HOMEWORK 7 CHECKLIST: DUE (DELAYED TO) FRIDAY NOVEMBER 22 (INCLUDES EXTRA CREDIT CHALLENGE PROBLEMS AT THE END, DUE WEDNESDAY NOVEMBER 27)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 3.2.8 (Page 131): Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The middle vector is the average of the first and third vector, so we write that fact, then move everything to one side. *Remember that we prefer to move everything to one side and get a* 0 *on the other side, since this helps express linear relations in a relatively standard form.* There are also deeper relations for "moving everything to one side" that will become clearer (or not) as we proceed.

Note that the linear relation is not unique, but is almost so: all nontrivial linear relations are scalar multiples of each other. "Essentially" there is only one linear relation. *This is a feature specific to this particular example.* We could imagine other situations where there are many different kinds of nontrivial linear relations between a given collection of vectors.

(2) Exercise 3.2.16 (Page 131) (was 3.2.14 in the 4th Edition): Use paper and pencil to identify the redundant vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The convention here is that we call a vector redundant if it is redundant relative to all the preceding vectors.

Note that each time we introduce a new nonzero coordinate, the new vector *cannot* be redundant on the preceding ones. In this case, each vector has a new nonzero coordinate relative to the preceding vectors, so it cannot be redundant.

(3) Exercise 3.2.19 (Page 131): Use paper and pencil to identify the redundant vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$$

Use the same criteria as for the preceding question. Note that the criterion does not quite work in reverse, but the triangular/echelon nature of this set of vectors means that it's good enough here.

(4) Exercise 3.2.30 (Page 131) (was 3.2.28 in the 4th Edition): Find a basis of the image of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

First, note that the set of *all* column vectors is a spanning set for the image of this matrix. However, *a priori*, there may be some redundancies. So, what we need to do is remove the redundancies. Our final answer will therefore be a subset of the set of column vectors of the matrix.

What subset? To find out, we convert to rref and use that to identify the leading variable columns. Then, pick the column vectors for the leading variable columns *from the original matrix*. Thus, we use rref to identify which variables are leading and which variables are non-leading, but we must ultimately select columns from the original matrix.

(5) Exercise 3.2.28 (Page 131) (was 3.2.30 in the 4th Edition): Find a basis of the image of the matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Same remarks as the preceding question, but note that this matrix is already in rref.

(6) Exercise 3.2.34 (Page 132): Consider the 5×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} \\ | & | & | & | \end{bmatrix}$$

We are told that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in the kernel of A. Write $\vec{v_4}$ as a linear combination of $\vec{v_1}$, $\vec{v}_2$, $\vec{v_3}$.

Checklist hint begins: Elements in the kernel of A are equivalent to linear relations between the columns of A. This is because of how matrix multiplication operates.

(7) Exercise 3.2.48 (Page 132): Express the plane V in $\mathbb{R}^3$ with equation $3x_1 + 4x_2 + 5x_3 = 0$ as the kernel of a matrix A and as the image of a matrix B.

It is the kernel of the *row* matrix of the coefficients $(3, 4, 5)$ because being in the kernel means that the dot product with each row is zero, and the given condition can be interpreted as saying that the dot product with the vector of coefficients is zero. Note that, going back to physical description, the vector with coordinates $3, 4, 5$ is the normal vector to the plane.

To write the plane as the image of a matrix, we need the columns of the image to be a spanning set for the plane. In other words, the columns of the matrix B are a spanning set for the kernel of A. Thus, we use the procedure to find a spanning set for the kernel of A, then write those as columns of the matrix B. If you do this efficiently, B will be a 3×2 matrix. You need two columns (at minimum) because the plane you are trying to span is two-dimensional. Another way of thinking of this is that you need to find two vectors that are linearly independent of each other, and are both orthogonal to the vector with coordinates $3, 4, 5$.

(8) Exercise 3.2.49 (Page 132): Express the line L in $\mathbb{R}^3$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the image of a matrix A and as the kernel of a matrix B.

This is similar in spirit to the preceding question. Note, however, that the roles of kernel and image are reversed, and we'll end up with the roles of rows and columns broadly interchanged. But think through it carefully.

(9) Exercise 3.3.23 (Page 143): Find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis of the kernel of A.

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

We've already seen how to do this, but some aspects bear repetition.

