TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY DECEMBER 2: SIMILARITY OF LINEAR TRANSFORMATIONS (APPLIED)

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

25 people took this 16-question quiz. The score distribution was as follows:

- Score of 4: 3 people
- Score of 5: 4 people
- Score of 6: 3 people
- Score of 8: 4 people
- Score of 9: 4 people
- Score of 10: 4 people
- Score of 12: 3 people

The mean score was 7.76.

The question-wise answers and performance review were as follows:

(1) Option (A): 15 people
(2) Option (B): 16 people
(3) Option (A): 15 people
(4) Option (B): 20 people
(5) Option (D): 10 people
(6) Option (B): 12 people
(7) Option (A): 3 people
(8) Option (B): 19 people
(9) Option (B): 15 people
(10) Option (B): 6 people
(11) Option (C): 10 people
(12) Option (E): 8 people
(13) Option (D): 6 people
(14) Option (C): 11 people
(15) Option (D): 16 people
(16) Option (A): 12 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz corresponds to material discussed in the lecture notes titled `Coordinates`. It also corresponds to Section 3.4 of the text.

Recall that $n \times n$ matrices A and B are termed *similar* if there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. The relation of matrices being similar is an *equivalence relation*.

Recall that $n \times n$ matrices A and B are termed *quasi-similar* if there exist $n \times n$ matrices C and D such that $A = CD$ and $B = DC$. Recall that similar matrices are always quasi-similar, but quasi-similar matrices need not be similar. However, for *invertible* matrices, similarity and quasi-similarity are equivalent.

Also, note that if A and B are quasi-similar matrices, then A and B have the same trace. However, the converse is not true: it is possible to have two matrices A and B that have the same trace but are not quasi-similar.

For these questions, assume $n > 1$, because a lot of phenomena are much simpler in the case $n = 1$ and you may be misled if you look only at that case.

Note also that the trace of a square matrix is defined as the sum of its diagonal entries.
The *determinant* of a 2×2 matrix, denoted det, is defined as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The following are some important facts about the determinant:

- The determinant of a 2×2 diagonal matrix is the product of the diagonal entries.
- The determinant of a 2×2 matrix is zero if and only if the matrix is non-invertible.
- The determinant of the product of two 2×2 matrices is the product of the determinants.
- The determinant of the inverse of an invertible 2×2 matrix is the reciprocal of the determinant.
- If A and B are similar 2×2 matrices, they have the same determinant.
- If A and B are quasi-similar 2×2 matrices, they have the same determinant.
- If the determinant of A is positive, then the linear transformation given by A is an orientation-preserving linear automorphism of $\mathbb{R}^2$.
- If the determinant of A is negative, then the linear transformation given by A is an orientation-reversing linear automorphism of $\mathbb{R}^2$.

(1) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Denote by I_n the $n \times n$ identity matrix. Which of the following holds?
  (A) A is similar to B if and only if $A - I_n$ is similar to $B - I_n$.
  (B) If A is similar to B, then $A - I_n$ is similar to $B - I_n$. However, $A - I_n$ being similar to $B - I_n$ does not imply that A is similar to B.
  (C) If $A - I_n$ is similar to $B - I_n$, then A is similar to B. However, A being similar to B does not imply that $A - I_n$ is similar to $B - I_n$.
  (D) A being similar to B does not imply that $A - I_n$ is similar to $B - I_n$. Also, $A - I_n$ being similar to $B - I_n$ does not imply that A is similar to B.
  Answer: Option (A)
  Explanation: We have that for any invertible matrix S, $S(B - I_n)S^{-1} = (SBS^{-1} - SI_nS^{-1} = SBS^{-1} - I_n$. In other words, if $A = SBS^{-1}$, then $A - I_n = S(B - I_n)S^{-1}$. Conversely, if $A - I_n = S(B - I_n)S^{-1}$, then $A = SBS^{-1}$. Thus, A is similar to B if and only if $A - I_n$ is similar to $B - I_n$, and the matrix used for similarity is the same in both cases.
  Performance review: 15 out of 25 got this. 7 chose (D), 2 chose (C), 1 chose (B).

Suppose f is a polynomial of degree r in one variable with real coefficients. For a $n \times n$ matrix X, we denote by $f(X)$ we mean the matrix we get by applying the polynomial to f, where constant terms are interpreted as scalar matrices. For instance, if $f(x) = x^2 + 3x + 5$, then $f(X) = X^2 + 3X + 5I_n$.

