TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 15: IMAGE AND KERNEL

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

27 people took this 18-question quiz. The score distribution was as follows:

- Score of 2: 3 people
- Score of 5: 3 people
- Score of 6: 1 person
- Score of 7: 4 people
- Score of 8: 2 people
- Score of 9: 2 people
- Score of 10: 3 people
- Score of 12: 2 people
- Score of 14: 3 people
- Score of 15: 2 people
- Score of 16: 1 person
- Score of 18: 1 peron

The mean score was 9.22.

The question-wise answers and performance review are below:

(1) Option (E): 18 people
(2) Option (D): 22 people
(3) Option (E): 18 people
(4) Option (D): 9 people
(5) Option (E): 16 people
(6) Option (D): 23 people
(7) Option (C): 9 people
(8) Option (E): 13 people
(9) Option (D): 12 people
(10) Option (E): 18 people
(11) Option (C): 12 people
(12) Option (B): 8 people
(13) Option (E): 7 people
(14) Option (C): 11 people
(15) Option (D): 12 people
(16) Option (B): 16 people
(17) Option (D): 15 people
(18) Option (C): 10 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The purpose of this quiz is to review in greater depth the ideas behind image and kernel. The goal of the first seven questions is to review the ideas of injectivity, surjectivity, and bijectivity in the context of arbitrary functions between sets. The purpose is two-fold: (i) to give a functions-based approach to justifying, intuitively and formally, facts about the effect of matrix multiplication on rank, and (ii) to hint at ways in which linear transformations behave better than other types of functions.

The corresponding lecture notes are titled `Image and kernel of a linear transformation` and the corresponding section of the text is Section 3.1.

Just as a reminder, a function $f : A \to B$ between sets A and B is said to be:

- *injective* if for every $b \in B$, there is *at most* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| \leq 1$ for all $b \in B$ (here $|f^{-1}(b)|$ denotes the size of the set $f^{-1}(b)$).
- *surjective* if for every $b \in B$, there is *at least* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| \geq 1$ for all $b \in B$.
- *bijective* if for every $b \in B$, there is *exactly* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| = 1$ for all $b \in B$.

(1) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. What can we say the relationship between the injectivity of $f \circ g$, the injectivity of f, and the injectivity of g?
  (A) $f \circ g$ is injective if and only if f and g are both injective.
  (B) If f and g are both injective, then $f \circ g$ is injective. However, $f \circ g$ being injective does not imply anything about the injectivity of either f or g.
  (C) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then at least one of f and g is injective, but we cannot conclusively say for any specific one of the two that it must be injective.
  (D) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then f is injective, but we do not have enough information to deduce whether g is injective.
  (E) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then g is injective, but we do not have enough information to deduce whether f is injective.

  Answer: Option (E)

  Explanation: See the lecture notes for more details (note that the roles of f and g are reversed in the lecture notes). The hard part is the direction from $f \circ g$ to g. To see this, note that if g has a collision $g(a_1) = g(a_2)$ (i.e., is non-injective) then that collision continues for $f \circ g$, i.e, we still have $f(g(a_1)) = f(g(a_2))$.

  Performance review: 18 out of 27 got this. 6 chose (D), 1 each chose (A), (B), and (C).

  Historical note (last time): 24 out of 26 got this. 1 each chose (A) and (D).

(2) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. What can we say the relationship between the surjectivity of $f \circ g$, the surjectivity of f, and the surjectivity of g?
  (A) $f \circ g$ is surjective if and only if f and g are both surjective.
  (B) If f and g are both surjective, then $f \circ g$ is surjective. However, $f \circ g$ being surjective does not imply anything about the surjectivity of either f or g.
  (C) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then at least one of f and g is surjective, but we cannot conclusively say for any specific one of the two that it must be surjective.
  (D) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then f is surjective, but we do not have enough information to deduce whether g is surjective.
  (E) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then g is surjective, but we do not have enough information to deduce whether f is surjective.

  Answer: Option (D)

  Explanation: See the lecture notes for more details (note that the roles of f and g are reversed in the lecture notes).

  Performance review: 22 out of 27 got this. 3 chose (A), 1 each chose (B) and (E).

