CLASS QUIZ SOLUTIONS: JANUARY 5: EXPONENTIAL GROWTH

MATH 153, SECTION 55 (VIPUL NAIK)

1. Performance review

29 people took this 6-question quiz. The scores ranged from 2 to 5 and the score distribution was as follows:

- Score of 2: 2 people.
- Score of 3: 2 people.
- Score of 4: 5 people.
- Score of 5: 20 people.

The median and modal scores were both 5. The mean score was 4.48.

Here is the question-wise performance (full solutions in the next section):

(1) Option (D): Everybody got this correct.
(2) Option (E): 4 people got this correct. *Make sure you read through the solution!*
(3) Option (B): 23 people got this correct.
(4) Option (E): 22 people got this correct.
(5) Option (E): 28 people got this correct.
(6) Option (C): 24 people got this correct.

2. Solutions

(1) A species of unicellular micro-organisms doubles in number every one hour at room temperature and remains constant when placed in a refrigerator. Given that the initial number of micro-organisms in a dish is N_0, and the dish is kept at room temperature for A hours and in a refrigerator for B hours, what is the **total number** of micro-organisms at **the end**? (5 points)
(A) $N_0 \cdot 2^{A-B}$
(B) $N_0 \cdot 2^{A+B}$
(C) $N_0 \cdot 2^{AB}$
(D) $N_0 \cdot 2^{A}$
(E) $N_0 \cdot 2^{B}$

Answer: Option (D)

Explanation: The hours in the refrigerator play no role in changing the number of micro-organisms, so all that matters is the hours spent at room temperature. For every hour spent, the number doubles, so the number is $N_0 \cdot 2^A$.

Post-performance review: All 29 students got this correct.

Historical note: A version of this question (without option (C), so there were only four options) appeared in last year's 152 final. 27 out of 30 students got it correct.

(2) The population in the island of Andrognesia as a function of time is believed to be an exponential function. On January 1, 1984, the population was measured to be $3 * 10^5$ with a measurement error of up to 10^5 on either side, i.e., the population was measured to be between $2 * 10^5$ and $4 * 10^5$. On January 1, 1998, the population was measured to be $1.2 * 10^6$ with a measurememt error of up to $4 * 10^5$ on either side, i.e., the population was measured to be between $8 * 10^5$ and $1.6 * 10^6$. If the population is an exponential function of time (i.e., the increment in population per year is a fixed proportion of the population that year), what is the **range of possible values** of the population measured on January 1, 2012?
(A) Between $3.2 * 10^6$ and $6.4 * 10^6$
(B) Between $3.2 * 10^6$ and $1.28 * 10^7$

(C) Between $1.6 * 10^6$ and $3.2 * 10^6$
(D) Between $1.6 * 10^6$ and $6.4 * 10^6$
(E) Between $1.6 * 10^6$ and $1.28 * 10^7$

Answer: Option (E)

Explanation: Note first that $2012 - 1998 = 1998 - 1984 = 14$.

The key idea is that the lowest estimate occurs if the 1998 population was measured as low as possible *and* the rate of population growth estimated using the 1984 and 1998 populations is as low as possible. The lowest possible rate of growth we can measure occurs if we choose the highest possible 1984 value and the lowest possible 1998 value. Picking these, we obtain that the population estimate for 1984 is $4 * 10^5$ and the population estimate for 1998 is $8 * 10^5$. Since the multiplicative growth of the population depends on the time elapsed, the total population in 2012 will be the solution x to:

$$\frac{x}{8 * 10^5} = \frac{8 * 10^5}{4 * 10^5}$$

which solves to $x = 1.6 * 10^6$.

Similarly, the highest estimate will occur if we take the highest estimate possible for the 1998 population and the lowest estimate possibly for the 1984 population.

Post-performance review: 4 out of 29 students got this correct. It is not clear by looking at the explanations given whether these people got it correct for the correct reasons. 22 people chose (A), which is the correct choice if you match the low estimates with each other and the high estimates with each other. 2 people chose (B) (possibly trying to get (A), but made a calculation error?) and 1 person left the question blank.

Action point: This is a *real-life error*! Making such an error in real life can be very costly to you (and to those whose lives depend upon your decisions), so internalize this concept. Please!

The key concept is that when one measurement depends on two others, the worst case scenario happens when the errors in the other two measurements are correlated in the *worst possible manner*.

Here's another example of this kind of error. Suppose a test measures student knowledge of calculus. The test is used at the beginning of a calculus course and then again at the end of Calculus class. The test has a margin of error δ in measurement of current student knowledge. If we're using the test to measure *gain in student knowledge* by doing final knowledge minus initial knowledge, the maximum possible error for the gain measure is 2δ – the worst case occurs when the initial and final measurement errors are in opposite directions.

In the real world, because we are working over probability distributions, the margins of error do not double; rather, under suitable nice assumption, the expected margin of error multiplies by a factor of $\sqrt{2}$. But it certainly *does increase*.

