DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 11: GAUSS-JORDAN ELIMINATION (ORIGINALLY DUE WEDNESDAY OCTOBER 9, BUT POSTPONED)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

28 people took this 6-question quiz. The score distribution was as follows:

- Score of 3: 3 people
- Score of 4: 7 people
- Score of 5: 12 people
- Score of 6: 6 people

The mean score was 4.75.

The question-wise answers and performance review are as follows:

(1) Option (B): 28 people (everybody!)
(2) Option (B): 26 people
(3) Option (A): 28 people (everybody!)
(4) Option (B): 15 people
(5) Option (C): 17 people
(6) Option (D): 19 people

Note: This quiz was not administered the last time I taught the course, so there is no previous performance to compare against.

2. Solutions

PLEASE DO *NOT* DISCUSS ANY QUESTIONS

The quiz covers basics related to Gauss-Jordan elimination (notes titled `Gauss-Jordan elimination`, corresponding section in the book Section 1.2). Explicitly, the quiz covers:

- Setting up linear systems and interpreting the coefficient matrix in terms of the setup.
- Knowledge of the permissible rules for manipulating linear systems.
- Metacognition of the process of Gauss-Jordan elimination and its eventual result, the reduced row-echelon form, as well as the interpretation in terms of the solution set.

The questions are fairly easy questions if you understand the material. But it's important that you be able to answer them, otherwise what we study later will not make much sense.

(1) *Do not discuss this!*: The row operations that we can perform on the augmented matrix of a linear system include adding or subtracting another row. However, they do not include multiplying another row. In other words, suppose we start with:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 7 & 6 \end{array}\right]$$

What we're not allowed to do is multiply row 2 by row 1 and obtain:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 14 & 30 \end{array}\right]$$

What's the most compelling reason for our not being allowed to perform this operation?

(A) The row operations arise from the corresponding operations on equations. For the "multiplication of rows" operation to be legitimate, it must correspond to multiplication of the corresponding equations, and multiplying equations is not a legitimate operation.
(B) The row operations arise from the corresponding operations on equations. However, the "multiplication of rows" operation does not correspond to any legitimate operation on equations. Note that it does not correspond to multiplying the equations, because that is not how multiplication of linear polynomials work (in fact, if we multiplied the equations, we would end up with an equation that is not linear).

Answer: Option (B)

Explanation: In point of fact, both (A) and (B) are true in different ways, but (B) is more compelling. Let's first understand why (B) is true. We will then turn to why (A) is (somewhat) true.

Every augmented matrix hides behind it a linear system. Let's call the variables x_1 and x_2. The original linear system is:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 7x_2 &= 6 \end{aligned}$$

Now, if we multiply the two equations, we get:

$$(x_1 + 2x_2)(2x_1 + 7x_2) = 30$$

This simplifies to:

$$2x_1^2 + 11x_1x_2 + 14x_2^2 = 30$$

Note that this is quite different from the linear equation that is represented by the row obtained by multiplying the two rows. That equation is:

$$2x_1 + 14x_2 = 30$$

So (B) is the main reason why the operation makes no sense.

It's worth noting that to a lesser extent, (A) is also an issue. Multiplying equations is legitimate: if two equations hold, their product also holds. However, replacing one of the equations by the product equation could lead to a potential loss of information. This loss of information occurs if the other equation being multiplied has both sides equal to zero (basically, multiplication by zero throws away information). Note that that problem does not occur with this linear system.

Performance review: All 28 got this.

(2) *Do not discuss this!*: Consider a model where the functional form is linear in the parameters (though not necessarily in the inputs). We can use (input, output) pairs to set up a system of linear equations in the parameters. Given enough such equations, we can determine the values of the parameters.

What is the relation between the coefficient matrix and the parameters and (input, output) pairs?
(A) The columns of the coefficient matrix correspond to the (input, output) pairs and the rows correspond to the parameters.
(B) The rows of the coefficient matrix correspond to the (input, output) pairs and the columns correspond to the parameters.

Answer: Option (B)

Explanation: Each (input, output) pair gives an equation. Since the functional form is linear in the parameters, the equation is a linear equation. Each equation corresponds to a row of the augmented matrix (the input part affects the left side of the equation, and hence the coefficient matrix row, while the output part is the constant on the right side of the equation, i.e., the augmenting value).

The parameters are the variables that we are trying to solve for. These thus correspond to the columns.

Performance review: 26 out of 28 got this. 2 chose (A).

(3) *Do not discuss this!*: Consider a model where the functional form is linear in the parameters (though not necessarily in the inputs). We can use (input, output) pairs to set up a system of linear equations in the parameters. Given enough such equations, we can determine the values of the parameters.

