TAKE-HOME CLASS QUIZ: DUE OCTOBER 19: SEQUENCES

MATH 153, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE MAKE SURE TO ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE

(1) *Forward difference operators and partial sums*: Recall that for a function $g : \mathbb{N} \to \mathbb{R}$, the forward difference operator of g, denoted Δg, is defined as the function $(\Delta g)(n) = g(n+1) - g(n)$. Suppose we have two functions $f, g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = \sum_{k=1}^{n} f(k)$. What is the relationship between Δg and f? *This is a discrete version of the fundamental theorem of calculus.*
(A) $(\Delta g)(n) = f(n)$ for all $n \in \mathbb{N}$
(B) $(\Delta g)(n) = f(n+1)$ for all $n \in \mathbb{N}$
(C) $(\Delta g)(n+1) = f(n)$ for all $n \in \mathbb{N}$
(D) $(\Delta g)(n) = f(n+2)$ for all $n \in \mathbb{N}$
(E) $(\Delta g)(n+2) = f(n)$ for all $n \in \mathbb{N}$

Your answer: ______________________

(2) Which of the following is the correct definition of $\lim_{x \to \infty} f(x) = L$ for L a finite number?
(A) For every $\varepsilon > 0$ there exists $a \in \mathbb{R}$ such that if $0 < |x - L| < \varepsilon$ then $f(x) > a$.
(B) For every $\varepsilon > 0$ there exists $a \in \mathbb{R}$ such that if $x > a$ then $|f(x) - L| < \varepsilon$.
(C) For every $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that if $x > a$ then $|f(x) - L| < \varepsilon$.
(D) For every $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that if $0 < |x - L| < \varepsilon$ then $f(x) > a$.
(E) There exists $a \in \mathbb{R}$ and $\varepsilon > 0$ such that if $x > a$ then $|f(x) - L| < \varepsilon$.

Your answer: ______________________

(3) *Horizontal asymptote and limit of derivative*: Suppose $\lim_{x \to \infty} f'(x)$ is finite. Which of the following is true (be careful about f versus f' when reading the choices)?
(A) If $\lim_{x \to \infty} f'(x)$ is zero, then $\lim_{x \to \infty} f(x)$ is finite.
(B) If $\lim_{x \to \infty} f(x)$ is finite, then $\lim_{x \to \infty} f'(x)$ is zero.
(C) If $\lim_{x \to \infty} f(x)$ is finite, then $\lim_{x \to \infty} f(x)$ is zero.
(D) All of the above.
(E) None of the above.

Your answer: ______________________

(4) *Convergent sequence and limit of forward difference operator*: Suppose $f : \mathbb{N} \to \mathbb{R}$ is a function (so we can think of it as a sequence). Which of the following is true? Here $(\Delta f)(n) = f(n+1) - f(n)$.
(A) If $\lim_{n \to \infty} (\Delta f)(n)$ is zero, then $\lim_{n \to \infty} f(n)$ is finite.
(B) If $\lim_{n \to \infty} f(n)$ is finite, then $\lim_{n \to \infty} (\Delta f)(n)$ is zero.
(C) If $\lim_{n \to \infty} f(n)$ is finite, then $\lim_{n \to \infty} f(n)$ is zero.
(D) All of the above.
(E) None of the above.

Your answer: ______________________

(5) *Function iteration converges at infinity*: Suppose (a_n) is a sequence whose terms are given by the relation $a_n = f(a_{n-1})$, with a_1 specified separately and f is a continuous function on $\mathbb{R}$. Further, suppose we know that $\lim_{n\to\infty} a_n = L$ for some finite L. What can we conclude is true about L?
(A) $f(L) = L$
(B) $f(L) = 0$
(C) $f'(L) = L$
(D) $f'(L) = 0$
(E) $f''(L) = 0$

Your answer: ____________________________

(6) *Equilibrium at infinity*: Suppose a function y of time t satisfies the differential equation $y' = f(y)$ for all time t, where f is a continuous function on $\mathbb{R}$. Further, suppose we know that $\lim_{t\to\infty} y = L$ for some finite L. What can we conclude is true about L? *Note: Although the question is conceptually similar to the preceding question, you have to reason about the question differently. Note 2: We haven't yet seen differential equations in the course yet, and we will return to this question when we do. This particular question does not require much knowledge of differential equations per se.*
(A) $f(L) = L$
(B) $f(L) = 0$
(C) $f'(L) = L$
(D) $f'(L) = 0$
(E) $f''(L) = 0$

Your answer: ____________________________

(7) A sequence a_n is found to satisfy the recurrence $a_{n+1} = 2a_n(1 - a_n)$. Assume that a_1 is strictly between 0 and 1. What can we say about the sequence (a_n)?
(A) It is monotonic non-increasing, and its limit is 0.
(B) It is monotonic non-decreasing, and its limit is 1.
(C) From a_2 onward, it is monotonic non-decreasing, and its limit is $1/2$.
(D) From a_2 onward, it is monotonic non-increasing, and its limit is $1/2$.
(E) It is either monotonic non-decreasing or monotonic non-increasing everywhere, and its limit is $1/2$.

