CLASS QUIZ SOLUTIONS: NOVEMBER 21: ONE-ONE FUNCTIONS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people attempted this quiz. The score distribution was as follows:

- Score of 1: 3 people.
- Score of 2: 6 people.
- Score of 3: 3 people.

The mean score was 2.
Here are the problem wise solutions and scores:

(1) Option (B): 11 people
(2) Option (C): 2 people
(3) Option (E): 9 people
(4) Option (C): 0 people. *Whoops!*
(5) Option (E): 2 people

2. Solutions

(1) For one of these function types for a continuous function from $\mathbb{R}$ to $\mathbb{R}$, it is *possible* to also be a one-to-one function. What is that function type?
(A) Function whose graph has mirror symmetry about a vertical line.
(B) Function whose graph has half turn symmetry about a point on it.
(C) Periodic function.
(D) Function having a point of local minimum.
(E) Function having a point of local maximum.

Answer: Option (B)

Explanation: Think $f(x) = x$, $f(x) = x^3$, or $f(x) = x - \sin x$.

For all the others, the violation of one-to-one is clear: periodic functions repeat completely after an interval, functions whose graph has mirror symmetry have the same value at equal distances from the axis of mirror symmetry, and the existence of a local extremum implies that values very close to that are attained both on the immediate left and the immediate right of the extremum.

Performance review: 11 out of 12 got this correct. 1 left the question blank.

Historical note (last year): Everybody got this correct!

(2) (**) Suppose f, g, and h are continuous one-to-one functions whose domain and range are both $\mathbb{R}$. **What can we say** about the functions $f + g$, $f + h$, and $g + h$?
(A) They are all continuous one-to-one functions with domain $\mathbb{R}$ and range $\mathbb{R}$.
(B) At least two of them are continuous one-to-one functions with domain $\mathbb{R}$ and range $\mathbb{R}$ – however, we cannot say more.
(C) At least one of them is a continuous one-to-one function with domain $\mathbb{R}$ and range $\mathbb{R}$ – however, we cannot say more.
(D) Either all three sums are continuous one-to-one functions whose domain and range are both $\mathbb{R}$, or none is.
(E) It is possible that none of the sums is a continuous one-to-one function whose domain and range are both $\mathbb{R}$; it is also possible that one, two, or all the sums are continuous one-to-one functions whose domain and range are both $\mathbb{R}$.

Answer: Option (C)

Explanation: Since f, g, and h are all continuous one-to-one functions with domain and range $\mathbb{R}$, each one of them is either increasing or decreasing. We consider various cases:

- If all three functions are increasing, so are all the pairwise sums, and hence, all the sums $f+g$, $f+h$, and $g+h$ are increasing. Further, the domain and range of all three pairwise sums is $\mathbb{R}$.
- If all three functions are decreasing, so are all the pairwise sums, and hence, all the sums $f+g$, $f+h$, and $g+h$ are decreasing. Further, the domain and range of all three pairwise sums is $\mathbb{R}$
- If two of the functions are increasing and the third function is decreasing, then we know for certain that the sum of the two increasing functions is increasing and hence one-to-one. But the sum of either of the increasing functions with the decreasing function may be increasing, decreasing, or neither. For instance, if $f(x) = g(x) = x$ and $h(x) = -x$, then $f+h$ and $g+h$ are both the zero function, which is neither increasing nor decreasing, and hence not one-to-one.
- If two of the functions are decreasing and the third function is increasing, then we know for certain that the sum of the two dcereasing functions is decreasing and hence one-to-one. We cannot say anything for sure about the other two sums, for the same reasons as in the previous case.

It's clear from all these that (C) is the right option.

Performance review: 2 out of 12 got this. 6 chose (E), 2 chose (A), 1 each chose (B) and (D).

Historical note (last year): 2 out of 15 people got this correct. 8 people chose (A), 1 person chose (B), and 4 people chose (E).

Action point: Please make sure you understand this solution really well! This kind of question should not trip you in the future.

(3) (**) Suppose f is a one-to-one function with domain a closed interval $[a, b]$ and range a closed interval $[c, d]$. Suppose t is a point in (a, b) such that f has left hand derivative l and right-hand derivative r at t, with both l and r nonzero. What is the left hand derivative and right hand derivative to f^{-1} at $f(t)$?

(A) The left hand derivative is $1/l$ and the right hand derivative is $1/r$.
(B) The left hand derivative is $-1/l$ and the right hand derivative is $-1/r$.
(C) The left hand derivative is $1/r$ and the right hand derivative is $1/l$.
(D) The left hand derivative is $-1/r$ and the right hand derivative is $-1/l$.
(E) The left hand derivative is $1/l$ and the right hand derivative is $1/r$ if $l > 0$, otherwise the left hand derivative is $1/r$ and the right hand derivative is $1/l$.

