TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FEBRUARY 24: SEQUENCES AND SERIES, MISCELLANEOUS STUFF

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

The score distribution was as follows:

- Score of 3: 1 person
- Score of 6: 1 person
- Score of 7: 3 people
- Score of 8: 3 people
- Score of 9: 2 people
- Score of 13: 1 person

The mean score was 7.73. The question wise answers and performance summary are below:

(1) Option (B): 9 people
(2) Option (B): 11 people
(3) Option (B): 8 people
(4) Option (B): 8 people
(5) Option (A): 6 people
(6) Option (B): 2 people
(7) Option (C): 5 people
(8) Option (B): 4 people
(9) Option (D): 5 people
(10) Option (E): 6 people
(11) Option (E): 4 people
(12) Option (A): 5 people
(13) Option (D): 2 people
(14) Option (D): 6 people
(15) Option (C): 4 people

2. Solutions

(1) *Forward difference operators and partial sums*: Recall that for a function $g : \mathbb{N} \to \mathbb{R}$, the forward difference operator of g, denoted Δg, is defined as the function $(\Delta g)(n) = g(n+1) - g(n)$. Suppose we have two functions $f, g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = \sum_{k=1}^{n} f(k)$. What is the relationship between Δg and f? *This is a discrete version of the fundamental theorem of calculus.*
(A) $(\Delta g)(n) = f(n)$ for all $n \in \mathbb{N}$
(B) $(\Delta g)(n) = f(n+1)$ for all $n \in \mathbb{N}$
(C) $(\Delta g)(n+1) = f(n)$ for all $n \in \mathbb{N}$
(D) $(\Delta g)(n) = f(n+2)$ for all $n \in \mathbb{N}$
(E) $(\Delta g)(n+2) = f(n)$ for all $n \in \mathbb{N}$

Answer: Option (B)

Explanation: We have:

$$g(n+1) = \sum_{k=1}^{n+1} f(k) = f(1) + f(2) + \cdots + f(n) + f(n+1)$$

and:

$$g(n) = \sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n)$$

Subtracting, we obtain:

$$(\Delta g)(n) = f(n+1)$$

Performance review: 9 people got this correct. 1 each chose (A) and (C).

(2) Which of the following is the correct definition of $\lim_{x\to\infty} f(x) = L$ for L a finite number?
   (A) For every $\epsilon > 0$ there exists $a \in \mathbb{R}$ such that if $0 < |x - L| < \epsilon$ then $f(x) > a$.
   (B) For every $\epsilon > 0$ there exists $a \in \mathbb{R}$ such that if $x > a$ then $|f(x) - L| < \epsilon$.
   (C) For every $a \in \mathbb{R}$ there exists $\epsilon > 0$ such that if $x > a$ then $|f(x) - L| < \epsilon$.
   (D) For every $a \in \mathbb{R}$ there exists $\epsilon > 0$ such that if $0 < |x - L| < \epsilon$ then $f(x) > a$.
   (E) There exists $a \in \mathbb{R}$ and $\epsilon > 0$ such that if $x > a$ then $|f(x) - L| < \epsilon$.

   Answer: Option (B)

   Explanation: Straightforward unraveling of the definition.

   Performance review: 11 people got this.

   Historical note (last year): 21 out of 25 people got this correct. 1 person each chose options (A), (C), (D), and (E).

(3) *Horizontal asymptote and limit of derivative*: Suppose $\lim_{x\to\infty} f'(x)$ is finite. Which of the following is true (be careful about f versus f' when reading the choices)?
   (A) If $\lim_{x\to\infty} f'(x)$ is zero, then $\lim_{x\to\infty} f(x)$ is finite.
   (B) If $\lim_{x\to\infty} f(x)$ is finite, then $\lim_{x\to\infty} f'(x)$ is zero.
   (C) If $\lim_{x\to\infty} f(x)$ is finite, then $\lim_{x\to\infty} f(x)$ is zero.
   (D) All of the above.
   (E) None of the above.

   Answer: Option (B)

   Explanation: If f is going to a finite value, its derivative cannot have a nonzero limit, because that would mean a roughly linear behavior. Since the derivative has a limit, it must go to zero.

   Note that *if we were not explicitly told that the derivative has a limit*, then option (E) would be the correct option.

   Note that option (A) is wrong, because we can construct counterexamples such as ln and the square root function, where the derivative of the function goes to zero but the function does not have a finite limit at the point.

   Performance review: 8 out of 11 people got this correct. 2 chose (A), 1 had an ambiguous entry.

   Historical note (last year): 8 out of 25 people got this correct. 10 people chose (A), 5 people chose (E), and 2 people chose (C).

   Action point: Make sure you understand this and get it right in the future!

