CLASS QUIZ SOLUTIONS: SEPTEMBER 30: LIMITS

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1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 2: 8 people.
- Score of 3: 4 people.

The median score was 2 and the mean score was 2.33.
Here is the information by question:

(1) Option (A): 11 people got this correct.
(2) Option (D): 8 people got this correct. *Although many of you got this correct, it's likely that some of you did so through educated guesswork, so I recommend you go through the solution, which is not completely straightforward.*
(3) Option (B): 9 people got this correct.

More details in the next section.

2. Solutions

(1) (**) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0,1)$?

(A) $f(x) := \{ \begin{array}{rl} x, & 0 < x \le 1/2 \\ 2x, & 1/2 < x < 1 \end{array}$

(B) $f(x) := \{ \begin{array}{rl} x, & 0 < x < 1/2 \\ 2x, & 1/2 \le x < 1 \end{array}$

(C) $f(x) := \{ \begin{array}{rl} x, & 0 < x \le 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{array}$

(D) $f(x) := \{ \begin{array}{rl} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \le x < 1 \end{array}$

(E) All of the above

Answer: Option (A)

Explanation: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, $1/2$)? and (ii) is the point (in all cases, $1/2$) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at $1/2$. Namely the function x and $2x - (1/2)$ both take the value $1/2$ at the domain point $1/2$. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function).

This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at $1/2$: the former gives $1/2$ and the latter gives 1. In other words, the function has a jump discontinuity at $1/2$. Thus, (ii) becomes relevant: is $1/2$ included in the left or the right definition?

For option (A), $1/2$ is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \to 1/2^-} f(x)$. On the other hand, $\lim_{x \to 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous.

For option (B), $1/2$ is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at $1/2$.

Performance review: 11 out of 12 people got this correct. 1 chose (C).

Historical note (last year): 6 out of 13 people got this correct. 6 people chose option (E) – I'm not sure why this option was so popular. 1 person chose option (C) but was quite close to choosing (A).

Action point: Please read through the lecture notes on Chalk (title "informal introduction to limits" – these roughly correspond to today's lecture), as well as any notes you took during class discussion today, very carefully, till you are completely confusion-free on the issues of left and right limits and continuity. This is the kind of question that, once you are thorough with the definitions, you should be able to get correctly. In other words, while the question is "hard" today, I expect it to be in the "moderately easy" category by the time you take the first midterm.

(2) (**) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both left continuous on $(0,1)$. Which of the following is *not* guaranteed to be left continuous on $(0,1)$?

(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$
(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D)

Explanation: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail.

For $c \in (0,1)$, we want to know whether:

$$\lim_{x \to c^-} f(g(x)) \stackrel{?}{=} f(g(c))$$

We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false.

We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f, the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say:

$$f(x) := \left\{ \begin{array}{ll} 1/3, & 0 < x \le 1/2 \\ 2/3, & 1/2 < x < 1 \end{array} \right.$$

and

$$g(x) := 1 - x$$

Note that both functions have range a subset of $(0,1)$.

Composing, we obtain that:

$$f(g(x)) = \left\{ \begin{array}{ll} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \le x < 1 \end{array} \right.$$

f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

Performance review: 8 out of 12 got this correct. 4 chose (E).

Historical note (last year): 4 out of 13 people got this correct. 9 people chose option (E). This is understandable, because if you look only at the obvious examples (all of which are increasing functions), you are likely to think that $f \circ g$ must be left continuous. If you got this question right for the right reasons, congratulate yourself.

Action point: We will emphasize the moral of this problem in a class in the near future. When we discuss the theorems involving limits, we will note that the theorems on sums, differences, products, etc. also hold for one-sided limits (i.e., each of the theorems holds for left hand limits and each of the theorems holds for right hand limits). However, the theorem on compositions for limits does not hold for one-sided limits, unless we make additional assumptions. I hope you will never forget this point (or at least, not till the end of the winter quarter).

(3) (*) Consider the function:

$$f(x) := \{ \begin{array}{rl} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{array}$$

What is the set of all points at which f is continuous?

(A) $\{0, 1\}$
(B) $\{-1, 1\}$
(C) $\{-1, 0\}$
(D) $\{-1, 0, 1\}$
(E) f is continuous everywhere

Answer: Option (B)

Explanation: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c, we need that: (i) the limit for the function as defined for rationals exists at c, (ii) the limit for the function as defined for irrationals exists at c, and (iii) the two limits are equal.

Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c. This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals.

The limit for rationals exists for all c and equals the value c. The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

Performance review: 9 out of 12 got this correct. 3 chose (E).

Historical note (last year): 5 out of 13 people got this correct. 3 people chose (D), 3 people chose (A), and 1 person each chose (C) and (E). Some of the people who chose (D) wrote "all rationals", so they probably thought that the correct answer is "all rationals" but it was not one of the options.

Action point: Getting this correct requires a thorougher definition of limit than the purely intuitive one. Like the $\sin(1/x)$-based functions, these functions are fascinating precisely because of the lack of clarity in what it means for such a function to have a limit. In a couple of weeks, after you have dealt more with functions defined differently for the rationals and irrationals, *and* seen the $\epsilon - \delta$ definition of limit, you will be in a much better position to tackle this question. By the time of the first midterm, this question should be in the "moderately easy" category.