CLASS QUIZ SOLUTIONS: NOVEMBER 5: SERIES SUMMATION

MATH 153, SECTION 59 (VIPUL NAIK)

1. Performance review

44 people took this quiz. The mean score was 4.77, median score was 5, and modal score was 7. The score distribution was as follows:

- Score of 1: 1 person
- Score of 2: 1 person
- Score of 3: 9 people
- Score of 4: 9 people
- Score of 5: 8 people
- Score of 6: 6 people
- Score of 7: 10 people

The question-wise answers were as follows:

(1) Option (B): 36 people
(2) Option (D): 28 people
(3) Option (A): 20 people
(4) Option (D): 36 people
(5) Option (E): 27 people
(6) Option (E): 34 people
(7) Option (C): 31 people

2. Solutions

(1) Consider the function $f(x) := \sum_{n=1}^\infty \frac{x^n}{n(n+2)}$ defined on the closed interval $[-1, 1]$. What are the values of $f(1)$ and $f(-1)$?
(A) $f(1) = 3/4$ and $f(-1) = 1/4$
(B) $f(1) = 3/4$ and $f(-1) = -1/4$
(C) $f(1) = 3/4$ and $f(-1) = -3/4$
(D) $f(1) = 1/4$ and $f(-1) = 3/4$
(E) $f(1) = 1/4$ and $f(-1) = -1/4$

Answer: Option (B)

Explanation: $f(1) = \sum_{n=1}^\infty \frac{1}{n(n+2)}$. Rewrite each summand as:

$$\frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

The terms cancel (telescoping) and we are left with:

$$\frac{1}{2}\left(1 + \frac{1}{2}\right)$$

Simplifying, we get $3/4$.

Similarly, for $f(-1)$, the summands are:

$$\frac{(-1)^n}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

Because the term repetition is every two steps, the n^{th} summand and $(n+2)^{th}$ summand have the same outer sign, so cancellation still proceeds as before. We are left with:

$$\frac{1}{2}\left[\frac{(-1)^1}{1} + \frac{(-1)^2}{2}\right]$$

Simplifying, we get $-1/4$.

"Alternative" approach: Even if you're unable to do these summations, you can estimate the sums quickly using the first few terms and rule out all the other possibilities. For $f(1)$, we have:

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots$$

The sum of the first three terms is $21/40$, which is slightly more than $1/2$, so $3/4$ is the only viable option.

For $f(-1)$, we have:

$$-\frac{1}{3} + \frac{1}{8} - \frac{1}{15} + \dots$$

The sum of the first three terms is $-11/30$, which bounds the alternating sum from *below*. On the other hand, the sum of the first two terms, $-5/24$, bounds the alternating sum from *above*. Among the given options, the only possibility is $-1/4$.

Performance review: 36 out of 44 got this correct. 8 chose (A).

Historical note (two years ago): 16 out of 26 got this correct. 5 chose (C), 3 chose (A), and 1 each chose (D) and (E).

(2) Given that we have the following: $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$ for all $-1 < x < 1$ and the series converges absolutely in the interval, what is an explicit expression for the summation $\sum_{n=1}^{\infty} x^n/(n(n+1))$ for $x \in (-1,1) \setminus \{0\}$?
(A) $1 + \ln(1-x)$
(B) $1 - \ln(1-x)$
(C) $1 + \frac{(1+x)\ln(1-x)}{x}$
(D) $1 + \frac{(1-x)\ln(1-x)}{x}$
(E) $1 + \frac{(x-1)\ln(1-x)}{x}$

Answer: Option (D)

Explanation: We telescope and rewrite the summands as:

$$x^n\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

Due to absolute convergence, we can split the summation across the $-$ sign and get:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^n}{n+1}$$

The first sum is $-\ln(1-x)$. We now calculate the second sum. If $x \neq 0$, we can multiply and divide by x to obtain:

$$\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1}$$

Choose $m = n+1$ to rewrite the right side as $\frac{1}{x}\sum_{m=2}^{\infty} \frac{x^m}{m}$. Note that the sum starts from 2, so adding and subtracting the case $m = 1$ gives:

$$\frac{1}{x}\left[\frac{-x^1}{1} + \sum_{m=1}^{\infty} \frac{x^m}{m}\right]$$

This becomes:

$$\frac{1}{x}\left[-\ln(1-x) - x\right]$$

which becomes:

$$\frac{-\ln(1-x)}{x} - 1$$

Plugging this back into the original, we get:

$$-\ln(1-x) - \left[\frac{-\ln(1-x)}{x} - 1\right]$$

This simplifies to $1 + \ln(1-x)(-1 + (1/x))$ which simplifies to option (D).

