CLASS QUIZ SOLUTIONS: NOVEMBER 30: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 7-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 3: 2 people
- Score of 4: 2 people
- Score of 5: 2 people
- Score of 6: 4 people

The mean score was 4.17 out of 7. Here are the problem answers:

(1) (A): 7 people
(2) (B): 8 people
(3) (C): 6 people
(4) (D): 8 people
(5) (A): 10 people
(6) (B): 5 people
(7) (B): 6 people

2. Solutions

(1) What is the limit $\lim_{x \to \infty} \left[\left(\int_0^x \sin^2 \theta d\theta \right) / x \right]$?
(A) $1/2$
(B) 1
(C) $1/\pi$
(D) $2/\pi$
(E) $1/(2\pi)$

Answer: Option (A)

Explanation: The average over a period is 1/2. Thus, this is also the limit of the average over intervals of arbitrarily large length. See the notes on the mean value of a periodic function over a period.

Performance review: 7 out of 11 got this correct. 3 chose (B) and 1 chose (D).

Historical note (last year): 13 out of 16 people got this correct. 1 person each chose (C) and (E) and 1 person left the question blank.

(2) Consider the substitution $u = -1/x$ for the integral $\int \frac{dx}{x^2+1}$. What is the **new integral**?
(A) $\int \frac{du}{u(u^2+1)}$
(B) $\int \frac{du}{u^2+1}$
(C) $\int \frac{u du}{u^2+1}$
(D) $\int \frac{u^2 du}{u^2+1}$
(E) $\int \frac{u^2 du}{(u^2+1)^2}$

Answer: Option (B)

Explanation: Setting $u = -1/x$, we get $x = -1/u$, so $dx/du = 1/u^2$. Plugging in, we get:

$$\int \frac{(1/u^2)\, du}{(-1/u)^2 + 1} = \int \frac{du}{1 + u^2}$$

Performance review: 8 out of 11 got this correct. 2 chose (D), 1 chose (A).

Historical note (last year): 8 out of 16 people got this correct. 6 people chose (D), 1 person chose (C), and 1 person left the question blank.

Action point: Those who chose (D) either made a calculation error or forgot the relative derivative $1/u^2$ in the numerator. Please make sure you review u-substitutions and get this kind of question correct in the future.

(3) *Hard*: What is the **value** of $c \in (0, \infty)$ such that $\int_0^c \frac{dx}{x^2+1} = \lim_{a \to \infty} \int_c^a \frac{dx}{x^2+1}$?

(A) $\frac{1}{\sqrt{3}}$

(B) $\frac{1}{\sqrt{2}}$

(C) 1

(D) $\sqrt{2}$

(E) $\sqrt{3}$

Answer: Option (C)

Explanation: By the previous question, we get:

$$\lim_{a \to \infty} \int_c^a \frac{dx}{x^2+1} = \int_{-1/c}^0 \frac{du}{u^2+1}$$

Because the integrand is even, we get that:

$$\lim_{a \to \infty} \int_c^a \frac{dx}{x^2+1} = \int_0^{1/c} \frac{dx}{x^2+1}$$

Thus, from the given data, we get:

$$\int_0^c \frac{dx}{x^2+1} = \int_0^{1/c} \frac{dx}{x^2+1}$$

This gives:

$$\int_c^{1/c} \frac{dx}{x^2+1} = 0$$

But the function in question is a positive function, hence the only way the above can hold is if $c = 1/c$, giving $c = 1$ (since c is positive).

Note that the problem can also be solved using the "fact" that arctan is an indefinite integral, so we note that the integral from 0 to 1, as well as the integral from 1 to ∞, are both $\pi/4$. However, that solution requires a knowledge of the antiderivative and of the properties of inverse trigonometric functions, whereas this proof does not require any development of that theory.

Performance review: 6 out of 11 got this correct. 3 chose (E), 1 chose (B), 1 chose (D).

Historical note (last year): 8 out of 16 people got this correct. 6 people chose (B), for reasons unclear. 1 person chose (A) and 1 person left the question blank.

(4) Suppose f is a continuous nonconstant even function on $\mathbb{R}$. Which of the following is **true**?

(A) Every antiderivative of f is an even function.

(B) f has exactly one antiderivative that is an even function.

(C) Every antiderivative of f is an odd function.

(D) f has exactly one antiderivative that is an odd function.

(E) None of the antiderivatives of f is either an even or an odd function.

