TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY NOVEMBER 22: LINEAR DYNAMICAL SYSTEMS

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

24 people took this 12-question quiz. The score distribution was as follows:

- Score of 3: 3 people
- Score of 4: 5 people
- Score of 5: 6 people
- Score of 6: 4 people
- Score of 7: 3 people
- Score of 10: 3 people

The mean score was about 5.58.

The question-wise answers and performance review were as follows:

(1) Option (A): 21 people
(2) Option (D): 11 people
(3) Option (E): 8 people
(4) Option (E): 10 people
(5) Option (A): 11 people
(6) Option (B): 13 people
(7) Option (C): 14 people
(8) Option (B): 15 people
(9) Option (A): 9 people
(10) Option (C): 7 people
(11) Option (A): 8 people
(12) Option (A): 7 people

2. Solutions

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz covers a topic that we will not be able to get to formally in the course due to time constraints. The corresponding section of the book is Section 7.1, and there is more relevant material discussed in the later sections of Chapter 7. However, you do not need to read those sections in order to attempt this quiz. Also, simply mastering the computational techniques in those sections of the book will not help you much with the quiz questions.

The questions here consider a linear dynamical system. Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let A be the matrix of T, so that A is a $n \times n$ matrix. For any positive integer r, the matrix A^r is the matrix for the linear transformation T^r (note here that T^r refers to the r-fold *composite* of T). The goal is to determine, starting off with an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how the following sequence behaves:

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \dots$$

More explicitly, each term of the sequence is obtained by applying T to the preceding term. In other words, the sequence is:

$$\vec{x}, T(\vec{x}), T(T(\vec{x})), T(T(T(\vec{x}))), \dots$$

(1) What is the necessary and sufficient condition on A such that for *every* choice of $\vec{x} \in \mathbb{R}^n$, the sequence described above eventually reaches, and stays at, the zero vector? Note that if it reaches the zero vector, it must do so in at most n steps. Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Answer: Option (A)

Explanation: First, note that if A is a nilpotent matrix, there exists a positive integer r such that A^r is the zero matrix, which is equivalent to requiring that T^r be the zero linear transformation. In particular, this means that $T^r(\vec{x})$ is the zero vector for all initial vectors $\vec{x}$, so every sequence eventually reaches the zero vector.

Other direction: From the fact stated, if the sequence reaches the zero vector, it must do so in n steps, so that this means that if the condition holds for every $\vec{x} \in \mathbb{R}^n$, then T^n sends every vector to the zero vector. In particular, this means that A^n is the zero matrix, so A is nilpotent.

Performance review: 21 out of 24 got this. 3 chose (E).

Historical note (last time): 21 out of 24 got this. 2 chose (E), 1 chose (B).

(2) What is the necessary and sufficient condition on A such that there *exists* a nonzero vector $\vec{x} \in \mathbb{R}^n$ for which the sequence described above eventually reaches, and stays at, the zero vector? Note that if it reaches the zero vector, it must do so in at most n steps. Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Answer: Option (D)

Explanation: Suppose $\vec{x}$ is a nonzero vector and r is the smallest positive integer such that the vector $A^r\vec{x}$ is the zero vector. Then, $A(A^{r-1}\vec{x}) = 0$ but $A^{r-1}\vec{x} \neq 0$, so that $A^{r-1}\vec{x}$ is a nonzero vector in the kernel of T. Thus, T has a nonzero kernel, so it must be non-invertible, hence A must be a non-invertible matrix.

Conversely, if A is non-invertible, there is a nonzero vector, say $\vec{x}$, in the kernel of A. This vector can be used.

Performance review: 11 out of 24 got this. 7 chose (A), 5 chose (E), 1 chose (B).

Historical note (last time): 5 out of 24 got this. 15 chose (A), 3 chose (C), and 1 chose (E).

