TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY FEBRUARY 18: PARTIAL DERIVATIVES

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 3: 1 person.
- Score of 4: 1 person.
- Score of 5: 2 people.
- Score of 6: 1 person.
- Score of 7: 2 people.
- Score of 8: 3 people.
- Score of 9: 5 people.
- Score of 11: 5 people.
- Score of 12: 3 people.
- Score of 13: 3 people.

The question wise answers and performance review were as follows:

(1) Option (D): 17 people.
(2) Option (A): 3 people.
(3) Option (A): 15 people.
(4) Option (E): 16 people.
(5) Option (A): 18 people.
(6) Option (E): 4 people.
(7) Option (C): 2 people.
(8) Option (B): 0 people.
(9) Option (C): 8 people.
(10) Option (C): 20 people.
(11) Option (E): 19 people.
(12) Option (D): 21 people.
(13) Option (B): 24 people.
(14) Option (D): 25 people.
(15) Option (B): 14 people.
(16) Option (D): 15 people.
(17) Option (E): 15 people.

2. Solutions

(1) For this and the next question, consider the function on $\mathbb{R}^2$ given as:

$$f(x, y) := \begin{cases} 1, & x \text{ rational or } y \text{ rational} \\ 0, & x \text{ and } y \text{ both irrational} \end{cases}$$

What can we say about the subset S of $\mathbb{R}^2$ defined as the set of points where f_x is defined?

(A) S is the set of points for which at least one coordinate is rational.
(B) S is the set of points for which both coordinates are rational.
(C) S is the set of points for which the x-coordinate is rational.
(D) S is the set of points for which the y-coordinate is rational.
(E) S is the set of points for which at least one coordinate is irrational.

Answer: Option (D)

Explanation: f_x means we take the derivative with respect to x holding y constant. Consider the point (x_0, y_0). If y_0 is rational, then $f(x, y_0) = 1$ on the entire line $y = y_0$. Thus, $f_x(x_0, y_0)$ is the derivative of a constant function, hence is 0. In particular, it is well defined.

On the other hand, if y_0 is irrational, then there are points (x, y_0) for x arbitrarily close to x_0 for which x is rational, giving $f(x, y_0) = 1$, and also points where x is irrational, giving $f(x, y_0) = 0$. Thus, f is not continuous in x at (x_0, y_0), and hence $f_x(x_0, y_0)$ does not exist.

Performance review: 17 out of 26 people got this. 4 chose (A), 2 each chose (C) and (E), 1 chose (B).

(2) With f as in the previous question, what is the subset T of $\mathbb{R}^2$ at which the second-order mixed partial derivative f_{xy} is defined?
(A) T is the empty subset.
(B) T is the set of points for which both coordinates are rational.
(C) T is the set of points for which the x-coordinate is rational.
(D) T is the set of points for which the y-coordinate is rational.
(E) T is the set of points for which both coordinates are irrational.

Answer: Option (A)

Explanation: We have $f_{xy} = (f_x)_y$. In order for this to be defined at (x_0, y_0), a necessary condition is that f_x be defined at (x_0, y_0), and also that it be defined at (x_0, y) for y close to y_0. Note that the former condition holds only if y_0 is rational. However, the latter condition is never true, because for any value of y_0, there are values y arbitrarily close to y_0 that are rational, and also values y that are irrational.

Performance review: 3 out of 26 people got this. 19 chose (B), 3 chose (D), 1 chose (C).

(3) For this and the next three questions, consider the function on $\mathbb{R}^2$ given as:

$$g(x, y) := \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

What can we say about the subset U of $\mathbb{R}^2$ defined as the set of points where g_x is defined?
(A) U is the empty subset.
(B) U is the set of points for which both coordinates are rational.
(C) U is the set of points for which the x-coordinate is rational.
(D) U is the set of points for which the y-coordinate is rational.
(E) U is the whole plane $\mathbb{R}^2$.

Answer: Option (A)

Explanation: At any point (x_0, y_0), there are x-values arbitrarily close to x_0 that are rational, and x-values arbitrarily close to x_0 that are irrational. Thus, g is not continuous in x at any point, so g_x does not exist anywhere.

