TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY OCTOBER 11: LINEAR SYSTEMS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

29 people took this 10-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 3: 2 people
- Score of 4: 2 people
- Score of 5: 4 people
- Score of 6: 3 people
- Score of 7: 7 people
- Score of 8: 3 people
- Score of 9: 3 people
- Score of 10: 4 people

The mean score was 6.69.

The question-wise answers and performance review were as follows:

(1) Option (B): 12 people
(2) Option (A): 20 people
(3) Option (B): 17 people
(4) Option (C): 15 people
(5) Option (D): 19 people
(6) Option (D): 19 people
(7) Option (B): 27 people
(8) Option (A): 23 people
(9) Option (C): 22 people
(10) Option (B): 20 people

2. Solutions

The quiz questions here, although not hard *per se*, are conceptually demanding: answering them requires a clear understanding of multiple concepts and an ability to execute them conjunctively. Even if you feel that you've understood the material as presented in class, you will need to think through each question carefully. Some of the questions are related to similar homework problems (Homeworks 1 and 2), and they test a conceptual understanding of the solutions to these problems. You might want to view them in conjunction with the homework problems. Other questions sow the seeds of ideas we will explore later. The quiz should seem relatively easier when you review it later, assuming that you work hard on attempting the questions right now and read the solutions once they're put up.

(1) (*) Rashid and Riena are trying to study a function f of two variables x and y. Rashid is convinced that the function is linear (i.e., it is of the form $f(x, y) := ax + by + c$) but has no idea what a, b, and c are. Riena thinks a linear model is completely out-of-place. Rashid is eager to find a, b, and c, whereas Riena is eager to disprove Rashid's linear model. Unfortunately, all they have is a black box that will output the value of the function for a given input pair (x, y), and that black box can only be called three times. What should Rashid and Riena try for?
   (A) Rashid and Riena would both like to provide three input pairs that are non-collinear as points in the xy-plane

(B) Rashid would like to provide three input pairs that are non-collinear, while Riena would like to provide three input pairs that are collinear as points in the xy-plane.
(C) Rashid and Riena would both like to provide three input pairs that are collinear as points in the xy-plane.
(D) Rashid would like to provide three input pairs that are collinear, while Riena would like to provide three input pairs that are non-collinear as points in the xy-plane.
(E) Both Rashid and Riena are indifferent regarding how the three input pairs are picked.

Answer: Option (B)

Explanation: Rashid wants to pick three independent constraints, i.e., pick three linear equations that are independent of each other. This would give him three independent equations in three variables, allowing him to uniquely determine the values of the parameters a, b, and c. Thus, he wishes to pick three non-collinear points. If he picked three collinear points, one of his equations would be redundant (deducible from the other two).

Riena, on the other hand, wants to pick three points with some redundancy between them. If she picks three collinear points, then, assuming Rashid's model is correct, the values of the outputs at two of the three points determine the value of the output at the third. If it turns out that the actual value differs from the predicted value, this *falsifies* Rashid's model.

The reason Riena wants them to be collinear is thus the same as the reason Rashid wants them to be non-collinear. Rashid already thinks he knows the predicted result from the collinear case, so he is not interested in it. Riena wants to show Rashid that his prediction is wrong, so she cares about it. The non-collinear case would not be interesting to Riena because it offers no evidence against Rashid.

Note that if 4 or more inputs can be queried, then both Rashid's and Riena's concerns can be addressed. *Scarcity breeds conflict.*

More about this is discussed in the notes on `Hypothesis testing, rank, and overdetermination` that we intend to discuss next week.

Not clear to you?: Pick some function of two variables that is not linear, and put yourself in Riena's shoes. How would you convinced Rashid that the function is not linear? What types of inputs would you choose? Similarly, put yourself in Rashid's shoes and try to determine what types of inputs will help you determine uniqueness.

Performance review: 12 out of 29 got this. 7 each chose (A) and (D), 2 chose (E), 1 chose (C).

Historical note (last time): 18 out of 29 got this. 6 chose (D), 4 chose (C), 1 chose (A).

(2) (*) Let m and n be natural numbers with $m \geq 3$. We are given a bunch of numbers $x_1 < x_2 < \cdots < x_m$ and another bunch of numbers $y_1, y_2, \ldots, y_m$. We want to find a continuous function f on $[x_1, x_m]$, such that $f(x_i) = y_i$ for all $1 \leq i \leq m$, and such that the restriction of f to any interval of the form $[x_i, x_{i+1}]$ (for $1 \leq i \leq m-1$) is a polynomial of degree $\leq n$. What is the smallest value of n for which we are guaranteed to be able to find such a function f?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (A)

Explanation: The brief explanation is that we can construct a piecewise linear function: a function whose graph comprises straight line segments joining (x_i, y_i) to (x_{i+1}, y_{i+1}).

