CLASS QUIZ SOLUTIONS: JANUARY 20: MIXED BOWL OF HARD NUTS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 14-question quiz. The score distribution was as follows:

- Score of 4: 1 person
- Score of 7: 2 people
- Score of 10: 1 person
- Score of 11: 1 person
- Score of 12: 1 person
- Score of 13: 1 person
- Score of 14: 4 people

The mean score was 10.9. The problem wise answers as as follows:

(1) Option (B): 9 people
(2) Option (B): 9 people
(3) Option (B): 9 people
(4) Option (D): 7 people
(5) Option (D): 10 people
(6) Option (B): 9 people
(7) Option (B): 8 people
(8) Option (A): 7 people
(9) Option (C): 8 people
(10) Option (A): 8 people
(11) Option (C): 9 people
(12) Option (E): 10 people
(13) Option (C): 10 people
(14) Option (E): 7 people

2. Solutions

(1) Which of the following statements is **always true**?
  (A) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).
  (B) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).
  (C) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.
  (D) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.
  (E) None of the above.

  Answer: Option (B)

  Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

The other choices:

Option (A): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1-x^2}$ on $(-1,1)$ has range $(0,1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1,1]$.

Option (C): The same counterexample as for option (A) works.

Option (D): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0,1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

Performance review: 9 out of 11 got this. 1 chose (A) and 1 chose (C).

Historical note (last year): 16 out of 28 people got this correct. 7 people chose (A), 2 people each chose (C) and (E), and 1 person chose (D).

(2) For which of the following specifications is there **no continuous function** satisfying the specifications?

(A) Domain $(0,1)$ and range $(0,1)$
(B) Domain $[0,1]$ and range $(0,1)$
(C) Domain $(0,1)$ and range $[0,1]$
(D) Domain $[0,1]$ and range $[0,1]$
(E) None of the above, i.e., we can get a continuous function for each of the specifications.

Answer: Option (B)

Explanation: By the extreme value theorem, any continuous function on a closed bounded interval must attain its maximum and minimum, and hence its image cannot be an open interval.

The other choices:

For options (A) and (D), we can pick the identity functions $f(x) := x$ on the respective domains.

For option (C), we can pick the function $f(x) := \sin^2(2\pi x)$ on the domain $(0,1)$.

Performance review: 9 out of 11 got this. 2 chose (C).

Historical note (last year): 21 out of 28 people got this correct. 5 people chose (C) and 2 people chose (E).

(3) Suppose f is a continuously differentiable function from the open interval $(0,1)$ to $\mathbb{R}$. Suppose, further, that there are exactly 14 values of c in $(0,1)$ for which $f(c) = 0$. What can we say is **definitely true** about the number of values of c in the open interval $(0,1)$ for which $f'(c) = 0$?

(A) It is at least 13 and at most 15.
(B) It is at least 13, but we cannot put any upper bound on it based on the given information.
(C) It is at most 15, but we cannot put any lower bound (other than the meaningless bound of 0) based on the given information.
(D) It is at most 13.
(E) It is at least 15.

Answer: Option (B)

Explanation: Suppose the zeros of f are $a_1 < a_2 < \cdots < a_{14}$. By Rolle's theorem, there is *at least one* zero of f' between each a_i and a_{i+1}. There may be more, since Rolle's theorem gives only a *lower* bound. This gives thirteen solutions c to $f'(c) = 0$.

Note that in order to apply Rolle's theorem, it is enough to be given that f is differentiable, so the additional hypothesis that f' is continuous is not necessary.

Performance review: 9 out of 11 got this. 2 chose (A).

Historical note (last year): 19 out of 28 people got this correct. 5 people chose (A), 2 people chose (C), and 1 person each chose (D) and (E).

(4) Consider the function $f(x) := \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x - (1/7), & 1/2 < x \le 1 \end{array}$. Define by $f^{[n]}$ the function obtained by iterating f n times, i.e., the function $f \circ f \circ f \circ \cdots \circ f$ where f occurs n times. What is the smallest n for which $f^{[n]} = f^{[n+1]}$? *Earlier score:* 3/16

(A) 1
(B) 2
(C) 3

(D) 4
(E) 5
Answer: Option (D)
Explanation: We need to iterate f enough times that everything gets inside $[0, 1/2]$, after which it becomes stable. Note that each time, the value goes down by $1/7$. Thus, for any $x \leq 1$, we need at most four steps to bring it in $[0, 1/2]$, with the upper bound of 4 being attained for 1.
Performance review: 7 out of 11 got this. 3 chose (C), 1 chose (A).
Historical note (last year): 10 out of 28 people got this correct. 9 people chose (C), 5 people chose (B), 2 people chose (A), and 2 people left the question blank.

