HOMEWORK 2 CHECKLIST: DUE MONDAY OCTOBER 14

MATH 196, SECTION 57 (VIPUL NAIK)

You may not get the graded homework back in time for the midterm, so please keep a copy of it for your own review.

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 1.2.8 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work. The variables are x_1, x_2, x_3, x_4, and x_5, even though x_1 and x_3 do not appear anywhere in the system.

$$\begin{aligned} x_2 + 2x_4 + 3x_5 &= 0 \\ 4x_4 + 8x_5 &= 0 \end{aligned}$$

Checklist hint begins: Multiply the second equation by 1/4. One more row subtraction and we are in reduced row-echelon form. Note that there are two leading variables and three non-leading variables (including variables that don't appear at all in the expressions for the leading variables). Express the leading variables in terms of non-leading variables.

(2) Exercise 1.2.9 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

$$\begin{aligned} x_4 + 2x_5 - x_6 &= 2 \\ x_1 + 2x_2 + x_5 - x_6 &= 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2 \end{aligned}$$

Checklist hint begins: Swap the second equation with the first. Now, row reduce. At best, you will get three leading variables, hence three non-leading variables.

(3) Exercise 1.2.10 (Page 18): Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

Checklist hint begins: This involves a fairly lengthy process of Gauss-Jordan elimination. Grit your teeth and do it! You might want to see the "Human computer tips" section in the notes on Gauss-Jordan elimination (Section 4 of the notes). In particular note that rather than dividing the first equation by four, it's better to subtract the first equation from the second, then interchange the equations, and proceed from there.

It will turn out in this case that the fourth equation is redundant (all zeros in the fourth row of the coefficient matrix *and* in the augmenting column). Note that we do *not* know that in advance, and you should not assume that, but it will *transpire* after you faithfully execute the many steps of row reduction.

(4) Exercise 1.2.33 (Page 19) (was Exercise 1.2.31 in the 4th Edition): Find the polynomial of degree 4 whose graph goes through the points $(1, 1)$, $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$. Graph this polynomial.

Checklist hint begins: You will get a system of five linear equations in five variables. Alternatively, you can use the Lagrange interpolation formula (look it up if you wish).

(5) Exercise 1.3.2 (Page 33): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Checklist hint begins: Based on the upper triangular nature of the matrix, we can see that the Gauss-Jordan elimination will give the identity matrix. Hence ...

(6) Exercise 1.3.3 (Page 33): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Checklist hint begins: The fact that the rows are identical tells us something ...

(7) Exercise 1.3.4 (Page 33): Find the rank of the matrix:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Checklist hint begins: Note that the middle row is the arithmetic mean of the first and third row. This should tell us something about one row being redundant, hence the rank is ...

(8) Exercise 1.3.19 (Page 33): Compute the matrix-vector product:

$$\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Checklist hint begins: Just do it! Your answer should be a column vector with three entries.

For the next four problems, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(9) Exercise 1.3.29 (Page 35): Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.

Checklist hint begins: Set up a diagonal matrix of the form:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Now, the matrix-vector product gives equations:

$$\begin{aligned} 5a &= 2 \\ 3b &= 0 \\ -9c &= 1 \end{aligned}$$

Solve. We will get a unique diagonal matrix.

(10) Exercise 1.3.30 (Page 35): Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.

Checklist hint begins: The general format of an equation of rank one is that all the rows are either zero or multiples of each other. There are lots of possibilities for the first and third row, though the middle row must be zero.

(11) Exercise 1.3.31 (Page 35): Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$, where all the entries of A on and above the diagonal are nonzero.

Checklist hint begins: Set up a general upper-triangular matrix, then find values that work. There is a lot of flexibility in choosing entries. Explicitly, if we have:

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Then, we have flexibility in choose a and b (but c is then fixed). Similarly, we have flexibility in choosing d (but e is then fixed). f is known in advance. We should just be careful not to choose a and b so badly that c turns out to be 0. Similar remarks apply for other choices.

(12) Exercise 1.3.32 (Page 35): Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.

Checklist hint begins: Set up a general matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

We get three equations in the nine variables. Note that there is a lot of flexibility here in terms of choosing the variables. We can choose two of the three variables in a given row almost arbitrarily and then compute the third variable. We just need to be sure not to hit on zero values for the variables.

