TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FEBRUARY 15: INTERPLAY OF CONTINUOUS AND DISCRETE

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this quiz. The score distribution was as follows:

- Score of 4: 1 person
- Score of 5: 1 person
- Score of 6: 7 people
- Score of 7: 2 people

The mean score was 5.91. The question wise answers and performance review were as follows:

(1) Option (C): 11 people (everybody)
(2) Option (D): 11 people (everybody)
(3) Option (B): 9 people
(4) Option (B): 2 people (the groupthink folk all got it wrong!)
(5) Option (D): 2 people (the groupthink folk all got it wrong!)
(6) Option (E): 11 people (everybody)
(7) Option (D): 9 people
(8) Option (D): 10 people

2. Solutions

(1) Consider a function f defined on all real numbers. Consider also the sequence $a_n = f(n)$ defined for n a natural number. Which of the following is true?
(A) $\lim_{x \to \infty} f(x)$ is finite if and only if $\lim_{n \to \infty} a_n$ is finite, and if so, both limits are equal.
(B) $\lim_{x \to \infty} f(x)$ is finite if and only if $\lim_{n \to \infty} a_n$ is finite, but the limits need not be equal.
(C) If $\lim_{x \to \infty} f(x)$ is finite, then $\lim_{n \to \infty} a_n$ is finite, but the converse is not true. Moreover, if both limits are finite, they must be equal.
(D) If $\lim_{n \to \infty} a_n$ is finite, then $\lim_{x \to \infty} f(x)$ is finite, but the converse is not true. Moreover, if both limits are finite, they must be equal.
(E) It is possible for either of the limits $\lim_{x \to \infty} f(x)$ and $\lim_{n \to \infty} a_n$ to be finite, but for the other one not to be finite. Moreover, even if both limits exist, they need not be equal.

Answer: Option (C)

Explanation: The key idea is that the values at the natural numbers only form a part of the behavior of the function. If the function as a whole has a finite limit L at infinity, then that means that for every ϵ there exists a value of A such that $|f(x) - L| < \epsilon$ for all real $x > A$.

This in turn forces that all the values that the function takes at *integers* bigger than A is also within ϵ-distance of L. Thus, $\lim_{n \to \infty} a_n = L$.

The converse is not true because the function outside of the integers could behave in a completely different way. For instance, take $f(x) = \sin(\pi x)$. We get $a_n = 0$ for all n. $\lim_{n \to \infty} a_n = 0$ but $\lim_{x \to \infty} f(x)$ does not exist.

See the lecture notes on the interplay between continuous and discrete.

Performance review: All 11 people got this.

(2) Consider a function $f : \mathbb{R} \to \mathbb{R}$. Restricting the domain of f to the natural numbers, obtain a sequence whose n^{th} member a_n is defined as $f(n)$. Which of the following statements is **false** about the relationship between f and the sequence (a_n)?
(A) If f is an increasing function, then (a_n) form an increasing sequence.

(B) If f is a decreasing function, then (a_n) form a decreasing sequence.
(C) If f is a bounded function, (i.e., its range is a bounded set) then (a_n) form a bounded sequence.
(D) If f is a periodic function, then (a_n) form a periodic sequence.
(E) If f has a limit at infinity, then (a_n) is a convergent sequence.

Answer: Option (D)

Explanation: (A), (B), (C), and (E) are immediately true (see the lecture notes for more information). As for option (D), the problem with it is that f may not have an *integer* period even though it is periodic. For instance, if we set $f = \sin$, then f is periodic, but its period is 2π which has no multiple that is an integer, on account of π being irrational.

Performance review: All 11 people got this.

(3) We are given a sequence $a_1, a_2, \dots, a_n, \dots$ of real numbers. The goal is to find a *continuous* function f on all of $\mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$. Which of the following is true?
(A) There is a unique choice of f that works.
(B) There exist infinitely many different choices of f that work.
(C) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero, one, or infinite.
(D) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero or one. It can never be infinite.
(E) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be one or infinite. It can never be zero.

Answer: Option (B)

Explanation: Imagine the graph of the sequence, i.e., we plot the points (n, a_n) in the coordinate plane for all $n \in \mathbb{N}$. The goal is to find a continuous function whose graph passes through all these points. We could do this in many ways. For instance, for each pair of adjacent points, we could join them up by a line segment or some other continuous curve. And to the left of 1 we could do any of a number of things.

