TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY NOVEMBER 25: SUBSPACE, BASIS, DIMENSION, AND ABSTRACT SPACES: APPLICATIONS TO CALCULUS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

22 people took this 13-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 4: 3 people
- Score of 5: 2 people
- Score of 6: 2 people
- Score of 7: 1 person
- Score of 8: 5 people
- Score of 9: 3 people
- Score of 10: 2 people
- Score of 11: 3 people

The mean score was about 7.41.

The question-wise answers and performance review are as follows:

(1) Option (A): 18 people
(2) Option (A): 15 people
(3) Option (E): 17 people
(4) Option (D): 13 people
(5) Option (C): 15 people
(6) Option (E): 10 people
(7) Option (E): 12 people
(8) Option (E): 13 people
(9) Option (E): 9 people
(10) Option (D): 1 person
(11) Option (C): 15 people
(12) Option (D): 13 people
(13) Option (B): 12 people

REVIEW NOTE: Please make sure to read the corresponding lecture notes on abstract vector spaces rather than simply going over the quiz solutions.

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz builds on the November 8 and November 20 quizzes that apply ideas we are learning about linear transformations to the calculus setting. The November 8 quiz went over some basic ideas related to differentiation as a linear transformation. The November 20 quiz explored the ideas in greater depth. We now look at questions that apply the ideas of basis, dimension, and subspace to the calculus setting.

We begin by recalling some notation and facts we already saw in earlier quizzes. Denote by $C(\mathbb{R})$ (or alternatively by $C^0(\mathbb{R})$) the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with pointwise addition and scalar multiplication. Note that the elements of this vector space, which we would ordinarily call "vectors", are now *functions*.

For k a positive integer, denote by $C^k(\mathbb{R})$ the subspace of $C(\mathbb{R})$ comprising those continuous functions that are at least k times *continuously* differentiable. Note that $C^{k+1}(\mathbb{R})$ is a subspace of $C^k(\mathbb{R})$, so we have a descending chain of subspaces:

$$C(\mathbb{R}) = C^0(\mathbb{R}) \supseteq C^1(\mathbb{R}) \supseteq C^2(\mathbb{R}) \supseteq \dots$$

The intersection of these spaces is the vector space $C^\infty(\mathbb{R})$, defined as the subspace of $C(\mathbb{R})$ comprising those functions that are *infinitely* differentiable.

We had also noted that:

- The kernel of differentiation is the vector space of constant functions.
- The kernel of k times differentiating is the vector space of polynomials of degree at most $k - 1$.
- The fiber of any function for differentiation is a translate of the space of constant functions. That's what explains the $+C$ when you perform indefinite integration.

Note: For finite-dimensional spaces, a linear transformation T from a vector space to itself is injective if and only if it is surjective. This follows from dimension and rank considerations: T is injective if and only its kernel is zero, which happens if and only if the matrix has full column rank, which happens if and only if the matrix has full row rank (because the matrix is a square matrix), which happens if and only if T is surjective. The rank-nullity theorem provides an equivalent explanation. We had also seen that if $T : \mathbb{R}^m \to \mathbb{R}^n$ is injective, then $m \le n$, and if $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective, then $m \ge n$. In particular, we cannot have a surjective map from a proper subspace to the whole space.

With infinite-dimensional spaces, however, we can have funny phenomena. Examples of these phenomena are strewn across the quizzes.

- We can have a map from an infinite-dimensional vector space to itself that is injective but not surjective.
- We can have a map from an infinite-dimensional vector space to itself that is surjective but not injective.
- We can have a surjective map from a proper subspace to the whole space (for instance, differentiation $C^1(\mathbb{R}) \to C(\mathbb{R})$ is surjective, even though $C^1(\mathbb{R})$ is a proper subspace of $C(\mathbb{R})$).
- We can have an injective map from a space to a proper subspace.

Note that we will use the terms *subspace* and *vector subspace* synonymously with *linear subspace* in this quiz.

