TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY NOVEMBER 20: IMAGE AND KERNEL: APPLICATIONS TO CALCULUS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

25 people took this 7-question quiz. The score distribution was as follows:

- Score of 1: 1 person
- Score of 2: 3 people
- Score of 3: 5 people
- Score of 4: 3 people
- Score of 5: 9 people
- Score of 6: 3 people
- Score of 7: 1 person

The question-wise answers and performance review were as follows:

(1) Option (B): 4 people
(2) Option (B): 8 people
(3) Option (C): 18 people
(4) Option (B): 19 people
(5) Option (C): 14 people
(6) Option (A): 18 people
(7) Option (A): 23 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The goal of this quiz is to use the setting of calculus to practice our skill of understanding linear transformations, specifically their injectivity, surjectivity, bijectivity, kernel and image. It builds on the November 8 quiz, but goes further. Please refer back to the November 8 quiz for the definitions of vector space, subspace, and linear transformation.

Please read these questions *very* carefully. For the first few questions, the interpretation of the question in the language of calculus is provided. Please refer to that if the linear algebra-based description is unclear.

(1) Let $\mathbb{R}[x]$ denote the vector space of all polynomials in one variable with real coefficients, with the usual addition and scalar multiplication of polynomials. There is an obvious linear transformation from $\mathbb{R}[x]$ to $C^\infty(\mathbb{R})$ that sends any polynomial to the function it describes, e.g., the polynomial $x^2 + 1$ gets sent to the function $x \mapsto x^2 + 1$. What can you say about this map $\mathbb{R}[x] \to C^\infty(\mathbb{R})$?

Please note: We are *not* talking here about whether the polynomial functions themselves are injective or surjective as functions from $\mathbb{R}$ to $\mathbb{R}$. Rather, we are talking about whether the mapping from *the set of polynomials* (which itself is a vector space over the reals) to *the set of infinitely differentiable functions* (which itself is another vector space).

(A) The map is neither injective nor surjective, i.e., different polynomials may define the same function, and not every infinitely differentiable function can be expressed using a polynomial.
(B) The map is injective but not surjective, i.e., different polynomials always define different functions, and not every infinitely differentiable function can be expressed using a polynomial.
(C) The map is surjective but not injective, i.e., different polynomials may define the same function, and every infinitely differentiable function can be expressed using a polynomial.

(D) The map is bijective, i.e., different polynomials always define different functions, and every infinitely differentiable function can be expressed using a polynomial.

Answer: Option (B)

Explanation: The map is linear, so to prove injectivity, it suffices to show that the kernel is zero. In other words, it suffices to show that if $p(x) \in \mathbb{R}[x]$ is in the kernel, then p is the zero polynomial.

Suppose p is in the kernel. Then, this means that $p(x) = 0$ for all $x \in \mathbb{R}$. This means that *every* real number is a root of p. But a nonzero polynomial can have only finitely many roots (the number of roots is at most equal to the degree of the polynomial), so this forces p to be the zero polynomial.

The map is not surjective because there exist lots of infinitely differentiable functions that are not polynomials. Examples include exponential and trigonometric functions.

Performance review: 4 out of 25 got this. 12 chose (C), 7 chose (D), 2 chose (A).

Historical note (last time): 3 out of 26 got this. 10 chose (D), 8 chose (C), and 5 chose (A).

(2) Denote by $\mathbb{R}[[x]]$ the vector space of all *formal power series* in one variable with real coefficients, with coefficient-wise addition and scalar multiplication. Explicitly, an element $a \in \mathbb{R}[[x]]$ is of the form:

$$a = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots$$

where $a_i \in \mathbb{R}$ for $i \in \mathbb{N}_0$. Addition is coefficient-wise, i.e., if:

$$a = \sum_{i=0}^{\infty} a_i x^i, b = \sum_{i=0}^{\infty} b_i x^i$$

Then we have:

$$a + b = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and for any real number λ, we have:

$$\lambda a = \sum_{i=0}^{\infty} (\lambda a_i) x^i$$

Note that a formal power series may have any radius of convergence. The radius of convergence could range from being 0 (which means that the formal power series converges only at the point $\{0\}$) to being ∞ (which means that the formal power series converges on all of $\mathbb{R}$). In other words, a formal power series need not define an actual function on $\mathbb{R}$.

