CLASS QUIZ SOLUTIONS: OCTOBER 21: MAX-MIN PROBLEMS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

Note: The quiz was submitted in class on October 26.
12 students took this quiz. The score distribution was as follows:

- Score of 3: 2 persons.
- Score of 4: 5 persons.
- Score of 6: 4 persons.
- Score of 7: 1 person.

The mean score was 4.75. Here are the problem solutions and problem wise scores:

(1) Option (B): Everybody
(2) Option (B): Everybody
(3) Option (D): Everybody
(4) Option (A): 6 people
(5) Option (A): 6 people
(6) Option (C): 4 people
(7) Option (C): 5 people

2. Solutions

(1) Consider all the rectangles with perimeter equal to a fixed length $p > 0$. Which of the following **is true** for the unique rectangle which is a square, compared to the other rectangles?
(A) It has the largest area and the largest length of diagonal.
(B) It has the largest area and the smallest length of diagonal.
(C) It has the smallest area and the largest length of diagonal.
(D) It has the smallest area and the smallest length of diagonal.
(E) None of the above.

Answer: Option (B)

Explanation: We can see this easily by doing calculus, but it can also be deduced purely by thinking about how a square and a long thin rectangle of the same perimeter compare in terms of area and diagonal length.

Performance review: Everybody got this correct.

Historical note (last year): Everybody got this correct.

Historical note (two years ago): This question appeared on last year's 151 final, and 31 out of 33 people got it correct.

(2) Suppose the total perimeter of a square and an equilateral triangle is L. (We can choose to allocate all of L to the square, in which case the equilateral triangle has side zero, and we can choose to allocate all of L to the equilateral triangle, in which case the square has side zero). Which of the following statements **is true** for the sum of the areas of the square and the equilateral triangle? (The area of an equilateral triangle is $\sqrt{3}/4$ times the square of the length of its side).
(A) The sum is minimum when all of L is allocated to the square.
(B) The sum is maximum when all of L is allocated to the square.
(C) The sum is minimum when all of L is allocated to the equilateral triangle.
(D) The sum is maximum when all of L is allocated to the equilateral triangle.
(E) None of the above.

Answer: Option (B)

Quick explanation: The problem can also be solved using the rough heuristic that works for these kinds of problems: the maximum occurs when everything is allocated to the most efficient use, but the minimum typically occurs somewhere in between.

Full explanation: Suppose x is the part allocated to the square. Then $L - x$ is the part allocated to the equilateral triangle. The total area is:

$$A(x) = x^2/16 + (\sqrt{3}/4)(L-x)^2/9$$

Differentiating, we obtain:

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(L-x) = x\left(\frac{1}{8} + \frac{\sqrt{3}}{18}\right) - \frac{\sqrt{3}}{18}L$$

We see that $A'(x) = 0$ at

$$x = \frac{L(\sqrt{3}/18)}{(1/8) + (\sqrt{3}/18)}$$

This number is indeed within the range of possible values of x.

Further, $A'(x) > 0$ for x greater than this and $A'(x) < 0$ for x less than this. Thus, this point is a local minimum and the maximum must occur at one of the endpoints. We plug in $x = 0$ to get $(\sqrt{3}/36)L^2$ and we plug in $x = L$ to get $L^2/16$. Since $1/16 > \sqrt{3}/36$, we obtain the the maximum occurs when $x = L$, which means that all the perimeter goes to the square.

Performance review: Everybody got this correct.

Historical note (last year): 9 out of 15 people got this correct. 2 people each chose (E) and (C), 1 person chose (D), and 1 person left the question blank.

Historical note: This question appeared in a 152 midterm last year, and 20 of 29 people got it correct. This is a somewhat better showing than you lot, but that midterm occurred after several homeworks, lectures, and two review sessions covering max-min problems. Also, in that midterm, option (E) wasn't there, so things became a little easier.

(3) Suppose x and y are positive numbers such as $x + y = 12$. For **what values** of x and y is x^2y maximum?

