TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY FEBRUARY 22: MULTI-VARIABLE INTEGRATION

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

24 people took this quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 3: 5 people
- Score of 4: 7 people
- Score of 5: 7 people
- Score of 6: 2 people
- Score of 7: 2 people

The question wise answers and performance review are as follows:

(1) Option (C): 14 people
(2) Option (A): 19 people
(3) Option (C): 17 people
(4) Option (D): 14 people
(5) Option (A): 7 people
(6) Option (C): 14 people
(7) Option (C): 20 people
(8) Option (D): 1 person

2. Solutions

The following setup is for the first five questions only.

Suppose F is a function of two real variables, say x and t, so $F(x,t)$ is a real number for x and t restricted to suitable open intervals in the real number. Suppose, further, that F is jointly continuous (whatever that means) in x and t.

Define $f(t) := \int_0^\infty F(x,t)\,dx$. Here, while doing the integration, t is treated as a constant. x, the variable of integration, is being integrated on $[0,\infty)$.

Suppose further that f is defined and continuous for t in $(0,\infty)$.

In the next few questions, you are asked to compute the function f explicitly given the function F, for $t \in (0,\infty)$.

(1) *Do not discuss!* $F(x,t) := e^{-tx}$. Find f. *Last time:* 15/19 *correct*
  (A) $f(t) = e^{-t}/t$
  (B) $f(t) = e^{t}/t$
  (C) $f(t) = 1/t$
  (D) $f(t) = -1/t$
  (E) $f(t) = -t$

  Answer: Option (C)

  Explanation: The integral becomes $[-e^{-tx}/t]_0^\infty$. Plugging in at ∞ gives 0 and plugging in at 0 gives $-1/t$. Since the value at 0 is being subtracted, we eventually get $1/t$.

  Note that the answer must be positive for the simple reason that we are integrating a positive function from left to right across an interval.

  Performance review: 14 out of 24 people got this. 9 chose (D), 1 chose (E).

Historical note (last time): 15 out of 19 people got this correct. 3 people chose (D) and 1 person chose (B).

(2) *Do not discuss!* $F(x, t) := 1/(t^2 + x^2)$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/(2t)$
(B) $f(t) = \pi/t$
(C) $f(t) = 2\pi/t$
(D) $f(t) = \pi t$
(E) $f(t) = 2\pi t$

Answer: Option (A)

Explanation: We get $[(1/t)\arctan(x/t)]_0^\infty$. The evaluation at ∞ gives $\pi/(2t)$ and the evaluation at 0 gives 0. Subtracting, we get $\pi/(2t)$.

Performance review: 19 out of 24 people got this. 3 chose (B) and 2 chose (C).

Historical note (last time): 13 out of 19 people got this correct. 2 people each chose (B), (C), and (D).

(3) *Do not discuss!* $F(x, t) := 1/(t^2 + x^2)^2$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/t^3$
(B) $f(t) = \pi/(2t^3)$
(C) $f(t) = \pi/(4t^3)$
(D) $f(t) = \pi/(8t^3)$
(E) $f(t) = 3\pi/(8t^3)$

Answer: Option (C)

Explanation: Put in $\theta = \arctan(x/t)$. Substitute, and we get $(1/t^3)\int_0^{\pi/2} \cos^2\theta\, d\theta$. Integrating, we get $[\theta/2t^3 + \sin(2\theta)/4t^3]_0^{\pi/2}$. The trigonometric part vanishes between limits, and we are left with $\pi/(4t^3)$

Performance review: 17 out of 24 people got this. 6 chose (B), 1 chose (D).

Historical note (last time): 13 out of 19 people got this correct. 3 people chose (D) and 1 each chose (A), (B), and (E).

(4) (*) *You can discuss this!* $F(x, t) = \exp(-(tx)^2)$. Use that $\int_0^\infty \exp(-x^2)\, dx = \sqrt{\pi}/2$. Find f. *Last time:* 7/19 *correct*
(A) $f(t) = t^2\sqrt{\pi}/2$
(B) $f(t) = t\sqrt{\pi}/2$
(C) $f(t) = \sqrt{\pi}/2$
(D) $f(t) = \sqrt{\pi}/(2t)$
(E) $f(t) = \sqrt{\pi}/(2t^2)$

Answer: Option (D)

Explanation: Put $u = tx$, get a $1/t$ on the outside, giving $(1/t)\int_0^\infty \exp(-u^2)\, du$.

Performance review: 14 out of 24 got this, 7 chose (B), 2 chose (E), 1 chose (C).

Historical note (last time): 7 out of 19 people got this correct. 7 people chose (E), 2 each chose (A) and (B), and 1 chose (C).

(5) (**) *You can discuss this!* (could confuse you if you don't understand it): In the same general setup as above (but with none of these specific Fs), which of the following is a *sufficient* condition for f to be an increasing function of t? *Last time:* 3/19 *correct*
(A) $t \mapsto F(x_0, t)$ is an increasing function of t for every choice of $x_0 \geq 0$.
(B) $x \mapsto F(x, t_0)$ is an increasing function of x for every choice of $t_0 \in (0, \infty)$.
(C) $t \mapsto F(x_0, t)$ is a decreasing function of t for every choice of $x_0 \geq 0$.
(D) $x \mapsto F(x, t_0)$ is a decreasing function of x for every choice of $t_0 \in (0, \infty)$.
(E) None of the above.