(5) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both right continuous on $(0,1)$. Which of the following is *not* guaranteed to be right continuous on $(0,1)$? *Earlier scores:* 3/11*,* 9/14
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be right continuous functions
Answer: Option (D)
Explanation: See the explanation for Question 2 on the October 1 quiz. Note that that quiz uses left continuity, but the example can be adapted to right continuity.
Performance review: 10 out of 11 got this. 1 chose (B).
Historical note (last year): 20 out of 28 people got this correct. 4 people chose (C), 2 people chose (B), 1 person chose (E), and 1 person left the question blank.

(6) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing functions from $\mathbb{R}$ to $\mathbb{R}$? *Earlier scores:* 1/15*,* 9/16
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.
Answer: Option (B)
Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.
The other choices:
Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.
Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.
Performance review: 9 out of 11 got this. 2 chose (C).
Historical note (last year): 18 out of 28 people got this correct. 6 people chose (E) and 4 people chose (C).

(7) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero) *Earlier score:* 6/16
(A) $k - 2$
(B) $k - 1$
(C) k
(D) $k + 1$

(E) There is no bound in terms of k.

Answer: Option (B)

Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F-G$ have k^{th} derivative equal to zero. For $k=1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k-1$.

Performance review: 8 out of 11 got this. 2 chose (E), 1 chose (D).

Historical note (last year): 10 out of 28 people got this correct. 5 people each chose (D) and (E) and 4 people each chose (A) and (C).

Action point: This is a question you really *should* get correct!

(8) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy). *Earlier score:* 3/16

(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 7 out of 11 got this. 3 chose (D), 1 chose (E).

Historical note (last year): 10 out of 28 people got this correct. 6 people chose (B), 5 people chose (E), 4 people chose (D), 2 people chose (C), and 1 person left the question blank.

(9) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? *Earlier score:* 4/16

(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 8 out of 11 got this. 3 chose (D).

Historical note (last year): 15 out of 28 people got this correct. 9 people chose (D), 2 people chose (E), and 1 person each chose (A) and (B).

(10) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? *Earlier scores:* 0, 10/16

(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.

(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 8 out of 11 got this. 2 chose (E), 1 chose (D).

Historical note (last year): 11 out of 28 people got this correct. 8 people chose (E), 5 people chose (C), and 2 people each chose (B) and (D).

(11) Consider the four functions $\sin(\sin x)$, $\sin(\cos x)$, $\cos(\sin x)$, and $\cos(\cos x)$. Which of the following statements are true about their periodicity? *Earlier score:* 5/16
(A) All four functions are periodic with a period of π.
(B) All four functions are periodic with a period of 2π.
(C) $\cos(\sin x)$ and $\cos(\cos x)$ have a period of π, whereas $\sin(\sin x)$ and $\sin(\cos x)$ have a period of 2π.
(D) $\sin(\sin x)$ and $\sin(\cos x)$ have a period of π, whereas $\cos(\sin x)$ and $\cos(\cos x)$ have a period of 2π.
(E) $\sin(\sin x)$ has a period of 2π, the other three functions have a period of π.

Answer: Option (C)

Explanation: Since the inner functions in all cases have a period of 2π, it is clear that all the four functions have a period of at most 2π, in fact, the period of each divides 2π. The crucial question is which of them have the smaller period π.

Let's look at $\sin \circ \sin$ first. We have:

$$\sin(\sin(x + \pi)) = \sin(-\sin x) = -\sin(\sin x)$$

So, we see that that value at $x + \pi$ is the negative, and hence usually not the equal, of the value at x. Similarly:

$$\sin(\cos(x + \pi)) = \sin(-\cos x) = -\sin(\cos x)$$

On the other hand, for the functions that have a cos on the outside, the negative sign on the inside gets eaten up by the even nature of the outer function. For instance:

$$\cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x)$$

and:

$$\cos(\cos(x + \pi)) = \cos(-\cos x) = \cos(\cos x)$$

Now, this is not a proof that π is strictly the smallest period for these functions, but that can be proved using other methods. In any case, given the choices presented, it is now easy to single out (D) as the only correct answer.

The key feature here is that both sin and cos (viewed as the inner functions of the composition) have *anti-period* π: their value gets negated after an interval of π.

The outer function cos is even, hence it converts an anti-period for the inner function into a period for the overall function. The outer function sin is odd, so it keeps anti-periods anti-periods.

Performance review: 9 out of 11 got this. 1 chose (B), 1 chose (D).

Historical note (last year): 14 out of 28 people got this correct. 6 people chose (D), 3 people chose (B), 2 people each chose (A) and (E), and 1 person left the question blank.