Aside: If you remember sequences and series from single-variable calculus, you will recall that the radius of convergence is the reciprocal of the exponential growth rate of coefficients. In particular, if the coefficients *grow superexponentially*, the radius of convergence is zero. On the other hand, if the coefficients *decay superexponentially*, the radius of convergence is ∞. If the coefficients have exponential growth, the radius of convergence is less than 1. If the coefficients have exponential decay, the radius of convergence is greater than 1. Finally, if the coefficients grow or decay subexponentially, the radius of convergence is 1.

Note that $\mathbb{R}[x]$ can be viewed as a subspace of $\mathbb{R}[[x]]$ by thinking of each polynomial as a formal power series where there are only finitely many nonzero coefficients.

Let Ω be the subset of $\mathbb{R}[[x]]$ comprising those formal power series that converge globally, i.e., the radius of convergence is ∞. Note that Ω is a sub*space* of $\mathbb{R}[[x]]$.

What is the relation between $\mathbb{R}[x]$ and Ω?

Note that by *proper* subspace we mean a subspace that is not equal to the whole space.

(A) $\mathbb{R}[x] = \Omega$, i.e., a power series is globally convergent if and only if it is a polynomial (i.e., it has only finitely many nonzero coefficients).

(B) $\mathbb{R}[x]$ is a proper subspace of Ω, i.e., every polynomial is a globally convergent power series, but there exist globally convergent power series that are not polynomials.

(C) Ω is a proper subspace of $\mathbb{R}[x]$, i.e., every globally convergent power series is a polynomial, but there are polynomials that are not globally convergent power series.

(D) $\mathbb{R}[x]$ and Ω are incomparable, i.e., there exist polynomials that are not globally convergent power series and there exist globally convergent power series that are not polynomials.

Answer: Option (B)

Explanation: The power series given by a polynomial converges, because it is a finite sum.

However, there do exist globally convergent power series that are not polynomials. Specifically, any power series where the coefficients decay super-exponentially. An example is the power series of the exponential function.

Performance review: 8 out of 25 got this. 8 chose (C), 5 chose (D), 4 chose (A).

Historical note (last time): 12 out of 26 got this. 11 chose (C), 2 chose (A), 1 chose (D).

(3) The *Taylor series operator* can be viewed as a linear transformation from $C^\infty(\mathbb{R})$ to $\mathbb{R}[[x]]$. This operator sends any infinitely differentiable function to its Taylor series centered at 0. Explicitly, the operator is:

$$f \mapsto \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

What can we say about the kernel of this linear transformation?

(A) The kernel is the set of functions f satisfying $f(0) = 0$

(B) The kernel is the set of functions f satisfying $f'(0) = 0$

(C) The kernel is the set of functions f such that f and all its derivatives take the value 0 at 0.

(D) The kernel is the set of polynomial functions.

(E) The kernel is the set of functions that have globally convergent power series.

Answer: Option (C)

Explanation: This is direct from the definition.

Performance review: 18 out of 25 got this. 4 chose (B), 2 chose (E), 1 chose (A).

Historical note (last time): 5 out of 26 got this. 10 chose (B), 9 chose (A), 1 each chose (D) and (E).

(4) Which of the following is the best explanation for why we put the $+C$ when performing indefinite integration?

(A) The kernel of differentiation is a zero-dimensional space (namely, the zero function only), hence the fibers (inverse images or pre-images) for differentiation are all zero-dimensional spaces, i.e., single functions.

(B) The kernel of differentiation is a one-dimensional space (namely, the vector space of constant functions), hence the fibers (inverse images or pre-images) for differentiation are all one-dimensional spaces, i.e., lines that are translates of the space of constant functions.

(C) The image of differentiation is a zero-dimensional space (namely, the zero function only), hence the fibers (inverse images or pre-images) for differentiation are all zero-dimensional spaces, i.e., single functions.

(D) The image of differentiation is a one-dimensional space (namely, the vector space of constant functions), hence the fibers (inverse images or pre-images) for differentiation are all one-dimensional spaces, i.e., lines that are translates of the space of constant functions.

Answer: Option (B)

Explanation: This is obvious once you think about it.

