CLASS QUIZ SOLUTIONS: NOVEMBER 11: WHOPPERS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 1: 2 people
- Score of 2: 3 people
- Score of 3: 3 people
- Score of 4: 2 people
- Score of 8: 2 people

The mean score was 3.42. Here are the problem wise answers:

(1) Option (C): 3 people.
(2) Option (A): 8 people. *This was an exact replica, so good performance was expected.*
(3) Option (D): 3 people.
(4) Option (C): 4 people.
(5) Option (B): 5 people.
(6) Option (C): 4 people.
(7) Option (B): 7 people. *This was an exact replica, so good performance was expected.*
(8) Option (A): 6 people. *This was an exact replica, so good performance was expected.*
(9) Option (D): 1 person.

2. Solutions

(1) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x^2 = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x^4$? *Similar to (but trickier than) October 4 Question 2: only 1 person got that right.*
(A) A
(B) A^2
(C) A^3
(D) $A^2 g(A)$
(E) $g(A)/A^2$

Answer: Option (C)

Explanation: First, note that since $g(x)/x^2 \to A$ as $x \to 0$, we must have $g(x) \to 0$ as $x \to 0$. In particular, $g(0) = 0$.

Now, consider:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \cdot \frac{(g(x))^2}{x^4}$$

Splitting the limit, we get:

$$\lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \left(\frac{g(x)}{x^2} \right)^2$$

Setting $u = g(x)$ for the first limit, and using the fact that as $x \to 0$, $u \to 0$ we see that the first limit is A. For the second limit, pulling the square out yields that the second limit is A^2. The overall limit is thus $A \cdot A^2 = A^3$.

We can also use an actual example to solve this problem. For instance, consider the extreme case where $g(x) = Ax^2$ (identically). In this case, $g(g(x)) = A(Ax^2)^2 = A^3x^4$. Thus, $g(g(x))/x^4 = A^3$, and the limit is thus A^3.

Even more generally, if $\lim_{x\to 0} g(x)/x^n = A$, then $\lim_{x\to 0} g(g(x))/x^{n^2} = A^{n+1}$.

Performance review: 3 out of 12 got this correct. 7 chose (D), 1 chose (B), 1 chose (E).

Historical note (last year): 4 out of 16 people got this correct. 5 people chose (A), 4 people chose (B), 2 people chose (D), and 1 person chose (E).

Also, this appeared in one of the error-spotting exercises for Midterm 1.

(2) Which of the following statements is **always true**? *Exact replica of a past question.*

(A) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).

(B) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).

(C) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.

(D) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.

(E) None of the above.

Answer: Option (A)

Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

The other choices:

Option (B): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1 - x^2}$ on $(-1, 1)$ has range $(0, 1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1, 1]$.

Option (C): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0, 1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

Option (D): The same counterexample as for option (B) works.

Performance review: 8 out of 12 got this correct. 1 each chose (B), (C), (D), and (E).

Historical note (last year): 9 out of 16 people got it correct. 3 people chose (C), 2 people chose (D), and 1 person each chose (A) and (E).

Historical note (last year, previous quiz): When the question appeared in a previous quiz, 2 out of 11 people got it correct.

(3) Suppose f is a continuously differentiable function on $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following implications is **false**? *Similar to (but trickier than) October 13 question 4.*

(A) If f has mirror symmetry about $x = c$, f' has half turn symmetry about $(c, f'(c))$.

(B) If f has half turn symmetry about $(c, f(c))$, f' has mirror symmetry about $x = c$.

(C) If f' has mirror symmetry about $x = c$, f has half turn symmetry about $(c, f(c))$.

(D) If f' has half turn symmetry about $(c, f'(c))$, f has mirror symmetry about $x = c$.

(E) None of the above, i.e., they are all true.

Answer: Option (D)

Explanation: We can construct a number of examples, but instead of doing that, we make a general note. If f has mirror symmetry about $x = c$, then not only must f' have half turn symetry about $(c, f'(c))$, we must *also* have $f'(c) = 0$. Therefore, in any situation where f' has half turn symmetry about a point where it does not take the value 0, f will not have mirror symmetry about that point. More details below.

Option (A) is true. In fact, the following stronger claim is true: if f has mirror symmetry about $x = c$ and f' is defined on all of $\mathbb{R}$, then $f'(c) = 0$ and $f'(c+h) + f'(c-h) = 0$ for all $h \in \mathbb{R}$. This can be proved as follows. By mirror symmetry:

$$f(c+h) = f(c-h)$$

This is true as an identity in h. Thus, differentiating both sides with respect to h and using the chain rule, we get:

$$f'(c+h) = -f'(c-h)$$

The negative sign arises due to the chain rule.

