# CLASS QUIZ SOLUTIONS: JANUARY 7: INVERSE TRIGONOMETRIC FUNCTIONS

MATH 153, SECTION 55 (VIPUL NAIK)

## 1. Performance review

28 people took this 4-question quiz. The scores are as follows:

- Score of 0: 1 person.
- Score of 1: 2 people.
- Score of 2: 7 people.
- Score of 3: 7 people.
- Score of 4: 11 people.

The mean score was 2.89, the median score was 3, and the modal score was 4.

Here is the question-wise performance (full solutions in the next section):

(1) Option (B): 18 people got this correct.
(2) Option (C): 22 people got this correct.
(3) Option (B): 20 people got this correct.
(4) Option (A): 21 people got this correct.

## 2. Solutions

(1) What is the domain of $\arcsin \circ \arcsin$? Here, *domain* refers to the maximal possible subset of $\mathbb{R}$ on which the function is defined.
   (A) $[-1, 1]$
   (B) $[-\sin 1, \sin 1]$
   (C) $[-\arcsin 1, \arcsin 1]$
   (D) $[-\sin(2/\pi), \sin(2/\pi)]$
   (E) $[-\arcsin(2/\pi), \arcsin(2/\pi)]$

   *Answer*: Option (B)

   *Explanation*: For $x$ to be in the domain of this function, we need to be able to apply arcsin twice to $x$. Thus, the result obtained by applying arcsin once to $x$ must be in the domain of arcsin, which is $[-1, 1]$. Thus, $x$ must be in the inverse image of $[-1, 1]$ under arcsin, which is $[\sin(-1), \sin 1]$. Since sin is odd, this can be rewritten as $[-\sin 1, \sin 1]$.

   *Post-performance review*: 18 people got this correct. 5 people chose (A), 3 people chose (C), 1 person chose (D), and 1 person chose (E).

(2) Suppose $f$ is a polynomial with degree at least one and positive leading coefficient. Consider the function $g(x) := \arctan(f(x))$. What can we say about the horizontal asymptotes of the graph $y = g(x)$?
   (A) The horizontal asymptote is $y = \pi/2$ both for $x \to +\infty$ and for $x \to -\infty$, regardless of $f$.
   (B) The horizontal asymptote is $y = \pi/2$ for $x \to +\infty$ and $-\pi/2$ for $x \to -\infty$, regardless of $f$.
   (C) The horizontal asymptote is $y = \pi/2$ for $x \to +\infty$, and as $x \to -\infty$, it is $y = \pi/2$ if $f$ has even degree and $y = -\pi/2$ if $f$ has odd degree.
   (D) The horizontal asymptote is $y = f(\pi/2)$ both for $x \to +\infty$ and for $x \to -\infty$.
   (E) The horizontal asymptote is $y = f(\pi/2)$ for $x \to +\infty$ and as $x \to -\infty$, it is $y = f(\pi/2)$ if $f$ has even degree and $y = f(-\pi/2)$ if $f$ has odd degree.

   *Answer*: Option (C)

   *Explanation*: The key observation is that as $u \to +\infty$, $\arctan u \to \pi/2$, and as $u \to -\infty$, $\arctan u \to -\pi/2$. The question now is what happens to $f(x)$ as $x$ approaches $\pm\infty$.

Since $f$ has positive degree and positive leading coefficient, $f(x) \to +\infty$ as $x \to +\infty$. The behavior as $x \to -\infty$ depends on whether the degree of $f$ is even or odd. If the even, then it goes to $+\infty$, and if the latter, then it goes to $-\infty$.

Combining these observations gives us (C).

*Post-performance review*: 22 people got this correct. 2 people chose (A), 2 people chose (B), 2 people chose (E).

(3) Consider the function $f(x) := \arcsin(\sin x)$ on the domain $[\pi/2, 3\pi/2]$. Which of the following is $f(x)$ equal to on that domain?
(A) $\pi + x$
(B) $\pi - x$
(C) $x - \pi$
(D) $(3\pi/2) - x$
(E) $x - (\pi/2)$

*Answer*: Option (B)

*Explanation*: Note that $\sin(f(x)) = \sin(\arcsin(\sin x)) = \sin x$. We note that of the options given here, $\pi - x$ is the only option satisfying this constraint. Additionally, we need to check that $x \mapsto \pi - x$, sends the interval $[\pi/2, 3\pi/2]$ to the interval $[-\pi/2, \pi/2]$, which it does.

*Post-performance review*: 20 people got this correct. 5 people chose (C), 1 person chose (A), 1 person chose (D), 1 person chose (E).

(4) Consider the function $f(x) := \arccos(\sin x)$ on all of $\mathbb{R}$. What can we say about the function $f$?
(A) $f$ is periodic, continuous, and piecewise linear.
(B) $f$ is periodic and continuous but is not piecewise linear.
(C) $f$ is continuous and piecewise linear but not periodic.
(D) $f$ is periodic but not continuous.
(E) $f$ is continuous but not periodic or piecewise linear.

*Answer*: Option (A)

*Explanation*: First, note that the function *does* make sense on all of $\mathbb{R}$: sin is defined everywhere and has range $[-1, 1]$, and this is precisely the domain of arccos. Since both sin and arccos are continuous on their respective domains, the composite function is also continuous.

Since sin has a period of $2\pi$, the composite is also periodic. (In this case, the period turns out to be exactly $2\pi$, though in general the period of a composite function could be smaller than that of the inner function).

It remains to justify piecewise linearity. Working out the definition reveals this. On the interval $[2n\pi - \pi/2, 2n\pi + \pi/2]$, the correct definition is $2n\pi + \pi/2 - x$. On an interval $[2n\pi + \pi/2, 2n\pi + 3\pi/2]$, the correct definition is $x - 2n\pi - \pi/2$. Thus, the definition is piecewise linear on intervals of length $\pi$, and each of the pieces has slope 1. In fact, the graph of the function has a sawtooth-like shape.

*Post-performance review*: 21 people got this correct. 5 people chose (B), 1 person chose (C), and 1 person chose (D).