test, we can show that $\sum_{k=1}^{\infty} 1/k^2$ converges as follows. The ratio of the $(k+1)^{th}$ term to the k^{th} term is $k^2/(k+1)^2$. This is clearly less than 1. By the ratio test, since the ratio is less than 1, the series converges.

3. Absolute and conditional convergence

Words ...

(1) *Not for discussion*: When discussing the convergence of a series, we can throw out all the terms that are zero.
(2) *Not for discussion*: A series $\sum a_k$ is termed *absolutely convergent* if the series $\sum |a_k|$ converges. Note that for a series of nonnegative terms, being absolutely convergent is equivalent to being convergent.
(3) Suppose a series $\sum a_k$ is absolutely convergent. Then, the positive terms converge (say, to P) and the negative terms converge (say, to N). Moreover, $\sum a_k$ is the sum $P + N$, and $\sum |a_k| = |P| + |N|$.
(4) If a series is absolutely convergent, then it is convergent and *every rearrangement* of the series converges to the same sum.
(5) If a series is not absolutely convergent but is convergent, it is termed *conditionally convergent*, and the positive terms add up to $+\infty$, the negative terms add up to $-\infty$, and both the positive terms and the negative terms go to 0.
(6) The Riemann series rearrangement theorem states that a series that is conditionally convergent but not absolutely convergent can be rearranged to give any real number as its sum. It can also be rearranged to give a sum of $+\infty$ and it can also be rearranged to give a sum of $-\infty$. It can also be rearranged so that the sequence of partial sums oscillates between any two fixed locations. [Recall that you heard some non-symbolic, purely didactic reasoning for this in class. Please review this reasoning from the class notes.]
(7) The alternating series theorem states that a series whose terms have alternating signs, and have magnitudes *monotonically* decreasing to zero, must converge. Moreover, the point to which the series converges is the least upper bound of the decreasing sequence of even-numbered partial sums and the greatest lower bound of the increasing sequence of odd-numbered partial sums. [Recall the interpretation of this in terms of jumping along the number line.]

(8) (Questions: What happens if the magnitudes go to zero but not monotonically? What happens if the series is not alternating? What happens if the magnitudes decrease monotonically but not to zero?)

Error-spotting exercises ...

(1) Consider the series $\sum(-1)^k k^3/(k^2+1)$. We know that the terms are alternating in sign. Hence, by the alternating series theorem, the summation converges.

(2) Consider the series $1-1/2+1/3-1/4+1/5-\dots$. We know that the series converges by the alternating series theorem, and by Abel's theorem, it converges to $\ln 2$. Suppose $A = 1+1/2+1/3+1/4+\dots$. Then $A/2 = 1/2+1/4+1/6+\dots$. Thus $A - A/2 = 1+1/3+1/5+1/7+\dots$. So we get:

$$A/2 = 1+1/3+1/5+1/7+\dots$$

We also had:

$$A/2 = 1/2+1/4+1/6+1/8+\dots$$

Subtracting, we get:

$$0 = 1-1/2+1/3-1/4+1/5-1/6+\dots$$

But we already noted that the sum is $\ln 2$. Thus, $0 = \ln 2$.

4. Taylor series

4.1. **Taylor series at 0.** Words ...

(1) Suppose f is a function defined and n times differentiable at 0. Then, the n^{th} Taylor polynomial of f is:

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!}$$

(2) The degree of the n^{th} Taylor polynomial is $\leq n$. Note that it is exactly n if and only if $f^{(n)}(0) \neq 0$.

(3) The number of nonzero terms in the n^{th} Taylor polynomial is at most $n+1$, but it could be substantially less, depending on how many of the $n+1$ numbers $f(0)$, $f'(0)$, $\dots$, $f^{(n)}(0)$ are nonzero.

(4) For $m < n$, the m^{th} Taylor polynomial is the truncation to terms of degree $\leq m$ of the n^{th} Taylor polynomial.

(5) The Taylor series is the infinite sum:

$$\sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

The Taylor polynomials are thus truncations of the Taylor series.

