CLASS QUIZ SOLUTIONS: FRIDAY JANUARY 11: POLAR COORDINATES

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

27 people took this 3-question quiz. The score distribution was as follows:

- Score of 1: 1 person
- Score of 2: 8 people
- Score of 3: 18 people

The answers and performance review for individual questions are:

(1) Option (B): 26 people.
(2) Option (D): 22 people.
(3) Option (A): 23 people.

2. Solutions

(1) Consider a straight line that does not pass through the pole in a polar coordinate system. The equation of such a line in the polar coordinate system can be expressed as $r = F(\theta)$. What kind of function is F?
   (A) $F(\theta)$ is a linear combination of $\sin\theta$ and $\cos\theta$
   (B) $F(\theta)$ is the reciprocal of a linear combination of $\sin\theta$ and $\cos\theta$.
   (C) $F(\theta)$ is a linear combination of $\tan\theta$ and $\cot\theta$.
   (D) $F(\theta)$ is the reciprocal of a linear combination of $\tan\theta$ and $\cot\theta$.
   (E) $F(\theta)$ is a linear combination of $\sec\theta$ and $\csc\theta$.

   Answer: Option (B)

   Explanation: Consider the corresponding Cartesian coordinate system. The Cartesian equation of a straight line is of the form $ax + by = c$. Since the line does not pass through the origin/pole, $c \neq 0$. Set $x = r\cos\theta$ and $y = r\sin\theta$, and we get $r(a\cos\theta + b\sin\theta) = c$. Rearranging, we obtain that $r = c/(a\cos\theta + b\sin\theta)$. We could rewrite as $r = 1/((a/c)\cos\theta + (b/c)\sin\theta)$.

   Performance review: 26 out of 27 got this. 1 chose (C).

   Historical note (last time): 8 out of 21 people got this correct. 6 chose (C), 5 chose (A), 1 each chose (D) and (E).

(2) Consider the curve $r = \sin^2\theta$. Which of the following symmetries does the curve enjoy? Please see Options (D) and (E) before answering.
   (A) Mirror symmetry about the polar axis
   (B) Mirror symmetry about an axis perpendicular to the polar axis (what would be the y-axis if the polar axis is the x-axis)
   (C) Half turn symmetry about the pole
   (D) All of the above
   (E) None of the above

   Answer: Option (D)

   Explanation: We use the fact that $\sin^2\theta = \sin^2(-\theta)$ to deduce mirror symmetry about the polar axis. We use that $\sin^2\theta = \sin^2(\pi - \theta)$ to deduce mirror symmetry about the y-axis. Finally, we use that $\sin^2\theta = \sin^2(\pi + \theta)$ to deduce half turn symmetry about the pole.

   Performance review: 22 out of 27 got this. 5 chose (B).

   Historical note (last time): 10 out of 21 people got this correct. 6 chose (B), 5 chose (A).

(3) Which of the following is the correct expression for the length of the part of the curve $r = F(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, with $\alpha < \beta$?

(A) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2}\,d\theta$
(B) $\int_\alpha^\beta |F(\theta) + F'(\theta)|\,d\theta$
(C) $\int_\alpha^\beta |F(\theta) - F'(\theta)|\,d\theta$
(D) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 + 4F(\theta)F'(\theta)}\,d\theta$
(E) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 - 4F(\theta)F'(\theta)}\,d\theta$

Answer: Option (A)

Explanation: There are many ways of seeing this, including a direct justification in polar coordinates, but we provide an easy explanation using the Cartesian coordinates. We know that:

$$x = F(\theta)\cos\theta, \qquad y = F(\theta)\sin\theta$$

We get:

$$\frac{dx}{d\theta} = F'(\theta)\cos\theta - F(\theta)\sin\theta, \qquad \frac{dy}{d\theta} = F'(\theta)\sin\theta + F(\theta)\cos\theta$$

Squaring and adding, we know that the $2F(\theta)F'(\theta)\cos\theta\sin\theta$ term cancels between the two expressions, and we are left with:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (F'(\theta))^2\cos^2\theta + (F(\theta))^2\sin^2\theta + (F'(\theta))^2\sin^2\theta + (F(\theta))^2\cos^2\theta = [(F'(\theta))^2 + (F(\theta))^2](\cos^2\theta + \sin^2\theta)$$

Using $\cos^2\theta + \sin^2\theta = 1$, we get the desired result.

Performance review: 23 out of 27 got this. 2 chose (D), 1 each chose (B) and (E).

Historical note (last time): 14 out of 21 people got this correct. 4 chose (D), 3 chose (E).