TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 9: TAYLOR SERIES AND POWER SERIES

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

40 people took this quiz. The score distribution was as follows:

- Score of 3: 3 people
- Score of 4: 4 people
- Score of 5: 2 people
- Score of 6: 5 people
- Score of 7: 2 people
- Score of 8: 4 people
- Score of 9: 5 people
- Score of 10: 15 people

The question wise answers and performance reviews were as follows:

(1) Option (C): 32 people
(2) Option (C): 34 people
(3) Option (C): 28 people
(4) Option (D): 35 people
(5) Option (E): 35 people
(6) Option (E): 29 people
(7) Option (C): 29 people
(8) Option (B): 30 people
(9) Option (D): 28 people
(10) Option (E): 26 people

2. Solutions

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We denote by $C^k(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are at least k times continuously differentiable on all of $\mathbb{R}$. Note that for $k \geq l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k.

We say that a function f is (locally) analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

(1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which $\lim_{x \to c} |f(x)|/|x - c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f?
Two years ago: 11/26 *correct*
(A) The order of any zero of f must be between 0 and 1.
(B) The order of any zero of f must be between 1 and 2.
(C) The order of any zero of f, if finite, must be a positive integer.
(D) The order of any zero of f must be exactly 1.

(E) The order of any zero of f must be ∞.

Answer: Option (C)

Explanation: We consider two cases. First, that for every positive integer k, we have $f^{(k)}(c) = 0$. In that case, we can verify using the LH rule that $f(x)/(x-c)^k \to 0$ for every positive integer k, and hence, there is no finite least upper bound and hence no finite order.

Next, suppose there is a smallest k such that $f^{(k)}(c) \neq 0$. Suppose $f^{(k)}(c) = \lambda$. This k must be greater than 0, because we are given that $f^{(0)}(c) = f(c) = 0$. We can show by a k-fold application of the LH rule that $\lim_{x \to c} f(x)/(x-c)^k = \lambda/k!$ which is a finite nonzero number. By suitable chaining, we can therefore show that $\lim_{x \to c} |f(x)|/|x-c|^\beta = 0$ for all $\beta \in (0, k)$. Thus, the order of the zero at f is precisely k, whuch is a positive integer.

Performance review: 32 out of 40 got this. 7 chose (E), 1 chose (B).

Historical note (last year): 3 out of 11 people got this. 3 chose (B), 4 chose (E), and 1 left the question blank.

Historical note (two years ago): 11 out of 26 people got this correct. 7 chose (A), 3 each chose (B) and (E), 1 chose (D), and 1 left the question blank.

(2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on $\mathbb{R}$, what can we say about the Taylor polynomials about 0? *Two years ago: 8/26 correct*

(A) No Taylor polynomial is defined for f.
(B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
(C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
(D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
(E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

Answer: Option (C)

Explanation: The fraction power $x^{4/3}$ can be differentiated once but not twice about 0. The rest of the expression for f is polynomial. Thus, f is once differentiable but not twice differentiable at 0. Hence, we cannot define $P_2(f)$. $P_0(f)$ is just $f(0)$, which is 1, and $P_1(f)(x) = f(0) + f'(0)x = 1 + x$. Alternatively, $P_1(f)(x)$ is simply the truncation of f to the parts of degree at most 1.

Performance review: 34 out of 40 got this. 3 chose (D), 2 chose (E), and 1 chose (B).

Historical note (last year): 2 out of 11 got this correct. 7 chose (E), 1 each chose (B) and (D).

Historical note (two years ago): 8 out of 26 people got this correct. 5 each chose (A) and (B), 4 each chose (D) and (E).

(3) Consider the function $F(x, p) = \sum_{n=1}^{\infty} x^n/n^p$. For fixed p, this is a power series in x. What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0, \infty)$? *Two years ago: 4/26 correct*

(A) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
(B) The interval of convergence is $(-1, 1)$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
(C) The interval of convergence is $[-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
(D) The interval of convergence is $(-1, 1]$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
(E) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1)$ for $p > 1$.

