TAKE-HOME QUIZ SOLUTIONS: FEBRUARY 14: SEQUENCES AND MISCELLANEA

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 3: 3 people.
- Score of 4: 2 people.
- Score of 5: 2 people.
- Score of 6: 1 person.
- Score of 7: 1 person.
- Score of 8: 3 people.
- Score of 9: 6 people.
- Score of 10: 2 people.
- Score of 11: 2 people.
- Score of 13: 2 people.
- Score of 14: 2 people.

The mean score was 8.23 and median score was 9. The question wise answer choices and performance were as follows:

(1) Option (E): 10 people. *Please review this solution!*
(2) Option (E): 11 people. *Please review this solution!*
(3) Option (D): 20 people. *Everybody should get this correct!*
(4) Option (D): 20 people. *Everybody should get this correct!*
(5) Option (C): 19 people. *Everybody should get this correct!*
(6) Option (C): 6 people.
(7) Option (E): 6 people.
(8) Option (B): 17 people.
(9) Option (A): 10 people.
(10) Option (B): 13 people.
(11) Option (B): 23 people. *Everybody should get this correct!*
(12) Option (B): 17 people.
(13) Option (E): 14 people.
(14) Option (D): 10 people.
(15) Option (B): 9 people. *Please review this solution!*
(16) Option (D): 9 people. *Please review this solution!*

2. Solutions

(1) Consider the sequence $a_n = 2a_{n-1} - \alpha$, with $a_1 = \beta$, for α, β real numbers. What can we say about this sequence for sure?
  (A) (a_n) is eventually increasing for all values of α, β.
  (B) (a_n) is eventually decreasing for all values of α, β.
  (C) (a_n) is eventually constant for all values of α, β.
  (D) (a_n) is either increasing or decreasing, and which case occurs depends on the values of α and β.
  (E) (a_n) is increasing, decreasing, or constant, and which case occurs depends on the values of α and β.

Answer: Option (E)

Explanation: See the answer explanation for the next question. Note that the *constant* case could arise when $\alpha = \beta = 1$.

Performance review: 10 out of 26 people got this correct. 13 people chose (D), which is pretty close to correct, and 3 people chose (A).

(2) *This is a generalization of the preceding question.* Suppose f is a continuous increasing function on $\mathbb{R}$. Define a sequence recursively by $a_n = f(a_{n-1})$, with a_1 chosen separately. What can we say about this sequence for sure?
(A) (a_n) is eventually increasing regardless of the choice of a_1.
(B) (a_n) is eventually decreasing regardless of the choice of a_1.
(C) (a_n) is eventually constant regardless of the choice of a_1.
(D) (a_n) is either increasing or decreasing, and which case occurs depends on the value of a_1 and the nature of f.
(E) (a_n) is increasing, decreasing, or constant, and which case occurs depends on the value of a_1 and the nature of f.

Answer: Option (E)

Explanation: Since f is increasing, if $f(a_1) < a_1$, then $f(f(a_1)) < f(a_1)$. we can inductively show that if $f(a_1) < a_1$, then (a_n) is decreasing. If $f(a_1) = a_1$, then (a_n) is constant. If $f(a_1) > a_1$, then (a_n) is increasing. Any of these cases may occur (as we can see using specific example from the previous problem). So, (a_n) is either increasing, decreasing or constant, but which case occurs depends on the value of a_1 and the nature of f.

For a function $f : \mathbb{R} \to \mathbb{R}$ and a particular element $a \in \mathbb{R}$, define $g : \mathbb{N} \to \mathbb{R}$ by $g(n) = f(f(\dots(f(a))\dots))$ with the f occurring $n-1$ times. Thus, $g(1) = a$, $g(2) = f(a)$, and so on. Choose the right expression for g for each of these choices of f.

Performance review: 11 ouot of 26 people got this correct. 6 people chose (D), 6 people chose (A), 2 people chose (B), and 1 person left the question blank.

(3) $f(x) := x + \pi$.
(A) $g(n) := a + n\pi$.
(B) $g(n) := a + n\pi - 1$.
(C) $g(n) := a + n(\pi - 1)$.
(D) $g(n) := a + \pi(n - 1)$.
(E) $g(n) := \pi + n(a - 1)$.

Answer: Option (D)

Explanation: Straightforward summation/induction/observation. If you aren't able to arrive at the expression, just plug in and check the values $n = 1$ and $n = 2$.

Performance review: 20 out of 26 people got this correct. 3 people chose (C), 2 people chose (B), and 1 person chose (A).

Action point: This is the kind of question that everybody should be able to get correct in the future!

(4) $f(x) := mx$, $m \neq 0$.
(a) $g(n) := mna$.
(b) $g(n) := m^n a$.
(c) $g(n) := n^m a$.
(d) $g(n) := m^{n-1} a$.
(e) $g(n) := n^{m-1} a$.

Answer: Option (D)

Explanation: Straightforward summation/induction/observation. If you aren't able to arrive at the expression, just plug in and check the values $n = 1$ and $n = 2$.

Performance review: 20 out of 26 people got this correct. 5 people chose (B) and 1 person chose (A).