(4) *Convergent sequence and limit of forward difference operator*: Suppose $f : \mathbb{N} \to \mathbb{R}$ is a function (so we can think of it as a sequence). Which of the following is true? Here $(\Delta f)(n) = f(n+1) - f(n)$.
   (A) If $\lim_{n\to\infty}(\Delta f)(n)$ is zero, then $\lim_{n\to\infty} f(n)$ is finite.
   (B) If $\lim_{n\to\infty} f(n)$ is finite, then $\lim_{n\to\infty}(\Delta f)(n)$ is zero.
   (C) If $\lim_{n\to\infty} f(n)$ is finite, then $\lim_{n\to\infty} f(n)$ is zero.
   (D) All of the above.
   (E) None of the above.

   Answer: Option (B)

   Explanation: If $\lim_{n\to\infty} f(n) = L$, then for every $\epsilon > 0$ there exists a large enough value n_0 such that for $n > n_0$, we have $|f(n) - L| < \epsilon$. Using the triangle inequality, we get that $|f(n+1) - f(n)| < 2\epsilon$, so $\lim_{n\to\infty}(\Delta f)(n) = 0$.

   This is pretty much the sequence analogue of the preceding question, and the square root and logarithm functions provide counterexamples to (A). The key difference between the sequence version and the continuous version is that in the latter case, we need to explicitly assume that the derivative converges and *then* can show it must go to zero, but in the sequence case, convergence does not need to be assumed.

Performance review: 8 out of 11 got this correct. 3 chose (A).

(5) *Function iteration converges at infinity*: Suppose (a_n) is a sequence whose terms are given by the relation $a_n = f(a_{n-1})$, with a_1 specified separately and f is a continuous function on $\mathbb{R}$. Further, suppose we know that $\lim_{n\to\infty} a_n = L$ for some finite L. What can we conclude is true about L?
(A) $f(L) = L$
(B) $f(L) = 0$
(C) $f'(L) = L$
(D) $f'(L) = 0$
(E) $f''(L) = 0$

Answer: Option (A)

Explanation: This was one of your homework problems (Advanced 3 on HW 7).

Performance review: 6 out of 11 people got this correct. 4 chose (D), 1 chose (C).

(6) *Equilibrium at infinity*: Suppose a function y of time t satisfies the differential equation $y' = f(y)$ for all time t, where f is a continuous function on $\mathbb{R}$. Further, suppose we know that $\lim_{t\to\infty} y = L$ for some finite L. What can we conclude is true about L? *Note: Although the question is conceptually similar to the preceding question, you have to reason about the question differently.*
(A) $f(L) = L$
(B) $f(L) = 0$
(C) $f'(L) = L$
(D) $f'(L) = 0$
(E) $f''(L) = 0$

Answer: Option (B)

Explanation: We in particular have that $\lim_{t\to\infty} y' = \lim_{t\to\infty} f(y) = f(L)$, hence $\lim_{t\to\infty} y'$ is finite. Applying the previous question, we obtain that this limit must be 0, so we get $f(L) = 0$.

Performance review: 2 got this correct, 4 chose (D), 3 chose (C), and 2 chose (E).

Historical note (last year): 11 out of 25 people got this correct. 6 people chose (D), 5 people chose (C), and 3 people chose (A).

Those who chose option (A) probably used the discrete analogy. While the analogy works qualitatively, the conclusion was wrongly applied. Here, f is not the function being iterated (which is the discrete setup) but rather, it is the derivative of a changing value. When the value becomes constant, f must become zero.

(7) A sequence a_n is found to satisfy the recurrence $a_{n+1} = 2a_n(1 - a_n)$. Assume that a_1 is strictly between 0 and 1. What can we say about the sequence (a_n)?
(A) It is monotonic non-increasing, and its limit is 0.
(B) It is monotonic non-decreasing, and its limit is 1.
(C) From a_2 onward, it is monotonic non-decreasing, and its limit is 1/2.
(D) From a_2 onward, it is monotonic non-increasing, and its limit is 1/2.
(E) It is either monotonic non-decreasing or monotonic non-increasing everywhere, and its limit is 1/2.

Answer: Option (C)

Explanation: Whatever the value of a_1, $0 < a_2 \leq 1/2$. Once we are in this interval, we see that $f(x) \geq x$ for all x in the interval, and $f(x)$ is also in the interval. Thus, the sequence is monotonic non-decreasing from a_2 onward, and is bounded from above by 1/2. It converges to its greatest lower bound, which we know must be fixed under f. Hence, it must converge to 1/2, which is the only positive number fixed under f.

(More details can be worked out using algebra/calculus).

Performance review: 5 out of 11 got this correct. 4 chose (D), 1 each chose (B) and (E).

Historical note (last year): 8 out of 25 people got this correct. 8 people chose (D), 4 chose (A), 3 chose (E), 2 chose (B).