  Historical note (last time): 23 out of 26 got this. 2 chose (E), 1 chose (A).

(3) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. Suppose $f \circ g$ is bijective. What can we say about f and g individually?
  (A) Both f and g must be bijective.
  (B) Both f and g must be injective, but neither of them need be surjective.
  (C) Both f and g must be surjective, but neither of them need be injective.

(D) f must be injective but need not be surjective. g must be surjective but need not be injective.
(E) f must be surjective but need not be injective. g must be injective but need not be surjective.

Answer: Option (E)

Explanation: The composite $f \circ g$ is bijective, so it is both injective and surjective. The results of the previous two questions now take care of things.

Performance review: 18 out of 27 got this. 4 chose (A), 2 each chose (C) and (D), 1 left the question blank.

Historical note (last time): 23 out of 26 got this. 2 chose (D), 1 chose (A).

(4) $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. Suppose both f and g are surjective. Further, suppose that for every $b \in B$, $g^{-1}(b)$ has size m (for a fixed positive integer m) and for every $c \in C$, $f^{-1}(c)$ has size n (for a fixed positive integer n). Then, what can we say about the sizes of the fibers (i.e., the inverse images of points in C) under the composite $f \circ g$?

(A) The size is $\min\{m, n\}$
(B) The size is $\max\{m, n\}$
(C) The size is $m + n$
(D) The size is mn
(E) The size is m^n

Answer: Option (D)

Explanation: For any $c \in C$, $(f \circ g)^{-1}(c) = g^{-1}(f^{-1}(c))$ is the union, for all $b \in f^{-1}(c)$, of the sets $g^{-1}(b)$. Each of these sets has size m, and there is a total of n such sets, so we get a total of mn elements. *Remember, as you learned in kindergarten, that multiplication is repeated addition.*

Performance review: 9 out of 27 got this. 7 chose (A), 6 chose (B), 3 chose (E), 1 chose (C), 1 left the question blank.

Historical note (last time): 21 out of 26 got this. 4 chose (A), 1 chose (B).

(5) **PLEASE READ THIS VERY CAREFULLY AND CONSIDER A WIDE VARIETY OF POLYNOMIAL EXAMPLES**: Suppose f is a polynomial function of degree $n > 2$ from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the fibers of f, i.e., the sets of the form $f^{-1}(x)$, $x \in \mathbb{R}$?

Hint: At the one extreme, consider a polynomial of the form x^n. Consider the sizes of the fibers $f^{-1}(0)$ and $f^{-1}(x)$ for a positive value of x (the fiber size for the latter will depend on whether n is even or odd). Alternatively, consider a polynomial of the form $(x-1)(x-2)\dots(x-n)$. Consider the size of the fiber $f^{-1}(0)$.

(A) Every fiber has size n.
(B) The minimum of the sizes of fibers is exactly n, but every fiber need not have size n.
(C) The maximum of the sizes of fibers is exactly n, but every fiber need not have size n.
(D) The minimum of the sizes of fibers is at least n, but need not be exactly n.
(E) The maximum of the sizes of fibers is at most n, but need not be exactly n.

Answer: Option (E)

Explanation: Suppose we are trying to calculate the size of the fiber $f^{-1}(x)$ for a particular value of x. This is equivalent to solving the equation $f(t) = x$ in the variable t. This is a polynomial equation of degree n, so it has at most n roots. Thus, the size of each fiber is at most n. Thus, the maximum of the sizes of the fibers is at most n.

For $n = 2$, the maximum of the fiber sizes is always 2. However, for $n \geq 3$, there are examples where the maximum of the sizes of the fibers is less than n. Specifically, consider the example of x^n. The maximum of the fiber sizes here is 1 if n is odd and 2 if n is even. In both cases, it is less than n for $n \geq 3$.

Performance review: 16 out of 27 got this. 6 chose (C), 3 chose (B), 1 chose (A), 1 left the question blank.

Historical note (last time): 1 out of 26 got this (!!!). 16 chose (C), 5 chose (B), 3 chose (D), 1 chose (A).

(6) Suppose f is a continuous injective function from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the nature of f?

(A) f must be an increasing function on all of $\mathbb{R}$.
(B) f must be a decreasing function on all of $\mathbb{R}$.

