TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 18: LINEAR SYSTEMS: RANK AND DIMENSION CONSIDERATIONS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

27 people took this 8-question quiz. The score distribution was as follows:

- Score of 3: 1 person
- Score of 4: 4 people
- Score of 5: 7 people
- Score of 6: 6 people
- Score of 7: 8 people
- Score of 8: 1 person

The question-wise answers and performance review are as follows:

(1) Option (E): 24 people
(2) Option (B): 21 people
(3) Option (E): 10 people
(4) Option (E): 24 people
(5) Option (B): 24 people
(6) Option (B): 20 people
(7) Option (A): 17 people
(8) Option (C): 14 people

2. Solutions

The questions here consider a wide range of theoretical and practical settings where linear systems appear, and prompt you to think about the notion of rank and its relationship with whether we can uniquely acquire the information that we want. It relates approximately with the material in the `Linear systems and matrix algebra` notes (the corresponding section in the book is Section 1.3).

(1) (*) Let m and n be positive integers. It turns out that *almost all* $m \times n$ matrices over the real numbers have a particular rank. What is that rank? (Unfortunately, it is beyond our current scope to define "almost all").
   (A) m (regardless of whether m or n is bigger)
   (B) n (regardless of whether m or n is bigger)
   (C) $(m+n)/2$
   (D) $\max\{m,n\}$
   (E) $\min\{m,n\}$

   Answer: Option (E)

   Explanation: In some sense, this is the only viable option, because we know that the rank is at *most* $\min\{m,n\}$. The intuitive reason why the theoretical maximum is usually attained is that "random things are most likely to be unrelated."

   Performance review: 24 out of 27 got this. 2 chose (A), 1 chose (D).

   Historical note (last time): 23 out of 25 got this. 1 each chose (C) and (D).

(2) (*) A container has a mix of two known gases that do not react with each other. The temperature and pressure of the container are known. Assume that $PV = nRT$. The volume of the container is also known, and so is the total mass of the gases in the container. Under what conditions can we

predict the amount (say, in the form of the number of moles) of each gas that is present from this information?

(A) It is possible if both gases have the same molecular mass, because in that case, the coefficient matrix of the linear system has full rank 2.
(B) It is possible if both gases have different molecular masses, because in that case, the coefficient matrix of the linear system has full rank 2.
(C) It is possible if both gases have the same molecular mass, because in that case, the coefficient matrix of the linear system has rank 1.
(D) It is possible if both gases have different molecular masses, because in that case, the coefficient matrix of the linear system has rank 1.
(E) It is not possible to deduce the amount of each gas from the given information.

Answer: Option (B)

Explanation: Let n_1 and n_2 be the number of moles of each gas. Let m_1 and m_2 be their respective molecular masses. Then, we have two equations, where M is the total mass:

$$\begin{aligned} n_1 + n_2 &= PV/(RT) \\ n_1 m_2 + n_2 m_2 &= M \end{aligned}$$

The coefficient matrix for this is:

$$\begin{bmatrix} 1 & 1 \\ m_1 & m_2 \end{bmatrix}$$

This has rank one iff $m_1 = m_2$, and rank two otherwise.

If the coefficient matrix has rank one, then the second equation is redundant (note that it must be consistent since we are getting these numbers from an actual situation). In other words, we get a one-dimensional solution space, with a freely varying parameter. The nonnegativity of the number of moles of each gas does constrain the parameter to a closed bounded interval instead of all reals, but it still has infinitely many candidate values.

Performance review: 21 out of 27 got this. 4 chose (A), 2 chose (C).

Historical note (last time): 17 out of 25 got this. 4 chose (C), 3 chose (E), 1 chose (A).

(3) (*) A container has a mix of three known gases with no reactions between the gases. The temperature and pressure of the container are known. Assume that $PV = nRT$. The volume of the container is also known, and so is the total mass of the gases in the container. Under what conditions can we predict the amount (say, in the form of the number of moles) of each gas that is present from this information?

(A) It is possible if all three gases have the same molecular mass, because in that case, the coefficient matrix of the linear system has full rank 3.
(B) It is possible if all three gases have different molecular masses, because in that case, the coefficient matrix of the linear system has full rank 3.
(C) It is possible if all three gases have the same molecular mass, because in that case, the coefficient matrix of the linear system has rank 2.
(D) It is possible if all three gases have different molecular masses, because in that case, the coefficient matrix of the linear system has rank 2.
(E) It is not possible to deduce the amount of each gas from the given information.