(2) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Suppose f is a polynomial of degree r in one variable, where $r \geq 2$. Which of the following holds?
  (A) A is similar to B if and only if $f(A)$ is similar to $f(B)$.
  (B) If A is similar to B, then $f(A)$ is similar to $f(B)$. However, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.
  (C) If $f(A)$ is similar to $f(B)$, then A is similar to B. However, A being similar to B does not imply that $f(A)$ is similar to $f(B)$.
  (D) A being similar to B does not imply that $f(A)$ is similar to $f(B)$. Also, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.
  Answer: Option (B)
  Explanation: For the forward direction, note that $A = SBS^{-1}$ implies that $f(A) = Sf(B)S^{-1}$. For the breakdown of the reverse direction, see the explanation for Q8 of the November 27 quiz. This covers the $f(x) = x^2$ case. Similar examples can be constructed for other polynomials.
  Performance review: 16 out of 25 got this. 4 chose (D), 3 chose (A), 2 chose (C).

(3) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Suppose f is a polynomial of degree r in one variable, where $r = 1$. Which of the following holds?
(A) A is similar to B if and only if $f(A)$ is similar to $f(B)$.
(B) If A is similar to B, then $f(A)$ is similar to $f(B)$. However, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.
(C) If $f(A)$ is similar to $f(B)$, then A is similar to B. However, A being similar to B does not imply that $f(A)$ is similar to $f(B)$.
(D) A being similar to B does not imply that $f(A)$ is similar to $f(B)$. Also, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.

Answer: Option (A)

Explanation: Degree one polynomials differ from higher degree polynomials in that we can recover the original matrix from a degree one polynomial of it. Thus, we can deduce that $A = SBS^{-1}$ if and only if $f(A) = Sf(B)S^{-1}$. Special cases of this were covered in Question 1 of this quiz and Questions 5 and 6 of the November 27 quiz.

Performance review: 15 out of 25 got this. 4 chose (D), 3 each chose (B) and (C).

Suppose p and q are real numbers (possibly equal, possibly distinct). The diagonal matrices:

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

and

$$B = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

are similar. Explicitly, the two matrices are similar under the change-of-basis transformation that interchanges the coordinates, i.e., if we set:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then:

$$S = S^{-1}$$

and we have:

$$B = S^{-1}AS$$

Moreover, the only diagonal matrices similar to A are A and B (in the special case that $p = q$, we get $A = B$ is a scalar matrix, so A is the only diagonal matrix similar to A).

(4) What is the necessary and sufficient condition on p and q such that the matrix $A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is similar to $-A$?
(A) $p = q$
(B) $p = -q$
(C) $p = 1/q$
(D) $p = -1/q$
(E) $p + q = 1$

Answer: Option (B)

Explanation: We have:

$$-A = \begin{bmatrix} -p & 0 \\ 0 & -q \end{bmatrix}$$

Now, for A to be similar to $-A$, we have one of these two conditions:

$$-A = A \text{ or } -A = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

The first case ($-A = A$) gives us that $p = q = 0$, so that A is the zero matrix. The second case gives us that $-p = q$ and $-q = p$. Both of these are equivalent to $p = -q$. We now notice that the first case $p = q = 0$ is subsumed within the second case, so that $p = -q$ describes the necessary and sufficient condition.

Performance review: 20 out of 25 got this. 2 each chose (A) and (D), 1 chose (C).

(5) Which of the following is a necessary and sufficient condition on p and q so that the matrix $A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is invertible and similar to $-A^{-1}$?
(A) $p = q$
(B) $p = -q$
(C) $p = 1/q$
(D) $p = -1/q$
(E) $p + q = 1$

Answer: Option (D)

Explanation: We have:

$$-A^{-1} = \begin{bmatrix} -1/p & 0 \\ 0 & -1/q \end{bmatrix}$$

For this to be similar to A, we must have:

$$-A^{-1} = A \text{ or } -A^{-1} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

The first case gives $-1/p = p$ and $-1/q = q$, solving to $p^2 = -1$ and $q^2 = -1$, which is not possible. Thus, the first case is ruled out.

