CLASS QUIZ SOLUTIONS: OCTOBER 5: LIMITS AT AND TO INFINITY

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

46 people took this 2-question quiz. The score distribution was as follows:

- Score of 0: 4 people
- Score of 1: 12 people
- Score of 2: 30 people

The question-wise answers and performance are as follows:

(1) Option (C): 35 people
(2) Option (E): 37 people

2. Solutions

(1) The graph $y = f(x)$ of a function f defined on all reals has a horizontal asymptote $y = c$ as x approaches $+\infty$. Which of the following is the correct definition of this?
   (A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
   (B) For every $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
   (C) For every $\varepsilon > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
   (D) For every $\delta > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
   (E) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - c| < \varepsilon$.

   Answer: Option (C)

   Explanation: The neighborhood of c (picked by the skeptic) is the interval $(c - \varepsilon, c + \varepsilon)$, and it is parametrized by its radius ε. The neighborhood of $+\infty$ (picked by the prover) is the interval (a, ∞), and it is parametrzed by its lower endpoint a. The skeptic then picks x in the neighborhood specified by the prover, i.e., $f(x) > a$, and then they check whether $f(x)$ is in the chosen neighborhood of c.

   Performance review: 35 out of 46 got this correct. 4 each chose (B) and (E). 3 chose (A).

(2) Which of the following is the correct definition of $\lim_{x \to c^-} f(x) = -\infty$ (in words: the left hand limit of f at c is $-\infty$)?
   (A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
   (B) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) > a$.
   (C) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) < a$.
   (D) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) > a$.
   (E) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) < a$.

   Answer: Option (E)

   Explanation: The neighborhood of $-\infty$ chosen by the skeptic is $(-\infty, a)$, and it is parameterized by its upper endpoint a. The prover picks the parameter δ for the left side δ "half-neighborhood" of c, namely $(c - \delta, c)$. The skeptic then picks x in this half-neighborhood, and they then check whether $f(x) \in (-\infty, a)$. Translating the interval conditions into inequality notation, we get the definition as stated.

   Performance review: 37 out of 46 got this correct. 3 each chose (B), (C), and (D).