Option (B) is true and the justification is similar to that for option (A). Namely:

$$f(c+h) + f(c-h) = 2f(c)$$

Differentiating both sides with respect to h gives:

$$f'(c+h) - f'(c-h) = 0$$

Option (C) requires more justification. The idea is that:

$$f(c+h) - f(c) - \int_c^{c+h} f'(t)\,dt$$

and:

$$f(c) - f(c-h) = \int_{c-h}^{c} f'(t)\,dt$$

Now, since f' has mirror symmetry about c, the two definite integrals are equal, so we get:

$$f(c+h) - f(c) = f(c) - f(c-h)$$

which is precisely the condition for half turn symmetry of f about $(c, f(c))$.

Option (D) is false, because, as mentioned earlier, for f to have mirror symmetry given f' having half turn symmetry, we need the additional condition that $f'(c) = 0$.

For instance, any cubic polynomial has half turn symmetry, but most polynomials of degree four do not have mirror symmetry. In fact, a polynomial of degree four has mirror symmetry iff its derivative cubic has the property that its point of inflection is a critical point.

Additional note: As we differentiate things, we obtain more and more symmetry. Here is the overall summary (where each step is true assuming that we can differentiate):

Half turn symmetry $\stackrel{\text{diff}}{\to}$ Mirror symmetry $\stackrel{\text{diff}}{\to}$ Half turn symmetry about point on x-axis.

Further, these arrows are reversible: any antiderivative of a function with half turn symmetry about a point on the x-axis has mirror symmetry about the same point, and any antiderivative of that has half turn symmetry for a point with the same x-coordinate.

The even and odd functions are special cases where the x-coordinate is 0, so in that special case:

Half turn symmetry about point on y-axis $\stackrel{\text{diff}}{\to}$ Even function $\stackrel{\text{diff}}{\to}$ Odd function

Performance review: 3 out of 12 got this correct. 5 chose (E), 3 chose (C), 1 left the question blank.

Historical note (last year): 2 out of 16 people got it correct. 7 people chose (E), 5 people chose (C), and 2 people chose (A).

(4) Consider the function $f(x) := \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x - (1/5), & 1/2 < x \le 1 \end{array}$. Define by $f^{[n]}$ the function obtained by iterating f n times, i.e., the function $f \circ f \circ \cdots \circ f$ where f occurs n times. What is the smallest n for which $f^{[n]} = f^{[n+1]}$? *Similar to a question on the previous midterm.*

(A) 1
(B) 2
(C) 3

(D) 4
(E) 5
Answer: Option (C)
Explanation: We need to iterate f enough times that everything gets inside $[0, 1/2]$, after which it becomes stable. Note that each time, the value goes down by 0.2. Thus, for any $x \leq 1$, we need at most three steps to bring it in $[0, 1/2]$, with the upper bound of 3 being attained for 1.
Performance review: 4 out of 12 got this correct. 3 chose (D), 2 each chose (A) and (E), 1 chose (B).
Historical note (last year): 3 out of 16 people got this correct. 8 people chose (A), 3 people chose (B), and 2 people chose (D).
Action point: It seems that most people did not figure out how composition works. Please make sure you understand at least one question like this at some point in your life.

(5) With f as in the previous question, what is the set of points in $(0, 1)$ where $f \circ f$ is not continuous?
(A) 0.5 only
(B) 0.5 and 0.7
(C) 0.5, 0.7, and 0.9
(D) 0.7 and 0.9
(E) 0.9 only
Answer: Option (B)
Explanation: The piecewise definition is:

$$(f \circ f)(x) = \{ \begin{array}{rl} x, & 0 \leq x \leq 0.5 \\ x - (1/5), & 0.5 < x \leq 0.7 \\ x - (2/5), & 0.7 < x \leq 1 \end{array}$$

We see that that points of discontinuity are 0.5 and 0.7.
Note that if we considered $f \circ f \circ f$ instead, 0.9 would also be a point of discontinuity, since, the definition to the right of 0.9 would be $x - (3/5)$.
Performance review: 5 out of 12 got this correct. 4 chose (C), 2 chose (A), 1 chose (D).
Historical note (last year): 4 out of 16 people got this correct. 9 people chose (A), 2 people chose (C), and 1 person chose (D).
Action point: Same as for previous problem – make sure you understand this clearly at least once in your life.

(6) Consider the graph of the function $f(x) := x \sin(1/(x^2 - 1))$. What can we say about the vertical and horizontal asymptotes? *This resembles a future whopper.*
(A) The graph has vertical asymptotes at $x = +1$ and $x = -1$ and horizontal asymptote (in both directions) $y = 0$.
(B) The graph has vertical asymptotes at $x = +1$ and $x = -1$ and horizontal asymptote (in both directions) $y = 1$.
(C) The graph has no vertical asymptotes and horizontal asymptote (in both directions) $y = 0$.
(D) The graph has no vertical asymptotes and horizontal asymptote (in both directions) $y = 1$.
(E) The graph has no vertical or horizontal asymptotes.
Answer: Option (C)
Explanation: The points where f is undefined are $x = \pm 1$. At both these points, the limit is undefined, but the function is bounded, because the x-part has a finite limit and the $\sin(1/(x^2 - 1))$ part is bounded in $[-1, 1]$. Thus, the function cannot have a vertical asymptote (in fact, it is oscillatory with no limit at both these points).
For the horizontal asymptote, we rewrite the limit at $+\infty$ as:

$$\lim_{x \to \infty} \frac{x}{x^2 - 1} \lim_{x \to \infty} (x^2 - 1) \sin(1/(x^2 - 1))$$

