CLASS QUIZ SOLUTIONS: JANUARY 11: HYPERBOLIC FUNCTIONS

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 5-question quiz. The score distribution was as follows:

- Score of 1: 3 people
- Score of 2: 4 people.
- Score of 3: 1 person
- Score of 4: 1 person
- Score of 5: 2 people

The answers by question number are:

(1) Option (B): 6 people
(2) Option (E): 6 people
(3) Option (D): 4 people
(4) Option (C): 8 people
(5) Option (C): 4 people

2. Solutions

(1) What is the limit $\lim_{x\to\infty}(\cosh x)/e^x$?
  (A) 0
  (B) 1/2
  (C) 1
  (D) 2
  (E) The limit does not exist.
  Answer: Option (B)
  Explanation: We have:

$$\lim_{x\to\infty}\frac{\cosh x}{e^x} = \lim_{x\to\infty}\frac{e^x + e^{-x}}{2e^x} = \frac{1}{2} + \lim_{x\to\infty}\frac{e^{-2x}}{2} = \frac{1}{2}$$

  The key thing to note is that as $x \to \infty$, e^{-2x} tends to 0.
  Performance review: 6 out of 11 people got this. 2 chose (A), 3 chose (E).
  Historical note (last year): 21 out of 28 students got this correct. 3 people chose (C), 3 people chose (E), and 1 person chose (A).

(2) What is the limit $\lim_{x\to-\infty}(\cosh x)/e^x$?
  (A) 0
  (B) 1/2
  (C) 1
  (D) 2
  (E) The limit does not exist.
  Answer: Option (E)
  Explanation: We have

$$\lim_{x\to-\infty}\frac{\cosh x}{e^x} = \lim_{x\to-\infty}\frac{e^x + e^{-x}}{2e^x} = \frac{1}{2} + \lim_{x\to-\infty}\frac{e^{-2x}}{2}$$

  As $x \to -\infty$, $e^{-2x} \to \infty$, so the overall limit is undefined.

Alternatively, we can note that as $x \to -\infty$, $\cosh x \to +\infty$ and $e^x \to 0$, so the quotient goes off to infinity.

Performance review: 6 out of 11 got this. 4 chose (A), 1 chose (B).

Historical note (last year): 24 out of 28 people got this correct. 3 people chose (C) and 1 person chose (B).

(3) Consider the function $y = f(x)$ where $f(x) := \arctan(\sinh x)$. Which of the following does $\cosh x$ necessarily equal?
(A) $\sin y$
(B) $\cos y$
(C) $\cot y$
(D) $\sec y$
(E) $\csc y$

Answer: Option (D)

Explanation: We have the relationship $\tan y = \sinh x$. Squaring and adding 1 to both sides, we get:

$$\sec^2 y = \cosh^2 x$$

Now, we note that cosh is always positive, and for $y \in (-\pi/2, \pi/2)$, which it must be to be arctan of something, $\sec y$ is also positive. Thus, taking square roots on both sides yields:

$$\sec y = \cosh x$$

Performance review: 4 out of 11 got this. 3 chose (C), 2 chose (B), 1 each chose (A) and (E).

Historical note (last year): 19 out of 28 people got this correct. 8 people chose (C) and 1 person chose (A).

(4) Consider the function $y = f(x)$ where $f(x) := \arctan(\sinh x)$ (same as in the previous question). The function is a one-to-one increasing function on its domain. What are its domain and range?
(A) The domain and range are both equal to $\mathbb{R}$
(B) The domain and range are both equal to the open interval $(-\pi/2, \pi/2)$
(C) The domain equals $\mathbb{R}$ and the range equals the open interval $(-\pi/2, \pi/2)$
(D) The domain equals the open interval $(-\pi/2, \pi/2)$ and the range equals $\mathbb{R}$
(E) The domain equals the open interval $(-\pi/2, \pi/2)$ and the range equals the closed interval $[-\pi/2, \pi/2]$

Answer: Option (C)

Explanation: The sinh function has domain and range both $\mathbb{R}$. The arctan function has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Thus, the composite has domain $\mathbb{R}$ (any real input is permisible) and range equal to $(-\pi/2, \pi/2)$.

Performance review: 8 out of 11 got this. 3 chose (D).

Historical note (last year): 24 out of 28 people got this correct. 3 people chose (D) and 1 person chose (B).

(5) sinh is a one-to-one function with domain and range both equal to $\mathbb{R}$. Hence, it must have an inverse function with domain and range both equal to $\mathbb{R}$. What is this inverse function?
(A) $x \mapsto (\ln(x) - \ln(-x))/2$
(B) $x \mapsto (1/2)\ln(x^2 + 1)$
(C) $x \mapsto \ln[x + \sqrt{x^2 + 1}]$
(D) $x \mapsto \ln[x - \sqrt{x^2 + 1}]$
(E) $x \mapsto \ln[\sqrt{x^2 + 1} - x]$

Answer: Option (C)

Explanation: If $x = \sinh t$, then $\cosh^2 t = x^2 + 1$. Taking square roots, and using that cosh is always positive, we get $\cosh t = \sqrt{x^2 + 1}$. Thus, $\exp(t) = \sinh(t) + \cosh(t) = x + \sqrt{x^2 + 1}$. Taking ln both sides, we get $t = \ln[x + \sqrt{x^2 + 1}]$.

Performance review: 4 out of 11 got this. 3 each chose (A) and (D), 1 chose (B).

Historical note (last year): 15 out of 28 people got this correct. 1 person appears to have missed the question because it was printed on the back side page. 6 people chose (B), 3 people chose (A), 2 people chose (E), and 1 person chose (D).