TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY DECEMBER 4: MATRIX TRANSPOSE: PRELIMINARIES

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

25 people took this 7-question quiz. The score distribution was as follows:

- Score of 4: 4 people
- Score of 5: 5 people
- Score of 6: 4 people
- Score of 7: 12 people

The question-wise answers and performance review were as follows:

(1) Option (A): 25 people (everybody)
(2) Option (D): 25 people (everybody)
(3) Option (A): 23 people
(4) Option (C): 22 people
(5) Option (A): 18 people
(6) Option (D): 18 people
(7) Option (A): 18 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The following questions are related to material from parts of Chapter 5 that we are glossing over. You do not need to read that chapter, because we are using a very limited part of it in a very limited fashion and we've included all relevant definitions in the quiz. However, if you want to understand some of the constructs in more detail, please do read the chapter.

Note: Due to limited class time, I'm making this a take-home class quiz, but in an ideal world, this would have been a diagnostic in-class quiz.

For a $n \times m$ matrix A, denote by A^T (spoken as *A-transposed* and called the *transpose of* A) the $m \times n$ matrix whose $(ij)^{th}$ entry is defined as the $(ji)^{th}$ entry of A. In other words, the roles of rows and columns are interchanged when we transition from A to A^T. The ^{T} should not be interpreted as an exponent letter. Note that whereas A describes a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, A^T describes a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Note, however, that although the domain and co-domain for A and A^T are interchanged with each other, A and A^T are not in general inverses of each other.

(1) Suppose A is a $n \times m$ matrix and A^T is the tranpose of A. Under what conditions does the sum $A + A^T$ make sense (i.e., exist as a matrix)?
(A) $A + A^T$ makes sense if and only if $m = n$.
(B) $A + A^T$ makes sense if and only if $m < n$.
(C) $A + A^T$ makes sense if and only if $m > n$.
(D) $A + A^T$ makes sense regardless of whether $m = n$, $m < n$, or $m > n$.

Answer: Option (A)

Explanation: A is a $n \times m$ matrix whereas A^T is a $m \times n$ matrix. We know that $A + A^T$ makes sense if and only if A and A^T have the same number of rows and also have the same number of columns. Both of these conditions are equivalent to requiring that $m = n$.

Performance review: All 25 got this correct.

(2) Suppose A is a $n \times m$ matrix and A^T is the transpose of A. Under what conditions does the product AA^T make sense (i.e., exist as a matrix)?
(A) AA^T makes sense if and only if $m = n$.
(B) AA^T makes sense if and only if $m < n$.
(C) AA^T makes sense if and only if $m > n$.
(D) AA^T makes sense regardless of whether $m = n$, $m < n$, or $m > n$.

Answer: Option (D)

Explanation: A is a $n \times m$ matrix and A^T is a $m \times n$ matrix. The product AA^T makes sense because the number of columns in A equals the number of rows in A^T, so the product AA^T is a $n \times n$ matrix.

Performance review: All 25 got this correct.

(3) Suppose A is a $n \times m$ matrix and A^T is the transpose of A. Under what conditions do both AA^T and A^TA exist *and* have the same number of rows as each other and the same number of columns as each other (note that they still need not be equal)?
(A) This happens if and only if $m = n$.
(B) This happens if and only if $m < n$.
(C) This happens if and only if $m > n$.
(D) This happens always, regardless of whether $m = n$, $m < n$, or $m > n$.

Answer: Option (A)

Explanation: The product AA^T is a $n \times n$ matrix (see the explanation for Question 2). For similar reasons, the product A^TA is a $m \times m$ matrix. These have the same size if and only if $m = n$.

Performance review: 23 out of 25 got this. 2 chose (D).

(4) Suppose A is a $n \times n$ matrix such that $A^T = A^{-1}$. We describe this condition by saying that A is an *orthogonal* $n \times n$ matrix. Which of the following is a correct characterization of a matrix being orthogonal? Please see Option (C) before answering, and select the option that best reflects your view.
(A) Every row vector of A is a unit vector, and any two distinct rows of A are orthogonal.
(B) Every column vector of A is a unit vector, and any two distinct columns of A are orthogonal.
(C) Both of the above work, i.e., they are equivalent to each other and to the condition that $A^T = A^{-1}$.

Answer: Option (C)

Explanation: The condition that $A^T = A^{-1}$ can be interpreted in two equivalent ways: $AA^T = I_n$ and $A^TA = I_n$.

With the $AA^T = I_n$ interpretation, we see that the dot product of each row of A with the corresponding column of A^T is 1, and the dot product of each row of A with a different column of A^T is 0. Since the "corresponding column of A^T" agrees with the original row of A, we obtain that the dot product of each row of A with itself is 1 (i.e., each row of A is a unit vector) and the dot product of any two distinct rows of A is 0, i.e., any two distinct rows of A are orthogonal.

With the $A^TA = I_n$ interpretation, we obtain the analogous result for columns, because we are now dealing with dot products between rows of A^T and columns of A.

Performance review: 22 out of 25 got this. 2 chose (B), 1 chose (A).

A square matrix A is termed *symmetric* if $A = A^T$ and *skew-symmetric* if $A = -A^T$.

The following facts are true and can be easily verified:

- Suppose A and B are matrices such that $A + B$ makes sense. Then, $(A + B)^T = A^T + B^T$.
- Suppose A and B are matrices such that AB makes sense. Then, $(AB)^T = B^TA^T$. Note that the order of multiplication flips over. The rule is similar to the rule for inverses, even though the transpose is *not* the same as the inverse.
- For any matrix A, $(A^T)^T = A$.

