CLASS QUIZ SOLUTIONS: OCTOBER 19: INCREASE/DECREASE AND MAXIMA/MINIMA

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took the quiz. The score distribution was as follows:

- Score of 2: 8 people
- Score of 3: 3 people

The mean score was 2.08. The problem wise performance was:

1. Option (D): 10 people
2. Option (B): 6 people
3. Option (B): 2 people
4. Option (B): 7 people

2. Solutions

(1) Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?
(A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
(B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
(C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
(D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
(E) None of the above is true in general.

Answer: Option (D)

Explanation: If $f(x) \le f(c)$ for all $a \le x < c$, then all difference quotients from the left are nonnegative. The limiting value, which is the left-hand derivative, is thus also nonnegative. See the lecture notes for more details.

The other choices: Options (A) and (B) predict the wrong sign. Option (C) is incorrect because even though the difference quotients are all strictly positive, their limiting value could be 0. For instance, $\sin x$ on $[0, \pi/2]$ or x^3 on $[-1, 0]$.

Performance review: 10 out of 11 got this correct. 1 person chose (E).

Historical note (last year): 8 people got this correct. 5 people chose option (B) and 2 people chose option (E). It is likely that the people who chose option (B) made a sign computation error.

Action point: On the plus side, most of you seem to have understood the fact that strict inequality does not guarantee strict positivity or negativity of the one-sided derivative. But, please sort out your sign issues while the quarter is still young! Getting the right sign is a good sign for the future.

(2) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing function from $\mathbb{R}$ to $\mathbb{R}$?
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.

Answer: Option (B)

Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$.

However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

The other choices:

Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.

Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.

Performance review: 6 out of 11 got this correct. 2 chose (C) and 3 chose (E).

Historical note (last year): Only 1 person got this correct! 8 people chose option (E), 4 people chose option (C), 1 person chose option (D), and 1 person chose (A)+(B). Note that you'll always have exactly one correct answer option.

From the rough work shown by a few people, it seems that a lot of people were trying to reason this problem using derivatives. Using derivatives is *not* a sound approach to tackling this problem because it is not given that the function is differentiable or even continuous. Nonetheless, it is possible to obtain the correct answer using the flawed approach of derivatives, and it is sad that so few of you did so.

Others seem to have used examples. With examples, you should have found the counterexample rather easily, if you'd chosen $f(x) = g(x) = x$. However, most of you don't seem to have considered a sufficiently wide range of examples and to have settled with a few random ones. This is *not* the right way to use examples. When searching for counterexamples, you should look systematically and try to vary the essential features in a meaningful manner. More on this if we get time to cover this material in problem session.

Action point: Please, please make sure you understand this kind of problem so well that in the future, you're puzzled that this ever confused you. Unlike formulas for differentiating complicated functions, which you may forget a few years after doing calculus, the reasoning methods for these kinds of questions should stick with you for a lifetime.

(3) Suppose f is a continuous function defined on an open interval (a, b) and c is a point in (a, b). Which of the following implications is **true**?

(A) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$.

(B) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$, then c is a point of local minimum for f.

(C) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-decreasing on $(c - \delta, c)$ and non-increasing on $(c, c + \delta)$.

(D) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-decreasing on $(c - \delta, c)$ and non-increasing on $(c, c + \delta)$, then c is a point of local minimum for f.

(E) All of the above are true.

Answer: Option (B).

Explanation: Since f is continuous, being non-increasing on $(c - \delta, c)$ implies being non-increasing on $(c - \delta, c]$. Similarly on the right side. In particular, this means that $f(c) \le f(x)$ for all $x \in (c - \delta, c + \delta)$, establishing c as a point of local minimum.

The other choices: Options (C) and (D) have the wrong kind of increase/decrease. Option (A) is wrong, though counterexamples are hard to come by. The reason Option (A) is wrong is the core of the reason that the first-derivative test does not always work: the function could be oscillatory very close to the point c, so that even though c is a point of local minimum, the function does not steadily become non-increasing to the left of c. The example discussed in the lecture notes is $|x|(2+\sin(1/x))$.

Performance review: 2 out of 11 got this. 6 chose (A) and 1 each chose (C), (D), and (E).

Historical note (last year): 5 people got this correct. 5 people chose (A), which is the converse of the statement. 2 people chose (D) and 1 person each chose (C) and (E). Thus, most people got the sign/direction part correct but messed up on which way the implication goes.

Action point: This is tricky to get when the lecture material is still very new to you. However, you should consistently get this kind of question correct once you have reviewed and mastered the lecture material.

(4) Suppose f is a continuously differentiable function on $\mathbb{R}$ and f' is a periodic function with period h. (Recall that periodic derivative implies that the original function is a sum of ...). Suppose S is the set of points of local maximum for f, and T is the set of local maximum values. Which of the following is **true in general** about the sets S and T?

(A) The set S is invariant under translation by h (i.e., $x \in S$ if and only if $x + h \in S$) and all the values in the set T are in the image of the set $[0, h]$ under f.

(B) The set S is invariant under translation by h (i.e., $x \in S$ if and only if $x + h \in S$) but all the values in the set T need not be in the image of the set $[0, h]$ under f.

(C) Both the sets S and T are invariant under translation by h.

(D) Both the sets S and T are finite.

(E) Both the sets S and T are infinite.

Answer: Option (B).

Explanation: For a differentiable function, whether a point is a point of local maximum or not depends only on the derivative behavior near the point. Since the derivative is periodic with period h, S is invariant under translation by h.

The set of values T may be finite or infinite. If f itself is periodic, then the set of maximum values over a single period is the same as the set of maximum values overall. Thus, for instance, in the case of the function $f(x) := \sin x$, $T = \{1\}$ and $S = \{\pi/2 + 2n\pi : n \in \mathbb{Z}\}$. On the other hand, for the function $f(x) := 2\sin x - x$, the derivative is $f'(x) = 2\cos x - 1$, which is zero at $2n\pi \pm \pi/3$. The local maxima are attained at $2n\pi + \pi/3$ (as we can see from either derivative test). However, the set of local maximum *values* is infinite – in fact, each n gives a different local maximum value, and the set of local maximum values is infinite and unbounded. For instance, for $n = 0$, the local maximum value is $f(\pi/3) = \sqrt{3} - \pi/3$. For $n = 1$, the local maximum value is $f(7\pi/3) = \sqrt{3} - (7\pi/3)$. And so on.

Performance review: 7 out of 11 got this correct. 3 chose (E) and 1 chose (C).

Historical note (last year): 3 people got this correct. 3 people each chose (A) and (D), 4 people chose (C), and 2 people chose (E).

Action point: This was a fairly hard problem. However, it should become easy after we've seen a little more about functions with periodic derivative.