TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY FEBRUARY 1: LIMITS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

27 people took this 14-question quiz. The score distribution was as follows:

- Score of 8: 2 people.
- Score of 9: 3 people.
- Score of 10: 2 people.
- Score of 11: 4 people.
- Score of 12: 5 people.
- Score of 13: 5 people.
- Score of 14: 6 people.

The question wise answers and performance review were as follows:

(1) Option (A): All 27 people.
(2) Option (D): 22 people.
(3) Option (C): 26 people.
(4) Option (C): 20 people.
(5) Option (B): 21 people.
(6) Option (C): 23 people.
(7) Option (B): 26 people.
(8) Option (C): 26 people.
(9) Option (E): 21 people.
(10) Option (D): 21 people.
(11) Option (A): 18 people.
(12) Option (E): 22 people.
(13) Option (A): 19 people.
(14) Option (B): 24 people.

2. Solutions

(1) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0,1)$? If all are examples, please select Option (E).

(A) $f(x) := \begin{cases} x, & 0 < x \leq 1/2 \\ 2x, & 1/2 < x < 1 \end{cases}$

(B) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x, & 1/2 \leq x < 1 \end{cases}$

(C) $f(x) := \begin{cases} x, & 0 < x \leq 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{cases}$

(D) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \leq x < 1 \end{cases}$

(E) All of the above

Answer: Option (A)

Explanation: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, $1/2$)? and (ii) is the point (in all cases, $1/2$) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at $1/2$. Namely the function x and $2x - (1/2)$ both take the value $1/2$ at the domain point $1/2$. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function).

This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at $1/2$: the former gives $1/2$ and the latter gives 1. In other words, the function has a jump discontinuity at $1/2$. Thus, (ii) becomes relevant: is $1/2$ included in the left or the right definition?

For option (A), $1/2$ is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \to 1/2^-} f(x)$. On the other hand, $\lim_{x \to 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous.

For option (B), $1/2$ is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at $1/2$.

Performance review: All 27 got this.

Historical note (last time): 47 out of 49 people got this correct. 1 person each chose (B) and (C).

(2) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both left continuous on $(0,1)$. Which of the following is *not* guaranteed to be left continuous on $(0,1)$? Please see Option (E) before answering.
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D)

Explanation: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail.

For $c \in (0,1)$, we want to know whether:

$$\lim_{x \to c^-} f(g(x)) \stackrel{?}{=} f(g(c))$$

We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false.

We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f, the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say:

$$f(x) := \begin{cases} 1/3, & 0 < x \leq 1/2 \\ 2/3, & 1/2 < x < 1 \end{cases}$$

and

$$g(x) := 1 - x$$

Note that both functions have range a subset of $(0,1)$.

Composing, we obtain that:

$$f(g(x)) = \begin{cases} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \leq x < 1 \end{cases}$$

f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

Performance review: 22 out of 27 got this. 2 chose (E), 1 each chose (C), 1 chose (B).

Historical note (last time): 20 out of 49 people got this correct. 26 people chose (E), 2 chose (C), 1 chose (B).

(3) Which of these is the correct interpretation of $\lim_{x \to c} f(x) = L$ in terms of the definition of limit? Please see Option (E) before answering.
   (A) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
   (B) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
   (C) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
   (D) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.
   (E) None of the above

   Answer: Option (C)

   Explanation: α plays the role of ε and β plays the role of δ.

   Performance review: 26 out of 27 got this. 1 chose (B).

   Historical note (last time): 44 out of 49 people got this correct. 3 people chose (A), 2 chose (B).

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x \to c} f(x)$ exists)? If all, please select Option (E).
   (A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
   (B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
   (C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
   (D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
   (E) All of the above.

   Answer: Option (C)

   Explanation: Our statement should be that *every* c has no limit. In other words, for *every* c and *every* L, it is *not* true that $\lim_{x \to c} f(x) = L$. That's exactly what option (C) says.

   Performance review: 20 out of 27 got this. 4 chose (A), 1 each chose (B), (D), and (E).

   Historical note (last time): 20 out of 49 people got this correct. 15 chose (B), 7 chose (E), 4 chose (D), 3 chose (A).

(5) In the usual $\varepsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\varepsilon > 0$, then which of the following is true? Please see Option (E) before answering.
   (A) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
   (B) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
   (C) Every larger value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
   (D) Every larger value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
   (E) None of the above statements need always be true.