Answer: Option (E)

Explanation: We have two equations in three variables. Explicitly, if the total mass is M, the number of moles of the three gases are n_1, n_2, and n_3, and the molecular masses are m_1, m_2, and m_3, then we have:

$$\begin{aligned} n_1 + n_2 + n_3 &= PV/(RT) \\ n_1m_1 + n_2m_2 + n_3m_3 &= M \end{aligned}$$

The coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

This matrix can have rank either one or two. The rank is one if all three gases have the same molecular mass. The rank is two otherwise. Note that in either case, the rank is less than three, i.e., the system does not have full column rank. Thus, it will not be possible to solve the system and uniquely determine the values of n_1, n_2, and n_3.

What if we include the nonnegativity constraints? Even in the presence of these constraints, a unique solution is not possible if there are nonzero amounts of all gases in the actual solution.

Performance review: 10 out of 27 got this. 10 chose (B), 4 chose (C), 3 chose (A).

Historical note (last time): 10 out of 25 got this. 13 chose (B), 1 each chose (A) and (C).

The branch of chemistry called quantitative analysis has historically used stoichiometric methods to determine the proportions of various chemicals present in a given mix. The idea is to use information about the amounts needed and produced in various reactions to estimate the quantities of chemicals present (the possible chemicals are first identified via "qualitative analysis" techniques). We generally find that these conditions give linear systems, and the coefficient matrices of these systems have (or can be written in a manner as to have) small integer entries.

(4) (*) Consider a situation where we have a material that is a mix (in fixed proportion) of three known chemicals X, Y, and Z. Our goal is to find the amount of X, Y, and Z present. Suppose we want to set up a collection of experiments so that the coefficient matrix is diagonal, i.e., we are effectively solving a diagonal system of equations and can recover the quantities of each of X, Y, and Z. Which of the following is the best approach? Assume that we can measure, for each reagent, the amount of the reagent that gets used up for the reaction(s) to proceed to completion, but cannot isolate or separate the outputs from each other.

(A) Choose a single reagent that reacts with all of X, Y, and Z.
(B) Choose a single reagent that reacts with only one of X, Y, and Z.
(C) Choose three separate reagents, each of which reacts with *all* of X, Y, and Z.
(D) Choose three separate reagents, each of which reacts only with X.
(E) Choose three separate reagents, one of which reacts only with X, one of which reacts only with Y, and one of which reacts only with Z.

Answer: Option (E)

Explanation: Consider a matrix with the columns indexed by X, Y, and Z and with the rows indexed by the reagents. The entry in each cell of the matrix is the amount of the row reagent needed to react with a unit amount of the column substance. It turns out that this is the coefficient matrix of the system of simultaneous linear equations that we construct.

In order to get a diagonal matrix, we need to have three reagents (so the number of rows equals the number of columns) and further, we need to choose them so that the off-diagonal entries are zero, i.e., our first reagent should react only with X, our second reagent should react only with Y, and our third reagent should react only with Z.

Performance review: 24 out of 27 got this. 2 chose (C), 1 chose (A).

Historical note (last time): 22 out of 25 got this. 1 each chose (A), (B), and (D).

(5) (*) Suppose we are given an aqueous solution with two known dissolved substances. There are two different types of reactions. One is an acid-base reaction and the other is a redox reaction. For both reactions, we can use titrations (separately) to deduce the quantity of reagent needed. What type of system should we expect to get if only one of the solutes participates in the redox reaction but both participate in the acid-base reaction?

(A) A diagonal system, i.e., the coefficient matrix is a diagonal matrix.

(B) A triangular system, i.e., the coefficient matrix is a triangular matrix (whether it is upper or lower triangular depends on the order in which we write the rows).
(C) A system of rank one, i.e., the coefficient matrix has rank one.

Answer: Option (B)

Explanation: The coefficient matrix has columns corresponding to the solutes, and rows corresponding to the reagents, with the matrix entries describing the amount of the row reagent consumed per unit of the solute. The row for the redox reagent has one entry zero. If we write that solute on the left, and that row as the second row, we get a matrix of the form:

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

This is upper triangular.

Performance review: 24 out of 27 got this. 2 chose (C), 1 chose (A).

Historical note (last time): 20 out of 25 got this. 4 chose (A), 1 chose (C).

