PROBLEMS DISCUSSED IN THE AMTI CAMP

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Abstract. This is a collection of problems that were discussed with the AMTI students. A complete discussion of what was covered in class has been attempted here. The documentation may prove valuable to the students as well as to others interested in Olympiad related teaching/learning activities.

1. The equal parts problems

1.1. The problem statement.

Problem 1. *Suppose there is a sequence of $2n+1$ numbers with the property that if we remove any number, the remaining $2n$ can be divided into two parts such that both parts have the same sum. Prove that all the numbers are equal.*

This is an interesting problem because the problem statement does not make clear what the kind of *numbers* are. In fact, the result holds when the numbers are nonnegative integers, when they are integers, when they are rationals and when they are real numbers. (It holds in an even more general scenario).

It is important to note that it is only true that the remaining $2n$ numbers *can* be divided into two equal parts. and it is not in general true that *every* partition into two parts of size n each will have equal sum.

1.2. **The translation and scaling properties.** Let us call a sequence of length $2n+!$ satisfying the conditions of the problem to be a sequence with the **equal parts property**. Our goal is to show that any sequence with the equal parts property has all its members equal.

If $\{a_1, a_2 \dots a_{2n+1}\}$ is such a sequence then, for any k, $\{a_1+k, a_2+k \dots a_{2n+1}+k\}$ is also such a sequence. The proof of this fact relies on the observation that the same partition that we use for $\{a_1, a_2 \dots a_{2n+1}\}$ also works for $\{a_1+k, a_2+k \dots a_{2n+1}+k\}$.

Changing all the numbers by an amount k shall be called a **translation** by k. Thus, we can say that translations preserve the equal parts property. A sequence has the equal parts property iff[1] a translate of the sequence has the equal parts property.

Further, multiplying all the members of a sequence also preserves the equal parts property. If $m \neq 0$, then $\{a_1, a_2 \dots a_{2n+1}\}$ has the equal parts property if and only if $\{ma_1, ma_2 \dots ma_{2n+1}\}$ has the equal parts property.

1.3. **The proof for nonnegative integers.** The proof for nonnegative integers rests on the following simple observation –

Observation 1. *All the members of a sequence with the equal parts property have the same* **parity**. *That is, if any one of them is odd, so are all the others, and if any one of them is even, so are all the others.*

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[1]iff means if and only if

This was proved in class and follows from the fact that the parity of any member of the sequence equals the parity of the sum because the sum of the parities of the remaining elements is even.

Suppose our initial sequence is of nonnegative integers with the equal parts property. Then there are three cases :

- All its elements are 0. In this case, we stop.
- All of its elements are odd. In this case, they must all be at least 1. Thus, subtracting 1 from each of them gives a sequence of even terms, which again has the equal parts property (because the equal parts property is invariant under translation).
- All of its elements are even, and at least one of them is greater than 0. Thus, dividing all of them by 2 still gives a sequence with the equal parts property and at least one of the terms decreases in size.

If we apply this process iteratively to the original sequence, the second and third step always decrease the size of at least one element. Because the starting size was finite, we cannot keep applying these indefinitely and must eventually land up with all the entries of the sequence being 0. But then, working backwards, we can conclude that all the elements in the sequence we started out with were equal.

1.4. **Generalizing to integers and rationals.** The problem for integers can be reduced to the problem for nonnegative integers simply by translating so that all the integer entries become nonnegative. This translation is always possible (for instance, we can translate by a positive integer whose magnitude is greater than the largest magnitude of a negative integer in the sequence). Since translations preserve the equal parts property, we are done.[2]

For rationals, all we need to do is clear the denominator of the rational numbers. That is, we take the least common multiple of the denominators of all the rationals in the sequence, and multiply by this number. This gives us a collection of integers, which has the equal parts property iff the original sequence of rationals has the equal parts property. Thus, the problem for rationals reduces to the problem for integers.

In a similar way, the problem for reals reduces to a problem for rationals, though this proof requires a little more care.

1.5. **Where it does not hold.** This was not discussed in class!

We define the set of integers modulo a positive integer t as the set of all possible remainders under division by t. We define an operation of addition on this set by setting $a + b$ as the remainder of $a + b$ (as integers) modulo t

The problem statement is not true if we consider addition modulo some odd number. That is, given an odd number greater than 3, that is, an odd number of the form $2m + 1$ where m is an integer greater than 1, it is always possible to find $2m + 1$ integers with the equal parts property modulo $2m + 1$, but such that they are not equal. In fact, the sequence $\{0, 1, 2 \dots (2m)\}$ will do. (This is not very easy to prove)

Modulo 3, it is not possible to find 3 numbers with the equal parts property which are not all equal. However, the sequence $\{0, 0, 1, 2, 2\}$ is an example of a sequence of 5 numbers modulo 3 which are not all equal but satisfy the equal parts property.