(3) What is the necessary and sufficient condition on A such that for *every* choice of $\vec{x} \in \mathbb{R}^n$, the sequence described above returns to $\vec{x}$ after a finite and positive number of steps? Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Answer: Option (E)

Explanation: It is definitely a *necessary* condition that A be invertible, otherwise there would be a nonzero vector in its kernel for which the sequence could never return to the vector. However, this is not sufficient. Consider the case where $n = 1$ and the matrix A is $[2]$. This is invertible, but applying it repeatedly to a nonzero vector can never get us back to the original vector.

Performance review: 8 out of 24 got this. 16 chose (B).

Historical note (last time): 5 out of 24 got this. 16 chose (B), 2 chose (C), 1 chose (A).

(4) What is the necessary and sufficient condition on A such that there *exists* a nonzero vector $\vec{x} \in \mathbb{R}^n$ for which the sequence described above returns to $\vec{x}$ after a finite and positive number of steps? Please see Option (E) before answering.

(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Answer: Option (E)

Explanation: Similar to the preceding question, except now that A does not even need to be invertible.

Performance review: 10 out of 24 got this. 7 chose (C), 6 chose (B), 1 chose (A).

Historical note (last time): 3 out of 24 got this. 11 chose (A), 9 chose (B), 1 chose (C).

(5) Suppose $n = 2$ and T is a rotation by an angle that is a rational multiple of π. What can we say about the range of the sequence

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \dots$$

starting from a nonzero vector $\vec{x}$?

(A) The range is finite, i.e., there are only finitely many distinct vectors in the sequence.
(B) The range is infinite and forms a dense subset of the circle centered at the origin and with radius equal to the length of the vector $\vec{x}$. However, it is not the entire circle.
(C) The range is infinite and is the entire circle centered at the origin and with radius equal to the length of the vector $\vec{x}$.
(D) The range is infinite and forms a dense subset of the line of the vector $\vec{x}$ (excluding the origin), but is not the entire line (excluding the origin).
(E) The range is infinite and is the entire line of the vector $\vec{x}$, excluding the origin.

Answer: Option (A)

Explanation: If the angle of rotation for T is θ, then the angle of rotation for T^r is $r\theta$. This is because angles of rotation add up when we compose the rotations.

Since the angle of rotation θ is a rational multiple of π, it is of the form $p\pi/q$ where p and q are integers with $q \neq 0$. Then, $2q\theta = 2\pi p$. This implies that T^{2q} is rotation by an integer multiple of 2π, and hence, is the identity transformation. In particular, this means that for any nonzero vector $\vec{x}$, the sequence $\vec{x}, T(\vec{x}), T^2(\vec{x}), \dots$ returns to $\vec{x}$ at $T^{2q}(\vec{x})$. Beyond that point, it will just cycle the same set of vectors.

Performance review: 11 out of 24 got this. 9 chose (C), 3 chose (B), 1 chose (D).

Historical note (last time): 4 out of 24 got this. 12 chose (B), 5 chose (C), 2 chose (E), 1 chose (D).

(6) Suppose $n = 2$ and T is a rotation by an angle that is a irrational multiple of π. What can we say about the range of the sequence

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \dots$$

starting from a nonzero vector $\vec{x}$?

(A) The range is finite, i.e., there are only finitely many distinct vectors in the sequence.
(B) The range is infinite and forms a dense subset of the circle centered at the origin and with radius equal to the length of the vector $\vec{x}$. However, it is not the entire circle.
(C) The range is infinite and is the entire circle centered at the origin and with radius equal to the length of the vector $\vec{x}$.
(D) The range is infinite and forms a dense subset of the line of the vector $\vec{x}$ (excluding the origin), but is not the entire line (excluding the origin).
(E) The range is infinite and is the entire line of the vector $\vec{x}$, excluding the origin.

