CLASS QUIZ SOLUTIONS: NOVEMBER 28: LOGARITHM AND EXPONENTIAL

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 6-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 4: 6 people
- Score of 5: 4 people

The mean score was 3.83. The answers and performance review are as follows:

(1) Option (B): 3 people
(2) Option (A): 9 people
(3) Option (A): 10 people
(4) Option (C): 2 people
(5) Option (B): 11 people
(6) Option (B): 11 people

2. Solutions

(1) Consider the function $f(x) := \exp(5 \ln x)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (B)
   Explanation: $\exp(5 \ln x) = (\exp(\ln x))^5 = x^5$ is a polynomial in x.
   Performance review: 3 people got this correct. 3 chose (C), 3 chose (D), 1 chose (E).

(2) Consider the function $f(x) := \ln(5 \exp x)$ for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (A)
   Explanation: $\ln(5 \exp x) = \ln 5 + \ln(\exp x) = \ln 5 + x$, which is a linear function.
   Performance review: 9 people got this correct. 2 people chose (B).

(3) Consider the function $f(x) := \ln((\exp x)^5)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (A)
   Explanation: $\ln((\exp x)^5) = \ln(\exp(5x)) = 5x$ is a linear function of x.
   Performance review: 10 people got this correct. 1 chose (C).

(4) Consider the function $f(x) := \exp((\ln x)^5)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
(A) As a linear function
(B) As a polynomial function but faster than a linear function
(C) Faster than a polynomial function but slower than an exponential function
(D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
(E) Faster than an exponential function
Answer: Option (C)
Explanation: This is a little tricky, so we break it down into two parts.
First, note that any polynomial function (with positive leading coefficient) grows like x^n for n a positive integer. The log of that grows like $\ln(x^n) = n \ln x$, i.e., as a linear function of $\ln x$. This is slower than $(\ln x)^5$. Thus, the polynomial function grows slower than $\exp((\ln x)^5)$.
Second, note that an exponential function in x is something like $\exp x$ or $\exp(kx)$, and x or kx grows faster than any power of $\ln x$, so the function $\exp x$ grows faster than $\exp((\ln x)^5)$.
Performance review: 2 people got this correct. 6 chose (E), 2 chose (D), 1 chose (B).

(5) *Consumption smoothing*: A certain measure of happiness is found to be a logarithmic function of consumption, i.e., the happiness level H of a person is found to be of the form $H = a + b \ln C$ where C is the person's current consumption level, and a and b are positive constants independent of the consumption level.
The person has a certain total consumption C_{tot} to be split within two years, year 1 and year 2, i.e., $C_{tot} = C_1 + C_2$. Thus, the person's happiness level in year 1 is $H_1 = a + b \ln C_1$ and the person's happiness level in year 2 is $H_2 = a + b \ln C_2$. How would the person choose to split consumption between the two years to maximize average happiness across the years?
(A) All the consumption in either one year
(B) Equal amount of consumption in the two years
(C) Consume twice as much in one year as in the other year
(D) Consumption in the two years is in the ratio $a : b$
(E) It does not matter because any choice of split of consumption level between the two years produces the same average happiness
Answer: Option (B)
Explanation: Basically, happiness is logarithmic in consumption, so if consumption is unequal, then it can be distributed from the higher consumption year to the lower consumption year. The *fractional* loss in the higher consumption year is lower than the *fractional* gain in the lower consumption year. The nature of logarithmis means that the *absolute* loss in the higher consumption year is lower than the *absolute* gain in the lower consumption year. The process continues till consumption in both years is exactly equal.
We can also do this formally. We are basically using the fact that the logarithm function is concave down.
Performance review: Everybody got this correct.

(6) *Income inequality and subjective well being*: Subjective well being *across* individuals is found to be logarithmically related to income. Every doubling of income is found to increase an individuals' measured subjective well being by 0.3 points on a certain scale. *Holding total income across two individuals constant*, how should that income be divided between the two individuals to maximize their average subjective well being?
(A) All the income goes to one person
(B) Both earn the exact same income
(C) One person earns twice as much as the other
(D) One person earns 0.3 times as much as the other
(E) It does not matter because the average subjective well being is independent of the distribution of income.
Answer: Option (B)

Explanation: The logic is *exactly* the same as the preceding question, except that instead of an individual distributing consumption between years, income is being "distributed" between individuals in the same year, and instead of happiness, we are measuring subjective well being.

Performance review: Everybody got this correct.