Performance review: 19 out of 25 got this. 4 chose (D), 2 chose (A).

Historical note (last time): 11 out of 26 got this. 7 chose (C), 4 chose (D), 3 chose (A), 1 chose (E).

(5) When finding all functions f on $\mathbb{R}$ such that $f''(x) = g(x)$ for some known continuous function g on $\mathbb{R}$, we get a general description of the form $G(x) + C_1 x + C_2$ where C_1, C_2, are arbitrary real numbers. Which of the following is the best explanation for this?

(A) The kernel of the operation of differentiating twice is precisely the set of constant functions.

(B) The kernel of the operation of differentiating twice is precisely the set of nonconstant linear functions.
(C) The kernel of the operation of differentiating twice is the union of the set of constant functions and the set of nonconstant linear functions.
(D) The image of the operation of differentiating twice is precisely the set of constant functions.
(E) The image of the operation of differentiating twice is precisely the set of nonconstant linear functions.

Answer: Option (C)

Explanation: Note that the kernel is the set of functions of the form $x \mapsto C_1x + C_2$ where $C_1, C_2 \in \mathbb{R}$. Note that C_1 and C_2 are allowed to be equal to zero. In the case that $C_1 = 0$, we get constant functions, and in the case $C_1 \neq 0$, we get nonconstant linear functions. The kernel includes both types.

Performance review: 14 out of 25 got this. 6 chose (A), 2 each chose (B) and (D), 1 chose (E).

Historical note (last time): 5 out of 26 got this. 11 chose (A), 6 chose (D), 3 chose (B), 1 chose (E).

(6) Consider a second-order homogeneous linear differential equation of the form:

$$y'' + p_1(x)y' + p_2(x)y = 0$$

where x is the independent variable, y is the dependent variable, and p_1 and p_2 are known functions. We are trying to find global solutions, i.e., functions defined on all of $\mathbb{R}$. One way of thinking of this is to consider the linear transformation L that sends a function y of x to $L(y) = y'' + p_1(x)y' + p_2(x)y$, a new function of x. Which of the following best describes what we are trying to do?

(A) L is a linear transformation $C^2(\mathbb{R}) \to C(\mathbb{R})$, and the solution space we are interested in is the kernel of L.
(B) L is a linear transformation $C^2(\mathbb{R}) \to C(\mathbb{R})$, and the solution space we are interested in is the image of L.
(C) L is a linear transformation $C(\mathbb{R}) \to C^2(\mathbb{R})$, and the solution space we are interested in is the kernel of L.
(D) L is a linear transformation $C(\mathbb{R}) \to C^2(\mathbb{R})$, and the solution space we are interested in is the image of L.

Answer: Option (A)

Explanation: L makes sense for functions in $C^2(\mathbb{R})$ in so far as it requires differentiating twice. It does not make sense for other functions. The image of L could (a priori) land anywhere in $C(\mathbb{R})$. We are interested in the kernel of L, i.e., the functions whose image is 0, because the left side of the differential equation is precisely computing $L(y)$.

Performance review: 18 out of 25 got this. 4 chose (B), 2 chose (C), 1 chose (D).

Historical note (last time): 5 out of 26 got this. 9 chose (B), 8 chose (C), 4 chose (D).

(7) Consider a second-order non-homogeneous linear differential equation of the form:

$$y'' + p_1(x)y' + p_2(x)y = q(x)$$

where x is the independent variable, y is the dependent variable, and p_1, p_2, and q are known functions. We are trying to find global solutions, i.e., functions defined on all of $\mathbb{R}$. One way of thinking of this is to consider the linear transformation L that sends a function y of x to $L(y) = y'' + p_1(x)y' + p_2(x)y$, a new function of x. Which of the following best describes what we are trying to do?

(A) We are trying to find the inverse image under L of $q(x)$, and we know this is a translate of the solution space of the corresponding homogeneous linear differential equation (the one from the preceding question).
(B) We are trying to find the image under L of $p_1(x)$, and we know this is a translate of the solution space of the corresponding homogeneous linear differential equation (the one from the preceding question).

Answer: Option (A)
Explanation: This follows from the fibers being translates of the kernel.
Performance review: 23 out of 25 got this. 2 chose (B).