The more sophisticated explanation is that for each pair of adjacent points, we have to choose a functional form between the two points, and there are two input-output pairs constraining the functional form (namely, the two endpoints). We thus need a functional form with two parameters. The linear functional form works well for that purpose. If we were to count the *total* number of parameters, it would work out to $2(m-1)$ (2 parameters for each of the $m-1$ intervals (x_1, x_2), (x_2, x_3), $\ldots$, (x_{m-1}, x_m)). The number of equations would also work out to $2(m-1)$. The number of parameters equals the number of equations. However, the equality of the number of parameters and the number of equations in the abstract is not *completely sufficient* to argue that we can always find a solution, because of concerns about redundant equations. In this case, it is sufficient, based on the explanation offered earlier. It is still a good working heuristic, though.

Note on subtle difference in goals from previous situations: The situation of this question, and the next three questions, is different in a subtle but important way from earlier situations. Broadly, consider two types of situations.

(a) *Prediction/forecasting/extrapolation situation*: The situation where we are trying to model some pre-existing phenomenon and we have a model with a general functional form (involving not-yet-known parameters) that we believe applies to the phenomenon. We are trying to find the parameters using input-output pairs. In this case, we are trying to find information about a function that, in some sense, already exists. This means that out job isn't just to find *some function that fits the known data*, but to find *the right function*. In this situation, having more input-output pairs (and hence equations) than parameters (that become our new variables) is desirable.

(b) *Engineering solution situation*: The situation where our goal is just to find some function that fits the existing data, rather than to find a particular function. In this case, having more variables than equations, resulting in the system being underdetermined (i.e., the solution being non-unique and having degrees of freedom) is a *good* thing.

The type of situation we are dealing with in these questions is type (b). We are interested in just finding some function satisfying certain constraints, not in finding a specific function that already exists.

Note that all situations that involve prediction and trend forecasting fall under (a) rather than (b). Situation (b) arises in cases where we are solving specific engineering problems to come up with one-time constructs that satisfy constraints (such as the roller-coaster ride example in the book). From the social science perspective, the more common situation is situation (a), where we are trying to predict or forecast for an existing model that we do not yet fully understand.

Not clear to you?: Make a picture with points marked for the values of the function at $x_1, x_2, \dots, x_m$. Now notice that we can make a graph that uses a straight line segment for each interval between adjacent x_is.

Performance review: 20 out of 29 people got this. 8 chose (B), 1 chose (C).

Historical note (last time): 17 out of 29 got this. 10 chose (C), 2 chose (D).

(3) (*) Let m and n be natural numbers with $m \geq 3$. We are given a bunch of numbers $x_1 < x_2 < \dots < x_m$ and another bunch of numbers $y_1, y_2, \dots, y_m$. We want to find a continuous function f on $[x_1, x_m]$, such that $f(x_i) = y_i$ for all $1 \leq i \leq m$, and such that the restriction of f to any interval of the form $[x_i, x_{i+1}]$ (for $1 \leq i \leq m-1$) is a polynomial of degree $\leq n$. In addition, we want to make sure that f is differentiable on the open interval (x_1, x_m). What is the smallest value of n for which we are guaranteed to be able to find such a function f?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (B)

Explanation: We can take the first piece to be linear (simply join the points (x_1, y_1) and (x_2, y_2)). For each subsequent piece, we have three constraints. For the i^{th} piece, the constraints include the value at x_i, the value at x_{i+1}, and the derivative at x_i (which must equal the corresponding derivative from the preceding piece). In order to be able to fit all three constraints, we need to use $n = 2$ so as to have three parameters. It is also straightforward to check that the system we get this way is not redundant.

Here is the accounting regarding parameters and equations. The number of equations is $2(m-1) + (m-2)$. The $2(m-1)$ equations arise from the values at the endpoints of each of the $m-1$ intervals. The extra $m-2$ equations arise from the equality of the formal expressions for the left hand derivative and right hand derivative at each of the $m-2$ transition points $x_2, x_3, \dots, x_{m-1}$. The total number of equations is therefore $3m-4$. Whatever value of n we choose, the total number of parameters is $(n+1)(m-1)$ (we need $n+1$ parameters for the piece definition on each of the $m-1$ intervals (x_1, x_2), (x_2, x_3), $\dots$, (x_{m-1}, x_m)). We want to choose n such that the number of parameters is greater than or equal to the number of equations, and the smallest value that works is $n = 2$, in which case we get $(n+1)(m-1) = 3m-3 \ge 3m-4$. Note that we have one extra parameter over the number of equations, so we have a bit of "slack" here, at least in principle. Again, the crude comparison of parameters and equations is not conclusive because of the potential for redundancy, so we need an explanation of the sort offered in the preceding paragraph to be sure. Note that the slight excess of parameters over equations corresponds to the fact that for the piece definition in the very first interval (x_1, x_2), we have some flexibility.

