# TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 1: MATRIX MULTIPLICATION AND INVERSION: ABSTRACT BEHAVIOR PREDICTION

MATH 196, SECTION 57 (VIPUL NAIK)

## 1. Performance review

26 people took this 6-question quiz. The score distribution was as follows:

- Score of 0: 1 person
- Score of 1: 3 people
- Score of 2: 3 people
- Score of 3: 7 people
- Score of 4: 5 people
- Score of 5: 6 people
- Score of 6: 1 person

The mean score was 3.3.

The question-wise answers and performance review are as follows:

(1) Option (B): 12 people
(2) Option (C): 12 people
(3) Option (E): 21 people
(4) Option (E): 19 people
(5) Option (A): 16 people
(6) Option (E): 6 people

*Note*: Question 6 erroneously had "5 points" printed in front of it in the print copy handed out to students. That was because the question was copy-pasted to the quiz from a previous year's test. We are not giving it additional weight.

*Note on comparison with last time*: Last time, I gave these questions (all except Question 6) *before* I gave the "linear transformations and finite state automata" questions. This might explain why the overall performance was somewhat worse last time compared to this time.

## 2. Solutions

**PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.**

This quiz tests for *abstract behavior prediction* related to the structure of matrices defined based on the operations of matrix multiplication and inversion. It is based on part of the `Matrix multiplication and inversion` notes and is related to Sections 2.3 and 2.4. It does not, however, test all aspects of that material.

To understand this abstract behavior, we will consider *nilpotent*, *invertible*, and *idempotent* matrices.

(1) Suppose $A$ and $B$ are $n \times n$ matrices such that $B$ is invertible. Suppose $r$ is a positive integer. What can we say that $(BAB^{-1})^r$ definitely equals?
(A) $A^r$
(B) $BA^rB^{-1}$
(C) $B^rA^rB^{-r}$
(D) $B^rAB^{-r}$
(E) $BAB^{-1-r}$

*Answer*: Option (B)

*Explanation*: Write:

$$BAB^{-1}BAB^{-1}\dots BAB^{-1}$$

Each $B^{-1}$ and subsequent $B$ multiply to the identity matrix, which disappears. So, we are left with:

$$BAA\dots AB^{-1}$$

where $A$ appears $r$ times. We thus get $BA^rB^{-1}$.

This is related to some deep facts about group structure. We will hint at these later in the course, but will not be able to appreciate the full depth of these.

*Performance review*: 12 out of 26 got this. 10 chose (A), 4 chose (C).

*Historical note (last time)*: 8 out of 26 got this. 13 chose (A), 5 chose (C).

(2) Suppose $A$ and $B$ are $n \times n$ matrices ($n$ not too small) such that $(AB)^2 = 0$. What is the smallest $r$ for which we can conclude that $(BA)^r$ is definitely 0?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

*Answer*: Option (C)

*Explanation*: We have:

$$(BA)^3 = BABABA = B(ABAB)A = B(AB)^2A = B(0)A = 0$$

Thus, $(BA)^3$ is definitely 0. There are examples of matrices $A$ and $B$ such that $(AB)^2 = 0$ but $(BA)^2 \neq 0$. The smallest examples are $3 \times 3$. For instance:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We obtain:

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have $(AB)^2 = 0$ but $(BA)^2 \neq 0$. However, $(BA)^3 = 0$.

*How did we construct this example?*: Building on the "linear transformations and finite state automata" framework, the idea is to choose $f, g : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$ (with $f(0) = g(0) = 0$) such that $f \circ g$ is a function whose composite with itself sends everything to zero, whereas $g \circ f$ is a function whose composite with itself does not send everything to zero. The above matrices arise from one such example:

$$f(0) = 0, f(1) = 0, f(2) = 1, f(3) = 2, \qquad g(0) = 0, g(1) = 1, g(2) = 2, g(3) = 0$$

The composites are:

$$(f \circ g)(0) = 0, (f \circ g)(1) = 0, (f \circ g)(2) = 1, (f \circ g)(3) = 0$$

and:

$$(g \circ f)(0) = 0, (g \circ f)(1) = 0, (g \circ f)(2) = 1, (g \circ f)(3) = 2$$

Notice that $f \circ g$ sends 2 to 1 and everything else to 0, hence its composite with itself sends everything to zero. On the other hand, the composite of $g \circ f$ with itself sends 3 to 1, and therefore does not send everything to zero.

In symbols, $A = M_f$, $B = M_g$, $AB = M_{f \circ g}$, and $BA = M_{g \circ f}$.

*Performance review*: 12 out of 26 got this. 12 chose (B), 1 each chose (A) and (E).

*Historical note (last time)*: 6 out of 26 got this. 16 chose (B), 3 chose (A), 1 chose (E).

