MR Forward Dynamics

$\because \quad \tau = ID(model, q, \dot{q}, \ddot{q})$

$\tau = M(\theta)\ddot{q} + C(\theta, \dot{\theta})$

$\therefore \quad C(\theta, \dot{\theta}) = ID(model, q, \dot{q}, 0)$

then get $M(\theta)$.

for $i = 1 \rightarrow N_B$

$\tau_i = M(\theta) \cdot \ddot{q}_i + C(\theta, \dot{\theta}) = ID(model, q, \dot{q}, \ddot{q}_i)$

$\ddot{q}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ element

$\therefore M(\theta) = \begin{bmatrix} \tau_1 - C, & \tau_2 - C, & \cdots & \tau_n - C \end{bmatrix}$

$\therefore \quad \ddot{q} = M^{-1} \cdot (\tau - C)$

RBDA Joint Space Inertia Matrix for FD

(this is exactly the same as MR)

The goal is the same, find $H(\theta)$ @ q

Def: Differential inverse dynamics function ID_δ

$$ID_\delta(\text{model}, q, \ddot{q}) = ID(\text{model}, q, \dot{q}, \ddot{q}, f^x) - ID(\text{model}, q, \dot{q}, 0, f^x)$$

$$\delta_\alpha = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow \alpha^{th} \text{ element}$$

$$H \cdot \delta_\alpha = ID_\delta(\text{model}, q, \delta_\alpha)$$

each time get on column of H matrix.

Composite Rigid Body Algorithm

(fastest way to form mass matrix H)

Idea : from kinetic energe, we got $H_{ij} = \sum_{k \in \nu(i) \cap \nu(j)} S_i^T I_k S_j$

$$I_i^c = \sum_{j \in \nu(i)} I_j = I_i + \sum_{j \in \mu(i)} I_j^c$$

$$H_{ij} = \begin{cases} S_i^T \cdot I_i^c \cdot S_j , & i \in \nu(j) , \ i \geq j \\ S_i^T \cdot I_j^c \ S_j , & j \in \nu(i) , \ j \leq i \\ 0 , & \text{no belonging relatinship} \end{cases}$$

transform H_{ij} from i to j frame

$$\left. \begin{aligned} I_i^c &\to {}^jX_i^* I_i^c \, {}^iX_j \\ S_j &\to S_j \\ S_i &\to {}^jX_i S_i \end{aligned} \right\} \quad S_i^T \cdot I_i^c \cdot S_j \Rightarrow S_i^T \cdot {}^jX_i^T \cdot \underbrace{{}^jX_i^*}_{= {}^iX_j^T} \cdot I_i^c \, {}^iX_j S_j$$

$$\Rightarrow S_i^T \cdot I_i^c \cdot {}^iX_j \cdot S_j$$

$\lambda(i)$: parent of link i ; $\mu(i)$: children of link i ; $\nu(i)$: sub tree of i

CH 7 of RBDA : ABA FD

when consider other link effect, the considered body is *handle*,
the considered system is *Articulated Body System*
用关节连接的

E.o.M of handle:

$$f = I^A \cdot a + p^A \quad \Leftarrow \quad a = \Phi^A f + b^A$$

I^A is articulated body inertia, combine of I in the system, map M^6 to F^6 (motion acc) (force)

p^A is bias force of combined system.

Φ^A is articulated body inverse inertia. $\Phi^A = \Phi^{AT} \geq 0$, Φ^A is invertable if handle has 6 DoF.

b^A is bias acceleration

to construct I^A: Projection or Assembly

1. Projection Method

in spatial frame, the handle obeys

$$\left.\begin{array}{l} v = J \cdot \dot{q} \Rightarrow a = \dot{J} \cdot \dot{q} + J \cdot \ddot{q} \\ \begin{cases} \tau = J^T f \\ \tau = H \ddot{q} + C \end{cases} \end{array}\right\} \quad \begin{aligned} a &= J H^{-1} J^T \cdot f + \overset{?}{\dot{J}} \dot{q} - J H^{-1} \cdot C \\ &= \Phi^A \cdot f + b^A \end{aligned}$$

if Φ^A invertible. $\Rightarrow$ $\quad f = (J H^{-1} J^T)^{-1} \cdot a - I^A b^A$
($\Leftrightarrow$ handle has 6 DoF) $\quad = I^A \cdot a + p^A$

where f is spatial force acting on handle
v & a are spatial velocity & acceleration of handle.