2. Problems for your own review, not for submission

You should know the answers to these, but do not submit them. This will keep the grading workload to a realistic minimum. I will put up hints for these in the checklist.

For the exercises here from Section 1.2, the exercise numbers in the 4th Edition are 2 lower than those in the 5th Edition. Exercises 1.2.22-28 are Exercises 1.2.20-26 in the old edition. The exercise numbers for the exercises from Section 1.3 are the same across editions.

(1) Exercise 1.2.22 (Page 19): We say that two $n \times m$ matrices in reduced row-echelon are of the same type if they contain the same number of leading 1's in the same positions. For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

Checklist hint begins: Be sure to include the possibility of getting zero rows. One possibility is that both rows are zero. The second row being zero admits two possibilities for the type of the first row. If the second row is nonzero, we have only one type.

(2) Exercise 1.2.23 (Page 19): How many types of 3×2 matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

Checklist hint begins: Again, make cases. The third row is forced to be zero, so all the action happens in the first two rows. It's pretty similar to the preceding question.

(3) Exercise 1.2.24 (Page 19): How many types of 2×3 matrices in reduced row-echelon form are there? *Please explain your answer briefly, and include a representative for each type.*

Checklist hint begins: Again, make cases.

(4) Exercise 1.2.26 (Page 19): Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

Checklist hint begins: Yes. Recall that each of the elementary row operations has a natural inverse operation that is a row operation of the same type. Swapping is self-inverse. Multiplication by λ can be reversed using multiplication by $1/\lambda$. Adding λ times a row to another can be reversed by adding $-\lambda$ times the row.

(5) Exercise 1.2.27 (Page 19): Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer (this relies on the preceding exercise).

Checklist hint begins: Reverse the operations one at a time. Note that the reversal will happen in reverse order, i.e., the last operation to be done will be the first to be reversed. When in reverse gear, you traverse backwards.

(6) Exercise 1.2.28 (Page 19): Consider an $n \times m$ matrix A? Can you transform the reduced row-echelon form of A into A by a sequence of elementary row operations? This relies on the preceding exercise.

Checklist hint begins: Yes, from the preceding exercise.

(7) Exercise 1.3.22 (Page 35): Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

Checklist hint begins: Full rank, so identity matrix. Think through this.

(8) Exercise 1.3.23 (Page 35): Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

Checklist hint begins: Unique solution means full column rank. The last row is zero, and the submatrix comprising the top three rows give an identity matrix.

(9) Exercise 1.3.24 (Page 35): Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

Checklist hint begins: In this case, full rank, so no worries regarding redundancy/inconsistency, even if you change the last column. Thus, the conclusion carries over to the other equation.

(10) Exercise 1.3.25 (Page 35): Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

Checklist hint begins: It may be inconsistent or it may have infinitely many solutions. Depends on the vector $\vec{c}$.

(11) Exercise 1.3.26 (Page 35): Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions to $A\vec{x} = \vec{c}$?

Checklist hint begins: The rref of A has the identity matrix as its top 3×3 and the last row is zero. If the last coordinate of the vector we get on doing the row operations to $\vec{c}$ is zero, then the system is consistent and we get a unique solution. if it is nonzero, the system is inconsistent and we get no solutions.

(12) Exercise 1.3.41 (Page 35): How many solutions do most systems of three linear equations with three unknowns have?

Checklist hint begins: "Most" matrices have rank equal to the minimum of the number of rows and number of columns. So, in this case, the rank is expected to be 3, and the dimension of the solution space is thus $3 - 3 = 0$. Thus, we have a unique solution.

(13) Exercise 1.3.42 (Page 35): How many solutions do most systems of three linear equations with four unknowns have?

Checklist hint begins: Intuitively, we'd expect the dimension to be $4 - 3 = 1$, i.e., we expect a solution space with one parameter, so we expect infinitely many solutions.

(14) Exercise 1.3.43 (Page 35): How many solutions do most systems of four linear equations with three unknowns have?

Checklist hint begins: The system is "overdetermined" and the dimension of the solution space is expectationally $3 - 4 = -1$. Of course, it won't ever actually be -1, but there usually won't be any solution.

