HOMEWORK 1 CHECKLIST: DUE MONDAY OCTOBER 7

MATH 196, SECTION 57 (VIPUL NAIK)

REMINDER: Please submit routine and advanced problems *separately* on the due date of the homework. A grader (not me!) is assigned to grade the routine problems. I'll be grading the advanced problems myself.

The bulk of your homework will be the routine problems and these should be your first priority. There may be weeks where the homework does not contain any advanced problems.

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 1.1.8 (Page 5): Find all solutions of the linear system using elimination as discussed in Section 1.1. Then check your solutions.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 10z &= 0 \end{aligned}$$

Checklist hint begins: A system of linear equations where all the constants are 0 is termed a *homogeneous system of linear equations*. One thing to remember about homogeneous systems is that the "all variables equal zero" solution is always a solution to the system. If the system is precisely determined, this should be the unique solution.

In our case, $x = 0, y = 0, z = 0$ is a solution to the system. However, we do not *a priori* know whether it is the only solution. There may be some redundancy in the system, in which case there would be other solutions. It may be possible, for instance, to have a "line's" worth of solutions if one of the equations is deducible from the other two.

(2) Exercise 1.1.10 (Page 5): Find all solutions of the linear system using elimination as discussed in Section 1.1. Then check your solutions.

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 7z &= 2 \\ 3x + 7y + 11z &= 8 \end{aligned}$$

Checklist hint begins: Solve this using the usual methods of elimination of variables.

Please remember that, with each transformation that you do, you must not throw out the information present in your original equations.

(3) Exercise 1.1.17 (Page 5): Find all solutions of the linear system:

$$\begin{aligned} x + 2y &= a \\ 3x + 5y &= b \end{aligned}$$

where a and b are arbitrary constants.

Checklist hint begins: Solve this using the usual methods. Your final answer should describe x and y both as expressions (linear, as it will turn out) in terms of a and b.

(4) Exercise 1.1.26 (Page 6) (was Exercise 1.1.22 in the 4th edition): Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t)$$

The equation could described a forced damped oscillator, as we will see in Chapter 9 (well, we actually *won't* see it in the course, but you are welcome to read it up). We are told that the differential equation has a solution of the form

$$x(t) = a\sin(t) + b\cos(t)$$

Find a and b, and graph the solution.

Checklist hint begins: Calculate dx/dt and d^2x/dt^2 in terms of a, b, and t. Now, plug these expressions into the differential equation. Now, bring everything to one side and simplify to get:

$$(\text{Some linear expression in } a \text{ and } b)\sin(t) + (\text{Some other linear expression in } a \text{ and } b)\cos(t) = 0$$

This equation holds *identically* in t, i.e., it holds for all values of t.

We now wish to argue that the coefficients of $\cos(t)$ and $\sin(t)$ are both zero: If we set $t = 0$, then $\sin t = 0$ and $\cos t = 1$. Plugging into the equation gives that the coefficient of $\cos t$ is 0. Similarly, if we set $t = \pi/2$, then $\cos t = 0$ and $\sin t = 1$. Plugging into the equation gives that the coefficient of $\sin t$ is 0.

We thus get a system of two simultaneous linear equations in the two variables a and b. Solve to get the values of a and b.

To graph the function, use the fact that any expression of the form $a\sin(t) + b\cos(t)$ can be written in the form $k\sin(t+\theta)$ for suitable constants k and θ based on some simple trigonometric manipulation. Thus, we will get a sinusoidal curve.

Additional note: Plugging in other values of t would give other linear equations in a and b, but these would be redundant relative to the two equations we have already obtained. To see this, note that once we already have that the coefficient of $\cos t$ is 0 and the coefficient of $\sin t$ is 0, the identity holds for all t, so additional values of t do not provide additional information.

(5) Exercise 1.1.20 (Page 5) (was Exercise 1.1.25 in the 4th Edition): Consider the linear system

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (k^2 - 5)z &= k \end{aligned}$$

where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does this system have infinitely many solutions? For which value(s) of k is the system inconsistent?

Checklist hint begins: Consider the difference of the first and third equation. This gives an equation in z. The nature of the equation depends on the value of k. Note that this equation can be kept alongside the first two equations, and the original third equation can be discarded, to give an equivalent system with precisely the same solutions.

There is a value of k for which the left side of the equation becomes 0 and the right side is nonzero, hence the equation has no solution for z. Thus, for that value of z, there are no solutions.

There is a value of k for which the left side and right side are both zero, regardless of the value of z. Thus, all z solve this equation. We now need to reason that for each such value of z, the system has unique solutions for x and y. Thus, we get a totality of infinitely many solutions.

For values of k other than these two, there is a unique value of z dependent on k, and plugging this into the other equations gives a unique value of x and a unique value of y, all in terms of k.

(6) Exercise 1.1.34 (Page 7) (was Exercise 1.1.32 in the 4th Edition): Find all the polynomials $f(t)$ of degree ≤ 2 whose graph runs through the points $(1,1)$ and $(2,0)$ and such that $\int_1^2 f(t)\,dt = -1$.

Checklist hint begins: This is a problem type where we need to determine the parameters of a function from data including (input,output)-pairs. First, we describe the general form of the function: $f(t) := at^2 + bt + c$. The goal is now to find a, b, and c. Each (input,output)-pair gives one equation. We thus get two equations this way in the three variables. The integration condition gives the third equation.

