# HOMEWORK 2: DUE MONDAY OCTOBER 14

MATH 196, SECTION 57 (VIPUL NAIK)

You may not get the graded homework back in time for the midterm, so please keep a copy of it for your own review.

## 1. Routine problems

*Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.*

(1) Exercise 1.2.8 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work. The variables are $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$, even though $x_1$ and $x_3$ do not appear anywhere in the system.

$$\begin{aligned} x_2 + 2x_4 + 3x_5 &= 0 \\ 4x_4 + 8x_5 &= 0 \end{aligned}$$

(2) Exercise 1.2.9 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

$$\begin{aligned} x_4 + 2x_5 - x_6 &= 2 \\ x_1 + 2x_2 + x_5 - x_6 &= 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2 \end{aligned}$$

(3) Exercise 1.2.10 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

(4) Exercise 1.2.33 (Page 19) (was Exercise 1.2.31 in the 4th Edition): Find the polynomial of degree 4 whose graph goes through the points $(1, 1)$, $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$. Graph this polynomial.

(5) Exercise 1.3.2 (Page 34): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(6) Exercise 1.3.3 (Page 34): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(7) Exercise 1.3.4 (Page 34): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(8) Exercise 1.3.19 (Page 35): Compute the matrix-vector product:

$$\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

For the next four problems, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(9) Exercise 1.3.29 (Page 35): Find a *diagonal* matrix $A$ such that $A\vec{x} = \vec{y}$.

(10) Exercise 1.3.30 (Page 35): Find a matrix $A$ of rank 1 such that $A\vec{x} = \vec{y}$.

(11) Exercise 1.3.31 (Page 35-36): Find an *upper triangular* matrix $A$ such that $A\vec{x} = \vec{y}$, where all the entries of $A$ on and above the diagonal are nonzero.

(12) Exercise 1.3.32 (Page 36): Find a matrix $A$ with all nonzero entries such that $A\vec{x} = \vec{y}$.

## 2. Problems for your own review, not for submission

You should know the answers to these, but do not submit them. This will keep the grading workload to a realistic minimum. I will put up hints for these in the checklist.

For the exercises here from Section 1.2, the exercise numbers in the 4th Edition are 2 lower than those in the 5th Edition. Exercises 1.2.22-28 are Exercises 1.2.20-26 in the old edition. The exercise numbers for the exercises from Section 1.3 are the same across editions.

(1) Exercise 1.2.22 (Page 19): We say that two $n \times m$ matrices in reduced row-echelon are of the same type if they contain the same number of leading 1's in the same positions. For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are of the same type. How many types of $2 \times 2$ matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

(2) Exercise 1.2.23 (Page 19): How many types of $3 \times 2$ matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

(3) Exercise 1.2.24 (Page 19): How many types of $2 \times 3$ matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

(4) Exercise 1.2.26 (Page 19): Suppose matrix $A$ is transformed into matrix $B$ by means of an elementary row operation. Is there an elementary row operation that transforms $B$ into $A$? Explain.

(5) Exercise 1.2.27 (Page 19): Suppose matrix $A$ is transformed into matrix $B$ by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms $B$ into $A$? Explain your answer (this relies on the preceding exercise).

(6) Exercise 1.2.28 (Page 19): Consider an $n \times m$ matrix $A$. Can you transform the reduced row-echelon form of $A$ into $A$ by a sequence of elementary row operations? This relies on the preceding exercise.

(7) Exercise 1.3.22 (Page 35): Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

(8) Exercise 1.3.23 (Page 35): Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

(9) Exercise 1.3.24 (Page 35): Let $A$ be a $4 \times 4$ matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

(10) Exercise 1.3.25 (Page 35): Let $A$ be a $4 \times 4$ matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

(11) Exercise 1.3.26 (Page 35): Let $A$ be a $4 \times 3$ matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

(12) Exercise 1.3.41 (Page 36): How many solutions do most systems of three linear equations with three unknowns have?

(13) Exercise 1.3.42 (Page 36): How many solutions do most systems of three linear equations with four unknowns have?

(14) Exercise 1.3.43 (Page 36): How many solutions do most systems of four linear equations with three unknowns have?

(15) Exercise 1.3.44 (Page 36): Consider an $n \times m$ matrix $A$ with more rows than columns ($n > m$). Show that there is a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

(16) Exercise 1.3.46 (Page 36): Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

where $a$, $d$, and $f$ are nonzero, and $b$, $c$, and $e$ are arbitrary numbers.

## 3. Advanced problems

(1) Exercise 1.2.34 (Page 20) (*Cubic splines*) (was Exercise 1.2.32 in the 4th Edition): This is an extremely long problem with pictures, so I recommend you read it from the book.

(2) Exercise 1.2.39 (Page 20) (Leontief input-output model for three variables) (was Exercise 1.2.37 in the 4th Edition): Please see the diagram and the labels in the diagram from the book.

(3) Exercise 1.2.40 (Page 20-21) (was Exercise 1.2.38 in the 4th Edition): This covers the Leontief input-output model in $n$ variables. The question is long, so I recommend you read it from the book.

(4) Exercise 1.2.44 (Page 22) (was Exercise 1.2.42 in the 4th Edition): This problem involves estimating the amount of traffic on one-way streets. Please read the problem from the book.

(5) Exercise 1.3.47 (Page 36): A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

(a) All homogeneous systems are consistent.

(b) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

(c) If $\vec{x_1}$ and $\vec{x_2}$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x_1} + \vec{x_2}$ is a solution as well.

(d) If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and $k$ is an arbitrary constant, then $k\vec{x}$ is a solution as well.

(6) Exercise 1.3.48 (Page 36-37): Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

(a) If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

(b) If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

(c) Now suppose $A$ is $2 \times 2$ matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$ (see the book for the figure: it has a line through the origin for the solutions of $A\vec{x} = \vec{0}$ and $\vec{x}_1$ depicted as a vector (not along the line) with tail at $\vec{0}$).

If you are puzzed by the generality of this problem think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \qquad \text{and} \qquad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$