[2] "We are done" is a mathematician's fancy way of saying that the proof (or at least the hard part of it), is over.

1.6. **Higher questions.** A structure with a general law of addition that satisfies many of the nice properties of addition is termed an **Abelian group**. Given any Abelian group, we can talk of the sequences in that group which have the equal parts property. We can ask for those Abelian groups where every sequence with the equal parts property must have all the terms equal. The integers, rationals and reals are examples of such Abelian groups. The integers modulo some odd number, on the other hand, are examples of Abelian groups where there are sequences with the equal parts property where all the terms are *not* equal.

Yet another question would be the *infinite analogue* of this problem. Consider a sequence of nonnegative terms such that the corresponding series is absolutely convergent. Suppose that whenever we remove one element from this sequence, the remaining elements can be divided into two parts with equal sums such that the frequency of occurrence of terms in both parts approaches 1. Then are all the terms 0?

1.7. **An alternative proof.** This proof was brought to my notice by Anupam Prakash, an IMO 2004 medalist. The proof relies on some elementary knowledge of linear equations and their solution using matrices.

For each element in the sequence, we have a way of partitioning the remaining elements into parts of size n each with the same sum. Suppose for each such element, we fix such a way. Then we get a matrix of size $2n+1$ that when multiplied by a vector corresponding to the given tuple, gives the value zero.

Clearly, the space of solutions is at least one dimensional. (because the vector with all entries equal is a solution). We need to show that, in fact, it is exactly one dimensional. This is done by proving that the matrix has rank ten. For this, it suffices to show that any minor has rank ten.

Take the minor obtained by deleting the first row and first column. This minor has zeroes along the diagonal and 1s and -1s elsewhere. The determinant of this is congruent, modulo 2, to the number of **derangements**[3] of $2n$. The problem thus reduces to proving that the number of derangements of $2n$ is odd.

1.8. **Generalizations for each.** The above proof goes through for any field of characteristic 0 or 2, so it is indeed very general. However, the earlier proof is easier to generalize in other ways.

The following statement can be generalized easily using my earlier proof :

> Given a sequence of $mn+1$ numbers such that given any number, the remaining mn can be divided into m parts of size n each such that all the sums are equal, prove that all the elements are equal.

On the other hand, the following follows directly from the matrix proof :

> Suppose that in a sequence of $2n+1$ elements, removing any one allows us to partition the remaining into parts with the same sum (though not necessarily equal in size). Then knowing how such a partition is determined for each element removed tells us the ratios of the elements.

[3] a derangement is a permutation that does not have any fixed point

2. The statue problem

2.1. **The real world motivation.** A statue is at some height from the ground, and is erected vertically. At what distance from the foot of the statue must an observer stand such that the statue subtends the maximum angle at her eye? [4]

This was the original problem formulation, as I had read in the Penguin Dictionary of Curious and Interesting Geometry.

Yet another variant of this is : from what distance must a shooter take aim in a game like basketball so as to have the best possible vision of the hoop?

2.2. **The precise formulation of the problem.**

Problem 2. *Given a line segment (which we call the* **object***) and a line completely on one side of it, but not parallel to it (which we call the* **line of vision***),* **find** *the point(s) on the line at which the angle subtended by the line has maximum magnitude. Give a geometrical construction to locate this point.*

Here , we drop the assumption that the line segment must be perpendicular to the line. Th perpendicularity assumption simplifies matters considerably.

2.3. **The tangent circle.** Extend the object line segment to make a line. This line divides the plane into two parts. In each part, we use the following observation :

Observation 2. *The angle made by a point on the circle at a chord is greater than the angle made by a point outside the circle, and less than the angle made by a point inside the circle, provided that all the points lie on the same side of the line formed by producing (extending) the chord.*

The line formed from the object cuts the line of vision at some point, and on both sides of this point, the maximum is obtained by taking the **point of contact** of a circle which passes through the two points and touches the line. However, only one of these is a **global maximum**, while the other is a **local maximum** – it is maximum in some neighbourhood of itself. [5]

2.4. **A geometrical construction.** The construction relies on the concept of **power of a point**, or, in other words, the result that the product of the lengths from a point on a secant to the points where the secant meets the circle is the square of the length of the tangent segment from that point to the circle.

The procedure is as follows :

(1) **Construction of an arbitrary circle through A and B** : Let A and B denote the endpoints of the statue. Draw the perpendicular bisector of AB. Taking any point O on this perpendicular bisector, make a circle passing through A (and hence, also through B). Call this Γ_1.
(2) **Construction of tangent** : Let AB (produced) meet the line of vision at P. Make a circle with OP diameter and let it intersect Γ_1 at points Q_1 and Q_2. Then, PQ_1 and PQ_2 are the tangents from P to Γ_1.
(3) **Determination of the points for maximum angle** : Make a circle centered at P with radius PQ_1 am-d let it meet the line of vision at points T_1 and T_2.

