CLASS QUIZ SOLUTIONS: JANUARY 9: INVERSE TRIGONOMETRIC FUNCTIONS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this 5-question quiz. The scores are as follows:

- Score of 2: 3 people
- Score of 3: 6 people
- Score of 4: 1 person
- Score of 5: 2 people

The mean score was 3.17.

Here is the question-wise performance (full solutions in the next section):

(1) Option (B): 5 people
(2) Option (B): 12 people (everybody!)
(3) Option (C): 6 people
(4) Option (B): 9 people
(5) Option (A): 6 people

2. Solutions

(1) What is the domain of $\arcsin \circ \arcsin$? Here, *domain* refers to the maximal possible subset of $\mathbb{R}$ on which the function is defined.
(A) $[-1, 1]$
(B) $[-\sin 1, \sin 1]$
(C) $[-\arcsin 1, \arcsin 1]$
(D) $[-\sin(2/\pi), \sin(2/\pi)]$
(E) $[-\arcsin(2/\pi), \arcsin(2/\pi)]$

Answer: Option (B)

Explanation: For x to be in the domain of this function, we need to be able to apply arcsin twice to x. Thus, the result obtained by applying arcsin once to x must be in the domain of arcsin, which is $[-1, 1]$. Thus, x must be in the inverse image of $[-1, 1]$ under arcsin, which is $[\sin(-1), \sin 1]$. Since sin is odd, this can be rewritten as $[-\sin 1, \sin 1]$.

Performance review: 5 out of 12 got this. 5 chose (A), 1 each chose (D) and (E).

Historical note (last year): 18 out of 28 people got this correct. 5 people chose (A), 3 people chose (C), 1 person chose (D), and 1 person chose (E).

(2) One of these five functions has a horizontal asymptote as $x \to +\infty$ and a horizontal asymptote as $x \to -\infty$, with the limiting values for $+\infty$ and $-\infty$ being *different*. Identify the function.
(A) $f(x) := \ln |x|$.
(B) $f(x) := \arctan x$.
(C) $f(x) := e^{-x}$.
(D) $f(x) := e^{-x^2}$.
(E) $f(x) := (\sin x)/(x^2 + 1)$.

Answer: Option (B)

Explanation: For arctan, the limit as $x \to \infty$ is $\pi/2$ and the limit as $x \to -\infty$ is $-\pi/2$. These are finite and unequal.

$\ln |x|$ has no horizontal asymptotes, since it goes to ∞ as $x \to \pm\infty$. e^{-x} goes to 0 as $x \to \infty$ but goes to $+\infty$ as $x \to -\infty$. e^{-x^2} goes to 0 as $x \to \pm\infty$. $(\sin x)/(x^2 + 1)$ also goes to 0 as $x \to \pm\infty$.

Performance review: Everybody got this correct!

Historical note: In a 153 midterm two years ago, 11 out of 15 students got it correct. There were only four options presented in that variant.

(3) Suppose f is a polynomial with degree at least one and positive leading coefficient. Consider the function $g(x) := \arctan(f(x))$. What can we say about the horizontal asymptotes of the graph $y = g(x)$?

(A) The horizontal asymptote is $y = \pi/2$ both for $x \to +\infty$ and for $x \to -\infty$, regardless of f.

(B) The horizontal asymptote is $y = \pi/2$ for $x \to +\infty$ and $-\pi/2$ for $x \to -\infty$, regardless of f.

(C) The horizontal asymptote is $y = \pi/2$ for $x \to +\infty$, and as $x \to -\infty$, it is $y = \pi/2$ if f has even degree and $y = -\pi/2$ if f has odd degree.

(D) The horizontal asymptote is $y = f(\pi/2)$ both for $x \to +\infty$ and for $x \to -\infty$.

(E) The horizontal asymptote is $y = f(\pi/2)$ for $x \to +\infty$ and as $x \to -\infty$, it is $y = f(\pi/2)$ if f has even degree and $y = f(-\pi/2)$ if f has odd degree.

Answer: Option (C)

Explanation: The key observation is that as $u \to +\infty$, $\arctan u \to \pi/2$, and as $u \to -\infty$, $\arctan u \to -\pi/2$. The question now is what happens to $f(x)$ as x approaches $\pm\infty$.

Since f has positive degree and positive leading coefficient, $f(x) \to +\infty$ as $x \to +\infty$. The behavior as $x \to -\infty$ depends on whether the degree of f is even or odd. If the even, then it goes to $+\infty$, and if the latter, then it goes to $-\infty$.

Combining these observations gives us (C).

Performance review: 6 out of 12 got this. 4 chose (E), 2 chose (B).

Historical note (last year): 22 out of 28 people got this correct. 2 people chose (A), 2 people chose (B), 2 people chose (E).

(4) Consider the function $f(x) := \arcsin(\sin x)$ on the domain $[\pi/2, 3\pi/2]$. Which of the following is $f(x)$ equal to on that domain?

(A) $\pi + x$

(B) $\pi - x$

(C) $x - \pi$

(D) $(3\pi/2) - x$

(E) $x - (\pi/2)$

Answer: Option (B)

Explanation: Note that $\sin(f(x)) = \sin(\arcsin(\sin x)) = \sin x$. We note that of the options given here, $\pi - x$ is the only option satisfying this constraint. Additionally, we need to check that $x \mapsto \pi - x$, sends the interval $[\pi/2, 3\pi/2]$ to the interval $[-\pi/2, \pi/2]$, which it does.

Performance review: 9 out of 12 got this. 1 each chose (A), (D), and (E).

Historical note (last year): 20 out of 28 got this correct. 5 people chose (C), 1 person chose (A), 1 person chose (D), 1 person chose (E).

(5) Consider the function $f(x) := \arccos(\sin x)$ on all of $\mathbb{R}$. What can we say about the function f?

(A) f is periodic, continuous, and piecewise linear.

(B) f is periodic and continuous but is not piecewise linear.

(C) f is continuous and piecewise linear but not periodic.

(D) f is periodic but not continuous.

(E) f is continuous but not periodic or piecewise linear.

Answer: Option (A)

Explanation: First, note that the function *does* make sense on all of $\mathbb{R}$: sin is defined everywhere and has range $[-1, 1]$, and this is precisely the domain of arccos. Since both sin and arccos are continuous on their respective domains, the composite function is also continuous.

Since sin has a period of 2π, the composite is also periodic. (In this case, the period turns out to be exactly 2π, though in general the period of a composite function could be smaller than that of the inner function).

It remains to justify piecewise linearity. Working out the definition reveals this. On the interval $[2n\pi - \pi/2, 2n\pi + \pi/2]$, the correct definition is $2n\pi + \pi/2 - x$. On an interval $[2n\pi + \pi/2, 2n\pi + 3\pi/2]$, the correct definition is $x - 2n\pi - \pi/2$. Thus, the definition is piecewise linear on intervals of length π, and each of the pieces has slope 1. In fact, the graph of the function has a sawtooth-like shape.

Performance review: 6 out of 12 got this. 3 chose (B), 1 each chose (C), (D), and (E).

Historical note (last year): 21 out of 28 got this correct. 5 people chose (B), 1 person chose (C), and 1 person chose (D).