(1) Suppose V is a vector subspace of the vector space $C^\infty(\mathbb{R})$. We know that differentiation is linear. How is that information computationally useful?
  (A) It tells us that knowing how to differentiate all functions in any spanning set for V tells us how to differentiate any function in V (assuming we know how to express any function in V as a linear combination of the functions in the spanning set).
  (B) It tells us that knowing how to differentiate all functions in any linearly independent set in V tells us how to differentiate any function in V.

  Answer: Option (A)

  Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

  Given the knowledge of the derivatives of all functions in a spanning set for V, we can differentiate any function in V as follows: first, express it as a linear combination of the functions in the spanning set. Now, use the linearity of differentiation to express its derivative as the corresponding linear combination of the derivatives.

  For instance, suppose we know that the derivative of sin is cos and the derivative of exp is exp. Then the derivative of the function:

$$f(x) = 2 \sin x + 5 \exp(x)$$

  is:

$$f'(x) = 2\sin' x + 5\exp'(x) = 2\cos x + 5\exp(x)$$

Note that it is the fact of the functions *spanning* V that is crucial in allowing us to be able to write *any* function in V as a linear combination of the functions.

Performance review: 18 out of 22 people got this. 4 chose (B).

Historical note (last time): 20 out of 26 got this. 6 chose (B).

(2) Suppose V is a vector subspace of the vector space $C^\infty(\mathbb{R})$. We know that differentiation is linear. How is that information computationally useful?

(A) It tells us that knowing the antiderivatives of all functions in any spanning set for V tells us the antiderivative of every function in V (assuming we know how to express any function in V as a linear combination of the functions in the spanning set).

(B) It tells us that knowing the antiderivatives of all functions in any linearly independent set in V tells us the antiderivative of every function in V.

Answer: Option (A)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

The reasoning is similar to that for differentiation, except that we put in the obligatory $+C$ of indefinite integration to account for the fact that the kernel of differentiation is the one-dimensional space of constant functions.

For instance, suppose we know that an antiderivative of sin is $-\cos$ and an antiderivative of exp is exp. Then, the indefinite integral of the function:

$$f(x) = 2\sin x + 5\exp(x)$$

is:

$$\int f(x)\,dx = 2(-\cos x) + 5\exp x + C$$

Performance review: 15 out of 22 people got this. 7 chose (B).

Historical note (last time): 23 out of 26 got this. 3 chose (B).

We now consider two related vector spaces. $\mathbb{R}[x]$ is defined as the vector space of polynomials with real coefficients in the single variable x, with the usual addition and scalar multiplication. There is a natural injective homomorphism from $\mathbb{R}[x]$ to $C^\infty(\mathbb{R})$ that sends any polynomial to the same polynomial viewed as a function.

$\mathbb{R}(x)$ is defined as the vector space of all rational functions where the numerator and denominator are both polynomials with the denominator nonzero, up to equivalence (i.e., two rational functions $p_1(x)/q_1(x)$ and $p_2((x)/q_2(x)$ are equivalent if $p_1(x)q_2(x) = q_1(x)p_2(x)$). Addition and scalar multiplication are defined the usual way. Note that there is a natural injective homomorphism from $\mathbb{R}[x]$ to $\mathbb{R}(x)$ that sends any polynomial $p(x)$ to the rational function $p(x)/1$.

Also note that $\mathbb{R}(x)$ does not map to $C^\infty(\mathbb{R})$, for the reason that a rational function, viewed *qua* function, is not necessarily defined everywhere. Specifically, if written in simplified form, it is not defined at the set of roots of its denominator.

Note that both $\mathbb{R}[x]$ and $\mathbb{R}(x)$ are infinite-dimensional vector spaces, i.e., they do not have finite spanning sets.

(3) Which of the following is *not* a basis for $\mathbb{R}[x]$? Please see Option (E) before answering.

