# DIFFERENTIAL EQUATIONS: AN INTRODUCTION

MATH 153, SECTION 55 (VIPUL NAIK)

**Corresponding material in the book**: Section 9.2.

**What students should already know**: The prime and Leibniz notation for derivatives, the meaning of differentiation, implicit differentiation, and integration.

**What students should definitely get**: What a differential equation means, what a solution to a differential equation means, how to solve a multiplicatively separable first-order differential equation, how to solve an initial value problem.

**What students should hopefully get**: The notion of parameters as degrees of freedom, the notion of constraints as pinning these down, the basic concerns in differential equation manipulation.

## 1. Understanding differential equations and solutions

1.1. **Differentiation: the two interpretations.** We have dealt with two interpretations of differentiation that it would be useful to recall at this stage. One interpretation is in terms of functions. Here, we think of a function $f$ as a black box that takes as input a variable $x$ and outputs a variable $f(x)$, that we may choose to call $y$. $f'$ is a new function, i.e., a new black box, that takes as input $x$ and gives an output called $f'(x)$, that we may also call $y'$.

In this interpretation, it is the function, rather than the inputs and outputs to it, that takes on primal importance. The disadvantage of this approach is that it does not allow us to go beyond functions.

The second interpretation is to view a function as a *relation* between two *quantities* – the *input quantity* and the *output quantity*. The function describes the nature of the dependence of the output quantity upon the input quantity. Under this approach, we denote the derivative as $dy/dx$, the Leibniz notation.

The Leibniz notation $dy/dx$ arises from the fact that the derivative is the *limit* of the *difference quotient*:

$$\frac{dy}{dx} = \lim \frac{\Delta y}{\Delta x}$$

The focus here is *not* on the function that relates $x$ to $y$, but on the variables $x$ and $y$. The advantage of this approach is that we can apply this approach even when neither of the variables is a function of the other. For instance, we could do something called *implicit differentiation*, which allows us to find $dy/dx$ when $x$ and $y$ are entangled. For instance, given:

$$y^2 + \sin(xy) = x^3 \cos(x + y)$$

We differentiate and get:

$$2y\frac{dy}{dx} + \cos(xy)\left[x\frac{dy}{dx} + y\right] = 3x^2\cos(x + y) - x^3\sin(x + y)\left[1 + \frac{dy}{dx}\right]$$

We can collect terms and obtain an expression for $dy/dx$ in terms of $x$ and $y$.

1.2. **A differential equation.** Consider two variables $x$ (the so-called independent variable) and $y$ (the so-called dependent variable). A *differential equation* is an equation involving the variables $x$, $y$, and first and higher derivatives of $y$ with respect to $x$. For instance, here's a differential equation.

$$x + yy' + xy\sin(y') = 0$$

Here, $y'$ is shorthand for $dy/dx$. Thus, this differential equation can also be written as:

$$x + y\frac{dy}{dx} + xy\sin\left(\frac{dy}{dx}\right) = 0$$

If we want to get $y = f(x)$, the above can be rewritten as:

$$x + f(x)f'(x) + xf(x)\sin(f'(x)) = 0$$

Another way of putting this is that a differential equation is something of the form $F(x, y, y', y'', \dots) = 0$ where $F$ is some expression in many variables.

Before proceeding further, however, we must understand what a differential equation *means*, and how it differs from an ordinary equation.

(1) A *functional solution* or *function solution* is a function $y = f(x)$ such that, taking derivatives the usual way, we find that the differential equation is satisfied for *all* $x$. More specifically, a *function* $y = f(x)$ solves the differential equation if $F(x, f(x), f'(x), f''(x), \dots) = 0$ for all $x$.
(2) A *solution* to a differential equation is a relation $R(x, y)$ between $x$ and $y$ such that if we consider the set $R(x, y) = 0$, and use implicit differentiation to find the higher derivatives, these satisfy the condition $F \equiv 0$. A solution may differ from a functional solution in the sense that $y$ may not be written *explicitly* as a function of $x$, and, in fact, globally, may not give a unique function of $x$.
(3) More pictorially, a solution to a differential equation is a curve in the plane $\mathbb{R}^2$ with the property that the differential equation holds at all points on the curve. What this means is that at any point on the curve, if we calculate the higher derivatives based on their geometric interpretations, we obtain a bunch of stuff that satisfies the differential equation.

