TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FEBRUARY 8: LIMITS AT INFINITY AND IMPROPER INTEGRAL

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 11-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 4: 1 person
- Score of 6: 2 people
- Score of 7: 1 person
- Score of 8: 1 person
- Score of 9: 1 person
- Score of 10: 1 person
- Score of 11: 3 people

The mean score was about 7.7.

The question wise answers and performance were as follows:

(1) Option (B): 10 people
(2) Option (B): 10 people
(3) Option (E): 7 people
(4) Option (D): 6 people
(5) Option (B): 5 people
(6) Option (D): 8 people
(7) Option (C): 10 people
(8) Option (A): 11 people
(9) Option (C): 7 people
(10) Option (D): 7 people
(11) Option (A): 4 people

2. Solutions

(1) If $\lim_{x\to\infty} f(x) = L$ for some finite L, this tells us that the graph of f has a:
   (A) vertical asymptote
   (B) horizontal asymptote
   (C) vertical tangent
   (D) horizontal tangent
   (E) vertical cusp

   Answer: Option (B)

   Explanation: Just by definition.

   Performance review: 10 out of 11 got this correct. 1 chose (D).

   Historical note (last year): 23 out of 26 people got this correct. 1 person each chose (A), (C), and (D).

   Action point: Everybody should get this correct!

(2) If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = M$, where both L and M are finite, then:
   (A) $L = 0$ but M need not be zero
   (B) $M = 0$ but L need not be zero
   (C) Both L and M must be zero.
   (D) Neither L nor M need be zero.

(E) At least one of L and M must be zero, but it could be either one.

Answer: Option (B)

Explanation: If M were finite and nonzero, f would have to go to $+\infty$ if M were positive and to $-\infty$ if M were negative, because f would be growing/decaying at a rate that was bounded from both above and below by linear functions.

Consider the following $\epsilon - \delta$ definition of limit at ∞: $\lim_{x\to\infty} f(x) = L$ if for all $\epsilon > 0$, there exists $a \in \mathbb{R}$ such that for all $x > a$, $|f(x) - L| < \epsilon$.

Performance review: 10 out of 11 got this correct. 1 chose (D).

Historical note (last year): 17 out of 26 people got this correct. 6 people chose (D), 2 people chose (E), and 1 person chose (C).

(3) What is the smallest a that can be picked for the function $f = \arctan$ with L being its limit at ∞ and $\epsilon = \pi$?

(A) $\sqrt{3}$
(B) 1
(C) 0
(D) -1
(E) There is no smallest a. Any $a \in \mathbb{R}$ will do.

Answer: Option (E)

Explanation: The function arctan has range $(-\pi/2, \pi/2)$, which is within the interval $(\pi/2 - \pi, \pi/2 + \pi)$. Thus, for all real numbers, the value of the arctan function is within the specified range.

Performance review: 7 out of 11 got this correct. 2 chose (B), 1 chose (C) and 1 chose (D).

Historical note (last year): 14 out of 26 people got this correct. 5 people chose (B), 3 chose (C), 2 each chose (A) and (D).

(4) What is the smallest a that can be picked for the function $f = \arctan$ with L being its limit at ∞ and $\epsilon = \pi/6$?

(A) $1/2$
(B) $1/\sqrt{3}$
(C) 1
(D) $\sqrt{3}$
(E) 2

Answer: Option (D)

Explanation: We need a such that for $x > a$, $\arctan x \in (\pi/2 - \pi/6, \pi/2 + \pi/6) = (\pi/3, 2\pi/3)$. The right value of a is thus $\tan(\pi/3) = \sqrt{3}$.

Performance review: 6 out of 11 got this correct. 2 chose (E), 1 each chose (A), (B), and (C).

Historical note (last year): 10 out of 26 people got this correct. 11 people chose (B), indicating that either they took $\tan(\pi/3)$ wrong, or they computed $\tan(\pi/6)$ and did not perform the subtraction step $\pi/2 - \pi/6$. 4 people chose (A), 1 person chose (C), and 1 person left the question blank.

