TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY NOVEMBER 27: SIMILARITY OF LINEAR TRANSFORMATIONS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

23 people took this 12-question quiz. The score distribution was as follows:

- Score of 4: 3 people
- Score of 5: 3 people
- Score of 6: 6 people
- Score of 7: 5 people
- Score of 8: 3 people
- Score of 9: 3 people

The mean score was about 6.48.

The question-wise answers and performance review are as follows:

(1) Option (D): 21 people
(2) Option (A): 4 people
(3) Option (E): 3 people
(4) Option (E): 17 people
(5) Option (A): 19 people
(6) Option (A): 22 people
(7) Option (A): 11 people
(8) Option (B): 9 people
(9) Option (B): 5 people
(10) Option (A): 6 people
(11) Option (B): 18 people
(12) Option (D): 14 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz corresponds to material discussed in the lecture notes titled `Coordinates`. It also corresponds to Section 3.4 of the text.

Recall that $n \times n$ matrices A and B are termed *similar* if there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. The relation of matrices being similar is an *equivalence relation* (please refer to the notes for an explanation of the terminology).

For these questions, assume $n > 1$, because a lot of phenomena are much simpler in the case $n = 1$ and you may be misled if you look only at that case. In other words, just because an equality is true for 1×1 matrices, do not assume it is always true. On the other hand, if you can find *counterexamples* to a statement for 1×1 matrices, you can probably use that to construct counterexamples for all sizes of matrices by using scalar matrices.

(1) Which of the following can we say about two (possibly equal, possibly distinct) similar $n \times n$ matrices A and B? Please see Options (D) and (E) before answering.
   (A) A is invertible if and only if B is invertible.
   (B) A is nilpotent if and only if B is nilpotent.
   (C) A is idempotent if and only if B is idempotent.
   (D) All of the above.

(E) None of the above.

Answer: Option (D)

Explanation: Suppose A is similar to B. Then, there exists an invertible matrix S such that $A = SBS^{-1}$, or equivalently, $B = S^{-1}AS$.

- Option (A): If A is invertible, then so is B, and $B^{-1} = S^{-1}A^{-1}S$. Conversely, if B is invertible, so is A, and $A^{-1} = SB^{-1}S^{-1}$.
- Option (B): We know that for any positive integer r, $A^r = SB^rS^{-1}$ and $B^r = S^{-1}A^rS$. If B is nilpotent, then there exists a positive integer r such that $B^r = 0$, so $A^r = SB^rS^{-1} = S(0)S^{-1} = 0$, so $A^r = 0$. Conversely, if A is nilpotent, then there exists a positive integer r such that $A^r = 0$, so $B^r = S^{-1}A^rS = S^{-1}(0)S = 0$.
- Option (C): We know that $SB^2S^{-1} = A^2$ and $S^{-1}A^2S = B^2$, so $A^2 = A$ if and only if $B^2 = B$.

Performance review: 21 out of 23 people got this. 1 each chose (A) and (E).

Historical note (last time): 17 out of 19 got this. 1 each chose (A) and (B).

(2) Which of the following can we say about two (possibly equal, possibly distinct) similar $n \times n$ matrices A and B? Please see Options (D) and (E) before answering.

(A) A is scalar if and only if B is scalar.
(B) A is diagonal if and only if B is diagonal.
(C) A is upper triangular if and only if B is upper triangular.
(D) All of the above.
(E) None of the above.

Answer: Option (A)

Explanation: If A is scalar, then it commutes with every matrix. In particular, A commutes with S, so $B = S^{-1}AS = AS^{-1}S = A$. Thus, $A = B$, and so B is also scalar. Similarly, if B is scalar, then it commutes with S, so $A = SBS^{-1} = BSS^{-1} = B$, so A is also scalar. In other words, A is scalar if and only if B is scalar, and in the event this happens, they are both equal.

The other options fail for reasons described below:

- Option (B): It is possible for A to be diagonal and B to not be diagonal. The idea is to use a matrix S that sends the standard basis vectors to vectors that are not standard basis vectors.
- Option (C): It is possible for A to be upper triangular and B to not be. For instance, consider:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Performance review: 4 out of 23 people got this. 10 chose (D), 7 chose (E), 1 each chose (B) and (C).

Historical note (last time): 11 out of 19 got this. 4 chose (D), 2 each chose (B) and (E).

(3) Suppose A_1, A_2, B_1, B_2 are $n \times n$ matrices such that A_1 is similar to B_1 and A_2 is similar to B_2. Which of the following is *definitely* true? Please see Options (D) and (E) before answering.