Answer: Option (C)

Explanation: The radius of convergence is 1 for obvious reasons. Convergence at the boundary point -1 follows from the alternating series theorem. At the boundary point 1, we get a p-series, which converges if and only if $p > 1$.

Performance review: 28 out of 40 got this. 5 each chose (A) and (D), 1 chose (B), and 1 left the question blank.

Historical note (last year): 5 out of 11 got this correct. 3 chose (A), 2 chose (D), 1 chose (B).

Historical note (two years ago): 4 out of 26 people got this correct. 10 chose (B), 7 chose (A), 3 chose (E), 2 chose (D).

(4) Which of the following functions of x has a power series $\sum_{k=0}^{\infty} x^{4k}/(4k)!$? *Two years ago: 9/26 correct*

(A) $(\sin x + \sinh x)/2$
(B) $(\sin x - \sinh x)/2$
(C) $(\sinh x - \sin x)/2$

(D) $(\cosh x + \cos x)/2$
(E) $(\cosh x - \cos x)/2$

Answer: Option (D)

Explanation: Take the Taylor series and add. Also, use that both cosh and cos are globally analytic.

Performance review: 35 out of 40 got this. 3 chose (A), 1 each chose (B) and (C).

Historical note (last year): 5 out of 11 got this correct. 3 chose (C), 2 chose (A), 1 chose (B).

Historical note (two years ago): 9 out of 26 people got this correct. 5 chose (B), 4 each chose (A), (C), and (E).

(5) What is the sum $\sum_{k=0}^{\infty}(-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$. *Two years ago:* 12/26 *correct*
(A) $\cos x$
(B) $\sin x$
(C) $\cos(x^2)$
(D) $\cosh(x^2)$
(E) $\exp(-x^2)$

Answer: Option (E)

Explanation: Put $u = -x^2$, and we get $\sum_{k=0}^{\infty} u^k/k!$.

Performance review: 35 out of 40 got this. 5 chose (C).

Historical note (last year): 1 out of 11 got this correct. 7 chose (C), 2 chose (D), 1 chose (B).

Historical note (two years ago): 12 out of 26 people got this correct. 7 chose (C), 3 each chose (A) and (D), 1 chose (B).

(6) Define an operator R from the set of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b, $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$?
(A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
(B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
(C) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
(D) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
(E) $R(a+b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Answer: Option (E)

Explanation: $R(a + b) \geq \min\{R(a), R(b)\}$ because if both a and b converge, so does $a + b$. Let $c = a + b$. We also then get that $R(a) \geq \min\{R(b), R(c)\}$ and $R(b) \geq \min\{R(a), R(c)\}$ because $a = c - b$ and $b = c - a$.

Juggling these possibilities, we find that of the three numbers $R(a)$, $R(b)$, and $R(a+b)$, the smaller two of the three numbers must be equal. This forces option (E).

This is a type of hyperbolic geometry – all "triangles" must be isosceles.

Performance review: 29 out of 40 got this. 5 chose (D), 3 each chose (B) and (C).

Historical note (last year): 3 out of 11 got this correct. 4 chose (B) 3 chose (D), 1 chose (C).

Historical note (two years ago): 3 out of 26 people got this correct. 11 chose (C), 7 chose (D), 3 chose (A), 1 chose (B).

(7) Which of the following is/are true? *Two years ago:* 5/26 *correct*
(A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on $\mathbb{R}$.
(B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence (which may not be all of $\mathbb{R}$).
(C) If we start with a power series about 0 that converges everywhere in $\mathbb{R}$, then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
(D) All of the above.

(E) None of the above.

Answer: Option (C)

Explanation: See the lecture notes. A counterexample to (A) is arctan, and a counterexample to (B) is e^{-1/x^2}.

Performance review: 29 out of 40 got this. 6 chose (D), 4 chose (B), 1 chose (A).

Historical note (last year): 9 out of 11 got this correct. 1 chose (B) and 1 chose (E).

