CLASS QUIZ SOLUTIONS: NOVEMBER 16: VOLUME

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 11 question quiz. The score distribution was as follows:

- Score of 4: 1 person.
- Score of 6: 2 people.
- Score of 7: 3 people.
- Score of 8: 2 people.
- Score of 10: 3 people.

The mean score was 6.92. The problem wise answers were as follows:

(1) Option (B): 11 people
(2) Option (D): 5 people
(3) Option (B): 10 people
(4) Option (A): 8 people
(5) Option (B): 8 people.
(6) Option (C): 6 people.
(7) Option (C): 4 people.
(8) Option (C): 9 people.
(9) Option (E): 7 people.
(10) Option (A): 8 people.
(11) Option (B): 7 people.

2. Solutions

(1) Oblique cylinder:Right cylinder::
   (A) Rectangle:Square
   (B) Parallelogram:Rectangle
   (C) Disk:Circle
   (D) Triangle:Rectangle
   (E) Triangle:Square

   Answer: Option (B).

   Explanation: A right cylinder is obtained by translating a region of the plane along a direction perpendicular to the plane, while an oblique cylinder is obtained by translating along some direction, not necessarily perpendicular to the plane. Similarly, a rectangle is obtained by translating a line segment along a direction perpendicular to the line segment, while a parallelogram is obtained by translating a line segment along some direction, not necessarily perpendicular to the line segment.

   Performance review: Everybody got this correct.

   Historical note (last year): 14 out of 16 people got this correct. 2 people chose (D).

(2) Right circular cone:Right circular cylinder::
   (A) Triangle:Square
   (B) Rectangle:Square
   (C) Isosceles triangle:Equilateral triangle
   (D) Isosceles triangle:Rectangle
   (E) Isosceles triangle:Square

Answer: Option (D).

Explanation: There are two ways of seeing this. One is that if we look at a cross section containing the axis of symmetry, the cross section of a right circular cone is an isosceles triangle and the cross section of a right circular cylinder is a rectangle. Another way of thinking about this is that a rectangle is obtained by translating a line segment of fixed length in a perpendicular direction, and an isosceles triangle is obtained by translating it in a perpendicular direction while shirinking it symmetrically in a fixed proportion. Similarly, a right circular cylinder is obtained by translating a fixed disk, and a right circular cone is obtained by translating it while shrinking it.

Performance review: 5 out of 11 got this. 4 chose (A), 1 chose (E), 1 left the question blank.

Historical note (last year): 13 out of 16 people got this correct. 2 people chose (A) and 1 person chose (E).

(3) Circular disk:Circle::
   (A) Hollow cylinder:Solid cylinder
   (B) Solid cylinder:Hollow cylinder
   (C) Cube:Cuboid (cuboid is a term for rectangular prism)
   (D) Cube:Square
   (E) Cube:Sphere

   Answer: Option (B).

   Explanation: The circular disk is the region enclosed by the circle. Similarly, the solid cylinder is the region enclosed by the hollow cylinder.

   Performance review: 10 out of 11 people got this. 1 chose (D).

   Historical note (last year): 8 out of 16 people got this correct. 4 people chose (A) (the inverted option), 3 people chose (D), and 1 person chose (E).

(4) Circular disk:Line segment::
   (A) Solid sphere:Circular disk
   (B) Circle:Rectangle
   (C) Sphere:Cube
   (D) Cube:Right circular cylinder
   (E) Square:Triangle

   Answer: Option (A).

   Explanation: There are many many ways of seeing this, none of which is obvious.

   In terms of cross sections, a line segment is a one-dimensional cross section (i.e., intersection with a line) of a circular disk, while a circular disk is a two-dimensional cross section (i.e., intersection with a plane) of a solid sphere. Thus, a circular disk is to a solid sphere what a line segment is to a circular disk.

   Better, a line segment can be described as the set of points on a line (one-dimensional space) of distance at most a certain length from a fixed point. A circular disk is the analogue in two dimensions, and a solid sphere is the analogue in three dimensions. Thus, a solid sphere is to a circular disk what a circular disk is to a line segment.

