TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY JANUARY 16: THREE DIMENSIONS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this quiz. The score distribution is as follows:

- Score of 2: 2 people
- Score of 3: 2 people
- Score of 4: 2 people
- Score of 5: 6 people
- Score of 6: 11 people
- Score of 7: 2 people

The mean score was about 5.12.

Here are the question-wise answers and performance summary (more details in the next section):

(1) Option (C): 16 people
(2) Option (B): 20 people
(3) Option (A): 19 people
(4) Option (C): 20 people
(5) Option (C): 24 people
(6) Option (A): 5 people
(7) Option (C): 24 people

2. Solutions

(1) (*) Consider the subset of $\mathbb{R}^3$ given by the condition $(x^2 + y^2 - 1)(y^2 + z^2 - 1)(x^2 + z^2 - 1) = 0$. What kind of subset is this?

(A) It is a sphere centered at the origin and of radius 1.
(B) It is the union of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
(C) It is the union of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.
(D) It is the intersection of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
(E) It is the intersection of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.

Answer: Option (C)

Explanation: Each of the individual conditions gives a cylinder about one of the coordinate axes of radius 1. For instance, $x^2 + y^2 - 1 = 0$ gives the cylinder of radius 1 about the z-axis. For the product to be zero, one or more of the conditions must be satisfied, so we get the union.

Performance review: 16 out of 25 got this. 7 chose (B), 2 chose (D).

Historical note (last time): 12 out of 24 people got this correct. 3 chose (B), 4 chose (D), 3 chose (E), 2 chose (A).

(2) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that AC and BC have equal length (i.e., C is equidistant from A and B)? *Didn't appear last time*

(A) Sphere
(B) Plane

(C) Circle
(D) Line
(E) Two points
Answer: Option (B)
Explanation: This is the plane perpendicular to the line segment AB and intersecting the line segment at its midpoint. It is the analogue in three dimensions of the perpendicular bisector in two dimensions.
Performance review: 20 out of 25 got this. 3 chose (A), 2 chose (C).

(3) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right triangle with AB as its hypotenuse?
(A) Sphere (minus two points)
(B) Plane
(C) Circle (minus two points)
(D) Line
(E) Square
Answer: Option (A)
Explanation: By some elementary geometry, we know that this is the sphere with diameter AB. However, the points A and B themselves need to be excluded.
Note that if we were in a plane, we would get merely the circle with diameter AB minus two points. This seems to have been the most popular incorrect option chosen.
—em Performance review: 19 out of 25 got this. 3 chose (C), 2 chose (E), 1 chose (D).
Historical note (last time): 15 out of 24 people got this correct. 7 chose (C), 1 each chose (B) and (E).

(4) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right isosceles triangle with AB as its hypotenuse? *Didn't appear last time.*
(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Square
Answer: Option (C)
Explanation: It arises as the intersection of spheres centered at A and B of radius equal to $|AB|/\sqrt{2}$.
Performance review: 20 out of 25 got this. 4 chose (E), 1 chose (B).

(5) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is equilateral?
(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Two points
Answer: Option (C)
Explanation: It arises as an intersection of spheres centered at A and B with radius equal to AB. Alternatively, pick any one choice of C. The set of all choices can be obtained by revolving this point about the line of AB, and we get a circle.
Performance review: 24 out of 25 got this. 1 chose (D).
Historical note (last time): 23 out of 24 people got this correct (way to go, folks!). 1 person chose (B).

(6) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that $|AC|/|BC| = \lambda$ for λ a fixed positive real number not equal to 1? *Didn't appear last time.*

(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Square

Answer: Option (A)

Explanation: Use distance formula, simplify. Similar questions appear on your homework.

Performance review: 5 out of 25 got this. 7 chose (C), 6 each chose (B) and (D), 1 chose (E).

(7) Consider the parametric curve in three dimensions given by the coordinate description $t \mapsto (\cos t, \sin t, \cos(2t))$, with $t \in \mathbb{R}$. We can consider the *projections* of this curve onto the xy-plane, yz-plane, and xz-plane, which are basically what we get by dropping perpendiculars from the curve to these planes. What is the correct description of the curves obtained by doing the three projections?

(A) The projections on the xy-plane and yz-plane are both parts of parabolas, and the projection on the xz-plane is a circle.
(B) The projections on the xy-plane and yz-plane are both circles, and the projection on the xz-plane is a part of a parabola.
(C) The projection on the xy-plane is a circle, and the projections on the yz-plane and xz-plane are both parts of parabolas.
(D) The projection on the xy-plane is a part of a parabola, the projection on the xz-plane and yz-plane are both circles.
(E) All the three projections are circles.

Answer: Option (C)

Explanation: The projection on the xy-plane is just $t \mapsto (\cos t, \sin t)$, which is the unit circle. The projection on the xz-plane is $t \mapsto (\cos t, \cos(2t))$ and we have the quadratic relation $\cos(2t) = 2(\cos t)^2 - 1$, subject to domain restriction $\cos t \in [-1, 1]$. Thus, we get a part of a parabola. The projection on the yz-plane is $t \mapsto (\sin t, \cos(2t))$ and we have the quadratic relation $\cos(2t) = 1 - 2(\sin t)^2$, subject to domain restriction $\sin t \in [-1, 1]$. Thus, we get a part of a parabola.

Performance review: 24 out of 25 got this. 1 chose (A).

Historical note (last time): 17 out of 24 people got this correct. 4 chose (B) and 3 chose (D).