(A) $A_1 + A_2$ is similar to $B_1 + B_2$.
(B) $A_1 - A_2$ is similar to $B_1 - B_2$.
(C) A_1A_2 is similar to B_1B_2.
(D) All of the above.
(E) None of the above.

Answer: Option (E)

Explanation: The key problem in each case is that the matrix we use for the similarity between A_1 and B_1 is not the same as the matrix we use for the similarity between A_2 and B_2. If the matrix were the same, then all the conclusions stated above would hold.

For instance, consider the case that:

$$A_1 = A_2 = B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that A_1 and B_1 are similar on account of being equal. A_2 and B_2 are similar using the (self-inverse) similarity matrix:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Consider the sums:

$$A_1 + A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B_1 + B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

These are not similar, because the latter is the identity matrix and hence is not similar to anything else.

Consider the differences:

$$A_1 - A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1 - B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are not similar, because the former is the zero matrix, which is not similar to any other matrix.

Finally, consider the products:

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_1 B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The latter is the zero matrix, hence is not similar to any other matrix.

Performance review: 3 out of 23 people got this. 11 chose (C), 9 chose (D).

Historical note (last time): 12 out of 19 got this. 3 each chose (C) and (D), 1 chose (A).

(4) Suppose A_1, A_2, B_1, B_2 are $n \times n$ matrices such that A_1 is similar to B_1 and A_2 is similar to B_2. Which of the following is *definitely* true? Please see Options (D) and (E) before answering.
 (A) $A_1 + B_1$ is similar to $A_2 + B_2$.
 (B) $A_1 - B_1$ is similar to $A_2 - B_2$.
 (C) $A_1 B_1$ is similar to $A_2 B_2$.
 (D) All of the above.
 (E) None of the above.

Answer: Option (E)

Explanation: There isn't even an *a priori* reason why any of the options should be true, unlike for the previous question where at least *a priori* the options are plausible. For Options (A) and (C), the following 1×1 counterexample works: $A_1 = B_1 = [1]$, $A_2 = B_2 = [2]$. For Option (B), we cannot use 1×1 counterexamples, because in the 1×1 case, we'd have $A_1 - B_1 = A_2 - B_2 = [0]$. We can, however, use 2×2 counterexamples. Explicitly, consider the example:

$$A_1 = A_2 = B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Here, we have:

$$A_1 - B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 - B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Clearly, $A_1 - B_1$ is *not* similar to $A_2 - B_2$.

Performance review: 17 out of 23 people got this. 3 each chose (C) and (D).

(5) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
 (A) A is similar to B if and only if $-A$ is similar to $-B$.
 (B) If A is similar to B, then $-A$ is similar to $-B$. However, $-A$ being similar to $-B$ does not imply that A is similar to B.
 (C) If $-A$ is similar to $-B$, then A is similar to B. However, A being similar to B does not imply that $-A$ is similar to $-B$.
 (D) A being similar to B does not imply that $-A$ is similar to $-B$. Also, $-A$ being similar to $-B$ does not imply that A is similar to B.

Answer: Option (A)

Explanation: We have that for any invertible matrix S, $S(-B)S^{-1} = -(SBS^{-1})$. In other words, if $A = SBS^{-1}$, then $-A = S(-B)S^{-1}$. Conversely, if $-A = S(-B)S^{-1}$, then $A = SBS^{-1}$. Thus, A is similar to B if and only if $-A$ is similar to $-B$, and the matrix used for similarity is the same in both cases.

Note that invertibility of A or B is not necessary for this question, and the inclusion of the adjective "invertible" in the original print version of the quiz was based on an erroneous copy-paste. However, the question is correct even with the "invertible" assumption.

Performance review: 19 out of 23 people got this. 2 chose (D), 1 chose (B), 1 left the question blank.

(6) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
 (A) A is similar to B if and only if $2A$ is similar to $2B$.
 (B) If A is similar to B, then $2A$ is similar to $2B$. However, $2A$ being similar to $2B$ does not imply that A is similar to B.
 (C) If $2A$ is similar to $2B$, then A is similar to B. However, A being similar to B does not imply that $2A$ is similar to $2B$.
 (D) A being similar to B does not imply that $2A$ is similar to $2B$. Also, $2A$ being similar to $2B$ does not imply that A is similar to B.

Answer: Option (A)

Explanation: We have that for any invertible matrix S, $S(2B)S^{-1} = 2(SBS^{-1})$. In other words, if $A = SBS^{-1}$, then $2A = S(2B)S^{-1}$. Conversely, if $2A = S(2B)S^{-1}$, then $A = SBS^{-1}$. Thus, A is similar to B if and only if $2A$ is similar to $2B$, and the matrix used for similarity is the same in both cases.

