CLASS QUIZ SOLUTIONS: OCTOBER 24: CONCAVE, INFLECTIONS, TANGENTS, CUSPS, ASYMPTOTES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 students took the quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 3 people
- Score of 4: 5 people
- Score of 5: 2 people

The mean score was 3.6.
Here are the problem wise answers:

(1) Option (C): 9 people
(2) Option (A): 5 people
(3) Option (C): 6 people
(4) Option (C): 11 people
(5) Option (B): 3 people
(6) Option (A): 6 people
(7) Option (E): 3 people

2. Solutions

(1) Consider the function $f(x) := x^3(x-1)^4(x-2)^2$. Which of the following **is true**?
(A) 0, 1, and 2 are all critical points and all of them are points of local extrema.
(B) 0, 1, and 2 are all critical points, but only 0 is a point of local extremum.
(C) 0, 1, and 2 are all critical points, but only 1 and 2 are points of local extrema.
(D) 0, 1, and 2 are all critical points, and none of them is a point of local extremum.
(E) 1 and 2 are the only critical points.

Answer: Option (C)

Quick explanation: 0, 1, and 2 are all roots of f with multiplicity greater than one, hence they are also roots of the derivative. Moreover, their multiplicity in the derivative is one less than their multiplicity in the original function.

To see which ones are local extrema, just think of the functions x^3, $(x-1)^4$, and $(x-2)^2$ as isolated functions.

Alternatively, since all these are critical points where the function is also zero, we can, instead of using the derivative test, directly compute the sign of the function to the left and the right of each critical points. For those critical points where there is an even power, the sign of the original function is the same on both sides close to the point. For thos ecritical points where there is an odd power, the sign of the original function flips.

Full explanation: Try it yourself! You need to calculate the first derivative, see where it is zero, etc.

Performance review: 9 out of 12 got this correct. 2 chose (B), 1 chose (D).

Historical note (last year): 11 out of 15 people got this correct, which is a good showing. Other choices were (B) (2), (A) (1), and (E) (1).

However, many people did tedious derivative computations for this question. Please try to understand the intuition behind how this problem can be solved without computing derivatives.

(2) Suppose f and g are continuously differentiable functions on $\mathbb{R}$. Suppose f and g are both concave up. Which of the following is **always true**?
(A) $f + g$ is concave up.
(B) $f - g$ is concave up.
(C) $f \cdot g$ is concave up.
(D) $f \circ g$ is concave up.
(E) All of the above.

Answer: Option (A)

Explanation: The sum of two increasing functions is increasing. Hence, if f' and g' are both increasing, so is the sum $f' + g' = (f + g)'$. The other options are false. In fact, for any two functions that are concave up, both the differences $f - g$ and $g - f$ cannot be concave up. As for products, consider the example of $f(x) = x^2$ and $g(x) = (x - 1)^2$, which are both concave up everywhere, but their product is not. As for composites, consider $f(x) = x^2$ and $g(x) = x^2 - 2x$. The composite is $(x^2 - 2x)^2$, which is not concave up everywhere.

Performance review: 5 out of 12 got this correct. 3 chose (D), 2 each chose (C) and (E).

Historical note (last year): 8 out of 15 people got this correct. Other choices were (E) (5), (C) (1), and (D) (1).

(3) Consider the function $p(x) := x(x - 1) \dots (x - n)$, where $n \geq 1$ is a positive integer. How many points of inflection does p have?
(A) $n - 3$
(B) $n - 2$
(C) $n - 1$
(D) n
(E) $n + 1$

Answer: Option (C)

Explanation: p has degree $n + 1$, since it is the product of $n + 1$ linear polynomials. Thus, p'' has degree $n - 1$, and hence can have at most $n - 1$ roots. Thus, the number of points of inflection of p is at most $n - 1$. If we can locate $n - 1$ points of inflection, we will be done.

Note that since p has zeros at $0, 1, 2, \dots, n$. By the extreme value theorem, there exists a local extreme value for p between any two consecutive zeros, and this gives a root of p'. By degree considerations, there must be exactly one root in each interval $(i - 1, i)$. There are thus n distinct roots of p', each giving a local extreme value of p, located in the intervals $(0, 1), (1, 2), \dots, (n - 1, n)$. Call these roots $a_1, a_2, \dots, a_n$. Then $a_1 < a_2 < \dots < a_n$. Applying the extreme value theorem again on the intervals $[a_{i-1}, a_i]$, we see that there is at least one local extremum for p', and hence a zero for p'', in that interval. Since p'' has degree $n - 1$, there must be exactly one local extremum on each interval. Since local extrema of the derivative correspond to points of inflection, we have found $n - 1$ distinct points of inflection, and we are done.

Performance review: 6 out of 12 got this correct. 3 each chose (B) and (E).

Historical note (last year): 7 out of 15 people got this correct. Other choices were (B) (3), (D) (3), (A) (1), and (E) (1). Those people who chose (D) typically made the error of doing $(n + 1) - 1$ instead of $(n + 1) - 2$.

(4) Suppose f is a polynomial function of degree $n \geq 2$. What can you say about the sense of concavity of the function f for **large enough inputs**, i.e., as $x \to +\infty$? (Note that if $n \leq 1$, f is linear so we do not have concavity in either sense).
(A) f is eventually concave up.
(B) f is eventually concave down.
(C) f is eventually either concave up or concave down, and which of these cases occurs depends on the sign of the leading coefficient of f.
(D) f is eventually either concave up or concave down, and which of these cases occurs depends on whether the degree of f is even or odd.
(E) f may be concave up, concave down, or neither.