(5) Suppose $f(x) := p(x)/q(x)$ is a rational function in reduced form (i.e., the numerator and denominator are relatively prime) and $\lim_{x\to c} f(x) = \infty$. Which of the following can you conclude about f?

(A) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is even.
(B) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is even.
(C) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is odd.
(D) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is odd.
(E) $x - c$ does not divide either $p(x)$ or $q(x)$.

Answer: Option (B)

Explanation: We need $x - c$ to divide $q(x)$ for the denominator to blow up as $x \to c$. The power needs to be even to get the *same* sign of infinity for both left-sided and right-sided approach. $1/x^2$ at $c = 0$ is one example.

Performance review: 5 out of 11 got this correct. 5 chose (C), 1 chose (A).

Historical note (last year): 9 out of 26 people got this correct. 7 chose (A), 4 chose (C), 3 chose (D), 2 chose (E), and 1 left the question blank.

Action point: Please review this solution, make sure you understand it, and if you were convinced of another answer, debug the reasoning or examples that misled you.

(6) Suppose $f(x) := p(x)/q(x)$ is a rational function in reduced form (i.e., the numerator and denominator are relatively prime) and $\lim_{x \to c^-} f(x) = \infty$ and $\lim_{x \to c^+} f(x) = -\infty$. Which of the following can you conclude about f?

(A) $x - c$ divides $p(x)$, and the largest r such that $(x-c)^r$ divides $p(x)$ is even.
(B) $x - c$ divides $q(x)$, and the largest r such that $(x-c)^r$ divides $q(x)$ is even.
(C) $x - c$ divides $p(x)$, and the largest r such that $(x-c)^r$ divides $p(x)$ is odd.
(D) $x - c$ divides $q(x)$, and the largest r such that $(x-c)^r$ divides $q(x)$ is odd.
(E) $x - c$ does not divide either $p(x)$ or $q(x)$.

Answer: Option (D)

Explanation: We need $x - c$ to divide $q(x)$ for the denominator to blow up as $x \to c$. The power needs to be even to get *opposite* signs of infinity for left-sided and right-sided approach. $-1/x$ at $c = 0$ is one example.

Performance review: 8 out of 11 got this correct. 1 each chose (A), (C), and (E).

Historical note (last year): 9 out of 26 people got this correct. 8 people chose (C), 5 people chose (E), 2 people chose (A), 1 chose (B), and 1 left the question blank.

Action point: Please review this solution, make sure you understand it, and if you were convinced of another answer, debug the reasoning or examples that misled you.

Suppose F is a function of two real variables, say x and t, so $F(x,t)$ is a real number for x and t restricted to suitable open intervals in the real number. Suppose, further, that F is jointly continuous (whatever that means) in x and t.

Define $f(t) := \int_0^\infty F(x,t)\,dx$. Here, while doing the integration, t is treated as a constant. x, the variable of integration, is being integrated on $[0, \infty)$.

Suppose further that f is defined and continuous for t in $(0, \infty)$. *Note that similar computations we did in the midterm review session involved integration from $-\infty$ to ∞.*

In the next few questions, you are asked to compute the function f explicitly given the function F, for $t \in (0, \infty)$.

(7) $F(x,t) := e^{-tx}$. Find f.

(A) $f(t) = e^{-t}/t$
(B) $f(t) = e^t/t$
(C) $f(t) = 1/t$
(D) $f(t) = -1/t$
(E) $f(t) = -t$

Answer: Option (C)

Explanation: The integral becomes $[-e^{-tx}/t]_0^\infty$. Plugging in at ∞ gives 0 and plugging in at 0 gives $-1/t$. Since the value at 0 is being subtracted, we eventually get $1/t$.

Note that the answer must be positive for the simple reason that we are integrating a positive function from left to right across an interval.

Performance review: 10 out of 11 got this correct. 1 chose (A).

Historical note (last year): 17 out of 25 people got this correct. 4 chose (A), 3 chose (D), and 1 chose (E).