When we convert the matrix to rref, this *changes the image but does not change the kernel*. If our sole goal is to find the kernel, we can forget the original matrix completely. Note also that the spanning set for the kernel that we learned to construct a while back (with one vector corresponding to each non-leading variable) is a basis, i.e., there are no nontrivial linear relations between the vectors there. The *dimension of the kernel is equal to the number of non-leading variables.*

As for the image, the purpose of converting to rref is to find out which of the original column vectors are redundant. Specifically, the columns corresponding to non-leading variables are redundant, and the columns corresponding to leading variables are irredundant. Having identified the appropriate columns, we look at these columns in the original matrix, and these form a basis for the image. The *dimension of the image is equal to the number of leading variables.*

(10) Exercise 3.3.25 (Page 143): Find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis of the kernel of A.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$$

Same remarks apply as for the preceding question.

(11) Exercise 3.3.27 (Page 143): Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}$$

If we have n vectors in $\mathbb{R}^n$, then they form a basis for $\mathbb{R}^n$ if and only if they are linearly independent. So in this case, we need to verify that the matrix with these vectors as column vectors has full rank (in this case, rank 4).

(12) Exercise 3.3.49 (Page 145) (was 3.3.47 in the 4th Edition): Consider the problem of fitting a cubic through m given points $P_1(x_1, y_1), \dots, P_m(x_m, y_m)$ in the plane. A cubic is a curve in the plane that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 = 0$. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ define the same cubic.

Find all cubics through the given points:

$$(0,0), (1,0), (2,0), (3,0), (0,1), (0,2), (0,3), (1,1), (2,2), (2,1)$$

Note: You could use Exercises 3.3.44 and/or 3.3.45 if you wish, but if doing so, you should do those exercises as well.

Checklist hint begins (this is self-contained and does not use past exercises in the book, though it follows a similar approach: This goes back to the old days of parameter determination using input-output pairs, except that instead of input-output pairs, we have relational pairs. But the thrust is the same: for any pair (x_i, y_i) of points, plugging them in gives a linear equation in the coefficients $c_1, c_2, \dots, c_{10}$.

Since there are 10 points given on the purported cubic, we get a system of 10 equations in the 10 variables. We can write this in terms of matrices: there is a 10×10 coefficient matrix, and an augmenting column that is all zeros.

Now, note that there is one solution we know the system has for sure: the solution with $c_1 = c_2 = c_3 = \dots = c_{10} = 0$. Note, however, that this does *not* define a cubic. It gives the trivial equation $0 = 0$ which is not interesting.

What would be of interest is other solutions. Note also that solution vectors that are scalar multiples of each other represent the same cubic. With this in mind, try solving the system, and note that:

- If we get no non-leading variables, that means the zero vector is the only solution, and therefore, that there are no cubics.

- If we get one non-leading variable, that means the solution space is one-dimensional, so all the nonzero solution vectors are scalar multiples of each other. This implies that there is a unique cubic.
- If we get two or more non-leading variables, we have infinitely many solutions.

Let's now try to understand how to solve this system efficiently. A generic 10×10 is hard to solve, but this particular system is relatively nice. Let's see why.

The $(0,0)$ point gives rise to the equation:

$$c_1 = 0$$

Explicitly, the first row of the coefficient matrix has a 1 in the c_1-column and 0s in the other columns.

Once we've figured out this value, we can in essence get rid of the c_1-column and the $(0,0)$-row. We thus have a 9×9 matrix. The columns of the matrix correspond to the variables $c_2, c_3, \dots, c_{10}$ and the rows correspond to the nine other points on the alleged cubic.

Note that the points $(1,0), (2,0), (3,0)$ have the property that their corresponding rows have nonzero entries only for the c_2, c_4, and c_7 coordinates. This is on account of y being equal to 0. Let's look at the matrix with the rows for $(1,0), (2,0), (3,0)$ and the columns for c_2, c_4, and c_7. We can solve the linear system with this as coefficient matrix to find the values of c_2, c_4, and c_7. The matrix in question is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix}$$

Solve to get the rref of this. It will turn out to be the identity matrix, and hence, we get ...

We can use a similar approach for the points $(0,1), (0,2), (0,3)$. These give nonzero coefficients for the variables c_3, c_6, and c_{10}. The coefficient matrix turns out to be the same, and we get ...

Finally, we plug these in and use the remaining three points.

The upshot is that ...