(6) (*) Suppose we are given an aqueous solution with two known dissolved substances. There are two different types of reactions. One is an acid-base reaction and the other is a redox reaction. For both reactions, we can use titrations (separately) to deduce the quantity of reagent needed.Suppose we are given an aqueous solution with two known dissolved substances. Suppose both solutes participate in both reactions. What should we desire if we want to use the data from the two titrations to determine the amounts of each of the substances?

(A) The proportions in which the two substances react should be the same for the two reactions.
(B) The proportions in which the two substances react should differ for the two reactions.
(C) It does not matter; we will be able to determine the amounts of each of the substances in both cases.
(D) It does not matter; we will not be able to determine the amounts of each of the substances in either case.

Answer: Option (B)

Explanation: If they react in the same proportions, then the coefficient matrix will have rank one, i.e., the equations we get from the two reactions will convey the same algebraic information.

Performance review: 20 out of 27 got this. 4 chose (A), 2 chose (D), 1 left the question blank.

Historical note (last time): 12 out of 25 got this. 7 chose (C), 4 chose (D), 2 chose (A).

(7) *Do not discuss this!*: A consumer price index is obtained from a "goods basket" by multiplying the price of each good in the basket by a fixed weight, and then adding up all the price X weight products. The weights are kept fixed, but the prices vary from year to year. Thus, the consumer price index value itself fluctuates from year to year.

What is a good way of modeling this?

(A) The prices of the various goods in various years are stored in a matrix, the weights used in the index are stored in a vector, and the consumer price index values arise as the output vector of the matrix-vector product.
(B) The weights used in the index are stored in a matrix, the prices of the various goods in various years are stored in a vector, and the consumer price index values arise as the output vector of the matrix-vector product.
(C) The prices of the various goods in various years are stored in a matrix, the consumer price index values are stored as a vector, and the weights used in the index arise as the output vector of the matrix-vector product.
(D) The weights used in the index are stored in a matrix, the consumer price index values are stored in a vector, and the prices of the various goods in various years arise as the output vector of the matrix-vector product.
(E) The consumer price index values are stored in a matrix, the prices of the various goods in various years are stored in a vector, and the weights used in the index arise as the output vector of the matrix-vector product.

Answer: Option (A)

Explanation: The weight vector is fixed. There are many price vectors, one for each year. It makes sense to store these as different rows of a matrix, that is then multiplied with the weight vector to give the vector of consumer price index values. The intuition is that the fixed vector is chosen as the vector and the many different vectors that need to be dotted (in the sense of "taking the dot product") with the fixed vector are taken as rows of a matrix.

Performance review: 17 out of 27 got this. 10 chose (B).

Historical note (last time): 4 out of 25 got this. 20 chose (B), 1 chose (E).

(8) Amelia wants to choose a healthy balanced diet. She has access to 30 different types of foods. There are 400 different nutrients that she wants a good amount of. Each of the foods that Amelia consumes offers a positive amount of each nutrient per unit foodstuff. Amelia is interested in meeting the daily value requirements for all nutrients. For some nutrients, her daily value requirements specify only a minimum. For some nutrients, both a minimum and a maximum are specified. Assume that the total amount of any nutrient can be obtained by adding up the amounts obtained from each of the foodstuffs Amelia consumes. Amelia wants to determine how much of each foodstuff she should consume. How should she model the situation?

(A) The matrix with information on the nutritional contents of the foodstuffs is a 400×400 matrix, and the vector of amounts of each foodstuff consumed is a 400×1 column vector.
(B) The matrix with information on the nutritional contents of the foodstuffs is a 30×30 matrix, and the vector of amounts of each foodstuff consumed is a 30×1 column vector.
(C) The matrix with information on the nutritional contents of the foodstuffs is a 400×30 matrix, and the vector of amounts of each foodstuff consumed is a 30×1 column vector.
(D) The matrix with information on the nutritional contents of the foodstuffs is a 30×400 matrix, and the vector of amounts of each foodstuff consumed is a 400×1 column vector.
(E) The matrix with information on the nutritional contents of the foodstuffs is a 400×400 matrix, and the vector of amounts of each foodstuff consumed is a 30×1 column vector.

Answer: Option (C)

Explanation: The input food vector is 30×1, the output nutrient vector is 400×1, so the matrix must be 400×30.

Performance review: 14 out of 27 got this. 12 chose (D), 1 left the question blank.

Historical note (last time): 20 out of 25 got this. 2 each chose (A) and (B), 1 chose (E).