Since we have three equations in three variables, and these are all linear, the system is probably precisely determined and there should be a unique solution. But we won't be sure until we solve the system to find a, b, and c. Remember to plug back in and get the function.

(7) Exercise 1.1.40 (Page 7) (was Exercise 1.1.39 in the 4th Edition): Find the ellipse centered at the origin that runs through the points $(1,2)$, $(2,2)$, and $(3,1)$. Write your equation in the form $ax^2 + bxy + cy^2 = 1$.

Checklist hint begins: Similar remarks as for the preceding problem. Note that we just have relational pairs instead of (input,output)-pairs.

(8) Exercise 1.1.42a (Page 7) (was Exercise 1.1.40a in the 4th Edition): Solve the lower triangular system:

$$\begin{aligned} x_1 &= -3 \\ -3x_1 + x_2 &= 14 \\ x_1 + 2x_2 + x_3 &= 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 &= 33 \end{aligned}$$

by forward substituting, finding x_1 first, then x_2, then x_3, and finally x_4.

Checklist hint begins: Just do what's asked. Note that at each stage, the equation is a linear equation in one variable once we substitute the values of the preceding variables.

Note also that in each case, since the coefficient on the variable that we are solving for is 1, and all the other coefficients and the constant terms are integers, all variables have integer values in our solution.

(9) Exercise 1.1.48 (Page 8): A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrate per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates (per serving).

(a) If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?

(b) If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how many servings of each item should he consume?

Checklist hint begins: Your two variables are the respective numbers of servings. Form a system with two variables and two equations. Solve. If all goes well, you should get nonnegative values for the numbers (also, hopefully, the values should be integers).

What if one of the values is negative? That would mean that book foods are too heavily skewed in their nutrient mix to offer the mix the hermit needs (that doesn't happen in part (a)).

If the values are positive but non-integers, that suggests that the hermit must use partial servings to meet his needs (again, that doesn't happen in part (a)).

(10) Exercise 1.1.49 (Page 8) (was Exercise 1.1.47 in the 4th edition): I have 32 bills in my wallet, in the denominations of US$ 1, 5, and 10, worth $ 100 in total. How many do I have of each denomination?

Checklist hint begins: Suppose x, y, and z denote the number of bills with denominations US $ 1, 5, 10 respectively. We have the equations:

$$\begin{aligned} x + y + z &= 32 \\ x + 5y + 10z &= 100 \end{aligned}$$

We wish to solve this system for x, y, and z. As a system of equations in real variables, the system is underdetermined, since there are three variables and only two equations.

However, there is another constraint arising from the context: x, y, and z are all positive integers. We wish to use this constraint.

Manipulate the system to obtain that $y = 17 - (9z/4)$. Thus, z is a multiple of 4. Further, size constraints will force $z = 4$ (larger values of z produce negative values of y). Plug in and simplify to get the values of x and y.

Note that although we get a unique solution in this case, that is specific to the setup here. Underdetermined linear Diophantine systems (the general name for systems of linear equations with the variables constrained to be integers or something similar) can have numerous solutions. The solutions are typically computed similarly to those for the reals, but with some constraints on the inputs in terms of size and remainders in order to guarantee the constraint that all variables are integer-valued.

2. Advanced problems

(1) Exercise 1.1.24 (Page 6) (was Exercise 1.1.20 in the 4th Edition): The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906-1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1000 and 780, in millions of dollars per year.

What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand? One complication is that some of the output of each industry needs to be diverted to the other industry to help it produce stuff (e.g., A may be producing electricity). Suppose that industry B needs 10 cents worth of electrical power for each \$ 1 of output that B produces and industry A needs 20 cents worth of B's outputs for each \$ 1 of output A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.

(The book has pictorial depictions that you may find useful).

You can read more at:

`http://en.wikipedia.org/wiki/Input-output_model`

Checklist hint begins: Let a be the output of A and b be the output of B. We set up equations:

$$\begin{aligned}
\text{Output of } A &= (\text{Consumer demand quantity for } A) + (\text{Quantity demanded of } A\text{'s output that provides input to } B) \\
\text{Output of } B &= (\text{Consumer demand quantity for } B) + (\text{Quantity demanded of } B\text{'s output that provides input to } A)
\end{aligned}$$

The first terms on both right sides are known. The second terms are expressed proportionally to the output of the other industry. Thus, the quantity demanded of A's output that provides input to B will be $0.1b$. Note that we use 0.1 rather than 0.2 here. The first equation thus becomes:

$$a = 1000 + 0.1b$$

Make sure you understand this clearly, then formulate the second equation similarly.

Simplify and solve the linear system in a and b. In your final answer, a should be greater than 1000 and b should be greater than 780, reflecting the fact that some part of the output of both industries is used to feed the other industry.

Additional observations: Note that part of the industries' outputs is feeding each other. A produces stuff to feed B, part of which is produced to again feed A, and so on. This mutually fed stuff essentially does not get seen by consumers. The larger the interindustry dependence, the larger the fraction of output produced by industries that is just "for each other" rather than for consumers. In the limiting case of interindustry dependence values of 1, the industries cannot satisfy consumer demand regardless of how much they produce.