   Performance review: 8 out of 11 got this. 2 chose (A), 1 chose (D).

   Historical note (last year): 14 out of 16 got this correct. 1 person chose (D) and 1 person left the question blank. The number of correct answers was surprisingly large given that the logic of the analogy is far from obvious, but many people probably used the difference of dimensions.

(5) Suppose a filled triangle ABC in the plane is revolved about the side AB. Which of the following best describes the solid of revolution thus obtained if both the angles A and B are acute (ignoring issues of boundary inclusion/exclusion)?
   (A) It is a right circular cone.
   (B) It is the union of two right circular cones sharing a common disk as base.
   (C) It is the set difference of two right circular cones sharing a common disk as base.
   (D) It is the union of two right circular cones sharing a common vertex.
   (E) It is the set difference of two right circular cones sharing a common vertex.

   Answer: Option (B)

Explanation: Let D be the foot of the perpendicular from C to AB. Since both the angles A and B are cute, D lies on the line segment AB. Then, the triangle ABC is the union of the right triangles ACD and BCD, sharing a common side CD. Each of these right triangles gives as its solid of revolution a right circular cone, with the base disk being the disk corresponding to CD in both. Thus, the overall solid is the union of the two right circular cones with a common disk.

Performance review: 8 out of 11 got this. 2 chose (A), 1 chose (E).

Historical note (last year): 13 out of 16 people got this correct. 2 people chose (D) and 1 person chose (A).

(6) Suppose a filled triangle ABC in the plane is revolved about the side AB. Which of the following best describes the solid of revolution thus obtained if the angle A is obtuse (ignoring issues of boundary inclusion/exclusion)?

(A) It is a right circular cone.
(B) It is the union of two right circular cones sharing a common disk as base.
(C) It is the set difference of two right circular cones sharing a common disk as base.
(D) It is the union of two right circular cones sharing a common vertex.
(E) It is the set difference of two right circular cones sharing a common vertex.

Answer: Option (C) (this is not quite precise language, because we need to be careful about boundaries, but it is basically correct).

Explanation: Let D be the foot of the perpendicular from C to the line AB. Unlike the previous case, D does *not* lie on the line segment AB, because the angle A is obtuse. In fact, it lies on the A-side of the line segment. Thus, the triangle ABC is the set difference of the right triangles BCD and ACD, sharing a common side CD (modulo some boundaries getting re-included). The corresponding solid of revolution is the set difference of the corresponding right circular cones, both of which have a common base disk corresponding to the side CD.

Performance review: 6 out of 11 got this. 3 chose (E), 1 chose (B), 1 chose (D).

Historical note (last year): 9 out of 16 people got this correct. 5 people chose (D) and 1 person each chose (B) and (E).

(7) What is the volume of the solid of revolution obtained by revolving the filled triangle ABC about the side AB, if the length of the base AB is b and the height corresponding to this base is h?

(A) $(1/6)\pi b^{3/2}h^{3/2}$
(B) $(1/3)\pi b^2 h$
(C) $(1/3)\pi b h^2$
(D) $(2/3)\pi b^2 h$
(E) $(2/3)\pi b h^2$

Answer: Option (C).

Explanation: The region is a union or difference of two right circular cones, as seen in some earlier multiple choice questions. Both these cones have *radius* h. The sum or difference of the heights of these cones is b. Thus, the formula gives (C).

For the next two questions, suppose Ω is a region in a plane Π and ℓ is a line on Π such that Ω lies completely on one side of ℓ (in particular, it does not intersect ℓ). Let Γ be the solid of revolution obtained by revolving Ω about ℓ. Suppose further that the intersection of Ω with any line perpendicular to ℓ is either empty or a point or a line segment.

Performance review: 4 out of 11 got this. 3 chose (C), 3 chose (E), 1 left the question blank.