(A) $1, x, x^2, x^3, \dots$

(B) $1, x, x(x-1), x(x-1)(x-2), x(x-1)(x-2)(x-3), \dots$

(C) $1, x+1, x^2+x+1, x^3+x^2+x+1, \dots$

(D) $1, x, x^2-x, x^3-x^2, x^4-x^3, \dots$

(E) None of the above, i.e., each of them is a basis.

Answer: Option (E)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

Option (A) clearly is a basis: polynomials are by definition linear combinations of $1, x, x^2, \dots$ and the method of expressing a polynomial as a linear combination is unique. All the other options are equivalent to Option (A) in the following sense: if we think of how the span grows as we go from left to right, it's the same in all options. In each option, the span of the first n vectors is the same as the span of $1, x, x^2, \dots, x^{n-1}$. In particular, in all options, there are no redundant vectors, and the span of all vectors together is all of $\mathbb{R}[x]$. In other words, each option gives a basis.

Performance review: 17 out of 22 people got this. 2 each chose (B) and (C), 1 chose (D).

Historical note (last time): 16 out of 26 got this. 6 chose (C), 2 chose (D), 1 each chose (A) and (B).

Let's now revisit the topic of *partial fractions* as a tool for integrating rational functions. The idea behind partial fractions is to consider an integration problem with respect to a variable x with integrand of the following form:

$$\frac{a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}}{p(x)}$$

where p is a polynomial of degree n. For convenience, we may take p to be a monic polynomial, i.e., a polynomial with leading coefficient 1. For p fixed, the set of all rational functions of the form above forms a vector subspace of dimension n inside $\mathbb{R}(x)$. A natural choice of basis for this subspace is:

$$\frac{1}{p(x)}, \frac{x}{p(x)}, \dots, \frac{x^{n-1}}{p(x)}$$

The goal of partial fraction theory is to provide an *alternate basis* for this space of functions with the property that those basis elements are particularly easy to integrate (recurring to one of our earlier questions). Let's illustrate one special case: the case that p has n distinct real roots $\alpha_1, \alpha_2, \dots, \alpha_n$. The alternate basis in this case is:

$$\frac{1}{x - \alpha_1}, \frac{1}{x - \alpha_2}, \dots, \frac{1}{x - \alpha_n}$$

The explicit goal is to rewrite a partial fraction:

$$\frac{a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}}{p(x)}$$

in terms of the basis above. If we denote the numerator as $r(x)$, we want to write:

$$\frac{r(x)}{p(x)} = \frac{c_1}{x - \alpha_1} + \frac{c_2}{x - \alpha_2} + \dots + \frac{c_n}{x - \alpha_n}$$

The explicit formula is:

$$c_i = \frac{r(\alpha_i)}{\prod_{j \neq i} (\alpha_i - \alpha_j)}$$

Once we rewrite the original rational function as a linear combination of the new basis vectors, we can integrate it easily because we know the antiderivatives of each of the basis vectors. The antiderivative is thus:

$$\left(\sum_{i=1}^{n} \frac{r(\alpha_i)}{\prod_{j \neq i}(\alpha_i - \alpha_j)} \ln|x - \alpha_i|\right) + C$$

where the obligatory $+C$ is put for the usual reasons.

Note that this process only handles rational functions that are proper fractions, i.e., the degree of the numerator must be less than that of the denominator.

We now consider cases where p is a polynomial of a different type.

(4) Suppose p is a monic polynomial of degree n that is a product of pairwise distinct irreducible factors that are all either monic linear or monic quadratic. Call the roots for the linear polynomials $\alpha_1, \alpha_2, \dots, \alpha_s$ and call the monic quadratic factors $q_1, q_2, \dots, q_t$. Which of the following sets forms a basis for the vector space that we are interested in, namely all rational functions of the form $r(x)/p(x)$ where the degree of r is less than n? Please see Option (E) before answering.