Answer: Option (B)

Explanation: Since T is rotation by an *irrational* multiple θ of π, there is no positive integer multiple of θ that is also an integer multiple of 2π. Thus, T^r is not the identity map for any positive integer r, so there is no cycling back, so we do get an infinite set of vectors. All of them have the same length as $\vec{x}$, because rotations preserve length. Thus, they lie on the circle centered at the origin with length equal to the length of the vector $\vec{x}$. In fact, we can show that they form a dense subset of the circle, i.e., they come arbitrarily close to every point of the circle. Note, however, that we do not get all points on the circle. For instance, $-\vec{x}$ is not in the range, because achieving it would require a nonzero rational (in fact, odd integer) multiple of π, and no integer multiple of an irrational multiple of π is of that form.

Performance review: 13 out of 24 got this. 8 chose (C), 2 chose (D), 1 chose (A).

Historical note (last time): 9 out of 24 got this. 4 chose (A), 7 chose (C), 3 chose (D), 1 chose (E).

We return to generic n now.

(7) A nonzero vector $\vec{x}$ is termed an *eigenvector* for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with *eigenvalue* a real number $\lambda \in \mathbb{R}$ if $T(\vec{x}) = \lambda\vec{x}$. Note that λ is allowed to be 0. We sometimes conflate the roles of T and its matrix A, so that we call $\vec{x}$ an eigenvector for A and λ an eigenvalue for A.

If $\vec{x}$ is an eigenvector of T (or equivalently, of A) with eigenvalue λ, which of the following is true? We denote by I_n the identity transformation from $\mathbb{R}^n$ to itself.

(A) $\vec{x}$ must be in the kernel of the linear transformation $T + \lambda I_n$
(B) $\vec{x}$ must be in the image of the linear transformation $T + \lambda I_n$
(C) $\vec{x}$ must be in the kernel of the linear transformation $T - \lambda I_n$
(D) $\vec{x}$ must be in the image of the linear transformation $T - \lambda I_n$
(E) $\vec{x}$ must be in the kernel of the linear transformation λT

Answer: Option (C)

Explanation: We are trying to find the vectors $\vec{x}$ such that $T(\vec{x}) = \lambda\vec{x}$. This can be rewritten as $T\vec{x} = \lambda I_n\vec{x}$ and hence as $(T - \lambda I_n)\vec{x} = 0$, so that $\vec{x}$ is in the kernel of $T - \lambda I_n$.

Note that the other options are false for the following reasons:

- Option (A): Consider the case that $T = I_n$, $\vec{x}$ is any nonzero vector, and $\lambda = 1$. Then, $T + \lambda I_n = 2I_n$, the kernel of which is zero, so $\vec{x}$ is not in the kernel.
  In fact, Option (A) holds true if and only if $\lambda = 0$.
- Option (B): Consider the case that $T = 0$, $\vec{x}$ is any nonzero vector, and $\lambda = 0$. Then, $T+\lambda I_n = 0$, the image of which is zero, so $\vec{x}$ is not in the image.
  In fact, Option (B) holds true if $\lambda \neq 0$. If $\lambda \neq 0$, then $\vec{x} = (T + \lambda I_n)(\vec{x}/(2\lambda))$. For $\lambda = 0$, Option (B) may or may not hold. The example of T being the zero linear transformation gives a situation where it does not hold. Here is an example where it does hold. Consider:
  $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
  Then, $\vec{e}_1$ is an eigenvector for T with eigenvalue 0, and it *is* in the image of $T + \lambda I_n = T$, since $T(\vec{e}_2) = \vec{e}_1$.
- Option (D): Consider the case that $T = I_n$, $\vec{x}$ is any nonzero vector, and $\lambda = 1$. Then, $T - \lambda I_n = 0$, the image of which is zero, so $\vec{x}$ is not in the image.
  Option (D) is often true and often false, but its truth or falsehood does not have any direct relation with whether λ is zero or nonzero.
- Option (E): Consider the case that $T = I_n$, $\vec{x}$ is any nonzero vector, and $\lambda = 1$. Then, $\lambda T = I_n$, the kernel of which is zero, so $\vec{x}$ is not in the kernel.
  In fact, Option (E) holds true if and only if $\lambda = 0$.