What does this mean and how does this differ from an ordinary equation? Two important differences:

(1) Each solution to an ordinary equation (such as a polynomial equation) is a *number*. In contrast, each solution to a differential equation is a *function* or *relation* between two variables.
(2) When we are looking at an ordinary equation, such as $x^2 + x + 2 = 0$, we are looking at points where this equation holds. When we are looking at a differential equation, we are looking at curves such that the equation holds at all points on the curve.
(3) To check that an ordinary equation holds at a point, we evaluate at the point. However, to check whether a differential equation holds, we need to understand behavior locally, on a neighborhood. In other words, *it makes no sense to ask whether a differential equation holds for a given point* $(x_0, y_0)$*; it only makes sense to ask whether it holds on a given curve.*

Basically, a differential equation seeks to *find a function* that exhibits certain *local behavior* (in contrast with *pointwise behavior*) as described by an *expression involving the function and its derivatives*.

**Aside: Differential equations as functional equations.** A *functional equation* is an equation that asks for a function satisfying certain conditions. Specifically, a functional equation is an equation in terms of a function that we require to be true for *every* choice of value for all the letters in the equation, i.e., we require it to be an identity in all letter variables.[1]

For instance, the equation:

$$f(x) = f(-x) \ \forall \ x \in \mathbb{R}$$

has solution set precisely the set of all *even* functions. Similarly, the equation:

$$f(ax) = af(x) \ \forall a, x \in \mathbb{R}$$

has solution set precisely the set of all functions $f$ of the form $f(x) = \lambda x$, where $\lambda$ is a constant.

The desired solutions to functional equations are *functions*, and is does not make sense to ask whether a particular input-output pair satisfies a functional equation.

Differential equations are a *particular kind* of functional equations. Specifically, differential equations are functional equations involving derivative behavior all considered *at a single point.* [2]

[1] In mathematical jargon, the letter variables for numbers are typically quantified over all integers.

[2] There are more complicated functional equations involving derivatives that are *not* differential equations in the sense that we have talked about. Examples include *delay differential equations*, which relate the values of the function and its derivative at far-off points.

1.3. **Some examples of differential equations and solutions.** We begin with a simple differential equation:

$$\frac{dy}{dx} = 1$$

We claim that $y = x + 13$ is a solution to this differential equation. In the various jargon that we have introduced:

(1) The function $f(x) = x + 13$ satisfies the condition that $f'(x) = 1$.
(2) The curve $y = x + 13$ satisfies the condition $y' = 1$.

Both these statements are clearly true. Graphically, the curve $y = x + 13$ is a straight line with slope 1 and intercept 13. Since its slope is 1, $dy/dx = 1$ everywhere on the line.

However, this is not the only solution. Astute observers would have noted that there was nothing particularly auspicious about the number 13. In fact, for any constant $C$, $y = x + C$, or the function $f(x) = x + C$, solves this differential equation.

Thus, any line with slope 1 is a *solution curve* to this differential equation.

Note that every value of $C$ gives one solution, called a *particular solution.* Each such solution corresponds to a line with slope 1. Pictorially, we get a bunch of parallel lines that cover the entire plane.

Are these the only ones? Indeed, which brings us to the next topic.

1.4. **De ja vu.** When we first learn algebra in middle or high school, we are given examples such as:

> How many more apples need to be added to 3 apples to obtain 5 apples? Algebra version: Solve $3 + t = 5$.

At first, these seem like silly examples, because anybody who has grasped the concept of *subtraction* (the inverse operation to addition) can probably solve the problem without any knowledge of algebra. It is only after seeing harder examples of equations that people begin to appreciate the power of algebraic manipulation in solving problems that are too hard to manipulate with basic arithmetic.

