TAKE-HOME CLASS QUIZ SOLUTIONS: WEDNESDAY JANUARY 9: PARAMETRIC STUFF

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this 11-question quiz. The score distribution was as follows:

- Score of 4: 1 person.
- Score of 6: 3 people.
- Score of 7: 7 people.
- Score of 8: 1 person.
- Score of 9: 6 people.
- Score of 10: 4 people.
- Score of 11: 4 people.

The question-wise answers and performance review are below.

(1) Option (E): 19 people.
(2) Option (C): 13 people.
(3) Option (C): 25 people.
(4) Option (A): 24 people.
(5) Option (A): 20 people.
(6) Option (D): 25 people.
(7) Option (A): 25 people.
(8) Option (C): 20 people.
(9) Option (B): 11 people.
(10) Option (D): 12 people.
(11) Option (E): 23 people.

2. Solutions

(1) Consider the curve given by the parametric description $x = \cos t$, $y = \sin t$, where t varies over the interval $[a, b]$ with $a < b$. What is a necessary and sufficient condition on a and b for this curve to be the circle $x^2 + y^2 = 1$?

(A) $b - a = \pi$
(B) $b - a > \pi$
(C) $b - a = 2\pi$
(D) $b - a > 2\pi$
(E) $b - a \geq 2\pi$

Answer: Option (E)

Explanation: The curve is traced along the circle starting at $(\cos a, \sin a)$ and going around the circle till we reach b. In order to cover the whole circle, it is necessary that it make at least one full angle of 2π. Thus, the condition $b - a \geq 2\pi$.

Note that the equality case is valid because we are working with the *closed* interval $[a, b]$. If we were working with the open interval (a, b), then strict inequality would be the necessary and sufficient condition.

Performance review: 19 out of 26 got this. 7 chose C, which is the correct answer for the curve to *just* cover the circle but is the wrong choice for a necessary and sufficient condition.

Historical note (last time): 11 people got this correct. 9 people chose (C), 2 people chose (B), 1 person each chose (A) and (D).

(2) (*) Consider the curve given by the parametric description $x = \arctan t$ and $y = \arctan t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve?
   (A) It is the graph of the function arctan
   (B) It is the line $y = x$
   (C) It is a line segment (without endpoints) that is part of the line $y = x$
   (D) It is a half-line (with endpoint) that is part of the line $y = x$
   (E) It is a disjoint union of two half-lines that are both part of the line $y = x$

   Answer: Option (C)

   Explanation: Eliminating the parameter t, we get that $y = x$, but with the additional caveat that the value of x (hence also y) must be in the range of arctan. The range of arctan is the open interval $(-\pi/2, \pi/2)$, thus we get the corresponding line segment without endpoints joining the point with coordinates $(\pi/2, \pi/2)$ to the point with coordinates $(-\pi/2, -\pi/2)$.

   Performance review: 13 out of 26 got this. 10 chose (D), 2 chose (B), 1 chose multiple options.

   Historical note (last time): 8 people got this correct. 9 people chose (B), which would be the right idea *except for the issue of domain/range restrictions*. 4 chose (D), 2 chose (A), 1 chose (E).

(3) (*) Consider the curve given by the parametric description $x = \sin^2 t$ and $y = \cos^2 t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve?
   (A) It is the arc of the circle $x^2 + y^2 = 1$ comprising the first quadrant, i.e., when $x \geq 0$ and $y \geq 0$.
   (B) It is the entire circle $x^2 + y^2 = 1$
   (C) It is the line segment joining the points $(0, 1)$ and $(1, 0)$
   (D) It is the line $y = 1 - x$
   (E) It is a portion of the parabola $y = x^2$

   Answer: Option (C)

   Explanation: Eliminating the parameter, we obtain that $x + y = 1$. Further, we much have $x \geq 0$ and $y \geq 0$ since they are both squares. Subject to these conditions, any pair (x, y) works. This is thus the part of the line $x + y = 1$ which lies in the first quadrant. This can alternatively be described as the line segment joining the points $(0, 1)$ and $(1, 0)$.

   Performance review: 25 out of 26 got this. 1 chose (B).

   Historical note (last time): 5 people got this correct. 11 people chose (D), which would be the correct answer *except for the issue of domain/range restrictions*. 4 people each chose (A) and (B).