(C) f must be a constant function on all of $\mathbb{R}$.
(D) f must be either an increasing function on all of $\mathbb{R}$ or a decreasing function on all of $\mathbb{R}$, but the information presented is insufficient to decide which case occurs.
(E) f must be either an increasing function or a decreasing function or a constant function on all of $\mathbb{R}$, but the information presented is insufficient for deciding anything stronger.

Answer: Option (D)

Explanation: This is easy to see pictorially, though a rigorous proof would invoke the intermediate value theorem and the extreme value theorem.

Performance review: 23 out of 27 got this. 4 chose (E).

Historical note (last time): 21 out of 26 got this. 5 chose (E).

(7) **PLEASE READ THIS CAREFULLY, MAKE CASES, AND CHECK YOUR REASONING**: Suppose f, g, and h are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the functions $f+g$, $f+h$, and $g+h$?

Hint: Based on the preceding question, you know something about the nature of f, g, and h individually as functions, but there is some degree of ambiguity in your knowledge. Make cases based on the possibilities and see what you can deduce in the best and worst case.

(A) They are all continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$.
(B) At least two of them are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$. However, we cannot say more.
(C) At least one of them is a continuous bijective function from $\mathbb{R}$ to $\mathbb{R}$. However, we cannot say more.
(D) Either all three sums are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$, or none is.
(E) It is possible that none of the sums is a continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$; it is also possible that one, two, or all the sums are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$.

Answer: Option (C)

Explanation: Since f, g, and h are all continuous bijective functions $\mathbb{R} \to \mathbb{R}$, each one of them is either increasing or decreasing. Further, the functions that are increasing must have a limit of $-\infty$ at $-\infty$ and a limit of ∞ at ∞, whereas the functions that are decreasing must have a limit of ∞ at $-\infty$ and a limit of $-\infty$ at ∞. Thus:

- A sum of two continuous increasing surjective functions is also a continuous increasing surjective function, and hence is bijective: To see this, use the fact that the limit of the sum is the sum of the limits to deduce that for the sum, the limit at $-\infty$ is $-\infty$ and the limit at ∞ is ∞, so that the function must be surjective.
- A sum of two continuous decreasing surjective functions is also a continuous decreasing surjcetive function, hence is bijective. To see this, use the fact that the limit of the sum is the sum of the limits to deduce that for the sum, the limit at $-\infty$ is ∞ and the limit at ∞ is $-\infty$, so that the function must be surjective.

We consider various cases:

- If all three functions are increasing, so are all the pairwise sums, and hence, all the sums $f+g$, $f+h$, and $g+h$ are bijective.
- If all three functions are decreasing, so are all the pairwise sums, and hence, all the sums $f+g$, $f+h$, and $g+h$ are decreasing.
- If two of the functions are increasing and the third function is decreasing, then we know for certain that the sum of the two increasing functions is bijective. But the sum of either of the increasing functions with the decreasing function may be increasing, decreasing, or neither. For instance, if $f(x) = g(x) = x$ and $h(x) = -x$, then $f+h$ and $g+h$ are both the zero function, which is neither increasing nor decreasing, and hence not one-to-one.
- If two of the functions are decreasing and the third function is increasing, then we know for certain that the sum of the two dcereasing functions is bijective. We cannot say anything for sure about the other two sums, for the same reasons as in the previous case (specifically, we can just use the negatives of the functions in the preceding example).

It's clear from all these that (C) is the right option.

Performance review: 9 out of 27 got this. 10 chose (E), 4 chose (B), 3 chose (A), 1 chose (D).

Historical note (last time): 1 out of 26 got this. 15 chose (E), 8 chose (A), 2 chose (D).

The questions that follow tripped up students quite a bit last time, so I urge you to proceed with caution. You can do each of these questions in either of two ways:

- Using abstract, general reasoning.
- Constructing concrete examples.

While the former approach is one you should eventually be able to embrace without trepidation, feel free to rely on the latter approach for now. For this, consider matrices describing the linear transformations and use matrix multiplication to compute the composite where needed. Compute the kernel, image, and rank using the methods known to you. Take matrices such as those arising from finite state automata (as described in the "linear transformations and finite state automata" quiz) or their generalizations to rectangular matrices.

