DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: DUE MONDAY NOVEMBER 25: SUBSPACE, BASIS, AND DIMENSION

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

23 people took this 3-question quiz. The score distribution was as follows:

- Score of 0: 2 people
- Score of 1: 8 people
- Score of 2: 6 people
- Score of 3: 7 people

The mean score was 1.78.

The question-wise answers and performance review were as follows:

(1) Option (A): 15 people
(2) Option (E): 11 people
(3) Option (C): 15 people

2. Solutions

PLEASE DO NOT DISCUSS *ANY* QUESTIONS.

This quiz covers material related to the `Linear dependence, bases and subspaces` notes corresponding to Sections 3.2 and 3.3 of the text.

Keep in mind the following facts. Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. Suppose A is the matrix for T, so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$. Then, A is a $n \times m$ matrix. Further, the following are true:

- The dimension of the image of T equals the rank of A.
- The dimension of the kernel of T, called the *nullity* of A, is m minus the rank of A.

(1) *Do not discuss this!*: Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. What is the best we can say about the dimension of the image of T?
  (A) It is at least 0 and at most $\min\{m, n\}$. However, we cannot be more specific based on the given information.
  (B) It is at least 0 and at most $\max\{m, n\}$. However, we cannot be more specific based on the given information.
  (C) It is at least $\min\{m, n\}$ and at most $\max\{m, n\}$. However, we cannot be more specific based on the given information.
  (D) It is at least $\min\{m, n\}$ and at most $m + n$. However, we cannot be more specific based on the given information.
  (E) It is at least $\max\{m, n\}$ and at most $m + n$. However, we cannot be more specific based on the given information.

  Answer: Option (A)

  Explanation: The dimension of the image equals the rank of the matrix for T, which is a $n \times m$ matrix, hence is at most $\min\{m, n\}$. It is at least 0 for obvious reasons. It is easy to see that there exist examples for each possible dimension ranging from 0 to $\min\{m, n\}$.

  Performance review: 15 out of 23 people got this. 5 chose (E), 2 chose (B), 1 chose (C).

  Historical note (last time): 15 out of 26 got this. 5 chose (C), 3 each chose (B) and (D).

(2) *Do not discuss this!*: Suppose $T_1, T_2 : \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations. Suppose the images of T_1 and T_2 have dimensions d_1 and d_2 respectively. What can we say about the dimension of the image of $T_1 + T_2$? Assume that both m and n are larger than $d_1 + d_2$.

(A) It is precisely $|d_2 - d_1|$.
(B) It is precisely $\min\{d_1, d_2\}$.
(C) It is precisely $\max\{d_1, d_2\}$.
(D) It is precisely $d_1 + d_2$.
(E) Based on the information, it could be any integer r with $|d_2 - d_1| \leq r \leq d_1 + d_2$.

Answer: Option (E)

Explanation: The image of $T_1 + T_2$ is contained in the sum of the images of T_1 and T_2, hence its dimension is at most the dimension of the sum of the images of T_1 and T_2. The dimension of the sum of subspaces is at most equal to the sum of the dimensions (because we can take the union of the spanning sets). Thus, the dimension of the image of $T_1 + T_2$ is at most equal to $d_1 + d_2$.

For the lower bound of $|d_2 - d_1|$, note that the dimension of the image of T_1 is at most the sum of the dimensions of the images of $T_1 + T_2$ and of T_2 and also that the dimension of the image of T_2 is at most the sum of the dimensions of the images of $T_1 + T_2$ and of T_1. This gives lower bounds of $d_1 - d_2$ and $d_2 - d_1$ respectively on the dimension of the image of $T_1 + T_2$. The maximum of these is $|d_2 - d_1|$, which is the lower bound.

It is easy to construct examples of diagonal matrices with 0, 1 and -1 as the diagonal entries to realize any r with $|d_2 - d_1| \leq r \leq d_1 + d_2$.

Performance review: 11 out of 23 people got this. 7 chose (C), 3 chose (B), 2 chose (D).

Historical note (last time): 12 out of 26 got this. 8 chose (B), 2 each chose (A) and (C), 1 chose (D).

(3) *Do not discuss this!*: Suppose V_1 and V_2 are subspaces of $\mathbb{R}^n$. We define the sum $V_1 + V_2$ as the subset of $\mathbb{R}^n$ comprising all vectors that can be expressed as a sum of a vector in V_1 and a vector in V_2. Define $V_1 \cup V_2$ as the set-theoretic union of V_1 and V_2, i.e., the set of all vectors that are either in V_1 or in V_2. What can we say about these?
(A) $V_1 \cup V_2 = V_1 + V_2$ and it is a subspace of $\mathbb{R}^n$.
(B) $V_1 \cup V_2$ is contained in $V_1 + V_2$ and both are subspaces of $\mathbb{R}^n$.
(C) $V_1 \cup V_2$ is contained in $V_1 + V_2$, and $V_1 + V_2$ is a subspace of $\mathbb{R}^n$. $V_1 \cup V_2$ is generally not a subspace of $\mathbb{R}^n$ (though it might be in special cases).
(D) $V_1 \cup V_2$ contains $V_1 + V_2$, and both are subspaces of $\mathbb{R}^n$.
(E) $V_1 \cup V_2$ contains $V_1 + V_2$, and $V_1 \cup V_2$ is a subspace of $\mathbb{R}^n$. $V_1 + V_2$ is generally not a subspace of $\mathbb{R}^n$ (though it might be in special cases).

Answer: Option (C)

Explanation: $V_1 + V_2$ is defined as the set of all vectors that can be expressed as the sum of a vector in V_1 and a vector in V_2. In particular, it contains both V_1 and V_2. The reason it contains V_1 is that any vector in V_1 can be written as itself plus the *zero vector* of V_2. The reason it contains V_2 is that any vector in V_2 can be written as the *zero vector* of V_1 plus that vector itself.

Thus, $V_1 \cup V_2$ is contained in $V_1 + V_2$. They need not be equal. For instance, consider the case that V_1 is the span of $\vec{e}_1$ and V_2 is the span of $\vec{e}_2$ inside $\mathbb{R}^n$. The union of these is the set of vectors that are on either of the axes. It is a union of two perpendicular lines. The sum on the other hand is the plane spanned by the first two coordinates. The sum includes linear combinations where both spaces contribute nontrivially.

This also hints at why $V_1 \cup V_2$ does not need to be a subspace: it contains both subspaces, but not the vectors that are obtained by combining nonzero vectors from both subspaces. In fact, $V_1 \cup V_2$ is a subspace if and only if it equals $V_1 + V_2$, and this happens if and only if either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Performance review: 15 out of 23 got this. 5 chose (D), 2 chose (B), 1 chose (E).

Historical note (last time): 8 out of 26 got this. 10 chose (B), 4 chose (E), 2 chose (D), 1 chose (A).