   Answer: Option (B)

   Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ε also works for larger εs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

   Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

   Performance review: 21 out of 27 got this. 3 chose (E), 2 chose (D), 1 chose (C).

Historical note (last time): 31 out of 49 people got this correct. 6 each chose (A) and (E), 5 chose (C), 1 chose (D).

(6) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of εs and δs? Please see Option (E) before answering.

(A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(E) None of the above.

Answer: Option (C)

Explanation: The "open interval centered at L" describes the "$\varepsilon > 0$" part of the definition (where the open interval is the interval $(L - \varepsilon, L + \varepsilon)$). The "open interval centered at c" describes the "$\delta > 0$" part of the definition (where the open interval is the interval $(c - \delta, c + \delta)$). x being in the open interval centered at c (except the case $x = c$) is equivalent to $0 < |x - c| < \delta$, and $f(x)$ being in the open interval centered at L is equivalent to $|f(x) - L| < \varepsilon$.

Performance review: 23 out of 27 got this. 3 chose (D), 1 chose (B).

Historical note (last time): 24 out of 49 people got this correct. 12 chose (A), 8 chose (D), 3 chose (E), 2 chose (B).

(7) Consider the function:

$$f(x) := \begin{cases} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{cases}$$

What is the set of all points at which f is continuous?

(A) $\{0, 1\}$

(B) $\{-1, 1\}$

(C) $\{-1, 0\}$

(D) $\{-1, 0, 1\}$

(E) f is continuous everywhere

Answer: Option (B)

Explanation: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c, we need that: (i) the limit for the function as defined for rationals exists at c, (ii) the limit for the function as defined for irrationals exists at c, and (iii) the two limits are equal.

Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c. This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals.

The limit for rationals exists for all c and equals the value c. The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

Performance review: 26 out of 27 got this. 1 chose (A).

Historical note (last time): 39 out of 49 people got this correct. 7 chose (D), 2 chose (A), 1 chose (E).

(8) The graph $y = f(x)$ of a function f defined on all reals has a horizontal asymptote $y = c$ as x approaches $+\infty$. Which of the following is the correct definition of this?
(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(B) For every $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
(C) For every $\varepsilon > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
(D) For every $\delta > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(E) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - c| < \varepsilon$.

Answer: Option (C)

Explanation: The neighborhood of c (picked by the skeptic) is the interval $(c - \varepsilon, c + \varepsilon)$, and it is parametrized by its radius ε. The neighborhood of $+\infty$ (picked by the prover) is the interval (a, ∞), and it is parametrzed by its lower endpoint a. The skeptic then picks x in the neighborhood specified by the prover, i.e., $f(x) > a$, and then they check whether $f(x)$ is in the chosen neighborhood of c.

Performance review: 26 out of 27 got this. 1 chose (B).

Historical note (last time): 35 out of 46 got this correct. 4 each chose (B) and (E). 3 chose (A).

(9) Which of the following is the correct definition of $\lim_{x \to c^-} f(x) = -\infty$ (in words: the left hand limit of f at c is $-\infty$)?
(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(B) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) > a$.
(C) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) < a$.
(D) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) > a$.
(E) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) < a$.

Answer: Option (E)

Explanation: The neighborhood of $-\infty$ chosen by the skeptic is $(-\infty, a)$, and it is parameterized by its upper endpoint a. The prover picks the parameter δ for the left side δ "half-neighborhood" of c, namely $(c - \delta, c)$. The skeptic then picks x in this half-neighborhood, and they then check whether $f(x) \in (-\infty, a)$. Translating the interval conditions into inequality notation, we get the definition as stated.

Performance review: 21 out of 27 got this. 3 chose (C), 2 chose (D), 1 chose (B).

Historical note (last time): 37 out of 46 got this correct. 3 each chose (B), (C), and (D).

(10) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon - \delta$ definition for the statement "f is differentiable at c"?
(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$.
(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.
(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$
(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$
(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.

Answer: Option (D)

Explanation: We would like to say that there exists $L \in \mathbb{R}$ (where L will be the claimed value of $f'(c)$) such that:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$$

To do this, we must say that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have:

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|<\varepsilon$$

Rewriting the final inequality, we get option (D).

Performance review: 21 out of 27 got this. 3 chose (B), 1 each chose (A) and (C), 1 left the question blank.

Historical note (last time): 17 out of 44 got this. 13 chose (A), 12 chose (B), 1 each chose (C) and (E).

