TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY OCTOBER 30: LINEAR TRANSFORMATIONS AND FINITE STATE AUTOMATA

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

28 people took this 7-question quiz. The score distribution was as follows:

- Score of 0: 2 people
- Score of 1: 1 person
- Score of 2: 1 person
- Score of 3: 5 people
- Score of 4: 3 people
- Score of 5: 3 people
- Score of 6: 8 people
- Score of 7: 5 people

The mean score was 4.57.

The question-wise answers and performance review are below:

(1) Option (E): 19 people
(2) Option (B): 23 people
(3) Option (A): 21 people
(4) Option (C): 10 people
(5) Option (D): 17 people
(6) Option (B): 18 people
(7) Option (C): 20 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The purpose of this quiz is to explore in greater depth particular types of matrices, the corresponding linear transformations, and the relationship between operations on sets and similar operations on vector spaces. The material covered in the quiz will also prove to be a fertile source of *examples* and *counterexamples* for later content: in the future, when you are asked to come up with matrices that satisfy some very loosely stated conditions, the matrices of the type described here can be a place to begin your search.

Let n be a natural number greater than 1. Suppose $f : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ is a function satisfying $f(0) = 0$. Let T_f denote the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following for all $i \in \{1, 2, \dots, n\}$:

$$T_f(\vec{e}_i) = \begin{cases} \vec{e}_{f(i)}, & f(i) \neq 0 \\ \vec{0}, & f(i) = 0 \end{cases}$$

Let M_f denote the matrix for the linear transformation T_f. M_f can be described explicitly as follows: the i^{th} column of M_f is $\vec{0}$ if $f(i) = 0$ and is $\vec{e}_{f(i)}$ if $f(i) \neq 0$.

Note that if $f, g : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ are functions with $f(0) = g(0) = 0$, then $M_{f \circ g} = M_f M_g$ and $T_{f \circ g} = T_f \circ T_g$.

We will also use the following terminology:

- A $n \times n$ matrix A is termed *idempotent* if $A^2 = A$.
- A $n \times n$ matrix A is termed *nilpotent* if there exists a positive integer r such that $A^r = 0$.

- A $n \times n$ matrix A is termed a *permutation matrix* if every row contains one 1 and all other entries 0, *and* every column contains one 1 and all other entries 0.

(1) What condition on a function $f : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ (satisfying $f(0) = 0$) is equivalent to requiring M_f to be idempotent?
(A) $(f(x))^2 = x$ for all $x \in \{0, 1, 2, \dots, n\}$
(B) $f(x^2) = x$ for all $x \in \{0, 1, 2, \dots, n\}$
(C) $(f(x))^2 = f(x)$ for all $x \in \{0, 1, 2, \dots, n\}$
(D) $f(f(x)) = x$ for all $x \in \{0, 1, 2, \dots, n\}$
(E) $f(f(x)) = f(x)$ for all $x \in \{0, 1, 2, \dots, n\}$

Answer: Option (E)

Explanation: As noted above, $M_f^2 = M_f M_f = M_{f \circ f}$. In order to have $M_f^2 = M_f$, we need $M_{f \circ f} = M_f$. Since the matrix determines the function, we get $f \circ f = f$, so $f(f(x)) = f(x)$ for all $x \in \{0, 1, 2, \dots, n\}$.

Performance review: 19 out of 28 got this. 5 chose (C), 3 chose (D), 1 chose (B).

Historical note (last time): 12 out of 26 got this. 9 chose (C), 5 chose (D).

(2) What condition on a function $f : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ (satisfying $f(0) = 0$) is equivalent to requiring M_f to be nilpotent?
(A) Composing f enough times with itself gives the identity function (i.e., the function that sends everything to itself).
(B) Composing f enough times with itself gives the function that sends everything to 0.
(C) Composing f enough times with itself gives the function that sends everything to 1.
(D) Multiplying f enough times with itself gives the identity function (i.e., the function that sends everything to itself).
(E) Multiplying f enough times with itself gives the function that sends everything to 0.

Answer: Option (B)

Explanation: The matrix M_f^r is the same as $M_{f \circ \dots \circ f}$ where f is composed with itself r times. Thus, $M_f^r = 0$ if and only if $M_{f \circ \dots f} = 0$, which means that the r-fold composite of f is the function that sends everything to zero.

Performance review: 23 out of 28 got this. 3 chose (A), 2 chose (E).