(8) Suppose f is a continuous function on $\mathbb{R}$ and (a_n) is a sequence satisfying the recurrence $f(a_n) = a_{n+1}$ for all n. Further, suppose the limit of the a_ns for odd n is L and the limit of the a_ns for even n is M. What can we say about L and M?
(A) $f(L) = L$ and $f(M) = M$
(B) $f(L) = M$ and $f(M) = L$
(C) $f(L) = f(M) = 0$
(D) $f'(L) = f'(M) = 0$
(E) $f'(L) = M$ and $f'(M) = L$

Answer: Option (B)

Explanation: Each even indexed term is obtained by applying f to the preceding odd indexed term, and each odd indexed term is obtained by applying f to the preceding even indexed term. Taking appropriate limits, we get the desired conclusion.

Performance review: 4 out of 11 people got this correct. 3 chose (A), 2 chose (D), 1 each chose (C) and (E).

Historical note (last year): 7 out of 25 people got this correct. 8 chose (A), 4 each chose (C) and (D), 1 chose (E), and 1 left the question blank.

Action point: After understanding the solution, you should not forget the idea!

(9) Consider a function f on the natural numbers defined as follows: $f(m) = m/2$ if m is even, and $f(m) = 3m + 1$ if m is odd. Consider a sequence where a_1 is a natural number and we define $a_n := f(a_{n-1})$. It is conjectured (see *Collatz conjecture*) that (a_n) is eventually periodic, regardless of the starting point, and that there is only one possibility for the eventual periodic fragment. Which of the following can be the eventual periodic fragment?
(A) $(1, 2, 3)$
(B) $(1, 3, 2)$
(C) $(1, 2, 4)$
(D) $(1, 4, 2)$
(E) $(1, 3, 4)$

Answer: Option (D)

Explanation: Can be seen by applying the definition.

Performance review: 5 out of 11 people to this correct. 3 chose (C), 2 chose (E), 1 chose (A).

Historical note (last year): 13 out of 25 people got this correct. 5 chose (C), 3 chose (B), 2 chose (A), 1 chose (E), and 1 left the question blank.

Those who chose (C) probably overlooked the issue of the cyclic ordering of elements within the periodic sequence.

Extra credit: Prove the Collatz conjecture.

(10) For which of the following properties p of sequences of real numbers does p equal *eventually* p?
(A) Monotonicity
(B) Periodicity
(C) Being a polynomial sequence (i.e., given by a polynomial function)
(D) Being a constant sequence
(E) Boundedness

Answer: Option (E)

Explanation: If a sequence is eventually bounded, then that means that excluding the first few terms gives a bounded sequence. But throwing back these finitely many terms, which have a fixed maximum and minimum, will still give a bounded sequence.

Performance review: 6 out of 11 people got this correct. 2 each chose (A) and (D), 1 chose (C).

Historical note (last year): 10 out of 25 people got this correct. 5 each chose (A) and (D), 2 each chose (B) and (C), and 1 left the question blank.

Action point: You should definitely understand, appreciate, and remember this one!

The remaining questions are based on a rule which we call the *degree difference rule*. This states the following. Consider a rational function $p(x)/q(x)$, and suppose $a \in \mathbb{R}$ is such that q has no roots in $[a, \infty)$. Then, the improper integral $\int_a^\infty \frac{p(x)}{q(x)}\,dx$ is finite if and only if the degree of q minus the degree of p is *at least* two. The same rule applies to $\int_{-\infty}^\infty \frac{p(x)}{q(x)}\,dx$ if q has no zero.

The degree difference rule has a slight variation: we can apply it to situation where p and q are not quite polynomials, but rather their growth rates are of the same order as that of some polynomial or power function. For instance, $(x^2+1)^{3/2}$ has "degree" three with this more liberal interpretation.

Consider a probability distribution on $\mathbb{R}$ with density function f. In particular, this means that $\int_{-\infty}^\infty f(x)\,dx = 1$. Further, assume that f has mean zero and is an even function, i.e., the probability distribution is symmetric about zero.

The *mean deviation* of the distribution is defined as $\int_{-\infty}^\infty |x| f(x)\,dx$. On account of the fact that f is an even function, this can be rewritten as $2\int_0^\infty x f(x)\,dx$.

The *standard deviation* of the distribution, denoted σ, of f is defined as $\sqrt{\int_{-\infty}^\infty x^2 f(x)\,dx}$.

The *kurtosis* of the distribution is defined as $-3 + (\int_{-\infty}^\infty x^4 f(x)\,dx)/\sigma^4$. Note that the kurtosis does not make sense if the standard deviation is infinite.

(11) Consider the distribution with density function $f(x) := (x^2+1)^{-1}/\pi$. (We divide by π so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?