[4] One of the factors that affects how "big" we perceive an object to be is the angle it subtends at the eye. The observer's aim in this case is that the angle should be maximized so that her view is best.

[5] Every global maximum must be a local maximum but the converse does not hold. The local maximum need not even be the second biggest value or in any way close to the global maximum.

Then, T_1 and T_2 and the two local maxima. The point which is on the side with which AB makes an acute angle, gives the global maximum.

The proof that T_1 and T_2 are indeed the required points stems from the fact that the power of P with respect to Γ_1 and with respect to the final circle that we want (passing through A and B and tangent to the line of vision) is the same, viz $PA.PB$. Hence the tangent length is also the same for both circles. For this reason, determining the tangent length for any one circle gives us the tangent length for the other.

2.5. **Related questions.** Some related questions are :

- What happens if the object is parallel to the line of vision?
- In the general case, which of the sides will give the larger angle?
- Suppose the object is not a line segment but some other curve. Can we still use the same idea?

3. Area bisectors problem

3.1. **Center of symmetry.** Given any simple closed curve, it divides the plane into two regions – the interior, and the exterior. The interior has a finite area. Given any line, there is a line parallel to that line that bisects the area.

If we now look at all these area bisectors, then we might ask : do they all pass through the same point? Intuitively, as we change the direction of the area bisector, it rotates somewhat about a point. However, the **instantaneous center of rotation** might be constantly changing.

We have the following sufficient condition for area bisectors :

Problem 3. *Suppose, for a simple closed curve, there is a point O such that for every P on the curve, the point Q such that O is the midpoint of PQ is also on the curve. Prove that every line through PQ bisects the area of the interior.*

Lacking a formal concept of area, we cannot prove this. However, we can intuitively understand why it is true. Roughly, the two pieces into which any line divides the interior have the same shape and size – they can be **superimposed** on each other via a rotation of π. Thus, they are **congruent**, and congruent figures have the same area.

Such a point, if it exists, is termed a **center of symmetry**. The operation taking P to Q described above is termed a **half turn** about O and can also be accomplished by a rotation via an angle of π.

3.2. **Polygons with a center of symmetry.** The following is a characterization of polygons in the plane with a center of symmetry:

Problem 4. *Show that a polygon in a plane has a center of symmetry iff it has an even number of sides and opposite sides are always parallel and equal.*

Such polygons form a generalization of **parallelograms**.

3.3. **Some implications.** This was not discussed in class!

Given two closed curves both with centers of symmetry, the line joining the centers of symmetry bisects the areas of both curves. If both of them have the *same* center of symmetry, then any line through that bisects the area of both.

A famous result in topology (that is beyond our current scope for rigourous proof) going by the name of the **Borsuk Ulam theorem** states that given any two simple closed curves, there is a line that bisects the areas of the interiors of both the curves.

3.4. **For triangles.** If we look at the collection of area bisectors of a triangle in all directions, then these lines are not concurrent (Even for an equilateral triangle). We know that three of these lines, namely the three medians, pass through the centroid, but the area bisectors parallel to the sides do not pass through the centroid.

The collection of all area bisectors enclose a small curve around the centroid (that is, they have an envelope which is a small curve around the centroid).

3.5. **Higher dimensions.** This was not discussed in class!

Given a solid figure, we can talk of planes acting as **volume bisectors**. Again, given any plane, there is a plane parallel to that that acts as a volume bisector. Further, if there is a center of symmetry, then every plane through that is a volume bisector.

The three dimensional analogue of the Borsuk Ulam theorem is called the **ham sandwich theorem** – given three solids (say, a piece of ham, and two loaves) there is a single plane (that is, a single knife cut) that simultaneously bisects the volumes of all three.

3.6. **Perimeter bisectors.** This was not discussed in class!

Let's get back to planes again.

In case of a center of symmetry, every line through that is not just an area bisector, but also a **perimeter bisector**, that is, it divides the perimeter (or the length of the bounding curve) into two equal halves. However, in general, area bisectors and perimeter bisectors look quite different. Thus, in the case of an equilateral triangle, the three medians function as both area bisectors and perimeter bisectors, and in the case of an isosceles triangle, the median to the unequal side serves as a perimeter bisector. In general, however, the medians are area bisectors but not perimeter bisectors.

4. The colliding particles problem

4.1. **Some mechanics of moving particles.** Let us first build up the scenario.

Suppose we have a system of particles moving along a straight line, in either direction, all at the same speed. Then, two of them can only collide **head on**, that is, if they are moving in opposite directions towards each other. After this collision, many things can happen.

