CLASS QUIZ SOLUTIONS: MARCH 9: TAYLOR SERIES AND POWER SERIES

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 2: 4 people
- Score of 3: 6 people
- Score of 4: 6 people
- Score of 5: 3 people
- Score of 6: 2 people
- Score of 7: 3 people
- Score of 9: 1 person
- Score of 10: 1 person

The mean score was 4.5.

Here are the answers and performance summary by question. A (*) in front of a question in the summary below indicates that the correct option was not the single most commonly chosen option for the question, i.e., there was at least one wrong option chosen more frequently for that question:

(1) Option (C): 11 people.
(2) Option (C): 8 people
(3) (*) Option (C): 3 people
(4) Option (A): 7 people
(5) Option (D): 10 people
(6) Option (A): 10 people
(7) Option (D): 11 people
(8) (*) Option (C): 4 people.
(9) Option (D): 9 people
(10) Option (E): 12 people
(11) (*) Option (E): 3 people.
(12) (*) Option (C): 5 people.
(13) Option (D): 12 people
(14) Option (B): 12 people

Performance on this quiz was not good, partly because some of the material tested on the quiz was not covered in class prior to the quiz. You may consider re-attempting the quiz after Friday's review and see if the answers make more sense.

2. Solutions

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic function sare in $C^\infty(\mathbb{R})$.

(1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which $\lim_{x \to c} |f(x)|/|x-c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f?
 (A) The order of any zero of f must be between 0 and 1.
 (B) The order of any zero of f must be between 1 and 2.
 (C) The order of any zero of f, if finite, must be a positive integer.
 (D) The order of any zero of f must be exactly 1.
 (E) The order of any zero of f must be ∞.

Answer: Option (C)

Explanation: We consider two cases. First, that for every positive integer k, we have $f^{(k)}(c) = 0$. In that case, we can verify using the LH rule that $f(x)/(x-c)^k \to 0$ for every positive integer k, and hence, there is no finite least upper bound and hence no finite order.

Next, suppose there is a smallest k such that $f^{(k)}(c) \neq 0$. Suppose $f^{(k)}(c) = \lambda$. This k must be greater than 0, because we are given that $f^{(0)}(c) = f(c) = 0$. We can show by a k-fold application of the LH rule that $\lim_{x \to c} f(x)/(x-c)^k = \lambda/k!$ which is a finite nonzero number. By suitable chaining, we can therefore show that $\lim_{x \to c} |f(x)|/|x-c|^\beta = 0$ for all $\beta \in (0,k)$. Thus, the order of the zero at f is precisely k, whuch is a positive integer.

Performance review: 11 out of 26 people got this correct. 7 chose (A), 3 each chose (B) and (E), 1 chose (D), and 1 left the question blank.

(2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on $\mathbb{R}$, what can we say about the Taylor polynomial about 0?
 (A) No Taylor polynomial is defined for f.
 (B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
 (C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
 (D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
 (E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

Answer: Option (C)

Explanation: The fraction power $x^{4/3}$ can be differentiated once but not twice about 0. The rest of the expression for f is polynomial. Thus, f is once differentiable but not twice differentiable at 0. Hence, we cannot define $P_2(f)$. $P_0(f)$ is just $f(0)$, which is 1, and $P_1(f)(x) = f(0) + f'(0)x = 1 + x$. Alternatively, $P_1(f)(x)$ is simply the truncation of f to the parts of degree at most 1.

Performance review: 8 out of 26 people got this correct. 5 each chose (A) and (B), 4 each chose (D) and (E).

(3) (*) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0?
 (A) $f_1(x) := \{ \begin{matrix} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{matrix}$
 (B) $f_2(x) := \{ \begin{matrix} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{matrix}$
 (C) $f_3(x) := \{ \begin{matrix} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{matrix}$
 (D) $f_4(x) := \{ \begin{matrix} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$
 (E) All of the above.

Answer: Option (C)

Explanation: This answer is explained more in the lecture notes.

Why the other options are wrong:

Option (A): This is in fact globlally analytic, and is given by the power series $1 - x^2/3! + x^4/5! - \dots$

Option (B): This is not continuous at 0. The left hand limit at 0 is $+\infty$.

Option (D): This is not continuous at 0. The limit at 0 is not defined.

Performance review: 3 out of 26 people got this correct. 11 chose (D), 6 each chose (A) and (E).

Action point: We will discuss the example in option (C) during Friday's review session. After that, hopefully what's happening will become clear.