The first limit is 0. As for the second limit, putting $t = 1/(x^2 - 1)$, we see that $t \to 0^+$ as $x \to \infty$, so the second limit becomes $\lim_{t \to 0^+} (\sin t)/t = 1$. The overall limit at $+\infty$ is thus 0. A similar argument works for $-\infty$. Note that since $x^2 - 1$ has even degree, we get $\lim_{t \to 0^+} (\sin t)/t$ in this case as well.

Performance review: 4 out of 12 got this correct. 4 chose (A), 2 chose (E), 1 each chose (B) and (D).

Historical note (last year): 3 out of 16 people got this correct. 6 people chose (A), 5 people chose (B), and 1 person each chose (D) and (E).

Action point: Since the most commonly chosen incorrect answer was (A), it seems that most people figured out the horizontal asymptotes correctly. However, the vertical asymptotes confused people – not surprisingly, because they confused me too when I dug up this question from last year. Once you read the solution, however, you should be able to understand it.

(7) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing functions from $\mathbb{R}$ to $\mathbb{R}$? *An exact replica of a past question.*
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.

Answer: Option (B)

Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

The other choices:

Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.

Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.

Performance review: 7 out of 12 got this correct. 4 chose (C), 1 chose (E).

Historical note (last year): 9 out of 16 people got this correct. 3 people chose (C) and 2 people each chose (D) and (E).

Historical note (last year, previous quiz): When this question appeared earlier on October 20, only 1 out of 15 people got it correct.

(8) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? *Exact replica of a previous question.*
(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (D).

Historical note (last year): 10 out of 16 people got this correct. 2 people each chose (B), (D), and (E).

Historical note (last year, previous quiz): Nobody got it correct.

Action point: It's possible that many of you just remembered/revisited the question and saw that the correct answer option is (A). However, you should make sure you understand *why* the correct answer option is (A).

(9) Consider the four functions $\sin(\sin x)$, $\sin(\cos x)$, $\cos(\sin x)$, and $\cos(\cos x)$. Which of the following statements are true about their periodicity?

(A) All four functions are periodic with a period of 2π.
(B) All four functions are periodic with a period of π.
(C) $\sin(\sin x)$ and $\sin(\cos x)$ have a period of π, whereas $\cos(\sin x)$ and $\cos(\cos x)$ have a period of 2π.
(D) $\cos(\sin x)$ and $\cos(\cos x)$ have a period of π, whereas $\sin(\sin x)$ and $\sin(\cos x)$ have a period of 2π.
(E) $\sin(\sin x)$ has a period of 2π, the other three functions have a period of π.

Answer: Option (D)

Explanation: Since the inner functions in all cases have a period of 2π, it is clear that all the four functions have a period of at most 2π, in fact, the period of each divides 2π. The crucial question is which of them have the smaller period π.

Let's look at $\sin \circ \sin$ first. We have:

$$\sin(\sin(x+\pi)) = \sin(-\sin x) = -\sin(\sin x)$$

So, we see that that value at $x+\pi$ is the negative, and hence usually not the equal, of the value at x. Similarly:

$$\sin(\cos(x+\pi)) = \sin(-\cos x) = -\sin(\cos x)$$

On the other hand, for the functions that have a cos on the outside, the negative sign on the inside gets eaten up by the even nature of the outer function. For instance:

$$\cos(\sin(x+\pi)) = \cos(-\sin x) = \cos(\sin x)$$

and:

$$\cos(\cos(x+\pi)) = \cos(-\cos x) = \cos(\cos x)$$

Now, this is not a proof that π is strictly the smallest period for these functions, but that can be proved using other methods. In any case, given the choices presented, it is now easy to single out (D) as the only correct answer.

The key feature here is that both sin and cos (viewed as the inner functions of the composition) have *anti-period* π: their value gets negated after an interval of π.

The outer function cos is even, hence it converts an anti-period for the inner function into a period for the overall function. The outer function sin is odd, so it keeps anti-periods anti-periods.

Performance review: 1 out of 12 got this correct. 5 chose (A), 4 chose (B), 1 chose (C), 1 chose (E).

Historical note (last year): 5 out of 16 people got this correct. 1 person left the question blank. 7 people chose (A), 1 person chose (B), and 2 people chose (C).

Action point: This is fairly tricky to get at first sight, but you should be able to read and understand the solution.