Note on subtle difference in goals from previous situations: See the note at the end of the answer to Question 2.

Not clear to you?: Make a picture similar to the preceding question, but notice now that from the second interval onward, you have a constraint on the initial slope.

Performance review: 17 out of 29 got this. 10 chose (C), 2 chose (A).

Historical note (last time): 25 out of 29 got this. 2 chose (C), 1 each chose (A) and (D).

(4) (*) Let m and n be natural numbers with $m \ge 3$. We are given a bunch of numbers $x_1 < x_2 < \dots < x_m$ and another bunch of numbers $y_1, y_2, \dots, y_m$. We want to find a continuous function f on $[x_1, x_m]$, such that $f(x_i) = y_i$ for all $1 \le i \le m$, and such that the restriction of f to any interval of the form $[x_i, x_{i+1}]$ (for $1 \le i \le m-1$) is a polynomial of degree $\le n$. In addition, we want to make sure that f is differentiable on the open interval (x_1, x_m). In addition, we are told the value of the right hand derivative of f at x_1 and the left hand derivative of f at x_m. What is the smallest value of n for which we are guaranteed to be able to find such a function f?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (C)

Explanation: This is similar to the cubic spline question on your advanced homework.

The key reason why this differs from the preceding question is that, here, we have a specification of the one-sided derivatives at the endpoint. The specification at x_1 constrains how we start, but that is stil acceptable. We'll need something quadratic to begin with, but we can keep choosing quadratics all the way, since at each piece we need to satisfy three conditions. The problem occurs in the final stage, i.e., the interval $[x_{m-1}, x_m]$. Here, we have *four* constraints: the values at the endpoints, and the derivatives at both endpoints. So, we need a functional form with four parameters. In other words, we need a cubic for the last piece.

It would be acceptable to use quadratics for all except the last piece, and a cubic for the last piece. The options available to us, however, require us to declare a single degree that works for all, so the smallest available is 3.

Using the accounting for parameters and equations similar to the preceding question, we obtain $3m-2$ equations (the $3m-4$ equations of the preceding question, plus 2 additional equations for the one-sided derivative conditions at the endpoints). We therefore need to choose n such that $(n+1)(m-1) \ge 3m-2$. The value $n = 2$ *just* falls short: it would give $3m-3$ parameters, and this corresponds to the fact that if we tried to fit a quadratic, we could run into trouble in the very last piece. The value $n = 3$, on the other hand, offers us considerable slack.

Note on subtle difference in goals from previous situations: See the note at the end of the answer to Question 2.

Performance review: 15 out of 29 got this. 7 chose (B), 3 chose (A), 2 chose (D), 1 chose (E), 1 left the question blank.

Historical note (last time): 11 out of 29 got this. 10 chose (B), 6 chose (D), 2 chose (A).

(5) (*) Let k, m, and n be natural numbers with $m \ge 3$. We are given a bunch of numbers $x_1 < x_2 < \dots < x_m$ and another bunch of numbers $y_1, y_2, \dots, y_m$. We want to find a continuous function f on $[x_1, x_m]$, such that $f(x_i) = y_i$ for all $1 \le i \le m$, and such that the restriction of f to any interval of

the form $[x_i, x_{i+1}]$ (for $1 \le i \le m-1$) is a polynomial of degree $\le n$. In addition, we want to make sure that f is at least k times differentiable on the open interval (x_1, x_m). What is the smallest value of n for which we are guaranteed to be able to find such a function f?

(A) $k-2$
(B) $k-1$
(C) k
(D) $k+1$
(E) $k+2$

Answer: Option (D)

Explanation: For the part $[x_1, x_2]$, we can simply fit a straight line. For each subsequent piece, we have $k+1$ constraints: the left and right endpoint values, and all the derivative values up to the k^{th} derivative at the left endpoint. This gives a total of $k+2$ constraints. A polynomial of degree $k+1$ has the necessary number of parameters, namely $k+2$. We can also use Taylor polynomials to see that the system can always be solved.