Performance review: 15 out of 26 got this. 10 chose (C), 1 chose (E).

(4) With g as in the preceding question, what can we say about the subset V of $\mathbb{R}^2$ defined as the set of points where g_y is defined?
(A) V is the empty subset.
(B) V is the set of points for which both coordinates are rational.
(C) V is the set of points for which the x-coordinate is rational.
(D) V is the set of points for which the y-coordinate is rational.
(E) V is the whole plane $\mathbb{R}^2$.

Answer: Option (E)

Explanation: Note that g depends *only* on x, hence it is independent of y. Thus, g_y is identically the zero function, and is defined everywhere.

Performance review: 16 out of 26 got this. 5 chose (D), 2 each chose (A) and (C), 1 chose (B).

(5) With g as in the preceding question, what can we say about the subset W of $\mathbb{R}^2$ defined as the set of points where g_{xy} is defined?
(A) W is the empty subset.
(B) W is the set of points for which both coordinates are rational.

(C) W is the set of points for which the x-coordinate is rational.
(D) W is the set of points for which the y-coordinate is rational.
(E) W is the whole plane $\mathbb{R}^2$.

Answer: Option (A)

Explanation: This follows from g_x not being defined anywhere.

Performance review: 18 out of 26 got this. 4 chose (B), 2 chose (E), 1 each chose (C) and (D).

(6) With g as in the preceding question, what can we say about the subset X of $\mathbb{R}^2$ defined as the set of points where g_{yx} is defined?
(A) X is the empty subset.
(B) X is the set of points for which both coordinates are rational.
(C) X is the set of points for which the x-coordinate is rational.
(D) X is the set of points for which the y-coordinate is rational.
(E) X is the whole plane $\mathbb{R}^2$.

Answer: Option (E)

Explanation: This follows from g_y being the zero function everywhere.

Performance review: 18 out of 26 got this. 4 chose (E), 3 chose (B), 1 chose (C).

(7) For this and the next two questions, consider the function on $\mathbb{R}^2$ given as:

$$h(x,y) := \begin{cases} 1, & x \text{ an integer or } y \text{ an integer} \\ 0, & x \text{ not an integer and } y \text{ not an integer} \end{cases}$$

What can we say about the subset A of $\mathbb{R}^2$ defined as the set of points where h_{xy} is defined?
(A) A is the empty set.
(B) A is the set of points whose x-coordinate is an integer.
(C) A is the set of points whose x-coordinate is not an integer.
(D) A is the set of points whose y-coordinate is an integer.
(E) A is the set of points whose y-coordinate is not an integer.

Answer: Option (C)

Explanation: We first note that if both coordinates are non-integers, then h is identiically the zero function *at and around* the point, so all first and higher order partials are zero. In particular, $h_x = h_{xy} = 0$ at all points with both coordinate non-integers.

Suppose now that the y-coordinate is an integer. In this case, h is identically 1 on lines of the form $y = y_0$, y_0 an integer. Thus, h_x is zero on these lines.

Suppose now that the x-coordinate is an integer but the y-coordinate is a non-integer. In this case, the function h takes the value 1 at the point, but takes the value 0 if we vary the x-coordinate even slightly. Thus, h_x is not defined at such points.

Thus, overall, h_x is defined and equal to zero at all points where either the y-coordinate is an integer or both coordinates are non-integers. It is not defined precisely at the points where the x-coordinate is an integer and the y-coordinate is a non-integer.

Of the points where h_x is defined, h_{xy} is not defined at points where both coordinates are integers, because slightly perturbing the y-coordinate gets a point where h_x is undefined. h_{xy} is defined and equal to zero at all other points, namely the points where the x-coordinate is a non-integer.

Performance review: 2 out of 26 got this. 11 chose (D), 8 chose (A), 4 chose (B), 1 left the question blank.

