CLASS QUIZ: JANUARY 14: INTEGRATION BY PARTS

MATH 153, SECTION 55 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

(1) Suppose f is an elementarily expressible and infinitely differentiable function on the positive reals (so all derivatives of f are also elementarily expressible). An antiderivative for $f''(x)/x$ is **not equivalent** up to elementary functions to **which one** of the following?
(A) An antiderivative for $x \mapsto f''(e^x)$, domain all of $\mathbb{R}$.
(B) An antiderivative for $x \mapsto f'(e^x/x)$, domain positive reals.
(C) An antiderivative for $x \mapsto f'''(x)(\ln x)$, domain positive reals.
(D) An antiderivative for $x \mapsto f'(1/x)$, domain positive reals.
(E) An antiderivative for $x \mapsto f(1/\sqrt{x})$, domain positive reals.

Your answer: ______________________

(2) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is [ADDED: *guaranteed to be*] **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ______________________

(3) Suppose f is a continuous function on $(0,\infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0,\infty)$ is [ADDED: *guaranteed to be*] **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________________

(4) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.
(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Your answer: ____________________________

(5) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin \sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0,1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**?
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.
(D) f_2 and f_3.
(E) f_2 and f_4.

Your answer: ____________________________

(6) Of the five functions below, four of them have antiderivatives that are equivalent up to elementary functions, i.e., an antiderivative for any one of them can be used to provide an antiderivative for the other three. The fifth function has an antiderivative that is **not equivalent** to any of these. Identify the fifth function.

(A) $x \mapsto e^{e^x}$, domain all reals
(B) $x \mapsto \ln(\ln x)$, domain $(1, \infty)$
(C) $x \mapsto e^x/x$, domain $(0, \infty)$
(D) $x \mapsto 1/(\ln x)$, domain $(1, \infty)$
(E) $x \mapsto 1/(\ln(\ln x))$, domain (e, ∞)

Your answer: ______________________________

(7) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?
(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2 e^{-x^2}$
(E) $x \mapsto x^4 e^{-x^2}$

Your answer: ______________________________

(8) Which of the following has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of the function $f(x) := e^x/x, x > 0$?
(A) $x \mapsto e^x/\sqrt{x}, x > 0$
(B) $x \mapsto e^x/x^2, x > 0$
(C) $x \mapsto e^x(\ln x), x > 0$
(D) $x \mapsto e^{1/\sqrt{x}}, x > 0$
(E) $x \mapsto e^{1/x}, x > 0$

Your answer: ______________________________

(9) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?
(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not

compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\ldots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Your answer: ____________________________