

HYBRID TAKE-HOME PLUS CLASS QUIZ: MARCH 7: TAYLOR SERIES AND POWER SERIES

MATH 153, SECTION 55 (VIPUL NAIK)

Your name (print clearly in capital letters): _____

PLEASE ATTEMPT THIS QUIZ BEFORE CLASS. I WILL GIVE YOU TIME TO REVIEW AND UPDATE YOUR SOLUTIONS IN CLASS, BUT THIS WILL NOT BE SUFFICIENT TO ATTEMPT ALL QUESTIONS FROM SCRATCH.

FEEL FREE TO DISCUSS ALL QUESTIONS, BUT ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE.

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from \mathbb{R} to \mathbb{R} that are *infinitely* differentiable *everywhere* in \mathbb{R} .

We denote by $C^k(\mathbb{R})$ the space of functions from \mathbb{R} to \mathbb{R} that are at least k times continuously differentiable on all of \mathbb{R} . Note that for $k \geq l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k .

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c . We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on \mathbb{R} .

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

- (1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which $\lim_{x \rightarrow c} |f(x)|/|x - c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f ?
Last year: 11/26 correct

- (A) The order of any zero of f must be between 0 and 1.
- (B) The order of any zero of f must be between 1 and 2.
- (C) The order of any zero of f , if finite, must be a positive integer.
- (D) The order of any zero of f must be exactly 1.
- (E) The order of any zero of f must be ∞ .

Your answer: _____

- (2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on \mathbb{R} , what can we say about the Taylor polynomial about 0? *Last year: 8/26 correct*
- (A) No Taylor polynomial is defined for f .
 - (B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
 - (C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
 - (D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
 - (E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

Your answer: _____

- (3) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0? *Last year: 3/26 correct*
- (A) $f_1(x) := \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$
 - (B) $f_2(x) := \begin{cases} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

- (C) $f_3(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
- (D) $f_4(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$
- (E) All of the above.

Your answer: _____

- (4) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic?

Last year: 7/26 correct

- (A) $x \mapsto \ln(1 + x^2)$
- (B) $x \mapsto \ln(1 + x)$
- (C) $x \mapsto \ln(1 - x)$
- (D) $x \mapsto \exp(1 + x)$
- (E) $x \mapsto \exp(1 - x)$

Your answer: _____

- (5) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic? *Last year: 10/26 correct*

- (A) x
- (B) x^2
- (C) $x + 1$
- (D) $x^2 + 1$
- (E) e^x

Your answer: _____

- (6) Consider the rational function $1/\prod_{i=1}^n(x - \alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_i s, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_i s? *Last year: 10/26 correct*

- (A) It is the minimum of the distances from c to the α_i s.
- (B) It is the second smallest of the distances from c to the α_i s.
- (C) It is the arithmetic mean of the distances from c to the α_i s.
- (D) It is the second largest of the distances from c to the α_i s.
- (E) It is the maximum of the distances from c to the α_i s.

Your answer: _____

- (7) What is the interval of convergence of the Taylor series for arctan about 0? *Last year: 11/26 correct*

- (A) $(-1, 1)$
- (B) $[-1, 1)$
- (C) $(-1, 1]$
- (D) $[-1, 1]$
- (E) All of \mathbb{R}

Your answer: _____

- (8) Consider the function $F(x, p) = \sum_{n=1}^{\infty} x^n/n^p$. For fixed p , this is a power series in x . What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0, \infty)$? *Last year: 4/26 correct*

- (A) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
- (B) The interval of convergence is $(-1, 1)$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
- (C) The interval of convergence is $[-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.

- (D) The interval of convergence is $(-1, 1]$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
 (E) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1)$ for $p > 1$.

Your answer: _____

- (9) Which of the following functions of x has a power series $\sum_{k=0}^{\infty} x^{4k}/(4k)!$? *Last year: 9/26 correct*
 (A) $(\sin x + \sinh x)/2$
 (B) $(\sin x - \sinh x)/2$
 (C) $(\sinh x - \sin x)/2$
 (D) $(\cosh x + \cos x)/2$
 (E) $(\cosh x - \cos x)/2$

Your answer: _____

- (10) What is the sum $\sum_{k=0}^{\infty} (-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$. *Last year: 12/26 correct*
 (A) $\cos x$
 (B) $\sin x$
 (C) $\cos(x^2)$
 (D) $\cosh(x^2)$
 (E) $\exp(-x^2)$

Your answer: _____

- (11) Define an operator R from the space of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b , $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$? *Last year: 3/26 correct*
 (A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
 (B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
 (C) $R(a + b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a + b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
 (D) $R(a + b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a + b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
 (E) $R(a + b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a + b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Your answer: _____

- (12) Which of the following is/are true? *Last year: 5/26 correct*
 (A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on \mathbb{R} .
 (B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence (which may not be all of \mathbb{R}).
 (C) If we start with a power series about 0 that converges everywhere in \mathbb{R} , then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
 (D) All of the above.
 (E) None of the above.

Your answer: _____

- (13) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$? *Last year: 12/26 correct*
 (A) $\exp(x)$ from below and $\cosh(2x)$ from above.

- (B) $\exp(x)$ from below and $\cosh(x^2)$ from above.
- (C) $\exp(x/2)$ from below and $\exp(x)$ from above.
- (D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.
- (E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Your answer: _____

- (14) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \rightarrow \infty$?

Last year: 12/26 correct

- (A) f grows polynomially in x .
- (B) f grows faster than any polynomial function but slower than any exponential function of x .
- (C) f grows like an exponential function of x .
- (D) f grows faster than any exponential function but slower than any doubly exponential function of x .
- (E) f grows like a doubly exponential function of x .

Your answer: _____