(3) A radioactive substance has a half-life of 3 years. **Determine the integer** n such that 90% of the substance decays within somewhere between $n - (1/2)$ and $n + (1/2)$ years.
(A) 5
(B) 10
(C) 15
(D) 20
(E) 25

Answer: Option (B)

Explanation: Since 90% of the radioactive substance decays, the fraction left is 0.1. Let x be the time taken for this to happen. Then:

$$\frac{\ln 0.5}{3} = \frac{\ln 0.1}{x}$$

Simplifying, we obtain that:

$$x = \frac{3 \ln 10}{\ln 2}$$

Since $\ln 10 \approx 2.3$ and $\ln 2 \approx 0.69$, we see that $x \approx 10$. Thus, the closest integer to x is $n = 10$.

We can work this out even if we do not remember $\ln 10$ and $\ln 2$. In order to do this, we need to recognize that:

$$x = 3\log_2(10)$$

Now, $\log_2(10) > 3$ since $2^3 = 8 < 10$, so $3\log_2(10) > 9$. Also, $\log_2(10) < 4$ since $2^4 = 16 > 10$, so $3\log_2(10) < 12$. Thus, the closest integer should be somewhere betwee 9 and 12. For further estimation, we verify that $2^{1/3} \approx 1.26$, so $2^{10/3} \approx 8(1.26)$ which is slightly greater than 10. Thus, $\log_2(10)$ is just slightly less than $10/3$, so $3\log_2(10)$ is just slightly less than 10, so $n = 10$ works.

Another way of seeing this is that $2^{10} = 1024$ (a standard fact in computer storage measurements) which is close to 10^3 (in fact, $1KB$ actually means 1024 bytes, not 1000 bytes). Thus, $2^{10/3} \approx 10$, which is what we need.

Post-performance review: 23 out of 29 people got this correct. 4 people chose (C), 1 person chose (A), and 1 person left the question blank.

(4) A, B, and C are three species of unicellular micro-organisms. Under specified conditions, species A doubles in number every 2 hours, species B triples in number every 3 hours, and species C quadruples (i.e., becomes 4 times) in number every 4 hours. Assume that they start off in the same quantities at the beginning. What can we say about their relative rates of growth?

(A) They are all growing at the same rate.
(B) Species A is growing fastest, species C is growing slowest, and species B is growing at an intermediate rate.
(C) Species A is growing slowest, species C is growing fastest, and species B is growing at an intermediate rate.
(D) Species A and C are both growing at the same rate, which is faster than the rate at which species B is growing.
(E) Species A and C are both growing at the same rate, which is slower than the rate at which species B is growing.

Answer: Option (E)

Explanation: If a population multiplies by a factor of q in a time period t, the rate of growth is $(\ln q)/t$. We thus need to compare $(\ln 2)/2$, $(\ln 3)/3$, and $(\ln 4)/4$.

First, note that since $\ln 4 = 2\ln 2$, $(\ln 2)/2 = (\ln 4)/4$. Thus A and C grow at the same rate. This makes sense: doubling every 2 hours is the same as quadrupling every 4 hours.

It remains to compare the value against $(\ln 3)/3$. We know that $\ln 3$ is between 1.09 and 1.1, so $(\ln 3)/3$ is between 0.363 and 0.367. $\ln 2$ is between 0.69 and 0.7, so $(\ln 2)/2$ is between 0.345 and 0.35. Thus, $(\ln 3)/3$ is the bigger number.

The comparison can be carried out without needing to know the ln values. For this, consider the question: what happens after 6 hours? Species A has doubled thrice, so it has multiplied by a factor of $2^3 = 8$. Species B has tripled twice, so it has multiplied by a factor of $3^2 = 9$. Thus, after six hours, species B is more numerous than species A. Hence, it must have a higher growth rate.

Post-performance review: 22 out of 29 people got this correct. 3 people chose (D), 2 people chose (B), and 1 person each chose (A) and (C).

(5) A species of bacteria doubles in number every hour. It takes 9 hours for a given initial quantity of this species to fill up a petri dish volume. How many hours from the start did the species occupy half the petri dish volume (assume that the volume occupied is proportional to the quantity)?

(A) 1 hour from the beginning
(B) 3 hours from the beginning
(C) 4.5 hours from the beginning
(D) 6 hours from the beginning
(E) 8 hours from the beginning

Answer: Option (E)

Explanation: Since the species doubles in number every hour, it must have doubled in the last hour. Thus, at the beginning of the last hour, it must have occupied half the petri dish volume. This is 8 hours from the beginning, since $9 - 1 = 8$.

Post-performance review: 28 out of 29 people got this correct. 1 person chose (C).

Note: This is a standard trick/conceptual question. Variants of this are used to test people's "intuitive" understanding of exponential growth.

(6) Suppose the populations in two countries A and B are growing exponentially at possibly different rates. Which of the following statements is **false**?

(A) If the initial population of A is more, and the exponential population growth rate of A is greater, then the population of A will always be greater than that of B.

(B) If the initial population of A is more, and the exponential population growth rate of B is greater, then the population of B will eventually overtake the population of A.

(C) If the initial population of A is more, and the exponential population growth rates of A and B are equal, then the populations of A and B will eventually become equal.

(D) All of the above.

(E) None of the above.

Answer: Option (C)

Explanation: With equal growth rates, the populations do not become equal – rather, the original proportion of the populations remains preserved forever. For instance, if the original population of A was 11 times the original population of B, then at any later instant of time, it will continue to be 11 times.

Post-performance review: 24 out of 29 people got this correct. 3 people chose (B), 1 person chose (A), and 1 person left the question blank.