HOMEWORK 3: DUE WEDNESDAY OCTOBER 23

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 2.1.4 (Page 53): Find the matrix of the linear transformation:

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3 \end{aligned}$$

(2) Exercise 2.1.5 (Page 53): Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}$$

Find the matrix A of T.

(3) Exercise 2.1.32 (Page 54): Find an $n \times n$ matrix A such that $A\vec{x} = 3\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

(4) Exercise 2.1.35 (Page 55): In the example about the French coast guard in Section 2.1, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

- When the actual position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$, they radio $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$.
- When the actual position is $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$, they radio $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

(5) Exercise 2.1.58 (was 2.1.50 in the 4th Edition) (Page 57): A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

(a) You can skip this.

(b) Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix}$$

for any piece of jewelry the goldsmith makes.

(c) Is the matrix A in part (b) invertible? If so, find the inverse. You can use the result from Question 1 of the advanced homework for this. (You can skip the rest of part (c)).

(6) Exercise 2.1.59 (was 2.1.51 in the 4th Edition) (Page 57): The conversion matrix $C = \frac{5}{9}(F - 32)$ from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

(a) Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$. (The second row of A will be $\begin{bmatrix} 0 & 1 \end{bmatrix}$.)

(b) Is the matrix A in part (a) invertible? If so, find the inverse. You can use the result from Question 1 of the advanced homework for this. Use the result to write a formula expressing F in terms of C.

(7) Exercise 2.1.62 (was 2.1.54 in the 4th Edition) (Page 57-58): Consider an arbitrary currency exchange matrix A (see Exercises 60 and 61 from the book, which were 52 and 53 in the 4th Edition).

(a) What are the diagonal entries a_{ii} of A?

(b) What is the relationship between a_{ij} and a_{ji}?

(c) What is the relationship between a_{ik}, a_{kj}, and a_{ij}?

(d) What is the rank of A? What is the relationship between A and $\operatorname{rref}(A)$?

2. Problems for your own review

(1) Exercise 2.1.40 (Page 55): Describe all linear transformations from $\mathbb{R}$ $(= \mathbb{R}^1)$ to $\mathbb{R}$. What do their graphs look like?

(2) Exercise 2.1.41 (Page 55): Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. What do their graphs look like?

(3) Exercise 2.1.44 (Page 56): The cross product of two vectors in $\mathbb{R}^3$ is given by:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

3. Advanced problems

(1) Exercise 2.1.13 (Page 54): Prove the following facts:

(a) The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. *Hint from the book*: Consider the cases $a \neq 0$ and $a = 0$ separately.

(b) If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The formula here is worth memorizing.

(2) Exercise 2.1.39 (Page 55): Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m)$$

where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ are the standard vectors in $\mathbb{R}^m$.

(3) Exercise 2.1.45 (Page 56): Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$ where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$, and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L(T(\vec{x}))$ linear as well? *Please justify your answer.* [The transformation $\vec{z} = L(T(\vec{x}))$ is called the *composite* of T and L.]