For instance, you might try taking a matrix such as $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This describes a linear transformation $\mathbb{R}^5 \to \mathbb{R}^4$ and has rank three. The dimension of the kernel (inside $\mathbb{R}^5$) is 2 (explicitly, the kernel is precisely the set of vectors in $\mathbb{R}^5$ whose first three coordinates are zero) and the dimension of the image (inside $\mathbb{R}^4$) is 3 (explicitly, the image is precisely the set of vectors in $\mathbb{R}^4$ whose fourth coordinate is 0).

(8) *This is the analogue for linear transformations of Question 1*: Suppose m, n, p are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. The product AB is a $m \times p$ matrix. Denote by T_A, T_B, and T_{AB} respectively the linear transformations corresponding to A, B, and AB. We have $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^p \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$. Note that $T_{AB} = T_A \circ T_B$.

Recall that a matrix has full column rank if and only if the corresponding linear transformation is injective.

Which of the following describes correctly the relationship between A having full column rank (i.e., rank n), B having full column rank (i.e., rank p), and AB having full column rank (i.e., rank p)?

(A) AB has full column rank (i.e., rank p) if and only if A and B both have full column rank (ranks n and p respectively).

(B) If A and B both have full column rank, then AB has full column rank. However, AB having full column rank does not imply anything (separately or jointly) regarding whether A or B has full column rank.

(C) If A and B both have full column rank, then AB has full column rank. If AB has full column rank, then at least one of A and B has full column rank, but we cannot definitively say for any particular one of A and B that it must have full column rank.

(D) If A and B both have full column rank, then AB has full column rank. AB having full column rank implies that A has full column rank, but it does not tell us for sure that B has full column rank.

(E) If A and B both have full column rank, then AB has full column rank. AB having full column rank implies that B has full column rank, but it does not tell us for sure that A has full column rank.

Answer: Option (E)

Explanation: This is a special case of Question 1 (note that the letters do not match). Essentially, T_A plays the role of f and T_B plays the role of g.

An alternative way of thinking of this is that the rank of a product is less than or equal to the rank of each of the matrices being multiplied. If the rank of AB is p (full column rank), then that means that the rank of B is at least p. Since the number of columns of B equals p, this forces B to have full column rank p.

Note that A need not have full column rank. For instance, consider:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Performance review: 13 out of 27 got this. 6 chose (D), 3 each chose (B) and (C), 2 chose (A).

(9) *This is the analogue for linear transformations of Question 2*: Suppose m, n, p are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. The product AB is a $m \times p$ matrix. Denote by T_A, T_B, and T_{AB} respectively the linear transformations corresponding to A, B, and AB. We have $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^p \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$. Note that $T_{AB} = T_A \circ T_B$.

Recall that a matrix has full row rank if and only if the corresponding linear transformation is surjective.

Which of the following describes correctly the relationship between A having full row rank (i.e., rank m), B having full row rank (i.e., rank n), and AB having full row rank (i.e., rank m)?

(A) AB has full row rank if and only if A and B both have full row rank.
(B) If A and B both have full row rank, then AB has full row rank. However, AB having full row rank does not imply anything (separately or jointly) regarding whether A or B has full row rank.
(C) If A and B both have full row rank, then AB has full row rank. If AB has full row rank, then at least one of A and B has full row rank, but we cannot definitively say for any particular one of A and B that it must have full row rank.
(D) If A and B both have full row rank, then AB has full row rank. AB having full row rank implies that A has full row rank, but it does not tell us for sure that B has full row rank.
(E) If A and B both have full row rank, then AB has full row rank. AB having full row rank implies that B has full row rank, but it does not tell us for sure that A has full row rank.

Answer: Option (D)

Explanation: This follows from Question 2. We can also think of it in terms of the rank of a product being less than or equal to the ranks of the individual matrices. This forces the rank of A to be at least m, and therefore exactly m.

Also, for an example of a situation where B does not have full row rank, we can use the same example as in the preceding question.

Performance review: 12 out of 27 got this. 6 chose (E), 4 chose (A), 3 chose (B), 2 chose (C).

