TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY MARCH 11 (POSTPONED TO MARCH 13): MAX-MIN VALUES: TWO-VARIABLE VERSION

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this 7-question quiz. The score distribution was as follows:

- Score of 1: 2 people.
- Score of 2: 1 person.
- Score of 3: 6 people.
- Score of 4: 4 people.
- Score of 5: 7 people.
- Score of 6: 5 people.

The question wise answers and performance information are below:

1. Option (B): 23 people.
2. Option (B): 14 people.
3. Option (C): 3 people.
4. Option (C): 19 people.
5. Option (D): 7 people.
6. Option (E): 21 people.
7. Option (E): 16 people.

2. Solutions

(1) Suppose $F(x,y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are differentiable functions of one variable, defined for all real numbers. What can we say about the critical points of F in its domain $\mathbb{R}^2$?
   (A) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *or* y_0 is a critical point for g.
   (B) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *and* y_0 is a critical point for g.
   (C) F has a critical point at (x_0, y_0) iff $x_0 + y_0$ is a critical point for $f + g$, i.e., the function $x \mapsto f(x) + g(x)$.
   (D) F has a critical point at (x_0, y_0) iff $x_0 y_0$ is a critical point for fg, i.e., the function $x \mapsto f(x)g(x)$.
   (E) None of the above.

   Answer: Option (B)

   Explanation: $F_x(x_0, y_0) = f'(x_0)$ and $F_y(x_0, y_0) = g'(y_0)$. In order for both of these to be 0, we must have $f'(x_0) = g'(y_0) = 0$. Thus, x_0 is a critical point for f and y_0 is a critical point for g.

   Performance review: 23 out of 25 people got this. 2 chose (C).

   Historical note (last time): 12 out of 17 people got this correct. 3 chose (C) and 2 chose (A).

(2) Suppose $F(x,y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both differentiable functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following is true?
   (A) F has a critical point at (x_0, y_0) if and only if x_0 is a critical point for f and y_0 is a critical point for g.
   (B) If x_0 is a critical point for f and y_0 is a critical point for g, then (x_0, y_0) is a critical point for F. However, the converse is not necessarily true, i.e., (x_0, y_0) may be a critical point for F even without x_0 being a critical point for f and y_0 being a critical point for g.
   (C) If (x_0, y_0) is a critical point for F, then x_0 must be a critical point for f and y_0 must be a critical point for g. However, the converse is not necessarily true.

(D) (x_0, y_0) is a critical point for F if and only if *at least* one of these is true: x_0 is a critical point for f and y_0 is a critical point for g.

(E) None of the above.

Answer: Option (B)

Explanation: We have $F_x(x_0, y_0) = f'(x_0)g(y_0)$ and $F_y(x_0, y_0) = f(x_0)g'(y_0)$. We see that if $f'(x_0) = 0$ and $g'(y_0) = 0$, then F has a critical point at (x_0, y_0). However, F could also have a critical point for other reasons: for instance, $f(x_0) = f'(x_0) = 0$ (and other cases).

Performance review: 14 out of 25 got this. 4 chose (D), 3 each chose (A) and (C), 1 chose (E).

Historical note (last time): 9 out of 17 people got this correct. 5 chose (A), 2 chose (D), 1 chose (C).

(3) Consider a homogeneous polynomial $ax^2 + bxy + cy^2$ of degree two in two variables x and y. Assume that at least one of the numbers a, b, and c is nonzero. What can we say about the local extreme values of this polynomial on $\mathbb{R}^2$?

(A) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.

(B) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.

(C) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained only at the origin.

(D) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.

(E) If $b^2 - 4ac = 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.

Answer: Option (C)

Explanation: See discussion of homogeneous quadratic in the relevant lecture notes.

Performance review: 3 out of 25 got this. 10 chose (B), 7 chose (D), 4 chose (A), 1 chose (E).

Historical note (last time): 5 out of 17 people got this correct. 4 chose (A), 4 chose (B), 2 chose (D), 2 chose (E).

A subset of $\mathbb{R}^n$ is termed *convex* if the line segment joining any two points in the subset is completely within the subset. A function f of two variables defined on a closed convex domain is termed *quasiconvex* if given any two points P and Q in the domain, the maximum of f restricted to the line segment joining P and Q is attained at one (possibly both) of the endpoints P or Q.

There are many examples of quasiconvex functions, including linear functions (which are quasi-convex but not strictly quasiconvex) and all convex functions.

(4) What can we say about the maximum of a continuous quasiconvex function defined on the circular disk $x^2 + y^2 \leq 1$?

(A) It must be attained at the center of the disk, i.e., the origin $(0, 0)$.

(B) It must be attained somewhere in the interior of the disk, but we cannot be more specific with the given information.

(C) It must be attained somewhere on the boundary circle $x^2 + y^2 = 1$. However, we cannot be more specific than that with the given information.

(D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

(E) It could be attained at any point. We cannot be specific at all.

Answer: Option (C)

Explanation: For any point in the interior of the circular disk, the point can be put on a chord with endpoints on the boundary circle. The maximum of the function restricted to this chord occurs at one of the boundary points, hence, the value at any point in the interior is equaled or exceeded by some point in the boundary. Thus, the maximum is attained at some point in the boundary.

However, no point in the boundary can be put in the interior of a line segment joining two other points, i.e., each point in the boundary is extreme. So we cannot narrow things down further.

Performance review: 19 out of 25 got this. 3 chose (D). 1 each chose (A), (B), and (E).

Historical note (last time): 8 out of 17 people got this correct. 5 chose (B), 3 chose (E), 1 chose (D).

(5) What can we say about the maximum of a continuous quasiconvex function defined on the square region $|x| + |y| \leq 1$? This is the region bounded by the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
 (A) It must be attained at the center of the square, i.e., the origin $(0, 0)$.
 (B) It must be attained somewhere in the interior of the square, but we cannot be more specific with the given information.
 (C) It must be attained somewhere on the boundary square $|x| + |y| \leq 1$. However, we cannot be more specific than that with the given information.
 (D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
 (E) It could be attained at any point. We cannot be specific at all.

Answer: Option (D)

Explanation: We can first push to the boundary the same way as for the circular disk. Now, unlike the circular disk, we note that any point in the boundary that is to one of the vertices is on a line segment joining two adjacent vertices, so we can push out to the vertices.

Performance review: 13 out of 25 got this. 7 chose (D), 3 chose (B), 1 eacch chose (A) and (E).

Historical note (last time): 3 out of 17 people got this correct. 8 chose (C), 3 chose (B), 2 chose (A), 1 chose (E).

(6) Suppose $F(x, y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are continuous functions of one variable, defined for all real numbers. Which of the following statements about local extrema of F is **false**?
 (A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
 (B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).
 (C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
 (D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).
 (E) None of the above, i.e., they are all true.

Answer: Option (E)

Explanation: This is discussed in the lecture notes.

Performance review: 21 out of 25 got this. 2 chose (B). 1 each chose (A) and (D).

(7) Suppose $F(x, y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both continuous functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following statements about local extrema is **true**?
 (A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
 (B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).
 (C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
 (D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).

(E) None of the above, i.e., they are all false.

Answer: Option (E)

Explanation: We also need information about the signs of $f(x_0)$ and $g(y_0)$ since that affects the conclusion. All the options above would be *true* (rather than false) if both $f(x_0)$ and $g(y_0)$ are known to be positive.

Performance review: 16 out of 25 got this. 3 chose (A), 2 each chose (B), (C), and (D).