CLASS QUIZ SOLUTIONS: FRIDAY JANUARY 18: VECTORS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

? people took this 6-question quiz. The score distribution was as follows:

- Score of 3: 3 people.
- Score of 4: 3 people.
- Score of 5: 17 people.
- Score of 6: 1 person.

The mean score was about 4.5. The question wise solutions and performance summary are below:

(1) Option (C): 21 people.
(2) Option (E): 23 people.
(3) Option (C): 20 people.
(4) Option (E): 21 people.
(5) Option (D): 3 people.
(6) Option (C): 24 people.

2. Solutions

(1) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ with the property that the dot product of any two distinct elements of S is zero. What is the maximum possible size of S?
(A) 1
(B) 2
(C) 3
(D) 4
(E) There is no finite bound on the size of S

Answer: Option (C)

Explanation: The given property indicates that any two elements of S are mutually orthogonal vectors. In $\mathbb{R}^3$, there can be at most three mutually orthogonal directions, because the space is three-dimensional. (This can be proved formally, but we won't bother here).

Performance review: 21 out of 24 people got this. 3 chose (B).

Historical note (last year): 14 out of 23 people got this correct. 6 chose (E), 2 chose (B), 1 chose (D).

(2) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ such that the cross product of any two distinct elements of S is the zero vector. What is the maximum possible size of S?
(A) 1
(B) 2
(C) 3
(D) 4
(E) There is no finite bound on the size of S

Answer: Option (E)

Explanation: The given condition means that any two elements of S are scalar multiples of each other. We can easily construct an infinite set with this property: consider *all* nonzero scalar multiples of a fixed nonzero vector. There are infinitely many such multiples, because there are infinitely many nonzero reals, and each such multiple is different.

Performance review: 23 out of 24 people got this. 1 person chose (B).

Historical note (last year): 17 out of 23 people got this correct. 3 chose (A), 2 chose (B), 1 chose (C).

(3) Suppose a and b are vectors in $\mathbb{R}^3$. Which of the following is/are true?
(A) If both a and b are nonzero vectors, then $a \times b$ is a nonzero vector.
(B) If $a \times b$ is a nonzero vector, then $a \cdot (a \times b)$ is a nonzero real number.
(C) If $a \times b$ is a nonzero vector, then $a \times (a \times b)$ is a nonzero vector.
(D) All of the above
(E) None of the above

Answer: Option (C)

Explanation: If $a \times b$ is a nonzero vector, then it is in particular orthogonal to both a and b. Further, it also means that neither a nor b is zero. Thus, a and $a \times b$ are mutually orthogonal nonzero vectors, and in particular are not scalar multiples of each other. Thus, the cross product $a \times (a \times b)$ is nonzero.

Performance review: 20 out of 24 people got this. 2 each chose (D) and (E).

Historical note (last year): 6 out of 23 people got this correct. 11 chose (E), 4 chose (D), 1 chose (B), 1 left the question blank.

(4) (*) Suppose a, b, c, and d are vectors in $\mathbb{R}^3$, with $a \times b \neq 0$ and $c \times d \neq 0$. What does $(a \times b) \times (c \times d) = 0$ mean?
(A) Both the vectors a and b are perpendicular to both the vectors c and d.
(B) a and b are perpendicular to each other and c and d are perpendicular to each other.
(C) a and c are perpendicular to each other and b and d are perpendicular to each other.
(D) The plane spanned by a and b is perpendicular to the plane spanned by c and d.
(E) a, b, c, and d are all coplanar.

Answer: Option (E)

Explanation: Since $a \times b$ and $c \times d$ are both nonzero, but their cross product is zero, we conclude that they are both scalar multiples of each other. In particular, they are in the same line. Further, we also obtain that a, b, c, and d are individually nonzero.

We know that a and b both lie in the plane orthogonal to $a \times b$. Similarly, c and d both lie in the plane orthogonal to $c \times d$. Because $a \times b$ and $c \times d$ are in the same line, we obtain that, in fact, the plane of a and b is the same as the plane of c and d.

Performance review: 21 out of 24 people got this. 2 chose (D), 1 chose (A).

Historical note (last year): 9 out of 23 people got this correct. 6 chose (D), 4 chose (C), 3 chose (B), 1 chose (A).