Answer: Option (A)

Explanation: If F is increasing in t for every value of x_0, then that means that as t gets bigger, the function F being integrated gets bigger everywhere in x, i.e., if $t_1 < t_2$, then $F(t_1, x_0) < F(t_2, x_0)$ for every $x_0 \geq 0$. The integral for the larger value t_2 must therefore also be bigger. (We looked at this stuff in Section 5.8 of the book).

Performance review: 7 out of 24 got this. 9 chose (C), 5 chose (B), 2 chose (D), 1 chose (E).

Historical note (last time): 3 out of 19 people got this correct. 9 chose (B), 4 chose (E), 2 chose (C), 1 chose (D).

(end of the setup)

(6) *Do not discuss!* Suppose f is a homogeneous polynomial of degree $d > 0$. Define g as the following function on positive reals: $g(a)$ is the double integral of f on the square $[0, a] \times [0, a]$. Assuming that $g(a)$ is not identically the zero function, which of these best describes the nature of $g(a)$? *Last time: 12/19 correct.*

(A) A constant times a^d
(B) A constant times a^{d+1}
(C) A constant times a^{d+2}
(D) A constant times a^{2d+1}
(E) A constant times a^{2d+2}

Answer: Option (C)

Explanation: Each monomial is of the form a constant times $x^p y^q$ where $p + q = d$. Integrating this as a multiplicatively separable function gives $x^{p+1}y^{q+1}$ times a constant. Evaluating between limits gives a^{d+2} times a constant. This is the form of the double integral of each monomial, and hence the double integral of the sum is also of the same form.

Performance review: 14 out of 24 people got tihs. 5 each chose (B) and (E).

Historical note (last time): 12 out of 19 people got this correct. 3 people chose (D), 2 each chose (A) and (E).

(7) (*) *You can discuss this!* Suppose $g(x, y)$ and $G(x, y)$ are continuous functions of two variables and $G_{xy} = g$. How can the double integral $\int_s^t \int_u^v g(x, y)\, dy\, dx$ be described in terms of the values of G? *Last time: 8/19 correct*

(A) $G(v, t) + G(u, s) - G(u, t) - G(v, s)$
(B) $G(v, t) - G(v, s) + G(u, t) - G(u, s)$
(C) $G(t, v) + G(s, u) - G(t, u) - G(s, v)$
(D) $G(t, v) - G(s, v) + G(t, u) - G(s, u)$
(E) $G(t, v) + G(v, t) - G(s, u) - G(u, s)$

Answer: Option (C)

Explanation: Note that the integration is over $[s, t] \times [u, v]$, i.e., the rectangular region with corner points (s, u), (s, v), (t, u), and (t, v). Recall that for the double integral, we put positive signs on the two extreme points (top right and bottom left) and negative signs on the other two points (top left and bottom right). See more in the lecture notes.

Performance review: 20 out of 24 people got this. 2 chose (D), 1 each chose (A) and (B).

Historical note (last time): 8 out of 19 people got this correct. 6 people chose (D), 2 each chose (A) and (E), 1 chose (B).

(8) (**) *You can discuss this!* Suppose f is an elementarily integrable function, but $f(x^k)$ is not elementarily integrable for any integer $k > 1$ (examples are sin, exp, cos). For which of the following types of regions D are we *guaranteed to be able* to compute, in elementary function terms, the double integral $\int_D \int f(x^2)\, dA$ over the region (note that f is just a function of x, but we treat it as a function of two variables)? Please see Option (E) before answering and select that if applicable. *Last time: 1/19 correct.*

(A) A rectangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, and (a, b), with $a, b > 0$.
(B) A triangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, with $a, b > 0$.
(C) A triangle with vertices $(0, 0)$, $(0, b)$, (a, b), with $a, b > 0$.
(D) A triangle with vertices $(0, 0)$, $(a, 0)$, (a, b), with $a, b > 0$.
(E) All of the above

Answer: Option (D)

Explanation: For such a triangle, we integrate on y inner and x outer. For a fixed value of x, the y-value ranges from 0 to bx/a, so the integral becomes:

$$\int_0^a \int_0^{bx/a} f(x^2)\,dy\,dx$$

$f(x^2)$ pulls out of the inner integral and the inner integral just gives (bx/a), so we get:

$$\int_0^a \frac{b}{a} x f(x^2)\,dx$$

This can be done by the substitution $u = x^2$ and the knowledge that f is elementarily integrable.

All the other integrations stumble because they require knowledge of an antiderivative of $f(x^2)$.

Performance review: 1 out of 24 people got this. 7 chose (A), 6 each chose (B) and (E), and 4 chose (C).

Historical note (last time): 1 person got this correct. 8 chose (E), 5 chose (A), 3 chose (B), 2 chose (C).