Action point: This is the kind of question that everybody should be able to get correct in the future!

(5) $f(x) := x^2$.

(A) $g(n) := a^{2^n} - 1$.
(B) $g(n) := a^{2^n - 1}$.
(C) $g(n) := a^{2^{n-1}}$.
(D) $g(n) := a^{{2^n}^{-1}}$.
(E) $g(n) := (a^{2^n})^{-1}$.

Answer: Option (C)

Explanation: Each time we square, the exponent gets multiplied by 2. Thus, the exponent itself is growing like 2^{n-1} (it starts out at 1).

Performance review: 19 out of 26 people got this correct. 5 people chose (B), indicating a lack of care in keeping track of exponent towers. 1 person each chose (A) and (E).

Action point: This is the kind of question that everybody should be able to get correct in the future!

(6) One of these sequences can *not* be obtained using the procedure described in the previous questions (i.e., iterated application of a function). Identify this sequence. Only the first five terms of the sequence are presented:
(A) $1, 2, 3, 3, 3$
(B) $1, 2, 3, 2, 3$
(C) $1, 2, 3, 2, 1$
(D) $1, 2, 3, 4, 5$
(E) $1, 2, 3, 4, 3$

Answer: Option (C).

Explanation: For a sequence obtained by function iteration, it must be true that the successor of an element is uniquely determined by that element. For the sequence with first five terms $1, 2, 3, 2, 1$, we note that at one place in the sequence, 2 is followed by 3, but at another place, 2 is followed by one. This is not possible, because $f(2)$ cannot be both 3 and 1.

Performance review: 6 out of 26 people got this correct. 14 people chose (A), 3 people chose (B), 2 chose (E), and 1 chose (D).

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Identify which of these definitions is *not* correct for $\lim_{x \to c} f(x) = L$, where c and L are both finite real numbers.
(A) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta) \setminus \{c\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.
(B) For every $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c - \delta_1, c + \delta_2) \setminus \{c\}$, then $f(x) \in (L - \epsilon_1, L + \epsilon_2)$.
(C) For every $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta) \setminus \{c\}$, then $f(x) \in (L - \epsilon_1, L + \epsilon_2)$.
(D) For every $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c - \delta_1, c + \delta_2) \setminus \{c\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.
(E) None of these, i.e., all definitions are correct.

Answer: Option (E)

Explanation: Although the usual $\epsilon - \delta$ definition uses centered intervals, i.e., intervals centered at the points c and L, this is not a necessary aspect of the definition. So, instead of taking centered intervals $(c - \delta, c + \delta)$ or $(L - \epsilon, L + \epsilon)$, we could consider open intervals that have different amounts on the left and on the right. Thus, all four definitions are correct.

Performance review: 6 out of 26 people got this correct. 9 people chose (C), 4 people each chose (A) and (D), 2 chose (B), and 1 left the question blank.

Action point: Revisit this question at the end of the week, after we have covered related ideas in class.

(8) In the usual $\epsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\epsilon > 0$, then which of the following is true?
(A) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.
(B) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.

(C) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.
(D) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.
(E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ϵ also works for larger ϵs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 17 out of 26 people got this correct. 5 people chose (A), 3 chose (C), and 1 chose (D).

(9) In the usual $\epsilon - \delta$ definition of limit, we find that the value $\delta = 0.2$ for $\epsilon = 0.7$ for a function f at 0, and the value $\delta = 0.5$ works for $\epsilon = 1.6$ for a function g at 0. What value of δ *definitely* works for $\epsilon = 2.3$ for the function $f + g$ at 0?
(A) 0.2
(B) 0.3
(C) 0.5
(D) 0.7
(E) 0.9

Answer: Option (A)

Explanation: We choose the *smaller* of the δs to guarantee that *both* f and g are within their respective ϵ-distances of the targets – 0.7 in the case of f and 1.6 in the case of g. Now, the triangle inequality guarantees that $f + g$ is within 2.3 of its proposed limit.

Performance review: 10 out of 26 people got this correct. 9 people chose (D) – *naive addition* – which does not really make sense. 4 chose (B), 2 chose (E), and 1 chose (C).

(10) The sum of limits theorem states that $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ *if* the right side is defined. One of the choices below gives an example where the left side of the equality is defined and finite but the right side makes no sense. Identify the correct choice.
(A) $f(x) := 1/x$, $g(x) := -1/(x + 1)$, $c = 0$.
(B) $f(x) := 1/x$, $g(x) := (x - 1)/x$, $c = 0$.
(C) $f(x) := \arcsin x$, $g(x) := \arccos x$, $c = 1/2$.
(D) $f(x) := 1/x$, $g(x) = x$, $c = 0$.
(E) $f(x) := \tan x$, $g(x) := \cot x$, $c = 0$.

Answer: Option (B)

Explanation: $f + g$ is the constant function 1, so it has a limit. On the other hand, both f and g have one-sided limits of $\pm\infty$.

For options (A), (D), and (E), one of the function f and g has a finite limit, and the other has an infinite or undefined limit, and the sum has an infinite or undefined limit. Option (C) is a case where f, g, and $f + g$ all have finite limits.

