Impedance Control: An Approach to Manipulation:

Part I—Theory

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Manipulation fundamentally requires the manipulator to be mechanically coupled to the object being manipulated; the manipulator may not be treated as an isolated system. This three-part paper presents an approach to the control of dynamic interaction between a manipulator and its environment. In Part I this approach is developed by considering the mechanics of interaction between physical systems. Control of position or force alone is inadequate; control of dynamic behavior is also required. It is shown that as manipulation is a fundamentally nonlinear problem, the distinction between impedance and admittance is essential, and given the environment contains inertial objects, the manipulator must be an impedance. A generalization of a Norton equivalent network is defined for a broad class of nonlinear manipulators which separates the control of motion from the control of impedance while preserving the superposition properties of the Norton network. It is shown that components of the manipulator impedance may be combined by superposition even when they are nonlinear.

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这句话不理解

Introduction

Understanding movement and manipulation and how they may best be controlled is a basic endeavour in several different fields. Understanding the strategies adopted by the central nervous system in the control of movement is one of the fundamental problems of neurophysiology; development of artificial limbs to rehabilitate people with functional disabilities requires an understanding of both how the human normally controls and commands movement and how this may best be implemented in a prosthesis or an orthosis; and the use of robots for industrial automation has focused attention on the problems of manipulation by machine.

The work presented here is an attempt to define a unified and general approach to the control of manipulation. The approach developed encompasses and includes the simple positioning or transporting tasks typically performed by robots and/or prostheses. It also builds on this capability, extending it to facilitate the application of robots and/or prostheses to tasks involving static and dynamic interactions between the manipulator and its environment. It will be shown (in Parts II and III) that the approach can lead to a simplification of some problems in manipulator control.

By any reasonable definition, manipulation fundamentally requires mechanical interaction with the object(s) being manipulated, and a useful classification of manipulatory tasks is by the magnitude of the mechanical work exchanged between the manipulator and its environment. In some cases the interaction forces are negligible, the instantaneous mechanical work done by the manipulator is negligible, ($dW = \mathbf{F} \cdot d\mathbf{X} = 0$) and for control purposes the manipulator may be treated as an isolated system, with its motion (e.g., position, velocity, acceleration) as the controlled variable(s). 情况1 Generally, applications of industrial robots to date have been based on position control, and some of the more successful 无接触 applications have been restricted to this case; examples are 施力 spray-painting and welding [28].

In other situations the manipulator encounters constraints in its environment and the interaction forces are not negligible. Although the manipulator is kinematically coupled 情况2 to its environment, dynamic interaction is still absent. Along the tangent to a pure (i.e., frictionless) kinematic constraint the interaction forces are zero ($\mathbf{F} = 0$) whereas along the normal into the surface the motions are zero ($d\mathbf{X} = 0$) and in all directions the instantaneous mechanical work done is again negligible ($dW = \mathbf{F} \cdot d\mathbf{X} = 0$). In this case an appropriate control strategy is a combination of motion control along the tangent and force control along the normal [22]. This approach to manipulator control has been termed "compliance" or "force control" [15], is more correctly called "accommodation" [16], and is the topic of a considerable body of laboratory research, although it has not yet seen widespread industrial application.

The most general case (which includes the previous two as special instances) is that in which the dynamic interaction is neither zero nor negligible ($dW \neq 0$). A large class of manufacturing operations fall into this category: examples include drilling, reaming, routing, counterboring, grinding, bending, chipping, fettling—any task requiring work to be done on the environment. Many activities of daily living to be performed by an amputee using a prosthesis—basically any

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task involving the use of a tool—are also in this category. Because of the dynamic interaction, the manipulator may no longer be treated for control purposes as an isolated system. Strategies directed toward the control of a vector quantity such as position, velocity, or force will be inadequate as they are insufficient to control the mechanical work exchanged between the manipulator and its environment.

A solution to this problem is to modulate and control the dynamic behavior of the manipulator in addition to commanding its position or velocity. If the environment is regarded as a source of "disturbances" to the manipulator, then modulating the "disturbance response" of the manipulator will permit control of dynamic interactions [18]. One way to vary the dynamic behavior of a manipulator would be to vary the parameters and/or structure of a feedback controller [16, 30], but this is not the only way, nor always the best way. Exploiting the intrinsic properties of mechanical hardware can also provide a simple, effective, and reliable way of dealing with mechanical interaction [3, 4, 17, 31]. A unified framework in which to consider the action of both hardware and software in controlling dynamic interaction is desirable. In the following it is developed from some simple and physically reasonable assumptions.

Physical Equivalence

Throughout this paper it will be assumed that the complete controlled system is hierarchically organized: a high-level supervisory system plans movement task and presents a set of commands $\{c\}$ to a lower-level (real-time) controller which operates directly on the manipulator hardware. Seen from the perspective of the high-level supervisor the control is effectively open-loop. The high-level supervisor, while it may have access to sensory data, does not use that data in an immediate feedback control mode to modulate its commands to the lower-level controller during an ongoing movement. This arrangement is diagrammed in Fig. 1. This organization has been proposed as a general form of control and communication for man/machine systems [26]: it is commonly used for robots [2]; and there is some evidence that the mammalian motor control system is similarly organized [5].

The manipulator is some collection of physical structures, sensors, and actuators (hardware) combined with some set of control algorithms (software). A unified framework for considering the action of both hardware and software in the control of dynamic behavior can be obtained by making the reasonable assumption that no controller can make the manipulator appear to the environment as anything other than a physical system. This can be stated as the following postulate:[1]

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"It is impossible to devise a controller which will cause a physical system to present an apparent behavior to its en-

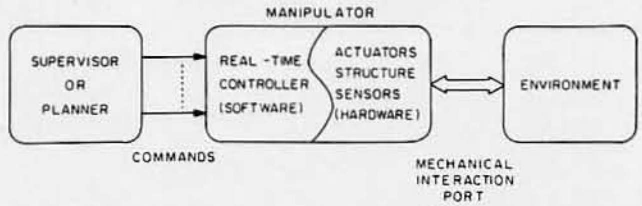

Fig. 1 A schematic diagram of the assumed hierarchical controller structure

没有控制器可使物理系统和普通物理系统与众不同

vironment which is distinguishable from that of a purely physical system."

The value of this postulate is that it is now possible to describe the complete controlled system as an equivalent physical system. Any of the several graphical techniques for describing physical systems may now be applied to the complete system, controller plus hardware. The constraints obeyed by physical systems are especially clearly represented by Paynter's bond graphs [14, 20, 23], and throughout this paper the formalism and terminology of bond graphs will be used.

Causality 因果

Several important constraints on the behavior of physical systems can be identified. Along each degree of freedom, instantaneous power flow between two or more physical systems (e.g., a physical system and its environment) is always definable as the product of two conjugate variables, an effort (e.g., a force, a voltage) and a flow (e.g., a velocity, a current) [20]. An obvious but important physical constraint is that no one system may determine both variables. Along any degree of freedom a manipulator may impress a force on its environment or impose a displacement or velocity on it, but not both.

effort → flow
非常核心的观点.
一个系统不能同时决定2个变量

Seen from the environment along any degree of freedom, physical systems come in only two types: admittances, which accept effort (e.g., force) inputs and yield flow (e.g., motion) outputs; and impedances, which accept flow (e.g., motion) inputs and yield effort (e.g., force) outputs. The concepts of impedance and admittance are familiar to designers of electrical systems as frequency-dependent generalizations of resistance or conductance and are usually regarded as equivalent and interchangeable representations of the same system. For a linear system operating at finite frequencies this is true, but manipulation is fundamentally a nonlinear problem, and for a nonlinear system it is not true; the two representations are in general not interchangeable.

→Admittance ↕ Impedance

For example, the constitutive equation for a point mass is fundamentally written with velocity as the output variable, defined as a function of momentum; momentum in turn is the integral of the input force. As the constitutive equation for a point mass is invertible the equations may also be written with

$$M = mv = \int_0^t F\,dt \Rightarrow v = \frac{1}{m}\int_0^t F\,dt \quad \ldots ①$$

[1] This bears some resemblance to the Turing test of Artificial Intelligence [29].

Nomenclature

- W = mechanical work
- $\mathbf{F}, F_1, F_2$ = force
- $\mathbf{X}, X_1, X_2$ = position
- L_1, L_2, L_3 = link lengths
- $\theta, \theta_1, \theta_2, \theta_3$ = angle
- T_1, T_2, T_3 = torque
- $L(\cdot)$ = linkage kinematic equations
- $:\{c\}$ = modulation by command set
- Se = effort (force) source
- Sf = flow (velocity) source
- Y = admittance
- Z = impedance
- Zo = nodic impedance
- Zn = nonnodic impedance
- $S(\cdot)$ = static force/displacement relation
- $\mathbf{X}_0$ = virtual position
- $\mathbf{V}_0$ = virtual velocity
- $\mathbf{f}$ = flow (velocity)
- t = time
- $\mathbf{z}$ = impedance state variables
- $Zs(\cdot)$ = impedance state equations
- $Zo(\cdot)$ = impedance output equations
- $\mathbf{y}$ = admittance state variables
- $Ys(\cdot)$ = admittance state equations
- $Yo(\cdot)$ = admittance output equations
- $\mathbf{V}$ = velocity

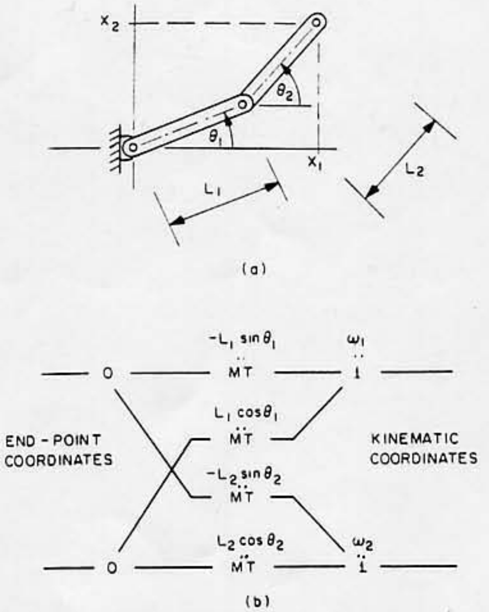

Fig. 2 (a) A planar two-member linkage and (b) a bond-graph of the associated kinematic transformations. Seen from the tip, this sytsem is properly described as an admittance.

$\int F dt = mv \Rightarrow F = m\frac{dv}{dt}$ ... ②，与①可互转

force as the output variable, defined as a function of the derivative of the input velocity variable. The only difference between the two representations of this linear element is that in the strictest sense differentiation is not a physically realizable operation as it is the limiting case of process which requires knowledge of the future. However, it is often a perfectly reasonable operation in a model (no worse than the assumption of the existence of lumped-parameter elements) although physically unrealizable infinite power flow may be predicted during transients.

However, the constitutive equation of a nonlinear dynamic element need not be invertible. The constitutive equation for any device which stores elastic energy is fundamentally written with force as the output variable, defined as a function of input displacement; displacement is in turn defined as the integral of input velocity. The constitutive equation may be nonmonotonic or even discontinuous; the only restriction is that the potential energy integral must be definable (the coenergy integral need not be). Real physical elastic devices exist which cannot be described in the derivative causal form with force as the input variable and motion as the output variable.

构成的

储能设备：$F = f(g(v))$

This inviolable causal contraint is not unique to energy storing elements. The real-world phenomenon of stiction is typically represented by a dissipative element with a noninvertible relation between force and velocity. A velocity may be imposed and a resulting force is defined but the converse is not true.

When more than one degree of freedom is considered, kinematic relations may impose a further causal constraint. Consider the planar linkage shown in Fig. 2(*a*). Assume that this system may interact with its environment across an interaction port at the tip of the linkage. A bond graph of the linkage showing the two independent power bonds associated with this point is shown in Fig. 2(*b*). The linkage equations are a transformation between kinematic variables $\{\theta_1, \theta_2\}$ and interaction port variables $\{X_1, X_2\}$:

$$X_1 = L_1 \cos\theta_1 + L_2 \cos\theta_2 \tag{1}$$

$$X_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2 \tag{2}$$

For every point in $\{\theta_1, \theta_2\}$ there is a corresponding point in $\{X_1, X_2\}$ but the transformation is, in general, not uniquely invertible and there exists a two-dimensional infinity of points in $\{X_1, X_2\}$ for which no point in $\{\theta_1, \theta_2\}$ exists. The latter problem could be eliminated by suitably restricting the range of points in $\{X_1, X_2\}$, and given a knowledge of the current joint angles the angular displacement corresponding to an end-point displacement could be uniquely defined.

