CLASS QUIZ SOLUTIONS: JANUARY 14: INTEGRATION BY PARTS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Update on Questions 2 and 3

Update added Sunday January 16: Slight phrasing correction in Questions 2 and 3. Since it is very minor, and most people got the sense of the questions, I plan to keep scoring for the questions as is. For both questions, the question should have asked for the largest k for *guaranteed* elementary integrability rather than just the largest k for elementary integrability. In other words, it may so happen that for specific choices of f, it is possible to integrate $x^k f(x)$ for some larger values of k, but this is not guaranteed in general.

A related clarification will be made in class on Wednesday January 19.

2. Performance review

27 people took this 9-question quiz. The score distribution was as follows:

- Score of 2: 2 people.
- Score of 3: 3 people.
- Score of 4: 10 people.
- Score of 5: 5 people.
- Score of 6: 4 people.
- Score of 8: 3 people.

The mean score was 4.67 and the mean and modal score were 4.

Here is the question-wise performance:

(1) Option (B): 10 people. *Please review this solution.*
(2) Option (B): 23 people.
(3) Option (C): 14 people. *Please review this solution.*
(4) Option (C): 22 people.
(5) Option (D): 17 people.
(6) Option (E): 7 people. *Please review this solution.*
(7) Option (A): 10 people. *Please review this solution.*
(8) Option (A): 5 people. *Please review this solution.*
(9) Option (E): 18 people.

3. Solutions

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

(1) Suppose f is an elementarily expressible and infinitely differentiable function on the positive reals (so all derivatives of f are also elementarily expressible). An antiderivative for $f''(x)/x$ is **not equivalent** up to elementary functions to **which one** of the following?
(A) An antiderivative for $x \mapsto f''(e^x)$, domain all of $\mathbb{R}$.
(B) An antiderivative for $x \mapsto f'(e^x/x)$, domain positive reals.
(C) An antiderivative for $x \mapsto f'''(x)(\ln x)$, domain positive reals.
(D) An antiderivative for $x \mapsto f'(1/x)$, domain positive reals.
(E) An antiderivative for $x \mapsto f(1/\sqrt{x})$, domain positive reals.

Answer: Option (B)

Explanation: We will show how an antiderivative for $f''(x)/x$ is equivalent to all the antiderivatives in options (A), (C), (D), and (E).

Option (A): Starting with $\int \frac{f''(x)}{x}\, dx$. Put $u = \ln x$. We get $\int f''(e^u)\, du$. Note that the domain now becomes all of $\mathbb{R}$. Replace the dummy variable u by the dummy variable x, and we get $\int f''(e^x)\, dx$.

Option (C): Let's start with $f'''(x)(\ln x)$. Integrate by parts taking $f'''(x)$ as the part to integrate. We get $\int f'''(x)(\ln x)\, dx = (\ln x)(f''(x)) - \int \frac{1}{x} f''(x)\, dx$. Thus, we see that the antiderivatives of $f'''(x)(\ln x)$ and $f''(x)/x$ add up to $f''(x)(\ln x)$, which is an elementarily expressible function, hence the antiderivatives are elementarily equivalent.

Option (D): Start with $\int f'(1/x)\, dx$. Put $u = 1/x$ to get $\int \frac{-1}{u^2} f'(u)\, du$. Now integrate by parts taking $-1/u^2$ as the part to integrate, and we obtain a relationship with the integral of $f''(u)/u$.

Option (E): Here, put $u = 1/\sqrt{x}$, so $x = 1/u^2$, giving $\int f(u)/u^3\, du$. Integrate by parts twice taking the rational function as the part to integrate each time. We get $f''(u)/u$ (up to constants).

Post-performance review: 10 out of 27 people got this correct. 8 people chose (C), 6 people chose (E), and 3 people chose (A).

(2) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is [ADDED: *guaranteed to be*] **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (B)

Explanation: Via integration by parts, integrating f m times is equivalent to finding antiderivatives for $f(x)$, $xf(x)$, and so on till $x^{m-1}f(x)$. In our case, f can be integrated 3 times, so the largest k is $3 - 1 = 2$.

Post-performance review: 23 out of 27 people got this correct. 2 people each chose (C) and (D).

Note: The original question was phrased somewhat inaccurately. It is true that $x^2 f(x)$ can be integrated and $x^3 f(x)$ cannot. *A priori*, we cannot say whether $x^4 f(x)$ and $x^5 f(x)$ can or cannot be integrated. They cannot be integrated *using the integration by parts approach*, but it may happen to be the case that they could be integrated by other methods, though this is rare in practical cases (and when that does happen, it is obvious).

(3) Suppose f is a continuous function on $(0, \infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0, \infty)$ is [ADDED: *guaranteed to be*] **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4

(E) 5

Answer: Option (C)

Explanation: Via the u-substitution $u = x^{1/k}$, we get $\int ku^{k-1} f(u)\, du$. Now using the previous question, the maximum value of $k-1$ possible is 2, so the maximum possible value is 3.

We can also do a direct integration by parts taking 1 as the second part.

Post-performance review: 14 out of 27 people got this correct. 4 people chose (D), 4 people chose (B), 3 people chose (A), 1 person chose (E), and 1 person left the question blank.

Note: The original question was phrased somewhat inaccurately. It is true that $f(x^{1/3})$ can be integrated and $f(x^{1/4})$ cannot. *A priori*, we cannot say whether $f(x^{1/5})$ can or cannot be integrated. It cannot be integrated *using the integration by parts approach*, but it may happen to be the case that they could be integrated by other methods, though this is rare in practical cases (and when that does happen, it is obvious).