If we account for the total number of parameters and equations, the numbers we get are as follows. We have $2(m-1)+k(m-2)$ equations. The $2(m-1)$ equations are from the values at the endpoins of the $m-1$ intervals. The $k(m-2)$ equations are from the equality of the first k derivatives of the piece functions at the transition points $x_2, x_3, \dots, x_{m-1}$. The total number of equations is $(k+2)(m-1)-k$. We have $(n+1)(m-1)$ parameters. Therefore, choosing $n = k+1$ gives more parameters than equations. On the other hand, choosing $n = k$ gives $(k+1)(m-1)$ parameters, so that the number of parameters - the number of equations is $k-(m-1)$. Since we are not given any concrete information about the sign relationship between k and $m-1$, we cannot be sure that this will work.

Note on subtle difference in goals from previous situations: See the note at the end of the answer to Question 2.

Performance review: 19 out of 29 got this. 5 chose (B), 3 chose (E), 2 chose (C).

Historical note (last time): 12 out of 29 got this. 14 chose (B), 2 chose (C), 1 chose (A).

The next few questions are framed deterministically, though similar real-world applications would be probabilistic, with some square roots floating around. Unfortunately, we do not have the tools yet to deal with the probabilistic versions of the questions.

(6) (*) A function f of one variable is known to be linear. We know that $f(1) = 1.5 \pm 0.5$ and $f(2) = 2.5 \pm 0.5$. Assume these are the full ranges, without any probability distribution known. Assuming nothing is known about how the measurement errors for f at different points are related, what can we say about $f(3)$?

(A) $f(3) = 3.5$ (exactly)
(B) $f(3) = 3.5 \pm 0.5$
(C) $f(3) = 3.5 \pm 1$
(D) $f(3) = 3.5 \pm 1.5$
(E) $f(3) = 3.5 \pm 2.5$

Answer: Option (D)

Explanation: The highest value occurs if we take $f(1) = 1$ and $f(2) = 3$, giving $f(3) = 5$. The lowest value occurs if we take $f(1) = f(2) = 2$, giving $f(3) = 2$. Overall, $f(3)$ is between 2 and 5, so 3.5 ± 1.5 is a reasonable description.

Graphical interpretation: Make a picture of the coordinate xy-plane with a vertical line segment joining the points with coordinates $(1,1)$ and $(1,2)$ (depicting the range of possibilities for $f(1)$) and another line segment joining the points with coordinates $(2,2)$ and $(2,3)$ (depicting the range of possibilities for $f(2)$). The actual value of $f(1)$ could be anywhere on the first line segment and the actual value of $f(2)$ could be anywhere on the second line segment. The slowest growth case is the case where $f(1) = 2$ and $f(2) = 2$, so in fact we get a constant function $f(x) = 2$ in this case, and the graph of this is the line $y = 2$, and $f(3) = 2$ in this case.. The fastest growth case is the case $f(1) = 1$ and $f(2) = 3$, and we get the function $f(x) = 2x - 1$ in this case, and the graph is the line $y = 2x - 1$. We get $f(3) = 5$ in this case. Thus, $f(3)$ could be any value between 2 and 5, or equivalently, the range of values for $f(3)$ is given by the line segment in the xy-plane joining the points $(3, 2)$ and $(3, 5)$.

The picture might remind you of eclipses. The region in the "shadow" so to speak is the penumbral region of the eclipse.

Performance review: 19 out of 29 got this. 6 chose (B), 3 chose (C), 1 chose (A).

Historical note (last time): 25 out of 29 got this. 3 chose (C), 1 chose (B).

(7) (*) A function f of one variable is known to be linear. We know that $f(1) = 1.5 \pm 0.5$ and $f(2) = 2.5 \pm 0.5$. Assume these are the full ranges, without any probability distribution known. Assume also that the measurement error for f at all points is the same in magnitude and sign. What can we say about $f(3)$?

(A) $f(3) = 3.5$ (exactly)
(B) $f(3) = 3.5 \pm 0.5$
(C) $f(3) = 3.5 \pm 1$
(D) $f(3) = 3.5 \pm 1.5$
(E) $f(3) = 3.5 \pm 2.5$

Answer: Option (B)

Explanation: $f(1) = 1.5+m$ where m is measurement error with $|m| \leq 0.5$. Since all measurement errors are equal, $f(2) = 2.5 + m$. Thus, $f(3) = 3.5 + m$, with $|m| \leq 0.5$, giving the answer.