Historical note: 10 out of 16 people got this correct. 2 people each chose (B) and (D) and 1 person each chose (A) and (E).

(8) (*) What is the intersection of Γ with Π (your answer should be always true)?

(A) It is precisely Ω.
(B) It is the union of Ω and a translate of Ω along a direction perpendicular to ℓ.
(C) It is the union of Ω and the reflection of Ω about ℓ.
(D) It is either empty or a rectangle whose dimensions depend on Ω.
(E) It is either empty or a circle or an annulus whose inner and outer radius depend on Ω.

Answer: Option (C).

Explanation: The intersection of Γ with π comprises those regions obtained by revolving Ω that land inside π. This is precisely Ω and the region obtained by revolving Ω by the *angle* π (180 degrees; unfortunately there's symbol overloading here), which is equivalent to the reflection of Ω about ℓ.

Performance review: 9 out of 11 got this correct. 2 chose (E).

Historical note (last year): 6 out of 16 people got this correct. 4 people chose (A), 3 people chose (E), 2 people chose (D), and 1 person chose (B).

(9) What is the intersection of Γ with a plane perpendicular to ℓ (your answer should be always true)?
(A) It is precisely Ω.
(B) It is the union of Ω and a translate of Ω along a direction perpendicular to ℓ.
(C) It is the union of Ω and the reflection of Ω about ℓ.
(D) It is either empty or a rectangle whose dimensions depend on Ω.
(E) It is either empty or a circle or an annulus whose inner and outer radius depend on Ω.

Answer: Option (E) (Oops, *circle* should have been *circular disk*)

Explanation: This is precisely the annulus procedure used to justify the washer method.

Note that the description given by option (D) requires Ω to satisfy the condition given in the correction, which essentially says that the slices perpendicular to ℓ are nice enough.

Performance review: 7 out of 11 go this correct. 2 chose (B) and 2 chose (C).

Historical note: 9 out of 16 people got this correct. 3 people chose (C), 2 people chose (B), 1 person chose (D), and 1 person left the question blank.

(10) (*) Consider a fixed equilateral triangle ABC. Now consider, for any point D outside the plane of ABC, the solid tetrahedron $ABCD$. This is the solid bounded by the triangles ABC, BCD, ACD, and ABD. The volume of this solid depends on D. What specific information about D completely determines the volume?
(A) The perpendicular distance from D to the plane of the triangle ABC.
(B) The minimum of the distances from D to points in the filled triangle ABC.
(C) The location of the point E in the plane of triangle ABC that is the foot of the perpendicular from D to ABC.
(D) The distance from D to the center of ABC (here, you can take the center as any of the notions of center since ABC is equilateral).
(E) None of the above.

Answer: Option (A).

Explanation: In fact, the volume is 1/3 times the area of the base (which is fixed) times this perpendicular distance.

Performance review: 8 out of 11 got this correct. 2 chose (D) and 1 chose (E).

Historical note (last year): 7 out of 16 people got this correct. 3 people chose (C), 3 people chose (D), and 1 person each chose (B) and (E).

(11) (**) For $r > 0$, consider the region $\Omega_r(a)$ bounded by the x-axis, the curve $y = x^{-r}$, and the lines $x = 1$ and $x = a$ with $a > 1$. Let $V_r(a)$ be the volume of the region obtained by revolving $\Omega_r(a)$ about the x-axis. What is the precise set of values of r for which $\lim_{a\to\infty} V_r(a)$ is finite?
(A) All $r > 0$
(B) $r > 1/2$
(C) $r > 1$
(D) $r > 2$
(E) No value of r

Answer: Option (B).

Explanation: $V_r(a) = \pi \int_1^a x^{-2r}\, dx$. The limit is finite iff $2r > 1$, which is equivalent to $r > 1/2$.

Performance review: 7 out of 11 got this correct. 3 chose (A) and 1 chose (C).

Historical note (last year): 3 out of 16 people got this correct. 7 people chose (C), 4 people chose (A), and 2 people chose (E).