(10) *This is the analogue for linear transformations of Question 3*: Suppose m and n are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times m$ matrix. The product AB is a $m \times m$ matrix. The corresponding linear transformations are $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^m \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^m \to \mathbb{R}^m$.

Suppose the square matrix AB has full rank m. What can we deduce about the ranks of A and B?

(A) Both A and B have full row rank, *and* both A and B have full column rank.
(B) Both A and B have full column rank, but neither of them need have full row rank.
(C) Both A and B have full row rank, but neither of them need have full column rank.
(D) A must have full column rank but need not have full row rank. B must have full row rank but need not have full column rank.
(E) A must have full row rank but need not have full column rank. B must have full column rank but need not have full row rank.

Answer: Option (E)

Explanation: Follows from Question 3. We can use the same example as for the preceding two questions.

Also note that in this case, we must have $m \leq n$, and therefore, the rank of both A and B, since it's $\leq \min\{m, n\}$ but also $\geq m$, must equal m. This means full row rank in the case of A, and full column rank in the case of B.

Performance review: 18 out of 27 got this. 5 chose (D), 2 chose (A), 1 each chose (B) and (C).

For the coming questions, we will denote vector spaces by letters such as U, V, and W. You can, however, consider them to be finite-dimensional vector spaces of the form $\mathbb{R}^n$. However, you should take care not to use a letter for the dimension of a vector space if the letter is already in use elsewhere in the question. Also, you should take care to use different letters for the dimensions of different vector spaces, unless it is given to you that the vector spaces have the same dimension. The results also hold for infinite-dimensional vector spaces, but you can work on all the problems assuming you are working in the finite-dimensional setting.

(11) *This is an analogue for linear transformations of Question 4*: Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations. The composite $T_2 \circ T_1$ is also a linear transformation, this time from U to W. Suppose the kernel of T_1 has dimension m and the kernel of T_2 has dimension n. Suppose both T_1 and T_2 are surjective. What can you say about the dimension of the kernel of $T_2 \circ T_1$?

Please note this carefully: Although this question is analogous to Question 4, the correct answer options differ for the two questions. Here is an intuitive explanation for the relationship between the questions. Question 4 asked about the *sizes* of the fibers. This question asks about the dimensions of the kernels. The fibers do correspond to the kernels. But the relationship between dimension and size is of a *logarithmic nature*. What we mean is that the dimension can be thought of as the logarithm of the size. This isn't literally true, because the size is infinite. But metaphorically, it makes sense, because, for instance, the dimension of $\mathbb{R}^p$ is the exponent p, and that comports with the laws of logarithms (similar to how the $\log_2(2^p) = p$).

(A) The dimension is $\min\{m, n\}$.
(B) The dimension is $\max\{m, n\}$.
(C) The dimension is $m + n$.
(D) The dimension is mn.
(E) The dimension is m^n.

Answer: Option (C)

Explanation: See the lecture notes for more.

Performance review: 12 out of 27 got this. 5 chose (D), 4 chose (A), 3 chose (B), 2 chose (E), 1 left the question blank.

Historical note (last time): 3 out of 26 got this. 13 chose (B), 7 chose (A), 3 chose (D).

(12) Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations. The composite $T_2 \circ T_1$ is also a linear transformation, this time from U to W. Suppose the kernel of T_1 has dimension m and the kernel of T_2 has dimension n. However, unlike the preceding question, we are not given any information about the surjectivity of either T_1 or T_2. The answer to the preceding question gives an (inclusive) *upper* bound on the dimension of the kernel of $T_2 \circ T_1$. Which of the following is the best *lower* bound we can manage in general?

(A) $|m - n|$
(B) m
(C) n
(D) $m + n$

Answer: Option (B)

Explanation: The kernel of $T_2 \circ T_1$ contains the kernel of T_1, so m is a lower bound on the dimension.

Performance review: 8 out of 27 got this. 11 chose (A), 4 chose (C), 3 chose (D), 1 left the question blank.

Historical note (last time): 11 out of 26 got this. 10 chose (A), 4 chose (C), 1 chose (D).

(13) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following is true? Please see Options (D) and (E) before answering and select the single option that best reflects your view.