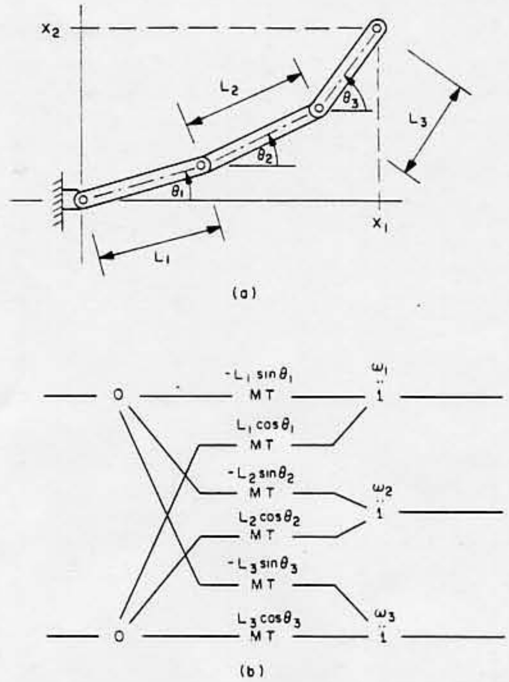

Fig. 3 (a) A planar three-member linkage and (b) a bond-graph of the associated kinematic transformations. Seen from the tip this system is properly described as an admittance.

However, consider the planar linkage shown in Fig. 3(*a*) and a corresponding bond graph shown in Fig. 3(*d*). The kinematic transformation equations are:

$$X_1 = L_1 \cos\theta_1 + L_2 \cos\theta_2 + L_3 \cos\theta_3 \tag{3}$$

$$X_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2 + L_3 \sin\theta_3 \tag{4}$$

Again, joint angles uniquely define end-point position but the converse is not true; even given a suitably restricted set of points in $\{X_1, X_2\}$ and a knowledge of the current joint angles, the end-point displacement does not provide sufficient information to determine the joint angular displacements.

In constrast, the corresponding transformation from forces applied at the interaction port to the resulting torques applied to the links is always well defined:

$$T_1 = -L_1 \sin\theta_1 F_1 + L_1 \cos\theta_1 F_2 \tag{5}$$

$$T_2 = -L_2 \sin\theta_2 F_1 + L_2 \cos\theta_2 F_2 \tag{6}$$

$$T_3 = -L_3 \sin\theta_3 F_1 + L_3 \cos\theta_3 F_2 \tag{7}$$

In fact, examination of the five-port bond graph of Fig. 3(*b*) will show that any combination of two efforts (forces or torques) may be impressed. Similarly, for the four-port bond graph of Fig. 2(*b*) any two efforts may be impressed. The kinematic transformations $\mathbf{X} = L(\theta)$ (equations (1), (2), (3) and (4)) are in fact part of the junction structure through which the various elements in a physical system interact[2] and impose a kinematic causal constraint which is related to but distinct from the conditions imposed by zero- and one-

[2]As an aside, it is the fact that in bond graphs functional relations are represented at graph nodes which makes the equivalence of transformers, gyrators and junctions clear. In contrast, in linear graphs [25] or Mason (signal flow) graphs [27] the junctions are implicit in the graph structure while transformers and gyrators masquerade as elements, and the equivalence is not clear. This is a strong reason for preferring bond-graphs over other methods for graphing physical dynamic systems. Paynter has pointed out some other more important reasons [21].

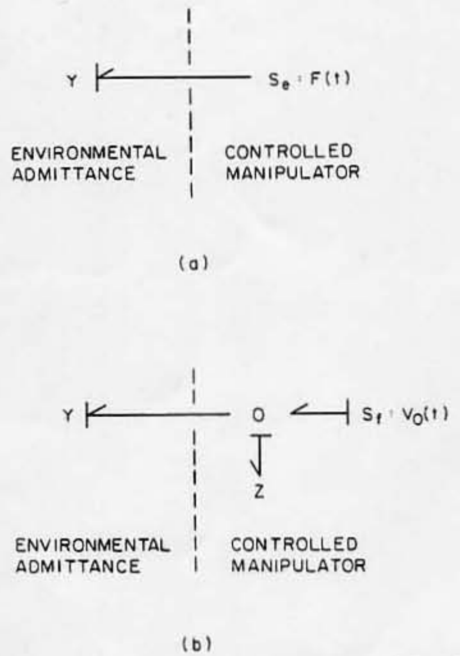

Fig. 4 Bond graph equivalent network representations of (a) pure force control and (b) impedance control

junctions [20]. Any one bond may be causally indifferent but its causality is constrained relative to the others.

The point of this discussion is that the distinction between admittance and impedance is fundamental: Real physical systems exist which can be described in one form and not the other. A spring with a nonmonotonic constitutive equation can only be described as an impedance; seen from an interaction port at its tip, the behavior of a kinematically constrained system such as the linkage of Fig. 3 can only be described as an admittance.

The most important consequence of dynamic interaction between two physical systems is that one must physically complement the other: Along any degree of freedom, if one is an impedance, the other must be an admittance and vice versa. Now, for almost all manipulatory tasks the environment at least contains inertias and/or kinematic constraints, physical systems which accept force inputs and which determine their own motion in response. However, as described above, while a constrained inertial object can always be pushed on, it cannot always be moved; These systems are properly described as admittances. Seen from the manipulator, the world is an admittance.

When a manipulator is mechanically coupled to its environment, to ensure physical compatibility with the environmental admittance, the manipulator should assume the behavior of an impedance. Because the mechanical interaction with the environment will change with different tasks, or even in the course of a single task—the manipulator may be coupled to the environment in one phase and decoupled from it in another—the behavior of the manipulator should be adaptable. Thus the controller should be capable of modulating the impedance of the manipulator as appropriate for a particular phase of a task.

Thus a general strategy for controlling a manipulator is to control its motion (as in conventional robot control) and in addition give it a "disturbance response" for deviations from that motion which has the form of an impedance. The dynamic interaction between manipulator and environment may then be modulated, regulated, and controlled by changing that impedance, and hence the approach described in this paper has been termed "impedance control" [1, 6–11].

Impedance Control, Force Control, and Compliance

If the environment as an admittance, then the manipulator must always impress a force on the environment. It might then be concluded that what is required in general is the control of a vector of interaction forces. Because the controlled manipulator corresponds to some equivalent physical system, it may be represented by a network of physical system elements such as a bond graph. An equivalent physical network representing *pure* force control along a single degree of freedom is shown in Fig. 4(*a*). In this graph the force commands from the high-level supervisor to the low-level controller are represented by an effort source, an ideal element which may impose any time-history of force on the rest of the system independent of its motion.

If it is assumed that at a minimum the manipulator should be capable of stably positioning a simple mass it can be seen that this network is an incomplete description of the necessary controller action: Stable positioning requires at a minimum a static relation between force and position; some spring-like element must be included in the equivalent physical network. The controller must specify a vector quantity such as the desired position, but it must also specify a quantity which is fundamentally different: a relationship, an impedance, which has properties similar to those of a second-rank, twice-covariant tensor; it operates on a contravariant vector of deviations from the desired position to produce a covariant vector of interface forces. In fact, linearized components of the impedance such as the stiffness and the viscosity are second-rank twice covariant tensors.

The simplest equivalent physical network representing impedance control is shown in Fig. 4(*b*). The position commanded by the high-level supervisor is represented by a flow source,[3] an ideal element which may impose any time history of velocity on the rest of the system. To prevent causal conflict between this element and the environmental admittance (which must experience an impressed effort) a zero-junction[4] is interposed between the two. The impedance coupled to this zero-junction represents the relation between force and motion commanded by the supervisor and includes both the static force/displacement relation plus the possible dynamic terms required to ensure controlled dynamic behavior.

The problems of controlling the mechanical interaction between a manipulator and its environment have been addressed by many researchers. The inadequacies of conventional position control are widely recognized and the alternatives are typically referred to as "force control," "compliance," "compliant motion control" or "fine motion control" [12, 13, 15, 19, 22, 30]. As discussed above, pure force control is also inadequate; however, the term is applied loosely to control strategies using force feedback in combination with other feedback variables such as position and/or velocity. The concept of tuning stiffness, damping, and other aspects of the dynamic behavior of a manipulator has been explored by several researchers [18, 19, 24, 30], and the two possible causal forms of manipulator behavior were discussed bv Nevins and Whitney [16]. However, they argued that when the manipulator was in contact with the environment the appropriate strategy was to "command a position or velocity and look at feedback forces" and this approach was used in their subsequent work [30] and that of many other researchers [12, 13, 19]. This is equivalent to

[3] In keeping with standard bond graph practice, the imposition of either a position or a velocity is represented by a flow source. The assumption is that the position is uniquely defined by the integral of the velocity. Either the velocity is known for the infinite past, or an initial position and the subsequent time-history of velocity are known [20].

[4] A zero-junction means that all systems connected to it experience the same effort whereas their flows sum to zero.

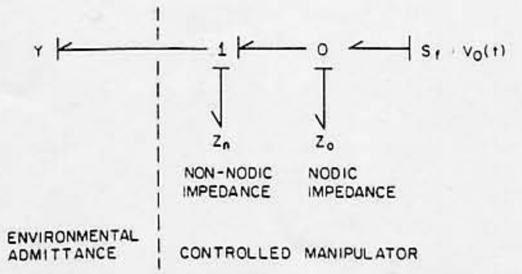

Fig. 5 A bond graph equivalent network representation of the minimum necessary structure of an impedance-controlled machine including both nodic (*Zo*) *and non-nodic* (*Zn*) impedance

giving the manipulator the behavior of an admittance, embodies an implicit assumption that the environment can be described as an impedance, and the approach might reasonably be termed "admittance control." As described above, because of the nature of kinematically constrained inertial objects, the environment is properly described as an admittance and the manipulator should be an impedance. This distinction is not merely one of terminology, but has important consequences, as discussed further below. First, the generality of impedance control is considered.

Generalized Equivalent Networks

Is the simple single-axis impedance controller represented by the equivalent network of Fig. 4(*b*) applicable to a general multi-axis manipulator? That network depicts the separation of the controller action into two distinct components, one (the flow source) representing the control of motion, the other (the impedance) representing the control of dynamic interaction. The separation of the controller action into a (vector) motion component and a impedance component (which has the properties of a tensor) can be achieved for a general class of nonlinear controlled manipulators but some further assumptions about the controller structure are required.

Figure 4(*b*) represents only the nodic component of the impedance seen at the interaction port. Nodicity refers to the invariance of the constitutive equation of an element under a change in the reference value (origin) of its argument. Consider again the static relation between force and position: The nodic component of this relation is the part which may be maintained invariant under a change in the coordinates of the interaction port, i.e., when the manipulator moves. It may be written in terms of a displacement of the end-point rather than an absolute position of the end-point. A general relation between force and position may include non-nodic components, relations which may only be written in terms of the position of the end point in some fixed reference frame. Examples of the latter include the constraints imposed by the finite workspace of a nonmobile manipulator. The non-nodic components should be coupled to a one-junction[5] shared by the manipulator and the environmental admittance. To include both of these components the minimum necessary controller structure is as shown in Fig. 5. However, in most practical situations the primary concern is to be able to specify positions of the workpiece in the workspace and to be able to control aspects of the behavoir of the workpiece at any of these positions. Accordingly, the immediate concern of this paper is with the nodic component of the impedance.

Equivalent networks of the Norton form (Fig. 4(*b*)) or the complementary Thevenin form are familiar to systems engineers, but they are normally applied only to linear systems under steady-state conditions [25]. With nonlinear systems (as usual) some difficulties are encountered. The basic concept underlying both Thevenin and Norton equivalent networks is the separation of unilateral power transmission effects from bilateral dynamic interaction effects. For any general physical system the equivalent source term seen at an interaction port is defined as that required to ensure zero power flow across the port. The differential equation relating port variables under conditions of zero net power flow is the impedance or admittance. Note that nonlinearity does not enter into these definitions. Unfortunately, the junction structure (common effort or common flow) and concomitant superposition properties of the Norton and Thevenin equivalent networks is only guaranteed for linear systems. This means that in a nonlinear system the separation of effects is possible, but reassembling the pieces is not necessarily easy.