(11) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon-\delta$ definition for the statement "f is not differentiable at c"?

(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x)-f(c)-L(x-c)| \geq |x-c|\varepsilon$.

(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x)-f(c)-L(x-c)| < |x-c|\varepsilon$.

(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x-c| < \delta$, we have $|f(x)-f(c)-L(x-c)| \geq |x-c|\varepsilon$

(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x-c| < \delta$, we have $|f(x)-f(c)-L(x-c)| < |x-c|\varepsilon$

(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x)-f(c)-L(x-c)| < |x-c|\varepsilon$.

Answer: Option (A)

Explanation: We would like to say that for every $L \in \mathbb{R}$, the statement:

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=L$$

is false. To do this, we must say that there exists $\varepsilon > 0$ such that the difference quotient (i.e., the expression on the left) cannot be trapped within the interval $(L-\varepsilon, L+\varepsilon)$. In other words, for every $\delta > 0$, there is some value of x such that $0 < |x-c| < \delta$ and:

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|\geq\varepsilon$$

Rewriting the final inequality, we get option (A).

Performance review: 18 out of 27 got this. 7 chose (C), 2 chose (D).

Historical note (administered in an earlier year): 5 out of 15 people got this correct. 4 people chose (C), 2 people chose (D), 2 people chose (E), 1 person chose (B), and 1 person left the question blank.

(12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Identify which of these definitions is *not* correct for $\lim_{x\to c} f(x) = L$, where c and L are both finite real numbers. If all are correct, please select Option (E).

(A) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.

(B) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.

(C) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.

(D) For every $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.

(E) None of these, i.e., all definitions are correct.

Answer: Option (E)

Explanation: Although the usual $\varepsilon-\delta$ definition uses centered intervals, i.e., intervals centered at the points c and L, this is not a necessary aspect of the definition. So, instead of taking centered intervals $(c-\delta, c+\delta)$ or $(L-\varepsilon, L+\varepsilon)$, we could consider open intervals that have different amounts on the left and on the right. Thus, all four definitions are correct.

Performance review: 22 out of 27 got this. 3 chose (D), 2 chose (C).

Historical note (last time): 33 out of 41 got this. 3 chose (C). 2 each chose (B) and (D), 1 left the question blank.

(13) In the usual $\varepsilon - \delta$ definition of limit, we find that the value $\delta = 0.2$ for $\varepsilon = 0.7$ for a function f at 0, and the value $\delta = 0.5$ works for $\varepsilon = 1.6$ for a function g at 0. What value of δ *definitely* works for $\varepsilon = 2.3$ for the function $f + g$ at 0?
(A) 0.2
(B) 0.3
(C) 0.5
(D) 0.7
(E) 0.9

Answer: Option (A)

Explanation: We choose the *smaller* of the δs to guarantee that *both* f and g are within their respective ε-distances of the targets – 0.7 in the case of f and 1.6 in the case of g. Now, the triangle inequality guarantees that $f + g$ is within 2.3 of its proposed limit.

Performance review: 19 out of 27 got this. 6 chose (D), 1 each chose (C) and (E).

Historical note (last time): 36 out of 41 got this. 3 chose (D), 1 each chose (C) and (E).

(14) The sum of limits theorem states that $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ *if* the right side is defined. One of the choices below gives an example where the left side of the equality is defined and finite but the right side makes no sense. Identify the correct choice.
(A) $f(x) := 1/x$, $g(x) := -1/(x + 1)$, $c = 0$.
(B) $f(x) := 1/x$, $g(x) := (x - 1)/x$, $c = 0$.
(C) $f(x) := \arcsin x$, $g(x) := \arccos x$, $c = 1/2$.
(D) $f(x) := 1/x$, $g(x) = x$, $c = 0$.
(E) $f(x) := \tan x$, $g(x) := \cot x$, $c = 0$.

Answer: Option (B)

Explanation: $f + g$ is the constant function 1, so it has a limit. On the other hand, both f and g have one-sided limits of $\pm\infty$.

For options (A), (D), and (E), one of the function f and g has a finite limit, and the other has an infinite or undefined limit, and the sum has an infinite or undefined limit. Option (C) is a case where f, g, and $f + g$ all have finite limits.

Performance review: 24 out of 27 got this. 2 chose (A), 1 chose (E).

Historical note (last time): 36 out of 41 got this. 2 each chose (A) and (E), 1 chose (C).