CLASS QUIZ SOLUTIONS: OCTOBER 28: INTEGRATION BASICS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

This quiz actually happened on November 4.
12 people took this quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 2 people
- Score of 2: 4 people
- Score of 3: 2 people
- Score of 4: 1 person

The mean score was 1.67.
Here are the problem-wise answers and performance:

(1) Option (C): 4 people
(2) Option (D): 7 people
(3) Option (D): 4 people
(4) Option (A): 5 people

2. Solutions

(1) Consider the function(s) $[0,1] \to \mathbb{R}$. **Identify the function**s for which the integral (using upper sums and lower sums) is not defined.

(A) $f_1(x) := \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1 \end{cases}$.

(B) $f_2(x) := \begin{cases} 0, & x \ne 0 \text{ and } 1/x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$.

(C) $f_3(x) := \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational} \end{cases}$

(D) All of the above

(E) None of the above

Answer: Option (C)

Explanation: For option (C), the lower sum for any partition is 0 and the upper sum is 1. Thus, the integral is not well-defined.

For option (A), the function is piecewise continuous with only jump discontinuities, hence the integral is well-defined: in fact, it is 1/2.

For (B), the integral is zero. We can see this by noting that the points where the function is 0 are all isolated points, so if in our partition the intervals surrounding each of these points is small enough, we can make the upper sums tend to zero. (This is hard to see. You should, however, be able to easily see that (A) has an integral and (C) does not. This forces the answer to be (C)).

Performance review: 4 out of 12 got this correct. 3 each chose (B) and (D), 2 chose (E).

Historical note (last year): Everybody got this correct.

(2) (**) Suppose $a < b$. Recall that a *regular partition* into n parts of $[a,b]$ is a partition $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ where $x_i - x_{i-1} = (b-a)/n$ for all $1 \le i \le n$. A partition P_1 is said to be a *finer partition* than a partition P_2 if the set of points of P_1 contains the set of points of P_2. Which of the following is a **necessary and sufficient condition** for the regular partition into m parts to be a *finer partition* than the regular partition into n parts? (Note: We'll assume that any partition is finer than itself).

(A) $m \leq n$
(B) $n \leq m$
(C) m divides n (i.e., n is a multiple of m)
(D) n divides m (i.e., m is a multiple of n)
(E) m is a power of n

Answer: Option (D)

Explanation: If n divides m, then the partition into m pieces is obtained by further subdividing the partition into m parts, with each part divided into n/m parts.

The other choices: Option (B) is a *necessary* condition but is not a sufficient condition. For instance, the regular partition of $[0, 1]$ into two parts corresponds to $\{0, 1/2, 1\}$ and the partition into three parts corresponds to $\{0, 1/3, 2/3, 1\}$. These partitions are incomparable, i.e., neither is finer than the other.

Performance review: 7 out of 12 got this correct. 4 chose(C), 1 chose (B).

Historical note (last year): 5 out of 15 people got this correct. 6 people chose (B), which is necessary but not sufficient, as indicated above. 2 people chose (C), which is the reverse option. 1 person chose (A) and 1 chose (A)+(D).

Action point: If you chose option (B), please make sure you understand the distinction between the options. Also review the concept of regular partitions and finer partition till you find this question obvious.

(3) (**) For a partition $P = x_0 < x_1 < x_2 < \cdots < x_n$ of $[a, b]$ (with $x_0 = a$, $x_n = b$) define the norm $\|P\|$ as the maximum of the values $x_i - x_{i-1}$. Which of the following **is always true** for any continuous function f on $[a, b]$?

(A) If P_1 is a finer partition than P_2, then $\|P_2\| \leq \|P_1\|$ (Here, *finer* means that, as a set, $P_2 \subseteq P_1$, i.e., all the points of P_2 are also points of P_1).
(B) If $\|P_2\| \leq \|P_1\|$, then $L_f(P_2) \leq L_f(P_1)$ (where L_f is the lower sum).
(C) If $\|P_2\| \leq \|P_1\|$, then $U_f(P_2) \leq U_f(P_1)$ (where U_f is the upper sum).
(D) If $\|P_2\| \leq \|P_1\|$, then $L_f(P_2) \leq U_f(P_1)$.
(E) All of the above.

Answer: Option (D).

Explanation: Option (D) is true for the rather trivial reason that any lower sum of f over any partition cannot be more than any upper sum of f over any partition. The norm plays no role.

Option (A) is incorrect because the inequality actually goes the other way: the finer partition has the smaller norm. Options (B) and (C) are incorrect because a smaller norm does not, in and of itself, guarantee anything about how the lower and upper sums compare.

Performance review: 4 out of 12 got this correct. 4 chose (A), 2 chose (B), 1 chose (C), 1 chose (E).

Historical note (last year): 4 out of 15 people got this correct. 8 people chose (C), presumably with the intuition that the smaller the norm of a partition, the smaller its upper sums. While this intuition is right in a broad sense, it is not correct in the precise sense that would make (C) correct. It is possible that a lot of people did not read (D) carefully, and stopped after seeing (C), which they thought was a correct statement. 1 person each chose (A), (E), and (C)+(D).

Historical note (two years ago): This question appeared in a 152 midterm two years ago, and 6 of 29 people got this right. Many people chose (C) in that test too (though I haven't preserved numerical information on number of wrong choices selected). *History repeats itself, despite my best efforts!* On the plus side, though, this is just a quiz after your first college exposure to the material, as opposed to what was a midterm last year, which happened after homeworks and a midterm review session.

Action point: Please make sure you understand very clearly the relation between the "finer" partition notion, norms of partitions, upper sums, and lower sums, to the point where you wondered how you could have ever got this question wrong.

(4) (**) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**?

(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 5 out of 12 got this correct. 3 chose (D), 2 chose (C), 2 chose (E).

Historical note (last year): Nobody got this correct. 14 people chose (E) and 1 person chose (B).

Action point: This was a devilish question, because answering it correctly requires a knowledge of (or at least intuition about) ideas that you have not yet encountered. Nonetheless, it is a question whose solution you should be able to understand after having read it. Please make sure you understand why (A) is correct, i.e., how the set of integers is different from the sets in options (B), (C), and (D).