Your answer: ____________________________

(8) Suppose f is a continuous function on $\mathbb{R}$ and (a_n) is a sequence satisfying the recurrence $f(a_n) = a_{n+1}$ for all n. Further, suppose the limit of the a_ns for odd n is L and the limit of the a_ns for even n is M. What can we say about L and M?
(A) $f(L) = L$ and $f(M) = M$
(B) $f(L) = M$ and $f(M) = L$
(C) $f(L) = f(M) = 0$
(D) $f'(L) = f'(M) = 0$
(E) $f'(L) = M$ and $f'(M) = L$

Your answer: ____________________________

(9) Consider a function f on the natural numbers defined as follows: $f(m) = m/2$ if m is even, and $f(m) = 3m + 1$ if m is odd. Consider a sequence where a_1 is a natural number and we define $a_n := f(a_{n-1})$. It is conjectured (see *Collatz conjecture*) that (a_n) is eventually periodic, regardless of the starting point, and that there is only one possibility for the eventual periodic fragment. Which of the following can be the eventual periodic fragment?
(A) $(1, 2, 3)$
(B) $(1, 3, 2)$

(C) $(1, 2, 4)$
(D) $(1, 4, 2)$
(E) $(1, 3, 4)$

Your answer: ____________________________

(10) For which of the following properties p of sequences of real numbers does p equal *eventually* p?
(A) Monotonicity
(B) Periodicity
(C) Being a polynomial sequence (i.e., given by a polynomial function)
(D) Being a constant sequence
(E) Boundedness

Your answer: ____________________________

The remaining questions are based on a rule which we call the *degree difference rule*. This states the following. Consider a rational function $p(x)/q(x)$, and suppose $a \in \mathbb{R}$ is such that q has no roots in $[a, \infty)$. Then, the improper integral $\int_a^\infty \frac{p(x)}{q(x)}\,dx$ is finite if and only if the degree of q minus the degree of p is *at least* two, or in other words, is strictly greater than one. The same rule applies to $\int_{-\infty}^\infty \frac{p(x)}{q(x)}\,dx$ if q has no zero.

The degree difference rule has a slight variation: we can apply it to situation where p and q are not quite polynomials, but rather their growth rates are of the same order as that of some polynomial or power function. For instance, $(x^2+1)^{3/2}$ has "degree" three with this more liberal interpretation. In this more liberal interpretation, we require that the degree difference (which could now be a non-integer), be strictly greater than one. For instance, a degree difference of $3/2$ means that the integral converges.

Consider a probability distribution on $\mathbb{R}$ with density function f. In particular, this means that $\int_{-\infty}^\infty f(x)\,dx = 1$. Further, assume that f has mean zero and is an even function, i.e., the probability distribution is symmetric about zero.

The *mean deviation* of the distribution is defined as $\int_{-\infty}^\infty |x|f(x)\,dx$. On account of the fact that f is an even function, this can be rewritten as $2\int_0^\infty xf(x)\,dx$.

The *standard deviation* of the distribution, denoted σ, of f is defined as $\sqrt{\int_{-\infty}^\infty x^2 f(x)\,dx}$.

The *kurtosis* of the distribution is defined as $-3 + (\int_{-\infty}^\infty x^4 f(x)\,dx)/\sigma^4$. Note that the kurtosis does not make sense if the standard deviation is infinite.

(11) Consider the distribution with density function $f(x) := (x^2+1)^{-1}/\pi$. (We divide by π so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?
(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Your answer: ____________________________

(12) Consider the distribution with density function $f(x) := (x^2+1)^{-3/2}/2$. (We divide by 2 so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?
(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Your answer: ____________________________

(13) Consider the distribution with density function $f(x) := (x^2+1)^{-2}/(\pi/2)$. (We divide by $\pi/2$ so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?
(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Your answer: ____________________________

(14) Consider the distribution with density function $f(x) := (x^2+1)^{-5/2}/(4/3)$. (We divide by $4/3$ so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?
(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Your answer: ____________________________

(15) Consider the distribution with density function $f(x) := (x^2+1)^{-3}/(3\pi/8)$. (We divide by $3\pi/8$ so that the integral on $(-\infty, \infty)$ is 1). Which of the following is true?
(A) The mean deviation is finite but the standard deviation is infinite.
(B) The standard deviation and kurtosis are finite, but the mean deviation is infinite.
(C) The standard deviation, mean deviation, and kurtosis are all finite.
(D) The standard deviation and mean deviation are finite, but the kurtosis is infinite.
(E) The standard deviation and mean deviation are both infinite.

Your answer: ____________________________