(8) With h as defined in the previous question, what can we say about the subset B of $\mathbb{R}^2$ defined as the set of points where h_x is defined but h_{xy} is not defined?
(A) B is the empty set.
(B) B is the set of points for which both coordinates are integers.
(C) B is the set of points for which both coordinates are non-integers.
(D) B is the set of points for which at least one coordinate is an integer.
(E) B is the set of points for which at least one coordinate is a non-integer.

Answer: Option (B)

Explanation: See the explanation for the preceding question.

Performance review: Nobody got this correct. 8 chose (D), 7 chose (E), 5 each chose (A) and (C). 1 left the question blank.

(9) With h as defined in the previous question, what can we say about the subset C of $\mathbb{R}^2$ defined as the set of points where both h_{xy} and h_{yx} are defined?

(A) C is the empty set.
(B) C is the set of points for which both coordinates are integers.
(C) C is the set of points for which both coordinates are non-integers.
(D) C is the set of points for which at least one coordinate is an integer.
(E) C is the set of points for which at least one coordinate is a non-integer.

Answer: Option (C)

Explanation: By the question before last, the set of points where h_{xy} is defined is the set of points where the x-coordinate is a non-integer. The function is symmetric in x and y, so analogous reasoning yields that the set of points where hyx is defined is the set of points where the y-coordinate is a non-integer. Intersecting the two sets, we get the set of points where both coordinates are non-integers.

Performance review: 8 out of 26 got this correct. 14 chose (B), 2 chose (A), 1 chose (D), 1 left the question blank.

(10) Students training for an examination can spend money either on purchasing textbooks or on private tuitions. A student's expected performance on the examination is a function of the money the student spends on textbooks and on tuition (viewed as separate variables). Two researchers want to consider the question of whether increased expenditure on textbooks leads to improved performance on the examination, and if so, by how much.

One researcher decides to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the expenditure on tuitions*, arguing that in order to determine the effect of changes in textbook expenditures, the other expenditures need to be kept constant.

The other researcher believes that since the student has a limited budget, it would be more realistic to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the total expenditure on both textbook and tuitions*. This is because the student is likely to allocate money away from tuition expenditures in order to spend money on textbooks.

Which of the following best describes what's happening?

(A) Both researchers are effectively computing the same quantity.
(B) The two quantities that the researchers are computing have a simple linear relationship, i.e., their sum or difference is a constant.
(C) The two quantities that the researchers are computing are meaningfully different and there is a relationship between them but that relationship involves other partial derivatives.

Answer: Option (C)

Explanation: Too tricky to review here, but you might want to watch this video and subsequent ones in the playlist:

`http://www.youtube.com/watch?v=tfH2iqt2E0E&list=PLC0bHnWu122kC1WBgr0H9PEbHTYnYev27&index=4`

Performance review: 20 out of 26 got this correct. 3 each chose (A) and (B).

(11) F is an everywhere twice differentiable function of two variables x and y. Which of the following captures the manner in which the inputs x and y *interact* with each other in the description of F?

(A) The difference $F_x - F_y$
(B) The quotient F_x/F_y.
(C) The product F_xF_y.
(D) The product $F_{xx}F_{yy}$.
(E) The mixed partial F_{xy}

Answer: Option (E)

Explanation: One extreme way of seeing this is that if F is additively separable (i.e., it is the sum of a function of x and a function of y) then $F_{xy}(x, y) = 0$. Thus, the stereotypical case in which the variables don't interact with each other is the case that the second-order mixed partial is zero.

Performance review: 19 out of 26 got this correct. 4 chose (D), 2 chose (B), 1 chose (C).

(12) F is a function of two variables x and y such that both F_x and F_y exist. Which of the following is generically true?

(A) In general, F_x depends only on x (i.e., it is independent of y) and F_y depends only on y. An exception is if F is multiplicatively separable.

(B) In general, F_x depends only on y (i.e., it is independent of x) and F_y depends only on x (i.e., it is independent of y). An exception is if F is multiplicatively separable.

(C) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on y and F_y depends only on x.

(D) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on x and F_y depends only on y.

