TAKE-HOME CLASS QUIZ: DUE FRIDAY NOVEMBER 2: SERIES CONVERGENCE

MATH 153, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE MAKE SURE TO ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE

(1) Suppose p is a polynomial that takes positive values on all positive integers. Consider the summation $\sum_{k=1}^{\infty} \frac{(k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.
   (A) The summation converges if and only if the degree of p is *at least* one
   (B) The summation converges if and only if the degree of p is *at least* two
   (C) The summation converges if and only if the degree of p is *at least* three
   (D) The summation converges if and only if the degree of p is *at most* two
   (E) The summation converges if and only if the degree of p is *at most* one

   Your answer: ____________________________

(2) Suppose p is a polynomial that takes positive values on all positive integers. Consider the summation $\sum_{k=1}^{\infty} \frac{(-1)^k(k^2+1)^{2/3}}{p(k)}$. Under what conditions does the summation converge? Note that the degree of p must be a nonnegative integer.
   (A) The summation converges if and only if the degree of p is *at least* one
   (B) The summation converges if and only if the degree of p is *at least* two
   (C) The summation converges if and only if the degree of p is *at least* three
   (D) The summation converges if and only if the degree of p is *at most* two
   (E) The summation converges if and only if the degree of p is *at most* one

   Your answer: ____________________________

(3) Which of the following series converges? Assume for all series that the starting point of summation is large enough that the terms are well defined. *Two years ago:* 11/25 *correct*
   (A) $\sum 1/(k \ln(\ln k))$
   (B) $\sum 1/(k \ln k)$
   (C) $\sum 1/(k(\ln(\ln k))^2)$
   (D) $\sum 1/(k(\ln k)(\ln(\ln k)))$
   (E) $\sum 1/(k(\ln k)(\ln(\ln k))^2)$

   Your answer: ____________________________

(4) Which of the following series converges? If all converge, select option (E). Assume for all series that the starting point of summation is large enough that the terms are well defined.
   (A) $\sum 1/(k \ln(k^2+1))$
   (B) $\sum 1/(k(\ln(k^2)+1))$
   (C) $\sum 1/((k \ln(k^2))+1)$
   (D) $\sum 1/(k((\ln k)^2+1))$
   (E) All of the above converge

   Your answer: ____________________________

(5) Which of the following series converges? *Two years ago:* 23/25 *correct*

(A) $\sum \frac{k+\sin k}{k^2+1}$
(B) $\sum \frac{k+\cos k}{k^3+1}$
(C) $\sum \frac{k^2-\sin k}{k+1}$
(D) $\sum \frac{k^3+\cos k}{k^2+1}$
(E) $\sum \frac{k}{\sin(k^3+1)}$

Your answer: ____________________________

(6) Consider the series $\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}$. What can we say about the sum of this series? *Two years ago:* 14/26 *correct*
(A) It is finite and strictly between 0 and 1.
(B) It is finite and equal to 1.
(C) It is finite and strictly between 1 and 2.
(D) It is finite and equal to 2.
(E) It is infinite.

Your answer: ____________________________

(7) For one of the following functions f on $(0, \infty)$, the integral $\int_0^\infty f(x)\,dx$ converges but $\int_0^\infty |f(x)|\,dx$ does not converge. What is that function f? (Note that this is similar to, but not quite the same as, the absolute versus conditional convergence notion for series).
(A) $f(x) = \sin x$
(B) $f(x) = \sin(\sin x)$
(C) $f(x) = (\sin \sqrt{x})/\sqrt{x}$
(D) $f(x) = (\sin x)/x$
(E) $f(x) = (\sin^3 x)/x^3$

Your answer: ____________________________

(8) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Since it is a series of positive terms, this means that the partial sums get arbitrarily large. What is the approximate smallest value of N such that $\sum_{n=1}^{N} \frac{1}{n} > 100$? *Two years ago:* 14/26 *correct*
(A) Between 90 and 110
(B) Between 2000 and 3000
(C) Between 10^{40} and 10^{50}
(D) Between 10^{90} and 10^{110}
(E) Between 10^{220} and 10^{250}

Your answer: ____________________________

(9) Consider the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Which of the following is **true** about the series?
(A) Every rearrangement of the series converges to $\pi^2/6$.
(B) Every rearrangement of the series converges to $\ln 2$.
(C) The series converges to $\pi^2/6$ and there is a rearrangement of the series that converges to $\ln 2$.
(D) The series converges to $\ln 2$ and there is a rearrangement of the series that converges to $\pi^2/6$.
(E) The series does not converge.