2. Assembly Method

to get I^A

Body i's Child j Articulated body I_j^A, p_j^A known.

Body i I_i, p_i known as Inertia & Bias force for single RB.

$\therefore$ Articulated Body A_i = Assembly Child Articulated Body one by one:

$$I_i^A = I_i + \sum I_j^a \ , \quad I_j^a = I_j^A - I_j^A \cdot S_j \cdot (S_j^T \cdot I_j^A \cdot S_j)^{-1} S_j^T I_j^A$$

$$p_i^A = p_i + \sum p_j^a \ , \quad p_j^a = p_j^A + I_j^a \cdot c_j + I_j^A \cdot S_j (S_j^T I_j^A S_j)^{-1} (\tau_j - S_j^T \cdot p_j^A)$$

Remark:

上标A表示 Articulated body 的实际量,

上标a表示换到下一 Articulated body 的过渡量.

for c_j, S_j, $\because a_j - a_i = S_j \cdot \ddot{q} + \dot{S}_j \dot{q} = S_j \cdot \ddot{q} + c_j$

$\therefore$ for known model, c_j, S_j are known

for p_i, $\because$ EoM: $f_{net} = \frac{d}{dt}(I v) = I \cdot a + (v \times^* I - I \cdot v \times) \cdot v$ (RBDA 2.14)

$= I \cdot a + v \times^* I \cdot v$

$\therefore f_{net} = \underset{\substack{\text{ext body} \\ \text{force}}}{f_i^x} + \underset{\text{gravity}}{f_g} + \underset{\text{joint force}}{f_i} = I a + v \times^* I \cdot v$

$\Rightarrow \quad f_i = I(a - g) + v \times^* I \cdot v - f_i^x = I_i a_i + p_i$

$\therefore$ choose $a_0 = -g$, $p_i = v \times^* I \cdot v - f_i^x$,

f_i^x is external force act on the handle.

ABA iterative equation

for Body 1 as the handle of A_1.

① τ_1, a_0, S_1, c_1 known from $1 \rightarrow N_B$

② I_i^A & p_i^A known from Assembly method, $N_B \rightarrow 1$

③
$$\begin{cases} \cancel{f_1} = I_1^A \cdot \cancel{a_1} + p_1^A \\ \cancel{a_1} = a_0 + c_1 + S_1 \ddot{q}_1 \\ S_1^T \cdot \cancel{f_1} = \tau_1 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{q}_1 = (S_1^T I_1^A S_1)^{-1} (\tau_1 - S_1^T I_1^A (a_0 + c_1) - S_1^T \cdot p_1^A) \\ a_1 = a_0 + c_1 + S_1 \cdot \ddot{q}_1 \end{cases}$$

$$\left.\begin{aligned} \ddot{q}_j &= (S_j^T I_j^A S_j)^{-1} \cdot (\tau_j - S_j^T I_j^A (a_i + c_j) - S_j^T \cdot p_j^A) \\ a_j &= a_i + c_j + S_j \cdot \ddot{q}_j \end{aligned}\right\} 1 \rightarrow N_B$$

where i is parent, j is child link.

Articulated Body Algorithm (Assembly method)

⓪ given $\tau, q, \dot{q}, S_i$

① iterative $1 \to N_B$, velocity $V_i = V_{\lambda(i)} + \underbrace{S_i \dot{q}}_{V_{Ji}}$

& $c_i = \underbrace{\overset{\circ}{S}_i \cdot \dot{q}}_{c_{Ji}} + V_i \times V_{Ji}$

& bias force $p_i = V_i \times^* I_i \cdot V_i - f_i^x$

② iterative $N_B \to 1$, Compute I_i^A, p_i^A, $i \in [1, N_B]$, use Assembly

③ iterative $1 \to N_B$, $\ddot{q}_i = (S_i^T \cdot I_i^A \cdot S_i)^{-1} \cdot (\tau_i - S_i^T \cdot I_i^A (a_{\lambda(i)} + c_i) - S_i^T \cdot p_i^A)$

$a_i = a_{\lambda(i)} + c_i + S_i \ddot{q}_i$, $a_0 = -a_g$

(treat G as acc rather than fext)

FD Conclusion

2 main branch

- solve $\ddot{q} = M^{-1}(\tau - c)$, 3 way of get M
  - Naive (MR)
  - Simple (Diff ID)
  - Composite
- propagation (iterative) : ABA
  $O(N_B)$
  - Projection (KDC)
  - Assemply (fastest)