In the same way, our first example of a differential equation is in fact an integration problem. Specifically, solving:

$$\frac{dy}{dx} = 1$$

is equivalent to performing the indefinite integration:

$$y = \int 1 \, dx$$

which gives the answer $y = x + C$ where $C$ is an arbitrary constant. Each value of $C$ gives a particular solution.

Let us elaborate the process a little further:

$$\frac{dy}{dx} = 1$$

Moving the $dx$ to the numerator on the other side, we obtain:

$$dy = dx$$

*Note that this is just formal manipulation, akin to the way you learned formal algebra when you got started*. Don't ponder about the intrinsic meaning of $dy$ and $dx$.

Integrating both sides, we get:

$$\int dy = \int dx$$

Which gives:

$$y = x + C$$

(Note that we can actually get positive constants with *both* indefinite integrals, but we can absorb the two constants into one).

This drawn-out process seems pointless for this specific example, *just like* algebra seemed pointless when you were solving $3 + t = 5$. Unlike middle school, however, where we waded through a lot of these silly examples before getting to more substantive examples of the use of algebra, we can now jump straight to harder situations where we see the machinery of differential equations and how it gets used.

**Aside: Yesterday's problem is today's solution.** In the elementary grades, addition, subtraction, multiplication and division were the problems, and numbers were the answers. In the middle grades, simple algebraic equations were the problems, and reducing them to a clearly stated arithmetic computation was the crux of the solution (the rest was trivial). When we learned differentiation, differentiating functions was the problem, but when we studied graphing and integration, differentiation was one of the many tools that was used in coming up with solutions.

This is the fundamental nature of mathematics: *yesterday's focus problems become the taken-for-granted tools of solution to today's focus problems.* Reducing the solution to today's focus problem to solving a bunch of yesterday's focus problems is almost as good at solving yesterday's problem.[3]

Until very recently in this course, integration was the *problem.* Now, integration is part of the *solution.* Once a differential equation is reduced to calculating a bunch of integrals, we're home. *Our goal in finding solution functions to differential equations is to reduce differential equations to (one or more) integration problems.*

## 2. Understanding and solving separable equations

### 2.1. Separable equations: a crash course.

A differential equation of the form:

$$\frac{dy}{dx} = f(x)$$

has as solution:

$$y = \int f(x)\, dx$$

where the right side can be computed by finding one antiderivative and then tacking on a $+C$.

This is a form of differential equation that *directly reduces to a single integration problem* – the calculus analogue to $3 + t = 5$.

Let's look at a slightly more complicated example:

$$\frac{dy}{dx} = f(x)g(y)$$

also written as:

$$y' = f(x)g(y)$$

In words, we say that $y'$ is a *multiplicatively separable function* of $x$ and $y$ – it is the product of a function that depends *only* on $x$ and a function that depends *only* on $y$.

We do some algebra-like manipulations whose aim is to *bring together on one side* all terms involving $y$ and *bring together on the other side* all terms involving $x$:

$$\frac{dy}{g(y)} = f(x)\, dx$$

We now put integral signs and carry out indefinite integration:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

[3] Assuming you remember how to solve yesterday's problems. Mathematical knowledge is cumulative, but an individual's mathematical knowledge is cumulative only if that individual actually accumulates knowledge.

Once we find antiderivatives, we put the $+C$ on just one side (because two additive constants can be absorbed into one).

For instance, consider:

$$\frac{dy}{dx} = (x^2 + 1)(y^2 + 4)$$

We proceed to get:

$$\int \frac{dy}{y^2 + 4} = \int (x^2 + 1)\, dx$$

This becomes:

$$\frac{1}{2} \arctan\left(\frac{y}{2}\right) = \frac{x^3}{3} + x + C$$

What the $+C$ means is that every *particular value* of $C$ gives a *particular solution*. For instance, when $C = 0$, we get the solution:

$$\frac{1}{2} \arctan\left(\frac{y}{2}\right) = \frac{x^3}{3} + x$$

The curve in the plane given by this solution (that we don't need to imagine) satisfies this differential equation.