Performance review: 22 out of 23 people got this. 1 chose (B).

(7) Suppose A and B are both invertible $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
 (A) A is similar to B if and only if A^{-1} is similar to B^{-1}.
 (B) If A is similar to B, then A^{-1} is similar to B^{-1}. However, A^{-1} being similar to B^{-1} does not imply that A is similar to B.
 (C) If A^{-1} is similar to B^{-1}, then A is similar to B. However, A being similar to B does not imply that A^{-1} is similar to B^{-1}
 (D) A being similar to B does not imply that A^{-1} is similar to B^{-1}. Also, A^{-1} being similar to B^{-1} does not imply that A is similar to B.

Answer: Option (A)

Explanation: Note that once we show one direction, the other direction follows, because the inverse operation is self-inverse: the inverse of the inverse is the inverse. This automatically narrows the space of possibilities to two: Option (A) and Option (D). To demonstrate that the correct answer is Option (A), we will show the forward implication: if A is similar to B, then A^{-1} is similar to B^{-1}.

Suppose A is similar to B. Then, there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. Then, $A^{-1} = (SBS^{-1})^{-1} = (S^{-1})^{-1}B^{-1}S^{-1} = SB^{-1}S^{-1}$ (note that the order of multiplication reverses when we take the inverse). Thus, A^{-1} and B^{-1} are also similar.

Performance review: 11 out of 23 people got this. 5 each chose (B) and (D), 1 each chose (C) and (E).

Historical note (last time): 14 out of 19 got this. 2 chose (B), 1 each chose (C), (D), and (E).

(8) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
 (A) A is similar to B if and only if A^2 is similar to B^2.
 (B) If A is similar to B, then A^2 is similar to B^2. However, A^2 being similar to B^2 does not imply that A is similar to B.
 (C) If A^2 is similar to B^2, then A is similar to B. However, A being similar to B does not imply that A^2 is similar to B^2.

(D) A being similar to B does not imply that A^2 is similar to B^2. Also, A^2 being similar to B^2 does not imply that A is similar to B.

Answer: Option (B)

Explanation: If A is similar to B, that means there exists a $n \times n$ invertible matrix S such that $A = SBS^{-1}$. Thus, $A^2 = (SBS^{-1})^2 = SB^2S^{-1}$, so that A^2 and B^2 are both similar.

However, A^2 being similar to B^2 does not imply that A is similar to B. For instance, consider:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note that both A^2 and B^2 are the zero matrix, so A^2 and B^2 are similar. However, A is not similar to B. In fact, B, being the zero matrix, is the only matrix in its similarity class, for obvious reasons.

Performance review: 9 out of 23 people got this. 10 chose (A), 4 chose (D).

Historical note (last time): 10 out of 19 got this. 5 chose (D), 2 chose (A), 1 each chose (C) and (E).

(9) Suppose A and B are $n \times n$ matrices (but they are not given to be similar and they are not given to be invertible). We say that A and B are *quasi-similar* (not a standard term!) if there exist $n \times n$ matrices C and D such that $A = CD$ and $B = DC$. What can we say is the relation between being similar and being quasi-similar?

(A) A and B are similar if and only if they are quasi-similar.

(B) If A and B are similar, they are quasi-similar. However, the converse is not necessarily true: A and B may be quasi-similar but not similar.

(C) If A and B are quasi-similar, they are similar. However, the converse is not necessarily true: A and B may be similar but not quasi-similar.

(D) Neither implies the other. A and B may be similar but not quasi-similar. Also, A and B may be quasi-similar but not similar.

Answer: Option (B)

Explanation: If A and B are similar, there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. In that case, we can set $C = SB$ and $D = D^{-1}$ to obtain that $A = CD$ and $B = DC$.

The converse is not necessarily true. To establish a counter-example, it suffices to construct matrices C and D such that $CD = 0$ but DC is nonzero. If we label CD as A and DC as B, we have constructed quasi-similar matrices that are not similar. Here are the examples:

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The products are:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

These are quasi-similar but not similar.

Performance review: 5 out of 23 people got this. 15 chose (D), 2 chose (C), 1 chose (E).

Historical note (last time); 10 out of 19 got this. 7 chose (D), 2 chose (C).

(10) With the notion of quasi-similar as defined in the preceding question, what can we say about the relation between being similar and being quasi-similar for $n \times n$ matrices A and B that are both given to be *invertible*?

(A) A and B are similar if and only if they are quasi-similar.

