CLASS QUIZ SOLUTIONS: NOVEMBER 23: MEMORY LANE

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 2: 5 people
- Score of 3: 3 people
- Score of 4: 2 people
- Score of 6: 1 person
- Score of 7: 1 person

The mean score was 3.33. The problem wise answers and performance are as follows:

(1) Option (B): 5 people
(2) Option (E): 7 people
(3) Option (D): 9 people
(4) Option (C): 7 people
(5) Option (D): 5 people
(6) Option (D): 2 people
(7) Option (A): 5 people

2. Solutions

(1) For which of the following specifications is there **no continuous function** satisfying the specifications?
(A) Domain $[0, 1]$ and range $[0, 1]$
(B) Domain $[0, 1]$ and range $(0, 1)$
(C) Domain $(0, 1)$ and range $[0, 1]$
(D) Domain $(0, 1)$ and range $(0, 1)$
(E) None of the above, i.e., we can get a continuous function for each of the specifications.

Answer: Option (B)

Explanation: By the extreme value theorem, any continuous function on a closed bounded interval must attain its maximum and minimum, and hence its image cannot be an open interval.

The other choices:

For options (A) and (D), we can pick the identity functions $f(x) := x$ on the respective domains.

For option (C), we can pick the function $f(x) := \sin^2(2\pi x)$ on the domain $(0, 1)$.

Performance review: 5 out of 12 got this. 6 chose (C), 1 chose (D).

Historical note (last year): 7 out of 14 people got this correct. 5 people chose (C) and 2 people chose (E).

Action point: Any question that involves feasible options for the range of a function should remind you of the *intermediate value theorem* and *extreme value theorem*. It seems likely that the people who got this question wrong (and perhaps some of the point who got it right too!) did not even think of the extreme value theorem.

(2) Suppose f and g are continuous functions on $\mathbb{R}$, such that f is continuously differentiable everywhere and g is continuously differentiable everywhere except at c, where it has a vertical tangent. What can we say is **definitely true** about $f \circ g$?
(A) It has a vertical tangent at c.
(B) It has a vertical cusp at c.
(C) It has either a vertical tangent or a vertical cusp at c.

(D) It has neither a vertical tangent nor a vertical cusp at c.

(E) We cannot say anything for certain.

Answer: Option (E).

Explanation: Consider $g(x) := x^{1/3}$. This has a vertical tangent at $c = 0$. If we choose $f(x) = x$, we get (A). If we choose $f(x) = x^2$, we get (B). If we choose $f(x) = x^3$, we get neither a vertical tangent nor a vertical cusp. Hence, (E) is the only viable option.

Performance review: 7 out of 12 got this correct. 4 chose (C), 1 chose (D).

Historical note (last year): 5 out of 14 people got this correct. Other choices were (A) (3), (C) (4), (B) (1),and (D) (1).

Historical note: In an earlier quiz where this question appeared, 3 out of 15 people got this correct. Other choices were (A) (7), (C) (4), and (D) (1). The main thing that people had trouble with was thinking of possibilities for f that could play the role of converting the vertical tangent behavior of the original function g into vertical cusp or "neither" behavior for the composite function.

Action point: Performance this time was a little better than earlier, but it seems that many of you either did not read the original solution or it did not register properly in your minds. Well, there's always a second chance! Take it this time.

(3) Consider the function $p(x) := x^{2/3}(x-1)^{3/5} + (x-2)^{7/3}(x-5)^{4/3}(x-6)^{4/5}$. For what values of x does the graph of p have a vertical cusp at $(x, p(x))$?

(A) $x = 0$ only.

(B) $x = 0$ and $x = 5$ only.

(C) $x = 5$ and $x = 6$ only.

(D) $x = 0$ and $x = 6$ only.

(E) $x = 0$, $x = 5$, and $x = 6$.

Answer: Option (D)

Explanation: This uses local behavior heuristics, both additive and multiplicative. We need the exponent on top to be p/q where $0 < p < q$ with p even and q odd.

Performance review: 9 out of 12 got this correct. 1 each chose (A), (C), and (E).

Historical note (last year): 3 out of 14 people got this correct. 5 people chose (E) (indcating that they probably forgot the condition that $p < q$), 4 people chose (C), 3 people chose (B), and 1 person chose (A).

Action point: Review the local behavior heuristics section of the review sheet for midterm 2. Or, if this was just a careless error about not noting that a particular number was bigger than 1, don't make the careless error again.

(4) Consider the function $f(x) := \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1 \end{array}$. What is $f \circ f$?

