DIAGNOSTIC IN-CLASS QUIZ SOLUTIONS: ORIGINALLY DUE FRIDAY NOVEMBER 15, DELAYED TO WEDNESDAY NOVEMBER 20: LINEAR DEPENDENCE, BASES, AND SUBSPACES

MATH 196, SECTION 57 (VIPUL NAIK)

1. Performance review

25 people took this 3-question quiz. The score distribution was as follows:

- Score of 0: 1 person
- Score of 1: 8 people
- Score of 2: 13 people
- Score of 3: 3 people

The question-wise answers and performance review were as follows:

(1) Option (E): 21 people
(2) Option (D): 14 people
(3) Option (C): 8 people

2. Solutions

PLEASE DO NOT DISCUSS *ANY* QUESTIONS

The purpose of this quiz is to review some basic ideas from part of the lecture notes titled `Linear dependence, bases, and subspaces`. The corresponding sections of the book are Sections 3.2 and 3.3.

(1) *Do not discuss this!*: Suppose S is a finite nonempty set of vectors in $\mathbb{R}^n$, and T is a nonempty subset of S. What can we say about S and T?
  (A) S is linearly dependent if and only if T is linearly dependent. S is linearly independent if and only if T is linearly independent.
  (B) If S is linearly dependent, then T is linearly dependent. If S is linearly independent, then T is linearly independent. However, we cannot deduce anything about the linear dependence or independence of S from the linear dependence or independence of T.
  (C) If T is linearly dependent, then S is linearly dependent. If T is linearly independent, then S is linearly independent. However, we cannot deduce anything about the linear dependence or independence of T from the linear dependence or independence of S.
  (D) If S is linearly dependent, then T is linearly dependent. If T is linearly independent, then S is linearly independent. We cannot make either of the two other deductions.
  (E) If T is linearly dependent, then S is linearly dependent. If S is linearly independent, then T is linearly independent. We cannot make either of the other two deductions.

  Answer: Option (E)

  Explanation: Any linear relation between the vectors in T is also a linear relation between the vectors in S, because all the vectors in T are vectors in S.

  The corect statement in the reverse direction is actually a logically equivalent statement, namely, the *contrapositive*. All it's saying is that if S is *not* linearly dependent, then T couldn't have been linearly dependent either, because if T had been linearly dependent, then S would have been too. Explicitly, given any implication $P \implies Q$, the implication (not Q) $\implies$ (not P), called the *contrapositive*, also holds. This is just a special case.

  For more details, see the lecture notes or textbook.

  Performance review: 21 out of 25 got this. 3 chose (B), 1 chose (D).

(2) *Do not discuss this!*: Suppose S is a finite set of vectors in $\mathbb{R}^n$. Consider the three statements: (i) S is linearly independent, (ii) S does not contain the zero vector, (iii) S does not contain any two vectors that are scalar multiples of one another. Which of the following options best describes the relationship between these statements?

(A) (i) is equivalent to (ii), and both imply (iii), but the reverse implication does not hold.
(B) (i) is equivalent to (iii), and both imply (ii), but the reverse implication does not hold.
(C) (i) is equivalent to (ii) and (iii) combined.
(D) (i) implies both (ii) and (iii), but (ii) and (iii), even if combined, do not imply (i).

Answer: Option (D)

Explanation: If either (ii) or (iii) is violated, we can obtain a linear relation within S, making S linearly dependent. Thus, the negation of (ii) or (iii) implies the negation of (i). Therefore, the contrapositive holds: (i) implies both (ii) and (iii).

However, (ii) and (iii), even if combined, do not imply (i). This is because there can be linear relations between the vectors that involve more than two vectors at a time. For instance, consider the set of vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. These three vectors are linearly dependent, because $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$. But none of the vectors equals zero, and no two vectors are scalar multiples of each other.

Performance review: 14 out of 25 got this. 6 chose (C), 5 chose (B).

(3) *Do not discuss this!*: Suppose V is a linear subspace of $\mathbb{R}^n$ for some n, and W is a linear subspace of V. Assume also that $W \neq V$, i.e., W is a *proper* subspace of V. Which of the following correctly describes the relationship between bases of V and bases of W?

(A) Given a basis of V, we can find a subset of that basis that is a basis of W. Also, given a basis of W, we can find a set containing that basis that is a basis of V.
(B) Given a basis of V, we can find a subset of that basis that is a basis of W. However, given a basis of W, we may not necessarily be able to find a set containing that basis that is a basis of V.
(C) Given a basis of V, we may not necessarily be able to find a subset of that basis that is a basis of W. However, given a basis of W, we can find a set containing that basis that is a basis of V.

Answer: Option (C)

Explanation: For a counterexample for the part about getting a basis for W from a basis for V, consider the case that $V = \mathbb{R}^2$ has the standard basis ($\vec{e}_1$ and $\vec{e}_2$) and W is the subspace spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly, there is no subset of the standard basis of V that forms a basis for W. In fact, none of the standard basis vectors of V is in W.

The idea behind showing the other direction is: start with a basis of W. Then, keep adding vectors one by one, making sure that each new vector being added is not in the span of the vectors so far. We will eventually obtain a basis of V.

Performance review: 8 out of 25 got this. 10 chose (A), 7 chose (B).