HOMEWORK 5 CHECKLIST: DUE MONDAY NOVEMBER 4

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 2.2.2 (Page 71): Find the matrix of a rotation through an angle of 60° in the counterclockwise direction.

This is a 2×2 matrix. The first column of the matrix describes the image of the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under the rotation. Think of unit circle trigonometry. The second column of the matrix describe the image of the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under the rotation. Think again of unit circle trigonometry.

(2) Exercise 2.2.6 (Page 71): Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

We can do this in many ways. The standard approach, which you already saw in Math 195 (multivariable calculus) (if you took that class) is to use the fact that for two vectors $\vec{v}$ and $\vec{w}$, we have:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. The projection of $\vec{v}$ on the line of $\vec{w}$ is $|\vec{v}| \cos \theta$ times a unit vector in the direction of $\vec{w}$. This simplifies to:

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$$

(3) Exercise 2.2.7 (Page 71): Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.

For any vector, the sum of the vector and its reflection equals *twice* the orthogonal projection. Thus, the reflection can be given as:

Reflection = Twice (orthogonal projection) - (Original vector)

For the next five questions, find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ described.

For all these questions, adopt the following approach: try to determine through direct reasoning where each of the standard basis vectors gets sent. Write these as the columns of the matrix.

(4) Exercise 2.2.19 (Page 72): The orthogonal projection onto the xy-plane.

Projecting onto a plane leaves the vectors that are already in the plane invariant. Thus, $\vec{e}_1$ gets sent to $\vec{e}_1$ and ...

Projecting onto a plane sends vectors perpendicular to the plane to the zero vector.

(5) Exercise 2.2.20 (Page 72): The reflection about the xz-plane.

Reflecting about a plane leaves the vectors that are already in the plane invariant. It sends vectors perpendicular to the plane to their negatives.

(6) Exercise 2.2.21 (Page 72): The rotations about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

The axis about which the rotation is happening doesn't move. In the plane perpendicular to it, we get a rotation matrix.

The final matrix looks block diagonal, with one 2×2 block describing a rotation by $\pi/2$ in the xy-plane, and the other 1×1 block describing an identity matrix for the z-axis.

(7) Exercise 2.2.22 (Page 72): The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

Similar to the preceding question. Note that the final answer will not look like a block diagonal matrix because the standard basis vectors appear in a bad order. But it'll have a similar look and feel.

Update: There's a little subtlety here that I missed, namely, there is an orientation issue, so the matrix restricted to x and z (first and third row, column) would actually be the rotation matrix for $-\theta$ rather than for θ. I'll ask the grader to overlook this subtlety while grading this homework because we didn't get adequate time to cover this issue in class.

(8) Exercise 2.2.23 (Page 72): The reflection about the plane $y = z$.

The x-axis remains fixed. The restriction to the yz-plane looks like the reflection about the *line* $y = z$. The matrix we obtain is block diagonal, with the top 1×1 block the identity matrix, and the bottom 2×2 matrix corresponding to the reflection about the $y = z$ line in the yz-plane.

2. Problems for your own review, not for submission

(1) Exercise 2.2.27 (Page 72): Please see this from the book.

(2) Exercise 2.2.28 (Page 72-73): Please see this from the book.

(3) Exercise 2.4.42 (Page 98): A square matrix is called a *permutation matrix* if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are I_n and

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

3. Advanced problems

(1) Exercise 2.2.29 (Page 73): Let T be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, and L be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that $L(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$ and $T(L(\vec{y})) = \vec{y}$ for all $\vec{y}$ in $\mathbb{R}^n$. If T is a linear transformation, show that L is as well. [*Hint*: $\vec{v} + \vec{w} = T(L(\vec{v})) + T(L(\vec{w})) = T(L(\vec{v}) + L(\vec{w}))$ since T is linear. Now apply L on both sides.]

The hint more or less says it.

(2) Exercise 2.2.37 (Page 73): The *trace* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the sum $a + d$ of its diagonal entries. What can you say about the trace of a 2×2 matrix that represents a/an

(a) orthogonal projection

It might be helpful to use that the matrix for orthogonal projection onto the axis of a unit vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ with $u_1^2 + u_2^2 = 1$ is:

$$\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

Note that this formula arises by interpreting in matrix terms the expression obtained in problem 2 of the routine homework. Explicitly, if we let $\vec{u} = \vec{w}/|\vec{w}|$, we obtain the formula.