Graphical interpretation: The vertical line segments are the same as before: one joins $(1, 1)$ and $(1, 2)$, the other joins $(2, 2)$ and $(2, 3)$. However, since we now know that the errors are the same, the lower bounding line connects the lowest ends of both vertical line segments, and the upper bounding line connects the upper ends of both vertical line segments. In other words, we get a pair of parallel lines. The lower line passes through $(3, 3)$ and the upper line passes through $(3, 4)$. The range of possibilities for $f(3)$ is therefore the set of values between 3 and 4, i.e., it is 3.5 ± 0.5.

Performance review: 27 out of 29 got this. 2 chose (C).

Historical note (last time): 26 out of 29 got this. 2 chose (D), 1 chose (C).

(8) (*) Suppose f is a linear function on a bounded interval $[a, b]$ but our measurement of outputs for given inputs has some measurement error (with the range of measurement error the same regardless of the input, and no known correlation between the magnitude of measurement error at different points). Assume we can get the outputs for any two specified inputs we desire, and we will then fit a line through the (input,output) pairs to get the graph of f. How should we choose our inputs?

(A) Choose the inputs as far as possible from each other, i.e., choose them as a and b.
(B) Choose the inputs to be as close to each other as possible, i.e., choose them to be nearby points but not equal to each other.
(C) It does not matter. Any choice of two distinct inputs is good enough.

Answer: Option (A)

Explanation: If the inputs are chosen close together, then even small errors in the input can cause large errors in the measurement of the slope. The error in slope is the signed difference of measurement errors divided by the distance between the inputs. The former is beyond our control, because we noted that the magnitude and sign of measurement error does not depend on where we choose the inputs. Thus, choosing the inputs as far as possible brings us a larger denominator, and therefore keeps the slope error at a minimum.

Performance review: 23 out of 29 got this. 4 chose (B), 2 chose (C).

Historical note (last time): 11 out of 29 got this. 15 chose (C), 3 chose (B).

(9) (*) f is a function of one variable defined on an interval $[a, b]$. You are trying to find an explicit function that fits f well. You initially try a straight line fit that works at the points a and b. It turns out that this fit systematically overestimates f for points in between (i.e., the actual function f is below the linear function) with the maximum magnitude of discrepancy occurring at the midpoint $(a + b)/2$. Based on this information, what kind of fit should you try to look for?

(A) Try to fit f using a logarithmic function
(B) Try to fit f using an exponential function
(C) Try to fit f using a quadratic function
(D) Try to fit f using a polynomial of degree at most 3

(E) Try to fit f using the reciprocal of a linear function

Answer: Option (C)

Explanation: Let L be the linear function obtained and let $g = f - L$. g is zero at both endpoints a and b, below zero in between, and has its minimum at the midpoint. These characteristics strongly suggest that g is a quadratic function with positive leading coefficient. Any quadratic function with positive leading coefficient that has zeros has its absolute minimum precisely at the midpoint between its zeros.

Since g is expected to be quadratic, $f = g + L$ is also expected to be quadratic.

Note that we could try fitting using a polynomial of degree at most 3. This, however, might run us into overfitting problems. As a general rule, we should try to fit using a function with as few parameters as possible and where the functional form is justified by broad theoretical considerations. If after fitting the quadratic, we discover some systemic errors that are best explained by a cubic type of discrepancy, we could then try a cubic.

Performance review: 22 out of 29 got this. 3 each chose (B) and (D), 1 chose (A).

Historical note (last time): 25 out of 29 got this. 3 chose (B), 1 chose (D).

(10) (*) Recall the Leontief input-output model. Recall that the GDP is defined as the total money value of all the *final* goods and services produced in the economy, which in this case means only those that go into meeting consumer demand, not interindustry demand (note that we are assuming away the existence of investment and government spending, which complicate the GDP calculation). Assuming that the unit prices of the goods are constant (a very unrealistic assumption given that price itself responds to supply and demand, but fortunately it does not affect the conclusion we draw here) what might be a way of increasing GDP while keeping the magnitude of output of each industry the same?

(A) Increase interindustry dependence, i.e., increase the amount needed from each industry that is necessary to produce a given amount in another industry.

(B) Reduce interindustry dependence, i.e., reduce the amount needed from each industry that is necessary to produce a given amount in another industry.

(C) Changes in interindustry dependence have no effect.

Answer: Option (B)

Explanation: The lower the interindustry dependence, the larger the share of the industries' output that can be used to meet consumer demand, i.e., contribute to GDP.

Performance review: 20 out of 29 got this. 6 chose (C), 3 chose (A).

Historical note (last time): 26 out of 29 got this. 3 chose (C).