DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 8 (DELAYED TO MONDAY NOVEMBER 11): IMAGE AND KERNEL (COMPUTATIONAL)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

26 people took this 3-question quiz. The score distribution was as follows:

- Score of 0: 2 people
- Score of 1: 5 people
- Score of 2: 8 people
- Score of 3: 11 people

The mean score was 2.08.

The question-wise answers and performance review were as follows:

(1) Option (E): 20 people
(2) Option (A): 15 people
(3) Option (B): 19 people

Note: Performance on these questions was notably better than last year, even though last year these questions were in the 'please feel free to discuss" category. Seems like people are understanding the image and kernel better this year!

2. Solutions

PLEASE DO NOT DISCUSS *ANY* QUESTIONS.

(1) *Do not discuss this!*: Consider the linear transformation $\text{Avg} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$\text{Avg} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (x+y)/2 \\ (x+y)/2 \end{bmatrix}$$

What can we say about the kernel and image of Avg? Note that in our descriptions of the kernel and the image below, we use x to denote the first coordinate of the vector and y to denote the second coordinate of the vector.

Note: One way you can do that is to write the matrix for Avg, but in this particular situation, it's easiest to just do things directly.

(A) The kernel is the zero subspace and the image is all of $\mathbb{R}^2$
(B) The kernel is the line $y = x$ and the image is also the line $y = x$
(C) The kernel is the line $y = x$ and the image is the line $y = -x$
(D) The kernel is the line $y = -x$ and the image is also the line $y = -x$
(E) The kernel is the line $y = -x$ and the image is the line $y = x$

Answer: Option (E)

Explanation: The kernel must satisfy that both coordinates of the output are zero, so $(x+y)/2 = 0$ and $(x+y)/2 = 0$. The solution is $y = -x$.

The image must satisfy that both coordinates are equal, so it is the line $y = x$.

Explicitly, the matrix in question is:

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

The rref for this is:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The kernel is generated by the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The image is generated by $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Performance review: 20 out of 26 got this. 5 chose (D), 1 chose (B).

Historical note (last time): 10 out of 26 got this. 7 chose (B), 6 chose (D), 3 chose (C).

(2) *Do not discuss this!*: Consider the *average of other two* linear transformation $\nu : \mathbb{R}^3 \to \mathbb{R}^3$ given as follows:

$$\nu = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} (y+z)/2 \\ (z+x)/2 \\ (x+y)/2 \end{bmatrix}$$

What can we say about the kernel and image of ν?

Note that in our descriptions of the kernel and the image below, we use x to denote the first coordinate of the vector, y to denote the second coordinate of the vector, and z to denote the third coordinate of the vector.

Note: This can both be reasoned directly (without any knowledge of linear algebra) or alternatively it can be done by writing the matrix of ν and computing its rank, image, and kernel.

(A) The kernel is the zero subspace and the image is all of $\mathbb{R}^3$
(B) The kernel is the line $x = y = z$ (one-dimensional) and the image is the plane $x + y + z = 0$ (two-dimensional)
(C) The kernel is the plane $x + y + z = 0$ (two-dimensional) and the image is the line $x = y = z$ (one-dimensional)
(D) The kernel is the plane $x = y = z$ (two-dimensional) and the image is the line $x + y + z = 0$ (one-dimensional)
(E) The kernel is the line $x + y + z = 0$ (one-dimensional) and the image is the plane $x = y = z$ (two-dimensional)

Answer: Option (A)

Explanation: We can check that if all three outputs are zero, then $x = y = z = 0$. Alternatively, we can verify that the matrix:

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

has full rank 3. Its inverse is the matrix:

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Intuitively, x is the sum of the second and third output minus the first output, and so on.

Performance review: 15 out of 26 got this. 6 chose (C), 5 chose (B).

Historical note (last time): 4 out of 26 got this. 11 chose (E), 7 chose (D), 2 each chose (B) and (C).

(3) *Do not discuss this!*: Consider the *difference of other two* linear transformation $\mu : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$\mu = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} y - z \\ z - x \\ x - y \end{bmatrix}$$

What can we say about the kernel and image of μ?

Note that in our descriptions of the kernel and the image below, we use x to denote the first coordinate of the vector, y to denote the second coordinate of the vector, and z to denote the third coordinate of the vector.

Note: This can both be reasoned directly (without any knowledge of linear algebra) or alternatively it can be done by writing the matrix of μ and computing its rank, image, and kernel.

(A) The kernel is the zero subspace and the image is all of $\mathbb{R}^3$

(B) The kernel is the line $x = y = z$ (one-dimensional) and the image is the plane $x + y + z = 0$ (two-dimensional)

(C) The kernel is the plane $x + y + z = 0$ (two-dimensional) and the image is the line $x = y = z$ (one-dimensional)

(D) The kernel is the plane $x = y = z$ (two-dimensional) and the image is the line $x + y + z = 0$ (one-dimensional)

(E) The kernel is the line $x + y + z = 0$ (one-dimensional) and the image is the plane $x = y = z$ (two-dimensional)

Answer: Option (B)

Explanation: The kernel must satisfy that all the three output coordinates are 0. This means that $y - z = 0$, $z - x = 0$, and $x - y = 0$, so we get $x = y = z$. The image must satisfy that the sum of the three coordinates is zero, so it lies in the plane $x + y + z = 0$. A little more effort can show that it is equal to the entire plane.

The matrix for the linear transformation μ is:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

We can row reduce this to get the rref:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rref shows that the linear transformation has rank two. The third variable is non-leading, so the kernel is generated by the vector:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Explicitly, the kernel is the line $x = y = z$.

For computing the image: the first two variables are leading variables, so the image of the linear transformation is the space spanned by the first two columns of the original matrix, i.e., it is the subspace spanned by the vectors:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

We want to express this as a plane. The appropriate approach to determining the equation of the plane is to take the cross product of the vectors, albeit that is a construct specific to three dimensions. Alternatively, we can solve another linear system. Doing this from sratch is somewhat beyond our current scope, but it is relatively easy to figure out the plane from the collection of options presented.

Performance review: 19 out of 26 got this. 4 chose (D), 2 chose (A), 1 chose (E).

Historical note (last time): 8 out of 26 got this. 15 chose (D), 2 chose (C), and 1 chose (E).