Answer: Option (E)

Explanation: Although it isn't necessary to note this, a one-to-one function that satisfies the intermediate value property is continuous, so even though f is not explicitly given to be continuous, it is in fact continuous on its domain.

If $l > 0$, then, since we are dealing with a one-to-one function, the function is increasing throughout, and so $r \ge 0$ as well. Since we know $r \ne 0$, we conclude that $r > 0$ strictly. The upshot is that as $x \to t^-$, $f(x) \to f(t)^-$ and as $x \to t^+$, $f(x) \to f(t)^+$. Thus, when we pass to the inverse function, the roles of left and right remain the same.

On the other hand, if $l < 0$, then as $x \to t^-$, $f(x) \to f(t)^+$, and hence the roles of left and right get interchanged.

Performance review: 9 out of 12 got this. 2 chose (C) and 1 chose (A).

Historical note (last year): 6 out of 15 people got this correct. 6 people chose (A) and 3 people chose (C).

Action point: One-sided derivatives and increasing/decreasing functions are a potent mix. We've tried and failed to understand this mix many times in the past. But this might well be the time it finally clicks! Here's a repetition: when we apply an increasing function, *left remains left* and *right remains right*. But when we apply a decreasing function, *left becomes right* and *right becomes left*. Keep chanting this again and again until you understand, appreciate, and *believe* it.

(4) (**) Which of these functions is one-to-one?

(A) $f_1(x) := \begin{cases} x, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

(B) $f_2(x) := \{ \begin{array}{cc} x, & x \text{ rational} \\ x^3, & x \text{ irrational} \end{array}$

(C) $f_3(x) := \{ \begin{array}{cc} x, & x \text{ rational} \\ 1/(x-1), & x \text{ irrational} \end{array}$

(D) All of the above

(E) None of the above

Answer: Option (C)

Explanation: Option (A) is easy to rule out: $\sqrt{2}$ and $-\sqrt{2}$ map to the same thing. Option (B) is a little harder to rule out, because the function is one-to-one within each piece, i.e., no two rationals map to the same thing and no two irrationals map to the same thing. However, a rational and an irrational can map to the same thing. For instance, 2 and $2^{1/3}$ both map to 2.

For option (C), note that not only is the map one-to-one in each piece, but also, the image of the rationals stays inside the rationals and the image of the irrationals stays inside the irrationals. In particular, this means that a rational number and an irrational number cannot map to the same thing, so the function is globally one-to-one.

Performance review: Nobody got this correct! 10 chose (C), 2 chose (A).

Historical note (last year): 2 out of 15 people got this correct. 7 people chose (E), 4 people chose (B), and 2 people chose (D). It is possible that some of those who chose (E) lost hope after looking at (A) and (B) and concluded that checking (C) is pointless too.

Action point: This one should not trip you in the future either!

(5) (**) Consider the following function $f : [0,1] \to [0,1]$ given by $f(x) := \{ \begin{array}{cc} \sin(\pi x/2), & 0 \le x \le 1/2 \\ \sqrt{x}, & 1/2 < x \le 1 \end{array}$.

What is the correct expression for $(f^{-1})'(1/2)$?

(A) It does not exist, since the two-sided derivatives of f at $1/2$ do not match.

(B) $\sqrt{2}$

(C) $2\sqrt{2}/\pi$

(D) $4/\pi$

(E) $4/(\sqrt{3}\pi)$

Answer: Option (E)

Explanation: We use:

$$(f^{-1})'(1/2) = \frac{1}{f'(f^{-1}(1/2))}$$

By inspection, we see that $f^{-1}(1/2)$ must be between 0 and $1/2$. Thus, we must solve $\sin(\pi x/2) = 1/2$. This gives $\pi x/2 = \pi/6$ (considering domain restrictions) so $x = 1/3$. Thus, we get:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/3)}$$

The expression for the derivative is $(\pi/2)\cos(\pi x/2)$, which evaluated at $1/3$ gives $(\pi\sqrt{3})/4$. Taking the reciprocal, we get $4/(\pi\sqrt{3})$.

Note that (A) is a sophisticated distractor in the sense that if you naively consider:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/2)}$$

You will wrongly conclude (A). (B) and (C) are the one-sided derivative at $f(1/2)$, so these too are attractive propositions for the naive.

Performance review: 2 out of 12 got this. 4 chose (A), 3 chose (D), 2 chose (C), 1 chose (B).

Historical note (last year): 1 out of 15 people got this correct. 7 people chose (A), 5 people chose (C), and 2 people chose (B).

Action point: I emphasized this point in class: $(f^{-1})'(x) = 1/(f'(f^{-1}(x)))$, not $1/(f'(x))$. However, the sting of getting it wrong on a quiz might be a greater spur to remember this fact forever. Make sure you never fall for this error again!