Performance review: 14 out of 24 got this. 4 chose (E), 3 chose (B), 2 chose (A), 1 chose (D).

Historical note (last time): 17 out of 24 got this. 4 chose (A), 2 chose (B), 1 chose (E).

(8) As above, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Use the terminology of eigenvector and eigenvalue from the preceding question. Which of the following is a characterization of the situation that A is a diagonal matrix?

(A) Every nonzero vector in $\mathbb{R}^n$ is an eigenvector for T.
(B) Every standard basis vector in $\mathbb{R}^n$ is an eigenvector for T.

(C) Every vector with at least one zero coordinate in $\mathbb{R}^n$ is an eigenvector for T.
(D) T has a unique eigenvector (up to scalar multiples, i.e., all eigenvectors of T are scalar multiples of each other).
(E) T has no eigenvector.

Answer: Option (B)

Explanation: Suppose the matrix of T has c_i as the i^{th} diagonal entry. Then, $T(\vec{e_i})$ is the i^{th} column of the diagonal matrix, which has 0s everywhere except in the diagonal entry, which is c_i. Thus, $T(\vec{e_i}) = c_i\vec{e_i}$, so that all standard basis vectors are eigenvectors.

Conversely, if every standard basis vector is an eigenvector, then for each i, $T(\vec{e_i}) = c_i\vec{e_i}$ for some c_i, which forces the i^{th} column to have the value c_i in the diagonal position and the value 0 elsewhere. This gives a diagonal matrix.

Performance review: 15 out of 24 got this. 5 chose (D), 2 chose (A), 1 each chose (C) and (E).

Historical note (last time): 13 out of 24 got this. 5 chose (E), 3 chose (D), 2 chose (C), 1 chose (A).

(9) As above, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Use the terminology of eigenvector and eigenvalue from the preceding question. Which of the following is a characterization of the situation that A is a scalar matrix (i.e., a diagonal matrix with all diagonal entries equal)?
(A) Every nonzero vector in $\mathbb{R}^n$ is an eigenvector for T.
(B) Every standard basis vector in $\mathbb{R}^n$ is an eigenvector for T.
(C) Every vector with at least one zero coordinate in $\mathbb{R}^n$ is an eigenvector for T.
(D) T has a unique eigenvector (up to scalar multiples, i.e., all eigenvectors of T are scalar multiples of each other).
(E) T has no eigenvector.

Answer: Option (A)

Explanation: If the matrix for T is scalar with scalar value λ, then $T(\vec{x}) = \lambda\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^n$. Thus, all nonzero vectors are eigenvectors.

To establish the converse, we need to show that if every nonzero vector of T is an eigenvector, then all of them have the *same* eigenvalue. Note that two vectors in the same line have the same eigenvalue, so the statement is trivial for $n = 1$.

For $n \geq 2$, we already know that the matrix of T is diagonal on account of the preceding question. We want to show that all diagonal entries are equal to each other. Consider the i^{th} and j^{th} diagonal entries. Let's say these are c_i and c_j respectively. Then, $T(\vec{e_i} + \vec{e_j}) = T(\vec{e_i}) + T(\vec{e_j})$. This simplifies to $c_i\vec{e_i} + c_j\vec{e_j}$. For this to be a multiple of $\vec{e_i} + \vec{e_j}$, we need that $c_i = c_j$. Since this is true for every pair of i and j, we get that all the diagonal entries are equal and that the matrix is a scalar matrix.

Performance review: 9 out of 24 got this. 8 chose (B), 7 chose (D).

Historical note (last time): 8 out of 24 got this. 11 chose (B), 2 each chose (C) and (E), 1 chose (D).