Historical note (two years ago): 5 out of 26 people got this correct. 10 chose (B), 5 each chose (D) and (E), and 1 chose (A).

(8) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \to \infty$? *Two years ago:* 12/26 *correct*

(A) f grows polynomially in x.

(B) f grows faster than any polynomial function but slower than any exponential function of x (i.e., any function of the form $x \mapsto e^{mx}, m > 0$).

(C) f grows like an exponential function of x, i.e., it can be sandwiched between two exponentially growing functions of x.

(D) f grows faster than any exponential function but slower than any doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.

(E) f grows like a doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.

Answer: Option (B)

Explanation: Note that *any* power series with infinitely many positive coefficients (and no negative coefficients) must grow faster than a polynomial, which, after all, has finite degree. Note that this logic does not work for power series that have a mix of positive and negative coefficients, such as the power series for the sin and cos functions.

The rough reason that growth is strictly slower than an exponential function is that the denominators are growing much faster than $k!$. Recall that if the denominators are $k!$, we get precisely the exponential function. This can be made more precise.

Performance review: 30 out of 40 got this. 5 chose (A), 3 chose (C), 1 each chose (D) and (E).

Historical note (last year): 5 out of 11 got this correct. 4 chose (D), 1 each chose (A) and (C).

Historical note (two years ago): 12 out of 26 people got this correct. 6 chose (D), 4 chose (C), 2 each chose (A) and (E).

(9) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$? *Two years ago:* 12/26 *correct*

(A) $\exp(x)$ from below and $\cosh(2x)$ from above.

(B) $\exp(x)$ from below and $\cosh(x^2)$ from above.

(C) $\exp(x/2)$ from below and $\exp(x)$ from above.

(D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.

(E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Answer: Option (D)

Explanation: We use the fact that:

$$k! \leq (k!)^2 \leq (2k)!$$

for all $k \geq 0$, with both inequalities strict if $k \geq 2$.

We thus get:

$$\frac{x^k}{k!} \geq \frac{x^k}{(k!)^2} \geq \frac{x^k}{(2k)!}$$

for $x > 0$, with both inequalities strict if $k \geq 2$. Summing up, we get:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} > \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} > \sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$$

The left most expression is e^x. For the right most expression, put $u = \sqrt{x}$, and we get $\cosh u$, so $\cosh \sqrt{x}$. Thus option (D) is the right choice.

Performance review: 28 out of 40 got this. 7 chose (C), 4 chose (B), 1 chose (A).

Historical note (last year): 1 out of 11 got this correct. 6 chose (C) and 4 chose (B).

Historical note (two years ago): 12 out of 26 people got this correct. 5 each chose (A) and (C), 3 chose (B), and 1 chose (E).

(10) Consider the function $f(x) := \max\{0, x\}$. What can we say about the Taylor series of f at various points?

(A) The Taylor series of f at any point is the zero series.

(B) The Taylor series of f at any point simplifies to x.

(C) The Taylor series of f at any point other than zero converges to f globally. However, the Taylor series is not defined at 0.

(D) The Taylor series of f at any point is either the zero series or simplifies to x.

(E) The Taylor series of f at any point other than the point 0 is either the zero series or simplifies to x. However, the Taylor series is not defined at 0.

Answer: Option (E)

Explanation: A piecewise description of f is:

$$f(x) := \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

The Taylor series is not defined at 0. The reason is that the function is not differentiable at 0, because the left hand derivative $f'_-(0)$ is 0 and the right hand derivative $f'_+(0)$ is 1.

At any other point, the Taylor series corresponds globally to the piece function for that point. So, the Taylor series at any positive real number is just the x function and the Taylor series at any negative real number is just the 0 function. This ties in with the idea that the Taylor series is completely determined by the local behavior of the function and cannot see changes in the function definition far from the point.

Performance review: 26 out of 40 got this. 6 chose (C), 4 chose (D), 3 chose (A), 1 chose (B).