CLASS QUIZ SOLUTIONS: WEDNESDAY FEBRUARY 6: MULTIVARIABLE LIMIT COMPUTATIONS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 0: 1 person.
- Score of 1: 2 people.
- Score of 2: 9 people.
- Score of 3: 10 people.
- Score of 4: 4 people.

The question-wise answers and performance review were as follows:

(1) Option (A): 18 people
(2) Option (D): 24 people
(3) Option (B): 9 people
(4) Option (B): 15 people

2. Solutions

(1) Consider the function $f(x,y) := x\sin(1/(x^2+y^2))$, defined on all points other than the point $(0,0)$. What is the limit of the function at $(0,0)$?
   (A) 0
   (B) $1/\sqrt{2}$
   (C) 1
   (D) The limit is undefined, because the expression becomes unbounded around 0.
   (E) The limit is undefined, because the expression is oscillatory around 0.

   Answer: Option (A)

   Explanation: We know that $\sin(1/(x^2+y^2))$ is bounded in $[-1,1]$, and $x \to 0$ as we approach the origin. We thus get the product of something approaching 0 and something bounded, which is therefore 0.

   Performance review: 18 out of 26 got this. 7 chose (E), 1 chose (D).

   Historical note (last time): 8 out of 22 people got this correct. 12 chose (E). 1 each chose (C) and (D).

   Some failed to note the x on the outside. Others thought that the behavior of $\sin(1/(x^2+y^2))$ is like the behavior of $1/(x^2+y^2)$ near the origin. This is not true. Near the origin, $1/(x^2+y^2)$ approaches ∞ but sin of the same quantity is oscillatory. Note also that sin cannot be stripped off because the input to it is not going to 0.

(2) The typical $\varepsilon-\delta$ definition of limit in two dimensions makes use of open disks centered at the points on the domain and range side, where the open disk is the interior region bounded by a circle centered at the point. Which other geometric shapes can we use instead of a circle of specified radius centered at the point?
   (A) A square of specified side length centered at the point
   (B) An equilateral triangle of specified side length centered at the point
   (C) A regular hexagon of specified side length centered at the point
   (D) Any of the above
   (E) None of the above

Answer: Option (D)

Explanation: Basically, any shape that is bounded both from inside and from outside by a circle will do.

Performance review: 24 out of 26 got this. 1 each chose (B) and (E).

Historical note (last time): 12 out of 22 people got this correct. 6 chose (A), 4 chose (E).

(3) Here's a quick recap of the limit definition for a function of a vector variable. We say that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}$ satisfying $0 < |\mathbf{x} - \mathbf{c}| < \delta$, we have $|f(\mathbf{x}) - L| < \varepsilon$. We define $|\mathbf{x} - \mathbf{c}|$ as the Euclidean norm of $\mathbf{x} - \mathbf{c}$ where the Euclidean norm of a vector is the square root of the sum of the squares of its coordinates.

We could replace the Euclidean norm by other measurements. For instance, we could use:

(i) The *sum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

(ii) The *maximum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

(iii) The *minimum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

For any of (i) - (iii), we could replace $|\mathbf{x} - \mathbf{c}|$ in our current definition of limit with that notion. The question is: for which of the replacements will our new notion of limit be the same as the old one? The deeper idea here is that limit depends upon a concept of what it means for two points to be close. So another way of phrasing the question is: which of the notions (i)-(iii) capture the same notion of closeness as the usual Euclidean distance?

(A) All of (i), (ii), and (iii).
(B) (i) and (ii) but not (iii).
(C) (i) and (iii) but not (ii).
(D) Only (i).
(E) None of (i), (ii), or (iii).

Answer: Option (B)

Explanation: The sum of the absolute values of the coordinates of a vector is small if and only if all the absolute values of the coordinates are small. Similarly, the maximum of the absolute values of the coordinates is small if and only if all the coordinates are small. On the other hand, the minimum could be small even if some of the coordinates are very large. For this reason, the minimum does not capture the correct notion of "closeness" – in the notion of closeness it gives, very far-away points can appear close merely because they are close in one coordinate.

Performance review: 9 out of 26 people got this. 16 chose (C), 1 chose (D).

(4) Suppose f is a function of two variables x, y and is defined on the whole xy-plane. Consider three conditions: (i) f is continuous on the whole xy-plane, (ii) for every fixed value $x = x_0$, the function $y \mapsto f(x_0, y)$ is continuous in y for all $y \in \mathbb{R}$, (iii) for every fixed value $y = y_0$, the function $x \mapsto f(x, y_0)$ is continuous in x for all $x \in \mathbb{R}$, (iv) the function $t \mapsto f(p(t), q(t))$ is continuous for all $t \in \mathbb{R}$ whenever p and q are both constant or linear functions (in other words, the restriction of f to any straight line in $\mathbb{R}^2$ is continuous).

Which of the following correctly describes the implications between (i), (ii), (iii), and (iv)?

(A) (i) implies both (ii) and (iii), and (ii) and (iii) together imply (iv).
(B) (i) implies (iv), and (iv) implies both (ii) and (iii).
(C) (iv) implies (ii) and (iii), and (ii) and (iii) together imply (i).
(D) (iv) implies (i), and (i) implies both (ii) and (iii).
(E) (ii) and (iii) together imply (iv), and (iv) implies (i).

Answer: Option (B)

Explanation: Continuous implies continuous in every direction, linear and curved, hence (i) implies (iv) (we can also think of this as a result of composition of continuous functions being continuous). (iv) implies (ii) and (iii) because (ii) and (iii) are continuous from specific linear directions, namely the directions parallel to the coordinate axes.

That the reverse implicationa fail is covered in the lecture notes and in videos.

Performance review: 15 out of 26 got this. 4 chose (A), 3 each chose (D) and (E). 1 left the question blank.