(10) Suppose A is a strictly upper-triangular $n \times n$ matrix, i.e., all entries of A that are on or below the main diagonal are zero. T is the linear transformation corresponding to A. It will turn out that the only eigenvalue for T is 0. What can we say about the eigenvectors for T for this eigenvalue?
(A) All nonzero vectors in $\mathbb{R}^n$ are eigenvectors for T with eigenvalue 0.
(B) All standard basis vectors in $\mathbb{R}^n$ are eigenvectors for T with eigenvalue 0.
(C) The vector $\vec{e_1}$ is an eigenvector for T with eigenvalue 0. The information presented is not sufficient to determine whether any of the other standard basis vectors is an eigenvector.
(D) The vector $\vec{e_n}$ is an eigenvector for T with eigenvalue 0. The information presented is not sufficient to determine whether any of the other standard basis vectors is an eigenvector.
(E) At least one of the standard basis vectors is an eigenvector for T with eigenvalue 0. However, the information presented is not sufficient to say definitively for any particular standard basis vector that it is an eigenvector.

Answer: Option (C)

Explanation: The first column is the zero column, so the vector $\vec{e_1}$ gets mapped to zero. As for the remaining standard basis vectors, we do not have enough information to know whether they are sent to zero. This is because there are entries above the diagonal in those columns, and these are allowed to be nonzero.

For instance, consider:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This sends $\vec{e}_1$ to the zero vector and sends $\vec{e}_2$ to $\vec{e}_1$.

Performance review: 7 out of 24 got this. 5 each chose (A) and (D), 3 each chose (B) and (E), 1 left the question blank.

Historical note (last time): 5 out of 24 got this. 9 chose (A), 6 chose (E), 2 each chose (B) and (D).

(11) Suppose A is a strictly upper-triangular $n \times n$ matrix, i.e., all entries of A that are on or below the main diagonal are zero. T is the linear transformation corresponding to A. Which of the following is A guaranteed to be? Please see Options (D) and (E) before answering.
(A) Nilpotent
(B) Idempotent
(C) Invertible
(D) All of the above
(E) None of the above

Answer: Option (A)

Explanation: The image of T is contained in the span of the vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}$. The image of T^2 is in the span of $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-2}$. Each time we apply T, we lose the last basis vector. Thus, T^n is the zero transformation, so A^n is the zero matrix, so A is nilpotent.

For instance, consider:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The image of T is spanned by the column vectors. The first column is the zero column, so it is spanned by the other two column vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

This is the same as the span of $\vec{e}_1$ and $\vec{e}_2$.

The image of T^2 is thus the image of the span of these two vectors. This is the span of the first two columns of A. The first column is zero, so the image of T^2 is simply the span of the second column vector:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is the span of $\vec{e}_1$. The image of this under T again is the zero space. Thus, the image of T^3 is zero, so $A^3 = 0$.

Performance review: 8 out of 24 got this. 10 chose (E), 5 chose (C), 1 chose (D).

Historical note (last time): 7 out of 24 got this. 10 chose (C), 5 chose (E), 1 chose (D). 1 left the question blank.

(12) Consider the case $n = 2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by an angle that is *not* an *integer* multiple of π. What can we say about the set of eigenvectors and eigenvalues for T?
(A) T has no eigenvectors
(B) T has one eigenvector (up to scalar multiples) with eigenvalue 1
(C) T has one eigenvector (up to scalar multiples) and the eigenvalue depends on the angle of rotation

(D) T has two linearly independent eigenvectors (so that the set of all eigenvectors is obtained as the set of scalar multiples of either one of these vectors) with the same eigenvalue

(E) T has two linearly independent eigenvectors (so that the set of all eigenvectors is obtained as the set of scalar multiples of either one of these vectors) with distinct eigenvalues

Answer: Option (A)

Explanation: Every nonzero vector gets rotated, so no nonzero vector goes to a scalar multiple of itself. Note that the case that the angle of rotation is an integer multiple of π differs: in that case, the transformation is either the identity or the negative identity and hence is scalar, so all nonzero vectors are eigenvectors for it.

Performance review: 7 out of 24 got this. 6 chose (C), 4 each chose (B) and (D), 3 chose (E).

Historical note (last time): 6 out of 24 got this. 10 chose (E), 4 chose (C), 2 chose (D), 1 chose (B).