CLASS QUIZ SOLUTIONS: JANUARY 18: MATHEMATICAL INDUCTION (ACTUAL DATE JAN 20)

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 students took this 4-question quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 1 preson
- Score of 2: 2 people
- Score of 3: 2 people
- Score of 4: 3 people

The mean score was 2.1.

Here are the question wise solutions and performance:

(1) Option (C): 6 people
(2) Option (C): 6 people
(3) Option (D): 7 people
(4) Option (E): 4 people

2. Solutions

(1) Suppose S is a subset of the natural numbers with the property that $1 \in S$ and $k \in S \implies k+2 \in S$. What can we conclude is **definitely true** about S?
   (A) S contains all natural numbers.
   (B) S contains all natural numbers other than 2. It may or may not contain 2.
   (C) S contains all odd natural numbers.
   (D) S contains all even natural numbers.
   (E) S does not contain any natural number other than 1.

   Answer: Option (C)

   Explanation: Once we know that $1 \in S$, then we get $3 \in S$, and then $5 \in S$, and this way, we get $1, 3, 5, 7, \dots$ are all in S. However, there is no way to conclude anything about any of the even numbers.

   Performance review: 6 out of 11 got this correct. 2 each chose (A) and (B), 1 chose (D).

   Historical note (last year): 18 people got this correct. 6 people chose (B) and 5 people chose (A).

(2) Suppose S is a subset of the natural numbers with the property that whenever $k \in S$, we have $k + 5 \in S$. Which of these is the **smallest subset** T with the property that checking $T \subseteq S$ is sufficient to show that S is the set of all natural numbers?
   (A) $\{1, 2, 3\}$
   (B) $\{1, 2, 3, 4\}$
   (C) $\{1, 2, 3, 4, 5\}$
   (D) $\{1, 4\}$
   (E) $\{1, 3, 5\}$

   Answer: Option (C)

   Explanation: The fact that $k \in S \implies k + 5 \in S$ does not say anything about the numbers $1, 2, 3, 4, 5$ because subtracting 5 from any of these gives something that is not a natural number. Thus, we need to guarantee this subset to be in S. Once we have all these in S, everything else is automatically in S because it is of the form $5k + r$ where $r \in \{1, 2, 3, 4, 5\}$.

   This is like an extended/enhanced base case.

Performance review: 6 out of 11 got this correct. 3 chose (B), 1 each chose (D) and (E).

Historical note (last year): 21 people got this correct. 3 people chose (B), 3 people chose (D), and 2 people chose (A).

(3) Consider the function $f(x) := a \sin x + b \cos x$, with a, b nonzero reals. The n^{th} derivative of f is denote $f^{(n)}$. The association $n \mapsto f^{(n)}$ is periodic, i.e., there is a unique smallest positive integer h such that $f^{(n+h)} = f^{(n)}$ for all n. What is **this value** of h?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (D)

Explanation: For both sin and cos have derivative cycles of size four; hence, so does any nontrivial linear combination of these.

Performance review: 7 out of 11 got this. 2 chose (A), 1 each chose (B) and (C).

Historical note (last year): 25 people got this correct. 3 people chose (B) and 1 person chose (C).

(4) What is the **sum** $\sum_{k=2}^{n} \frac{1}{k^2-1}$ for a positive integer $n \geq 2$?

(A) $\frac{3}{2} - \frac{2n+3}{2(n+1)}$
(B) $\frac{3}{2} - \frac{2n+3}{n(n+1)}$
(C) $\frac{3}{4} - \frac{2n+1}{(n+1)(n+2)}$
(D) $\frac{3}{4} - \frac{2n-1}{2n(n-1)}$
(E) $\frac{3}{4} - \frac{2n+1}{2n(n+1)}$

Answer: Option (E)

Explanation: We give below the full proof by induction.

Base case for induction: This is the case $n = 2$. In this case, the left side is $1/(2^2 - 1) = 1/3$ and the right side is $3/4 - (2 \cdot 2 + 1)/(2 \cdot 2 \cdot (2 + 1)) = 3/4 - 5/12 = 1/3$. Thus, the two sides are equal for $n = 2$ and the base case is settled.

Inductive step: Suppose the statement is true for k. We want to show it is true for $k + 1$.

In other words, we are given that:

$$\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{k^2-1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)} \tag{*}$$

and we want to show that:

$$\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{k^2-1} + \frac{1}{(k+1)^2-1} = \frac{3}{4} - \frac{2(k+1)+1}{2(k+1)((k+1)+1)} \tag{**}$$

Let's do this. Add $1/((k+1)^2 - 1)$ to both sides of (*):

$$
\begin{aligned}
& \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{2k+1}{2k(k+1)}+\frac{1}{(k+1)^2-1} \\
\Longrightarrow\ & \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{2k+1}{2k(k+1)}+\frac{1}{k(k+2)} \\
\Longrightarrow\ & \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{(2k+1)(k+2)-2(k+1)}{2k(k+1)(k+2)} \\
\Longrightarrow\ & \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{2k^2+3k}{2k(k+1)(k+2)} \\
\Longrightarrow\ & \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{2k+3}{2(k+1)(k+2)} \\
\Longrightarrow\ & \frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{k^2-1}+\frac{1}{(k+1)^2-1} &=& \frac{3}{4}-\frac{2(k+1)+1}{2(k+1)((k+1)+1)}
\end{aligned}
$$

which is precisely what we want, namely (**). This completes the inductive step and we thus have the result by the principle of mathematical induction.

Performance review: 4 out of 11 got this. 3 chose (D), 2 each chose (A) and (B).

Historical note (last year): 18 people got this correct. 4 people chose (D), 2 people chose (A), 2 people chose (C), 2 people chose (B), and 1 left the question blank.