(4) Identify the parametric description which *does not* correspond to the set of points (x, y) satisfying $x^3 = y^5$.
   (A) $x = t^3$, $y = t^5$, for $t \in \mathbb{R}$
   (B) $x = t^5$, $y = t^3$, for $t \in \mathbb{R}$
   (C) $x = t$, $y = t^{3/5}$, for $t \in \mathbb{R}$
   (D) $x = t^{5/3}$, $y = t$, for $t \in \mathbb{R}$
   (E) All of the above parametric descriptions work

   Answer: Option (A)

   Explanation: The exponents are at the wrong places – if $x = t^3$, then $x^3 = t^9$ and if $y = t^5$, then $y^5 = t^{25}$ – these are certainly not equal.

   Performance review: 24 out of 26 got this. 2 chose (D).

   Historical note (last time): 16 people got this correct. 4 chose (E), 3 chose (B), 1 chose (C).

(5) (*) Consider the parametric description $x = f(t)$, $y = g(t)$ where t varies over all of $\mathbb{R}$. What is the necessary and sufficient condition for the curve given by this to be the graph of a function, i.e., to satisfy the vertical line test?
   (A) For any t_1 and t_2 satisfying $f(t_1) = f(t_2)$, we must have $g(t_1) = g(t_2)$.
   (B) For any t_1 and t_2 satisfying $g(t_1) = g(t_2)$, we must have $f(t_1) = f(t_2)$.
   (C) Both f and g are one-to-one functions.
   (D) For any t_1 and t_2, we must have $f(t_1) = f(t_2)$.
   (E) For any t_1 and t_2, we must have $g(t_1) = g(t_2)$.

   Answer: Option (A)

Explanation: We want that for a given x-value there is at most one y-value (the vertical line test). This means that if, at two times t_1 and t_2, the x-values $f(t_1)$ and $f(t_2)$ are equal to each other, the y-values $g(t_1)$ and $g(t_2)$ must also be equal to each other. This is option (A).

Performance review: 20 out of 26 got this. 3 each chose (B) and (C).

Historical note (last time): 10 people got this correct. 11 chose (C), which is a *sufficient* but not a necessary condition. 2 chose (B) and 1 chose (D).

(6) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \circ g)''$?
(A) $(f'' \circ g) \cdot g''$
(B) $(f'' \circ g) \cdot (f' \circ g') \cdot g''$
(C) $(f'' \circ g) \cdot (f' \circ g') \cdot (f \circ g'')$
(D) $(f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$
(E) $(f' \circ g') \cdot (f \circ g) + (f'' \circ g'')$

Answer: Option (D)

Explanation: This question is tricky because it requires the application of both the product rule and the chain rule, with the latter being used twice. We first note that:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Now, we differentiate both sides:

$$(f \circ g)'' = [(f' \circ g) \cdot g']'$$

The expression on the right side that needs to be differentiated is a product, so we use the product rule:

$$(f \circ g)'' = [(f' \circ g)' \cdot g'] + [(f' \circ g) \cdot g'']$$

Now, the inner composition $f' \circ g$ needs to be differentiated. We use the chain rule and obtain that $(f' \circ g)' = (f'' \circ g) \cdot g'$. Plugging this back in, we get:

$$(f \circ g)'' = (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$$

Remark: What's worth noting here is that in order to differentiate composites of functions, you need to use both composites *and* products (that's the chain rule). And in order to differentiate products, you need to use both products *and* sums (that's the product rule). Thus, in order to differentiate a composite twice, we need to use composites, products, *and* sums.

Performance review: 25 out of 26 got this. 1 chose (A).

Historical note (last time): 20 out of 21 people got this correct. 1 person left the question blank.

Historical note: I put this question in a quiz for Math 152 back in October 2010, and 14 of 14 people who took that quiz got it correct.

(7) Suppose $x = f(t)$ and $y = g(t)$ where f and g are both twice differentiable functions. What is d^2y/dx^2 in terms of f and g and their derivatives evaluated at t?
(A) $(f'(t)g''(t) - g'(t)f''(t))/(f'(t))^3$
(B) $(f'(t)g''(t) - g'(t)f''(t))/(g'(t))^3$
(C) $(g'(t)f''(t) - f'(t)g''(t))/(f'(t))^3$
(D) $(g'(t)f''(t) - f'(t)g''(t))/(g'(t))^3$
(E) None of the above

Answer: Option (A)

Explanation: See lecture notes.