(15) Exercise 1.3.44 (Page 35): Consider an $n \times m$ matrix A with more rows than columns ($n > m$). Show that there is a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

Checklist hint begins: Row reduce, then in the row reduced version, we want a nonzero output for the zero row. Reverse all the operations and get an original system.

(16) Exercise 1.3.46 (Page 35): Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

where a, d, and f are nonzero, and b, c, and e are arbitrary numbers.

Checklist hint begins: An upper triangular or lower triangular matrix with all diagonal entries nonzero must have full rank (in this case 3). We can see this explicitly: the row reductions will yield the identity matrix.

3. Advanced problems

(1) Exercise 1.2.34 (Page 19-20) (*Cubic splines*) (was Exercise 1.2.32 in the 4th Edition): This is an extremely long problem with pictures, so I recommend you read it from the book.

This is related to a recent collection of quiz questions (Questions 2-5 on Friday April 12) and the explanation there may be helpful conceptually.

(2) Exercise 1.2.39 (Page 20) (Leontief input-output model for three variables) (was Exercise 1.2.37 in the 4th Edition): Please see the diagram and the labels in the diagram from the book.

Review the input-output model question from HW1 and set up equations in a similar manner.

(3) Exercise 1.2.40 (Page 20-21) (was Exercise 1.2.38 in the 4th Edition): This covers the Leontief input-output model in n variables. The question is long, so I recommend you read it from the book.

Please read through the entire hint setup from the book.

(4) Exercise 1.2.42 (Page 22) (was Exercise 1.2.40 in the 4th Edition): This problem involves estimating the amount of traffic on one-way streets. Please read the problem from the book.

Checklist hint begins: Give variable letter names to the traffic at the four locations. At each intersection, outflow = inflow. Formulate an equation for each intersection. Now, solve the system.

Note that Dunster street is unusual: its one-way direction changes at the intersection with Mt. Auburn Street. All the other intersections have two inflow directions and two outflow directions. The Dunster-Mt. Auburn intersection has three inflow directions and one outflow direction.

It will turn out that, based on the information, we will be unable to estimate the traffic volume in the four streets. The system of four equations in four variables will turn out to have rank three. It is consistent, and we can express all the traffic volumes in terms of any one of them.

To get the maximum and minimum possible traffic volumes, use the constraint that all four values have to be nonnegative. Since we've got all four of them in terms of one variable, this gives rise to four inequalities in one variable. Find the common solutions to all four. This gives an interval. Plug in the endpoints of the interval to get the other traffic volumes at both extremes. These give the maximum and minimum traffic volumes (some volumes will be maximum in one solution, some will be maximum in the other solution).

(5) Exercise 1.3.47 (Page 35-36): A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

(a) All homogeneous systems are consistent.
The all coordinates zero vector is a solution.

(b) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

There is a positive number of non-leading variables, which hence function as parameters. These parameters can take arbitrary real values. Hence, we get infinitely many solutions.

(c) If $\vec{x_1}$ and $\vec{x_2}$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x_1} + \vec{x_2}$ is a solution as well.
This follows directly from the additive part of the linearity property of the matrix-vector product: $A(\vec{x_1} + \vec{x_2}) = A\vec{x_1} + A\vec{x_2}$.

(d) If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.
This follows directly from the scalar multiples part of the linearity property of the matrix-vector product: $A(k\vec{x}) = k(A\vec{x})$.

(6) Exercise 1.3.48 (Page 36-37): Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

(a) If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

(b) If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

(c) Now suppose A is 2×2 matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$ (see the book for the figure: it has a line through the origin for the solutions of $A\vec{x} = \vec{0}$ and $\vec{x}_1$ depicted as a vector (not along the line) with tail at $\vec{0}$).

If you are puzzed by the generality of this problem think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \qquad \text{and} \qquad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Checklist hint begins: For parts (a) and (b), use that $A(\vec{v}+\vec{w}) = A\vec{v}+A\vec{w}$ and $A(\vec{v}-\vec{w}) = A\vec{v}-A\vec{w}$. For part (c), think about translating a line by a fixed amount (something to do with parallel lines).