CLASS QUIZ SOLUTIONS: FEBRUARY 17: LIMIT, ORDER OF ZERO, LH RULE

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

The score distribution was as follows:

- Score of 0: 1 person
- Score of 1: 1 person
- Score of 2: 1 person
- Score of 3: 1 person
- Score of 4: 2 people
- Score of 5: 3 people
- Score of 6: 2 people

The mean score was 3.73. The question wise answers and performance review are below:

(1) Option (C): 6 people
(2) Option (C): 5 people
(3) Option (B): 6 people
(4) Option (D): 8 people
(5) Option (E): 9 people
(6) Option (A): 7 people

2. Solutions

(1) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x$?
(A) 0
(B) A
(C) A^2
(D) $g(A)$
(E) $g(A)/A$

Answer: Option (C)

Explanation: We have $\lim_{x \to 0} g(x) = \lim_{x \to 0}(g(x)/x) \lim_{x \to 0} x = A \cdot 0 = 0$.

Also, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x} = \lim_{x \to 0} \frac{g(g(x))}{g(x)} \lim_{x \to 0} \frac{g(x)}{x}$$

The second limit is A. For the first limit, note that as $x \to 0$, we also have $g(x) \to 0$, so the first limit can be rewritten as $\lim_{y \to 0} g(y)/y$, which is also equal to A. Hence, the overall limit is the product A^2.

Performance review: 6 out of 11 got this. 3 chose (B) and 2 chose (D).

Historical note (last quarter): 4 out of 12 go this correct. 3 each chose (A) and (E), 2 chose (D).

Historical note (last year): 1 out of 12 people got this correct. 5 people chose (D), 2 people each chose (B) and (E), 1 person chose (A), and 1 person left the question blank.

(2) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x^2 = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x^4$?
(A) A
(B) A^2
(C) A^3

(D) $A^2 g(A)$
(E) $g(A)/A^2$

Answer: Option (C)

Explanation: First, note that since $g(x)/x^2 \to A$ as $x \to 0$, we must have $g(x) \to 0$ as $x \to 0$. In particular, $g(0) = 0$.

Now, consider:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \cdot \frac{(g(x))^2}{x^4}$$

Splitting the limit, we get:

$$\lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \left(\frac{g(x)}{x^2} \right)^2$$

Setting $u = g(x)$ for the first limit, and using the fact that as $x \to 0$, $u \to 0$ we see that the first limit is A. For the second limit, pulling the square out yields that the second limit is A^2. The overall limit is thus $A \cdot A^2 = A^3$.

We can also use an actual example to solve this problem. For instance, consider the extreme case where $g(x) = Ax^2$ (identically). In this case, $g(g(x)) = A(Ax^2)^2 = A^3 x^4$. Thus, $g(g(x))/x^4 = A^3$, and the limit is thus A^3.

Even more generally, if $\lim_{x \to 0} g(x)/x^n = A$, then $\lim_{x \to 0} g(g(x))/x^{n^2} = A^{n+1}$.

Performance review: 5 out of 11 got this. 3 chose (D), 2 chose (E) 1 chose (B).

Historical note (last quarter): 3 out of 12 got this correct. 7 chose (D), 1 chose (B), 1 chose (E).

Historical note (last year): 4 out of 16 people got this correct. 5 people chose (A), 4 people chose (B), 2 people chose (D), and 1 person chose (E).

For the remaining questions, keep in mind that the *order of a zero* for a function f at a point c in its domain (where it's continuous) such that $f(c) = 0$ is defined as the lub of the set $\{\beta \geq 0 \mid \lim_{x \to c} |f(x)|/|x - c|^\beta = 0\}$.

If f is an infinitely differentiable function at c, then the order, if finite, must be a positive integer. If the order is a positive integer r, then the first $r - 1$ derivatives of f at c equal zero and the r^{th} derivative at c is nonzero (assuming f to be infinitely differentiable).

For convenience, we take $c = 0$ in the next three questions, i.e., all limits are being taken as $x \to 0$.