(6) The Taylor series for exp, sin, cos, sinh, and cosh are particularly easy to write down because the sequence of derivatives of these functions is periodic, hence so is the sequence of derivative values at 0. [Review the formulas]

(7) The Taylor series of a polynomial is the same polynomial.

(8) For exp, sin, cos, sinh, cosh, and polynomials, the Taylor series converges to the function everywhere.

(9) The Taylor series for an even function has nonzero coefficients only for even powers of x. In other words, the Taylor series for an even function is an even power series. Similarly, the Taylor series for an odd function has nonzero coefficients only for odd powers of x.

(10) The Taylor series of the derivative is the derivative (via term wise differentiation) of the Taylor series.

(11) The Taylor series operator is linear and multiplicative: the Taylor series for $f+g$ is the sum of the Taylor series for f and g, and the Taylor series for $f \cdot g$ is the product for the Taylor series for f and g. [Note: Multiplying two Taylor series is a pain in general. Howveer, if one of the functions is a polynomial, it is not too hard. For instance, xe^x, $x^2\sin(x^2)$, $(2x+1)\cos x$]

(12) Suppose g is a polynomial with zero constant term. Then, the Taylor series for $f \circ g$ can be obtained by taking the Taylor series for f, replacing x by $g(x)$ throughout, and simplifying. Consider, for instance, the Taylor series for $\sin(x^2)$ and $e^{-x^2/2}$.

(13) The n^{th} (Taylor) *remainder* R_n for a function f is defined as $f - P_n$, where P_n is the n^{th} Taylor polynomial for f. Taylor's theorem states that if f is at least $(n+1)$ times differentiable, the remainder R_n is given by $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$.

(14) The Lagrange formula is a corollary of Taylor's theorem, and it states that there exists c between 0 and x such that $R_n(x) = f^{(n+1)}(c)x^{n+1}/(n+1)!$. Here, c between 0 and x means $c \in [0, x]$ if $x > 0$ and $c \in [x, 0]$ if $x < 0$.

(15) A further corollary of the Lagrange formula (that we call the max-estimate here) states that $|R_n(x)|$ is at most $|x|^{n+1}/(n+1)!$ times the maximum value of $|f^{(n+1)}(t)|$ for t between 0 and x.

(16) The max-estimate can be used to justify that the Taylor series for exp, sin, and cos actually converge to the respective functions. This is done by showing that for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} R_n(x) = 0$.

(17) The zeroth Taylor polynomial for a function f is the constant function $f(0)$. The first Taylor polynomial is the constant/linear function $f(0) + f'(0)x$. This describes the tangent line to the function, and is the *best straight line approximation* to the function locally around 0. More generally, the n^{th} Taylor polynomial is the best approximation to the function around 0 among the polnyomials of degree $\leq n$.

Error-spotting exercises:

(1) The Taylor series for $\sin x$ is just x: The first derivative of sin is cos, the second derivative is $-\sin$. We see that the second derivative of sin is 0 at 0. Differentiating 0 further gives 0, so all higher derivatives are also zero. So, the Taylor series is just P_1, which is just the polynomial x.

(2) The Taylor series for $x^{34/3}$ is just the zero polynomial: We know that all higher derivatives of $x^{34/3}$ are powers of x, but since 34/3 is not an integer, we never get to x^0, and hence all the powers evaluated at 0 give the value 0. So the Taylor series is just 0.

(3) The Taylor polynomial P_2 for $e^x \sin x$ is the product of the Taylor polynomials P_2 for e^x and for $\sin x$.

4.2. **Taylor series in $x - a$.** It is an instructive exercise (and I urge you to do this) to translate all the statements about Taylor series around 0 to the corresponding statements about Taylor series around an arbitrary $a \in \mathbb{R}$. In particular, see if you can correctly translate (pun intended) Taylor's theorem, the Lagrange formula, and its max-estimate corollary. We will go over this further in the review session.

5. Power series

Words ...

