TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY MARCH 1: MAX-MIN VALUES: ONE-VARIABLE RECALL

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this 6-question quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 4 people
- Score of 4: 7 people
- Score of 5: 8 people
- Score of 6: 4 people

The question wise answers and performance review were as follows:

(1) Option (D): 18 people
(2) Option (B): 10 people
(3) Option (B): 25 people
(4) Option (B): 15 people
(5) Option (D): 24 people
(6) Option (A): 16 people

2. Solutions

(1) Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?
   (A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
   (B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
   (C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
   (D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
   (E) None of the above is true in general.

   Answer: Option (D)

   Explanation: If $f(x) \le f(c)$ for all $a \le x < c$, then all difference quotients from the left are nonnegative. The limiting value, which is the left-hand derivative, is thus also nonnegative. See the lecture notes for more details.

   The other choices: Options (A) and (B) predict the wrong sign. Option (C) is incorrect because even though the difference quotients are all strictly positive, their limiting value could be 0. For instance, $\sin x$ on $[0, \pi/2]$ or x^3 on $[-1, 0]$.

   Performance review: 18 out of 25 got this. 4 chose (E), 3 chose (C).

   Historical note 1: 10 out of 18 people got this. 4 chose (E), 3 chose (B), 1 chose (C).

   Historical note 2: This question appeared in a 152 quiz, which 15 people took. At the time, 8 people got this correct. 5 people chose option (B) and 2 people chose option (E). It is likely that the people who chose option (B) made a sign computation error.

(2) Suppose f is a continuous function defined on an open interval (a, b) and c is a point in (a, b). Which of the following implications is **true**?
   (A) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$.
   (B) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$, then c is a point of local minimum for f.

(C) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a,b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$.
(D) If there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a,b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$, then c is a point of local minimum for f.
(E) All of the above are true.

Answer: Option (B).

Explanation: Since f is continuous, being non-increasing on $(c-\delta, c)$ implies being non-increasing on $(c-\delta, c]$. Similarly on the right side. In particular, this means that $f(c) \le f(x)$ for all $x \in (c-\delta, c+\delta)$, establishing c as a point of local minimum.

Performance review: 10 out of 25 got this. 12 chose (A), 2 chose (E), 1 chose (D).

Historical note 1: 7 out of 18 people got this correct. 5 chose (A), 3 each chose (C) and (D).

The other choices: Options (C) and (D) have the wrong kind of increase/decrease. Option (A) is wrong, though counterexamples are hard to come by. The reason Option (A) is wrong is the core of the reason that the first-derivative test does not always work: the function could be oscillatory very close to the point c, so that even though c is a point of local minimum, the function does not steadily become non-increasing to the left of c. The example discussed in the lecture notes is $|x|(2+\sin(1/x))$.

Historical note 2: This question appeared in a Math 152 quiz. At the time, 5 out of 15 people got this correct. 5 people chose (A), which is the converse of the statement. 2 people chose (D) and 1 person each chose (C) and (E). Thus, most people got the sign/direction part correct but messed up on which way the implication goes.

(3) Consider all the rectangles with perimeter equal to a fixed length $p > 0$. Which of the following **is true** for the unique rectangle which is a square, compared to the other rectangles?
(A) It has the largest area and the largest length of diagonal.
(B) It has the largest area and the smallest length of diagonal.
(C) It has the smallest area and the largest length of diagonal.
(D) It has the smallest area and the smallest length of diagonal.
(E) None of the above.

Answer: Option (B)

Explanation: We can see this easily by doing calculus, but it can also be deduced purely by thinking about how a square and a long thin rectangle of the same perimeter compare in terms of area and diagonal length.

Performance review: All 25 got this.

Historical note 1: 13 out of 18 people got this correct. 3 chose (E) and 2 chose (A).

Historical note 2: This question appeared in a past 152 quiz, and everybody got this correct.

Historical note3: This question appeared in an earlier 151 final, and 31 out of 33 people got it correct.

(4) Suppose the total perimeter of a square and an equilateral triangle is L. (We can choose to allocate all of L to the square, in which case the equilateral triangle has side zero, and we can choose to allocate all of L to the equilateral triangle, in which case the square has side zero). Which of the following statements **is true** for the sum of the areas of the square and the equilateral triangle? (The area of an equilateral triangle is $\sqrt{3}/4$ times the square of the length of its side).
(A) The sum is minimum when all of L is allocated to the square.
(B) The sum is maximum when all of L is allocated to the square.
(C) The sum is minimum when all of L is allocated to the equilateral triangle.
(D) The sum is maximum when all of L is allocated to the equilateral triangle.
(E) None of the above.