(B) If A and B are similar, they are quasi-similar. However, the converse is not necessarily true: A and B may be quasi-similar but not similar.

(C) If A and B are quasi-similar, they are similar. However, the converse is not necessarily true: A and B may be similar but not quasi-similar.

(D) Neither implies the other. A and B may be similar but not quasi-similar. Also, A and B may be quasi-similar but not similar.

Answer: Option (A)

Explanation: We already proved that similar implies quasi-similar. We want to prove the reverse implication under the assumption that A and B are invertible. So, suppose $A = CD$ and $B = DC$ with A and B both invertible.

First, note that C is invertible. In fact, $C(DA^{-1})$ is the identity matrix.

Now, note that:

$$A = CD = CDCC^{-1} = C(DC)C^{-1} = CBC^{-1}$$

Thus, A and B are similar.

Performance review: 6 out of 23 people got this. 6 each chose (B) and (D), 5 chose (C).

Historical note (last time): 9 out of 19 got this. 7 chose (B), 2 chose (D), 1 chose (C).

(11) Suppose A and B are two $n \times n$ matrices. Which of the following best describes the relation between similarity and having the same rank?

(A) A and B are similar if and only if they have the same rank.

(B) If A and B are similar, then they have the same rank. However, it is possible for A and B to have the same rank but not be similar.

(C) If A and B have the same rank, then they are similar. However, it is possible for A and B to be similar but not have the same rank.

(D) A and B may be similar but have different ranks. Also, A and B may have the same rank but not be similar.

Answer: Option (B)

Explanation: Note that similar matrices represent the same linear transformation in different coordinates. In particular, this means that geometrically, the kernel and image remain the same, but they get re-labeled. Thus, the matrices must have the same rank.

Explicitly, if $A = SBS^{-1}$, then the image of A is the image of SBS^{-1}. Start with $\mathbb{R}^n$. Its image under SBS^{-1} can be computed by taking successive images under the linear transformations corresponding to S^{-1}, then B, then S. The first transformation, given by S^{-1}, is bijective from $\mathbb{R}^n$ to $\mathbb{R}^n$ on account of S being invertible. We then do B on the image. Since the image of S^{-1} is all of $\mathbb{R}^n$, the image of BS^{-1} is the same as the image of B. Then again, S is bijective. Therefore it has zero kernel. Thus, its restriction to the image of BS^{-1} sends that subspace of $\mathbb{R}^n$ to an equal-dimensional subspace of $\mathbb{R}^n$. The upshot is that the image of $SBS^{-1}1 = A$ has the same dimension as the image of B. Thus, A and B have the same rank.

However, it is possible for matrices having the same rank to not be similar. For instance, *any* two invertible $n \times n$ matrices have the same rank, namely n. However, they need not be similar. In fact, we can take two different scalar matrices with different scalar values, such as [1] and [2]. Or, we could take these two matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Performance review: 18 out of 23 people got this. 5 chose (A).

Historical note (last time): 15 out of 19 got this. 3 chose (D), 1 chose (A).

(12) Suppose A and B are two $n \times n$ matrices. Which of the following best describes the relation between quasi-similarity and having the same rank?

(A) A and B are quasi-similar if and only if they have the same rank.

(B) If A and B are quasi-similar, then they have the same rank. However, it is possible for A and B to have the same rank but not be quasi-similar.

(C) If A and B have the same rank, then they are quasi-similar. However, it is possible for A and B to be quasi-similar but not have the same rank.

(D) A and B may be quasi-similar but have different ranks. Also, A and B may have the same rank but not be quasi-similar.

Answer: Option (D)

Explanation: For an example of quasi-similar matrices that have different ranks, consider the example provided earlier of the zero matrix being quasi-similar to a nonzero matrix. Explicitly:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For an example of matrices that have the same rank that are not quasi-similar, consider:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Both A and B have rank one. However, they are not quasi-similar. This can be seen in either of two ways:

- Given two quasi-similar matrices, one is nilpotent if and only if the other is, and their nilpotencies differ by at most one. However, in the example above, A is idempotent and not nilpotent, while B is nilpotent. The reason is roughly that if $(CD)^r = 0$, then $(DC)^{r+1} = 0$, and conversely, if $(DC)^s = 0$, then $(CD)^{s+1} = 0$.
- Any two quasi-similar matrices have the same trace (as mentioned below). However, A has trace 1 while B has trace 0.

Performance review: 14 out of 23 people got this. 5 chose (B), 3 chose (A), 1 chose (C).

Historical note (last time): 7 out of 19 got this. 7 chose (B), 3 chose (A), 2 chose (C).