(1) The objects of interest here are power series, which are series of the form $\sum_{k=0}^{\infty} a_k x^k$. Note that for power series, we start by default with $k = 0$. If the set of values of the index of summation is not specified, assume that it starts from 0 and goes on to ∞. The exception is when the index of summation occurs in the denominator, or some other such thing that forces us to exclude $k = 0$.

(2) Also note that x^0 is shorthand for 1. When evaluating a power series at 0, we simply get a_0. *We do not actually do* 0^0.

(3) If a power series converges for c, it converges absolutely for all $|x| < |c|$. If a power series diverges for c, it diverges for all $|x| > |c|$.

(4) Given a power series $\sum a_k x^k$, the set of values where it converges is either 0, or $\mathbb{R}$, or an interval that (apart from the issue of inclusion of endpoints) is symmetric about 0. In particular, the interval could be of the following four forms: $(-c, c)$, $[-c, c]$, $[-c, c)$, and $(-c, c]$. The radius of convergence is c. Note that if the set of convergence is $\{0\}$, we say that the radius of convergence is 0, and if the power series converges everywhere, we say that the radius of convergence is ∞.

(5) Suppose a power series $\sum a_k x^k$ converges on an interval $(-c, c)$ to a function f. Then, f is infinitely differentiable on $(-c, c)$ and the power series for f' is obtained by differentiating the power series for f. In fact, the radius of convergence of the power series for f' is precisely the same as the radius of convergence of the power series for f. *On the other hand, the interval of convergence may differ*

– the power series for f may converge at boundary points where the power series for f' does not. An example is arctan, which has interval of convergence $[-1, 1]$, but whose derivative has interval of convergence $(-1, 1)$.

(6) Suppose a power series $\sum a_k x^k$ converges on an interval $(-c, c)$ to a function f. Then, term wise integration of this power series gives an antiderivative of f on $(-c, c)$. In particular, if we choose the power series with constant term 0, we get the unique antiderivative that takes the value 0 at 0.

(7) Abel's theorem states that if $\sum a_k x^k = f(x)$ on $(-c, c)$, f is left continuous at c, and $\sum a_k c^k$ exists, then $f(c) = \sum a_k c^k$. Similarly, if f is right continuous at $-c$, and $\sum a_k(-c)^k$ exists, then $f(-c) = \sum a_k(-c)^k$.

(8) We can also consider power series centered at a: $\sum a_k(x-a)^k$. Everything translates nicely.

Deeper elaboration ...

(1) We have two kinds of operators: one from functions to power series (which involves taking the Taylor series) and the other from power series to functions (which involves summing up). It turns out that, if we start with a power series with a nonzero (possibly infinite) radius of convergence, look at the function it converges to, and take the Taylor series of that function, we retrieve the original power series. *This follows from the differentiation theorem stated above, which states that the derivative of a power series converges to the derivative of the function that the power series converges to.*

(2) On the other hand, it is possible to start with a function f infinitely differentiable on $\mathbb{R}$, take the Taylor series, and have the Taylor series converge to some function other than f. An example is the function that is e^{-1/x^2} for all $x \neq 0$ and 0 at $x = 0$. The function is infinitely differentiable everywhere and all its derivatives at 0 take the value 0. Thus, its Taylor series is 0, which obviously converges to the zero function rather than the specified function.

(3) It is also possible to have a function f that is infinitely differentiable on all of $\mathbb{R}$ such that the Taylor series of f converges to f, but the radius of convergence of the Taylor series is finite. More generally, it is possible that the interval of convergence of the Taylor series is smaller than the domain of the function. Two important examples in this direction are the arctan function (infinitely differentiable on all of $\mathbb{R}$ but interval of convergence $[-1, 1]$) and the function $\ln(1 + x)$ (infinitely differentiable on $(-1, \infty)$ but interval of convergence $(-1, 1]$).

