HYBRID TAKE-HOME PLUS CLASS QUIZ SOLUTIONS: MONDAY NOVEMBER 12: TAYLOR SERIES AND POWER SERIES

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

42 people took this quiz. The score distribution was as follows:

- Score of 3: 6 people
- Score of 4: 6 people
- Score of 5: 5 people
- Score of 6: 6 people
- Score of 7: 19 people

The question wise answers were as follows:

(1) Option (C): 30 people
(2) Option (A): 28 people
(3) Option (D): 37 people
(4) Option (D): 28 people
(5) Option (A): 34 people
(6) Option (D): 40 people
(7) Option (D): 39 people

2. Solutions

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We denote by $C^k(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are at least k times continuously differentiable on all of $\mathbb{R}$. Note that for $k \ge l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k.

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

(1) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0? *Two years ago:* 3/26 *correct*
(A) $f_1(x) := \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$
(B) $f_2(x) := \begin{cases} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(C) $f_3(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(D) $f_4(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$
(E) All of the above.
Answer: Option (C)
Explanation: This answer is explained more in the lecture notes.
Why the other options are wrong:

Option (A): This is in fact globlally analytic, and is given by the power series $1-x^2/3!+x^4/5!-\dots$

Option (B): This is not continuous at 0. The left hand limit at 0 is $+\infty$.

Option (D): This is not continuous at 0. The limit at 0 is not defined.

Performance review: 30 out of 42 got this. 5 chose (D), 4 chose (E), 2 chose (B), 1 chose (A).

Historical note (last year): 1 out of 11 got this correct. 5 chose (D), 2 chose (B), 1 each chose (A) and (E).

Historical note (two years ago): 3 out of 26 people got this correct. 11 chose (D), 6 each chose (A) and (E).

(2) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic? *Two years ago:* 7/26 *correct*

(A) $x \mapsto \ln(1+x^2)$
(B) $x \mapsto \ln(1+x)$
(C) $x \mapsto \ln(1-x)$
(D) $x \mapsto \exp(1+x)$
(E) $x \mapsto \exp(1-x)$

Answer: Option (A)

Explanation: The function is in $C^\infty(\mathbb{R})$ because it can be differentiated infinitely often: the first derivative is $2x/(1+x^2)$, and each subsequent derivative is a rational function whose denominator is a power of $1+x^2$. Since $1+x^2$ does not vanish anywhere on $\mathbb{R}$, each derivative is defined and continuous on all of $\mathbb{R}$.

The radius of convergence of the power series is 1, basically because it is a power series where the coefficients are rational functions, and any such power series has radius of convergence 1 by the root test or ratio test. Thus, the function is not globally analytic.

Why the other options are wrong:

Option (B) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \le -1$.

Option (C) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \ge 1$.

Options (D) and (E) are globally analytic because exp is globally analytic.

Performance review: 28 out of 42 got this. 8 chose (B), 6 chose (C).

Historical note (last year): Nobody got this correct! 8 chose (B), 3 chose (C).

Historical note (two years ago): 7 out of 26 people got this correct. 6 chose (C), 5 chose (B), 4 chose (E), 3 chose (D), and 1 left the question blank.

(3) Suppose f and g are globally analytic functions and g is nowhere zero. Which of the following is *not necessarily* globally analytic?

(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$
(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$
(C) fg, i.e., the function $x \mapsto f(x)g(x)$
(D) f/g, i.e., the function $x \mapsto f(x)/g(x)$
(E) $f \circ g$, i.e., the function $x \mapsto f(g(x))$

Answer: Option (D)

Explanation: See the example for the next question.

Performance review: 37 out of 42 got this. 4 chose (E), 1 chose (A).

(4) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic? *Two years ago:* 10/26 *correct*

(A) x
(B) x^2
(C) $x+1$
(D) x^2+1
(E) e^x

Answer: Option (D)

Explanation: The reciprocal $1/(x^2+1)$ is a rational function all of whose derivatives are rational functions with denominator a power of x^2+1, hence defined and continuous derivatives. Hence it is is $C^\infty(\mathbb{R})$. Further, the power series expansion for it is like a geometric series, which has radius of convergence 1, hence it is not globally analytic.

Why the other options are wrong:

Options (A) and (B): The reciprocals are not in $C^\infty(\mathbb{R})$ because the functions $1/x$ and $1/x^2$ are not defined or continuous at 0.

Option (C): The reciprocal is not in $C^\infty(\mathbb{R})$ because the function $1/(x+1)$ is not defined or continuous at -1.

Option (E): The reciprocal, which is $\exp(-x)$, is globally analytic.

Performance review: 28 out of 42 got this. 8 chose (C), 4 chose (E), 2 chose (A).

Historical note (last year): 9 out of 11 got this correct. 1 each chose (A) and (E).

Historical note (two years ago): 10 out of 26 people got this correct. 5 chose (C), 4 each chose (A) and (E), 2 chose (B), 1 left the question blank.

(5) Consider the rational function $1/\prod_{i=1}^{n}(x-\alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_is, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_is? *Two years ago:* 10/26 *correct*

(A) It is the minimum of the distances from c to the α_is.
(B) It is the second smallest of the distances from c to the α_is.
(C) It is the arithmetic mean of the distances from c to the α_is.
(D) It is the second largest of the distances from c to the α_is.
(E) It is the maximum of the distances from c to the α_is.

Answer: Option (A)

Explanation: Since the Taylor series converges to the function on its interval of convergence, the interval of convergence must be contained in the domain of definition. In particular, it must exclude all the α_is. Hence, the radius of convergence cannot be more than the minimum of the distances from c to the α_is.

That it is exactly equal to the minimum can be shown by using the fact that we get a product of geometric series.

Performance review: 34 out of 42 got this. 6 chose (C), 2 chose (D).

Historical note (last year): 3 out of 11 got this correct. 4 chose (E), 2 chose (C), 1 chose (B).

Historical note (two years ago): 10 out of 26 people got this correct. 6 people chose (C), 5 chose (E), 2 each chose (B) and (D), 1 left the question blank.

(6) What is the interval of convergence of the Taylor series for arctan about 0? *Two years ago:* 11/26 *correct*

(A) $(-1,1)$
(B) $[-1,1)$
(C) $(-1,1]$
(D) $[-1,1]$
(E) All of $\mathbb{R}$

Answer: Option (D)

Explanation: For the boundary points, we use the alternating series theorem. See a more detailed discussion in the lecture notes.

Performance review: 40 out of 42 got this. 1 each chose (A) and (E).

Historical note (last year): 10 out of 11 got this correct. 1 chose (A).

Historical note (two years ago): 11 out of 26 people got this correct. 8 chose (A), 4 chose (C), 2 chose (B), 1 chose (E).

(7) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} 2^{\sqrt{k}} x^k$? Please keep in mind the square root in the exponent.

(A) 0
(B) $1/2$
(C) $1/\sqrt{2}$
(D) 1
(E) infinite

Answer: Option (D)

Explanation: The coefficients have subexponential growth, so the radius of convergence is 1.

Performance review: 39 out of 42 got this. 3 chose (B).