(A) All rational functions of the form $1/(x - \alpha_i), 1 \le i \le s$ together with all rational functions of the form $1/q_j(x), 1 \le j \le t$

(B) All rational functions of the form $1/(x - \alpha_i), 1 \le i \le s$ together with all rational functions of the form $q_j'(x)/q_j(x), 1 \le j \le t$

(C) All rational functions of the form $1/q_j(x), 1 \le j \le t$ together with all rational functions of the form $q_j'(x)/q_j(x), 1 \le j \le t$

(D) All rational functions of the form $1/(x - \alpha_i), 1 \le i \le s$ together with all rational functions of the form $1/q_j(x), 1 \le j \le t$ *and* all rational functions of the form $q_j'(x)/q_j(x), 1 \le j \le t$

(E) None of the above

Answer: Option (D)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

This should be familiar to you from the halcyon days of doing partial fractions. For instance, consider the example where $p(x) = (x - 1)(x^2 + x + 1)$. In this case, the basis is:

$$\frac{1}{x-1}, \frac{1}{x^2+x+1}, \frac{2x+1}{x^2+x+1}$$

Note that an easy sanity check is that the *size* of the basis should be n. This is clear in the above example with $n = 3$, but let's reason generically.

We have that:

$$p(x) = \left[\prod_{i=1}^{s}(x - \alpha_i)\right]\left[\prod_{j=1}^{t} q_j(x)\right]$$

By degree considerations, we get that:

$$s + 2t = n$$

Now, the vector space for which we are trying to obtain a basis has dimension n. This means that the basis we are looking for should have size n. Of the given options, Option (D) (which gives one basis element for each of the s linear factors and two basis elements for each of the t quadratic factors) is the most attractive.

Also recall that the reciprocals of the linear factors integrate to logarithms. The expressions of the form $1/q_j(x)$ integrate to an expression involving arctan. The expressions of the form $q_j'(x)/q_j(x)$ integrate to logarithms.

Performance review: 13 out of 22 people got this. 3 each chose (A) and (B), 2 chose (E), 1 chose (C).

Historical note (last time): 4 out of 26 got this. 13 chose (A), 5 chose (B), and 4 chose (C).

(5) Suppose $p(x) = (x - \alpha)^n$. Which of the following sets forms a basis for the vector space that we are interested in, namely all rational functions of the form $r(x)/p(x)$ where the degree of r is less than n? Please see Options (D) and (E) before answering.

(A) The single function $1/(x - \alpha)$

(B) The single function $1/(x-\alpha)^n$
(C) All the functions $1/(x-\alpha), 1/(x-\alpha)^2, \dots, 1/(x-\alpha)^n$
(D) Any of the above works
(E) None of the above works

Answer: Option (C)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

This is the obvious choice by size considerations. It also makes sense based on what you remember about partial fractions.

Performance review: 15 out of 22 people got this. 4 chose (D), 2 chose (B), 1 chose (A).

Historical note (last time): 14 out of 26 got this. 8 chose (D), 3 chose (B), and 1 chose (E).

We now recall our earlier discussion of the solution process for first-order linear differential equations. Consider a first-order linear differential equation with independent variable x and dependent variable y, with the equation having the form:

$$y' + p(x)y = q(x)$$

where $p, q \in C^\infty(\mathbb{R})$.

We solve this equation as follows. Let H be an antiderivative of p, so that $H'(x) = p(x)$.

$$\frac{d}{dx}\left(ye^{H(x)}\right) = q(x)e^{H(x)}$$

This gives:

$$ye^{H(x)} = \int q(x)e^{H(x)}\,dx$$

So:

$$y = e^{-H(x)} \int q(x)e^{H(x)}\,dx$$

The indefinite integration gives a $+C$, so overall, we get:

$$y = Ce^{-H(x)} + \text{particular solution}$$

It's now time to understand this in terms of linear algebra.

Define a linear transformation $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ as:

$$f(x) \mapsto f'(x) + p(x)f(x)$$

(6) The kernel of L is one-dimensional. Which of the following functions spans the kernel?
(A) $p(x)$
(B) $q(x)$
(C) $H(x)$
(D) $e^{H(x)}$
(E) $e^{-H(x)}$

Answer: Option (E)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 7**.