(A) $x \mapsto \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x^4, & 1/2 < x \le 1 \end{array}$

(B) $x \mapsto \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1 \end{array}$

(C) $x \mapsto \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1/\sqrt{2} \\ x^4, & 1/\sqrt{2} < x \le 1 \end{array}$

(D) $x \mapsto \{ \begin{array}{rl} x, & 0 \le x \le 1/\sqrt{2} \\ x^2, & 1/\sqrt{2} < x \le 1 \end{array}$

(E) $x \mapsto \{ \begin{array}{rl} x, & 0 \le x \le 1/\sqrt{2} \\ x^4, & 1/\sqrt{2} < x \le 1 \end{array}$

Answer: Option (C)

Explanation: If $0 \le x \le 1/2$, then $f(x) = x$, so $f(f(x)) = x$. If $1/2 < x \le 1$, then $f(x) = x^2$. What happens when we apply f to that depends on where x^2 falls. If $0 \le x^2 \le 1/2$, then $f(x^2) = x^2$, so $f(f(x)) = x^2$. This covers $1/2 < x \le 1/\sqrt{2}$. Otherwise $f(x^2) = x^4$, so $f(f(x)) = x^4$.

Performance review: 7 out of 12 got this correct. 3 chose (A), 1 chose (A), 1 chose (B).

Historical note (last year): 4 out of 14 people got this correct. 4 people chose (E), 4 people chose (A), 1 person chose (D), and 1 person left the question blank.

Action point: It seems that many people don't have the correct conceptual picture of how to compose functions with piecewise definitions. *You need to spend some time to understand this – please do!* We will talk briefly about this in one of the subsequent review opportunities.

(5) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both right continuous on $(0,1)$. Which of the following is *not* guaranteed to be right continuous on $(0,1)$?
(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$
(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be right continuous functions

Answer: Option (D)

Explanation: See the explanation for Question 2 on the October 1 quiz. Note that that quiz uses left continuity, but the example can be adapted to right continuity.

Performance review: 5 out of 12 got this correct. 4 chose (E), 3 chose (C).

Historical note (last year): 9 out of 14 people got the question correct. 3 people chose (E) and 1 person each chose (B) and (C).

(6) For a partition $P = x_0 < x_1 < x_2 < \cdots < x_n$ of $[a,b]$ (with $x_0 = a$, $x_n = b$) define the norm $\|P\|$ as the maximum of the values $x_i - x_{i-1}$. Which of the following **is always true** for any continuous function f on $[a,b]$? (5 points)
(A) If P_1 is a finer partition than P_2, then $\|P_2\| \le \|P_1\|$ (Here, *finer* means that, as a set, $P_2 \subseteq P_1$, i.e., all the points of P_2 are also points of P_1).
(B) If $\|P_2\| \le \|P_1\|$, then $L_f(P_2) \le L_f(P_1)$ (where L_f is the lower sum).
(C) If $\|P_2\| \le \|P_1\|$, then $U_f(P_2) \le U_f(P_1)$ (where U_f is the upper sum).
(D) If $\|P_2\| \le \|P_1\|$, then $L_f(P_2) \le U_f(P_1)$.
(E) All of the above.

Answer: Option (D).

Explanation: Option (D) is true for the rather trivial reason that any lower sum of f over any partition cannot be more than any upper sum of f over any partition. The norm plays no role.

Option (A) is incorrect because the inequality actually goes the other way: the finer partition has the smaller norm. Options (B) and (C) are incorrect because a smaller norm does not, in and of itself, guarantee anything about how the lower and upper sums compare.

Performance review: 2 out of 12 got this correct. 5 chose (C), 3 chose (B), 1 each chose (A) and (E).

Historical note (last year): 9 out of 14 people got this correct. 3 people chose (B) and 1 person each chose (A) and (C).

Historical note 1: In the previous quiz appearance, 4 out of 15 prople got this correct. 8 people chose (C), presumably with the intuition that the smaller the norm of a partition, the smaller its upper sums. While this intuition is right in a broad sense, it is not correct in the precise sense that would make (C) correct. It is possible that a lot of people did not read (D) carefully, and stopped after seeing (C), which they thought was a correct statement. 1 person each chose (A), (E), and (C)+(D).

Historical note 2: This question appeared in a 152 midterm two years ago, and 6 of 29 people got this right. Many people chose (C) in that test too (though I haven't preserved numerical information on number of wrong choices selected).

(7) A disk of radius r in the xy-plane is translated parallel to itself with its center moving in the yz-plane along the semicircle $y^2 + z^2 = R^2, y \ge 0$. The solid thus obtained can be thought of as a *cylinder of bent spine* with cross sections being disks of radius r along the xy-plane and the centers forming a semicircle of radius R in the yz-plane, with the z-value ranging from $-R$ to R. What is the volume of this solid?
(A) $2\pi r^2 R$
(B) $\pi^2 r^2 R$

(C) $2\pi r R^2$
(D) $\pi^2 r R^2$
(E) $\pi^2 R^3$

Answer: Option (A)

Explanation: For $-R \leq z \leq R$, the cross section in the xy-plane has constant area with value πr^2. Thus, the total volume is $\pi r^2 \times (R - (-R)) = \pi r^2 \times 2R = 2\pi r^2 R$.

Performance review: 5 out of 12 got this correct. 3 chose (C), 2 chose (B), 1 each chose (D) and (E).

Historical note (last year): 3 out of 14 people got this correct. 5 people chose (C), 4 people chose (B), and 2 people chose (E).

Action point: Make sure you understand this really really well! In particular, make sure you understand why this is *not*, repeat *not*, a solid of revolution but rather a *cylinder with bent spine*.