The superposition properties may be preserved by assuming that the structure of the manipulator controller is such that it is always capable of determining an equilibrium position of an unconstrained inertial object. If the system is not at equilibrium, assume the set of commands (which may in general vary with time) are "frozen" at their current instantaneous values and impose steady-state conditions. The manipulator behavior (assumed to be nodic) is then characterized by a static relation between force and position (modulated by the command set).

$$\mathbf{F} = S(\mathbf{X}):\{c\} \tag{8}$$

By assumption the manipulator is interacting with an unconstrained inertial object, thus at equilibrium in steady state the interface force is zero. Now assume that zero interface force defines an unique equilibrium position. That is, the class of impedances considered is restricted so that if the gradient of the static force/position relation is nonzero, zero force defines an unique position. As a result the command set always defines an equivalent equilibrium position.

$$\mathbf{X}_0 = \mathbf{X}_0:\{c\} \tag{9}$$

This is the position with respect to which the input displacements to the nodic impedance are measured. It may be thought of as the position toward which the manipulator is heading[6] at any point in time. The actual position of the manipulator end-point may, of course, be different and as the commands may change with time, the manipulator need never reach the position $\mathbf{X}_0$. Consequently, this position need not be restricted to lie within the workspace of the manipulator. It is a convenient fiction and is a summary statement of one consequence of the commands. To keep this distinction clear, $\mathbf{X}_0$ is referred to as a "virtual position" and its time history $\mathbf{X}_0(t)$ is referred to as a "virtual trajectory."

By defining the virtual trajectory the behavior of the controlled manipulator has been decomposed into a vector of port variables which may be commanded and a relation between port variables, an impedance, which may also be commanded. The value of this exercise is that by definition the two components may now be reassembled in the simple manner represented by a zero-junction. The superposition properties of the Norton equivalent network have been retained *without restriction to linear systems*.

The behavior of the manipulator may now be written as follows (assuming a state-determined system):

$$\mathbf{V}_0 = \mathbf{V}_0:\{c\} \qquad \text{Virtual Source} \tag{10}$$

$$\mathbf{f} = \mathbf{V}_0 - \mathbf{V} \qquad \text{Junction Equations} \tag{11}$$

$$d\mathbf{z}/dt = Zs(\mathbf{z}, \mathbf{f}):\{c\} \tag{12}$$

Nodic Impedance

$$\mathbf{F} = Zo(\mathbf{z}, \mathbf{f}):\{c\} \tag{13}$$

As before, following standard bond graph convention the imposition of a virtual position or a virtual trajectory has

[5]A one-junction means that all systems connected to it experience the same flow whereas their efforts sum to zero.

[6]Or, if the equilibrium point is unstable, away from which it is heading.

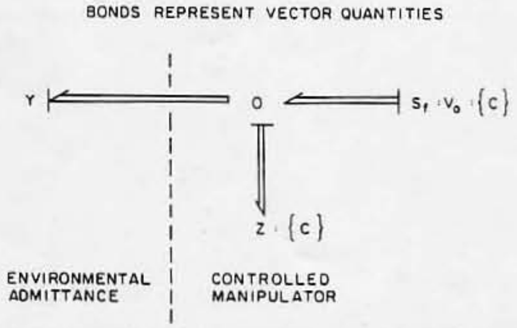

Fig. 6 A bond graph equivalent network representating a multiaxis manipulator with controlled nodic impedance interacting with an admittance-type environment. The bond graph for the manipulator is a generalized Norton equivalent network.

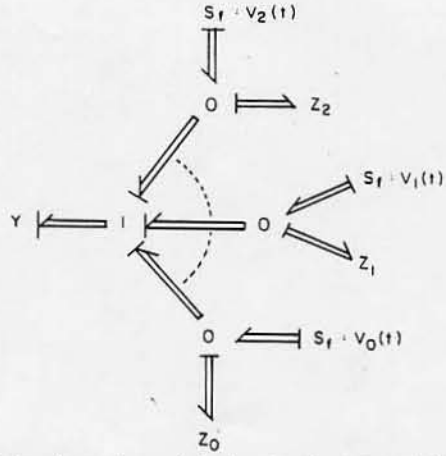

Fig. 7 A bond graph equivalent network representation of the superposition of multiple impedances coupled to an admittance. Each component of the total impedance is represented by a generalized Norton equivalent network. Non-nodic impedances may be included in this system by setting the corresponding virtual flow source to zero.

been represented by a flow source. Writing the environmental admittance in general form:

$$dy/dt = Ys(\mathbf{y},\mathbf{F}) \tag{14}$$

Admittance

$$\mathbf{V} = Yo(\mathbf{y}) \tag{15}$$

The two sets of equations may be combined to write the complete system equations in standard (integrable) form:

$$d\mathbf{z}/dt = Zs[\mathbf{z},(\mathbf{V}_0:\{c\} - Yo(\mathbf{y}))]:\{c\} \tag{16}$$

$$d\mathbf{y}/dt = Ys[\mathbf{y},Zo(\mathbf{z},[\mathbf{V}_0:\{c\} - Yo(\mathbf{y})])]:\{c\} \tag{17}$$

$$\mathbf{F} = Zo(\mathbf{z},[\mathbf{V}_0:\{c\} - Yo(\mathbf{y})]):\{c\} \tag{18}$$

$$\mathbf{V} = Yo(\mathbf{y}) \tag{19}$$

The purpose of the foregoing discussion was to demonstrate that a broad and useful class of nonlinear manipulator behaviors may be represented by a simple equivalent network. The only assumptions made were that the manipulator is sufficiently controllable to be able to determine an equilibrium position of an unconstrained inertial object such as a mass, that the port impedance is nodic, and that its static component is such that if its gradient is nonzero then zero force defines an unique position—not a restrictive set of assumptions. Thus a general class of manipulation problems have the same basic structure as Fig. 4(*b*). The behavior of a multiaxis impedance-controlled manipulator interacting with an admittance-type environment may be represented by the graph shown in Fig. 6, which is a generalization of a Norton equivalent network. Not only does this graph provide a compact representation of manipulation, the parallel with the standard Norton equivalent network is quite complete: The superposition properties of the Norton equivalent network have been preserved.

Superposition of Impedances

The most interesting consequence of the assumptions underlying impedance control is that if the dynamic behavior of the manipulator is dissected into a set of component impedances, these may be reassembled by simple addition *even when the behavior of any or all of the components is nonlinear*. This is a direct consequence of the assumption that the environment is an admittance. That admittance sums the forces applied to it and determines its motion in response, as represented by the one-junction of Fig. 5. The admittance also acts to sum any impedances coupled to it. All of the systems connected to the one-junction associated with the admittance experience the same input velocity; the total force they apply to the admittance is simply the sum of their individual force responses to the motion of the environmental admittance. Linearity of the impedances is not a consideration.

When the manipulator is decoupled from its environment the terms in the dynamic equations due to the environmental admittance disappear and in principle the manipulator alone need exhibit no inertial behavior. In practice the uncoupled manipulator still has inertia (albeit nonlinear and configuration-dependent). Because of the inevitable inertial dynamics of the isolated manipulator the superposition of impedances holds even when the manipulator is uncoupled from its environment as there is always an admittance to sum forces and impedances.

This simple observation has many important consequences, some of which will be pursued in the subsequent parts of this paper. One which is immediately apparent is that different controller actions aimed at simultaneously satisfying different task requirements may be superimposed. Each task component may be represented by a generalized Norton equivalent network, but referred to a different node (or virtual position) as shown in Fig. 7. Note that any non-nodic component of the manipulator behavior may be included in this equivalent network by associating it with a flow source identically equal to zero and thus the assumption of nodicity made earlier is not restrictive.

Summary

This paper has presented a unified approach to manipulation termed "impedance control." Because by its nature manipulation requires mechanical interaction between systems, the focus of the approach is on the characterization and control of interaction. To understand interaction concepts drawn from bond graph network analysis of dynamic systems are useful, particularly the concept of causality. By assuming that no control algorithm may make a physical system behave like anything other than a physical system the network concepts of bond graphs may be applied to describe the way the controller may modify the behavior of the manipulator. Several simple but fundamental observations may then be made: Command and control of a vector such as position or force is not enough to control dynamic interaction between systems; the controller must also command and control a relation between port variables. In the most common case in which the environment is an admittance (e.g., a mass, possibly kinematically constrained) that relation should be an impedance, a function, possibly nonlinear, dynamic, or even discontinuous, specifying the force produced in response to a motion imposed by the environment. Even more important, if the environment is an admittance, the total impedance coupled to it (due to the manipulator or anything else) is expressible as a sum of component impedances, even when the components are nonlinear.

Under a set of reasonable and unrestrictive assumptions the interaction port behavior of the manipulator may be decomposed into a vector motion component and an impedance component with some of the characteristics of a second-rank, twice-covariant tensor. The vector component may be expressed as a virtual trajectory towards which the controlled manipulator dynamics are trying to drive the interaction port. Its significance is that it permits the motion and impedance components of the manipulator behavior to be reassembled by superposition as depicted by the junction structure of a generalized Norton equivalent network. Note that no restrictive assumptions of small displacements or linearity were required.

Part II and III of this paper will discuss the implementation and application of impedance control.

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Impedance Control: An Approach to Manipulation:

Part II—Implementation

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This three-part paper presents an approach to the control of dynamic interaction between a manipulator and its environment. Part I presented the theoretical reasoning behind impedance control. In Part II the implementation of impedance control is considered. A feedback control algorithm for imposing a desired cartesian impedance on the end-point of a nonlinear manipulator is presented. This algorithm completely eliminates the need to solve the "inverse kinematics problem" in robot motion control. The modulation of end-point impedance without using feedback control is also considered, and it is shown that apparently "redundant" actuators and degrees of freedom such as exist in the primate musculoskeletal system may be used to modulate end-point impedance and may play an essential functional role in the control of dynamic interaction.

Introduction

Most successful applications of industrial robots to date have been based on position control, in which the robot is treated essentially as an isolated system. However, many practical tasks to be performed by an industrial robot or an amputee with a prosthesis fundamentally require dynamic interaction. The work presented in this three-part paper is an attempt to define a unified approach to manipulation which is sufficiently general to control manipulation under these circumstances.

In Part I this approach was developed by starting with the reasonable postulate that no controller can make the manipulator appear to the environment as anything other than a physical system. An important consequence of dynamic interaction between two physical systems such as a manipulator and its environment is that one must physically complement the other: Along any degree of freedom, if one is an impedance, the other must be an admittance and vice versa.

One of the difficulties of controlling manipulation stems from the fact that while the bulk of existing control theory applies to linear systems, manipulation is a fundamentally nonlinear problem. The familiar concepts of impedance and admittance are usually applied to linear systems and regarded as equivalent and interchangeable. As shown in Part I, for a nonlinear system, the distinction between the two is fundamental.

Now, for almost all manipulatory tasks the environment at least contains inertias and kinematic constraints, physical systems which accept force inputs and which determine their motion in response and are properly described as admittances. When a manipulator is mechanically coupled to such an environment, to ensure physical compatibility with the environmental admittance, something has to give, and the manipulator should assume the behavior of an impedance.

Thus a general strategy for controlling a manipulator is to control its motion (as in conventional robot control) and in addition give it a "disturbance response" for deviations from that motion which has the form of an impedance. The dynamic interaction between manipulator and environment may then be modulated, regulated, and controlled by changing that impedance.

This second part of the paper presents some techniques for controlling the impedance of a general nonlinear multiaxis manipulator.

Implementation of Impedance Control

A distinction between impedance control and the more conventional approaches to manipulator control is that the controller attempts to implement a dynamic relation between manipulator variables such as end-point position and force rather than just control these variables alone. This change in perspective results in a simplification of several control problems.

