CLASS QUIZ SOLUTIONS: NOVEMBER 4: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 2 people
- Score of 2: 5 people
- Score of 3: 2 people

The mean score was 1.5. Here are the problem wise answers and performance:

(1) Option (A): 7 people
(2) Option (B): 4 people
(3) Option (A): 3 people
(4) Option (C): 3 people
(5) Option (C): 1 person

2. Solutions

(1) Which of the following is an **antiderivative** of $x \cos x$?
 (A) $x \sin x + \cos x$
 (B) $x \sin x - \cos x$
 (C) $-x \sin x + \cos x$
 (D) $-x \sin x - \cos x$
 (E) None of the above

 Answer: Option (A).

 Explanation: Differentiating the function given in option (A) gives $x \cos x + \sin x - \sin x = x \cos x$.

 When we study integration by parts next quarter, we will see a constructive approach designed to arrive at the answer.

 Performance review: 7 out of 11 people got this. 1 chose (B), 4 chose (E).

 Historical note (last year): 11 out of 16 people got this correct. Remaining were 2 (B) and 3 (E).

 Action point: If you got this wrong, make sure you remember and are comfortable with the differentiation rules. The time is not yet ripe to forget those.

(2) (*) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
 (A) $k - 2$
 (B) $k - 1$
 (C) k
 (D) $k + 1$
 (E) There is no bound in terms of k.

 Answer: Option (B)

 Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F - G$ have k^{th} derivative equal to zero. For $k = 1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k - 1$.

Performance review: 4 out of 12 got this correct. 4 chose (E), 2 chose (C), 1 each chose (A) and (D).

Historical note (last year): 6 out of 16 people got this correct. Remaining were: 2 (A), 2 (C), 3 (D), 3 (E).

Action point: This is the kind of question you should *definitely* get right in the future. Please review the notes on repeated integration and finding functions with given k^{th} derivative. It seems like we didn't cover this well enough in class, which might be the reason for the not-so-good performance. We'll review these ideas in class Friday.

(3) (**) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).

(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 3 out of 12 got this correct. 4 chose (E), 3 chose (C), 2 chose (D).

Historical note (last year): 3 out of 16 people got this correct. Remaining were: 2 (B), 3 (C), 1 (D), 7 (E).

Action point: This is the kind of question that everybody should get correct in the future. Please make sure you understand the solution process for this question.

(4) (**) Suppose $F(x) = \int_0^x \sin^2(t^2)\,dt$ and $G(x) = \int_0^x \cos^2(t^2)\,dt$. Which of the following **is true**?

(A) $F + G$ is the zero function.
(B) $F + G$ is a constant function with nonzero value.
(C) $F(x) + G(x) = x$ for all x.
(D) $F(x) + G(x) = x^2$ for all x.
(E) $F(x^2) + G(x^2) = x$ for all x.

Answer: Option (C)

Explanation: $F(x) + G(x) = \int_0^x \sin^2(t^2) + \cos^2(t^2)\,dt = \int_0^x 1\,dt = x$.

Note that it is not possible to obtain closed expressions for F and G separately, and any attempt to do so is a waste of time.

Performance review: 3 out of 12 got this correct. 5 chose (B), 2 chose (D), 1 each chose (A) and (E).

Historical note (last year): 5 out of 16 people got this correct. Remaining were: 2 (A), 5 (B), 4 (D). It is likely that many people noted that $\sin^2(t^2) + \cos^2(t^2) = 1$ but then forgot to integrate it, hence (B) as a common wrong answer.

Action point: This one shouldn't trick you again!

(5) (**) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely?

(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.

(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.

(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 1 out of 12 got this correct. 10 chose (D) and 1 chose (E).

Historical note (last year): 4 out of 16 people got this correct. Remaining were: 8 (D), 4 (E). It seems that most people did not get the key idea for this question.

Action point: Once you have understood this question, you should be able to get any similar question correct in the future.