(3) Suppose $n > 1$. A $n \times n$ matrix $A$ is termed *nilpotent* if there exists a positive integer $r$ such that $A^r$ is the zero matrix. It turns out that if $A$ is nilpotent, then $A^n = 0$. Which of the following describes correctly the relationship between being invertible and being nilpotent for $n \times n$ matrices?
   (A) A matrix is nilpotent if and only if it is invertible.
   (B) Every nilpotent matrix is invertible, but not every invertible matrix is nilpotent.
   (C) Every invertible matrix is nilpotent, but not every nilpotent matrix is invertible.
   (D) An invertible matrix may or may not be nilpotent, and a nilpotent matrix may or may not be invertible.
   (E) A matrix cannot be both nilpotent and invertible.

   *Answer*: Option (E)

   *Explanation*: Suppose $A^r = 0$ and $A$ has inverse $A^{-1}$. We have $(A^{-1})^r A^r = I_n$ (where $I_n$ denotes the $n \times n$ identity matrix), but it also equals $(A^{-1})^r 0 = 0$, so $I_n = 0$ is a contradiction.

   *Performance review*: 21 out of 26 got this. 3 chose (B), 1 each chose (A) and (D).

   *Historical note (last time)*: 9 out of 26 got this. 9 chose (C), 6 chose (B), 2 chose (D).

(4) Suppose $A$ and $B$ are $n \times n$ matrices. Which of the following is true? Please see Option (E) before answering.
   (A) $AB$ is nilpotent if and only if $A$ and $B$ are both nilpotent.
   (B) $AB$ is nilpotent if and only if at least one of $A$ and $B$ is nilpotent.
   (C) If both $A$ and $B$ are nilpotent, then $AB$ is nilpotent, but $AB$ being nilpotent does not imply that both $A$ and $B$ are nilpotent.
   (D) If $AB$ is nilpotent, then both $A$ and $B$ are nilpotent. However, both $A$ and $B$ being nilpotent does not imply that $AB$ is nilpotent.
   (E) None of the above.

   *Answer*: Option (E)

   *Explanation*: Here is an example where $A$ and $B$ are both nilpotent but $AB$ is not:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

   Here is an example where neither $A$ nor $B$ is nilpotent but $AB$ is the zero matrix, and therefore, is nilpotent:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   *How did we construct these examples*: These examples fall out of the "linear transformations and finite state automata" framework. In fact, there are questions on that quiz that directly correspond to the construction of examples for this situation. Can you locate them?

   *Performance review*: 19 out of 26 got this. 5 chose (C), 1 each chose (A) and (B).

   *Historical note (last time)*: 6 out of 26 got this. 9 chose (D), 5 each chose (B) and (C), 1 chose (A).

(5) Suppose $A$ and $B$ are $n \times n$ matrices. Which of the following is true? Please see Option (E) before answering.
   (A) $AB$ is invertible if and only if $A$ and $B$ are both invertible.
   (B) $AB$ is invertible if and only if at least one of $A$ and $B$ is invertible.
   (C) If both $A$ and $B$ are invertible, then $AB$ is invertible, but $AB$ being invertible does not imply that both $A$ and $B$ are invertible.
   (D) If $AB$ is invertible, then both $A$ and $B$ are invertible. However, both $A$ and $B$ being invertible does not imply that $AB$ is invertible.
   (E) None of the above.

   *Answer*: Option (A)

   *Explanation*: If $A$ and $B$ are both invertible, then $(AB)^{-1} = B^{-1}A^{-1}$. if $AB$ is invertible with inverse $C$, then $C(AB) = I_n$, so $CA$ is an inverse for $B$, and $(AB)C = I_n$, so $BC$ is an inverse for $A$. We are using the (somewhat nontrivial fact) that if a square matrix has a one-sided inverse, that inverse is actually a two-sided inverse.

*Performance review*: 16 out of 26 got this. 5 each chose (C) and (D).

*Historical note (last time)*: 11 out of 26 got this. 8 chose (D), 3 each chose (C) and (E), 1 chose (B).

(6) Suppose $A$ and $B$ are $n \times n$ matrices. Which of the following is true? We call a $n \times n$ matrix *idempotent* if it equals its own square. Please see Option (E) before answering.

(A) $AB$ is idempotent if and only if $A$ and $B$ are both idempotent.
(B) $AB$ is idempotent if and only if at least one of $A$ and $B$ is idempotent.
(C) If both $A$ and $B$ are idempotent, then $AB$ is idempotent, but $AB$ being idempotent does not imply that both $A$ and $B$ are idempotent.
(D) If $AB$ is idempotent, then both $A$ and $B$ are idempotent. However, both $A$ and $B$ being idempotent does not imply that $AB$ is idempotent.
(E) None of the above.

*Answer*: Option (E)

*Explanation*: The following is an example where both $A$ and $B$ are idempotent but $AB$ is not idempotent:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The product is:

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The following is an example where neither $A$ nor $B$ is idempotent but $AB$ is idempotent:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The product matrix is:

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

*Performance review*: 6 out of 26 got this. 8 each chose (C) and (D), 3 chose (A), and 1 chose (B).

*Historical note (last time:* **final***)*: This question appeared in last year's final. 9 out of 30 got it correct at the time (note that students had significantly more exposure to the concepts by the time of the final). 12 chose (C), 4 each chose (A) and (D), 1 chose (B).