What is the relation between the inputs, the outputs, the coefficient matrix, and the augmenting column?

(A) The inputs correspond to the coefficient matrix and the outputs correspond to the augmenting column. In other words, knowing the values of the inputs allows us to write down the coefficient matrix. Knowing the values of the outputs allows us to write down the augmenting column.

(B) The outputs correspond to the coefficient matrix and the inputs correspond to the augment column. In other words, knowing the values of the outputs allows us to write down the coefficient matrix. Knowing the values of the inputs allows us to write down the augmenting column.

Answer: Option (A)

Explanation: Read the explanation for the preceding question.

Performance review: All 28 people got this.

(4) *Do not discuss this!*: Consider the following rule to check for consistency using the augmented matrix: the system is inconsistent if and only if there is a zero row of the coefficient matrix with a nonzero value for that row in the augmenting column. In what sense does this rule work?

(A) The rule can be applied to the augmented matrix directly in both the *if* and the *only if* direction.

(B) The rule can be applied to the augmented matrix only in the *if* direction in general. In the *only if* direction, the rule can be applied to the augmented matrix *after* we have reduced the system to a situation where the coefficient matrix is in row-echelon form (note: it's not necessary to reach reduced row-echelon form).

(C) The rule can be applied to the augmented matrix only in the *only if* direction in general. In the *if* direction, the rule can be applied to the augmented matrix *after* we have reduced the system to a situation where the coefficient matrix is in row-echelon form (note: it's not necessary to reach reduced row-echelon form).

(D) The rule can be applied in either direction only *after* we have reduced the system to a situation where the coefficient matrix is in row-echelon form (note: it's not necessary to reach reduced row-echelon form).

Answer: Option (B)

Explanation: The rule obviously applies in the *if* direction: if there is any row with zeros in the coefficient matrix and a nonzero augmenting value, that equation has no solutions. Therefore, the system as a whole is inconsistent.

It does not work in the *only if* direction in general. This is because the inconsistency may be more subtle: it may arise due to equations being inconsistent *when viewed together*, rather than any individual equation being inconsistent. For instance, consider:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 3 \end{aligned}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 2 & 2 & | & 3 \end{bmatrix}$$

Note that there is no zero row for the coefficient matrix. However, if we subtract twice the first row from the second row, we obtain:

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix}$$

The coefficient matrix is now in rref, and we can now see that the system is inconsistent based on the second row (the coefficient matrix is all zeros, and the augmenting entry is nonzero).

Performance review: 15 out of 28 got this. 6 chose (D), 3 each chose (A) and (C), and 1 left the question blank.

(5) *Do not discuss this!*: Which of the following is *not* a possibility for the number of solutions to a system of simultaneous linear equations? Please see Options (D) and (E) before answering.

(A) 0
(B) 1
(C) 2
(D) All of the above, i.e., none of them is a possibility
(E) None of the above, i.e., they are all possibilities

Answer: Option (C)

Explanation: 0 is a possibility that occurs when the system is inconsistent. 1 is a possibility that occurs when all the variables are leading variables and the system is consistent.

For there to be more than one solution, there must be a non-leading variable. This variable serves as a parameter that can take arbitrary real values. Thus, if there is more than one solution, there must be infinitely many solutions.

Performance review: 17 out of 28 got this. 9 chose (E), 1 each chose (B) and (D).

(6) *Do not discuss this!*: Which of the following describes the situation for a consistent system of simultaneous linear equations?

(A) The leading variables are the parameters used to describe the general solution, and the number of leading variables equals the number of nonzero equations in the reduced row-echelon form (here nonzero equation makes an equation that does not have a zero row in the augmented matrix).
(B) The non-leading variables are the parameters used to describe the general solution, and the number of non-leading variables equals the number of nonzero equations in the reduced row-echelon form (here nonzero equation makes an equation that does not have a zero row in the augmented matrix).
(C) The leading variables are the parameters used to describe the general solution, and the number of leading variables equals the value (number of variables) - (number of nonzero equations in the reduced row-echelon form) (here nonzero equation makes an equation that does not have a zero row in the augmented matrix).
(D) The non-leading variables are the parameters used to describe the general solution, and the number of non-leading variables equals the value (number of variables) - (number of nonzero equations in the reduced row-echelon form) (here nonzero equation makes an equation that does not have a zero row in the augmented matrix).

Answer: Option (D)

Explanation: The non-leading variables serve as parameters. Also, the number of *leading variables* is the number of nonzero rows in the reduced row-echelon form. The number of non-leading variables is thus the total number of variables minus this quantity.

Note that we are given that the system is consistent, so we do not have to worry about the solution set being empty.

Performance review: 19 out of 28 got this. 3 each chose (A), (B) and (C).