Answer: Option (B)

Quick explanation: The problem can also be solved using the rough heuristic that works for these kinds of problems: the maximum occurs when everything is allocated to the most efficient use, but the minimum typically occurs somewhere in between.

Full explanation: Suppose x is the part allocated to the square. Then $L - x$ is the part allocated to the equilateral triangle. The total area is:

$$A(x) = x^2/16 + (\sqrt{3}/4)(L-x)^2/9$$

Differentiating, we obtain:

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(L-x) = x\left(\frac{1}{8} + \frac{\sqrt{3}}{18}\right) - \frac{\sqrt{3}}{18}L$$

We see that $A'(x) = 0$ at

$$x = \frac{L(\sqrt{3}/18)}{(1/8) + (\sqrt{3}/18)}$$

This number is indeed within the range of possible values of x.

Further, $A'(x) > 0$ for x greater than this and $A'(x) < 0$ for x less than this. Thus, this point is a local minimum and the maximum must occur at one of the endpoints. We plug in $x = 0$ to get $(\sqrt{3}/36)L^2$ and we plug in $x = L$ to get $L^2/16$. Since $1/16 > \sqrt{3}/36$, we obtain the the maximum occurs when $x = L$, which means that all the perimeter goes to the square.

Performance review: 15 out of 25 got this. 6 chose (C), 2 each chose (D) and (E).

Historical note 1: 7 out of 18 people got this correct. 7 chose (E), 2 each chose (C) and (D).

Historical note 2: This question appeared in a Math 152 quiz, and at the time 9 out of 15 people got this correct. 2 people each chose (E) and (C), 1 person chose (D), and 1 person left the question blank.

Historical note 3: This question appeared in a 152 midterm last year, and 20 of 29 people got it correct. In that midterm, option (E) wasn't there, so things became a little easier.

(5) Suppose x and y are positive numbers such as $x + y = 12$. For **what values** of x and y is x^2y maximum?
(A) $x = 3$, $y = 9$
(B) $x = 4$, $y = 8$
(C) $x = 6$, $y = 6$
(D) $x = 8$, $y = 4$
(E) $x = 9$, $y = 3$

Answer: Option (D).

Quick explanation: This is a special case of the general Cobb-Douglas situation where we want to maximize $x^a(C-x)^b$. The general solution is to take $x = Ca/(a+b)$, i.e., to take x and $C - x$ in the proportion of a to b.

Full explanation: We need to maximize $f(x) := x^2(12-x)$, subject to $0 < x < 12$. Differentiating, we get $f'(x) = 3x(8-x)$, so 8 is a critical point. Further, we see that f' is positive on $(0,8)$ and negative on $(8,12)$, so f attains its maximum (in the interval $(0,12)$) at 8.

Performance review: 24 out of 25 got this. 1 chose (E).

Historical note 1: 11 out of 18 people got this correct. 5 chose (E), 2 chose (C).

Historical note 2: This question appeared in a Math 152 quiz. At the time, 12 out of 15 people got this correct. 2 people chose (E) and 1 person chose (B). Of the people who got this correct, some seem to have computed the numerical values and others seem to have used calculus. Some who did not show any work may have used the general result of the Cobb-Douglas situation.

(6) Consider the function $p(x) := x^2 + bx + c$, with x restricted to integer inputs. Suppose b and c are integers. The minimum value of p is attained either at a single integer or at two consecutive integers. Which of the following is a **sufficient condition** for the minimum to occur at two consecutive integers?
(A) b is odd
(B) b is even
(C) c is odd
(D) c is even
(E) None of these conditions is sufficient.

Answer: Option (A)

Explanation: The graph of f is symmetric about the half-integer axis value $-b/2$. It is an upward-facing parabola. For odd b, it attains its minimum among integers at the two consecutive integers $-b/2+1/2$ and $-b/2-1/2$. When b is even, the minimum is attained uniquely at $-b/2$, which is itself an integer. c being odd or even tells us nothing.

Performance review: 16 out of 25 got this. 4 chose (E), 2 each chose (B) and (D), 1 chose (C).

Historical note 1: 2 out of 18 people got this correct. 8 chose (E), 5 chose (B), 3 chose (C).

Historical note 2: This question appeared in a Math 152 quiz. At the time, 4 out of 15 people got this correct. 8 people chose (E) and 3 people chose (B).