(3) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the pointwise sum $f + g$ at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3
(D) 5
(E) 6

Answer: Option (B)

Explanation: The general rule is that when f and g both have zeros of *different* orders at a point, the order of zero of their sum is the minimum of the orders of zeros for the individual functions. We can interpret this result in terms of the limit definitions, or in terms of what's the first iterated derivative to take a nonzero value.

Performance review: 6 out of 11 got this. 4 chose (C), 1 chose (A).

(4) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the pointwise product fg at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3
(D) 5
(E) 6

Answer: Option (D)

Explanation: The order of zero for a product of two function is the sum of the orders of zeros for the two functions. This can be seen by thinking of the limit definition of order of zero.

Performance review: 8 out of 11 got this. 2 chose (E), 1 chose (C).

(5) If f has a zero of order 2 and g has a zero of order 3 at the point 0, what is the order of zero for the composite function $f \circ g$ at zero? (Example: $f(x) = \sin^2 x$ and $g(x) = \ln(1 + x^3)$).
(A) 1
(B) 2
(C) 3
(D) 5
(E) 6

Answer: Option (E)

Explanation: Roughly speaking, the order of zero of the composite function is the product of the orders of zeros. This is valid when $c = 0$, i.e., we are taking the order of zero at zero. Otherwise, the statement needs to be modified somewhat.

Performance review: 9 out of 11 got this. 1 each chose (C) and (D).

(6) The L'Hopital rule can be related with order of zero in the following manner: Every time the rule is applied to a $(\to 0)/(\to 0)$ form, the order of zero of the numerator and denominator go *down by one*. Repeated application hopefully yields a situation where either the numerator or the denominator has a nonzero value.

Assume that we start with a limit $\lim_{x \to c} f(x)/g(x)$ where both f and g are infinitely differentiable at c, and further, that $f(c) = g(c) = 0$. If the order of zero of f is d_f and the order of zero of g is d_g, which of the following is true?
(A) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a nonzero numerator and zero denominator, so the limit is undefined. If $d_g < d_f$, then we apply the LH rule d_g times to get a zero numerator and nonzero denominator, so the limit is zero.
(B) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a zero numerator and nonzero denominator, so the limit is undefined. If $d_g < d_f$, then we apply the LH rule d_g times to get a nonzero numerator and zero denominator, so the limit is zero.
(C) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a nonzero numerator and zero denominator, so the limit is zero. If $d_g < d_f$, then we apply the LH rule d_g times to get a zero numerator and nonzero denominator, so the limit is undefined.
(D) If $d_f = d_g$, then we need to apply the LH rule d_f times and we will then get a nonzero numerator and nonzero denominator, that we can evaluate to get the limit. If $d_f < d_g$, then we apply the LH rule d_f times to get a zero numerator and nonzero denominator, so the limit is zero. If $d_g < d_f$, then we apply the LH rule d_g times to get a nonzero numerator and zero denominator, so the limit is undefined.
(E) In all cases, we perform the LH rule $\min\{d_f, d_g\}$ times and obtain a nonzero numerator and nonzero denominator.

Answer: Option (A)

Explanation: Each time the LH rule is applied, the order of zero on the numerator goes down by one and the order of zero on the denominator goes down by one. Thus, we need to perform the LH rule $\min\{d_f, d_g\}$ times to reach a situation where either the numerator or the denominator gets a zero order of zero, which means (from the information we have) that it evaluates to something nonzero.

If $d_f = d_g$, then applying the LH rule d_f times yields a situation where both the numerator and denominator become nonzero.

If $d_f < d_g$ then we need to apply the LH rule d_f times. The denominator in this case still has a zero of order $d_g - d_f$, hence evaluates to zero. The numerator has a zero of order zero, i.e., it evaluates to something nonzero. The (nonzero)/(zero) form means that the limit is undefined.

If $d_g < d_f$, then we need to apply the LH rule d_g times. The numerator in this case still has a zero of order $d_f - d_g$, so is zero, whereas the denominator is nonzero. The (zero)/(nonzero) form means that the limit is zero.

Performance review: 7 out of 11 got this. 3 chose (D), 1 chose (B).