Reality check: There are two reality checks we can perform on the expression obtained: taking the limit as $x \to 1^-$, and $x \to 0$. We first consider the limit as x approaches 1 from the left.

In this case, the terms approach $1/(n(n+1))$, which is $(1/n) - (1/(n+1))$, which upon telescoping cancellation gives 1.

On the other hand, the limit:

$$\lim_{x \to 1^-} 1 + \frac{(1-x)\ln(1-x)}{x}$$

is also 1.

Let's now consider the case that x approaches 0.

In this case, the summation approaches 0 because all its terms approach zero.

For the expression, we have:

$$\lim_{x \to 0} \left(1 + \frac{(1-x)\ln(1-x)}{x}\right)$$

Take out the 1+ and we get:

$$1 + \lim_{x \to 0}(1-x) \lim_{x \to 0} \frac{\ln(1-x)}{x}$$

Simple stripping or LH rule gives that the remaining limit is -1, so the overall answer is $1 + (-1) = 0$, as desired.

Performance review: 28 out of 44 got this. 11 chose (E), 3 chose (C), 2 chose (B).

Historical note (two years ago): 10 out of 26 got this correct. 6 chose (C), 4 chose (A), 4 chose (E), 2 chose (B).

(3) Given that we have the following: $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$ for all $-1 < x < 1$ and the series converges absolutely in the interval, what is an explicit expression for the summation $\sum_{n=1}^{\infty} x^n/(n(n+2))$ for $x \in (-1, 1) \setminus \{0\}$?

(A) $\frac{1}{4} + \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}$
(B) $\frac{1}{4} + \frac{1}{2x} + \frac{(x^2-1)\ln(1-x)}{2x^2}$
(C) $\frac{1}{4} - \frac{1}{2x} + \frac{(x^2-1)\ln(1-x)}{2x^2}$
(D) $\frac{1}{4} - \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}$
(E) $\frac{1}{4} + \frac{1}{2x}$

Answer: Option (A)

Explanation: This is similar to the previous question – work it out yourself. It is character building.

Reality check: We now have three interesting limit cases to check: 1^-, -1^+, and 0. We know from Question 6 that the limits for 1^- and -1^+ should be $3/4$ and $-1/4$ respectively. Plugging in the values gives the same answer.

This leaves the case $x = 0$. Here, the limit of the summation should be 0, because all the terms tend to 0. Let's see if this is indeed the case:

$$\lim_{x \to 0} \left[\frac{1}{4} + \frac{1}{2x} + \frac{(1-x^2)\ln(1-x)}{2x^2}\right]$$

We can take the $1/4$ out and take $2x^2$ as a common denominator on the rest:

$$\frac{1}{4}+\lim_{x\to 0}\frac{x+(1-x^2)\ln(1-x)}{2x^2}$$

The limit is now a $\to 0/\to 0$ form, so we can use LH rule:

$$\frac{1}{4}+\lim_{x\to 0}\frac{1-2x\ln(1-x)-(1-x^2)/(1-x)}{4x}$$

Simplifying:

$$\frac{1}{4}+\lim_{x\to 0}\frac{1-2x\ln(1-x)-(1+x)}{4x}$$

Simplify further:

$$\frac{1}{4}+\lim_{x\to 0}\frac{x(-1-2\ln(1-x))}{4x}$$

Cancel the x and evaluate. We get $-1/4$ for the limit, and adding to the outer $+1/4$, we get 0, as expected.

Performance review: 20 out of 44 got this. 8 each chose (B), (C), and (D).

Historical note (two years ago): 7 out of 26 got this correct. 6 each chose (B), (C), (D), and 1 person left the question blank.