(4) Call a function *globally analytic* if it is defined on all of $\mathbb{R}$ and has a power series about 0 that converges to the function everywhere. Sine, cosine, the exponential function, and polynomial functions are all globally analytic. Moreover, globally analytic functions are closed under addition, subtraction, multiplication, and composition.

(5) Call a function C^∞ on $\mathbb{R}$ if it is defined and infinitely differentiable on all of $\mathbb{R}$. The space of C^∞ functions is closed under addition, subtraction, multiplication, and composition. Moreover, any globally analytic function is C^∞. The converse is not true.

(6) A function is termed *analytic about* 0 if its Taylor series converges to it on an interval of nonzero radius. Any function that is analytic about 0 is infinitely differentiable (C^∞) around 0. However, the function may well be C^∞ on a bigger interval than the interval on which the Taylor series converges.

(7) If f and g have Taylor series that both converge on $(-c, c)$ to the respective functions, the Taylor series for $f + g$ also converges on $(-c, c)$ to it.

Actions ...

(1) We can consider functions in the following decreasing order of the behavior as $x \to \infty$: double exponential (like e^{e^x}, e^{2^x}), exponential in higher powers of x (e^{x^λ}, $\lambda > 1$), factorial ($x!$, $\Gamma(x)$, x^x), exponential (a^x, $a > 1$), exponential in lower powers of x (e^{x^λ}, $0 < \lambda < 1$), exponential in higher powers of $\ln x$ ($e^{(\ln x)^\lambda}$, $\lambda > 1$), polynomial or power functions of x (x^λ, which we can again split into cases based on whether $\lambda > 1$, $\lambda = 1$, or $0 < \lambda < 1$), polynomials in $\ln x$, $\ln(\ln x)$, and so on down.

(2) There is often quite a bit of separation within each level of the hierarchy (allowing for further stratification). Anything at a higher level in the hierarchy beats anything at a lower level in the hierarchy, so that the quotients tend to ∞ or 0 depending on which one is higher.

(3) The decay rates of the reciprocal functions mirror the growth rates of the functions.

(4) We use the term *superexponential* for functions that grow faster than exponential functions (for instance, e^{e^x}, e^{x^2}, and $x!$), *exponential* for functions that grow exponentially, and *subexponential* for functions that grow smaller than exponential.
(5) In general, when adding, subtracting, and multiplying, the larger one dominates. Division by a superexponential function leads to superexponential decay.
(6) Consider a power series $\sum a_k x^k$. If the a_k grow superexponentially in k, then the series converges only at 0. If the a_k decay superexponentially in k (i.e., $1/a_k$ grow superexponentially in k), then the series converges everywhere. [Justify to yourself using the ratio and/or root test]
(7) For $\sum a_k x^k$, if the a_k grow or decay exponentially, then the radius of convergence is finite and nonzero, and equals $\lim_{k\to\infty} 1/|a_k|^{1/k}$ – in other words, the reciprocal of the limiting common ratio of the a_ks. This is because at exponential growth, the a_ks match the x^ks and can affect the radius of convergence.
(8) For $\sum a_k x^k$, if the a_k grow or decay subexponentially, they have no effect on the radius of convergence – it is still 1. More generally, if a_k is the product of an exponential and a subexponential function, only the exponential function affects the radius of convergence. *The subexponential component does affect whether the endpoints are included in the interval of convergence.*
(9) As regards endpoints, the following is a rough statement. Consider $\sum a_k x^k$. If the a_ks are growing or constant, the series diverges at ± 1, so the interval of convergence is $(-1, 1)$. If the a_ks are decaying at a rate that is linear or slower, then the series does not absolutely converge, but it may conditionally converge at one or both ends due to the alternating series theorem. If the a_ks are decaying at a rate that is $k^{-\lambda}$, $\lambda > 1$, then the series converges at both $+1$ and -1. Note that cases like $1/[k(1+(\ln k)^2)]$ are ambiguous, as discussed earlier.
(10) In particular, if $a_k = p(k)/q(k)$ where p and q are polynomials, the following can be said: if the degree of q is at least 2 greater than the degree of p, the interval of convergence is $[-1, 1]$. If the degree of q is equal to or less than the degree of p, the interval of convergence is $(-1, 1)$. If the degree of q is exactly one more than the degree of p, the interval of convergence is $[-1, 1)$. Note that the endpoint included may change under slight modifications of the situation, so you should also be aware of the reasoning process that leads to this conclusion. For instance, if there are only odd degree terms and nonnegative coefficients, we do not get any alternating series and the interval of convergence is $(-1, 1)$. On the other hand, if there are odd degree terms and alternating signs of coefficients among the odd degree terms, then the alternating series theorem applies at *both* ends -1 and 1.