Historical note (last time): 23 out of 26 got this. 2 chose (C), 1 chose (E).

(3) What condition on a function $f : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ (satisfying $f(0) = 0$) is equivalent to requiring M_f to be a permutation matrix?
(A) Composing f enough times with itself gives the identity function (i.e., the function that sends everything to itself).
(B) Composing f enough times with itself gives the function that sends everything to 0.
(C) Composing f enough times with itself gives the function that sends everything to 1.
(D) Multiplying f enough times with itself gives the identity function (i.e., the function that sends everything to itself).
(E) Multiplying f enough times with itself gives the function that sends everything to 0.

Answer: Option (A)

Explanation: f itself is bijective, which means it cycles around the coordinates. Repeating f enough times should bring everything back where it was.

Performance review: 21 out of 28 got this. 7 chose (D).

Historical note (last time): 23 out of 26 got this. 2 chose (C), 1 chose (B).

(4) Consider a function $f : \{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ with the property that $f(0) = 0$ and, for each $i \in \{1, 2, \dots, n\}$, $f(i)$ is either i or 0. Note that the behavior may be different for different values of i (so some of them may go to themselves, and others may go to 0). What can we say M_f must be?
(A) M_f must be the identity matrix.
(B) M_f must be the zero matrix.
(C) M_f must be an idempotent matrix.
(D) M_f must be a nilpotent matrix.
(E) M_f must be a permutation matrix.

Answer: Option (C)

Explanation: The function f given here satisfies the condition that $f(f(i)) = f(i)$ for all i. To see this, note that:

- if $f(i) = i$, then $f(f(i)) = f(i) = i$, i.e., we stay looped at i.
- if $f(i) = 0$, then $f(f(i)) = f(0) = 0$, i.e., once we reach 0, we stay there.

Performance review: 10 out of 28 got this. 9 chose (D), 7 chose (E), 2 chose (A).

Historical note (last time): 12 out of 26 got this. 6 chose (D), 5 chose (E), 3 chose (A).

(5) Which of the following pairs of candidates for $f, g : \{0, 1, 2\} \to \{0, 1, 2\}$ satisfies the condition that $M_f M_g = 0$ but $M_g M_f \neq 0$?
(A) $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 1$
(B) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 0$
(C) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 2$
(D) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 1$, $g(2) = 0$
(E) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 1$

Answer: Option (D)

Explanation: For this option, $f \circ g$ sends everything to 0. $g \circ f$ sends 0 and 1 to 0 and sends 2 to 1.

Performance review: 17 out of 28 got this. 5 chose (E), 2 each chose (A), (B), and (C).

Historical note (last time): 20 out of 26 got this. 2 each chose (A), (B), and (E).

(6) Which of the following pairs of candidates for $f, g : \{0, 1, 2\} \to \{0, 1, 2\}$ satisfies the condition that M_f and M_g are both nilpotent but $M_f M_g$ is not nilpotent?
(A) $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 1$
(B) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 0$
(C) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 2$
(D) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 1$, $g(2) = 0$
(E) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 1$

Answer: Option (B)

Explanation: $f \circ f$ is the function sending everything to 0. $g \circ g$ is also the function sending everything to 0. On the other hand, $f \circ g$ is the function that sends 1 to 1 and sends 0 and 2 to 0.

Performance review: 18 out of 28 got this. 3 each chose (A) and (E), 2 chose (D), 1 chose (C), and 1 left the question blank.

Historical note (last time): 22 out of 26 got this. 3 chose (A), 1 chose (C).

(7) Which of the following pairs of candidates for $f, g : \{0, 1, 2\} \to \{0, 1, 2\}$ satisfies the condition that neither M_f and M_g is nilpotent but $M_f M_g$ is nilpotent?
(A) $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 1$
(B) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 2$, $g(2) = 0$
(C) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 2$
(D) $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, whereas $g(0) = 0$, $g(1) = 1$, $g(2) = 0$
(E) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, whereas $g(0) = 0$, $g(1) = 0$, $g(2) = 1$

Answer: Option (C)

Explanation: M_f and M_g are both idempotent, since $f \circ f = f$ and $g \circ g = g$. However, $f \circ g$ is the function sending everything to 0, so $M_f M_g = 0$.

Performance review: 20 out of 28 got this. 3 chose (A), 2 chose (B), 1 each chose (D) and (E), 1 left the question blank.

Historical note (last time): 20 out of 26 got this. 3 chose (B), 1 each chose (A), (D), and (E).