This brings us to the second case. In this case, $-1/p = q$ and $-1/q = p$. Both of these are equivalent to $p = -1/q$. This is the correct answer.

Performance review: 10 out of 25 got this. 9 chose (B), 5 chose (C), 1 chose (E).

Consider the matrix:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

used above. We have $S = S^{-1}$. For a general matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have:

$$S^{-1}AS = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

In other words, it swaps the rows *and* swaps the columns. This observation may be useful for some of the following questions.

(6) For an angle θ with $-\pi \le \theta \le \pi$, the rotation matrix for θ is given as:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note that $R(-\pi) = R(\pi)$, but other than that equality, all the $R(\theta)$s are distinct.

Which of these describes the relation between the rotation matrices for different values of θ?

(A) All the rotation matrices $R(\theta)$, $-\pi < \theta \le \pi$, are similar to each other.

(B) The rotation matrix $R(\theta)$ is similar to itself and to the rotation matrix $R(-\theta)$. However, it is not in general similar to any other rotation matrix.

(C) No two different rotation matrices are similar.

(D) The rotation matrix $R(\theta)$ is similar to itself and to the rotation matrix $R(\pi - \theta)$ (or $R(-\pi - \theta)$, depending on which of the two angles lies within the specified range). However, it is not in general similar to any other rotation matrix.

Answer: Option (B)

Explanation: The matrix we can use for the similarity transformation is the following self-inverse matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This matrix essentially performs a reflection about the x-axis, and the net effect is to negate the angle of rotation. Explicitly:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Performance review: 12 out of 25 got this. 9 chose (D), 4 chose (A).

Historical note (last time): 9 out of 19 got this. 4 each chose (A) and (D), 2 chose (C).

(7) Consider the linear automorphisms of $\mathbb{R}^2$ that are given as *reflections* about lines in $\mathbb{R}^2$ through the origin. (Note that we need the line of reflection to pass through the origin for the automorphism to be *linear* rather than merely being *affine linear*). Which of these describes the relation between reflection matrices for different possible lines of reflection through the origin?

(A) All the reflection matrices are similar to each other.

(B) No two reflection matrices for different lines of reflection are similar.

(C) The reflection matrices for two different lines of reflection are similar if and only if the lines of reflection are perpendicular.

(D) The reflection matrices for two different lines of reflection are similar if and only if the lines are reflection make an angle that is a rational multiple of π.

Answer: Option (A)

Explanation: The rotation matrix that rotates one line to the other can be used as the matrix for similarity.

Performance review: 3 out of 25 got this. 12 chose (C), 6 chose (B), 4 chose (D).

Historical note (last time): 2 out of 19 people got this. 12 chose (C), 4 chose (D), 1 chose (B).

(8) Suppose m and n are positive integers with $m < n$. Denote by P_m the "orthogonal projection onto the first m coordinates" linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, defined as follows. This takes as input a n-dimensional vector, sends each of the first m coordinates to itself, and sends the remaining coordinates to zero. What is the trace of the matrix of P_m?

(A) 1

(B) m

(C) n

(D) $n - m$

(E) $m - n$

Answer: Option (B)

Explanation: The matrix is diagonal with the first m diagonal entries equal to 1 and the remaining $n - m$ diagonal entries equal to 0. The trace is thus m.

Performance review: 19 out of 25 got this. 3 each chose (D) and (E).

Historical note (last time): 9 out of 19 got this. 5 chose (D), 4 chose (E), 1 chose (C).

(9) It is a fact that if A, B are $n \times n$ matrices that describe orthogonal projections onto (possibly different) m-dimensional subspaces of $\mathbb{R}^n$, then A and B are similar. What can we say must be the trace of an orthogonal projection onto any m-dimensional subspace of $\mathbb{R}^n$?

(A) 1
(B) m
(C) n
(D) $n - m$
(E) $m - n$

Answer: Option (B)

Explanation: This follows from the preceding question.

Performance review: 15 out of 25 got this. 4 each chose (A) and (D), 2 chose (C).

Historical note (last time): 4 out of 19 got this. 6 chose (C), 5 chose (D), 4 chose (E).

(10) Suppose A, B and C are $n \times n$ matrices. Which of the following matrices is *not* guaranteed (based on the given information) to have the same trace as the product ABC? Please see (and read carefully) Options (D) and (E) before answering.