Error-spotting exercises ...

- Consider the power series $\sum_{k=0}^{\infty} x^k/2^{k^2}$. The radius of convergence for this is $\lim_{k\to\infty} 1/(1/(2^{k^2}))^{1/k}$, which is 2^k. So, the power series has radius of convergence 2^k.
- Consider the power series $\sum_{k=0}^{\infty} 2^{2^k} x^k$. The radius of convergence is $\lim_{k\to\infty} 1/(2^{2^k})^{1/k} = 1/2^2 = 1/4$.
- arctan is a function defined and infinitely differentiable on all of $\mathbb{R}$. So, the Taylor series of arctan must have radius of convergence equal to ∞.

6. Summation techniques

(1) For finite sums involving polynomials of small degree, use linearity and the formulas for summations of 1, k, k^2, k^3.
(2) For reciprocals of quadratic functions, use partial fractions and then look for telescoping when the quadratic can be factorized. If the quadratic cannot be factorized but is a perfect square, try to use $\zeta(2) = \pi^2/6$. If the quadratic has negative discriminant, there is no closed form expression.
(3) In general, look for telescoping wherever you go. This includes rational functions, logarithms (e.g., $\ln((k+1)/k)$).
(4) Sometimes, for higher degree rational functions, you can combine telescoping with known information about zeta functions.
(5) See if the summation is a geometric series in disguise, or combines two or more geometric series and some possibly anomalous terms.

(6) Sometimes, the summation is related to a geometric series via integration or differentiation. For instance $\sum kx^k$ is related to $\sum x^k$ via differentiation. Use the differentiation and integration theorems to use these to get closed forms.
(7) In some cases, the summation is a known series such as that for the exponential, sine, cosine, arc tangent or logarithm, with some modifications: it might involve a sum or difference of two such series, it might be arrived at by composing such a series with mx^n, it might be arrived by multiplying such a series with mx^n, it might be arrived at by integrating or differentiating such a series.
(8) To identify these possibilities better, here are some heuristics: factorials in denominator suggests exponentials or sine/cosine, and the nature of sign alternation helps decide which. Ordinary k in the denominator suggests logarithm or arc tangent, and the nature of sign alternation helps decide which. [Exponential and logarithm have a sign periodicity of at most 2, while sine, cosine and arc tangent have a sign periodicity of 4].

7. Approximations all in one place

To make life easier for you, we list here the various approximation techniques used.

(1) *Approximating a sum by an integral*: This is done using a numerical version of the integral test. This allows us to approximate the values of the zeta function. Please see the notes for how this is done. Note that this gives both an upper bound and a lower bound.
(2) *Approximating a function by Taylor polynomials*: For a function that is globally analytic or analytic about a point, we can approximate its value by Taylor polynomials. The higher the degree we allow for the Taylor polynomial, the better the approximation in general. The magnitude of possible error is determined using the max-estimate version of the Lagrange formula. *It is important to note that the error estimate could vary quite a bit from function to function. Some power series converge more slowly than others*. A good rule of thumb is that the more quickly the terms go to zero, the fewer the number of terms we need to take to get a good approximation.