Your answer: ____________________________

(10) Consider a series $\sum a_n$ whose terms have alternating signs, with the first term (i.e., a_1) positive in sign, such that $|a_n|$ form a decreasing sequence and $\lim_{n\to\infty} |a_n| = 1$. Let $b_k = \sum_{n=1}^{2k-1} a_n$ (so these are the sums of odd numbers of initial terms), with $b = \lim_{k\to\infty} b_k$, and $c_k = \sum_{n=1}^{2k} a_n$ (so these are sums of even numbers of initial terms), with $c = \lim_{k\to\infty} c_k$. Which of the following is **true** about the sequences b_k and c_k and the limits b and c? *Hint: This is similar to the alternating series theorem. Make a picture of the number line and hop on it.*
(A) b_k form an increasing sequence, c_k form a decreasing sequence, and $b = c$.
(B) b_k form an increasing sequence, c_k form a decreasing sequence, and $b - c = 1$
(C) b_k form a decreasing sequence, c_k form an increasing sequence, and $b - c = 1$
(D) b_k form an increasing sequence, c_k form a decreasing sequence, and $c - b = 1$
(E) b_k form a decreasing sequence, c_k form an increasing sequence, and $c - b = 1$

Your answer: ____________________________

For the next few questions, let $T : \mathbb{R} \to \mathbb{R}$ be a function defined as follows. For $a \in \mathbb{R}$, define:

$$T(a) := \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

Note that T is well defined because the summation on the right converges for every a.

(11) Which of the following function types does T have?
(A) Constant
(B) Linear
(C) Periodic
(D) Odd
(E) Even

Your answer: ____________________________

(12) Which of the following is true about T?
(A) T attains its absolute maximum at 0 and has no absolute minimum
(B) T attains its absolute maximum at 0 and its absolute minimum at 1 and -1
(C) T attains its absolute minimum at 0 and has no absolute maximum
(D) T attains its absolute minimum at 0 and its absolute maximum at 1 and -1
(E) T has no absolute maximum or absolute minimum

Your answer: ____________________________

(13) How can we express the summation:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 3n^2 + 2}$$

in terms of T?
(A) $T(1) - T(2)$
(B) $T(2) - T(1)$
(C) $T(1) + T(2)$
(D) $T(1) - T(\sqrt{2})$
(E) $T(\sqrt{2}) - T(1)$

Your answer: ____________________________

(14) Consider the function $F(x, p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. For what values of x and what values of p does this summation converge? *Two years ago:* 7/26 *correct*
(A) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| \geq 1$, it does not converge for any p.
(B) For $|x| \leq 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p.

(C) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $|x| = 1$, it converges if and only if $p > 1$.
(D) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any p. For $x = 1$, it converges if and only if $p > 0$. For $x = -1$, it converges if and only if $p > 1$.
(E) For $|x| < 1$, it converges for all $p \in \mathbb{R}$. For $|x| > 1$, it does not converge for any $p \in \mathbb{R}$. For $x = 1$, it converges if and only if $p > 1$. For $x = -1$, it converges if and only if $p > 0$.

Your answer: ____________________________

There is a result of calculus which states that, under suitable conditions, if $f_1, f_2, \ldots, f_n, \ldots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$. In other words, under suitable assumptions, we can differentiate a sum of countably many functions by differentiating each of them and adding up the derivatives.

We will not be going into what those assumptions are, but will consider some applications where you are explicitly told that these assumptions are satisfied.

(15) Consider the summation $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. Assume that the required assumptions are valid for this summation, so that $\zeta'(p)$ is the sum of the derivatives of each of the terms (summands) with respect to p. What is the correct expression for $\zeta'(p)$? *Two years ago:* 8/26 *correct*
(A) $\sum_{n=1}^{\infty} \frac{-p}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{-1}{(p+1)n^{p+1}}$
(C) $\sum_{n=1}^{\infty} \frac{p}{n^{p-1}}$
(D) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^p}$
(E) $\sum_{n=1}^{\infty} \frac{-\ln n}{n^{p+1}}$

Your answer: ____________________________

(16) Recall that we defined $F(x,p) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ with x and p both real numbers. Assume that, for a particular fixed value of p, the summation satisfies the conditions as a function of x for $|x| < 1$. What is its derivative with respect to x, keeping p constant? *Last year:* 14/26 *correct*
(A) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p+1}}$
(B) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^{p-1}}$
(C) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p+1}}$
(D) $\sum_{n=1}^{\infty} \frac{x^{n-1}\ln n}{n^{p+1}}$
(E) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{p-1}}$

Your answer: ____________________________