Performance review: 13 out of 26 people got this correct. 5 people chose (C), 3 chose (E), 2 chose (A), 1 chose (D), and 2 left the question blank.

(11) If $\lim_{x\to\infty} f(x) = L$ for some finite L, this tells us that the graph of f has a:
(A) vertical asymptote
(B) horizontal asymptote
(C) vertical tangent
(D) horizontal tangent
(E) vertical cusp

Answer: Option (B)

Explanation: Just by definition.

Performance review: 23 out of 26 people got this correct. 1 person each chose (A), (C), and (D).

Action point: Everybody should get this correct!

(12) If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = M$, where both L and M are finite, then:
(A) $L = 0$ but M need not be zero
(B) $M = 0$ but L need not be zero
(C) Both L and M must be zero.
(D) Neither L nor M need be zero.
(E) At least one of L and M must be zero, but it could be either one.

Answer: Option (B)

Explanation: If M were finite and nonzero, f would have to go to $+\infty$ if M were positive and to $-\infty$ if M were negative, because f would be growing/decaying at a rate that was bounded from both above and below by linear functions.

Consider the following $\epsilon - \delta$ definition of limit at ∞: $\lim_{x\to\infty} f(x) = L$ if for all $\epsilon > 0$, there exists $a \in \mathbb{R}$ such that for all $x > a$, $|f(x) - L| < \epsilon$.

Performance review: 17 out of 26 people got this correct. 6 people chose (D), 2 people chose (E), and 1 person chose (C).

(13) What is the smallest a that can be picked for the function $f = \arctan$ with L being its limit at ∞ and $\epsilon = \pi$?
(A) $\sqrt{3}$
(B) 1
(C) 0
(D) -1
(E) There is no smallest a. Any $a \in \mathbb{R}$ will do.

Answer: Option (E)

Explanation: The function arctan has range $(-\pi/2, \pi/2)$, which is within the interval $(\pi/2 - \pi, \pi/2 + \pi)$. Thus, for all real numbers, the value of the arctan function is within the specified range.

Performance review: 14 out of 26 people got this correct. 5 people chose (B), 3 chose (C), 2 each chose (A) and (D).

(14) What is the smallest a that can be picked for the function $f = \arctan$ with L being its limit at ∞ and $\epsilon = \pi/6$?
(A) $1/2$
(B) $1/\sqrt{3}$
(C) 1
(D) $\sqrt{3}$
(E) 2

Answer: Option (D)

Explanation: We need a such that for $x > a$, $\arctan x \in (\pi/2 - \pi/6, \pi/2 + \pi/6) = (\pi/3, 2\pi/3)$. The right value of a is thus $\tan(\pi/3) = \sqrt{3}$.

Performance review: 10 out of 26 people got this correct. 11 people chose (B), indicating that either they took $\tan(\pi/3)$ wrong, or they computed $\tan(\pi/6)$ and did not perform the subtraction step $\pi/2 - \pi/6$. 4 people chose (A), 1 person chose (C), and 1 person left the question blank.

(15) Suppose $f(x) := p(x)/q(x)$ is a rational function in reduced form (i.e., the numerator and denominator are relatively prime) and $\lim_{x\to c} f(x) = \infty$. Which of the following can you conclude about f?
(A) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is even.
(B) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is even.
(C) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is odd.
(D) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is odd.
(E) $x - c$ does not divide either $p(x)$ or $q(x)$.

Answer: Option (B)

Explanation: We need $x - c$ to divide $q(x)$ for the denominator to blow up as $x \to c$. The power needs to be even to get the *same* sign of infinity for both left-sided and right-sided approach. $1/x^2$ at $c = 0$ is one example.

Performance review: 9 out of 26 people got this correct. 7 chose (A), 4 chose (C), 3 chose (D), 2 chose (E), and 1 left the question blank.

Action point: Please review this solution, make sure you understand it, and if you were convinced of another answer, debug the reasoning or examples that misled you.

(16) Suppose $f(x) := p(x)/q(x)$ is a rational function in reduced form (i.e., the numerator and denominator are relatively prime) and $\lim_{x\to c^-} f(x) = \infty$ and $\lim_{x\to c^+} f(x) = -\infty$. Which of the following can you conclude about f?

(A) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is even.
(B) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is even.
(C) $x - c$ divides $p(x)$, and the largest r such that $(x - c)^r$ divides $p(x)$ is odd.
(D) $x - c$ divides $q(x)$, and the largest r such that $(x - c)^r$ divides $q(x)$ is odd.
(E) $x - c$ does not divide either $p(x)$ or $q(x)$.

Answer: Option (D)

Explanation: We need $x - c$ to divide $q(x)$ for the denominator to blow up as $x \to c$. The power needs to be even to get *opposite* signs of infinity for left-sided and right-sided approach. $-1/x$ at $c = 0$ is one example.

Performance review: 9 out of 26 people got this correct. 8 people chose (C), 5 people chose (E), 2 people chose (A), 1 chose (B), and 1 left the question blank.

Action point: Please review this solution, make sure you understand it, and if you were convinced of another answer, debug the reasoning or examples that misled you.