Performance review: 9 out of 11 got this. 1 chose (C), 1 chose (E).

(4) We are given a sequence $a_1, a_2, \dots, a_n, \dots$ of real numbers. The goal is to find an *infinitely differentiable* function f on all of $\mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$. Which of the following is true?
(A) There is a unique choice of f that works.
(B) There exist infinitely many different choices of f that work.
(C) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero, one, or infinite.
(D) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero or one. It can never be infinite.
(E) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be one or infinite. It can never be zero.

Answer: Option (B)

Explanation: The reasoning is similar to the previous problem, albeit there is more subtletly to this one. Stay tuned for more on this later in the course!

Performance review: 2 out of 11 got this. 8 chose (A), 1 chose (E).

Action point: Avoid the perils of groupthink. Follow your conscience.

(5) We are given a sequence $a_1, a_2, \dots, a_n, \dots$ of real numbers. The goal is to find a *polynomial* function f on all of $\mathbb{R}$ such that $f(n) = a_n$ for all $n \in \mathbb{N}$. Which of the following is true?
(A) There is a unique choice of f that works.
(B) There exist infinitely many different choices of f that work.
(C) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero, one, or infinite.
(D) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be zero or one. It can never be infinite.
(E) The number of possible choices of f depends on the sequence. Depending on the sequence, the number of possible choices of f may be one or infinite. It can never be zero.

Answer: Option (D)

Explanation: Imagine that there are two polynomials f and g that both satisfy $f(n) = g(n) = a_n$ for all $n \in \mathbb{N}$. Then, the polynomial $f - g$ is zero at all $n \in \mathbb{N}$. A polynomial can have infinitely many roots only if it is the zero polynomial, so $f - g = 0$ and $f = g$.

This shows that there is *at most* one polynomial function fitting the sequence. It is, however, possible for there to be no polynomial function. For instance, if we take a sequence that grows exponentially, such as $a_n = 2^n$, there will be no polynomial function fitting it.

Performance review: 2 out of 11 got this. 7 chose (B), 1 each chose (A) and (E).

For the remaining questions: For a function $f : \mathbb{N} \to \mathbb{R}$, define Δf as the function $n \mapsto f(n+1) - f(n)$. Denote by $\Delta^k f$ the function obtained by applying Δ k times to f.

(6) If $f(n) = n^2$, what is $(\Delta f)(n)$?
(A) 1
(B) n
(C) $2n - 1$
(D) $2n$
(E) $2n + 1$

Answer: Option (E)

Explanation: We get $f(n+1) - f(n) = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$.

Performance review: All 11 people got this.

(7) If f is expressible as a polynomial function of degree $d > 0$, what is the smallest k for which $\Delta^k f$ is identically the zero function? *Hint: Think of the analogous question using continuous derivatives. Although Δ differs from the continuous derivative, much of the qualitative behavior is the same.*
(A) $d - 2$
(B) $d - 1$
(C) d
(D) $d + 1$
(E) $d + 2$

Answer: Option (D)

Explanation: Every time we apply Δ, the degree of the polynomial goes down by one. After d applications to a polynomial of degree d, we get a constant polynomial. The $(d+1)^{th}$ application should therefore yield the zero polynomial.

Performance review: 9 out of 11 got this. 1 each chose (B) and (C).

(8) If f is a function such that $\Delta f = af$ for some positive constant a, and $f(1)$ is positive, which of the following best describes the nature of growth of f? *Hint: Think of the analogous differential equation using continuous derivatives. The precise solution is different but the nature of the solution is similar.*
(A) f grows like a sublinear function of n.
(B) f grows like a linear function of n.
(C) f grows like a superlinear but subexponential function of n.
(D) f grows like an exponential function of n.
(E) f grows like a superexponential function of n.

Answer: Option (D)

Explanation: The condition tells us that $f(n+1) - f(n) = af(n)$, so $f(n+1) = (a+1)f(n)$. Thus, each term is $(a+1)$ times its predecessor. Thus, the sequence grows exponentially, and the general term is $f(n) = (a+1)^{n-1} f(1)$.

Performance review: 10 out of 11 got this. 1 chose (B).