This is pretty obvious from the description of the general solution above. In particular, set $q(x) = 0$ and get the generic element of the kernel as:

$$y = Ce^{-H(x)}, C \in \mathbb{R}$$

This is spanned by $e^{-H(x)}$.

Performance review: 10 out of 22 people got this. 4 each chose (A) and (B), 2 each chose (C) and (D).

Historical note (last time): 17 out of 26 got thus. 5 chose (C), 2 each chose (A) and (B).

(7) I would like to argue that L is *surjective* as a linear transformation from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$. Why is that true?
 (A) The kernel of L is zero-dimensional.
 (B) The image of L is zero-dimensional.
 (C) The kernel of L is one-dimensional.
 (D) The image of L is one-dimensional.
 (E) For any q, we have a formula above that describes a solution function that maps to q.

Answer: Option (E)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 7**.

Obvious! Note that Option (C), while correct, is not an *explanation* for the surjectivity of the linear transformation, because the dimension of the kernel speaks to how far the linear transformation is from being injective, and, particularly in the infinite-dimensional case, does not provide any direct information regarding whether or not the linear transformation is surjective.

Performance review: 12 out of 22 people got this. 8 chose (C), 2 chose (A).

Historical note (last time): 2 out of 26 got this. 12 chose (C), 7 chose (D), 3 chose (A), 2 chose (B).

Let n be a nonnegative integer. Denote by P_n the vector space of all polynomials in one variable x that have degree $\leq n$. P_n is a subspace of $\mathbb{R}[x]$, which in turn can be viewed as a subspace of $C^\infty(\mathbb{R})$ through the natural injective map. For convenience and completeness, define P_{-1} to be the zero subspace.

Differentiation defines a linear transformation from $C^\infty(\mathbb{R})$ to itself.

(8) What are the kernel and image of the restriction of differentiation to P_n? The result should be valid for all positive integers n.
 (A) The kernel and image are both P_n
 (B) The kernel is the zero subspace and the image is P_n
 (C) The kernel is P_n and the image is the zero subspace
 (D) The kernel is P_{n-1} and the image is P_0 (the subspace of constant functions)
 (E) The kernel is P_0 and the image is P_{n-1}

Answer: Option (E)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

The only functions that differentiate to 0 are the constant functions, which is P_0. The derivative of any polynomial of degree $\leq n$ is a polynomial of degree $\leq n-1$. Further, *every* polynomial of degree $\leq n-1$ arises as the derivative of a polynomial of degree $\leq n$. So the image is P_{n-1}.

Performance review: 13 out of 22 people got this. 5 chose (B), 3 chose (D), 1 chose (C).

Historical note (last time): 8 out of 26 got this. 8 chose (D), 6 chose (B), 2 each chose (A) and (C).

(9) What are the kernel and image of the restriction of differentiation to all of $\mathbb{R}[x]$?
 (A) The kernel and image are both $\mathbb{R}[x]$
 (B) The kernel is the zero subspace and the image is $\mathbb{R}[x]$
 (C) The kernel is $\mathbb{R}[x]$ and the image is the zero subspace
 (D) The kernel is $\mathbb{R}[x]$ and the image is P_0 (the subspace of constant functions)
 (E) The kernel is P_0 and the image is $\mathbb{R}[x]$

Answer: Option (E)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

The only functions that differentiate to the zero function are the constant functions, so the kernel is P_0. The image is all of $\mathbb{R}[x]$, because every polynomial arises as the derivative of some polynomial.

Performance review: 9 out of 22 people got this. 9 chose (B), 2 chose (D), 1 chose (A), 1 left the question blank.

Historical note (last time): 9 out of 26 got this. 8 chose (B), 7 chose (D), 2 chose (C).

(10) We can use differentiation to define a linear transformation from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, where we differentiate a rational function using the quotient rule for differentiation and the known rules for differentiating polynomials. What can we say about this linear transformation?

