TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 4: LINEAR FUNCTIONS AND EQUATION-SOLVING (PART 1)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

27 people took this 9-question quiz. The score distribution was as follows:

- Score of 1: 2 people
- Score of 3: 1 person
- Score of 4: 2 people
- Score of 5: 4 people
- Score of 6: 3 people
- Score of 7: 8 people
- Score of 8: 5 people
- Score of 9: 2 people

The mean score was about 6.1.

The question-wise answers and performance review are below:

(1) Option (D): 20 people
(2) Option (C): 16 people
(3) Option (B): 21 people
(4) Option (A): 23 people
(5) Option (E): 9 people
(6) Option (D): 19 people
(7) Option (C): 26 people
(8) Option (C): 11 people
(9) Option (A): 20 people

2. Solutions

This quiz covers some basics involving linear functions and equation-solving (notes at `Linear functions: a primer` and `Equation-solving with a special focus on the linear case`). The quiz tests for the following:

- What it means to be (affine) linear, and in particular, the significance of the intercept as an additional parameter to track.
- The distinction between behavior relative to the variables (the inputs) and behavior relative to the parameters.
- Using the linear paradigm to study functional forms that are not themselves linear.
- A small taste of dealing with measurement uncertainty to obtain upper and lower bounds (not covered in the notes, so this is where your famed ability to think out of the box should manifest).
- Solving "triangular" systems of equations.

(1) A function f of 3 variables x, y, z defined everywhere is (affine) linear in the variables. (The "affine" is to indicate that the intercept may be nonzero). Based on the above information and some input-output pairs for f, we would like to determine f uniquely. What is the minimum number of input-output pairs that we would need in order to achieve this?
(A) 1
(B) 2
(C) 3

(D) 4
(E) 5
Answer: Option (D)
Explanation: The general expression for a linear function f of the variables x, y, and z is:

$$f(x, y, z) := ax + by + cz + d$$

There are four unknown parameters here (one coefficient for each variable, and one parameter d for the intercept). Our goal is to determine uniquely the values of the parameters. Since there are four parameters, we need four equations to determine them uniquely. Note that the equations themselves are linear. We should choose our four inputs in a manner that there are no linear dependencies between them (what this means will become clearer as we study more linear algebra).

Performance review: 20 out of 27 got this. 6 chose (C), 1 chose (E).

Historical note (last time; however, discussion was not permitted for this question last time): 7 out of 29 people got this. 20 chose (C), 2 chose (B).

(2) Which of the following gives an example of a function F of three variables x, y, z whose third-order mixed partial derivative F_{xyz} is zero everywhere, but for which none of the second-order mixed partial derivatives F_{xy}, F_{xz}, F_{yz} is zero everywhere?
(A) $\sin(xy) - z^2$
(B) $\cos(x^2 + y^2) - \sin(y^2 + z^2)$
(C) $e^{xy} + (y - z)^2 + 3xz$
(D) $x^2 + y^2 + z^2$
(E) xyz
Answer: Option (C)
Explanation: Look for functions of the form:

$$F(x, y, z) = f(x, y) + g(y, z) + h(x, z)$$

where f, g, and h are all nonzero and none of them is additively separable.
The only option fitting this description is Option (C).
As for the other options:
- Option (A): Both F_{xz} and F_{yz} are zero. This is because neither x nor y interacts with z.
- Option (B): F_{xz} is zero, because there is no interaction between x and z.
- Option (D): F is completely additively separable in terms of x, y, and z, so all the second-order mixed partials F_{xy}, F_{xz}, and F_{yz} are zero.
- Option (E): $F_{xyz} = 1$.

Performance review: 16 out of 27 got this. 7 chose (E), 3 chose (B), 1 chose (A).
Historical note (last time): 20 out of 29 people got this. 8 chose (E), 1 chose (D).

(3) Consider a function of the form $F(x, y) := Ca^x b^y$ where C, a, b are all positive reals that serve as parameters and x, y are restricted to the positive reals. We wish to study F using the paradigm of linear functions. What is the best way of doing this?
(A) Express $\ln(F(x, y))$ in terms of $\ln x$ and $\ln y$
(B) Express $\ln(F(x, y))$ in terms of x and y
(C) Express $F(x, y)$ in terms of $\ln x$ and $\ln y$
(D) Express $\ln(F(x, y))$ in terms of a^x and b^y
(E) Express $F(x, y)$ in terms of a^x and b^y
Answer: Option (B)
Explanation: We take logarithms to get:

$$\ln(F(x, y)) = \ln C + x \ln a + y \ln b$$

Note that a, b, and C are positive constants. Hence, $\ln C$, $\ln a$, and $\ln b$ are also constants. Thus, $\ln(F(x, y))$ is a linear function of x and y.