(5) (*) The *correlation* between two vectors in $\mathbb{R}^n$ is defined as the quotient of the dot product of the vectors by the product of their lengths. Suppose the correlation between vectors a and b is x and the correlation between b and c is y, and suppose x, y are both positive. What is the maximum possible value of the correlation between a and c given this information? *Hint: Geometrically if θ_{ab} is the angle between a and b, θ_{bc} is the angle between b and c, and θ_{ac} is the angle between a and c, then $|\theta_{ab} - \theta_{bc}| \leq \theta_{ac} \leq \theta_{ab} + \theta_{bc}$.*
(A) xy
(B) $\max\{1, xy\}$
(C) $\min\{1, xy\}$
(D) $xy + \sqrt{(1 - x^2)(1 - y^2)}$
(E) $xy - \sqrt{(1 - x^2)(1 - y^2)}$

Answer: Option (D)

Explanation: We have that $\theta_{ab} = \arccos x$ and $\theta_{bc} = \arccos y$. Thus, $x = \cos \theta_{ab}$ and $y = \cos \theta_{bc}$. Further, from the given data, both angles are acute angles.

The maximum possible correlation between a and c occurs when the angle between these vectors is minimum, which happens when all three vectors are coplanar and θ_{ab} and θ_{bc} move in opposite directions, so $\theta_{ac} = |\theta_{ab} - \theta_{bc}|$. This gives:

$$\cos \theta_{ac} = \cos |\theta_{ab} - \theta_{bc}| = \cos \theta_{ab} \cos \theta_{bc} + \sin \theta_{ab} \sin \theta_{bc}$$

Using $\sin\theta_{ab} = \sqrt{1-x^2}$ and $\sin\theta_{bc} = \sqrt{1-y^2}$, we get the result indicated.

Further note: The *expected* correlation between a and c is xy, and this occurs roughly if the correlation between a and b is not correlated to the correlation between b and c, which basically occurs when the plane of a and b is orthogonal to the plane of b and c. The maximum correlation is in the situation described above. The minimum correlation is when a, b, and c are coplanar and θ_{ab} and θ_{bc} go in the same direction. In this case, the correlation is $\cos(\theta_{ab} + \theta_{bc}) = xy - \sqrt{(1-x^2)(1-y^2)}$. Note that the minimum correlation case changes somewhat if $\theta_{ab} + \theta_{bc} > \pi$, because in that case, the minimum correlation is -1. But that case does not occur here because both angles are acute.

Performance review: 3 out of 24 people got this. 14 chose (E), 4 chose (B), 2 chose (A), 1 chose (C).

Historical note (last year): 5 out of 23 people got this correct. 7 chose (E), 7 chose (B), 2 each chose (A) and (C).

(6) If the correlation between nonzero vector v and nonzero vector w in $\mathbb{R}^n$ is c, then we say that the *proportion* of vector w *explained by* vector v is c^2. If $v_1, v_2, \dots, v_k$ are all pairwise orthogonal nonzero vectors, and c_i is the correlation between v_i and w, then $c_1^2 + c_2^2 + \dots + c_k^2 \le 1$, with equality occurring if and only if $k = n$. (This is all a result of the Pythagorean theorem). If $k < n$, then $1 - (c_1^2 + c_2^2 + \dots + c_k^2)$ is the *unexplained proportion* of w.

Suppose w is the *variation of beauty* vector, v_1 is the *variation of genes* vector, and v_2 is the *variation of make-up* vector. Assume that v_1 and v_2 are orthogonal (i.e., there is no correlation between genes and make-up choice). If the correlation between v_1 and w is 0.6 and the correlation between v_2 and w is 0.3, what proportion of the variation of beauty remains unexplained (i.e., is not explained by either genes or make-up)?

(A) 0.1
(B) 0.19
(C) 0.55
(D) 0.74
(E) 1

Answer: Option (C)

Explanation: We use the formula $1 - (0.6)^2 - (0.3)^2 = 1 - (0.36) - (0.09) = 0.55$.

In other words, genes explain 36% of the variance in beauty, make-up explains 9% of the variance, and the unexplained variance is 55%.

Note: The correlation values are bogus, this isn't a real world problem.

Performance review: All 24 people got this.

Historical note (last year): 17 out of 23 people got this correct. 4 chose (A), 1 chose (B), and 1 left the question blank.