CLASS QUIZ SOLUTIONS: OCTOBER 3: LIMITS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 0: 1 person
- Score of 1: 2 people
- Score of 2: 5 people
- Score of 3: 3 people
- Score of 4: 1 person

Here are the answers:

(1) Option (C): 8 people
(2) Option (C): 2 people. *Please review!*
(3) Option (B): 8 people
(4) Option (C): 7 people

2. Solutions

(1) Which of these is the correct interpretation of $\lim_{x \to c} f(x) = L$ in terms of the definition of limit?
   (a) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
   (b) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
   (c) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
   (d) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.

   Answer: Option (C)

   Explanation: α plays the role of ϵ and β plays the role of δ.

   Performance review: 8 out of 12 got this correct. 2 chose (B), 1 each chose (A) and (E).

   Historical note (last year): 9 out of 12 people got this correct. 2 people chose (B) and 1 person chose (D).

   Action point: If you got this correct, that means that you are not completely fixated on the letters ϵ and δ. This is good news, because it is important to concentrate on the substantive meaning rather than get caught up with a name. If you had difficulty with this, make sure you can understand it now.

(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x \to c} f(x)$ exists)?
   (A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \epsilon$.
   (B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.
   (C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.
   (D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \epsilon$.
   (E) All of the above.

   Answer: Option (C)

   Explanation: Our statement should be that *every* c has no limit. In other words, for *every* c and *every* L, it is *not* true that $\lim_{x \to c} f(x) = L$. That's exactly what option (C) says.

   Performance review: 2 out of 12 got this correct. 7 chose (B), 2 chose (D), 1 chose (A).

Historical note (last year): 10 out of 12 people got this correct. 1 person each chose (B) and (E).

Note on performance discrepancy with last year: I now remember that last year I had discussed the idea behind this very question in the same class as the quiz was administered, so students last year had an advantage in doing the quiz.

Action point: If you got this correct, great! Do make sure you understand this thoroughly – review the $\epsilon - \delta$ definition yet another time and try to understand how this follows from that definition.

(3) In the usual $\epsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\epsilon > 0$, then which of the following is true?

(A) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.

(B) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.

(C) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.

(D) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.

(E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ϵ also works for larger ϵs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 8 out of 12 got this correct. 3 chose (A), 1 chose (E).

Historical note (last year): 17 out of 26 people got this correct. 5 people chose (A), 3 chose (C), and 1 chose (D).

(4) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of ϵs and δs? *Not appeared in previous years*

(A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(E) None of the above.

Answer: Option (C)

Explanation: The "open interval centered at L" describes the "$\epsilon > 0$" part of the definition (where the open interval is the interval $(L-\epsilon, L+\epsilon)$). The "open interval centered at c" describes the "$\delta > 0$" part of the definition (where the open interval is the interval $(c - \delta, c + \delta)$). x being in the open interval centered at c (except the case $x = c$) is equivalent to $0 < |x - c| < \delta$, and $f(x)$ being in the open interval centered at L is equivalent to $|f(x) - L| < \epsilon$.

Performance review: 7 out of 12 got this correct. 2 chose (A), 1 each chose (B), (D), and (E).

Action point: You should master this way of thinking. This is actually the "correct" way of thinking of the definition. You'll see what I mean in future math classes (153 and beyond).