Most of our work to date [3, 6, 13, 14, 16] has focused on controlling the impedance of a manipulator as seen at its "port of interaction" with the environment, its end effector. A substantial body of literature has been published on methods for implementing a planned end effector cartesian path [5, 27, 28, 32, 34, 35]. The approach is widely used in the control of industrial manipulators and there is some evidence of a comparable strategy of motion control in biological systems [1, 24]. Following the lead from this prior work we have investigated ways of presenting the environment with a dynamic behavior which is simple when expressed in workspace (e.g., Cartesian) coordinates.

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The lowest-order term in any impedance is the static relation between output force and input displacement, a stiffness. If, in common with much of the current work on robot control, we assume actuators capable of generating commanded forces (or torques), Tact, sensors capable of observing actuator position (or angle), θ, and a purely kinematic relation (i.e., no structural elastic effects) between actuator position and end-point position[1], $\mathbf{X} = L(\theta)$, it is straightforward to design a feedback control law to implement in actuator coordinates a desired relation between end-point (interface) force, Fint, and position, $\mathbf{X}$. Defining the desired equilibrium position for the end-point in the absence of environmental forces (the virtual position) as $\mathbf{X}_0$, a general form for the desired force-position relation is:

$$\mathrm{Fint} = K[\mathbf{X}_0 - \mathbf{X}] \tag{1}$$

Compute the Jacobian, $\mathbf{J}(\theta)$:

$$d\mathbf{X} = \mathbf{J}(\theta)d\theta \tag{2}$$

From the principal of virtual work:

$$\mathrm{Tact} = \mathbf{J}^t(\theta)\mathrm{Fint} \tag{3}$$

The required relation in actuator coordinates is:

$$\mathrm{Tact} = \mathbf{J}^t(\theta)K\,[\mathbf{X}_0 - L(\theta)] \tag{4}$$

No restriction of linearity has been placed on the relation $K[\mathbf{X}_0 - \mathbf{X}]$, and the displacement of $\mathbf{X}$ from $\mathbf{X}_0$ need not be small. Note that in this equation the inverse Jacobian is not required.

Inverting the kinematic equations of a manipulator to determine the time-history of actuator (joint) positions required to produce a desired time-history of end-point positions has been described as one of the most difficult problems in robot control [28]. For some manipulators (e.g., those with nonintersecting wrist joint axes) no explicit (closed-form) algebraic solution may be possible. However, if $K[\mathbf{X}_0 - \mathbf{X}]$ is chosen so as to make the end-point sufficiently stiff, then a controller which implements equation (4) will accomplish Cartesian end-point position control and the need to solve the "inverse kinematics problem" has been completely eliminated. Only the forward kinematic equations for the manipulator need be computed. This is a direct consequence of the care which was taken to ensure that the desired behavior was compatible with the fundamental mechanics of manipulation and was expressed as an impedance.

[1] Throughout this paper, "position" will refer to both location and orientation, and "force" will refer to both force and moment.

Another important term in the manipulator impedance is the relation between force and velocity. Again, given the above assumptions, it is straightforward to define a feedback law to implement in actuator coordinates a desired relation between end-point force and end-point velocity such as:

$$\mathrm{Fint} = B[\mathbf{V}_0 - \mathbf{V}] \tag{5}$$

From the manipulator kinematics:

$$\mathbf{V} = \mathbf{J}(\theta)\omega \tag{6}$$

The required relation in actuator coordinates is:

$$\mathrm{Tact} = \mathbf{J}^t(\theta)B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega] \tag{7}$$

Again note that the relation $B[\mathbf{V}_0 - \mathbf{V}]$ need not be linear and that inversion of the Jacobian is not required.

The dynamic behavior to be imposed on the manipulator should be as simple as possible, but no simpler. The foregoing equations take no account of the inertial, frictional, or gravitational dynamics of the manipulator. Under some circumstances this may be reasonable, but in many situations these effects cannot be neglected. To ensure dynamic feasibility the choice of the impedance to be imposed should be based on the dominant dynamic behavior of the manipulator. The choice is a tradeoff between keeping the complexity of the controller within manageable limits while ensuring that imposed behavior adequately reflects the real dynamic behavior of the controlled system. As a result it depends both on the manipulator itself and on the environment in which it operates. For example, a manipulator intended for underwater applications will operate in a predominantly viscous environment and it may be reasonable to ignore inertial effects. In contrast, a manipulator intended for operation in a free-fall orbit will encounter a predominantly inertial environment. For terrestrial applications (which have been the main concern of our work) both gravitational and inertial effects are important, and the dominant dynamic behavior is that of a mass driven by motion-dependent forces, second order in displacement along each degree of freedom.

Nomenclature

- Y = admittance
- Z = impedance
- $:\{c\}$ = modulation by command set
- Sf = flow source
- Se = effort source
- Fext = external force
- Fint = interface force
- $\mathbf{X}$ = end-point position
- $\mathbf{X}_0$ = commanded (virtual) position
- $\mathbf{V}$ = end-point velocity
- $\mathbf{V}_0$ = commanded (virtual) velocity
- $K[\cdot]$ = force/displacement relation
- $B[\cdot]$ = force/velocity relation
- θ = actuator position or angle
- ω = actuator velocity
- $L(\cdot)$ = linkage kinematic equations
- $\mathbf{J}(\theta)$ = Jacobian
- M = inertia tensor in end-point coordinates
- m = mass
- I = inertia
- t = time
- $F(\cdot)$ = noninertial impedance
- Me = environmental inertia tensor
- $I(\theta)$ = inertia tensor in actuator coordinates
- $C(\theta,\omega)$ = inertial coupling torques
- $V(\omega)$ = velocity-dependent torques
- $S(\theta)$ = position-dependent torques
- $G(\theta,\omega)$ = accelerative coupling terms
- Tact = actuator force or torque
- Tint = interface torques
- $Y(\theta)$ = mobility tensor in actuator coordinates
- $W(\theta)$ = mobility tensor in end-point coordinates
- $\mathbf{h}$ = generalized momentum in actuator coordinates
- $\mathbf{p}$ = generalized momentum in end-point coordinates
- $H(\cdot)$ = Hamiltonian
- $\mathbf{T}, T_1, T_2$ = torque
- θ_1, θ_2 = absolute joint angle
- ρ_1, ρ_2 = relative joint angle
- L_1, L_2 = link lengths
- Ke = net stiffness due to elbow muscles
- Ks = net stiffness due to shoulder muscles
- Kt = net stiffness due to two-joint muscles
- Kx = stiffness tensor in end-point coordinates
- Ko = stiffness tensor in joint coordinates
- Ep = potential energy
- Ek = kinetic energy
- Ek^* = kinetic coenergy
- $\lambda_1 \lambda_2$ = eigenvalues

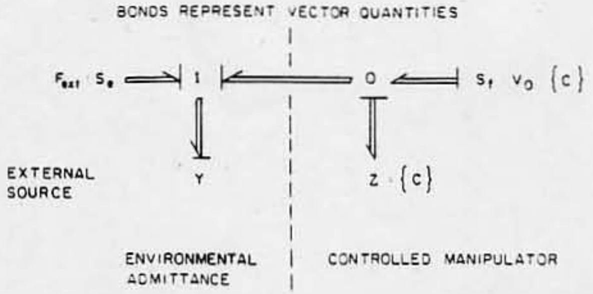

Fig. 1 A bond graph equivalent network representation of an impedance-controlled manipulator interacting with an environmental admittance. Each bond represents a vector of power flows along multiple degrees of freedom.

When the manipulator is decoupled from its environment the term due to the environmental admittance disappears, and in principle the manipulator alone need exhibit no mass-like behavior. In practice, the uncoupled manipulator still has inertia (albeit nonlinear and configuration-dependent). This means that the controlled system, both with the manipulator coupled to and uncoupled from its environment, can be represented by an admittance coupled to an impedance as shown in Fig. 1.

No physically realisable strategy can eliminate the inertial effects of a manipulator but the apparent inertia seen at the end effector can be modified. The approach we have taken to deal with inertial manipulator behavior is to "mask" the true nonlinear inertial dynamics of the manipulator and impose simpler dynamics, those of a rigid body. Most manipulatory tasks are fundamentally described in relative coordinates, that is, in terms of displacements and rotations with respect to a workpiece, tool or fixture whose location in the workspace is not known in advance with certainty. As a result, task planning and execution will be simplified if the end-point inertial behavior is modified to be that of a rigid body with an inertia tensor which remains invariant under translation and rotation of the coordinate axes. This is achieved if:

$$M = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{8}$$

This is the inertia tensor of a rigid body such as a cube of uniform density. This inertia tensor eliminates the angular velocity product terms in the Euler equations for the motion of a rigid body [8] and ensures that if the system is at rest the applied force and the resulting acceleration vectors are colinear.

To represent the dominant second-order behavior of the manipulator the noninertial interface forces are assumed to depend only on displacement, velocity and time:

$$\mathbf{F}\text{int} = F(\mathbf{X}, \mathbf{V}) - M d\mathbf{V}/dt \tag{9}$$

If the noninertial behavior to be imposed is nodic, it may be written in terms of a displacement from a commanded (time-varying) position $\mathbf{X}_0$:

$$\mathbf{F}\text{int} = F(\mathbf{X}_0 - \mathbf{X}, \mathbf{V}_0 - \mathbf{V}) - M d\mathbf{V}/dt \tag{10}$$

Although there may be cases in which coupled nonlinear viscoelastic behavior is useful, for simplicity the position- and velocity-dependent terms may be separated:

$$\mathbf{F}\text{int} = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] - M d\mathbf{V}/dt \tag{11}$$

All of the parameters in this expression are implicitly assumed to be functions of the set of control commands $\{c\}$ and of time.

This set of assumptions defines a target behavior which includes inertial effects. The first two terms are the position- and velocity-dependent impedances of equations (1) and (5). If the environment is a simple rigid body acted on by unpredictable (or merely unpredicted) forces, its dynamic equations are:

$$Me\, d\mathbf{V}/dt = \mathbf{F}\text{ext} + \mathbf{F}\text{int} \tag{12}$$

and the coupled equations of motion for the complete system of figure 1 are:

$$(Me + M) d\mathbf{V}/dt = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] + \mathbf{F}\text{ext} \tag{13}$$

Note that in this case both the coupled and uncoupled equations for the system have the same second-order form.

To implement the target behavior of equation (11), one approach we have used is to express the desired Cartesian coordinate impedance in actuator coordinates (the kinematic transformations between actuator coordinates and end-point coordinates provides sufficient information to do this) and then use a model of the manipulator dynamics to derive the required controller equations. Assuming that the kinematic, inertial, gravitational, and frictional effects provide an adequate model of the manipulator dynamics as follows:

$$I(\theta) d\omega/dt + C(\theta, \omega) + V(\omega) + S(\theta) = \mathbf{T}\text{act} + \mathbf{T}\text{int} \tag{14}$$

an expression for the required actuator torque as a function of actuator position and velocity and end-point force can be derived by straightforward substitution (see Appendix I):

$$\begin{aligned} \mathbf{T}\text{act} &= I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}K[\mathbf{X}_0 - L(\theta)] + S(\theta) \\ &+ I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega] + V(\omega) \\ &+ I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}\mathbf{F}\text{int} - \mathbf{J}^{t}(\theta)\mathbf{F}\text{int} \\ &- I(\theta)\mathbf{J}^{-1}(\theta)G(\theta, \omega) + C(\theta, \omega) \end{aligned} \tag{15}$$

This equation expresses the required behavior to be provided by the controller as a nonlinear impedance in actuator coordinates. It may be viewed as a nonlinear feedback law relating actuator torques to observations of actuator position, velocity and interface force. The input (command) variables are the desired cartesian position (and velocity) and the terms of the desired (possibly nonlinear) cartesian dynamic behavior characterized by M, $B[\cdot]$ and $K[\cdot]$.

The feasibility of this approach to cartesian impedance control has been investigated [6,16] by implementing this nonlinear control law to impose cartesian end-point dynamics on a servo-controlled, planar, two-link mechanism (similar to the nonlinear linkage in a SCARA[2] robot). A simple analysis estimating the computation required to implement this controller on a six-degree-of-freedom manipulator indicated that the computational burden is comparable to "exact" approaches to generating forward-path manipulator commands such as the recursive LaGrangian [17] and Newton-Euler [21] methods or the configuration space method [18].