Performance review: 21 out of 27 got this. 4 chose (A), 2 chose (D).
Historical note (last time): 26 out of 29 got this. 3 chose (A).

(4) Consider a function of the form $F(x,y) := Cx^a y^b$ where C, a, b are all positive reals that serve as parameters and x, y are restricted to the positive reals. We wish to study F using the paradigm of linear functions. What is the best way of doing this?
(A) Express $\ln(F(x,y))$ in terms of $\ln x$ and $\ln y$
(B) Express $\ln(F(x,y))$ in terms of x and y
(C) Express $F(x,y)$ in terms of $\ln x$ and $\ln y$
(D) Express $\ln(F(x,y))$ in terms of x^a and y^b
(E) Express $F(x,y)$ in terms of x^a and y^b
Answer: Option (A)
Explanation: We take logarithms to get:

$$\ln(F(x,y)) = \ln C + a \ln x + b \ln y$$

Note that C is a positive constant, so $\ln C$ is a constant. $\ln(F(x,y))$ is a linear function of $\ln x$ and $\ln y$.

Performance review: 23 out of 27 got this. 1 each chose (B), (C)., and (D), 1 left the question blank.

Historical note (last time): 28 out of 29 got this. 1 chose (B).

(5) (**) *This is a hard question!* The population in the island of Andrognesia as a function of time is believed to be an exponential function. On January 1, 1984, the population was measured to be $3 * 10^5$ with a measurement error of up to 10^5 on either side, i.e., the population was measured to be between $2 * 10^5$ and $4 * 10^5$. On January 1, 1998, the population was measured to be $1.2 * 10^6$ with a measurememt error of up to $4 * 10^5$ on either side, i.e., the population was measured to be between $8 * 10^5$ and $1.6 * 10^6$. If the population is an exponential function of time (i.e., the increment in population per year is a fixed proportion of the population that year), what is the **range of possible values** of the population measured on January 1, 2012? *Hint: Think of the umbral versus penumbral region for an eclipse*
(A) Between $3.2 * 10^6$ and $6.4 * 10^6$
(B) Between $3.2 * 10^6$ and $1.28 * 10^7$
(C) Between $1.6 * 10^6$ and $3.2 * 10^6$
(D) Between $1.6 * 10^6$ and $6.4 * 10^6$
(E) Between $1.6 * 10^6$ and $1.28 * 10^7$
Answer: Option (E)
Explanation: Note first that $2012 - 1998 = 1998 - 1984 = 14$.

The key idea is that the lowest estimate occurs if the 1998 population was measured as low as possible *and* the rate of population growth estimated using the 1984 and 1998 populations is as low as possible. The lowest possible rate of growth we can measure occurs if we choose the highest possible 1984 value and the lowest possible 1998 value. Picking these, we obtain that the population estimate for 1984 is $4 * 10^5$ and the population estimate for 1998 is $8 * 10^5$. Since the multiplicative growth of the population depends on the time elapsed, the total population in 2012 will be the solution x to:

$$\frac{x}{8 * 10^5} = \frac{8 * 10^5}{4 * 10^5}$$

which solves to $x = 1.6 * 10^6$.

Similarly, the highest estimate will occur if we take the highest estimate possible for the 1998 population and the lowest estimate possibly for the 1984 population.

This relates to the idea of linear models as follows. Consider a plot of the logarithm of the population with respect to time. Since the growth is exponential, this should be a linear plot. If we knew the precise values of the populations in 1984 and 1998, we could fit a straight line through those and use that to determine the population in 2012. Uncertainty regarding the values of the population in 1984 and 1998, however. means that instead of having points in the graph, we have an interval (represented by a vertical line segment) for the time coordinate value of 1984 and another interval (represented by another vertical line segment) for the time coordinate value of 1998.

The upper end estimate is obtained by making a line through the lower end of the 1984 estimate range and the upper end of the 1998 estimate range. The lower end estimate is obtained by making a line through the upper end of the 1984 estimate range and the lower end of the 1998 estimate range.

Performance review: 9 out of 27 got this. 14 chose (A), 3 chose (D), 1 chose (C).

Historical note (last time): 6 out of 29 got this. 15 chose (A), 6 chose (C), 1 each chose (B) and (D).