2. Problems for your own review, not for submission

(1) Exercise 3.2.35 (Page 132): Show that there is a nontrivial relation among the vectors $\vec{v}_1, \dots, \vec{v}_m$ if (and only if) at least one of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$.

We discussed this in Section 2.2 of the lecture notes.

(2) Exercise 3.2.36 (Page 132): Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly dependent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly dependent? How can you tell?

Yes, essentially the linear relation is preserved.

(3) Exercise 3.2.37 (Page 132): Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly independent? How can you tell?

No. There is a possibility of new linear relations getting introduced.

(4) Exercise 3.2.39 (Page 132): Consider some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$ and a vector $\vec{v}$ in $\mathbb{R}^n$ that is not contained in the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$. Are the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}$ linearly independent? Justify your answer.

Yes. Make two cases for a linear relation: coefficient on $\vec{v}$ is 0 and coefficient on $\vec{v}$ is nonzero.

(5) Exercise 3.2.40 (Page 132): Consider a $n \times p$ matrix A and a $p \times m$ matrix B. We are told that the columns of A are linearly independent and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? *Hint*: Exercise 3.1.51 is useful.

The linear independence of columns is equivalent to full column rank, which is equivalent to injectivity, so ...

(6) Exercise 3.2.41 (Page 132): Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \neq m$) such that $AB = I_m$ (We say that A is a *left inverse* of B). Are the columns of B linearly independent? What about the columns of A?

Since the composite is bijective, the map done first (the B map) must be injective and the map done second (the A map) must be surjective. Remember that injectivity is equivalent to full column rank.

(7) Exercise 3.3.39 (Page 144): We are told that a certain 5×5 matrix A can be written as

$$A = BC$$

where B is a 5×4 matrix and C is 4×5. Explain how you know that A is not invertible.

Checklist hint begins: The rank of A is at most 4, because the rank of a product is at most the minimum of the ranks. So ...

Alternatively, use the injective/surjective idea above: if A is invertible, then C must be injective, so full column rank, but that's not possible because it has more columns than rows.

(8) Exercise 3.3.40-43 (Page 144)

3. Advanced problems

(1) Exercise 3.2.50 (Page 132): Consider the two subspaces V and W of $\mathbb{R}^n$. Let $V + W$ be the set of all vectors in $\mathbb{R}^n$ of the form $\vec{v} + \vec{w}$, where $\vec{v}$ is in V and $\vec{w}$ is in W. Is $V + W$ necessarily a subspace of $\mathbb{R}^n$?

If V and W are two distinct lines in $\mathbb{R}^3$, what is $V + W$? Draw a sketch.

Checklist hint begins: *Yes* for the first part, see the discussion in the lecture notes on intersection and sum of subspaces (section 8 of image and kernel lecture notes).

For the second part, it's a plane, basically sums of all things in the two lines.

(2) Exercise 3.3.56 (Page 145) (was 3.3.54 in the 4th Edition): Consider the problem of fitting a cubic through m given points $P_1(x_1, y_1), \ldots, P_m(x_m, y_m)$ in the plane. A cubic is a curve in the plane that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 = 0$. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ define the same cubic.

Explain why fitting a cubic through the m points amounts to finding the kernel of a $m \times 10$ matrix A Give the entries of the i^{th} row of A.

Checklist hint begins: Explain why fitting a cubic through the m points amounts to finding the kernel of a $m \times 10$ matrix A Give the entries of the i^{th} row of A.

If you understood the final routine problem, this should be easy.

4. Extra credit challenge problems

Please turn these in for extra credit grading by **Wednesday, November 27**.

(1) Exercise 3.3.60 (Page 145) (was 3.3.58 in the 4th Edition): Please see this from the book.

(2) Exercise 3.3.69 (Page 145) (was 3.3.67 in the 4th Edition): Consider the subspaces V and W of $\mathbb{R}^n$. Show that

$$\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V + W)$$

For the definition of $V + W$, see Exercise 3.2.50.

You might wish to see the hint in the book. I believe the hint is available only in the (new) 5th Edition, and not in the 4th Edition. I've reproduced the hint below, but you'll need to look at the earlier exercise referenced in the hint.

Hint: Pick a basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ of $V \cap W$. Using Exercise 67 (65 in the 4th Edition, probably) construct bases $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ of V and $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ of W, Show that $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ is a basis of $V + W$. Demonstrating linear independence is somewhat challenging.