If the interface forces and torques in equations (11) and (15) are eliminated and the position- and velocity-dependent terms reduced to linear diagonal forms, this implementation of impedance control resembles the resolved acceleration method [22]. However, unlike the resolved acceleration method, the impedance control algorithm presented above is based on desired end-point behaviour which may be chosen rationally using approaches such as the optimisation technique presented in Part III. Furthermore, the impedance control algorithm includes terms for coping with external "disturbances." Without the external "disturbance" terms (which have no counterpart in the resolved acceleration algorithm) the manipulator is not capable of controlled

[2] Selective Compliance Assembly Robot Arm [23].

mechanical interaction with its environment. Note also that the above approach to defining the controller equations is not restricted to commanded linear behavior and can be applied equally well to achieve the more general coupled nonlinear behavior of equation (9).

It is not claimed that the above algorithm is the only way to achieve a desired end-point impedance. It is presented here only to demonstrate that a control law capable of modulating the end-point impedance of a manipulator may be formulated. The controller of equation (15) was designed by a technique which is similar to pole-placement methods [31] in that the desired behaviour and a model of the actual behaviour of the manipulator were compared algebraically to derive the controller equations. In common with most approaches to manipulator control the approach is based on a model which ignored many aspects of real manipulator performance, particularly the dynamics of the actuators and the transmission system. Furthermore, like many other approaches the method assumes that the Jacobian is invertible.

This technique is, of course, only one possible approach to the design of a controller for implementing a desired cartesian impedance, and, if one may draw from linear systems design experience without overstretching the analogy to pole-placements methods, it is not even likely to be the best. Other approaches to controller design such as the model-referenced adaptive control method [9] will probably be useful.

Impedance Modulation Without Feedback

Modulation of end-point impedance using feedback strategies is not the only way to control the dynamic behavior of a manipulator, nor is it always the best. This is particularly evident in a biological system. One of the most distinctive features of the primate neural control system is the unavoidable delay associated with neural transmission. The *shortest* time for information to get from peripheral sensors (e.g., in the muscles or skin) in the human arm to the higher levels of the central nervous system (e.g., the cortex) and back to the actuators of the arm is 70 milliseconds, and loop transmission delays of 100 to 150 milliseconds are typical [29]. This problem is further exacerbated if significant computation is required (the response time to a visual stimulus is somewhere between 200 and 250 milliseconds). The effectiveness of feedback control in the presence of a delay of this magnitude is severely limited, particularly in dealing with tasks involving dynamic interaction. Yet primates excel at controlling dynamic interactions; How do they do that?

One alternative to feedback which we have explored is the use of redundancies: "excess" actuators or "excess" skeletal degrees of freedom. From a purely kinematic standpoint the neuromuscular system is multiply redundant. For example, the kinematic chain connecting the wrist joint to the chest (clavicle, scapula, humerus, radius and ulna) has considerably more degrees of freedom than those required to specify the position (and orientation) of the hand in cartesian coordinates. These skeletal redundancies can serve to provide a measure of control over the inertial component of the end-point dynamics.

In considering the apparent inertial behaviour of the end-point it is useful to remember that an inertia is fundamentally an admittance; flow (velocity) is determined as a response to impressed effort (force). Dealing with kinematic redundancy is considerably simplified if the constitutive equations are written as a relation determining generalised velocity, ω, (e.g., the velocities of the manipulator joints) as a function of generalized momentum, $\mathbf{h}$:

$$\omega = Y(\theta)\mathbf{h} \tag{16}$$

$Y(\theta)$ is the inverse of the more commonly used inertia tensor, and to help distinguish the two, the term "mobility" is suggested. The elements of the mobility tensor in general will depend on the manipulator configuration.

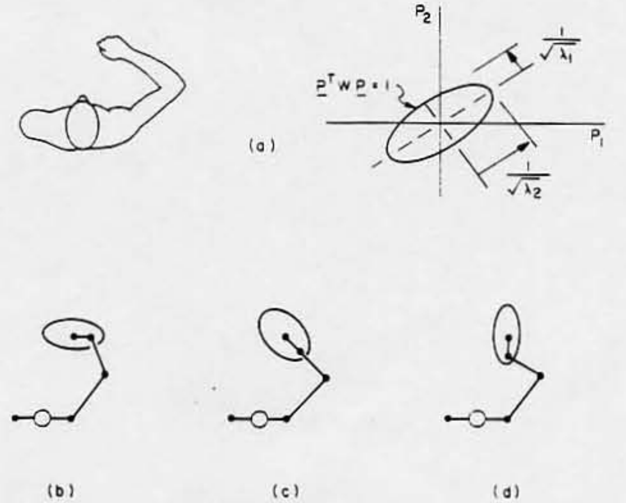

Fig. 2 A schematic representation of the influence of kinematic redundancies on the mobility (inverse effective mass) of the end-point of a planar linkage. The ellipsoid of gyration associated with the mobility tensor is shown in (*a*). The eigenvalues of the mobility tensor are inversely proportional to the effective mass in the direction of the corresponding eigenvectors and the square root of their ratio determines the ratio of the major and minor axes of the ellipsoid, which are colinear with the eigenvectors. For a planar, three-member linkage with links of uniform density and cross section and lengths in the ratio 1: 2: 3 the effect on the ellipsoid of gyration of changing the linkage configuration for a fixed position of the end-point is shown in (*b*), (*c*), and (*d*).

At any given configuration, the kinematic transformations between joint angles and end-point coordinates define not only the relations between generalized displacements, flows and efforts in the two coordinate frames, (see equations (2), (3), and (6)) they also define the relation between the generalised momenta in joint coordinates, $\mathbf{h}$, and end-point coordinates, $\mathbf{p}$, through the Jacobian (see Appendix II):

$$\mathbf{h} = \mathbf{J}^t(\theta)\mathbf{p} \tag{17}$$

Consequently, the mobility tensor in end-point coordinates $W(\theta)$ is related to the mobility in joint coordinates $Y(\theta)$ as follows:

$$\mathbf{V} = W(\theta)\mathbf{p} \tag{18}$$

$$W(\theta) = \mathbf{J}(\theta)\, Y(\theta)\mathbf{J}^t(\theta) \tag{19}$$

The physical meaning of the end-point mobility tensor is that if the system is at rest (zero velocity) then a force vector applied to the end-point causes an acceleration vector (not necessarily co-linear with the applied force) which is obtained by premultiplying the force vector by the mobility tensor (see Appendix II).

Note that the Jacobian in the above equation need not be square, and that the end-point mobility is configuration dependent. As a result, redundant degrees of freedom can be used to modulate the end-point mobility. Consider the simplified three-link model of the primate upper extremity (arm, forearm and hand, each considered to be rigid bodies, linked by simple pin-joints) moving in a plane as shown in Fig. 2. For simplicity, assume the links are rods of uniform density with lengths in the ratio of 1: 2: 3.

Any real linkage such as the skeleton is a generalised kinetic energy storage system. Kinetic energy is always a quadratic form in momentum:

$$Ek = \tfrac{1}{2}\mathbf{h}^t\, Y(\theta)\,\mathbf{h} \tag{20}$$

Thus the locus of deviations of the generalised momentum from zero for which the kinetic energy is constant is an ellipsoid, the "ellipsoid of gyration" [33]. It graphically

Table 1 Variation of apparent end-point mass with linkage configuration

Distal link orientation (degrees)	Effective mass X_1-direction (kgm)	Effective mass X_2-direction (kgm)
90	0.322	1.823
135	0.568	0.568
180	1.824	0.323

Link Lengths: 1, 2, 3 meters; Linear density: 1 kgm/m

represents the directional properties of the mobility tensor. The eigenvalues of the symmetric mobility tensor define the size and shape and the eigenvectors the orientation of the ellipsoid of gyration (see Appendix II). An ellipsoid of gyration can be associated with the mobility tensor in any coordinate frame, e.g., end-point coordinates (see Fig. 2(*a*)).

Figures 2(*b*) through 2(*d*) show the profound effect on the ellipsoid of gyration of changes in arm configuration while keeping the position of the end-point fixed. The inertial resistance to a force applied radially inward toward the shoulder (vertically downward in the figure) changes by almost a factor of six as the hand rotates through ninety degrees (see Table 1). In the configuration of Fig. 2(*d*) the applied force has to accelerate all three links; in that of Fig. 2(*b*) it primarily has to accelerate the distal link. Clearly, kinematic redundancies in a linkage provide a vehicle for changing the way the end-point will react to external disturbances without recourse to feedback strategies.

As an aside, an alternative representation of inertial behavior is via the ellipsoid of inertia [33]. Asada [4] has suggested its use as a tool for designing robot mechanisms. However, the ellipsoid of gyration is the more fundamental representation; it is readily obtained even when the Jacobian of the linkage is noninvertible. Also, while the matrix $Y(\theta)$ may never have zero eigenvalues, (assuming real links with nonzero mass) the matrix $W(\theta)$ may, because of the kinematics of the linkage. If the inertial behavior of the tip is written in the conventional (impedance) form:

$$\mathbf{p} = M(\theta)\mathbf{V} \tag{21}$$

there exist locations in the workspace for which the eigenvalues of the tensor $M(\theta)$ become infinite. Thus the end-point inertia tensor can not be defined for some linkage configurations. On the other hand the worst the eigenvalues of $W(\theta)$ will do is go to zero, which is easier to deal with computationally. Again, a reminder of the fact that the difference between impedance and admittance is fundamental.

Impedance Modulation Using Actuator Redundancies

It is also possible to modulate the position- and velocity-dependent components of end-point impedance without feedback by exploiting the intrinsic properties of the actuators, and again apparent redundancies are useful. Although a muscle is by no means thermodynamically conservative, it exhibits a static relation between force and length (for any given fixed level of neural input) similar to that of a mechanical spring, i.e., one which permits the definition of a potential function analogous to elastic energy.[3] Muscle force also exhibits a dependence on velocity similar to a mechanical damper. It has been shown that the mechanical impedance of a single muscle may be modulated by neural commands both in the presence and in the absence of neural feedback [7, 11, 12, 25, 26]. Simultaneously activating two or more muscles which oppose each other across a joint is one strategy which permits impedance to be modulated independent of joint torque [15, 20]. (This is what happens, for example, when one tenses the muscles of the arm without moving; the impedance of the limb increases.)

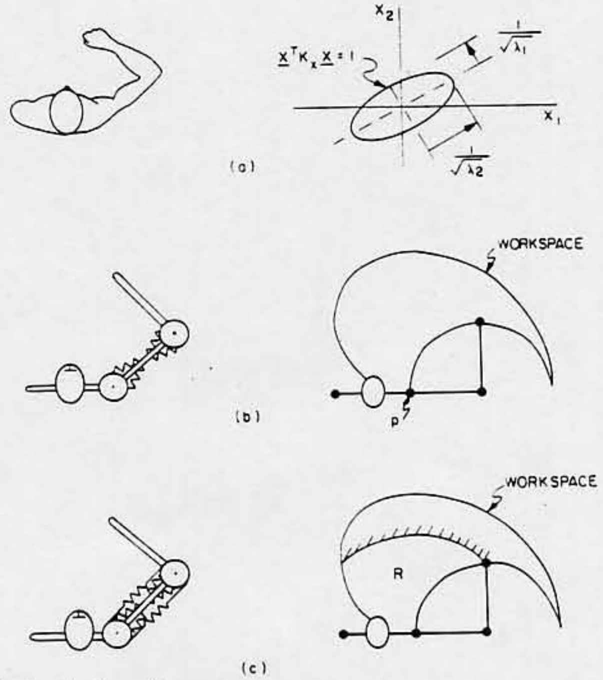

Fig. 3 A schematic representation of the influence of the polyarticular muscles of the primate upper extremity on the range of end-point stiffnesses which may be achieved without recourse to feedback strategies by simultaneous activation of opposing muscles. The ellipsoid associated with the symmetric differential stiffness tensor is shown in (a). The eigenvalues of the stiffness tensor are proportional to the stiffness in the direction of the corresponding eigenvectors and the square root of their ratio determines the ratio of the major and minor axes of the ellipsoid, which are colinear with the eigenvectors. Assuming the upper extremity may be modelled as a two-member linkage with equal link lengths, without biarticular muscles, a necessary condition to achieve an end-point stiffness with equal eigenvalues (hence a circular ellipsoid) is only satisfied at the point *p* on the workspace boundary as shown in (b). With biarticular muscles acting at equal moment arms about each joint an end-point stiffness with equal eigenvalues and a circular ellipsoid may be achieved throughout the region *R* shown in (c).