Apologies, the language of option (E) was confusing. Although the question did say "eventually", option (E) should ideally have read "f may be eventually concave up, concave down, or neither" rather than forcing you to infer this from the context.

Answer: Option (C).

Explanation: Note first that the sign of the leading coefficient of f'' is the same as the sign of the leading coefficient of f, because the leading coefficient gets multiplied by $n(n-1)$, which is positive for $n \geq 2$. If this sign is positive, then $f''(x) \to \infty$ as $x \to \infty$, and hence must be eventually positive, forcing f to be eventually concave up. If this sign is negative, then $f''(x) \to -\infty$ as $x \to \infty$, and hence must be eventually negative, forcing f to be eventually concave down.

Performance review: 11 out of 12 got this correct. 1 chose (D).

Historical note (last year): 12 out of 15 people got this correct. 2 people chose option (E). This is probably partly because of the confusing language, see the apology above.

(5) Suppose f is a continuously differentiable function on $[a, b]$ and f' is continuously differentiable at all points of $[a, b]$ except an interior point c, where it has a vertical cusp. What can we say is **definitely true** about the behavior of f at c?
(A) f attains a local extreme value at c.
(B) f has a point of inflection at c.
(C) f has a critical point at c that does not correspond to a local extreme value.
(D) f has a vertical tangent at c.
(E) f has a vertical cusp at c.

Answer: Option (B)

Explanation: A cusp for f' means that f' changes direction at c (either from increasing to decreasing or from decreasing to increasing). This in turn means that the sense of concavity of f changes at c. Hence, f has a point of inflection at c. Note that we cannot have the vertical tangent situation because f' is continuous and finite at c.

An example of such a function f would be $f(x) := x^{5/3}$. The first derivative $f'(x) = (5/3)x^{2/3}$ is everywhere defined and has a vertical cusp at $c = 0$. We note that the derivative switches from decreasing to increasing at $c = 0$, so the original function switches from concave down to concave up.

Performance review: 3 out of 12 got this correct. 6 chose (C), 2 chose (A), 1 chose (E).

Historical note (last year): 3 out of 15 people got this correct. Other options were (A) (4), (E) (3), (D) (3), and (C) (2). It seems that a lot of people did not make a clear enough distinction between the roles of f and f'.

Action point: This is the kind of question that is hard at first but at some stage should feel obvious to you. (Not immediately obvious, since it still requires you to read carefully, but the kind of thing that would not confuse you).

(6) Suppose f and g are continuous functions on $\mathbb{R}$, such that f attains a vertical tangent at a and is continuously differentiable everywhere else, and g attains a vertical tangent at b and is continuously differentiable everywhere else. Further, $a \neq b$. What can we say is **definitely true** about $f - g$?
(A) $f - g$ has vertical tangents at a and b.
(B) $f - g$ has a vertical tangent at a and a vertical cusp at b.
(C) $f - g$ has a vertical cusp at a and a vertical tangent at b.
(D) $f - g$ has no vertical tangents and no vertical cusps.
(E) $f - g$ has either a vertical tangent or a vertical cusp at the points a and b, but it is not possible to be more specific without further information.

Answer: Option (A)

Explanation: Note that $\lim_{x \to b}(f - g)'(x) = f'(b) - \lim_{x \to b} g'(x)$, which is an infinity of the sign opposite to that of g'. In particular, we hav a vertical tangent at b. Similarly, we have a vertical tangent at a.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (B).

Historical note (last year): 5 out of 15 people got this correct. Other choices were (E) (5), (C) (3), (B) (1), (D) (1). The main source of confusion here seems to have been that people did not realize if f has a vertical tangent at c, so does $-f$, because the whole picture flips over.

(7) Suppose f and g are continuous functions on $\mathbb{R}$, such that f is continuously differentiable everywhere and g is continuously differentiable everywhere except at c, where it has a vertical tangent. What can we say is **definitely true** about $f \circ g$?

(A) It has a vertical tangent at c.
(B) It has a vertical cusp at c.
(C) It has either a vertical tangent or a vertical cusp at c.
(D) It has neither a vertical tangent nor a vertical cusp at c.
(E) We cannot say anything for certain.

Answer: Option (E).

Explanation: Consider $g(x) := x^{1/3}$. This has a vertical tangent at $c = 0$. If we choose $f(x) = x$, we get (A). If we choose $f(x) = x^2$, we get (B). If we choose $f(x) = x^3$, we get neither a vertical tangent nor a vertical cusp. Hence, (E) is the only viable option.

Performance review: 3 out of 12 got this correct. 4 chose (C), 2 each chose (A) and (D), 1 chose (B).

Historical note (last year): 3 out of 15 people got this correct. Other choices were (A) (7), (C) (4), and (D) (1). The main thing that people had trouble with was thinking of possibilities for f that could play the role of converting the vertical tangent behavior of the original function g into vertical cusp or "neither" behavior for the composite function.

Action point: This is a devilishly tricky question that I don't expect you to get at all in your first try, but that I expect that you will remember for the rest of your life (or at least this course) after you've seen it.