For our purpose, we assume all the particles have identical behaviour and all collisions are **elastic collisions**. This in particular means that each particle will rebound back with the same speed, but in the opposite direction. [6]

4.2. **The problem formulation.**

Problem 5. *Consider a finite collection of particles moving in the interval* $[0, 1]$*, that is, the set of real numbers between, and including,* 0 *and* 1*. Suppose that the particles all move with the same speed* v *and undergo only elastic collisions with each other. Suppose further that when a particle hits either* 0 *or* 1*, it rebounds back with speed* v*. (The points* 0 *and* 1 *are thus the walls). Let us say, that at the initial time, one of the particles was at the point* $1/2$*. Then, determining the necessary and sufficient conditions for it to be at* $1/2$ *after time* $1/v$*.*

[6]We do not go into the meaning of elastic collisions here. Those who have studied mechanics will follow the allusion.

4.3. **Three crucial observations.**

- The **actual order** of the particles does not change.
- Let us say that when two particles collide, we interchange their identities. Then we can imagine as if they cross each other without colliding. Of course, the identities of the particles will get interchanged. However, as all the particles are identical in nature, we can assume as if the particles are moving without colliding, but continuously changing their identities.
- If all the particles moved independently without colliding, then the particle at x would be at $1 - x$ after time $1/v$.

4.4. **Piecing together these observations.** We can conclude that after time $1/v$, if a particle was initially at x, there will be *some* particle (not necessarily the same) at $1-x$. This suggests that the relative positioning of the points is simply reflected about $1/2$.

This means that if there were m particles to the left of $1/2$ and n to the right of $1/2$ at the beginning, there will finally be n particles to the left of $1/2$ and $m$particles to the right of $1/2$ at the end. But we must bear in mind that the *actual order* does not change. Hence, while initially the $(m+1)^{th}$ particle was at $1/2$, finally the $(n+1)^{th}$ particle will be at $1/2$. For them to be the same particle, ti is necessary that $m = n$.

This translates to the requirement that, at the start,the number of particles to the left of $1/2$ equals the number of particles to the right of $1/2$.

5. Anders' dilemna

5.1. **The problem statement.** The problem is as follows :

Problem 6 (Anders' dilemna)**.** *Anders needs to travel across an iceland, where no food supplies are available across the journey. There are unlimited supplies at either end of the iceland. Anders has some assistants to help him.*

Each person can carry enough supplies to last a fraction α of the journey (however, people keep consuming supplies even if waiting). The following operations are permitted : a person can move forwards, or backwards, or stay at the same place, two people at the same place can transfer supplies to each other, and a person can, on reaching the starting point, fill his supplies again and set off.

Given $\alpha < 1$, we need to find the minimum number of assistants with whose help Anders can safely reach the other side, such that all the assistants return safely to the starting side. Clearly, the more the number of assistants, the less the value of α needed. Another way of putting this problem is : given the number of assistants, find the smallest α for which Anders can complete his journey with his assistants returning safely to the starting point.

5.2. **The solution for one assistant.** For one assistant, the strategy is as follows :

(1) Anders and the assistant fill their supplies to α.
(2) They both travel (together) a fraction x of the journey. At this stage, the assistant transfers some of his supplies to Anders and sets back home.
(3) Anders completes the remaining $(1 - x)$ fraction of the journey.

Note that the assistant cannot transfer more than $\alpha - 2x$ (because he needs $2x$ to go and come back safely). Thus, after the exchange, Anders has $\min\{\alpha, \alpha - x + \alpha - 2x\}$. The system will have to satisfy the following constraints :

$$\begin{aligned} \alpha &\geq x \\ \min\{\alpha, 2\alpha - 3x &\geq 1 - x \end{aligned}$$

The first is redundant, and the second splits into two inequalities :

$$\begin{aligned} \alpha &\geq 1 - x \\ 2\alpha - 3x &\geq 1 - x \end{aligned}$$

Simplifying and adding, we get $\alpha \geq 3/4$.

The corresponding value of x is $1/4$. Putting these values, we find that the strategy.

Note the general approach :

1. We formulated a strategy qualitatively.
2. We gave quantitative variables (in this case, x, denoting the distance Anders and his assistant travel together)
3. We translated the constraints into a system of inequations between the quantitative variables.
4. We determined an objective function to be minimized, which was in this case α.

This problem is special because here, in the optimum case, all the inequations are *tight*, that is, equality holds in all the cases.

However, in most cases, this does not happen – in particular when the number of inequations is more than the number of variables.

The problems with coming up with a more general solution with a greater number of assistants is :

1. There is no clear cut qualitative formulation of the strategy which is optimal. Thus, it is necessary to experiment with multiple qualitative formulations.
2. There are a large number of quantitative variables.
3. The number of constraints is very huge.

We shall get a flavour of this when we try to solve the problem in the case of 2 assistants.