(4) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic?
(A) $x \mapsto \ln(1 + x^2)$
(B) $x \mapsto \ln(1 + x)$
(C) $x \mapsto \ln(1 - x)$
(D) $x \mapsto \exp(1 + x)$
(E) $x \mapsto \exp(1 - x)$

Answer: Option (A)

Explanation: The function is in $C^\infty(\mathbb{R})$ because it can be differentiated infinitely often: the first derivative is $2x/(1 + x^2)$, and each subsequent derivative is a rational function whose denominator is a power of $1 + x^2$. Since $1 + x^2$ does not vanish anywhere on $\mathbb{R}$, each derivative is defined and continuous on all of $\mathbb{R}$.

The radius of convergence of the power series is 1, basically because it is a power series where the coefficients are rational functions, and any such power series has radius of convergence 1 by the root test or ratio test. Thus, the function is not globally analytic.

Why the other options are wrong:

Option (B) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \le -1$.

Option (C) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \ge 1$.

Options (D) and (E) are globally analytic because exp is globally analytic.

Performance review: 7 out of 26 people got this correct. 6 chose (C), 5 chose (B), 4 chose (E), 3 chose (D), and 1 left the question blank.

(5) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic?
(A) x
(B) x^2
(C) $x + 1$
(D) $x^2 + 1$
(E) e^x

Answer: Option (D)

Explanation: The reciprocal $1/(x^2 + 1)$ is a rational function all of whose derivatives are rational functions with denominator a power of $x^2 + 1$, hence defined and continuous derivatives. Hence it is is $C^\infty(\mathbb{R})$. Further, the power series expansion for it is like a geometric series, which has radius of convergence 1, hence it is not globally analytic.

Why the other options are wrong:

Options (A) and (B): The reciprocals are not in $C^\infty(\mathbb{R})$ because the functions $1/x$ and $1/x^2$ are not defined or continuous at 0.

Option (C): The reciprocal is not in $C^\infty(\mathbb{R})$ because the function $1/(x + 1)$ is not defined or continuous at -1.

Option (E): The reciprocal, which is $\exp(-x)$, is globally analytic.

Performance review: 10 out of 26 people got this correct. 5 chose (C), 4 each chose (A) and (E), 2 chose (B), 1 left the question blank.

(6) Consider the rational function $1/\prod_{i=1}^{n}(x - \alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_is, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_is?
(A) It is the minimum of the distances from c to the α_is.
(B) It is the second smallest of the distances from c to the α_is.
(C) It is the arithmetic mean of the distances from c to the α_is.
(D) It is the second largest of the distances from c to the α_is.
(E) It is the maximum of the distances from c to the α_is.

Answer: Option (A)

Explanation: Since the Taylor series converges to the function on its interval of convergence, the interval of convergence must be contained in the domain of definition. In particular, it must exclude all the α_is. Hence, the radius of convergence cannot be more than the minimum of the distances from c to the α_is.

That it is exactly equal to the minimum can be shown by using the fact that we get a product of geometric series.

Performance review: 10 out of 26 people got this correct. 6 people chose (C), 5 chose (E), 2 each chose (B) and (D), 1 left the question blank.

(7) What is the interval of convergence of the Taylor series for arctan about 0?
(A) $(-1, 1)$
(B) $[-1, 1)$
(C) $(-1, 1]$
(D) $[-1, 1]$
(E) All of $\mathbb{R}$

Answer: Option (D)

Explanation: For the boundary points, we use the alternating series theroem. See a more detailed discussion in the lecture notes.

Performance review: 11 out of 26 people got this correct. 8 chose (A), 4 chose (C), 2 chose (B), 1 chose (E).

(8) (*) Consider the function $F(x, p) = \sum_{n=1}^{\infty} x^n/n^p$. For fixed p, this is a power series in x. What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0, \infty)$?
(A) The interval of convergence is $(-1, 1)$ for $0 < p \le 1$ and $[-1, 1]$ for $p > 1$.
(B) The interval of convergence is $(-1, 1)$ for $0 < p < 1$ and $[-1, 1]$ for $p \ge 1$.
(C) The interval of convergence is $[-1, 1)$ for $0 < p \le 1$ and $[-1, 1]$ for $p > 1$.
(D) The interval of convergence is $(-1, 1]$ for $0 < p < 1$ and $[-1, 1]$ for $p \ge 1$.
(E) The interval of convergence is $(-1, 1)$ for $0 < p \le 1$ and $[-1, 1)$ for $p > 1$.

Answer: Option (C)

Explanation: The radius of convergence is 1 for obvious reasons. Convergence at the boundary point -1 follows from the alternating series theorem. At the boundary point 1, we get a p-series, which converges if and only if $p > 1$.

Performance review: 4 out of 26 people got this correct. 10 chose (B), 7 chose (A), 3 chose (E), 2 chose (D).

