TAKE-HOME CLASS QUIZ SOLUTIONS: DUE OCTOBER 10: LEAST UPPER BOUND AXIOM

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

42 people took this 5-question quiz. The score distribution was as follows:

- Score of 2: 4 people
- Score of 3: 5 people
- Score of 4: 21 people
- Score of 5: 12 people

The mean score was slightly under 4. The question wise answers and performance are as follows:

(1) Option (A): 34 people
(2) Option (A): 41 people
(3) Option (D): 37 people
(4) Option (B): 18 people
(5) Option (D): 37 people

2. Solutions

(1) Which of the following is an **alternative characterization** of the least upper bound of a nonempty subset S of $\mathbb{R}$ that is bounded from above?
  (A) It is the greatest lower bound of the set of all upper bounds (in $\mathbb{R}$) of S.
  (B) It is the least upper bound of the set of all upper bounds (in $\mathbb{R}$) of S.
  (C) It is the greatest lower bound of the set of all lower bounds (in $\mathbb{R}$) of S.
  (D) It is the least upper bound of the set of all lower bounds (in $\mathbb{R}$) of S.
  (E) None of the above.

  Answer: Option (A)

  Explanation: If L is the least upper bound of S, then the set of all upper bounds of S is the set $[L, \infty)$, and the greatest lower bound of that set is L.

  This is in fact the idea behind one of the proofs for the fact that the least upper bound axiom and greatest lower bound axiom are equivalent.

  Performance review: 34 out of 42 got this. 7 chose (B), 1 chose (C).

(2) Consider the following four intervals where $a < b$ are both fixed real numbers: the open interval (a, b), the closed interval $[a, b]$, the left-closed right-open interval $[a, b)$, and the left-open right-closed interval $(a, b]$. Which of the following is **true** about the upper and lower bounds of these intervals?
  (A) All four intervals have the same greatest lower bound as each other. Also, all four intervals have the same least upper bound as each other.
  (B) All four intervals have greatest lower bounds and least upper bounds. However, the greatest lower bounds for $[a, b]$ and $[a, b)$, while equal to each other, differ from the greatest lower bound for $(a, b]$ and (a, b), which in turn are equal to each other. Similarly, the least upper bounds for $[a, b]$ and $(a, b]$, while equal to each other, differ from the least upper bound for $[a, b)$ and (a, b), which in turn are equal to each other.
  (C) The intervals $[a, b]$ and $[a, b)$ have a greatest lower bound and the intervals $(a, b]$ and (a, b) do not. Further, the intervals $[a, b]$ and $(a, b]$ have a least upper bound, and the intervals $[a, b)$ and (a, b) do not.
  (D) None of the intervals has a greatest lower bound or a least upper bound.

(E) $[a, b]$ is the only interval among the four intervals that has a greatest lower bound. It is also the only interval that has a least upper bound.

Answer: Option (A)

Explanation: All four intervals have greatest lower bound a. Also, all four intervals have least upper bound b. The difference between the intervals lies in whether the glb and lub are in the set or not in the set. The *value* of the glb is the same in all cases. Same for the lub.

Performance review: 41 out of 42 got this. 1 chose (B).

(3) The greatest lower bound of a sequence is defined as the greatest lower bound of the set of values that it takes (i.e., its range as a function). Similarly, we can define the least upper bound of a sequence. Which of the following is **true**?

(A) Any bounded monotonic sequence converges to its greatest lower bound.
(B) Any bounded monotonic sequence converges to its least upper bound.
(C) Any bounded monotonic sequence is convergent. It converges to its least upper bound if it is non-increasing (i.e., weakly decreasing) and to its greatest lower bound if it is non-decreasing (i.e., weakly increasing).
(D) Any bounded monotonic sequence is convergent. It converges to its greatest lower bound if it is non-increasing (i.e., weakly decreasing) and to its least upper bound if it is non-decreasing (i.e., weaky increasing).
(E) None of the above.

Answer: Option (D)

Explanation: One of these cases will be in a subsequent homework, with a $\varepsilon - n_0$ proof. The other case is similar. However, even without a proof, you should be able to see it intuitively: if it's non-increasing, the terms go down or stay constant, but being bounded from below, they cannot go arbitrarily down, and they have to converge to something, which is the glb. Analogously for lub.

Performance review: 37 out of 42 got this. 3 chose (C), 2 chose (E).

(4) Building on the definition of greatest lower bound and least upper bound of a sequence in the previous question, which of the following is **true**?

(A) A sequence is non-decreasing (i.e., weakly increasing) if and only if its first term equals its greatest lower bound.
(B) If a sequence is non-decreasing (i.e., weakly increasing), then its first term is its greatest lower bound, but the converse is not true in general.
(C) If the first term of a sequence is its greatest lower bound, then the sequence is non-decreasing (i.e., weakly increasing), but the converse is not true in general.
(D) All of the above.
(E) None of the above.

Answer: Option (B)

Explanation: If a sequence is non-decreasing, the first term is $\leq$ the second term, which is $\leq$ the third term, and so on. By the transitivity of $\leq$, the first term is $\leq$ all terms, so it is the minimum of the sequence. If a set does have a minimum, that minimum is the glb. So this shows one direction. To see that the other direction is not true, consider the sequence that alternates between going up by 2 and going down by 1:

$$0, 2, 1, 3, 2, 4, 3, \ldots$$

The first term is the greatest lower bound but the sequence is not a non-decreasing sequence.

Performance review: 18 out of 42 got this. 16 chose (A). 4 chose (D), 3 chose (C), 1 chose (E).

(5) Suppose S is a nonempty bounded subset of $\mathbb{R}$, so that it has a finite greatest lower bound and a finite least upper bound. Denote by $-S$ the set $\{-s : s \in S\}$, i.e., the set of negatives of elements of S. Which of the following is **true** about $-S$?

(A) The least upper bound of $-S$ equals the least upper bound of S, and the greatest lower bound of $-S$ equals the greatest lower bound of S.
(B) The least upper bound of $-S$ equals the greatest lower bound of S, and the greatest lower bound of $-S$ equals the least upper bound of S.

(C) The least upper bound of $-S$ equals the negative of the least upper bound of S, and the greatest lower bound of $-S$ equals the negative of the greatest lower bound of S.

(D) The least upper bound of $-S$ equals the negative of the greatest lower bound of S, and the greatest lower bound of $-S$ equals the negative of the least upper bound of S.

(E) None of the above.

Answer: Option (D)

Explanation: A formal proof can be given, but it basically follows from the fact that $a \leq b \implies -b \leq -a$. For a simple example, consider the two-point set $S = \{3, 5\}$ and figure out the lub and glb of S and $-S$.

Performance review: 37 out of 42 got this. 4 chose (B), 1 chose (A).