(6) Suppose, according to our model, a particular function $f(x, y)$ is of the form $f(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 y^2$ where a_1, a_2, a_3, a_4 are parameters. Our goal is to determine the values of the parameters a_1, a_2, a_3, a_4. We do this by collecting a number of (input,output) pairs for the function f and then setting up equations in terms of the parameters using the (input,output) pairs. What can we say about the nature of f and the nature of the system of equations that we will need to solve? *Note that "nonlinear" as used here simply means that the expression is not guaranteed to be linear, though it may turn out to be linear in some cases. Similarly, "non-polynomial" means not guaranteed to be polynomial, though it may turn out to be polynomial in some cases.*

(A) f is a linear function of x and y, hence we need to solve a linear system of equations to determine the parameters a_1, a_2, a_3, a_4.
(B) f is a nonlinear polynomial function of x and y, hence we need to solve a nonlinear polynomial system of equations to determine the parameters a_1, a_2, a_3, a_4.
(C) f is a linear function of x and y. However, we need to solve a nonlinear polynomial system of equations to determine the parameters a_1, a_2, a_3, a_4.
(D) f is a nonlinear polynomial function of x and y. However, we need to solve a linear system of equations to determine the parameters a_1, a_2, a_3, a_4.
(E) f is a nonlinear polynomial function of x and y. However, we need to solve a non-polynomial system of equations to determine the parameters a_1, a_2, a_3, a_4.

Answer: Option (D)

Explanation: f is polynomial in x and y, and it is not linear because one of its terms is $x^2 y^2$ (note that it may *happen* to be linear if $a_4 = 0$, but we do not know this in advance).

However, f is linear in the parameters, hence the system of equations that we get from input-output pairs is a linear system of equations.

Performance review: 19 out of 27 got this. 6 chose (B), 1 each chose (C) and (E).

Historical note (last time): 24 out of 29 got this. 4 chose (B), 1 chose (A).

(7) Consider the system of equations:

$$\begin{aligned} x^2 - x &= 2 \\ y^2 + xy &= x + 13 \end{aligned}$$

What is the number of solutions to this system for real x and y?

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8

Answer: Option (C)

Explanation: Solving the first equation, we get:

$$(x - 2)(x + 1) = 0$$

Thus, we have $x = 2$ or $x = -1$.

For each choice of x, we need to solve the second equation by plugging in that value of x.

For the choice $x = 2$, we have:

$$y^2 + 2y = 15$$

This simplifies to:

$$(y-3)(y+5) = 0$$

Thus, $y = 3$ or $y = -5$, so we get the solutions $x = 2, y = 3$ and $x = 2, y = -5$.

For the choice $x = -1$, we get:

$$y^2 - y = 12$$

This gives:

$$(y-4)(y+3) = 0$$

Thus, $y = 4$ or $y = -3$, so we get the solutions $x = -1, y = 4$ and $x = -1, y = -3$.

Overall, we have four solutions:

- $x = 2, y = 3$
- $x = 2, y = -5$
- $x = -1, y = 4$
- $x = -1, y = -3$

Performance review: 26 out of 27 got this. 1 chose (D).

Historical note (last time): 28 out of 29 got this. 1 chose (D).

(8) Consider the system of equations:

$$\begin{aligned} x^2 - x &= 2 \\ y^2 + xy &= x + 13 \\ z^2 &= xy \end{aligned}$$

What is the number of solutions to this system for real x, y, and z?

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8

Answer: Option (C)

Explanation: Recall from the preceding question that we have the following solutions to the first two equations:

- $x = 2, y = 3$
- $x = 2, y = -5$
- $x = -1, y = 4$
- $x = -1, y = -3$

For each of these, our goal is to find the corresponding z-values that work. Note that if xy is positive, there are two z-values. If $xy = 0$, there is a unique z-value, and if xy is negative, there are no z-values.

Of the four possible (x, y)-value pairs, only two give positive products. The other two give negative products. In both the positive product cases, we get 2 values of z, so we overall get four solutions as listed below:

- $x = 2, y = 3, z = \sqrt{6}$
- $x = 2, y = 3, z = -\sqrt{6}$
- $x = -1, y = -3, z = \sqrt{3}$
- $x = -1, y = -3, z = -\sqrt{3}$

Performance review: 11 out of 27 got this. 10 chose (B), 3 chose (D), 2 chose (E), 1 chose (A).

Historical note (last time): 25 out of 29 got this. 2 chose (D), 1 each chose (B) and (E).

(9) Consider the system of equations:

$$\begin{aligned} x^2 - x &= 2 \\ y^2 + xy &= x + 13 \\ z^2 &= x^2 - y^2 \end{aligned}$$

What is the number of solutions to this system for real x, y, and z?

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8

Answer: Option (A)

Explanation: The solutions to the first two equations are:

- $x = 2, y = 3$
- $x = 2, y = -5$
- $x = -1, y = 4$
- $x = -1, y = -3$

In all cases, $x^2 - y^2 < 0$. Thus, there is no z-value that works in any of the cases. Thus, there are no solutions to this system.

Performance review: 20 out of 27 got this. 5 chose (E), 1 each chose (C) and (D).

Historical note (last time): 22 out of 29 got this. 3 chose (D), 2 chose (C), 1 each chose (B) and (E).