Action point: This should probably be clearer to you after the material covered in class on Wednesday and further review that we'll do on Friday. In any case, please go through the solution and understand it clearly.

(9) Which of the following functions of x has a power series $\sum_{k=0}^{\infty} x^{4k}/(4k)!$?
(A) $(\sin x + \sinh x)/2$
(B) $(\sin x - \sinh x)/2$
(C) $(\sinh x - \sin x)/2$
(D) $(\cosh x + \cos x)/2$
(E) $(\cosh x - \cos x)/2$

Answer: Option (D)

Explanation: Take the Taylor series and add. Also, use that both cosh and cos are globally analytic.

Performance review: 9 out of 26 people got this correct. 5 chose (B), 4 each chose (A), (C), and (E).

(10) What is the sum $\sum_{k=0}^{\infty} (-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$.
(A) $\cos x$
(B) $\sin x$
(C) $\cos(x^2)$
(D) $\cosh(x^2)$
(E) $\exp(-x^2)$

Answer: Option (E)

Explanation: Put $u = -x^2$, and we get $\sum_{k=0}^{\infty} u^k/k!$.

Performance review: 12 out of 26 people got this correct. 7 chose (C), 3 each chose (A) and (D), 1 chose (B).

(11) (*) Define an operator R from the space of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b, $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$?

(A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
(B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
(C) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
(D) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
(E) $R(a+b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Answer: Option (E)

Explanation: $R(a + b) \geq \min\{R(a), R(b)\}$ because if both a and b, so does $a + b$. Let $c = a + b$. We also then get that $R(a) \geq \min\{R(b), R(c)\}$ and $R(b) \geq \min\{R(a), R(c)\}$ because $a = c - b$ and $b = c - a$.

Juggling these possibilities, we find that of the three numbers $R(a)$, $R(b)$, and $R(a+b)$, the smaller two of the three numbers must be equal. This forces option (E).

This is a type of hyperbolic geometry – all "triangles" must be isosceles.

Performance review: 3 out of 26 people got this correct. 11 chose (C), 7 chose (D), 3 chose (A), 1 chose (B).

Action point: Please make sure you review and understand this solution. The underlying idea is extremely important!

(12) (*) Which of the following is/are true?

(A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on $\mathbb{R}$.
(B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence.
(C) If we start with a power series about 0 that converges everywhere in $\mathbb{R}$, then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
(D) All of the above.
(E) None of the above.

Answer: Option (C)

Explanation: See the lecture notes. A counterexample to (A) is arctan, and a counterexample to (B) is e^{-1/x^2}.

Performance review: 5 out of 26 people got this correct. 10 chose (B), 5 each chose (D) and (E), and 1 chose (A).

Action point: This should become clearer to you after the material covered in class on Wednesday and the Friday review.

(13) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$?

(A) $\exp(x)$ from below and $\cosh(2x)$ from above.
(B) $\exp(x)$ from below and $\cosh(x^2)$ from above.
(C) $\exp(x/2)$ from below and $\exp(x)$ from above.
(D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.
(E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Answer: Option (D)

Explanation: We use the fact that:

$$k! \leq (k!)^2 \leq (2k)!$$

for all $k \geq 0$, with both inequalities strict if $k \geq 2$.

We thus get:

$$\frac{x^k}{k!} \geq \frac{x^k}{(k!)^2} \geq \frac{x^k}{(2k)!}$$

for $x > 0$, with both inequalities strict if $k \geq 2$. Summing up, we get:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} > \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} > \sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$$

The left most expression is e^x. For the right most expression, put $u = \sqrt{x}$, and we get $\cosh u$, so $\cosh \sqrt{x}$. Thus option (D) is the right choice.

Performance review: 12 out of 26 people got this correct. 5 each chose (A) and (C), 3 chose (B), and 1 chose (E).

(14) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \to \infty$?

(A) f grows polynomially in x.

(B) f grows faster than any polynomial function but slower than any exponential function of x.

(C) f grows like an exponential function of x.

(D) f grows faster than any exponential function but slower than any doubly exponential function of x.

(E) f grows like a doubly exponential function of x.

Answer: Option (B)

Explanation: Note that *any* power series wiht infinitely many positive coefficients must grow faster than a polynomial, which, after all, has finite degree.

The rough reason that growth is strictly slower than an exponential function is that the denominators are growing much faster than $k!$. Recall that if the denominators are $k!$, we get precisely the exponential function. This can be made more precise.

Performance review: 12 out of 26 people got this correct. 6 chose (D), 4 chose (C), 2 each chose (A) and (E).