(8) $F(x,t) := 1/(t^2 + x^2)$. Find f.

(A) $f(t) = \pi/(2t)$
(B) $f(t) = \pi/t$
(C) $f(t) = 2\pi/t$
(D) $f(t) = \pi t$
(E) $f(t) = 2\pi t$

Answer: Option (A)

Explanation: We get $[(1/t)\arctan(x/t)]_0^\infty$. The evaluation at ∞ gives $\pi/(2t)$ and the evaluation at 0 gives 0. Subtracting, we get $\pi/(2t)$.

Performance review: Everybody got this correct.

Historical note (last year): 17 out of 25 got this correct. 5 chose (B), 2 chose (D), 1 chose (C).

(9) $F(x,t) := 1/(t^2+x^2)^2$. Find f.
(A) $f(t) = \pi/t^3$
(B) $f(t) = \pi/(2t^3)$
(C) $f(t) = \pi/(4t^3)$
(D) $f(t) = \pi/(8t^3)$
(E) $f(t) = 3\pi/(8t^3)$

Answer: Option (C)

Explanation: Put in $\theta = \arctan(x/t)$. Substitute, and we get $(1/t^3)\int_0^{\pi/2}\cos^2\theta\,d\theta$. Integrating, we get $[\theta/2t^3 + \sin(2\theta)/4t^3]_0^{\pi/2}$. The trigonometric part vanishes between limits, and we are left with $\pi/(4t^3)$

Performance review: 7 out of 11 got this correct. 3 chose (B), 1 chose (D).

Historical note (last year): 15 out of 25 people got this correct. 5 chose (B), 2 chose (A), 1 each chose (D) and (E), 1 left the question blank.

(10) $F(x,t) = \exp(-(tx)^2)$. Use that $\int_0^\infty \exp(-x^2)\,dx = \sqrt{\pi}/2$. Find f.
(A) $f(t) = t^2\sqrt{\pi}/2$
(B) $f(t) = t\sqrt{\pi}/2$
(C) $f(t) = \sqrt{\pi}/2$
(D) $f(t) = \sqrt{\pi}/(2t)$
(E) $f(t) = \sqrt{\pi}/(2t^2)$

Answer: Option (D)

Explanation: Put $u = tx$, get a $1/t$ on the outside, giving $(1/t)\int_0^\infty \exp(-u^2)\,du$.

Performance review: 7 out of 11 got this correct. 2 chose (A), 2 chose (E).

Historical note (last year): 12 out of 25 people got the question correct. 6 chose (E), 3 chose (C), 2 each chose (A) and (B).

(11) In the same general setup as above (but with none of these specific Fs), which of the following is a *sufficient* condition for f to be an increasing function of t?
(A) $t \mapsto F(x_0,t)$ is an increasing function of t for every choice of $x_0 \geq 0$.
(B) $x \mapsto F(x,t_0)$ is an increasing function of x for every choice of $t_0 \in (0,\infty)$.
(C) $t \mapsto F(x_0,t)$ is a decreasing function of t for every choice of $x_0 \geq 0$.
(D) $x \mapsto F(x,t_0)$ is a decreasing function of x for every choice of $t_0 \in (0,\infty)$.
(E) None of the above.

Answer: Option (A)

Explanation: If F is increasing in t for every value of x_0, then that means that as t gets bigger, the function F being integrated gets bigger everywhere in x, i.e., if $t_1 < t_2$, then $F(t_1,x_0) < F(t_2,x_0)$ for every $x_0 \geq 0$. The integral for the larger value t_2 must therefore also be bigger. (We looked at this stuff in Section 5.8 of the book).

Performance review: 4 out of 11 got this correct. 4 chose (B), 1 each chose (C), (D), and (E).

Historical note (last year): 4 out of 25 got the question correct. 10 chose (B), 5 chose (E), 3 chose (C), 2 chose (D), and 1 left the question blank.

(A) was the "obvious" choice – people may have tried to seek more subtletly in the question than it had.

Action point: This should not trip *anybody* in the future.