CLASS QUIZ SOLUTIONS: OCTOBER 3: LIMITS AND CONTINUITY

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

49 people took this quiz.

The score distribution was as follows:

- Score of 2: 1 person.
- Score of 3: 12 people.
- Score of 4: 9 people.
- Score of 5: 12 people.
- Score of 6: 14 people.
- Score of 7: 1 person.

The mean score was 4.59.

The question-wise performance was as follows:

(1) Option (A): 47 people got this.
(2) Option (D): 20 people got this.
(3) Option (C): 44 people got this.
(4) Option (C): 20 people got this.
(5) Option (B): 31 people got this.
(6) Option (C): 24 people got this.
(7) Option (B): 39 people got this.

General comments on performance compared with previous years: Performance was somewhat better on the "easier" questions and about the same (within ordinary measurement error) on the "harder" questions. However, it's still quite impressive because in previous years, when I taught this material in Math 152, I spent considerably more class time explaining the concept of limit.

That said, I think there was a divide in performance between people who had reviewed the material prior to class (and tended to do quite well) and people who hadn't reviewed the material prior to class. I do recommend that you watch the limit video playlist if you haven't done so. Even if you've watched the playlist once, you may wish to watch parts of it again.

2. Solutions

You can access more limit-related quiz questions online at:
`http://calculus.subwiki.org/wiki/Quiz:Limit`

(1) (**) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0,1)$? *Previous year's performance*: 6/13 correct

(A) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x, & 1/2 < x < 1 \end{cases}$

(B) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x, & 1/2 \le x < 1 \end{cases}$

(C) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{cases}$

(D) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \le x < 1 \end{cases}$

(E) All of the above

Answer: Option (A)

Explanation: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, $1/2$)? and (ii) is the point (in all cases, $1/2$) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at $1/2$. Namely the function x and $2x - (1/2)$ both take the value $1/2$ at the domain point $1/2$. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function).

This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at $1/2$: the former gives $1/2$ and the latter gives 1. In other words, the function has a jump discontinuity at $1/2$. Thus, (ii) becomes relevant: is $1/2$ included in the left or the right definition?

For option (A), $1/2$ is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \to 1/2^-} f(x)$. On the other hand, $\lim_{x \to 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous.

For option (B), $1/2$ is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at $1/2$.

Performance review: 47 out of 49 people got this correct. 1 person each chose (B) and (C).

Historical note (last year): 11 out of 12 people got this correct. 1 chose (C).

Historical note (two years ago): 6 out of 13 people got this correct. 6 people chose option (E) – I'm not sure why this option was so popular. 1 person chose option (C) but was quite close to choosing (A).

(2) (**) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both left continuous on $(0,1)$. Which of the following is *not* guaranteed to be left continuous on $(0,1)$? *Previous year's performance*: 4/13 correct

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D)

Explanation: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail.

For $c \in (0,1)$, we want to know whether:

$$\lim_{x \to c^-} f(g(x)) \stackrel{?}{=} f(g(c))$$

We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false.

We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f, the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say:

$$f(x) := \begin{cases} 1/3, & 0 < x \le 1/2 \\ 2/3, & 1/2 < x < 1 \end{cases}$$

and

$$g(x) := 1 - x$$

Note that both functions have range a subset of $(0, 1)$.

Composing, we obtain that:

$$f(g(x)) = \begin{cases} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \leq x < 1 \end{cases}$$

f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

Performance review: 20 out of 49 people got this correct. 26 people chose (E), 2 chose (C), 1 chose (B).

Historical note (last year): 8 out of 12 got this correct. 4 chose (E).

Historical note (two years ago): 4 out of 13 people got this correct. 9 people chose option (E). This is understandable, because if you look only at the obvious examples (all of which are increasing functions), you are likely to think that $f \circ g$ must be left continuous. If you got this question right for the right reasons, congratulate yourself.

(3) Which of these is the correct interpretation of $\lim_{x \to c} f(x) = L$ in terms of the definition of limit? *Previous year's performance: 9/12 correct*

(A) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
(B) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
(C) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
(D) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.
(E) None of the above

Answer: Option (C)

Explanation: α plays the role of ε and β plays the role of δ.