(A) $x = 3$, $y = 9$
(B) $x = 4$, $y = 8$
(C) $x = 6$, $y = 6$
(D) $x = 8$, $y = 4$
(E) $x = 9$, $y = 3$

Answer: Option (D).

Quick explanation: This is a special case of the general Cobb-Douglas situation where we want to maximize $x^a(C-x)^b$. The general solution is to take $x = Ca/(a+b)$, i.e., to take x and $C - x$ in the proportion of a to b.

Full explanation: We need to maximize $f(x) := x^2(12-x)$, subject to $0 < x < 12$. Differentiating, we get $f'(x) = 3x(8-x)$, so 8 is a critical point. Further, we see that f' is positive on $(0, 8)$ and negative on $(8, 12)$, so f attains its maximum (in the interval $(0, 12)$) at 8.

Performance review: Everybody got this correct.

Historical note (last year): 12 out of 15 people got this correct. 2 people chose (E) and 1 person chose (B). Of the people who got this correct, some seem to have computed the numerical values and others seem to have used calculus. Some who did not show any work may have used the general result of the Cobb-Douglas situation.

(4) Consider the function $p(x) := x^2 + bx + c$, with x restricted to integer inputs. Suppose b and c are integers. The minimum value of p is attained either at a single integer or at two consecutive integers. Which of the following is a **sufficient condition** for the minimum to occur at two consecutive integers?

(A) b is odd

(B) b is even
(C) c is odd
(D) c is even
(E) None of these conditions is sufficient.

Answer: Option (A)

Explanation: The graph of f is symmetric about the half-integer axis value $-b/2$. It is an upward-facing parabola. For odd b, it attains its minimum among integers at the two consecutive integers $-b/2 + 1/2$ and $-b/2 - 1/2$. When b is even, the minimum is attained uniquely at $-b/2$, which is itself an integer. c being odd or even tells us nothing.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (B).

Historical note (last year): 4 out of 15 people got this correct. 8 people chose (E) and 3 people chose (B).

Action point: While this problem can be solved using calculus, it is much easier if you already know and remember important facts about the graphs of quadratic functions. Please review basic facts about quadratic functions.

(5) Consider a hollow cylinder with no top and bottom and total curved surface area S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).
(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (A)

Explanation: We have $h = S/2\pi r$. Thus, the volume is $rS/2$. We see that as $r \to \infty$, the volume goes to infinity, and as $r \to 0$, the volume tends to zero. Thus, the volume can be made arbitrarily large as well as arbitrarily small.

Performance review: 6 out of 12 got this correct. 4 chose (C), 2 chose (D).

Historical note (last year): 6 out of 15 people got this correct. 3 people chose (B) and 6 people chose (C).

Action point: Please make sure you understand this problem, and also how it differs in nature from the next two problems.

(6) Consider a hollow cylinder with a bottom but no top and total surface area (curved surface plus bottom) S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).
(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (C)

Quick explanation: We see that the radius cannot be expanded too much, otherwise the area of the bottom will itself exceed S. This puts a constraint on the total volume. On the other hand, the cylinder can be made arbitrarily thin and thus have arbitrarily small volume.

Full explanation: Try it yourself! There is a worked example in the book that essentially computes the maximum with specific numerical values – locate it!

Performance review: 4 out of 12 got this correct. 3 each chose (B) and (D), 2 chose (A).

Historical note (last year): 8 out of 15 people got this correct. 1 person chose (A), 4 people chose (B), and 2 people chose (D).

(7) Consider a hollow cylinder with a bottom and a top and total surface area (curved surface plus bottom and top) S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).

(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (C)

Quick explanation: Identical to the previous problem.

Full explanation: Try it yourself!

Performance review: 5 out of 12 got this correct. 4 chose (D), 2 chose (B), and 1 chose (A).

Historical note (last year): 5 out of 15 people got this correct. 8 people chose (D) and 2 people chose (A).

Action point: Please try to understand, both conceptually and computationally, why the qualitative conclusion for this problem is the same as for the previous problem.