(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Answer: Option (E)

Explanation: Ignoring constants, the integrals that need to be computed for the mean deviation and standard deviation are respectively:

$$\int_0^\infty \frac{x\,dx}{x^2+1}$$

and

$$\int_{-\infty}^\infty \frac{x^2\,dx}{x^2+1}$$

The degree difference for mean deviation is $2-1=1$ and the degree difference for standard deviation is $2-2=0$. By the degree difference rule, both these integrals diverge. It is also possible to compute the integrals explicitly and check that they diverge.

Performance review: 4 out of 11 people got this correct. 2 each chose (A), (B), and (C), 1 chose (D).

(12) Consider the distribution with density function $f(x) := (x^2+1)^{-3/2}/2$. (We divide by 2 so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?

(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Answer: Option (A)

Explanation: Ignoring constants, the integrals that need to be computed for the mean deviation and standard deviation are respectively:

$$\int_0^\infty \frac{x\,dx}{(x^2+1)^{3/2}}$$

and

$$\int_{-\infty}^\infty \frac{x^2\,dx}{(x^2+1)^{3/2}}$$

The "degree" of the denominator is 3 (obtained as $2 \times (3/2) = 3$).

The degree difference for the mean deviation is $3-1 = 2$. Thus, the integral for the mean deviation satisfies the degree difference rule, hence it converges.

The degree difference for the standard deviatin is $3 - 2 = 1$. Hence, the integral for the standard deviation does *not* satisfy the degree difference rule, hence it does not converge.

Bonus observation: If we were to actually calculate the mean deviation, we would get:

$$\frac{2}{2} \int_0^\infty \frac{x\,dx}{(x^2+1)^{3/2}}$$

This simplifies to:

$$\left[\frac{-1}{\sqrt{x^2+1}} \right]_0^\infty$$

This further simplifies to 1.

Performance review: 5 out of 11 got this correct. 4 chose (B), 1 each chose (C) and (E).

(13) Consider the distribution with density function $f(x) := (x^2+1)^{-2}/(\pi/2)$. (We divide by $\pi/2$ so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?

(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Answer: Option (D)

Explanation: Ignoring constants, the integrals that need to be computed for the mean deviation and standard deviation are respectively:

$$\int_0^\infty \frac{x\,dx}{(x^2+1)^2}$$

and

$$\int_{-\infty}^\infty \frac{x^2\,dx}{(x^2+1)^2}$$

The degree of the denominator is 4.

The degree difference for the mean deviation integral is $4-1 = 3$. Thus, the integral for the mean deviation satisfies the degree difference rule, hence it converges.

The degree difference for the standard deviation is $4 - 2 = 2$. Thus, the integral for the standard deviation converges as well.

Finally, we consider the integral for the kurtosis:

$$\int_{-\infty}^\infty \frac{x^4}{(x^2+1)^2}\,dx$$

The degree difference is $4 - 4 = 0$, so this integral does not converge.

Bonus observation: The mean deviation is $2/\pi$ and the standard deviation is 1.

If we were to actually calculate the mean deviation, we would get:

$$\frac{2}{\pi/2} \int_0^\infty \frac{x\,dx}{(x^2+1)^2}$$

This simplifies to:

$$\frac{2}{\pi} \left[\frac{-1}{x^2+1} \right]_0^\infty$$

This further simplifies to $2/\pi$. The mean deviation is thus $2/\pi$.

For the standard deviation, we need to compute:

$$\sqrt{\frac{1}{\pi/2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2}}$$

The integral simplifies to $\pi/2$, so the standard deviation is $\sqrt{1} = 1$.

Performance review: 2 out of 11 got this correct. 7 chose (C), 2 chose (B).

(14) Consider the distribution with density function $f(x) := (x^2+1)^{-5/2}/(4/3)$. (We divide by 4/3 so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?

(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Answer: Option (D)

Explnation: With reasoning similar to the previous questions, the degree differences for the mean deviation, standard deviation, and kurtosis are respectively $5-1=4$, $5-2=3$, and $5-4=1$. By the degree difference rule, the first two exist, but the third does not.

Performance review: 6 out of 11 got this correct. 2each chose (A) and (E), 1 chose (B).

(15) Consider the distribution with density function $f(x) := (x^2+1)^{-3}/(3\pi/8)$. (We divide by $3\pi/8$ so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?

(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Answer: Option (C)

Explanation: With reasoning similar to the previous questions, the degree differences for the mean deviation, standard deviation, and kurtosis are respectively $6-1=5$, $6-2=4$, and $6-4=2$. By the degree difference rule, all the integrals converge.

Performance review: 4 out of 11 people got this correct. 4 chose (B), 2 chose (D), 1 chose (E).