# HOMEWORK 8 CHECKLIST: DUE MONDAY DECEMBER 2

MATH 196, SECTION 57 (VIPUL NAIK)

## 1. Routine problems

*Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.*

(1) Exercise 3.4.1 (Page 159): Determine whether the vector $\vec{x}$ is in the span $V$ of the vectors $\vec{v}_1, \dots, \vec{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\vec{x}$ is in $V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathcal{B}}$:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since these are the standard basis vectors, the usual coordinates of $\vec{x}$ are its coordinates in this basis. Explicitly, $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$.

(2) Exercise 3.4.2 (Page 159; was 3.4.4 in the 4th Edition): Determine whether the vector $\vec{x}$ is in the span $V$ of the vectors $\vec{v}_1, \dots, \vec{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\vec{x}$ is in $V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathcal{B}}$:

$$\vec{x} = \begin{bmatrix} 23 \\ 29 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 46 \\ 58 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 61 \\ 67 \end{bmatrix}$$

In this case, it should be clear by inspection. In fact, we can express $\vec{x}$ using just one of the two vectors (which one?) so the other coordinate is zero.

(3) Exercise 3.4.18 (Page 159): Determine whether the vector $\vec{x}$ is in the span $V$ of the vectors $\vec{v}_1, \dots, \vec{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\vec{x}$ is in $V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathcal{B}}$:

$$\vec{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

You can set up the linear system with coefficient matrix given by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and augmenting column given by the vector $\vec{x}$.

(4) Exercise 3.4.25 (Page 160): Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The general formula is as follows: if $S$ is the matrix whose columns are given by the new basis vectors, then $B = S^{-1}AS$. See the book or lecture notes for more discussion.

(5) Exercise 3.4.27 (Page 160): Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Same remark as for preceding question.

(6) Exercise 3.4.37 (Page 160): Find a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix $B$ of the given linear transformation $T$ is diagonal: Orthogonal projection $T$ onto the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Choose a basis such that the first vector is the vector you are trying to project onto, and the second vector is a nonzero vector orthogonal to it. The matrix for the linear transformation $T$ *in* this new basis would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(7) Exercise 4.1.3 (Page 176): Is the subset of $P_2$ given here a subspace? If so, find a basis, where $p'$ is the derivative:

$$\{p(t) : p'(1) = p(2)\}$$

To check whether a subset is a subspace, you need to check that (a) it contains the zero vector (in this case, the zero polynomial), (b) it is closed under addition of vectors (in this case, addition of polynomials), (c) it is closed under multiplication by scalars. Of these three conditions, if there is one you should check first, it is (b). Usually, (b) is what will fail.

In this case, note that differentiation and evaluation are both linear, so the condition $p'(1) = p(2)$ is a linear constraint on the coefficients. Thus, the solution space is a subspace. Formally, we need to show that:

(a) The zero polynomial satisfies the condition: It does.

(b) If $p$ and $q$ satisfy the condition, so does $p + q$: Use the fact that $(p + q)' = p' + q'$ in the course of the proof.

(c) If $p$ satisfies the condition and $\lambda$ is a real number, then $\lambda p$ satisfies the condition: In the course of the proof, use that $(\lambda p)' = \lambda p'$.

Now, we move to the task of explicitly describing the space. First, note that the following is a *generic* description of an element of $P_2$:

$$p(t) = at^2 + bt + c$$

We can think of these polynomials as being described in the basis $t^2, t, 1$, in which case the coordinates for the above are $a, b, c$. The condition $p'(1) = p(2)$ gives a linear equation in $a, b, c$, which can be written as:

$$\begin{bmatrix} \text{A single row} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

We now want to find all solutions to this, i.e., the kernel. We know how to do that, and also how to get a basis for the kernel. Note that our basis description involves vectors written in coordinates. Finally, we need to re-convert the basis elements we have into their polynomial form.

Note that the dimension of this space is 2. The intuitive reason is that we are starting with $P_2$, a three-dimensional space, and we impose one linear constraint, so the dimension goes down by one, and we end up getting a two-dimensional space.