Performance review: 25 out of 26 got this. 1 chose (B).

Historical note (last time): 20 out of 21 people got this correct. 1 person chose (E).

(8) Which of the following pair of bounds works for the arc length for the portion of the graph of the sine function between $(a, \sin a)$ and $(b, \sin b)$ where $a < b$?
(A) Between $(b - a)/\sqrt{3}$ and $(b - a)/\sqrt{2}$
(B) Between $(b - a)/\sqrt{2}$ and $b - a$

(C) Between $(b-a)$ and $\sqrt{2}(b-a)$
(D) Between $\sqrt{2}(b-a)$ and $\sqrt{3}(b-a)$
(E) Between $\sqrt{3}(b-a)$ and $2(b-a)$

Answer: Option (C)

Explanation: The derivative function is cos, so the corresponding arc length formula gives:

$$\int_a^b \sqrt{1+\cos^2 x}\,dx$$

The integrand is always between 1 and $\sqrt{2}$, so the integral must be between $1\cdot(b-a)$ and $\sqrt{2}\cdot(b-a)$.

Performance review: 20 out of 26 got this. 4 chose (D), 1 each chose (A) and (B).

Historical note (last time): 15 out of 21 people got this correct. 2 chose (B), 2 left the question blank, 1 each chose (A) and (D).

(9) (*) Consider the parametric curve $x = e^t$, $y = e^{t^2}$. How does y grow in terms of x as $x \to \infty$?
(A) y grows like a polynomial in x.
(B) y grows faster than any polynomial in x but slower than an exponential function of x.
(C) y grows exponentially in x.
(D) y grows super-exponentially in x but slower than a double exponential in x.
(E) y grows like a double exponential in x.

Answer: Option (B)

Explanation: Note that a polynomial in x is still exponential in t, and not in t^2, i.e., it is too slow to be y. Thus y grows faster than any polynomial in x. On the other hand, an exponential in x is doubly exponential in t, which is faster in growth than e^{t^2}. Thus, option (B).

Performance review: 11 out of 26 got this. 9 chose (D), 3 chose (E), 2 chose (C), 1 chose (A).

Historical note (last time): 7 out of 21 people got this correct. 4 each chose (A) and (E), 3 chose (C), 1 chose (D), and 2 left the question blank.

(10) (*) We say that a curve is *algebraic* if it admits a parameterization of the form $x = f(t)$, $y = g(t)$, where f and g are rational functions and t varies over some subset of the real numbers. Which of the following curves is *not* algebraic?
(A) $x = \cos t$, $y = \sin t$, $t \in \mathbb{R}$
(B) $x = \cos t$, $y = \cos(3t)$, $t \in \mathbb{R}$
(C) $x = \cos t$, $y = \cos^2 t$, $t \in \mathbb{R}$
(D) $x = \cos t$, $y = \cos(t^2)$, $t \in \mathbb{R}$
(E) None of the above, i.e., they are all algebraic

Answer: Option (D)

Explanation: In all the other cases, we can elucidate an algebraic relationship between the variables.

For option (A), we can set both $\cos t$ and $\sin t$ as rational functions in $\tan(t/2)$. In fact, the rational functions in $\tan(t/2)$ approach works for options (B) and (C) as well, though there are simpler approaches in those cases. The approach does not work for option (D).

Performance review: 12 out of 26 got this. 9 chose (E), 3 chose (A), 1 each chose (B) and (C).

Historical note (last time): 11 out of 21 people got this correct. 8 chose (E), 1 chose (B), 1 left the question blank.

(11) (+) Suppose $x = f(t)$, $y = g(t)$, $t \in \mathbb{R}$ is a parametric description of a curve Γ and both f and g are continuous on all of $\mathbb{R}$. If both f and g are even, what can we conclude about Γ and its parameterization?
(A) Γ is symmetric about the y-axis
(B) Γ is symmetric about the x-axis
(C) Γ is symmetric about the line $y = x$
(D) Γ has half turn symmetry about the origin
(E) The parameterizations of Γ for $t \leq 0$ and for $t \geq 0$ both cover all of Γ, and in directions mutually reverse to each other.

Answer: Option (E)

Explanation: See lecture notes.

Performance review: 23 out of 26 got this. 2 chose (A), 1 chose (C).

Historical note (last time): 5 out of 21 people got this correct. 7 chose (A), 4 chose (D), 2 each chose (B) and (C), 1 left the question blank.