TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY FEBRUARY 6: MULTIVARIABLE FUNCTION BASICS CONTINUED

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 0: 4 people
- Score of 1: 6 people
- Score of 2: 9 people
- Score of 3: 5 people
- Score of 4: 2 people

The question-wise answers and performance review were as follows:

(1) Option (C): 12 people
(2) Option (B): 13 people
(3) Option (E): 15 people
(4) Option (B): 7 people

2. Solutions

(1) Suppose F is an additively separable function of two variables x and y that is defined everywhere, i.e., there exist functions f and g of one variable, both defined on all of $\mathbb{R}$, such that $F(x, y) = f(x) + g(y)$ for all $x, y \in \mathbb{R}$.

We call two curves *parallel* if there is a vector by which we can translate all the points in one curve to get precisely the other curve.

Consider the following three statements:

(i) All curves obtained as the intersections of the graph of F with planes parallel to the xy-plane are parallel to each other.

(ii) All curves obtained as the intersections of the graph of F with planes parallel to the xz-plane are parallel to each other.

(iii) All curves obtained as the intersections of the graph of F with planes parallel to the yz-plane are parallel to each other.

Which of the statements (i)-(iii) is/are necessarily true?

(A) All of (i), (ii), and (iii) are true.
(B) Both (i) and (ii) are true but (iii) need not be true.
(C) Both (ii) and (iii) are true but (i) need not be true.
(D) Both (i) and (iii) are true but (ii) need not be true.
(E) (i) is true but (ii) and (iii) need not be true.

Answer: Option (C)

Explanation:

(i): The intersections are level curves, but these need not be parallel to each other. In fact, they could be different sizes, such as with $x^2 + y^2$.

(ii): The intersection with a plane of the form $y = y_0$ gives the graph of a function $x \mapsto F(x, y_0) = f(x) + g(y_0)$. Note that all the functions whose graphs are obtained by such restrictions just look like the graph of f, translated to different z-heights and y-locations. Thus, they are parallel to one another. Note that we use additive separability in this reasoning.

(iii): Similar reasoning as with (ii).

Performance review: 12 out of 26 got this. 7 chose (A), 6 chose (E), 1 chose (B).

(2) Suppose f is a continuous function of two variables x and y, defined on the entire xy-plane. Suppose further that f is increasing in x for each fixed value of y, and that f is increasing in y for every fixed value of x. Which of the following is the most plausible description of the level curves of f in the xy-plane? *Note: You might wish to take an extremely simple example, e.g., an additively separable function where each of the pieces is the simplest possible increasing function you can think of.*

(A) They are all upward-sloping, i.e., they are of the form $y = g(x)$ with g an increasing function.
(B) They are all downward-sloping, i.e., they are of the form $y = g(x)$ with g a decreasing function.
(C) They look like closed loops (e.g., circles).
(D) They look like graphs of functions with a unique local and absolute minimum (such as the parabola $y = x^2$, though the actual picture may be different).
(E) They look like graphs of functions with a unique local and absolute maximum (such as the parabola $y = -x^2$, though the actual function may be different).

Answer: Option (B)

Explanation: Since f is increasing in x and in y, it means that an increase in the x-value must be compensated by a decrease in the y-value to keep the output constant.

The example $f(x, y) := x + y$ is illustrative. The level curves of these are downward-sloping straight lines.

Performance review: 13 out of 26 got this. 7 chose (A), 3 each chose (C) and (D).

(3) What do the level curves of the function $f(x, y) := \sin(x + y)$ look like for output value in $[-1, 1]$? Note that all these level curves are being considered as curves in the xy-plane. *Note: This builds upon the idea of Question 3 of the previous quiz.*

(A) Each level curve is a single line.
(B) Each level curve is a union of two intersecting lines.
(C) Each level curve is a union of two distinct parallel lines.
(D) Each level curve is a union of infinitely many concurrent lines (i.e., infinitely many lines, all passing through the same point).
(E) Each level curve is a union of infinitely many distinct parallel lines (i.e., infinitely many lines, all parallel to each other).

Answer: Option (E)

Explanation: For $C \in [-1, 1]$, we need to solve $\sin(x + y) = C$. We first find all solutions u to $\sin u = C$, which is a countably infinite subset of $\mathbb{R}$. Then, for each such u, we get the line $x + y = u$ in the xy-plane. All these lines are parallel to each other, and have slope -1.

Performance review: 15 out of 26 got this. 6 chose (A), 4 chose (D), 1 chose (B).

(4) Suppose f and g are both continuous functions of two variables x and y, both defined on all of $\mathbb{R}^2$, and such that $f(x, y) + g(x, y)$ is a constant C. What is the relation between the level curves of f and the level curves of g, all drawn in the xy-plane?

(A) Every level curve of f is a level curve of g and vice versa, with the same level value for both functions.
(B) Every level curve of f is a level curve of g and vice versa, but the value for which it is a level curve may be different for the two functions.
(C) The level curves of f need not be precisely the same as the level curves of g, but we can go from one set of level curves to the other via a parallel translation.
(D) Each level curve of f can be obtained by reflecting a suitable level curve of g about a suitable line in the xy-plane.
(E) Each level curve of f can be obtained by reflecting a suitable level curves of g about a suitable line in the xy-plane and then performing a suitable translation.

Answer: Option (B)

Explanation: Any level curve of the form $f(x, y) = k$ coincides with the level curve $g(x, y) = C - k$. Note that unless $k = C/2$, the level values for the two curves are different.

Performance review: 7 out of 26 got this. 8 chose (D), 6 chose (C), 3 chose (A), 2 chose (E).