(12) Suppose f is a one-to-one function with domain a closed interval $[a, b]$ and range a closed interval $[c, d]$. Suppose t is a point in (a, b) such that f has left hand derivative l and right-hand derivative r at t, with both l and r nonzero. What is the left hand derivative and right hand derivative to f^{-1} at $f(t)$? *Earlier score:* 6/15
(A) The left hand derivative is $1/l$ and the right hand derivative is $1/r$.
(B) The left hand derivative is $-1/l$ and the right hand derivative is $-1/r$.
(C) The left hand derivative is $1/r$ and the right hand derivative is $1/l$.
(D) The left hand derivative is $-1/r$ and the right hand derivative is $-1/l$.
(E) The left hand derivative is $1/l$ and the right hand derivative is $1/r$ if $l > 0$, otherwise the left hand derivative is $1/r$ and the right hand derivative is $1/l$.

Answer: Option (E)

Explanation: Although it isn't necessary to note this, a one-to-one function that satisfies the intermediate value property is continuous, so even though f is not explicitly given to be continuous, it is in fact continuous on its domain.

If $l > 0$, then, since we are dealing with a one-to-one function, the function is increasing throughout, and so $r \geq 0$ as well. Since we know $r \neq 0$, we conclude that $r > 0$ strictly. The upshot is that as $x \to t^-$, $f(x) \to f(t)^-$ and as $x \to t^+$, $f(x) \to f(t)^+$. Thus, when we pass to the inverse function, the roles of left and right remain the same.

On the other hand, if $l < 0$, then as $x \to t^-$, $f(x) \to f(t)^+$, and hence the roles of left and right get interchanged.

Performance review: 10 out of 11 got this. 1 chose (C).

Historical note (last year): 16 out of 28 people got this correct. 5 people chose (A), 3 people chose (B), and 2 people each chose (C) and (D).

(13) Which of these functions is one-to-one? *Earlier score:* 2/15
(A) $f_1(x) := \{ \begin{array}{rl} x, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{array}$
(B) $f_2(x) := \{ \begin{array}{rl} x, & x \text{ rational} \\ x^3, & x \text{ irrational} \end{array}$
(C) $f_3(x) := \{ \begin{array}{rl} x, & x \text{ rational} \\ 1/(x-1), & x \text{ irrational} \end{array}$
(D) All of the above
(E) None of the above

Answer: Option (C)

Explanation: Option (A) is easy to rule out: $\sqrt{2}$ and $-\sqrt{2}$ map to the same thing. Option (B) is a little harder to rule out, because the function is one-to-one within each piece, i.e., no two rationals map to the same thing and no two irrationals map to the same thing. However, a rational and an irrational can map to the same thing. For instance, 2 and $2^{1/3}$ both map to 2.

For option (C), note that not only is the map one-to-one in each piece, but also, the image of the rationals stays inside the rationals and the image of the irrationals stays inside the irrationals. In particular, this means that a rational number and an irrational number cannot map to the same thing, so the function is globally one-to-one.

Performance review: 10 out of 11 got this. 1 chose (B).

Historical note (last year): 13 out of 28 people got this correct. 6 people each chose (B) and (D), 2 people chose (E), and 1 person chose (A).

(14) Consider the following function $f : [0, 1] \to [0, 1]$ given by $f(x) := \{ \begin{array}{rl} \sin(\pi x/2), & 0 \leq x \leq 1/2 \\ \sqrt{x}, & 1/2 < x \leq 1 \end{array}$.
What is the correct expression for $(f^{-1})'(1/2)$? *Earlier score:* 1/15
(A) It does not exist, since the two-sided derivatives of f at 1/2 do not match.
(B) $\sqrt{2}$
(C) $2\sqrt{2}/\pi$
(D) $4/\pi$
(E) $4/(\sqrt{3}\pi)$

Answer: Option (E)
Explanation: We use:

$$(f^{-1})'(1/2) = \frac{1}{f'(f^{-1}(1/2))}$$

By inspection, we see that $f^{-1}(1/2)$ must be between 0 and $1/2$. Thus, we must solve $\sin(\pi x/2) = 1/2$. This gives $\pi x/2 = \pi/6$ (considering domain restrictions) so $x = 1/3$. Thus, we get:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/3)}$$

The expression for the derivative is $(\pi/2)\cos(\pi x/2)$, which evaluated at $1/3$ gives $(\pi\sqrt{3})/4$. Taking the reciprocal, we get $4/(\pi\sqrt{3})$.

Note that (A) is a sophisticated distractor in the sense that if you naively consider:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/2)}$$

You will wrongly conclude (A). (B) and (C) are the one-sided derivative at $f(1/2)$, so these too are attractive propositions for the naive.

Performance review: 7 out of 11 got this. 2 chose (C), 1 each chose (A) and (B).

Historical note (last year): 12 people got this correct. 9 people chose (A), 4 people chose (B), and 3 people chose (C).