(E) In general, either both F_x and F_y depend only on x or both F_x and F_y depend only on y.

Answer: Option (D)

Explanation: This should be straightforward if you understand what's going on. Otherwise, watch this video and the subsequent one:

`http://www.youtube.com/watch?v=2T7iFZVLtn0&list=PLC0bHnWu122kC1WBgr0H9PEbHTYnYev27&index=1`

Performance review: 21 out of 26 got this correct. 3 chose (A), 1 each chose (C) and (E).

(13) Consider a production function $f(L, K, T)$ of three inputs L (labor expenditure), K (capital expenditure), and T (technology expenditure). Suppose all mixed partials of f with respect to L, K, and T are continuous. Suppose we have the following signs of partial derivatives: $\partial f/\partial L > 0$, $\partial f/\partial K > 0$, $\partial^2 f/(\partial L \partial K) < 0$, and $\partial^3 f/(\partial L \partial K \partial T) > 0$. What does this mean?

(A) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital substitute for each other.

(B) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital substitute for each other, i.e., with more technology investment, labor and capital become more complementary.

(C) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital complement for each other.

(D) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital complement for each other.

(E) Increasing labor or capital decreases production.

Answer: Option (B)

Explanation: $\partial f/\partial L > 0$ shows that increasing labor increases production. $\partial f/\partial K > 0$ shows that increasing capital increases production. $\partial^2 f/\partial L \partial K < 0$ indicates that labor and capital substitute for each other, i.e., a small increase in capital reduces the marginal product of labor. Finally, $\partial^3 f/\partial L \partial K \partial T > 0$ indicates that $\partial^2 f/\partial L \partial K$ is increasing with T, i.e., getting less negative. So, although labor and capital substitute for each other, the degree to which they do so reduces as T increases. Roughly speaking, more technology reduces the antagonism between labor and capital.

Performance review: 24 out of 26 got this correct. 1 each chose (A) and (C).

Historical note (last time): 12 out of 20 people got this correct. 7 chose (A) and 1 chose (D). The people who chose (A) probably didn't note that an increase in the degree of substitution would mean a decrease in the derivative, rather than an increase.

(14) Analysis of usage of an online social network finds that the total time spent by people on the social network is $P^{1.3}L^{0.5}$ where P is the total number of people on the network and L is a number of processors used at the social network's server facility. Which of these is true?

(A) Increasing returns both on persons and on processors: every new person joining the network increases the average time spent *per person* (and not just the total time), and every new processor added to the server facility increases the average time spent per processor.

(B) Constant returns on persons, increasing returns on processors

(C) Constant returns on persons, decreasing returns on processors
(D) Increasing returns on persons, decreasing returns on processors
(E) Decreasing returns on persons, increasing returns on processors
Answer: Option (D)

Explanation: The short explanation is that the exponent on P is greater than 1, so the second partial derivative is positive, and the exponent of L is between 0 and 1, so the second partial derivative is negative.

The long explanation is just working it out.

Performance review: 25 out of 26 got this. 1 chose (C).

Historical note (last time): 14 out of 20 got this correct. 5 chose (A), 1 chose (B).

(15) *Not a calculus question, but has deep calculus interpretations – it is basically measuring the derivative of the $1/x$ function with respect to x*: A person travels fifty miles every day by car and the travel distance is fixed. The price of gasoline, which she uses to fuel her car, is also fixed. Which of the following increases in fuel efficiency result in the maximum amount of savings for her?
(A) From 11 to 12 miles per gallon
(B) From 12 to 14 miles per gallon
(C) From 20 to 25 miles per gallon
(D) From 36 to 54 miles per gallon
(E) From 50 to 100 miles per gallon
Answer: Option (B)

Explanation: The gain achieved by upgrading from a miles per gallon to b miles per gallon is $(50/a - 50/b)$ times the cost of a gallon. In particular, it is proportional to $1/a - 1/b$. It remains to compute the case where this difference is largest.

