CLASS QUIZ SOLUTIONS: OCTOBER 12: DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution is as follows:

- Score of 1: 6 people.
- Score of 2: 5 people.
- Score of 3: 1 person.

The mean score was 1.58.
Here are the problem-wise answers and scores:

(1) Option (C): 0 people
(2) Option (A): 4 people
(3) Option (E): 12 people. *Everybody got this correct!*
(4) Option (C): 3 people

2. Solutions

(1) (**) Suppose f is a differentiable function on $\mathbb{R}$. Which of the following implications is **false**?
 (A) If f is even, then f' is odd.
 (B) If f is odd, then f' is even.
 (C) If f' is even, then f is odd.
 (D) If f' is odd, then f is even.
 (E) None of the above, i.e., they are all true.

Answer: Option (C)

Explanation: The function $f(x) := 3x + 1$ has derivative $f'(x) = 3$, which is even, but the original function f is not odd.

The key idea is that being an odd function has an additional condition, namely, that $f(0) = 0$, and the derivative provides no control over the value at a point, because we ca nadd a constant to a function and still retain the same derivative.

Performance review: *Nobody* got this correct. 7 chose (E), 3 chose (A), 1 chose (B), 1 chose (D).

Historical note (last year): *Nobody* got this correct. 11 people chose (E), 2 people chose (A), and 1 person chose (D).

Action point: This is a tricky problem. The reason why you were all led astray is that you were simply using examples, but did not have a wide enough repertoire of examples. You needed to think of examples of functions f where $f(0) \neq 0$ – examples as given above. Alternatively, you can try to do the theoretical derivation and stumble on the key insight that way.

The problem will probably become easier to think about when we reach indefinite integration.

(2) (*) A function f on $\mathbb{R}$ is said to satisfy the *intermediate value property* if, for any $a < b \in \mathbb{R}$, and any d between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = d$. Which (one or more) of the following functions satisfies the intermediate value property?
 (A) $f(x) := \{ \begin{array}{rl} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$
 (B) $f(x) := \{ \begin{array}{ll} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{array}$
 (C) $f(x) := \{ \begin{array}{ll} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{array}$
 (D) All of the above

(E) None of the above

Answer: Option (A)

Explanation: The crucial thing we use is that the range of f is $[-1, 1]$, and it takes all values in the range on any nonempty open interval. From this, it is easy to see that f satisfies the intermediate value property.

The other choices:

Option (B): This is not correct. The function takes the values 0 and 1 only. In particular, it does not take any of the intermediate values between 0 and 1.

Option (C): This is not correct. For instance, suppose $a = \sqrt{2}$ and $b = 1.44$. Then $f(a) = 0$ and $f(b) = 1.44$. The value 0.264 lies between $f(a)$ and $f(b)$. But there is nothing between a and b to which applying f gives 0.264.

Remark: The intermediate value theorem can be interpreted as the statement that any continuous function (on an interval) satisfies the intermediate value property. The example (A), however, illustrates that the converse to the intermediate value theorem does not hold, i.e., that there are functions that satisfy the intermediate value property but are not continuous.

Performance review: 4 out of 12 people got this correct. 4 chose (D), 3 chose (E), 1 chose (C).

Historical note (last year): 7 out of 14 people got this correct. 4 people chose (D) and 3 people chose (E). It is likely that the people who chose (D) first tried and found that (A) works, and assumed (incorrectly) that the other options work similarly.

(3) Which (one or more) of the following functions have a period of π?

(A) $x \mapsto \sin^2 x$

(B) $x \mapsto |\sin x|$

(C) $x \mapsto \cos^2 x$

(D) $x \mapsto |\cos x|$

(E) All of the above

Answer: Option (E)

Explanation: We have $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = \cos x$. Thus, when we square or take the absolute value, we see that the function value repeats after an interval of π. It is also clear from the graph or by inspection that no smaller thing works as the period.

Performance review: *Everybody* got this correct.

Historical note (last year): 12 out of 14 people got this correct. 1 person chose (B) and (D) and 1 person chose (A).

(4) Suppose f is a function defined on all of $\mathbb{R}$ such that f' is a periodic function defined on all of $\mathbb{R}$. What can we conclude is **definitely true** about f?

(A) f must be a linear function.

(B) f must be a periodic function.

(C) f can be expressed as the sum of a linear and a periodic function, but f need not be either linear or periodic.

(D) f can be expressed as the product of a linear and periodic function, but f need not be either linear or periodic.

(E) f can be expressed as a composite of a linear and a periodic function, but f need not be either linear or periodic.

Answer: Option (C)

Explanation: The function $x + \sin x$ provides an example for option (C) that does not satisfy any of the descriptions of the other options. Thus, by eliminating other choices, we see that (C) is correct. A more formal explanation of why (C) works will have to wait till we reach the material leading up to indefinite integration.

Remark: Graphically, a sum of a linear function and a periodic function has a graph that repeats itself, but shifted over both vertically and horizontally. We'll talk quite a bit about such functions when we study graphing techniques. Another way of thinking of it is that the linear function represents the *secular trend* and the periodic function represents the *seasonal variation*. For instance, a graph of the daily sales revenue at a supermarket will have a secular trend (increasing, if the supermarket and its customer base are expanding over time, and decreasing if the customer base is shrinking) and a seasonal variation (spikes during Black Friday and Christmas season, lows at some times of the year). If the secular trend is linear, and the secular and seasonal trend interact additively, then the sales revenues are approximately represented as the sum of a linear and a periodic function. If they interact multiplicatively, then the sales revenues are approximately represented as the product of a linear and a periodic function.

Performance review: 3 out of 12 people got this correct. 9 chose (B).

Historical note (last year): 8 out of 14 people got this correct. 5 people chose (B), and 1 person chose (E).

Action point: It *is* true that if f is a periodic differentiable function, then f' is also periodic. However, that's not what the question is asking for. The question is asking for the reverse: if f' is a periodic function, what can we conclude about f? Is f also periodic? In fact, as the answer above makes clear, it is possible for a non-periodic function to have a periodic derivative.