Note that although the expression is not in the form of $y$ as a function of $x$, we can bring it in that form with some algebraic manipulation, to get:

$$y = 2 \tan\left[\frac{2x^3}{3} + 2x\right]$$

However, this kind of separation and writing things as functions is not always possible (I am glossing over *many* details here).

2.2. **Separable equations from the other side.** Suppose we start with a family of curves of the form:

$$F(x) + G(y) = C$$

where $C$ varies over $\mathbb{R}$. We want to find a differential equation that is satisfied by all curves in the family. We use implicit differentiation to get:

$$F'(x) + G'(y)y' = 0$$

which can also be written as:

$$y' = \frac{-F'(x)}{G'(y)}$$

Which is a (slightly differently written) version of the original thing we stated out with.

Basically, an expression where the derivative $y'$ is *multiplicatively separable* in $x$ and $y$ solves to get a situation where an *additively separable function* of $x$ and $y$ takes constant values, where each possible constant value gives a particular solution.

2.3. **Of circles.** Consider, for instance, the family of circles centered at the origin (a *concentric family*):

$$x^2 + y^2 = a^2$$

Differentiating and rearranging terms, we obtain the differential equation:

$$ydy = -xdx$$

In fractions, this becomes:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Conversely, solving this differential equation yields:

$$x^2 + y^2 = C$$

Note, however, something peculiar here. The claim in general is that each value of $C$ gives a particular solution. However, from the way sums of squares behave, we know that:

(1) For $C > 0$, we do get particular solutions – the circles we began with.
(2) For $C = 0$, we get a single point circle, which cannot be called much of a solution to the differential equation, because there isn't room to move around within the point.
(3) For $C < 0$, we get the empty set, which again cannot be called much of a solution.

In other words, many of the values of $C$ give empty sets as their curves, which could not be called solutions. Thus, in general, it is *not* correct to say that each value of $C$ gives a legitimate and nontrivial solution. On the other hand, the general theory we have developed so far is not strong enough to predict precisely which values of $C$ give legitimate solutions and which ones give degenerate (as in the case of the single point circle) or empty solution curves.

## 3. Initial value problem

3.1. **Terminology recall and improvement.** Every function or relation that *solves* a particular differential equation is called a *particular solution* and the corresponding curve in $\mathbb{R}^2$ is termed an *integral curve* or *solution curve*. We have seen that, in general, there could be more than one solution curve – in fact, there could be entire families of solution curves parametrized by constants $C \in \mathbb{R}$. A general expression that describes the entire family of solutions is termed a/the *general solution* to the differential equation.

Some other terminology: We say that the *order* of a differential equation is the largest $n$ such that the $n^{th}$ derivative of the dependent variable appears in the differential equation. A *polynomial differential equation* is a differential equation of the form $F(x, y, y', \dots)$ where $F$ looks like a polynomial in $y$ and in each of the derivatives. A *linear differential equation* is a differential equation of the form $F(x, y, y', \dots)$ where $F$ is a *linear* function of the variables $y$, $y'$, $\dots$. The *degree* of a (usually polynomial) differential equation is the power to which the highest order derivative is raised.

3.2. **The constants as parameters.** Remember that when we integrate a function once, we get a $+C$ in the solution. The $+C$ indicates that the integral is not a single unique function but rather a family of functions, all of which can be obtained by taking a *particular* antiderivative and adding any constant function to it. We can think of the set of possible antiderivatives as being parametrized by the real numbers.

When we integrate a function $k$ times, the general solution is of the form of (particular solution) plus (an arbitrary polynomial of degree less than $k$, i.e., degree at most $k - 1$). The *coefficients* of this polynomial are the $k$ constants, one arising from each integration. The set of solutions is thus parametrized by the set of possible $k$-tuples of real numbers.

In physics and chemistry (for instance, in statistical mechanics and thermodynamics), each *freely varying real parameter* is termed a *degree of freedom*.