The values of $1/a - 1/b$ are: Option (A):/ 1/132, Option (B): 1/84, Option (C): 1/100, Option (D): 1/108, Option (E): 1/100. Of these, the largest is the one with smallest denominator, i.e., 1/84. In other words, the maximum gain happens in going from 12 to 14.

This seems a little counter-intuitive at first. Looked at in terms of ratios, the gain from 50 to 100 is most impressive. Looked at in terms of differences in MPG values, again the gain from 50 to 100 is more impressive. However, these gains are not what we are measuring, because in the question, it is specified that the distance of travel is *fixed* and hence what matters is the absolute savings in cost.

Intuitively, what's happening is that while a gain from 50 to 100 halves the cost, that halving is occurring from an already fairly small cost base, so the quantitative savings are little. On the other hand, a jump from 12 to 14 is small in proportion but large in absolute terms because the base from which the savings are occurring is much larger. In fact, even a gain from 100 miles per gallon to infinite miles per gallon produces less in cost savings assuming fixed distance and fixed cost per gallon than a gain from 12 to 14.

Another way of thinking of this is in terms of the derivative of $1/x$. We know that as x increases, $1/x$ decreases. However, the derivative is not constant. When x is small, the derivative $-1/x^2$ is huge in magnitude, which means that small changes in x lead to large changes in $1/x$. When x is large, the derivative $-1/x^2$ is small in magnitude, which means that large changes in x lead to only small changes in $1/x$.

Thus, we can get three fairly different pictures depending on whether we measure things using x, $1/x$, or $\ln x$.

Performance review: 14 out of 26 people got this. 6 chose (E), 3 chose (A), 2 chose (D), 1 chose (C).

Historical note (last time): 2 out of 20 people got this correct. 7 chose (C), 6 chose (E), 4 chose (D), and 1 chose (A).

(16) For which of the following production functions $f(L, K)$ of labor and capital is it true that labor and capital can be complementary for some choices of (L, K), and substitutes for others? In other words, for which of these are labor and capital neither globally complements nor globally substitutes? Assume the domain $L > 0, K > 0$.
(A) $L^2 + LK + K^2$

(B) $L^2 - LK + K^2$
(C) $L^3 + L^2K + LK^2 + K^3$
(D) $L^3 + L^2K - LK^2 + K^3$
(E) $L^3 - L^2K - LK^2 + K^3$

Answer: Option (D)

Explanation: For option (D), the second mixed partial is $2(L - K)$ which is positive if $L > K$ and negative if $L < K$. For options (A) and (C), the second mixed partial is always positive, while for options (B) and (E), the second mixed partial is always negative.

Performance review: 15 out of 26 got this. 6 chose (E), 3 chose (B), 2 chose (C).

Historical note (last time): 8 out of 20 people got this correct. 4 each chose (B), (C) and (E).

(17) Consider the following Leontief-like production function $f(L, K) = (\min\{L, K\})^2$. Assume the domain $L > 0$, $K > 0$. What is the nature of returns and complementarity here?

(A) Positive increasing returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(B) Positive constant returns of the smaller of the inputs, positive increasing returns on the larger of the inputs
(C) Zero returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(D) Positive decreasing returns on the smaller of the inputs, zero returns on the larger of the inputs
(E) Positive increasing returns on the smaller of the inputs, zero returns on the larger of the inputs

Answer: Option (E)

Explanation: This is a "weak link" type of production function in the sense that the weakest link in the labor-capital nexus determines output. If $L < K$, then output is L^2, and if $K < L$, then output is K^2. This means that, at the margin, increasing the one which is already larger produces no gain in output. However, increasing the one which is smaller increases output as the square thereof. Since $2 > 1$, there are positive increasing returns on the smaller input.

A practical example of this is where "it takes two to tango" – for instance, if each unit of labor is a person and each unit if capital is a machine, and if there are more machines than people or vice versa, the extra machines/people are completely unused.

Performance review: 15 out of 26 got this. 5 chose (A), 4 chose (D), 1 each chose (B) and (C).

Historical note (last time): 10 out of 20 people got this correct. 3 each chose (A) and (D), 2 each chose (B) and (C).