TAKE-HOME CLASS QUIZ: DUE WEDNESDAY FEBRUARY 27: TAYLOR SERIES AND POWER SERIES

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

FEEL FREE TO DISCUSS ALL QUESTIONS, BUT ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE. DO NOT ENGAGE IN GROUPTHINK.

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We denote by $C^k(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are at least k times continuously differentiable on all of $\mathbb{R}$. Note that for $k \geq l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k.

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

(1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which $\lim_{x\to c} |f(x)|/|x-c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f? *Two years ago:* 11/26 *correct*
   (A) The order of any zero of f must be between 0 and 1.
   (B) The order of any zero of f must be between 1 and 2.
   (C) The order of any zero of f, if finite, must be a positive integer.
   (D) The order of any zero of f must be exactly 1.
   (E) The order of any zero of f must be ∞.

   Your answer: ______________________

(2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on $\mathbb{R}$, what can we say about the Taylor polynomials about 0? *Two years ago:* 8/26 *correct*
   (A) No Taylor polynomial is defined for f.
   (B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
   (C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
   (D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
   (E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

   Your answer: ______________________

(3) Consider the function $F(x, p) = \sum_{n=1}^\infty x^n/n^p$. For fixed p, this is a power series in x. What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0, \infty)$? *Two years ago:* 4/26 *correct*
   (A) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
   (B) The interval of convergence is $(-1, 1)$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
   (C) The interval of convergence is $[-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
   (D) The interval of convergence is $(-1, 1]$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
   (E) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1)$ for $p > 1$.

   Your answer: ______________________

(4) Which of the following functions of x has a power series $\sum_{k=0}^\infty x^{4k}/(4k)!$? *Two years ago:* 9/26 *correct*

(A) $(\sin x + \sinh x)/2$
(B) $(\sin x - \sinh x)/2$
(C) $(\sinh x - \sin x)/2$
(D) $(\cosh x + \cos x)/2$
(E) $(\cosh x - \cos x)/2$

Your answer: ____________________________

(5) What is the sum $\sum_{k=0}^{\infty} (-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$. *Two years ago:* 12/26 *correct*
(A) $\cos x$
(B) $\sin x$
(C) $\cos(x^2)$
(D) $\cosh(x^2)$
(E) $\exp(-x^2)$

Your answer: ____________________________

(6) Define an operator R from the set of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b, $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$?
(A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
(B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
(C) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
(D) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
(E) $R(a+b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Your answer: ____________________________

(7) Which of the following is/are true? *Two years ago:* 5/26 *correct*
(A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on $\mathbb{R}$.
(B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence (which may not be all of $\mathbb{R}$).
(C) If we start with a power series about 0 that converges everywhere in $\mathbb{R}$, then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
(D) All of the above.
(E) None of the above.

Your answer: ____________________________

(8) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \to \infty$? *Two years ago:* 12/26 *correct*
(A) f grows polynomially in x.
(B) f grows faster than any polynomial function but slower than any exponential function of x (i.e., any function of the form $x \mapsto e^{mx}, m > 0$).
(C) f grows like an exponential function of x, i.e., it can be sandwiched between two exponentially growing functions of x.
(D) f grows faster than any exponential function but slower than any doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.
(E) f grows like a doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.

Your answer: ____________________________

(9) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$? *Two years ago:* 12/26 *correct*

(A) $\exp(x)$ from below and $\cosh(2x)$ from above.
(B) $\exp(x)$ from below and $\cosh(x^2)$ from above.
(C) $\exp(x/2)$ from below and $\exp(x)$ from above.
(D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.
(E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Your answer: ____________________________

(10) Consider the function $f(x) := \max\{0, x\}$. What can we say about the Taylor series of f centered at various points?

(A) The Taylor series of f centered at any point is the zero series.
(B) The Taylor series of f centered at any point simplifies to x.
(C) The Taylor series of f centered at any point other than zero converges to f globally. However, the Taylor series centered at 0 is not defined.
(D) The Taylor series of f centered at any point is either the zero series or simplifies to x.
(E) The Taylor series of f centered at any point other than the point 0 is either the zero series or simplifies to x. However, the Taylor series is not defined at 0.

Your answer: ____________________________

(11) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0? *Two years ago:* 3/26 *correct*

(A) $f_1(x) := \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$
(B) $f_2(x) := \begin{cases} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(C) $f_3(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(D) $f_4(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$
(E) All of the above.

Your answer: ____________________________

(12) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic? *Two years ago:* 7/26 *correct*

(A) $x \mapsto \ln(1 + x^2)$
(B) $x \mapsto \ln(1 + x)$
(C) $x \mapsto \ln(1 - x)$
(D) $x \mapsto \exp(1 + x)$
(E) $x \mapsto \exp(1 - x)$

Your answer: ____________________________

(13) Suppose f and g are globally analytic functions and g is nowhere zero. Which of the following is *not necessarily* globally analytic?

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) fg, i.e., the function $x \mapsto f(x)g(x)$
(D) f/g, i.e., the function $x \mapsto f(x)/g(x)$
(E) $f \circ g$, i.e., the function $x \mapsto f(g(x))$

Your answer: ____________________________

(14) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic? *Two years ago:* 10/26 *correct*

(A) x
(B) x^2
(C) $x+1$
(D) x^2+1
(E) e^x

Your answer: ____________________

(15) Consider the rational function $1/\prod_{i=1}^{n}(x-\alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_is, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_is? *Two years ago:* 10/26 *correct*

(A) It is the minimum of the distances from c to the α_is.
(B) It is the second smallest of the distances from c to the α_is.
(C) It is the arithmetic mean of the distances from c to the α_is.
(D) It is the second largest of the distances from c to the α_is.
(E) It is the maximum of the distances from c to the α_is.

Your answer: ____________________

(16) What is the interval of convergence of the Taylor series for arctan about 0? *Two years ago:* 11/26 *correct*

(A) $(-1,1)$
(B) $[-1,1)$
(C) $(-1,1]$
(D) $[-1,1]$
(E) All of $\mathbb{R}$

Your answer: ____________________

(17) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} 2^{\sqrt{k}} x^k$? Please keep in mind the square root in the exponent.

(A) 0
(B) 1/2
(C) $1/\sqrt{2}$
(D) 1
(E) infinite

Your answer: ____________________