When solving a *first-order differential equation* (i.e., a differential equation where second or higher derivatives do not appear) what we hope to do is *separate* $x$ and $y$ (something that can be done in the case $y'$ is multiplicatively separable) and then integrate both pieces. We get constants at both places, but these constants can be merged into one constant. *The general idea is that when solving a first-order differential equation, we expect to have one free real parameter, or one degree of freedom, in the solution.* Similarly, when solving a differential equation of order $k$, we expect to have $k$ free parameters, or $k$ degrees of freedom, in the solution.

3.3. **Initial value specification for first-order differential equations.** As noted above, the general solution to a first-order differential equation contains a degree of freedom, typically described by a freely varying real parameter $C$. Geometrically, there is a family of solution curves, and this family is parametrized by a real number.

To find a *particular solution*, we need some piece of information that helps us narrow down to a particular choice of curve.

Information that tells us the location of *one point* on the desired solution curve is termed an *initial value condition* or an *initial value specification*. A problem that consists of a differential equation along with an initial value condition is termed an *initial value problem*.

Typically, an initial value specification helps us determine a specific value of $C$, i.e., it helps us pin down and destroy the one degree of freedom.

For instance, consider:

$$\frac{dy}{dx} = xy$$

Rearranging, we have:

$$\int \frac{dy}{y} = \int x\,dx$$

We obtain:

$$\ln|y| = \frac{x^2}{2} + C$$

Exponentiating both sides, we obtain:

$$|y| = e^{x^2/2}e^C$$

Note that we can pick a new constant $k = e^C \operatorname{sgn}(y)$ and obtain:

$$y = ke^{x^2/2}$$

This is the *general solution*. If, however, we are given that $y(1) = 1$ (in other words, the solution curve passes through $(1, 1)$), we get:

$$1 = ke^{1/2}$$

Thus, we obtain $k = 1/\sqrt{e}$. Plugging this back in, we get the particular solution that we are interested in:

$$y = \frac{1}{\sqrt{e}}e^{x^2/2}$$

or equivalently:

$$y = e^{(x^2-1)/2}$$

3.4. **Brief note on initial value specifications for higher orders.** As noted above, when solving a general differential equation of order $k$, we expect to obtain a general solution with $k$ degrees of freedom. To constrain these, we need $k$ pieces of information (in a rough sense). An *initial value condition* would provide these $k$ pieces of information by providing the information of a point and the first $k - 1$ derivative values at the point.

3.5. **An example of a second-order differential equation and a geometric interpretation.** Consider the second-order differential equation:

$$y'' = 0$$

In the Leibniz notation, this is:

$$\frac{d^2y}{dx^2} = 0$$

This basically involves *integrating twice*. We let $z = y'$, and get:

$$z' = 0$$

Solving, we obtain:

$$z = C_0$$

where $C_0$ is an arbitrary real constant. We now need to solve:

$$y' = C_0$$

which becomes:

$$\frac{dy}{dx} = C_0$$

Solving this yields:

$$y = C_0 x + C_1$$

Thus, the *general solution* involves two arbitrary real constants. Looking at the *solution curves* in the plane, we see that these are precisely all the non-vertical lines.

With first order differential equations, the integral curves typically do not intersect each other. In other words, in the case of first order differential equations, every point is on a unique integral curve (there are sometimes exceptions, such as some special points that lie on lots of curves, and there are sometimes situations where every point lies on two curves). On the other hand, for this second order differential equation, each point lies on an entire infinite family of curves.

Thus, simply specifying *one point* is not enough to determine an integral curve. On the other hand, if we specify one point on the curve *and* the value of the derivative at that point, that information together is enough to determine the integral curve. Thus, *higher order derivative information* is necessary to obtain a strong enough initial value condition to uniquely solve the problem.

The fact that we need *two* pieces of information to pinpoint a curve in this family is not surprising considering that there are two degrees of freedom, or two parameters.

3.6. **Other ways of pinning down degrees of freedom.** An initial value specification constrains degrees of freedom by providing a point and derivative information at that point. There are other ways of specifying a unique integral curve among the set of all possible integral curves, and one of these is to specify the value of the function at *multiple* points. For instance, in the case of the family of all non-vertical straight lines given by the second-order differential equation $y'' = 0$, specifying *two* points is enough to specify a line – a fact which we already learned in high school geometry.