(8) Exercise 4.1.20 (Page 176): Find a basis for the following space and determine its dimension: the space of all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$ such that $a + d = 0$.

We can eliminate either one of the parameters $a$ and $d$ by writing it as the negative of the other. $b$, $c$, and whichever of $a$ and $d$ we don't eliminate are the three parameters controlling our subspace (why's it a subspace? duh!). Note that we could also do this formally: first, flatten out the matrix to a column vector:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The condition $a + d = 0$ means that:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Thus, $b$, $c$, and $d$ are the non-leading variables whereas $a$ is a leading variable that can be written as $-d$. The basis vectors we get are:

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that we now need to re-convert these to matrices! So, for instance, our first vector becomes:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(9) Exercise 4.1.27 (Page 176): Find a basis for the following space and determine its dimension: the space of all $2 \times 2$ matrices $A$ that commute with $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Just set up a general matrix, and determine the condition for it to commute with this matrix. The answer will turn out to be the space of diagonal matrices, and there is a natural basis for that space!

(10) Exercise 4.1.28 (Page 176): Find a basis for the following space and determine its dimension: the space of all $2 \times 2$ matrices $A$ that commute with $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

This is similar to the preceding question in method, though different in answer.

(11) Exercise 4.2.23 (Page 184): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f(t)) = f(7) \text{ from } P_2 \text{ to } \mathbb{R}$$

$(f+g)(7)$ is *defined* as $f(7) + g(7)$ and likewise for the other condition for a transformation to be linear. Perform basic reasoning to see if it is injective and if it is surjective (it's an isomorphism only if it is both injective and surjective). Consider for instance the polynomial $t - 7$ and where it goes.

(12) Exercise 4.2.24 (Page 184): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f(t)) = f''(t)f(t) \text{ from } P_2 \text{ to } P_2$$

Consider the question of linearity first.

(13) Exercise 4.2.37 (Page 185): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f) = f + f' \text{ from } C^\infty \text{ to } C^\infty$$

Linearity follows from the fact that differentiation is linear. Finding the fibers over a point amounts to solving a first-order linear differential equation. Recall the solution method. You might also benefit by reviewing some related quiz questions from the November 20 and November 25 quizzes or reading Section 7 of the lecture notes on abstract vector spaces.

(14) Exercise 4.2.43 (Page 185): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f(t)) = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix} \text{ from } P_2 \text{ to } \mathbb{R}^3$$

Linearity is easy. Note that both $P_2$ and $\mathbb{R}^3$ are three-dimensional, so it is at minimum *possible* that this map is an isomorphism. If we use coordinates $1, t, t^2$ for $P_2$, then the matrix of the linear transformation is:

$$\begin{bmatrix} 1 & 5 & 5^2 \\ 1 & 7 & 7^2 \\ 1 & 11 & 11^2 \end{bmatrix}$$

We can perform row reduction to check that this matrix is invertible, i.e., it has full rank. Note that we do not need to actually *find* the inverse, so it suffices to do Gaussian elimination rather than Gauss-Jordan elimination, and reach an upper triangular matrix with nonzero entries everywhere on the diagonal, because once we reach there, we know that a few more steps will get us to the identity matrix.

(15) Exercise 4.2.45 (Page 185): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f(t)) = t(f(t)) \text{ from } P \text{ to } P$$

Here, $P$ is used for the vector space of all polynomials, with no specified bound on the degree.

First, check linearity. As for isomorphism, note the absence of surjectivity: polynomials with a nonzero constant term are not in the image.

(16) Exercise 4.2.49 (Page 185): Find out if the transformation here is linear, and if so, determine whether it is an isomorphism:

$$T(f(t)) = f(t^2) \text{ from } P \text{ to } P$$

Here, $P$ is used for the vector space of all polynomials, with no specified bound on the degree.

First, check linearity. What kind of polynomials are in the image?

(17) Exercise 4.2.66 (Page 185): Find the kernel and nullity of the linear transformation $T(f) = f - f'$ from $C^\infty$ to $C^\infty$.

Similar note as for Question 13 (Exercise 4.2.37), but now you actually need to *solve* the differential equation to find the kernel.