SOLVING DIFFERENTIAL EQUATIONS: HEAT EQUATION, WAVE EQUATION AND MORE

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ABSTRACT. This article describes some aspects of solving differential equations using the toolkit and ideas of functional analysis.

1. DIFFERENTIAL EQUATIONS AND FUNCTIONAL ANALYSIS

We shall here study differential equations that have the hope of modelling some physical law. The idea usually is that there is some physical quantity that is defined for every point in time and space, and whose value changes both with time and space. The "evolution" of this physical quantity with time and space is described by a partial differential equation.

Two basic principles that we can start out with are:

- Time-invariance: This principle states that translating the time origin should not change the physical law
- Spatial invariance, or translaton-invariance: This principle states that translating the spatial origin should not change the physical law

Assuming both spatial and time invariance restricts us to a class of partial differential equations called **autonomous partial differential equation**(defined). Loosely speaking, an autonomous partial differential equation is a partial differential equation that is an equation of the form $F(u, \partial u) = 0$, in other words, it is an equation relating the value of u and its mixed partials at a particlar point in space-time. The crucial feature is that space and time can appear in the differential equation *only* by differentiating in them. Thus:

$$tu = 0$$

is *not* autonomous,but $u = 0$ is.

There could be other desirable features of a physical law, for instance, dimension matching, rotation invariance, covariance with respect to scaling space and time, translation invariance of u and so on.

1.1. Evolution and flows.

The typical kind of PDE we come across everyday from physical situations is:

$$\frac{\partial u}{\partial t} = \text{Something involving } u\text{, and its } x\text{-partials}$$

In other words, the *rate of change* of u depends entirely on u and its mixed x-partials, where x is the spatial variable. This is a flow equation. Intuitively, if we are thinking of u as living inside a monstrous function space, what we're doing is providing a vector field in this function space and requiring that any solution is an "integral curve" of this vector field.

Our goal is the following: given an initial condition u_0 at time 0, find a solution $u(x,t)$ such that $u(x,0) = u_0(x)$. In other words, trace the integral curve of the vector field that at time 0, is the point u_0 (in the monstrous function space). The existence and uniqueness of solution to differential equations is thus the theory of existence and uniqueness to integral curves for vector fields.

In this and subsequent sections, we shall use subscript notation for partial derivatives. So u_{xy}, for instance, denotes a second derivative of u, a mixed partial in x and y.

1.2. **The solution operator.** Let's first define the notion of a solution operator.

Definition (Solution operator)**.** Suppose $u_t = F$ is an autonomous differential equation (Where the right side is a differential operator involving only partials in the spatial variables). A solution operator for this associates, to every t, a linear operator S_t such that for any function u_0, $S_t(u_0)(x) = u(x,t)$ where $u(x,t)$ is the solution corresponding to u_0.

In other words, the solution operator is an operator that takes the value at time 0 and outputs the value at time t. Ideally we'd like to be able to guarantee that solution operators exist, and can be written down explicitly.

If reasonable existence and uniqueness conditions hold, we get a solution operator for every t, and the solution operators form a one-parameter group of transformations.

We consider the particularly nice situation where the right side is a linear differential operator in u. The following is clear:

> When the differential operator is linear, so is the solution operator (by linear, we mean linear in u_0).

1.3. **A good family of kernels.** Our setup so far is that we have a linear differential operator F in terms of the spatial coordinates, and we want to solve the initial value problem:

$$u_t = Fu$$

where we are given the function u at time 0. More generally, we want to find a family S_t of solution operators for this differential equation.

Let's now define a family of good kernels:

Definition (Family of good kernels)**.** A collection of functions $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is termed a **family of good kernels**(defined):

- For each t, the map $x \mapsto \phi(x,t)$ integrates to 1
- The ϕs are Schwarz in x for each t, in a uniform manner.
- The integral operators obtained by integrating against $\phi(x,t)$ converge to the δ-operator

Now the following is an easy to check fact:

> If ϕ is a family of good kernels that itself solves the differential equation we want (i.e. $\phi_t = F\phi$), then the solution operators for the differential equation $u_t = Fu$ are given by integral operators determiend by ϕ.