(4) Suppose $p > 1$ and let $\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}$. What is the value of $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^p}$ (this is the sum of the p^{th} powers of all odd positive integers) in terms of $\zeta(p)$? *Hint: Write the sum of the odd-numbered terms as the total sum minus the sum of the even-numbered terms.*

(A) $2\zeta(p)-1$
(B) $\zeta(p)/3$
(C) $(2^p-1)\zeta(p)$
(D) $(1-2^{-p})\zeta(p)$
(E) $\zeta(p)/(2^p+1)$

Answer: Option (D)

Explanation: The sum of odd-positioned terms that we are interested in is the sum of all terms minus the sum of even-positioned terms:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}-\sum_{n=1}^{\infty}\frac{1}{(2n)^p}$$

We can pull out a factor of $1/2^p$ from the subtracted sum, to get:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}-\frac{1}{2^p}\sum_{n=1}^{\infty}\frac{1}{n^p}$$

This becomes:

$$\zeta(p)-\frac{1}{2^p}\zeta(p)$$

This simplifies to $(1-2^{-p})\zeta(p)$.

Performance review: 36 out of 44 got this. 6 chose (C), 2 chose (E).

(5) Suppose $p > 1$ and $\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}$. What is the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ in terms of $\zeta(p)$?

(A) $-\zeta(p)$
(B) $\zeta(p)/2$
(C) $\zeta(p)/3$
(D) $(2^{p-1}-1)\zeta(p)$
(E) $(2^{1-p}-1)\zeta(p)$

Answer: Option (E)

Explanation: This is the sum for even-positioned terms minus the sum of odd-positioned terms in the series for $\zeta(p)$. Plugging in the expressions obtained for these in the previous question, we get $2^{-p}\zeta(p) - (1 - 2^{-p})\zeta(p)$ which becomes $(2 \cdot 2^{-p} - 1)\zeta(p)$ which simplifies to option (E).

Performance review: 27 out of 44 got this. 14 chose (D), 1 each chose (A), (B), and (C).

There is a result of calculus which states that, under suitable conditions, if $f_1, f_2, \dots, f_n, \dots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f^{(r)}(x) = \sum_{n=1}^{\infty} f_n^{(r)}(x)$ for any positive integer r. In other words, under suitable assumptions, we can repeatedly differentiate a sum of countably many functions by repeatedly differentiating each of them and adding up the derivatives.

We will not be going into what those assumptions are, but will consider some applications where you are explicitly told that these assumptions are satisfied.

(6) Consider the summation $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. Assume that the required assumptions are valid for this summation, so that $\zeta''(p)$ is the sum of the second derivatives of each of the terms (summands) with respect to p. What is the correct expression for $\zeta''(p)$?

(A) $\sum_{n=1}^{\infty} \frac{(\ln p)^2}{n^p}$
(B) $\sum_{n=1}^{\infty} \frac{(\ln n)(\ln p)}{n^p}$
(C) $\sum_{n=1}^{\infty} \frac{-(\ln n)(\ln p)}{n^p}$
(D) $\sum_{n=1}^{\infty} \frac{-(\ln n)^2}{n^p}$
(E) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^p}$

Answer: Option (E)

Explanation: By Q15 of the previous quiz, the derivative of the n^{th} term with respect to p is $-\frac{\ln n}{n^p}$. Since $\ln n$ is independent of p, the $-\ln n$ can be pulled out, and differentiating again, we get $(-\ln n)\frac{-\ln n}{n^p}$ which becomes $\frac{(\ln n)^2}{n^p}$.

Performance review: 34 out of 44 got this. 3 each chose (C) and (D), 2 each chose (A) and (B).

(7) What can you say about the nature of the function ζ on the interval $(1, \infty)$?

(A) Increasing and concave up
(B) Increasing and concave down
(C) Decreasing and concave up
(D) Decreasing and concave down
(E) Decreasing, initially concave down, then concave up

Answer: Option (C)

Explanation: The "decreasing" part is clear from the expression: as p increases, all the terms being added decrease, so the sum also decreases. This can also be verified by noting that the derivative, which is $\sum_{n=1}^{\infty} \frac{-\ln n}{n^p}$, is negative. The "concave up" part is not so obvious, but the second derivative calculation of the previous question shows that it is positive.

Note that even without doing this derivative calculation, a little reflection would show that (C) is the only feasible option among the choices, though (E) is a bit of a contender at first glance.

Performance review: 31 out of 44 got this. 9 chose (B), 3 chose (D), 1 chose (E).