Performance review: 44 out of 49 people got this correct. 3 people chose (A), 2 chose (B).

Historical note (last year): 8 out of 12 got this correct. 2 chose (B), 1 each chose (A) and (E).

Historical note (two years ago): 9 out of 12 people got this correct. 2 chose (B) and 1 person chose (D).

Action point: If you got this correct, that means that you are not completely fixated on the letters ε and δ. This is good news, because it is important to concentrate on the substantive meaning rather than get caught up with a name. If you had difficulty with this, make sure you can understand it now.

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x \to c} f(x)$ exists)? *Previous year's performance: 10/12 correct*

(A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
(B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
(C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
(D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
(E) All of the above.

Answer: Option (C)

Explanation: Our statement should be that *every* c has no limit. In other words, for *every* c and *every* L, it is *not* true that $\lim_{x \to c} f(x) = L$. That's exactly what option (C) says.

Performance review: 20 out of 49 people got this correct. 15 chose (B), 7 chose (E), 4 chose (D), 3 chose (A).

Historical note (last year): 2 out of 12 people got this correct. 7 chose (B), 2 chose (D), 1 chose (A).

Historical note (two years ago): 10 out of 12 people got this correct. 1 person each chose (B) and (E). *I had discussed the question in class; that explains the high score.*

(5) In the usual $\varepsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\varepsilon > 0$, then which of the following is true? *Previous year's performance:* 17/26 *correct*

- (A) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
- (B) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
- (C) Every larger value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
- (D) Every larger value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
- (E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ε also works for larger εs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 31 out of 49 people got this correct. 6 each chose (A) and (E), 5 chose (C), 1 chose (D).

Historical note (last year): 8 out of 12 got this correct. 3 chose (A), 1 chose (E).

Historical note (two years ago): 17 out of 26 people got this correct. 5 people chose (A), 3 chose (C), and 1 chose (D).

(6) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of εs and δs?

- (A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.
- (B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.
- (C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.
- (D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.
- (E) None of the above.

Answer: Option (C)

Explanation: The "open interval centered at L" describes the "$\varepsilon > 0$" part of the definition (where the open interval is the interval $(L - \varepsilon, L + \varepsilon)$). The "open interval centered at c" describes the "$\delta > 0$" part of the definition (where the open interval is the interval $(c - \delta, c + \delta)$). x being in the open interval centered at c (except the case $x = c$) is equivalent to $0 < |x - c| < \delta$, and $f(x)$ being in the open interval centered at L is equivalent to $|f(x) - L| < \varepsilon$.

Performance review: 24 out of 49 people got this correct. 12 chose (A), 8 chose (D), 3 chose (E), 2 chose (B).

Historical note (last year): 7 out of 12 got this correct. 2 chose (A), 1 each chose (B), (D), and (E).

(7) (*) Consider the function:

$$f(x) := \begin{cases} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{cases}$$

What is the set of all points at which f is continuous? *Previous year's performance*: 5/13 correct

(A) $\{0, 1\}$
(B) $\{-1, 1\}$
(C) $\{-1, 0\}$
(D) $\{-1, 0, 1\}$
(E) f is continuous everywhere

Answer: Option (B)

Explanation: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c, we need that: (i) the limit for the function as defined for rationals exists at c, (ii) the limit for the function as defined for irrationals exists at c, and (iii) the two limits are equal.

Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c. This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals.

The limit for rationals exists for all c and equals the value c. The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

Performance review: 39 out of 49 people got this correct. 7 chose (D), 2 chose (A), 1 chose (E).

Historical note (last year): 9 out of 12 got this correct. 3 chose (E).

Historical note (two years ago): 5 out of 13 people got this correct. 3 people chose (D), 3 people chose (A), and 1 person each chose (C) and (E). Some of the people who chose (D) wrote "all rationals", so they probably thought that the correct answer is "all rationals" but it was not one of the options.