There are also considerably more skeletal muscles than joints, even beyond the antagonist pairing required to permit unidirectional muscle force to produce bidirectional joint torques. For example, the torque flexing the elbow joint (one of the simpler joints in the primate upper extremity) is generated by brachialis, brachioradialis, biceps capitus brevis, and biceps capitus longus. Does this complexity serve any purpose? If the control of end-point impedance of the limb without feedback is considered it will seen that these apparent actuator redundancies may have a functional role to play [13].

Consider the simplified two-link model of the primate upper limb (forearm and hand treated as a single rigid body, pin-jointed to the upper arm) moving in a horizontal plane as shown in Fig. 3. In the absence of feedback, the static component of the total end-point impedance will solely be due to the spring-like properties of the individual muscles. For each muscle, a potential function may be defined, and the combined effect of multiple muscles is to define a total potential function (which could be determined by adding the potential functions of the individual muscles). The total potential at any point is invariant under coordinate transformations and the total potential function may be expressed in any coordinate system by direct substitution.

Now, for simplicity, assume that the relations between muscle force and length and muscle length and joint rotation result in a linear torque/angle relation for each muscle. First consider the monoarticular (single-joint) muscles which generate torques about only a single joint: their combined

[3] Curiously, the force/length behaviour of most muscles is such that the co-energy integral is not defined and thus no compliance form is definable [29]: Muscles are impedances, not admittances.

effect is to define a diagonal stiffness tensor in relative angular coordinates:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks & 0 \\ 0 & Ke \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{22}$$

Each of the terms Ks and Ke may vary. For example, the stiffness about the human elbow can vary from about 1 Nm/rad. to more than 200 Nm/rad [20, 36].

When this stiffness tensor is expressed in end-point coordinates, because of the distortion due to the nonorthogonality of the kinematic transformations the end-point stiffness will no longer be diagonal, but the range of end-point stiffnesses which could be achieved without feedback using monoarticular muscles to change Ks and Ke is quite restricted. This is readily seen in the shape of the potential function corresponding to this stiffness. For small displacements the potential function is a quadratic form and its isopotential contours are ellipsoids which graphically represent the directional character of the stiffness tensor (see Fig. 3(a)).

To illustrate the nature of the problem, suppose it were desired to have the end-point equally stiff in all directions. This would correspond to a potential function with circular isopotentials. However, given only single joint muscles, throughout the useful workspace a potential function with circular isopotentials can not be achieved. For example, assuming links of equal length and joint ranges of 0 to 90 degrees for the shoulder and 0 to 180 degrees for the elbow, a necessary condition to achieve circular isopotentials is only satisfied at one point (point p in Fig. 3(*b*)) on the boundary of the workspace (see Appendix III). This is because to specify a symmetric second-rank tensor such as stiffness in two dimensions requires three parameters and the monoarticular muscles provide only two.

However, the biomechanical system abounds with polyarticular muscles – muscles which generate torques about more than one joint. The biceps and triceps muscles of the upper arm cross both the elbow joint and the shoulder joint and provide a mechanical coupling between shoulder and elbow rotations which radically increases the range of stiffnesses which may be achieved without feedback.

For simplicity assume the same linear relation between muscle-generated torque and angle for both joints. Now, including the two-joint muscles, the stiffness tensor in relative joint angle coordinates will have off-diagonal terms:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{23}$$

The term Kt represents the contribution of the two-joint muscles and, like Ke and Ks, it may vary. Now suppose again that it is desired to have the end-point equally stiff in all directions. As a result of the two-joint muscles, as shown in Appendix III, a potential field with circular isopotentials could be achieved without feedback (by varying Ke, Ks, and Kt) throughout a much larger region in the workspace (region R in Fig. 3(*c*)). In effect, the two-joint muscles provide a third parameter with which to modulate the stiffness tensor. Note that this is not peculiar to the specific set of simplifying assumptions made above: In general, the availability of polyarticular muscles dramatically increases the range of end-point impedances which could be achieved without feedback.

The point of this discussion is to demonstrate that impedance control is possible without depending on feedback strategies, by using to advantage the intrinsic behavior of the manipulator "hardware." Apparent redundancies in the musculoskeletal system, which are frequently seen as presenting a coordination problem which the biological controller has to solve, may in fact represent a solution to a problem: they may play a functional role in controlling the interaction between the limb and the environment during dynamic events sufficiently rapid to limit the effectiveness of feedback control.

Summary

In this part of the paper, techniques for implementing a desired impedance on a manipulator were considered. Feedback control algorithms for imposing Cartesian impedances up to second order on a general nonlinear manipulator were presented. Because care was taken to ask for a manipulator behaviour which is compatible with the fundamental mechanics of manipulation, (as outlined in Part I) the need to solve the "inverse kinematics problem" – generally regarded as fundamental to all robot control – was circumvented.

Techniques for modulating the end-point impedance of a manipulator without recourse to feedback were also discussed. Multiple actuators and "excess" linkage degrees of freedom may also be used to modulate end-point impedance and it is suggested that the apparent redundancies in the primate musculoskeletal system may in fact play an essential functional role in controlling interactive behavior. The hypothesis that impedance modulation is one of the prominent strategies of natural movement control provides the motivation for a research project to develop a cybernetically controlled prosthesis which will give an amputee the ability to change its impedance at will [2].

The modulation of end-point impedance without feedback may also be important for industrial robots. Feedback loop transmission delays are not just a biological problem; It is widely recognized that computation time is one of the limiting factors in the design of robot controllers. It could be argued that as computation becomes cheaper and faster, this problem will disappear, but one reasonable way of describing manipulation is as a series of "collisions" with objects in the environment [10]. During a collision dynamic events take place extremely rapidly and *any* feedback controller may encounter difficulties. Control of dynamic interaction without feedback is an interesting alternative and is currently under investigation [19].

A feature of impedance control is that different controller actions (aimed at satisfying different task requirements) may be superimposed. For example, suppose that a desired end-point position- and velocity-dependent behaviour is implemented on a manipulator using a feedback control strategy as outlined above in equations (4) and (7). At the same time kinematic redundancies in the manipulator are used to modulate the end-point mobility. At any given end-point position, $\mathbf{X}$, (which is determinable from the configuration, $\boldsymbol{\theta}$) the manipulator configuation may be chosen to best approximate a desired inertial behaviour (for example, the mobility normal to a kinematic constraint surface may be maximised). This configuration may then be used in the feedback law which implements the position- and velocity-dependent behaviour. As the equations never require inversion of the Jacobian, they can be applied to a manipulator with kinematic redundancies. Note that this approach to end-point control in the presence of kinematic redundancies is significantly different from the use of a generalised pseudoinverse [35].

Part III of this paper will discuss the application of impedance control.

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APPENDIX I

A Nonlinear Feedback Law for Impedance Control

Assume that the desired end-point behavior to be imposed on the manipulator is given by:

$$M d\mathbf{V}/dt - B[\mathbf{V}_0 - \mathbf{V}] - K[\mathbf{X}_0 - \mathbf{X}] = \mathbf{Fint}$$

Assume that an adequate model of the manipulator dynamics is:

$$I(\theta) d\omega/dt + C(\theta,\omega) + V(\omega) + S(\theta) = \mathbf{Tact} + \mathbf{Tint}$$

In this equation, $I(\theta)$ is the configuration-dependent inertia tensor for the manipulator, $C(\theta,\omega)$ are the inertial coupling terms (due to centrifugal and coriolis accelerations), $V(\omega)$ includes any velocity-dependent forces (e.g., frictional) and $S(\theta)$ includes any static configuration-dependent forces (e.g., gravitational). Any actuator dynamics have been neglected. The actuator forces (or torques) **Tact** are assumed to be the control input to the manipulator.

The equation for the desired behavior may be regarded as a specification of the desired end-point acceleration which is to result from an external force impressed on the manipulator admittance.

$$d\mathbf{V}/dt = M^{-1}K[\mathbf{X}_0 - \mathbf{X}] + M^{-1}B[\mathbf{V}_0 - \mathbf{V}] + M^{-1}\mathbf{Fint}$$

The corresponding acceleration in actuator coordinates is obtained by differentiating the kinematic transformations.

$$d\mathbf{V}/dt = \mathbf{J}(\theta) d\omega/dt + G(\theta,\omega)$$

where

$$G(\theta,\omega) = [d\{\mathbf{J}(\theta)\omega\}/d\theta]\omega$$

$$d\omega/dt = \mathbf{J}^{-1}(\theta)[d\mathbf{V}/dt - G(\theta,\omega)]$$

Each of the impedance terms in the desired end-point behavior may be expressed in actuator coordinates using the kinematic transformations

$$K[\mathbf{X}_0 - \mathbf{X}] = K[\mathbf{X}_0 - L(\theta)]$$

$$B[\mathbf{V}_0 - \mathbf{V}] = B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega]$$

For the purposes of controller design, each of these terms may be regarded as a component of a desired feedback law relating the control input **Tact** to the variables θ, ω and **Fint**, which are assumed to be accessible measurements. The complete control law is obtained by substitution.

$$\begin{aligned}
\mathrm{Tact} = {} & I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}K[\mathbf{X}_0 - L(\theta)] + S(\theta) \text{ (position terms)} \\
& + I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega] + V(\omega) \text{ (velocity terms)} \\
& + I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}\mathbf{Fint} - \mathbf{J}^t(\theta)\mathbf{Fint} \text{ (force terms)} \\
& - I(\theta)\mathbf{J}^{-1}(\theta)G(\theta,\omega) + C(\theta,\omega) \text{ (inertial coupling terms)}
\end{aligned}$$

Note that although this equation does require the inverse Jacobian, it does not require inversion of the kinematic equations. Only the forward kinematic equations need be computed. This will be important for those manipulators for which no explicit algebraic (closed form) solution to the inverse kinematic equations exists.

APPENDIX II

Generalized Inertial Systems and the Mobility Tensor

Any mechanical linkage is a generalized inertial system. The defining property of an inertial system is its ability to store kinetic energy, defined as the integral of (generalized) velocity with respect to (generalized) momentum [8]. At any configuration defined by the generalized coordinates the kinetic energy is a quadratic form in (generalized) momentum.

$$Ek = \tfrac{1}{2}\mathbf{h}^t Y(\theta)\mathbf{h}$$

From Hamilton's equations [30], the (generalized) velocity is the momentum gradient of the kinetic energy.

$$H(\mathbf{h},\theta) = Ek(\mathbf{h},\theta)$$

$$d\theta/dt = \omega = \nabla_{\mathbf{h}} H = Y(\theta)\mathbf{h}$$

Kinetic energy is commonly confused with kinetic coenergy. The two are not identical and are related by a Legendre transform [8].

$$Ek^* = \omega^t\mathbf{h} - Ek = \omega^t Y^{-1}\omega - \tfrac{1}{2}\omega^t Y^{-1} Y Y^{-1}\omega$$

$$Ek^* = \tfrac{1}{2}\omega^t Y^{-1}(\theta)\omega = \tfrac{1}{2}\omega^t I(\theta)\omega$$

At any configuration kinetic coenergy is a quadratic form in (generalized) velocity and its velocity gradient is the (generalized) momentum [8].

$$\mathbf{h} = I(\theta)\omega$$

For a generalized inertial system, Y is a symmetric, twice-contravariant tensor. To distinguish it from its inverse, the inertia tensor I, (symmetric, twice-covariant) Y will be termed the mobility tensor.