(5) Suppose A is a matrix. What can we say that the nature of the matrices $A + A^T$ and AA^T?
(A) $A + A^T$ is symmetric if it makes sense. AA^T is symmetric if it makes sense.

(B) $A + A^T$ is symmetric if it makes sense. AA^T is skew-symmetric if it makes sense.
(C) $A + A^T$ is skew-symmetric if it makes sense. AA^T is symmetric if it makes sense.
(D) $A + A^T$ is skew-symmetric if it makes sense. AA^T is skew-symmetric if it makes sense.

Answer: Option (A)

Explanation: For the sum: $(A+A^T)^T = A^T+(A^T)^T = A^T+A = A+A^T$. We use that $(A^T)^T = A$ while simplifying.

For the product: $(AA^T)^T = (A^T)^T A^T$ (we use the rule for transpose of a product). This simplifies to AA^T using the fact that $(A^T)^T = A$.

Performance review: 18 out of 25 got this. 5 chose (B), 2 chose (C).

(6) Suppose n is a positive integer. Consider the vector space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices. The subset comprising symmetric matrices is a linear subspace and the subset comprising skew-symmetric matrices is also a linear subspace. The subset comprising diagonal matrices is also a linear subspace. Which of the following best describes the containment relation between the subspaces of diagonal matrices, symmetric matrices, and skew-symmetric matrices?

(A) The subspace comprising all diagonal matrices is contained in the subspace comprising all skew-symmetric matrices, which in turn is contained in the subspace comprising all symmetric matrices.
(B) The subspace comprising all diagonal matrices is contained both in the subspace comprising all skew-symmetric matrices and in the subspace comprising all symmetric matrices. However, neither of the two latter subspaces is contained in the other.
(C) The subspace comprising all diagonal matrices is contained in the subspace comprising all skew-symmetric matrices, but neither of these subspaces is contained in the subspace comprising all symmetric matrices.
(D) The subspace comprising all diagonal matrices is contained in the subspace comprising all symmetric matrices, but neither of these subspaces is contained in the subspace comprising all skew-symmetric matrices.
(E) None of the three subspaces is fully contained in any of the others.

Answer: Option (D)

Explanation: The subspace of diagonal matrices is contained in the subspace of skew-symmetric matrices, because any diagonal matrix is symmetric. This can easily be seen by looking at such a matrix. In the 2×2 case, a diagonal matrix is a matrix of the form:

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

Clearly, such a matrix equals its own transpose.

On the other hand, the *only* symmetric matrix that is also skew-symmetric is the zero matrix. Explicitly, if A is a matrix satisfying both the conditions $A = A^T$ and $A + A^T = 0$, then we get $2A = 0$, hence $A = 0$. In particular, this also means that the only *diagonal* matrix that is skew-symmetric is the zero matrix. Thus, the space of diagonal matrices is not contained in the space of skew-symmetric matrices.

Performance review: 18 out of 25 got this. 2 each chose (B), (C), and (E), 1 chose (A).

(7) Suppose n is a positive integer. Consider the vector space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices. This vector space has dimension $n \times n = n^2$. What are the respective dimensions of the subspaces comprising symmetric and skew-symmetric matrices? *Hint*: Try the case $n = 1$ and then the case $n = 2$. In both cases, try to write down an explicit basis for each of the subspaces. You might want to revisit the preceding question in light of your improved understanding after solving this question.

(A) The subspace comprising all symmetric matrices has dimension $n(n + 1)/2$ and the subspace comprising all skew-symmetric matrices has dimension $n(n - 1)/2$.
(B) The subspace comprising all symmetric matrices has dimension $n(n - 1)/2$ and the subspace comprising all skew-symmetric matrices has dimension $n(n + 1)/2$.
(C) Both subspaces have dimension $n^2/2$.

Answer: Option (A)

Explanation: For a symmetric matrix A, we can freely choose all the diagonal entries (this gives n dimensions) and for each pair $i \neq j$ of distinct elements of $\{1, 2, \ldots, n\}$, we can choose the entry $a_{ij} = a_{ji}$. There are $(n^2 - n)/2$ dimensions arising from the latter choices. The total number of choices we have is therefore $n + (n^2 - n)/2 = n(n+1)/2$.

In the skew-symmetric case, the diagonal entries all need to be zero. For each pair $i \neq j$ of distinct elements of $\{1, 2, \ldots, n\}$, we can choose the entry $a_{ij} = -a_{ji}$. There are $(n^2 - n)/2 = n(n-1)/2$ such choices.

In the 1×1 case, all matrices are symmetric and only the zero matrix is skew-symmetric. This agrees with our counts: the dimension of the subspace of symmetric matrices is $n(n+1)/2 = 1(1+1)/2 = 1$ (as it should be, because that's the dimension of the whole space) and the dimension of the subspace of skew-symmetric matrices is $n(n-1)/2 = 1(1-1)/2 = 0$ (as it should be, because that's the dimension of the zero subspace).

In the 2×2 case, the following matrices form a basis for the subspace of symmetric matrices (the basis has size $2(2+1)/2 = 3$):

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The subspace of skew-symmetric matrices has dimension $2(2-1)/2 = 1$ and has a basis given by the following matrix:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In the 3×3 case, the following matrices form a basis for the subspace of symmetric matrices (there are $3(3+1)/2 = 6$ of them):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The following matrices form a basis for the subspace of skew-symmetric matrices (there are $3(3-1)/2 = 3$ of them):

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Performance review: 18 out of 25 got this. 5 chose (B), 2 chose (C).