HOMEWORK 5: DUE MONDAY NOVEMBER 4

MATH 196, SECTION 57 (VIPUL NAIK)

1. Routine problems

Please write your solutions clearly, show relevant steps, but be concise. Underline, highlight, or box your final answers to make life easy for the grader.

(1) Exercise 2.2.2 (Page 71): Find the matrix of a rotation through an angle of 60° in the counterclockwise direction.

(2) Exercise 2.2.6 (Page 71): Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

(3) Exercise 2.2.7 (Page 71): Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.

For the next five questions, find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ described.

(4) Exercise 2.2.19 (Page 72): The orthogonal projection onto the xy-plane.

(5) Exercise 2.2.20 (Page 72): The reflection about the xz-plane.

(6) Exercise 2.2.21 (Page 72): The rotations about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

(7) Exercise 2.2.22 (Page 72): The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

(8) Exercise 2.2.23 (Page 72): The reflection about the plane $y = z$.

2. Problems for your own review, not for submission

(1) Exercise 2.2.27 (Page 72): Please see this from the book.

(2) Exercise 2.2.28 (Page 72-73): Please see this from the book.

(3) Exercise 2.4.42 (Page 98): A square matrix is called a *permutation matrix* if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are I_n and

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

3. Advanced problems

(1) Exercise 2.2.29 (Page 73): Let T be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, and L be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that $L(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$ and $T(L(\vec{y})) = \vec{y}$ for all $\vec{y}$ in $\mathbb{R}^n$. If T is a linear transformation, show that L is as well. [*Hint*: $\vec{v} + \vec{w} = T(L(\vec{v})) + T(L(\vec{w})) = T(L(\vec{v}) + L(\vec{w}))$ since T is linear. Now apply L on both sides.]

(2) Exercise 2.2.37 (Page 73): The *trace* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the sum $a + d$ of its diagonal entries. What can you say about the trace of a 2×2 matrix that represents a/an

(a) orthogonal projection
(b) reflection about a line
(c) rotation
(d) (horizontal or vertical) shear

In three cases, given the exact value of the trace, and in one case, give an interval of possible values.

(3) Exercise 2.2.38 (Page 73): The *determinant* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$ (we have seen this quantity in Exercise 2.1.13 already, as Homework 3, Advanced Question 1). What can you say about the determinant of a (2×2) matrix that represents a/an
(a) orthogonal projection
(b) reflection about a line
(c) rotation
(d) (horizontal or vertical) shear

What do your answers tell you about the invertibility of these matrices?

(4) Exercise 2.3.29 (Page 85): Consider the matrix

$$D_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

We know that the transformation $T(\vec{x}) = D_\alpha(\vec{x})$ is a counterclockwise rotation through an angle of α.
(a) For two angles α and β, consider the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the linear transformations $\vec{y} = D_\alpha D_\beta(\vec{x})$ and $\vec{y} = D_\beta D_\alpha(\vec{x})$. Are the two transformations the same?
(b) Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \end{aligned}$$

Note: For the second identity, the use of $\pm$ along with $\mp$ indicates that the $+$ case on the left corresponds to the $-$ case on the right, and the $-$ case on the left corresponds to the $+$ case on the right.

(5) Exercise 2.4.32 (Page 98): Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad-bc = 1$ and $A = A^{-1}$.

(6) Exercise 2.4.41 (Page 98): Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ is invertible? Find the inverse if it exists.
(a) Reflection about a plane
(b) Orthogonal projection onto a plane
(c) Scaling by a factor of 5 [i.e., $T(\vec{v}) = 5\vec{v}$ for all vectors $\vec{v}$]
(d) Rotation about an axis

(7) Exercise 2.4.49 (Page 99-100): *Input-Output Analysis*. This is a very lengthy problem. Please see it from the book.