    In some cases, the Taylor approximation applies nicely only in a small range. For instance, the Taylor series for sin converges to sin globally, but the convergence is quick only for small values of x. The Taylor series for arctan converges to it only on $[-1, 1]$. However, we can use various identities such as those relating $\sin x$ and $\cos(\pi/2 - x)$ and those relating $\arctan x$ and $\arctan(1/x)$ to reduce to the case of a rapidly converging Taylor series.
(3) *Approximating an integral computation using Taylor polynomials*: Even computing the value of plain vanilla functions like sin at specific points requires the use of Taylor series. Miraculously, we can do with Taylor series what we cannot always do with functions – integrate term wise. This allows us to calculate definite integrals of globally and locally analytic functions by first integrating the Taylor series term wise and then using a Taylor polynomial approximation. Examples of functions that cannot be integrated in the language of elementary functions, but whose integrals can be computed to a fair degree of accuracy using this approach, are $(\sin x)/x$, $(e^x - 1)/x$, e^{-x^2}, $\sin(x^2)$, and similar functions.

    Note that, at least for cases where the power series is easy to write down, this approach is a lot less cumbersome than the approach of using upper and lower sums.
(4) *Trying to find where a function is zero*, particularly when there is no algebraic method to solve this: We use techniques like the intermediate value theorem and the mean value theorem to show the existence of zeros in certain intervals. We also use derivative behavior and other techniques to further narrow down the intervals under consideration.

8. Limit computations and order of zero

(1) Suppose f is a function that has a zero at a. The order r of the zero at a is the least upper bound of the set of values β such that $\lim_{x \to a} |f(x)|/|x - a|^\beta = 0$. For $\beta < r$, the limit is 0. For $\beta > r$, the limit is undefined (∞-types).
(2) To simplify notation, we concentrate on the case where $a = 0$ (the location of a does not matter, because we can always translate).

(3) If f is infinitely differentiable at 0 and takes the value 0 at 0, the order of f at 0 is the smallest n such that $f^{(n)}(0) \neq 0$. Note that it is possible, but rare, for a function to have a zero of order ∞ at 0 (an example is the e^{-1/x^2} function). We ignore such examples.
(4) In particular, for an infinitely differentiable function that is 0 at 0, the order of the zero (if finite) if always a positive integer. Moreover, if the order is r, them $\lim_{x \to 0} f(x)/x^r = f^{(r)}(0)/r!$, which is the corresponding Taylor coefficient.
(5) This is particularly intuitive for analytic functions, because if we replace a function by its Taylor series we readily see that the order of its zero is the lowest order with nonzero coefficient. We also see that the limit of the quotient by x^r is that coefficient.
(6) We can have a function with a zero at a point that has fractional order, but the function cannot be infinitely differentiable. For instance, consider $x^{p/q}$, where p and q are positive and q is odd, with q not dividing p. This is differentiable k times where k is the greatest integer less than p/q. On the other hand, the order of the zero is p/q, which is a fraction. What happens is that we cannot differentiate the $(k+1)^{th}$ time at 0, since that exceeds the order.
(7) We can also have a situation of a function f with zero of order r such that $\lim_{x \to \infty} f(x)/x^r$ is zero. Again, however, f cannot be infinitely differentiable. Examples include $x/(\ln x)$, which has a zero order order 1. Intuitively, the order of the zero here is 1^+. (Note that ln in the denominator has a positive effect near zero though it has a negative effect near ∞. Ponder why).
(8) *Important note on order of zero of log*: Note that $\ln x \to -\infty$ as $x \to 0^+$, so log hsa an *anti-zero* at zero of order infinitesimally less than 0, i.e., 0^-. Thus, for $r > 0$, $x^r \ln x$ has a zero at 0 of degree r^-.

Thus, the rule for logarithm near 0 *(where the zero has degree* 0^-*) is somewhat opposite to the rule for logarithm out to* ∞ *(where the growth is* 0^+*).*
(9) We can also have a situation of a function f with zero of order r such that $\lim_{x \to \infty} f(x)/x^r$ is undefined or infinite. Again, however, f cannot be infinitely differentiable. Examples include $x(\ln x)$, which has a zero order order 1 at 0. Intuitively, the order of the zero here is 1^-.