(A) The differentiation linear transformation is bijective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of a unique rational function.

(B) The differentiation linear transformation is injective but not surjective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of *at most one* rational function, but there do exist rational functions that are not expressible as the derivative of any rational function.

(C) The differentiation linear transformation is surjective but not injective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of *at least one* rational function, but there do exist rational functions that occur as derivatives of more than one rational function.

(D) The differentiation linear transformation is neither injective nor surjective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$.

Answer: Option (D)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

Differentiation is not injective because it has a nonzero kernel comprising constants. It is not surjective because there exist rational functions, such as $1/(x^2+1)$, that have no rational function antiderivative: the indefinite integral of $1/(x^2+1)$ is of the form $(\arctan x) + C$, $C \in \mathbb{R}$, and no possible function here is a rational function.

Performance review: 1 (?????) out of 22 people got this. 14 chose (C), 6 chose (B), 1 chose (A).

Historical note (last time): 16 out of 26 got this. 5 chose (B), 2 chose(C), 1 chose (A), 1 chose (E) (????), 1 non-attempt.

(11) Denote by $\mathbb{R}[[x]]$ the vector space of all formal power series with real coefficients in one variable, i.e., series of the form:

$$\sum_{i=0}^{\infty} a_i x^i$$

Formal differentiation defines a linear transformation from $\mathbb{R}[[x]]$ to itself. What can we say about this linear transformation?

(A) The formal differentiation linear transformation is bijective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.

(B) The formal differentiation linear transformation is injective but not surjective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.

(C) The formal differentiation linear transformation is surjective but not injective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.

(D) The formal differentiation linear transformation is neither injective nor surjective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.

Answer: Option (C)

Explanation: **MAKE SURE TO READ THE LECTURE NOTES ON ABSTRACT VECTOR SPACES, SECTION 5**.

The linear transformation is surjective because every formal power series can be integrated termwise to obtain another formal power series (note that we have flexibility in choosing the constant term). It is not injective, because constant formal power series are in the kernel.

Performance review: 15 out of 22 people got this. 5 chose (A), 2 chose (D).

Historical note (last time): 17 out of 26 got this. 3 each chose (A), (B), and (D).

(12) Consider the following two linear transformations $T_1, T_2 : \mathbb{R}[x] \to \mathbb{R}[x]$: T_1 is differentiation, and T_2 is multiplication by x. Which of the following is true?

(A) Both T_1 and T_2 are injective, but neither is surjective.
(B) Both T_1 and T_2 are surjective, but neither is injective.
(C) T_1 is injective but not surjective. T_2 is surjective but not injective.
(D) T_1 is surjective but not injective. T_2 is injective but not surjective.
(E) Neither T_1 nor T_2 is injective. Neither T_1 nor T_2 is surjective.

Answer: Option (D)

Explanation: We already discussed that the image of T_1 (differentiation) is all of $\mathbb{R}[x]$, and that it is not injective because it has constants in its kernel. T_2 is injective because $xp(x) = xq(x) \implies p(x) = q(x)$. However, it is not surjective because its image comprises only the polynomials with zero constant term.

Performance review: 13 out of 22 people got this. 4 chose (C), 3 chose (B), 2 chose (A).

Historical note (last time): 14 out of 26 got this. 7 chose (C), 3 chose (B), and 2 chose (E).

(13) Consider the linear transformations T_1 and T_2 of the preceding question. What can we say regarding whether T_1 and T_2 commute?

(A) T_1 and T_2 commute.
(B) T_1 and T_2 do not commute.

Answer: Option (B)

Explanation: Consider the input x. $T_1(T_2(x)) = T_1(x^2) = 2x$. On the other hand, $T_2(T_1(x)) = T_2(1) = x$.

Performance review: 12 out of 22 people got this. 10 chose (A).

Historical note (last time): 16 out of 26 got this. 7 chose (A), 3 chose (E) (?????).