(A) BCA
(B) CBA
(C) CAB
(D) None the above, i.e., they are all guaranteed to have the same trace as ABC.
(E) All of the above, i.e., none of them is guaranteed to have the same trace as ABC.

Answer: Option (B)

Explanation: First, let's note that the other two options don't work:

- Option (A): $ABC = A(BC)$ whereas $BCA = (BC)A$, so ABC and BCA are quasi-similar matrices. Therefore, they have the same trace.
- Option (C): $ABC = (AB)C$ and $CAB = C(AB)$, so ABC and CAB are quasi-similar matrices. Therefore, they have the same trace.

Consider the example:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The product ABC is the zero matrix, because AB is the zero matrix. Thus, ABC has trace zero. The product CBA, on the other hand, is the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Performance review: 6 out of 25 got this. 11 chose (D), 7 chose (E), 1 chose (C).

Historical note (last time): 4 out of 19 got this. 8 chose (D), 5 chose (E), 1 each chose (A) and (C).

(11) Which of the following gives a pair of matrices A and B that have the same trace as each other *and* the same determinant as each other, but that are *not* similar to each other?

(A) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
(B) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(C) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
(D) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
(E) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Answer: Option (C)

Explanation: The identity matrix is not similar to any non-identity matrix, because it is scalar, so conjugating it by anything leaves it as it is.

All other pairs of matrices are in fact similar:

- Option (A): We can use the coordinate interchange matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- Option (B): We can use the conjugating matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
- Option (D) We can use the coordinate interchange matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- Option (E): Use $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Performance review: 10 out of 25 got this. 7 chose (B), 5 chose (E), 2 chose (D), 1 left the question blank.

Historical note (last time): 6 out of 19 got this. 5 chose (B), 3 each chose (D) and (E), and 2 chose (A).

(12) Suppose A and B are 2×2 matrices. Which of the following correctly describes the relation between $\det A$, $\det B$, and $\det(A+B)$? Please see Option (E) before answering.
(A) $\det(A+B) = \det A + \det B$
(B) $\det(A+B) \leq \det A + \det B$, but equality need not necessarily hold.
(C) $\det(A+B) \geq \det A + \det B$, but equality need not necessarily hold.
(D) $|\det(A+B)| \leq |\det A| + |\det B|$, but equality need not necessarily hold.
(E) None of the above.

Answer: Option (E)

Explanation: The determinant does not interact in any meaningful manner with addition. In fact, for any (possibly equal, possibly distinct) real numbers p, q, and r, we can construct matrices A and B such that $\det A = p$, $\det B = q$, $\det(A+B) = r$. We can in fact just use 2×2 matrices to achieve this. The general description is tricky, so let's just give counter-examples for each part:

- Options (A), (B) and (D): Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note that $\det A = \det B = 0$, but $\det(A+B) = 1$.
- Option (C): Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, $\det A = \det B = 1$, but $\det(A+B) = 0$.

Performance review: 8 out of 25 got this. 7 chose (C), 5 chose (B), 3 chose (D), 2 chose (A).

Historical note (last time): 4 out of 19 got this. 5 chose (B), 4 chose (C), 3 each chose (A) and (D).

Let n be a natural number greater than 1. Suppose $f : \{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n\}$ is a function satisfying $f(0) = 0$. Let T_f denote the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following for all $i \in \{1, 2, \ldots, n\}$:

$$T_f(\vec{e}_i) = \begin{cases} \vec{e}_{f(i)}, & f(i) \neq 0 \\ \vec{0}, & f(i) = 0 \end{cases}$$

Let M_f denote the matrix for the linear transformation T_f. M_f can be described explicitly as follows: the i^{th} column of M_f is $\vec{0}$ if $f(i) = 0$ and is $\vec{e}_{f(i)}$ if $f(i) \neq 0$.

Note that if $f, g : \{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n\}$ are functions with $f(0) = g(0) = 0$, then $M_{f \circ g} = M_f M_g$ and $T_{f \circ g} = T_f \circ T_g$.

For the following questions, the discussion prior to Question 3 might be helpful. Note, however, that while that discussion gives one possible candidate for the matrix S of the similarity transformation, it is not the only possible candidate. For some but not all of the following questions, in the case that two matrices are similar, the matrix S described there works. In the case that they are not similar, the lack of similarity can be inferred from the traces not being equal, or from the determinants not being equal.