A knowledge of the geometric relation between coordinate frames is sufficient to transform any tensor from one frame to another. As the joint angles are a set of generalized coordinates, for any configuration of the linkage of Fig. 2 the end-point coordinates are related to the joint angles via the kinematic transformations.

$$\mathbf{X} = L(\theta)$$

Differentiating these transformations yields the relation between velocities (at any given configuration).

$$d\mathbf{X}/dt = \mathbf{V} = \mathbf{J}(\theta)\omega$$

$\mathbf{J}(\theta)$ in these equations is the configuration-dependent Jacobian. As the coordinate transformation does not store, dissipate or generate energy, incremental changes in energy are the same in all coordinate frames. This yields the relation between forces in each coordinate frame.

$$dEp = \mathbf{T}^t d\theta = \mathbf{F}^t d\mathbf{X} = \mathbf{F}^t\mathbf{J}(\theta)d\theta$$

At any given configuration

$$\mathbf{T} = \mathbf{J}^t(\theta)F$$

The same approach yields the relation between the momenta in each coordinate frame.

$$dEk = d\mathbf{h}^t\omega = d\mathbf{p}^t\mathbf{V} = d\mathbf{p}J(\theta)\omega$$

At any given configuration

$$\mathbf{h} = \mathbf{J}^t(\theta)\mathbf{p}$$

These relations may be used to express the mobility in end-point coordinates.

$$\mathbf{V} = \mathbf{J}\omega = \mathbf{J}Y\mathbf{J}^t\mathbf{p}$$

Denoting the end-point mobility by $W(\theta)$

$$W(\theta) = \mathbf{J}Y\mathbf{J}^t$$

$$\mathbf{V} = W(\theta)\mathbf{p}$$

The physical meaning of the mobility tensor is that if the system is at rest an applied force will produce an acceleration equal to the force vector premultiplied by the mobility tensor. At rest, $d\theta/dt = 0$ and hence:

$$d\mathbf{V}/dt = \mathbf{J}\omega/dt$$

$$d\omega/dt = Yd\mathbf{h}/dt$$

From the generalized Hamiltonian [30]:

$$d\mathbf{h}/dt = \mathbf{T} - \nabla_\theta H$$

At rest, $\mathbf{h} = 0$ hence $H(\mathbf{h},\theta) = Ek = 0$ and $\nabla_\theta H = 0$. Thus:

$$d\mathbf{h}/dt = \mathbf{T}$$

$$d\mathbf{V}/dt = \mathbf{J}Y\mathbf{J}^t\mathbf{F} = W\,\mathbf{F}$$

As the mobility tensor is symmetric it may be diagonalized by rotating the coordinate axes to coincide with its eigenvectors. A force applied in the direction of an eigenvector (when the system is at rest) results in an acceleration in the same direction equal to the applied force multiplied by the corresponding eigenvalue. The eigenvalues represent the inverse of the apparent mass or inertia seen by the applied force or torque.

Because the kinetic energy is a quadratic form in momentum, it may be represented graphically by an ellipsoid (see Fig. 2), the ellipsoid of gyration [33]. This may be thought of as the set of all momenta which produce the same kinetic energy (an isokinetic contour in momentum space). The lengths of the principal axes of the ellipsoid of gyration are inversely proportional to the square roots of the eigenvalues, proportional to the square roots of the associated apparent mass or inertia. The long direction of the ellipsoid of Fig. 2 is the direction of the greatest apparent inertia.

In the general case when the system is not at rest the relation between applied force and resulting motion is (in general) nonlinear and must be written in terms of a complete set of state equations for the inertial system. A convenient set of state variables are the Hamiltonian states, generalized position (e.g., θ) and generalized momentum ($\mathbf{h}$). The state and output equations are in the form of generalized admittance (see Part I) as follows.

State equations:

$$d\mathbf{h}/dt = -\nabla_\theta[\tfrac{1}{2}\mathbf{h}^t Y(\theta)\mathbf{h}] + \mathbf{J}^t(\theta)\mathbf{F}$$

$$d\theta/dt = \nabla_{\mathbf{h}}[\tfrac{1}{2}h^t Y(\theta)\mathbf{h}] = Y(\theta)\mathbf{h}$$

Output equations (position and velocity):

$$X = L(\theta)$$

$$V = \mathbf{J}(\theta)\,Y(\theta)\mathbf{h}$$

APPENDIX III

Effect of Actuator Redundancy on Range of Feasible Stiffness

The differential stiffness tensor in relative joint angle coordinates $\{\rho_1,\rho_2\}$ due to the combined stiffnesses of monoarticular actuators, Ks, Ke and biarticular actuators Kt, is:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

The transformation from relative joint angle coordinates $\{\rho_1, \rho_2\}$ to absolute joint angle coordinates $\{\theta_1, \theta_2\}$ is:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Hence the stiffness tensor in absolute joint angle coordinates is:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} Ks+Ke & -Ke \\ -Ke & Kt+Ke \end{bmatrix}$$

The differential transformation from absolute joint angle coordinates $\{\theta_1, \theta_2\}$ to Cartesian end-point coordinates $\{X_1, X_2\}$ is:

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 & -L_2 \sin\theta_2 \\ L_1 \cos\theta_1 & L_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

$$d\mathbf{X} = \mathbf{J}\, d\boldsymbol{\theta}$$

To achieve an isotropic end-point stiffness (for which the corresponding potential function will have circular isopotentials) its eigenvalues must be equal. For simplicity assume each eigenvalue is unity.

$$Kx = 1$$

The corresponding stiffness tensor in absolute angle coordinates is:

$$Ko = \mathbf{J}^t Kx \mathbf{J} = \mathbf{J}^t \mathbf{J}$$

$$Ko = \begin{bmatrix} L_1{}^2 & L_1 L_2 \cos(\theta_2 - \theta_1) \\ L_1 L_2 \cos(\theta_2 - \theta_1) & L_2{}^2 \end{bmatrix}$$

To achieve an isotropic end-point stiffness it is necessary for the actual joint coordinate stiffness to equal the desired joint coordinate stiffness. Assuming $L_1 = L_2 = 1$ it can be seen that in the absence of biarticular actuators, i.e., $Kt = 0$, this condition is not satisfied except at:

$$\theta_2 - \theta_1 = 180°$$

point p in figure 3b. In contrast, given bi-articular actuators, i.e., $Kt \neq 0$, isotropic stiffness can be achieved throughout the region R in Fig. 3(*c*) defined by:

$$90° < \theta_2 - \theta_1 < 180°$$

Impedance Control: An Approach to Manipulation:
Part III—Applications

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This three-part paper presents a unified approach to the control of a manipulator applicable to free motions, kinematically constrained motions, and dynamic interaction between the manipulator and its environment. In Part I the approach was developed from a consideration of the fundamental mechanics of manipulation. Part II presented techniques for implementing a desired manipulator impedance. In Part III a technique for choosing the impedance appropriate to a given application using optimization theory is presented. Based on a simplified analysis it is shown that if the task objective is to tradeoff interface forces and motion errors, the manipulator impedance should be proportional to the environmental admittance. An application of impedance control to unconstrained motion is presented. The superposition properties of nonlinear impedances are used to develop a real-time feedback control algorithm which permits a manipulator to avoid unpredictably moving objects without explicit path planning.

Introduction

The work presented in this three-part paper is an attempt to define an approach to manipulation which is sufficiently general to be applied both to the control of free motions and to the control of dynamic interaction between a manipulator and its environment. In Part I it was shown from a consideration of the mechanics of interaction that a general strategy is to control the motion of the manipulator and in addition control its dynamic behavior; controlling a vector quantity such as force or position alone is inadequate. To be compatible with the mechanics of an environment which in general will contain constrained inertial objects, the manipulator should exhibit the behavior of an impedance. It was also shown in Part I that for a broad class of nonlinear manipulators (basically those capable of positioning an unconstrained inertial object) the relation between the commanded motions and the commanded dynamic behavior could be represented by a generalized Norton equivalent network.

In Part II the implementation of a desired manipulator impedance either using a feedback strategy or using the intrinsic mechanics of the manipulator was discussed. We now turn to a consideration of a method for choosing an appropriate manipulator impedance. In this, the Norton equivalent network representation will prove to be of some value. We will also show how the superposition property of impedances leads to a simplification of a problem in manipulator control.

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Choosing an Appropriate Impedance

The manipulator impedance appropriate for a given situation depends on the task to be performed. In most manipulatory tasks there is a tradeoff to be made between allowable interface forces and allowable deviations from desired motions. Whether it has been rationally chosen or not, the manipulator impedance specifies a relation between interface forces and imposed motions. If the tradeoff implicit in the task is expressed as a performance index to be maximized or minimized which is a function of the interface forces and motions then the impedance appropriate for that task may be determined using optimization theory [10].

Because a general class of nonlinear manipulators can be represented by a generalized Norton equivalent network as shown in Fig. 1, considerable insight into manipulation can be gained by considering analogous (but simpler) systems with the same Norton network structure. Assume a manipulator interacts with a passive environment (no active energy source terms). For simplicity, consider a single degree-of-freedom and assume that both the manipulator impedance and the environmental admittance are simple linear dissipative elements. This simplified linear system has the same basic structure as a more general multiple degree-of-freedom nonlinear manipulator interacting with an environmental admittance. The following equations relate the port variables:

$$V = YF \tag{1}$$

$$F = Z(V_0 - V) \tag{2}$$

$$V = YZV_0/(1 + YZ) \tag{3}$$

$$F = ZV_0/(1 + YZ) \tag{4}$$

Now assume that one task is to minimize the transmission of power into the environmental admittance. Express this as an objective function to be optimized:

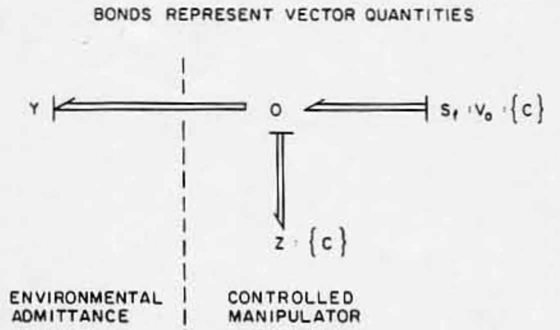

Fig. 1 A bond-graph equivalent network representation of an impedance-controlled manipulator interacting with an environmental admittance. Each bond represents a vector of power flows along multiple degrees of freedom. The bond graph for the manipulator is a generalized Norton equivalent network.

Objective: maximize $P = FV$ where P = power transmitted

$$P = YZ^2 V_0^2/(1+YZ)^2 \tag{5}$$

Maximizing the power transmitted requires the commanded motion V_0 to be maximized, or the commanded impedance Z to be maximized. Maximizing with respect to the admittance Y yields an equality condition:

$$ZY = 1 \tag{6}$$

or

$$Z\text{manipulator} = Z\text{environment} \tag{7}$$

The first two conditions state essentially that the machine should operate on the boundaries of its performance envelope. The third condition states that (after the first two conditions have been satisfied) the machine and environment impedances should be matched. This is a familiar result and is a design rule of great versatility, applicable in any situation in which a source is to impart maximum power to a load. Its applicability to robotic transport tasks has recently been shown [19].

For manipulation, another common task is to minimize deviations from desired motions while simultaneously minimizing interface forces. Assume this objective may be expressed as follows:

$$\text{Objective: minimize } Q = p(V_0 - V)^2 + F^2 \tag{8}$$

p is a weighting coefficient specifying an allowable tradeoff between interface forces and motion errors. Rewriting the objective using equations (3) and (4):

$$Q = (p + Z^2) V_0^2/(1+YZ)^2 \tag{9}$$

Minimizing this objective requires the commanded motion V_0 to be minimized or the environmental admittance Y to be maximized, two physically reasonable conditions. Minimizing with respect to the commanded impedance yields the following equality condition:

$$Z - pY = 0 \tag{10}$$

or

$$Z\text{manipulator} = pY\text{environment} \tag{11}$$

This condition may be considered as a designer's "rule of thumb" for manipulation, analogous to the impedance matching rule applicable to power transmission: "Make the manipulator impedance proportional to the environmental admittance." If the environment is unyielding (low admittance), the manipulator should accommodate the environment (low impedance); if the environment offers little resistance (high admittance), the manipulator may impose motion upon it (high impedance).

Although these results were obtained using an extreme simplification of the mechanics of manipulation, this simple static analysis captures the essence of the interaction between manipulator and environment, and yields an intuitively satisfying result: that manipulation (at least insofar as it is modeled by the cost function of equation (8)) and power transmission are fundamentally conflicting task requirements. In view of the fact that a manipulator must be versatile – it may be called upon to transmit power in one phase of a working cycle (e.g., transport a workpiece as fast as possible) and manipulate at another (e.g., assemble the workpiece to another) – a controllable mechanical impedance is imperative.