(A) If both T_1 and T_2 are injective, then $T_1 + T_2$ is injective.
(B) If both T_1 and T_2 are surjective, then $T_1 + T_2$ is surjective.
(C) If both T_1 and T_2 are bijective, then $T_1 + T_2$ is bijective.
(D) All of the above
(E) None of the above

Answer: Option (E)

Explanation: We can get counterexamples in one dimension: consider the situation where T_1 has matrix [1] and T_2 has matrix $[-1]$. Then, both T_1 and T_2 are bijective (hence also injective and surjective) but the sum $T_1 + T_2$, which is the zero map, is neither injective nor surjective.

Performance review: 7 out of 27 got this. 13 chose (D), 4 chose (B), 2 chose (C), 1 left the question blank.

Historical note (last time): 6 out of 26 got this. 19 chose (D), 1 chose (C).

(14) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following best describes the relation between the kernels of T_1, T_2, and $T_1 + T_2$?
- (A) The kernel of $T_1 + T_2$ equals the intersection of the kernel of T_1 and the kernel of T_2.
- (B) The kernel of $T_1 + T_2$ is contained inside the intersection of the kernel of T_1 and the kernel of T_2, but need not be equal to the intersection.
- (C) The kernel of $T_1 + T_2$ contains the intersection of the kernel of T_1 and the kernel of T_2, but need not be equal to the intersection.
- (D) The kernel of $T_1 + T_2$ is contained inside the sum of the kernel of T_1 and the kernel of T_2, but need not be equal to the sum.
- (E) The kernel of $T_1 + T_2$ contains the sum of the kernel of T_1 and the kernel of T_2, but need not be equal to the sum.

Answer: Option (C)

Explanation: We will show this in steps:
- Suppose $\vec{u}$ is in the intersection of the kernel of T_1 and the kernel of T_2. Then, $\vec{u}$ is in the kernel of $T_1 + T_2$: This is easy to see: $(T_1 + T_2)(\vec{u}) = T_1(\vec{u}) + T_2(\vec{u}) = 0 + 0 = 0$.
- It is possible to have a situation where the kernel of $T_1 + T_2$ does not contain the sum of the kernels of T_1 and T_2. For instance, consider the case that T_1 has matrix [0] and T_2 has matrix [1].
- It is possible to have a situation where the kernel of $T_1 + T_2$ is not even contained inside the sum of the kernels of T_1 and T_2, let alone the intersection: Consider T_1 to be the linear transformation with matrix [1] and T_2 to be the linear transformation with matrix $[-1]$. Both have zero kernels, so the sum of the kernels is also zero. But the sum $T_1 + T_2$ is a linear transformation with matrix [0], so its kernel is all of $\mathbb{R}$.

Performance review: 11 out of 27 got this. 6 chose (A), 4 each chose (B) and (D), 1 chose (E), 1 left the question blank.

Historical note (last time): 3 out of 26 got this. 10 chose (B), 6 each chose (A) and (D), 1 chose (E).

(15) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following best describes the relation between the images of T_1, T_2, and $T_1 + T_2$?
- (A) The image of $T_1 + T_2$ equals the intersection of the image of T_1 and the image of T_2.
- (B) The image of $T_1 + T_2$ is contained inside the intersection of the image of T_1 and the image of T_2, but need not be equal to the intersection.
- (C) The image of $T_1 + T_2$ contains the intersection of the image of T_1 and the image of T_2, but need not be equal to the intersection.
- (D) The image of $T_1 + T_2$ is contained inside the sum of the image of T_1 and the image of T_2, but need not be equal to the sum.
- (E) The image of $T_1 + T_2$ contains the sum of the image of T_1 and the image of T_2, but need not be equal to the sum.

Answer: Option (D)

Explanation: We show the following:
- Any vector in the image of $T_1 + T_2$ can be expressed as the sum of a vector in the image of T_1 and a vector in the image of T_2: Suppose $\vec{v}$ is in the image of $T_1 + T_2$. Thus, $\vec{v} = (T_1 + T_2)(\vec{u})$ which simplifies to $T_1(\vec{u}) + T_2(\vec{u})$, thus it is in the sum of the images.
- The image of $T_1 + T_2$ need not contain the intersection of the images of T_1 and T_2: We can again use the example of linear transformations with matrices [1] and $[-1]$ respectively.
- The image of $T_1 + T_2$ need not be contained in the intersection of the images of T_1 and T_2: We can again use the example of linear transformations with matrices [1] and [0].