(13) $n = 2$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which

case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).
(A) $f(0) = 0, f(1) = 1, f(2) = 0$
(B) $g(0) = 0, g(1) = 0, g(2) = 2$
(C) $h(0) = 0, h(1) = 1, h(2) = 1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Answer: Option (D)

Explanation: All three matrices are similar. Here is what the matrices look like:

- Option (A): $M_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Option (B): $M_g = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- Option (C): $M_h = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Clearly, they all have trace 1, rank 1, and determinant 0. Thus, we cannot *prima facie* rule out the possibility of their being similar. But to actually confirm that they are similar, it would help to demonstrate a matrix that accomplishes the similarity transformation.

The similarity of Options (A) and (B) is relatively easy: the two options are related in that they have interchanged roles of the first and second vector relative to each other. Thus, the following matrix works well for accomplishing similarlity:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Explicitly, $M_f = SM_gS^{-1}$. Note that $S^{-1} = S$.

The similarity between M_f and M_h is trickier. If we set:

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then:

$$X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then, $M_f = XM_hX^{-1}$.

Now that we have shown the similarity of M_f with M_g and the similarity of M_g with M_h, the fact that similarity is an equivalence relation tells us that all three matrices are similar.

Performance review: 6 out of 25 got this. 10 chose (C), 4 chose (E), 3 chose (B), 2 chose (A).

Historical note (last time): 4 out of 19 got this. 10 chose (C), 3 chose (A), 1 each chose (A) and (E).

(14) $n = 2$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).
(A) $f(0) = 0, f(1) = 0, f(2) = 1$
(B) $g(0) = 0, g(1) = 2, g(2) = 0$
(C) $h(0) = 0, h(1) = 2, h(2) = 1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Answer: Option (C)

Explanation: The three matrices are:

- Option (A): $M_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- Option (B): $M_g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
- Option (C): $M_h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

M_f and M_g are similar. In fact, we can use the matrix M_h itself:

$$M_f = M_h M_g M_h^{-1}$$

M_h is not similar to either M_f or M_g. We can see this, for instance, by noting that M_h has full rank, but neither M_f nor M_g do. Alternatively, note that M_h has determinant -1, unlike both M_f and M_g, that have determinant 0.

Performance review: 11 out of 25 got this. 5 chose (D), 4 chose (E), 3 chose (B), 2 chose (A).

Historical note (last time): 8 out of 19 got this. 4 each chose (B) and (D), 3 chose (E).

(15) $n = 3$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).

(A) $f(0) = 0, f(1) = 2, f(2) = 1, f(3) = 3$
(B) $g(0) = 0, g(1) = 1, g(2) = 3, g(3) = 2$
(C) $h(0) = 0, h(1) = 3, h(2) = 2, h(3) = 1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Answer: Option (D)

Explanation: In each case, the matrix interchanges two coordinates and leaves the third coordinate as is. *Which* two coordinates get interchanged just depends on how we label, and therefore, all the matrices are similar. The explicit descriptions are below:

- Option (A): $M_f = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Option (B): $M_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
- Option (C): $M_h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Performance review: 16 out of 25 got this. 4 chose (E), 3 chose (C), 2 chose (B).

Historical note (last time): 8 out of 19 got this. 4 chose (B), 3 chose (E), 2 each chose (A) and (C).

(16) $n = 3$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).

(A) $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$
(B) $g(0) = 0, g(1) = 2, g(2) = 3, g(3) = 1$
(C) $h(0) = 0, h(1) = 3, h(2) = 1, h(3) = 2$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Answer: Option (A)

Explanation: Let's first write out the matrices:

- Option (A): $M_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Option (B): $M_g = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- Option (C): $M_h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

M_f is the identity matrix, and therefore, cannot be similar to anything else. M_g and M_h both describe matrices that cycle the three coordinates, albeit in opposite orders. A re-labeling can change M_g to M_h. Explicitly, if:

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $SM_gS^{-1} = M_h$.

Performance review: 12 out of 25 got this. 6 chose (D), 4 chose (E), 2 chose (C), 1 chose (B).

Historical note (last time): 7 out of 19 got this. 3 each chose (B), (C), (D), and (E).