(b) reflection about a line
Recall that
Reflection = 2(Orthogonal projection) - (Original vector)
Thus:
Matrix for reflection = 2(Matrix for orthogonal projection) - (Identity matrix)
Trace is itself a linear functional (i.e., trace of sum is sum of traces, etc.) so:
Trace of matrix for reflection = 2(Trace of matrix for orthogonal projection) - (Trace of identity matrix)
The trace of the identity matrix is 2, and we can use part (a) to find the trace of the matrix for orthogonal projection. Now, subtract, and you have the answer.
You might wish to take a simple reflection, such as reflection about the x-axis, as a sanity check. That will not, however, constitute a proof or a confirmation of the fact in *general*.

(c) rotation
The trace here depends on the angle of rotation, and is obtained using a trigonometric function of that. The matrix for rotation by an angle θ is (this generalizes routine Question 1):

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(d) (horizontal or vertical) shear
The prototypical shear is of this form:

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

Final sanity check: In three of the four cases, you should have given the exact value of the trace, and in one case, give an interval of possible values.

(3) Exercise 2.2.38 (Page 73): The *determinant* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$ (we have seen this quantity in Exercise 2.1.13 already, as Homework 3, Advanced Question 1). What can you say about the determinant of a (2×2) matrix that represents a/an
(a) orthogonal projection
(b) reflection about a line
(c) rotation
(d) (horizontal or vertical) shear
What do your answers tell you about the invertibility of these matrices?
Checklist hint begins: See the hints for the corresponding parts for the preceding question to deduce the nature of the matrices, then use the formula $ad - bc$ to compute the determinant.
Note that a matrix is invertible if and only if its determinant is nonzero.
You can use the following sanity checks:

- The absolute value of the determinant equals the scaling factor for the area. In particular, if the transformation is area-preserving, then the determinant is either ± 1.
- The determinant is positive if the transformation is orientation-preserving, and negative if the transformation is orientation-reversing.

(4) Exercise 2.3.29 (Page 85): Consider the matrix

$$D_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

We know that the transformation $T(\vec{x}) = D_\alpha(\vec{x})$ is a counterclockwise rotation through an angle of α.

(a) For two angles α and β, consider the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the linear transformations $\vec{y} = D_\alpha D_\beta(\vec{x})$ and $\vec{y} = D_\beta D_\alpha(\vec{x})$. Are the two transformations the same?
Yes, the angles of rotation should add up (all angles are modulo 2π). Addition of angles is commutative (duh!).

(b) Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\end{aligned}$$

Note: For the second identity, the use of $\pm$ along with $\mp$ indicates that the $+$ case on the left corresponds to the $-$ case on the right, and the $-$ case on the left corresponds to the $+$ case on the right.
Use that the product of the matrices is the matrix for the sum of the angles $\alpha + \beta$. The $-$ case can be obtained once we also use facts about cos being even and sin being odd.
Please note the pairing convention: When $\pm$ and $\mp$ are used in the same equation, then the $+$ of $\pm$ pairs with the $-$ of $\mp$. Also, the $-$ of $\pm$ pairs with the $+$ of $\mp$. Explicitly, therefore, the cos angle sum formula is:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

(5) Exercise 2.4.32 (Page 98): Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A = A^{-1}$.

The condition $A = A^{-1}$ is more easily framed as saying that A^2 is the identity matrix. This gives a bunch of equations that we need to solve. Solve them.

Alternatively, use the explicit description of the inverse matrix computed in Advanced Homework 3, Question 1 combined with the fact that $ad - bc = 1$.

(6) Exercise 2.4.41 (Page 98): Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ is invertible? Find the inverse if it exists.
  (a) Reflection about a plane
  (b) Orthogonal projection onto a plane
  (c) Scaling by a factor of 5 [i.e., $T(\vec{v}) = 5\vec{v}$ for all vectors $\vec{v}$]
  (d) Rotation about an axis

Invertibility here can be assessed in two ways:
- Convert to rref to see if it has full rank (full rank 3 would mean that the rref is the identity matrix).
- Find explicitly, using direct reasoning, a linear transformation that acts as the inverse of the given linear transformation.

(7) Exercise 2.4.49 (Page 99-100): *Input-Output Analysis*. This is a very lengthy problem. Please see it from the book.

You're on your own here. Please read the book carefully, and feel free to use the Internet to read more about this subject (but the hint in the book should suffice for the specific question you are asked).