CLASS QUIZ SOLUTIONS: OCTOBER 7: LIMIT THEOREMS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 students took this quiz. The score distribution was as follows:

- Score of 2: 3 people
- Score of 3: 5 people
- Score of 4: 4 people

The mean score was 3.08. Here are the problem-wise answers and scores:

1. Option (A): 6 people
2. Option (C): 4 people
3. Option (D): 10 people
4. Option (D): 9 people
5. Option (B): 8 people

More details below.

2. Solutions

(1) (**) Which of the following statements is **always true**?

(A) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).

(B) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).

(C) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.

(D) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.

(E) None of the above.

Answer: Option (A)

Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

The other choices:

Option (B): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1 - x^2}$ on $(-1, 1)$ has range $(0, 1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1, 1]$.

Option (C): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0, 1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

Option (D): The same counterexample as for option (B) works.

Performance review: 6 out of 12 got this correct. 3 chose (C), 2 chose (D), 1 chose (E).

Historical note (last year): 2 out of 11 people got this correct. (C) was the most frequently chosen incorrect answer.

Action point: Please review the statement of the extreme value theorem, as well as understand why all the other examples are incorrect.

(2) (**) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x$?
(A) 0
(B) A
(C) A^2
(D) $g(A)$
(E) $g(A)/A$
Answer: Option (C)
Explanation: We have $\lim_{x \to 0} g(x) = \lim_{x \to 0}(g(x)/x) \lim_{x \to 0} x = A \cdot 0 = 0$.
Also, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x} = \lim_{x \to 0} \frac{g(g(x))}{g(x)} \lim_{x \to 0} \frac{g(x)}{x}$$

The second limit is A. For the first limit, note that as $x \to 0$, we also have $g(x) \to 0$, so the first limit can be rewritten as $\lim_{y \to 0} g(y)/y$, which is also equal to A. Hence, the overall limit is the product A^2.

Performance review: 4 out of 12 go this correct. 3 each chose (A) and (E), 2 chose (D).

Historical note (last year): 1 out of 12 people got this correct. 5 people chose (D), 2 people each chose (B) and (E), 1 person chose (A), and 1 person left the question blank.

(3) Suppose $I = (a, b)$ is an open interval. A function $f : I \to \mathbb{R}$ is termed *piecewise continuous* if there eixst points $a_0 < a_1 < a_2 < \cdots < a_n$ (dependent on f) with $a = a_0$ and $a_n = b$, such that f is continuous on each interval (a_i, a_{i+1}). In other words, f is continuous at every point in (a, b) except possibly the a_is.

Suppose f and g are piecewise continuous functions on the same interval I (with possibly different sets of a_is). Which of the following is/are guaranteed to be piecewise continuous on I?
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) All of the above
(E) None of the above
Answer: Option (D)
Explanation: We take the points where f is possibly discontinuous and the points where g is possible discontinuous, and we take the union of these sets of points. We get a new finite set of points. Note that everywhere except these points, both f and g are continuous, hence $f + g$, $f - g$, and $f \cdot g$ are all continuous.

A numerical illustration might help here. (Note, however, that there is nothing special about the numbers). Suppose $a = 1$ and $b = 2$. Let's say that f is continuous on $(1, 1.5)$ and $(1.5, 2)$, so it is possibly discontinuous at 1.5. Suppose g is continuous on $(1, \sqrt{2})$, $(\sqrt{2}, \sqrt{3})$ and $(\sqrt{3}, 2)$, so the points where it may be discontinuous are $\sqrt{2}$ and $\sqrt{3}$.

We now take the union of the points of discontinuity of f and g. We get the points 1.5, $\sqrt{2}$, and $\sqrt{3}$. Recall that $\sqrt{2} \approx 1.414 < 1.5$ while $\sqrt{3} \approx 1.732 > 1.5$, so rearranging in increasing order, we get $1 < \sqrt{2} < 1.5 < \sqrt{3} < 2$. We can now see that $f + g$, $f - g$ and $f \cdot g$ are all continuous on the intervals $(1, \sqrt{2})$, $(\sqrt{2}, 1.5)$, $(1.5, \sqrt{3})$ and $(\sqrt{3}, 2)$.

Memory lane: This is the idea of *breaking up the domains for two piecewise definned functions in the same manner* so as to be able to add, subtract, and multiply them. You have seen a problem with this theme in Homework 1, Problem 8 (Exercise 1.7.14 of the book, Page 46). There, goal was to compute $f + g$, $f - g$, and $f \cdot g$ with f and g given piecewise with different domain breakdowns. Here, our goal is to pontificate about continuity, but the idea is the same.

Future teaser: This idea of partitioning an interval into sub-intervals by choosing some points keeps coming up. Further, the idea of combining two partitions of the same interval into a finer partition that refines both of them will also come up. Specifically, both these ideas turn up when we try to define the integral of a continuous (or piecewise continuous) function on an interval.

Performance review: 10 out of 12 got this correct. 2 chose (E).

Historical note (last year): 9 out of 11 people got this correct. 1 person chose (C) and 1 person chose (E).

Action point: Whether or not you got this correct, make sure that you *now* understand the logic behind it. This idea is extremely important in the future.