Performance review: 12 out of 27 got this. 8 chose (B), 4 chose (C), 2 chose (E), 1 left the question blank.

Historical note (last time): 11 out of 26 got this. 11 chose (B), 2 chose (C), 1 each chose (A) and (E).

(16) Suppose T is a linear transformation from a vector space V to itself. Note that V may be an infinite-dimensional space, such as $C^\infty(\mathbb{R})$ (with T being differentiation), but for convenience, you can imagine V to be finite-dimensional (we will not reference the dimension of V in this question, however). Suppose the kernel of T has dimension n. What can you say from this information about the dimension of the kernel of T^r for a positive integer r?

(A) It is at least n and at most $n + r$.
(B) It is at least n and at most nr.
(C) It is at least $n + r$ and at most nr.
(D) It is at least $n + r$ and at most n^r.

Answer: Option (B)

Explanation: We know the kernel of a composite contains the kernel of the very first operation, so the dimension is at least n. But it could be bigger. Recall that the dimensions of the kernels could at worst add up, so the worst case scenario (which occurs if each time the kernel is also contained in the image) is that the total dimension is nr.

Performance review: 16 out of 27 got this. 9 chose (A), 1 chose (C), 1 left the question blank.

Historical note (last time): 13 out of 26 got this. 7 chose (A), 5 chose (C), 1 chose (D).

The next few questions deal with the relationship between the rows and columns of the matrix on the one hand, and the image and kernel of the linear transformation on the other hand.

(17) Suppose A is a $n \times m$ matrix and $T_A : \mathbb{R}^m \to \mathbb{R}^n$ is the corresponding linear transformation. Which of the following correctly describes the relationship between the rows and columns of A and the image and kernel of T_A?

(A) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ spanned by the columns of A.
(B) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the columns of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ spanned by the rows of A.
(C) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ comprising the vectors that are *orthogonal* to the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ comprising the vectors that are *orthogonal* to the columns of A.
(D) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ comprising the vectors that are *orthogonal* to the rows of A. The image of T_A is the subspace of $\mathbb{R}^n$ spanned by the columns of A.
(E) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ comprising the vectors that are *orthogonal* to the columns of A.

Answer: Option (D)

Explanation: The kernel is the set of vectors $\vec{x} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{0}$. The entries of $A\vec{x}$ are the dot products of the rows of A with $\vec{x}$. Therefore, a particular entry is zero if and only if the corresponding dot product is zero, i.e., $\vec{x}$ is orthogonal to the corresponding row of A. Thus, all the entries of the output vector are zero iff $\vec{x}$ is orthogonal to all rows of A (there were also similar questions in the preceding quiz).

The part of the statement about the image is a standard fact about the image that you have seen in lecture.

Performance review: 15 out of 27 got this. 5 chose (B), 4 chose (C), 2 chose (A), 1 left the question blank.

(18) Suppose A and B are $n \times m$ matrices, $T_A : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation corresponding to A, and $T_B : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation corresponding to B. Which of the following correctly describes the relation between the rows, columns, image and kernel? Please see Option (E) before answering.

(A) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same kernel as each other and also the same image as each other.
(B) If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same kernel as each other and also the same image as each other.
(C) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same kernel as each other. If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same image as each other.
(D) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same image as each other. If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same kernel as each other.
(E) None of the above.

Answer: Option (C)

Explanation: Interchanging the rows is a legitimate operation for computing the reduced row-echelon form, and the process does not affect the solution set for the system of linear equations, aka the kernel.

Also, the columns of the matrix are a spanning set for the image, so interchanging the columns should not affect the image.

However, interchanging the rows can alter the image. For instance:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

have the same set of rows, but different images.

Similarly, interchanging the columns can alter the kernel. For instance:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

have the same set of columns (interchanged) but different kernels.

Performance review: 10 out of 27 got this. 6 each chose (D) and (E), 3 chose (B), 1 chose (A), 1 left the question blank.