(4) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.

(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Answer: Option (C)

Explanation: If f is elementarily integrable, then $xf(x)$ is elementarily integrable iff f is twice elementarily integrable; this is easily seen using integration by parts. Of the function options given here, tan is the only function that is not twice elementarily integrable, because the first integration gives $-\ln|\cos x|$ which cannot be integrated. Of the others, note that sin, cos, and $\sin^2$ can be integrated using elementary functions infinitely many times. $\tan^2$ is twice elementarily integrable but no further: integrates the first time to $\tan x - x$, which integrates one more time to $-\ln|cosx| - x^2/2$, which cannot be integrated further.

Post-performance review: 22 out of 27 people got this correct. 4 people chose (E), and 1 person chose (D).

(5) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin\sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0,1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are the **two elementarily integrable functions**?

(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.
(D) f_2 and f_3.
(E) f_2 and f_4.

Answer: Option (D)

Explanation: Integration of f_3 is a standard procedure, so we say nothing about that. As for f_2, recall that integrating $f(x^{1/k})$ is equivalent to integrating $u^{k-1}f(u)$ where $u = x^{1/k}$, which in turn is equivalent to integrating f k times. Since sin can be integrated as many times as we wish, f_2 can be integrated.

The reason why f_1 and f_4 are not elementarily integrable is subtler but it's clear that none of the obvious methods work.

Post-performance review: 17 out of 27 people got this correct. 5 people chose (B), 3 people chose (A), and 2 people chose (C).

(6) Of the five functions below, four of them have antiderivatives that are equivalent up to elementary functions, i.e., an antiderivative for any one of them can be used to provide an antiderivative for the other three. The fifth function hais **not equivalent** to any of these. Identify the fifth function.

(A) $x \mapsto e^{e^x}$, domain all reals
(B) $x \mapsto \ln(\ln x)$, domain $(1, \infty)$
(C) $x \mapsto e^x/x$, domain $(0, \infty)$

(D) $x \mapsto 1/(\ln x)$, domain $(1, \infty)$
(E) $x \mapsto 1/(\ln(\ln x))$, domain (e, ∞)

Answer: Option (E)

Explanation: We show the equivalence of all the others:

(A) and (C): Starting with $\int e^{e^x}\,dx$, put $u = e^x$, to get $\int e^u/u\,du$. Note that the domain of definition transforms correctly.

(C) and (D): Starting with $\int e^x/x\,dx$, put $u = e^x$, to get $du/(\ln u)$. Note that the domain of definition transforms correctly.

(B) and (D): Start with $\int \ln(\ln x)\,dx$. Use integration by parts taking 1 as the part to integrate. We get $x\ln(\ln x) - \int \frac{1}{\ln x}\,dx$, establishing the equivalence.

Post-performance review: 7 out of 27 people got this correct. 14 people chose (C), 3 people chose (B), 2 people chose (D), 1 person chose (A).

(7) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?
(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2e^{-x^2}$
(E) $x \mapsto x^4e^{-x^2}$

Answer: Option (A)

Explanation: We show the equivalence with the others.

Option (D): We use integration by parts, writing $x^2e^{-x^2}$ as $x \cdot (xe^{-x^2})$ and taking xe^{-x^2} as the part to integrate, so that x is the part to differentiate. An antiderivative for xe^{-x^2} is $(-1/2)e^{-x^2}$, so we get:

$$\frac{-x}{2}e^{-x^2} - \int \frac{-1}{2}e^{-x^2}\,dx$$

We thus see that it reduces to $\int e^{-x^2}\,dx$.

Option (E), via reduction to option (D): We use integration by parts, taking x^3 as the part to differentiate and xe^{-x^2} as the part to integrate. One application of integration by parts reduces this to $\int x^2e^{-x^2}$, which is option (D).

Option (B), via reduction to option (D): Start with $\int e^{-x^{2/3}}\,dx$. Put $u = x^{1/3}$. The substitution gives (up to scalars) $\int u^2e^{-u^2}\,du$, which is option (D).

Option (C), via reduction to option (D): Start with $\int e^{-x^{2/5}}\,dx$. Put $u = x^{1/5}$. The substitution gives (up to scalars) $\int u^4e^{-u^2}\,du$, which is option (E).

Post-performance review: 10 out of 27 people got this correct. 7 people chose (D), 4 people chose (C), 4 people chose (E), and 2 people chose (B).

(8) Which of the following has an antiderivative that is not equivalent up to elementary functions to the antiderivative of the function $f(x) := e^x/x, x > 0$?
(A) $x \mapsto e^x/\sqrt{x}, x > 0$
(B) $x \mapsto e^x/x^2, x > 0$
(C) $x \mapsto e^x(\ln x), x > 0$
(D) $x \mapsto e^{1/\sqrt{x}}, x > 0$
(E) $x \mapsto e^{1/x}, x > 0$

Answer: Option (A)

Explanation: Option (B) is equivalent via one application of integration by parts. Option (C) is also equivalent via one application of integration by parts. Option (E) reduces to option (B) when we put $u = 1/\sqrt{x}$. Option (D) reduces to e^x/x^3, which is equivalent to option (B) via one application of integration by parts.

Post-performance review: 5 out of 27 people got this correct. 9 people chose (C), 10 people chose (D), and 3 people chose (E).

(9) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?

(A) There is no way of deducing Q from P because P is true and Q is false.

(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.

(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\dots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Answer: Option (E)

Explanation: Fill in yourself, it's been said often enough.

Post-performance review: 18 out of 27 people got this correct. 4 people chose (D), 2 people chose (C), 2 people chose (B), and 1 person chose (A).