(4) Suppose f and g are everywhere defined and $\lim_{x \to 0} f(x) = 0$. Which of these pieces of information is **not sufficient** to conclude that $\lim_{x \to 0} f(x)g(x) = 0$?
  (A) $\lim_{x \to 0} g(x) = 0$.
  (B) $\lim_{x \to 0} g(x)$ is a constant not equal to zero.
  (C) There exists $\delta > 0$ and $B > 0$ such that for $0 < |x| < \delta$, $|g(x)| < B$.
  (D) $\lim_{x \to 0} g(x) = \infty$, i.e., for every $N > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $g(x) > N$.
  (E) None of the above, i.e., they are all sufficient to conclude that $\lim_{x \to 0} f(x)g(x) = 0$.

Answer: Option (D)

Explanation: If $f(x) \to 0$ and $g(x) \to \infty$, then the limit of $f(x)g(x)$ is indeterminate. It may be 0, finite, infinite, or oscillatory. For instance, if $f(x) = x^2$ and $g(x) = 1/x^2$, then the limit of $f(x)g(x)$ is 1. Thus, we cannot conclude that the limit of the product is 0.

Memory lane: Routine Problem 5 on Homework 2 (Exercise 2.3.3 of the book, Page 79) explores a similar theme. The new ingredient here is that, in cases where $f(x)$ goes to zero and $g(x)$ does not have a limit but is still bounded, we *can* say that the product goes to zero.

The other choices:

Option (A) is sufficient because the limit of the sums is the sum of the limits.

Option (B) is sufficient for the same reason.

Option (C) is a little trickier to justify. Here, what we are saying is that $\lim_{x \to 0} f(x) = 0$ and, for x close enough to 0, g is bounded, though it need not have a limit. The bound here is B. In particular, what this is saying is that if $0 < |x| < \delta$, then $g(x)$ is between $-B$ and B.

Thus, we can see that:

$$-Bf(x) \leq f(x)g(x) \leq Bf(x) \; \forall \; 0 < |x| < \delta$$

We now note that both $-Bf(x)$ and $Bf(x)$ tend to 0 as $x \to 0$. Hence, by the pinching theorem, $f(x)g(x) \to 0$.

Examples for g: One example of such a function g is the Dirichlet function, i.e., $g(x)$ is 1 if x is rational and 0 if x is irrational. Clearly, the Dirichlet function is bounded near 0 (in fact, it is universally bounded). If we set $f(x) := x$, then $f(x)g(x) = \left\{ \begin{array}{ll} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{array} \right.$, and the limit of this function at 0 is 0. Incidentally, Advanced Problem 3 of Homework 2 (Exercise 2.2.54 of the book, Page 72) asked you to give an explicit $\epsilon - \delta$ proof of this fact.

Another example for g is the $\sin(1/x)$ function. This function oscillates between -1 and 1, hence does not converge to a limit as $x \to 0$. However, it is bounded. Thus, if we have $f(x) := x$, the function $x \sin(1/x)$ must converge to 0 as x goes to 0. This function appears in Advanced Problem 4 of Homework 3 (Exercise 3.6.67 of the book, Page 146).

Performance review: 10 out of 12 got this correct. 1 each chose (A), (C), and (E).

Historical note (last year): 8 out of 11 people got this correct. 1 person each chose (A), (B), and (C).

Action point: You should understand this, but don't have to worry too much about it for now. We will cover these issues in more detail later.

(5) f and g are functions defined for all real values. c is a real number. Which of these statements is ***not* necessarily true**?
  (A) If $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^-} g(x) = M$, then $\lim_{x \to c^-}(f(x) + g(x))$ exists and is equal to $L + M$.
  (B) If $\lim_{x \to c^-} g(x) = L$ and $\lim_{x \to L^-} f(x) = M$, then $\lim_{x \to c^-} f(g(x)) = M$.

(C) If there exists an open interval containing c on which f is continuous and there exists an open interval containing c on which g is continuous, then there exists an open interval containing c on which $f + g$ is continuous.
(D) If there exists an open interval containing c on which f is continuous and there exists an open interval containing c on which g is continuous, then there exists an open interval containing c on which the product $f \cdot g$ (i.e., the function $x \mapsto f(x)g(x)$) is continuous.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (B)

Explanation: This is the cliched fact that composition results do not hold for one-sided limits. The main reason is that when we compose, we need the inner function of the composition to approach the limit *from the correct side* in order for the result to go through. Thus, in this case, for instance, the result would be true if the function g were strictly increasing on the immediate left of c.

Memory lane: We already saw this fact in the October 1 quiz on limits, Problem 2. Please review the solution to that (where we've also given an explicit example).

The other choices:

Option (A) is the sum theorem for one-sided limits: the limit of the sum is the sum of the limits.

For options (C) and (D), note that if f is continuous on one open interval containing c and g is continuous on another open interval containing c, then *both* f and g are continuous on the *intersection* of the two open intervals containing c (which is also an open interval containing c). Thus, $f+g$ is also continuous on this intersection. This is analogous to the trick we often use of picking $\delta = \min\{\delta_1, \delta_2\}$ in $\epsilon - \delta$ proofs for piecewise functions.

Performance review: 8 out of 12 got this correct. 2 chose (C) and 2 chose (E).

Historical note (last year): 9 out of 11 people got this correct. 1 person each chose (A) and (E).