Answer: Option (D)

Explanation: The odd function will be the unique antiderivative that takes the value 0 at 0. Specifically, if F is an antiderivative of f, we can easily check that:

$$F(x) - F(0) = \int_0^x f(t)\, dt = \int_{-x}^0 f(t)\, dt = F(0) - F(-x)$$

Thus, we get that:

$$F(0) = \frac{F(x) + F(-x)}{2}$$

Thus, F has half-turn symmetry about $(0, F(0))$. It is odd iff $F(0) = 0$. We see that, among the family of antiderivatives, there is a unique one with the property.

Additional note: A little while ago, you proved that a cubic function enjoys half-turn symmetry about its point of inflection, and were supposed to give a computational proof thereof. The fact can actually be deduced without any computation using the fact that the derivative function, the quadratic, has *mirror symmetry* about the same x-value. The proof of that fact follows the same lines as the proof given above.

Performance review: 8 out of 11 got this correct. 3 chose (C).

Historical note (last year): 4 out of 16 people got this correct. 9 people chose (C), apparently forgetting the fact that an odd function must be 0 at 0. 1 person chose (A) and 2 people chose (B).

(5) Suppose f is a continuous nonconstant odd function on $\mathbb{R}$. Which of the following is **true**?
(A) Every antiderivative of f is an even function.
(B) f has exactly one antiderivative that is an even function.
(C) Every antiderivative of f is an odd function.
(D) f has exactly one antiderivative that is an odd function.
(E) None of the antiderivatives of f is either an even or an odd function.

Answer: Option (A)

Explanation: Fill this in yourself; it is similar to the previous exercise. Note that the key difference here is that an even function does not have to be 0 at 0, and adding a constant preserves the property of being even.

Performance review: 10 out of 11 got this correct. 1 chose (E).

Historical note (last year): 13 out of 16 people got this correct. 1 person each chose (B), (C), and (D).

(6) Suppose f is a continuous nonconstant periodic function on $\mathbb{R}$ with period h. Which of the following is **true**?
(A) Every antiderivative of f is a periodic function with period h, regardless of the choice of f.
(B) For some choices of f, every antiderivative of f is a periodic function; for all others, f has no periodic antiderivative.
(C) f has exactly one periodic antiderivative for every choice of f.
(D) For some choices of f, f has exactly one periodic antiderivative; for all others, f has no periodic antiderivative.
(E) Regardless of the choice of f, no antiderivative of f can be periodic.

Answer: Option (B)

Explanation: What this crucially depends on is the mean value of f over a period. If this mean value is 0 (e.g., for sin and cos), then every antiderivative is periodic. If the mean value is nonzero (e.g., $x \mapsto 1 + \sin x$ or $\sin^2$) then the antiderivative is (linear + periodic), and that mean value is the slope of the linear component of any antiderivative. For instance, $1 + \cos x$ has mean value 1, and its antiderivative, $x + \sin x$, has linear part x of slope 1 and periodic part $\sin x$.

Performance review: 5 out of 11 got this correct. 4 chose (A), 2 chose (D).

Historical note (last year): 5 out of 16 people got this correct. 9 people chose (A), suggesting that they didn't remember the ideas about non-periodic functions with periodic derivatives. 2 people chose (A).

Action point: Review the material on functions that are "periodic with shift" – discussed when we covered graphing of functions.

(7) Consider a continuous increasing function f defined on the nonnegative real numbers. Define $m_f(a)$, for $a > 0$, as the unique value $c \in [0, a]$ such that $f(c)$ is the mean value of f on the interval $[0, a]$.

If $f(x) := x^n$, n an integer greater than 1, what kind of function is m_f (your answer should be valid for all n)?
(A) $m_f(a)$ is a constant λ dependent on n but independent of a.
(B) It is a function of the form $m_f(a) = \lambda a$, where λ is a constant depending on n.

(C) It is a function of the form $m_f(a) = \lambda a^{n-1}$, where λ is a constant depending on n.
(D) It is a function of the form $m_f(a) = \lambda a^n$, where λ is a constant depending on n.
(E) It is a function of the form $m_f(a) = \lambda a^{n+1}$, where λ is a constant depending on n.

Answer: Option (B)

Explanation: The integral on the interval $[0, a]$ is $a^{n+1}/(n+1)$. The mean value is $a^n/(n+1)$. The value c is thus $(a^n/(n+1))^{1/n} = a/(n+1)^{1/n}$. Setting $\lambda = 1/(n+1)^{1/n}$, we see that option (B) works.

Performance review: 6 out of 11 got this correct. 4 chose (C), 1 chose (D).

Historical note (last year): 1 out of 16 people got this correct. 5 people chose (D), 4 people chose (E), 3 people chose (C), 2 people chose (A), and 1 person left the question blank. Most probably, people forgot the step of raising to the power of $1/n$, and of course, many people just guessed.

Action point: Make sure that you can solve the problem under fewer time constraints.