HOMEWORK 7: DUE (DELAYED TO) FRIDAY NOVEMBER 22 (INCLUDES EXTRA CREDIT CHALLENGE PROBLEMS AT THE END, DUE WEDNESDAY NOVEMBER 27)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 3.2.8 (Page 131): Find a nontrivial relation among the following vectors:

$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$$

(2) Exercise 3.2.16 (Page 131) (was 3.2.14 in the 4th Edition): Use paper and pencil to identify the redundant vectors:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

(3) Exercise 3.2.19 (Page 131): Use paper and pencil to identify the redundant vectors:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\\0\end{bmatrix}$$

(4) Exercise 3.2.30 (Page 131) (was 3.2.28 in the 4th Edition): Find a basis of the image of the matrix:

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 5\\1 & 3 & 7\end{bmatrix}$$

(5) Exercise 3.2.28 (Page 131) (was 3.2.30 in the 4th Edition): Find a basis of the image of the matrix:

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

(6) Exercise 3.2.34 (Page 132): Consider the 5×4 matrix

$$A = \begin{bmatrix}| & | & | & |\\ \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4}\\ | & | & | & |\end{bmatrix}$$

We are told that the vector $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in the kernel of A. Write $\vec{v_4}$ as a linear combination of $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$.

(7) Exercise 3.2.48 (Page 132): Express the plane V in $\mathbb{R}^3$ with equation $3x_1 + 4x_2 + 5x_3 = 0$ as the kernel of a matrix A and as the image of a matrix B.

(8) Exercise 3.2.49 (Page 132): Express the line L in $\mathbb{R}^3$ spanned by the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as the image of a matrix A and as the kernel of a matrix B.

(9) Exercise 3.3.23 (Page 143): Find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis of the kernel of A.

$$\begin{bmatrix}1 & 0 & 2 & 4\\ 0 & 1 & -3 & -1\\ 3 & 4 & -6 & 8\\ 0 & -1 & 3 & 1\end{bmatrix}$$

(10) Exercise 3.3.25 (Page 143): Find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis of the kernel of A.

$$\begin{bmatrix}1 & 2 & 3 & 2 & 1\\ 3 & 6 & 9 & 6 & 3\\ 1 & 2 & 4 & 1 & 2\\ 2 & 4 & 9 & 1 & 2\end{bmatrix}$$

(11) Exercise 3.3.27 (Page 143): Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\4\\8\end{bmatrix}, \begin{bmatrix}1\\-2\\4\\-8\end{bmatrix}$$

(12) Exercise 3.3.49 (Page 145) (was 3.3.47 in the 4th Edition): Consider the problem of fitting a cubic through m given points $P_1(x_1, y_1)$, ..., $P_m(x_m, y_m)$ in the plane. A cubic is a curve in the plane that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 = 0$. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ define the same cubic.

Find all cubics through the given points:

$$(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1), (2, 2), (2, 1)$$

Note: You could use Exercises 3.3.44 and/or 3.3.45 if you wish, but if doing so, you should do those exercises as well.

2. Problems for your own review, not for submission

(1) Exercise 3.2.35 (Page 132): Show that there is a nontrivial relation among the vectors $\vec{v}_1$, ..., $\vec{v}_m$ if (and only if) at least one of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_m$.

(2) Exercise 3.2.36 (Page 132): Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly dependent vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$, ..., $T(\vec{v}_m)$ necessarily linearly dependent? How can you tell?

(3) Exercise 3.2.37 (Page 132): Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$, ..., $T(\vec{v}_m)$ necessarily linearly independent? How can you tell?

(4) Exercise 3.2.39 (Page 132): Consider some linearly independent vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$ in $\mathbb{R}^n$ and a vector $\vec{v}$ in $\mathbb{R}^n$ that is not contained in the span of $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$. Are the vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$, $\vec{v}$ linearly independent? Justify your answer.

(5) Exercise 3.2.40 (Page 132): Consider a $n \times p$ matrix A and a $p \times m$ matrix B. We are told that the columns of A are linearly independent and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? *Hint*: Exercise 3.1.51 is useful.

(6) Exercise 3.2.41 (Page 132): Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \neq m$) such that $AB = I_m$ (We say that A is a *left inverse* of B). Are the columns of B linearly independent? What about the columns of A?

(7) Exercise 3.3.39 (Page 144): We are told that a certain 5×5 matrix A can be written as

$$A = BC$$

where B is a 5×4 matrix and C is 4×5. Explain how you know that A is not invertible.

(8) Exercise 3.3.40-43 (Page 144)

3. Advanced problems

(1) Exercise 3.2.50 (Page 132): Consider the two subspaces V and W of $\mathbb{R}^n$. Let $V + W$ be the set of all vectors in $\mathbb{R}^n$ of the form $\vec{v} + \vec{w}$, where $\vec{v}$ is in V and $\vec{w}$ is in W. Is $V + W$ necessarily a subspace of $\mathbb{R}^n$?

If V and W are two distinct lines in $\mathbb{R}^3$, what is $V + W$? Draw a sketch.

(2) Exercise 3.3.56 (Page 145) (was 3.3.54 in the 4th Edition): Consider the problem of fitting a cubic through m given points $P_1(x_1, y_1), \ldots, P_m(x_m, y_m)$ in the plane. A cubic is a curve in the plane that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 = 0$. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ define the same cubic.

Explain why fitting a cubic through the m points amounts to finding the kernel of a $m \times 10$ matrix A Give the entries of the i^{th} row of A.

4. Extra credit challenge problems

Please turn these in for extra credit grading by **Wednesday, November 27**.

(1) Exercise 3.3.60 (Page 145) (was 3.3.58 in the 4th Edition): Please see this from the book.

(2) Exercise 3.3.69 (Page 145) (was 3.3.67 in the 4th Edition): Consider the subspaces V and W of $\mathbb{R}^n$. Show that

$$\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V + W)$$

For the definition of $V + W$, see Exercise 3.2.50.

You might wish to see the hint in the book. I believe the hint is available only in the (new) 5th Edition, and not in the 4th Edition. I've reproduced the hint below, but you'll need to look at the earlier exercise referenced in the hint.

Hint: Pick a basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ of $V \cap W$. Using Exercise 67 (65 in the 4th Edition, probably) construct bases $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ of V and $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ of W, Show that $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ is a basis of $V + W$. Demonstrating linear independence is somewhat challenging.