The key steps/ideas of the proof are as follows:

- We know that φ solves the differential equation. We want to use this to show that something obtained by integrating against it satisfies the differential equation. Let's consider the definition of u:

$$u(x,t) = \int \phi(x-y,t)u_0(y)\,dy$$

- We now want to check that $u_t = Fu$. The key point is that we can use a corollary of the dominated convergence theorem to interchange the derivative and integral, if φ satisfies a Schwarz-like condition. After that, the proof for satisfying the differential equation is direct.
- We also need to show that as $t \to 0$, the function $x \mapsto u(x,t)$ approaches the function $u_0(x)$.

2. Weak solutions to PDEs

2.1. **Functions, measures and operators.** Before proceeding further, let us look at three related notions. We look at the three classes of ideas:

Function	Measure	Linear operator
Nonnegative integrable function	Positive finite measure	Positive linear operator
Real integrable function	Real finite measure	Real linear operator
Complex integrable function	Finite complex measure	Linear operator
Nonnegative L^1_{loc} function	Real measure	Real linear operator

Given an integrable function from a measure space to $\mathbb{C}$, we can associate to that a measure, as the measure obtained by integrating against it. For the measure to be finite, we require that the function be in L^1. If the function is real-valued, we get a real measure, and if the function is nonnegative, we get a nonnegative measure.

Any measure defines a linear functional, again by integrating against it. The correspondence between measures and operators is bijective; for further terminology, we also sometimes use the term "distribution" for the measure or associated operator. On the other hand, not every measure can be traced as coming from a function.

In fact, the Radon-Nikodym theorem tells us that any measure on $\mathbb{R}$ can be decomposed asa sum of two orthogonal pieces: a part that is absolutely continuous with respect to Lebesgue measure, and a singular part, which is in some sense "orthogonal" to Lebesgue measure. An example of a singular measure (which thus, cannot be obtained as coming from an integrable function) is the delta-measure, which associated measure 1 to the point 0 and measure 0 elsewhere.

2.2. **Weak convergence and weak solutions.** Weak convergence is a somewhat slippery concept, so don't be surprised of you don't get this straight at the start. The notion works well only for linear PDEs, which will be our focus for this subsection:

$$Fu = 0$$

here u is the function, F is the differential operator, and we need to find a "solution" u.

Now the nature of the differential operator F imposes conditions on the kind of us we can look for if we're interested in strong, or genuine, solutions. For instance, if F involves second partials, then honest solutions must at the very least be in C^2.

The notion of "weak solution" puts the lie to this. The idea, roughly speaking is as follows:

- Assume we had a strong solution.
- Pick a test function g
- Find a new PDE such that, within the domain of strong solutions for the original PDE, the two PDEs are equivalent.
- Now try to solve the new PDE within a bigger, or different, function space.

Let's take an example. Suppose we want to solve the differential equation:

$$\Delta u = f$$

on a domain $\Omega \subset \mathbb{R}^n$ whose closure is compact.

Now, if we pick a function g that is continuous with compact support, then we know that:

$$\int_\Omega (\Delta u)(g) = \int_\Omega (\nabla u) \cdot \nabla g = \int_\Omega u(\Delta g)$$

(The two steps there are applications of the product rule to $(\nabla u) \cdot g$. Thus, a "solution" function f should be a function such that:

$$\int_\Omega fg = \int_\Omega u(\Delta g)$$

for any function in $C^2_c(\Omega)$ (i.e. any twice-continuously differentiable function).

We can now try to look at solutions for the above equation, over a larger domain. In other words, we want f such that for *every* $g \in C^2_c(\Omega)$, the above two integrals are equal.

Note that we didn't have to go two