The simple analysis presented above demonstrates that the tradeoff implicit in the specification of most manipulatory tasks may be mapped directly onto a statement about the manipulator impedance. That analysis was purely static: algebraic equations related the port variables, not differential equations. In the following a method is presented for determining an appropriate impedance in a simple dynamic case.

Assume that the end-point inertial behavior of the manipulator has been modified to be that of a rigid body using (for example) the technique outlined in Part II. The nodic (noninertial) interface forces can be represented by a generalized Norton equivalent network as shown in Fig. 1 and are assumed to depend only on the displacement (and its rate of change) from a commanded time-varying (virtual) position, with the displacement- and velocity-dependent terms assumed to be separable. The dynamic equations for the interaction port behavior are:

$$\text{Fint} = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] - Md\mathbf{V}/dt \tag{12}$$

The environment will be assumed to be a rigid workpiece

Nomenclature

- Y = admittance
- Z = impedance
- Sf = flow source
- Se = effort source
- $:\{c\}$ = modulation by command set
- $\mathbf{V}_0$ = commanded (virtual) velocity
- $\mathbf{V}$ = velocity
- $\mathbf{X}_0$ = commanded (virtual) position
- $\mathbf{X}$ = position
- $\mathbf{F}$ = force
- P = power transmitted
- Q = objective function
- p = weighting coefficient
- **Fint** = interface force
- **Fext** = external force
- $K[\cdot]$ = force/displacement relation
- $B[\cdot]$ = force/velocity relation
- M = inertia tensor in end-point coordinates
- Me = environmental inertia tensor
- Ftol = force tolerance
- Xtol = position tolerance
- t = time
- k = stiffness
- b = viscosity
- m = mass
- m_m = manipulator mass
- m_e = environmental mass
- S = strength of Gaussian random process
- δ = Dirac delta function
- H = Pontryagin function
- $\lambda_1\ \lambda_2\ \lambda_3$ = LaGrange multipliers
- $E[\cdot]$ = expectation operator

Overbar also denotes expectation

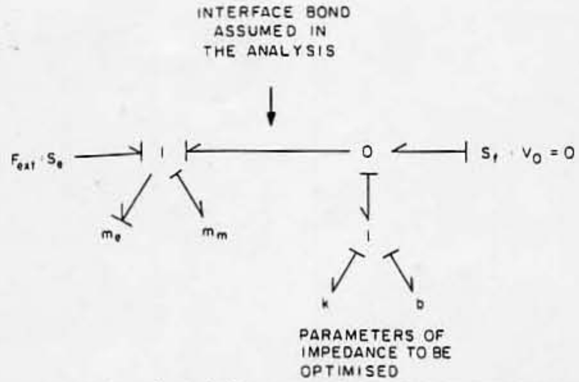

Fig. 2 A bond graph equivalent network showing the interface bond assumed in the derivation of the optimal dynamic impedance

acted on by unpredictable (or merely unpredicted) forces. Its dynamic equations are:

$$Me\, d\mathbf{V}/dt = \mathbf{F}\text{ext} + \mathbf{F}\text{int} \tag{13}$$

Both the isolated manipulator (**Fint** = 0) and the coupled system have the behavior of a mass driven by motion-dependent forces. The dynamic equations of the coupled system are:

$$(Me + M)d\mathbf{V}/dt = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] + \mathbf{F}\text{ext} \tag{14}$$

A further simplification is to assume that the position-dependent terms are curl-free[1]. A potential function is then definable which is analogous to stored elastic energy. A similar set of assumptions permit the velocity-dependent terms to be described as a dissipative potential field. Finally, the elastic and viscous terms will be assumed linear.

The combined inertia tensor, $Me + M$, for the manipulator and the workpiece will not in general be diagonal. However, it is symmetric and thus can be diagonalized by rotating the coordinate axes in which the task is described. The stiffness and viscosity tensors are to be chosen to suit the task. It will be assumed that the eigenvectors of the symmetric stiffness and viscosity tensors are colinear with those of the inertia tensor. Given this assumption, the general six degree-of-freedom problem decomposes into six single degree-of-freedom problems. Consequently, each degree of freedom may be dealt with separately as follows.

The task considered will be that of maintaining a fixed position in the face of perturbations from the environment. (These might be due to excitation forces from a power tool or due to the process of using the tool.) To reflect the paucity of a-priori information about the perturbations from the environment they will be modeled as a zero-mean, Gaussian, purely-random process of strength S. The tradeoff implicit in this task will be modeled as before (equation (8)) as the minimization of interface forces and position errors. For simplicity, the interface is assumed to be between the total inertia (controlled manipulator plus environment) and the elastic and viscous elements as shown in the equivalent network of Fig. 2. The inertial behavior of the manipulator has essentially been lumped with the admittance of the environment.

The objective function to be minimized is:

$$Q = \int_0^\infty \{(F/F\text{tol})^2 + [(X_0 - X)/X\text{tol}]^2\}\, dt \tag{15}$$

Writing the equations for a single degree of freedom in phase variable form:

$$d/dt \begin{bmatrix} X \\ V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} X \\ V \end{bmatrix} + \begin{bmatrix} 0 \\ k/m \end{bmatrix} X_0 + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F\text{ext} \tag{16}$$

$$F = [k \quad m] \begin{bmatrix} X \\ V \end{bmatrix} \tag{17}$$

In these equations m refers to the combined apparent mass of manipulator and workpiece along this degree of freedom. Because of the random forcing term the objective function (equation (15)) is a random variable and the optimum impedance is obtained by minimizing its expectation with respect to the parameters k and b of the manipulator impedance, subject to the dynamic constraints imposed by the system (equations (16) and (17)). The final simplifying assumption is to consider only steady state conditions (the method is readily generalized to the transient case using standard numerical techniques). The analysis is presented in Appendix I. Summarizing the results:

$$k_{\text{opt}} = F\text{tol}/X\text{tol} \tag{18}$$

$$b_{\text{opt}} = \sqrt{2(k_{\text{opt}} m)} \tag{19}$$

In this simple case the optimum stiffness is equal to the ratio of force tolerance, Ftol, to position tolerance, Xtol. With no penalty on velocity errors, the optimum damping is such as to yield a damping ratio of 0.707. A nonzero penalty on velocity errors would yield a more heavily damped system.

Viewed simply as an optimization problem, these results are the well-known solution to the second-order feedback regulator problem [13]. Their importance in this context is twofold: First, they demonstrate that a tradeoff modeled by an objective function such as equation (15) can be used to derive a specification of the appropriate manipulator impedance. Because of the assumptions permitting decoupling of the end-point behavior along each degree of freedom, these results can be applied to each degree of freedom in turn. Furthermore, the analytical technique can be applied to nonlinear systems [6, 9].

Second, and more important, the results are expressed in terms of the mechanical behavior of the end-point regardless of how that behavior is achieved. Although a large number of (gratuitous) assumptions were made in the derivation, none of them are impractical and the result expresses the required impedance command to the manipulator in terms of readily available mechanical quantities associated with the task. The optimal impedance may be implemented by any means, feedback or otherwise, permitted by a given manipulator design. As outlined in Part II, the primate neuromuscular system has the capacity to change its mechanical impedance by simultaneous activation of opposing muscles [6, 9, 14] and the above analytical technique has been used to derive a prediction of antagonist coactivation which has been shown to be consistent with experimental observation [6, 9].

In this simple analysis the external forces were almost completely unmodelled. The assumption of a purely random process is tantamount to an assumption of complete unpredictability. The analysis demonstrates that even with extremely little information about the environment, the interaction between manipulator and environment may be controlled so as to meet task specifications. Naturally, the

[1] For each component of the vector force field defined by $K[\cdot]$ and each component of $\mathbf{X}$, the crossed partial derivatives are identical.

more information about the environment that is available, the better one would expect the system performance to be. However, this suggests the tantalizing possibility that the impedance may be chosen to tradeoff performance against need for information about the environment. This is a topic for further research.

Obstacle Avoidance Using Superposition of Impedances

One useful and important consequence of the assumptions underlying impedance control is that if the dynamic behavior of the manipulator is dissected into a set of component impedances, these may be reassembled by simple addition even when the behavior of any or all of the components is nonlinear. This is a direct consequence of the assumption that the environment is an admittance, containing at least an inertia. That inertia acts to sum both forces applied to it and impedances coupled to it.

The additive property of impedances permits complicated tasks to be dealt with one piece at a time and all of the pieces combined by simple addition. We have taken advantage of this to implement a real-time feedback control law which drives the manipulator end-point to a target location while simultaneously preventing unwanted collision with unpredictably moving objects in the manipulator's workspace [1-3, 7, 8].

Obstacle avoidance is generally regarded as a problem in position control, specifically that of planning a collision-free path [15]. The approach we have taken is not to plan a path, but to specify an impedance which produces the desired behavior without explicit path planning. In the following example, recall that although the need for the manipulator to have the behavior of an impedance arose from considerations of the mechanical interaction between a manipulator and its environment, cases in which the mechanical work exchanged is negligible (e.g., free motions) may be treated as special (or degenerate) instances.

The primary difference between impedance control and the more conventional approaches is that the controller attempts to implement a dynamic relation between manipulator variables such as end-point position and force rather than just control these variables alone. That entire relation becomes the command to the manipulator which may be updated as often as practical considerations (such as speed of computation) dictate. In this sense, impedance control is an augmentation of conventional position control. Each command to the manipulator specifies a position (as in conventional control) and in addition specifies a relation determining the accelerating force to be applied to the total mechanical admittance in response to deviations of the actual position from the commanded position.

If the position- and velocity-dependent terms in the commanded impedance are each assumed to satisfy the requirements for the existence of a potential function then the manipulator behavior is simplified. It may be thought of as analogous to that of a sticky marble rolling on a continuously deformable surface. Varying the impedance varies the shape of the surface and the stickiness of the marble. Target acquisition and obstacle avoidance may now be dealt with separately as follows.

Successive target locations may be specified by means of a (time-varying) depression in the surface. Each single command has a position-dependent component which specifies a potential function which is a "valley" with its bottom at the target. This "valley" is depicted by a map of isopotential contours in Fig. 3(*a*).

Conversely, given an observation of the relative location (with respect to the end-point) of an obstacle (or any other region in the workspace to be avoided) that object may be avoided by specifying a (time-varying) bump in the deformable surface. Now each single command also contains a position-dependent component which specifies a potential field with an unstable equilibrium point at the location of the object to be avoided. The potential function is a "hill" centered over the obstacle (see Fig. 3(*b*)).

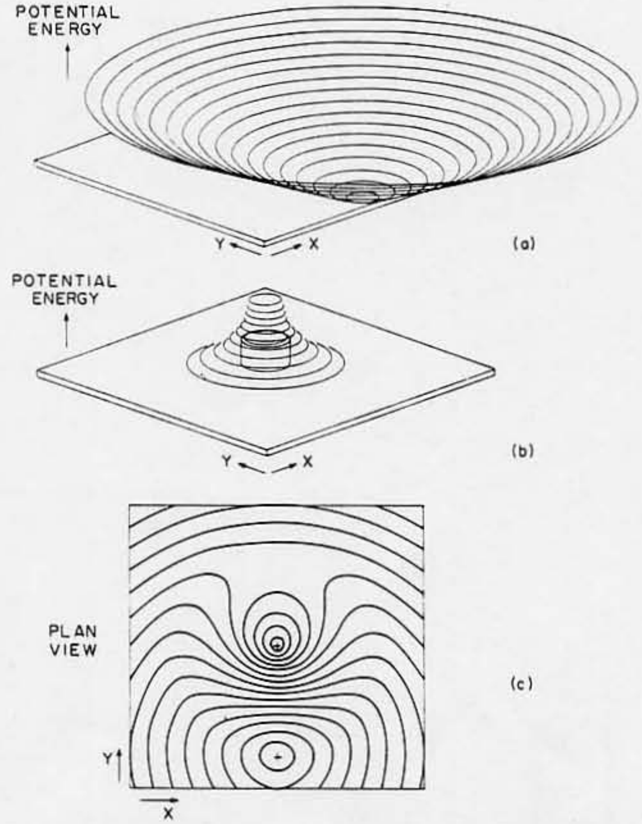

Fig. 3 A diagram of the potential functions corresponding to the static component of the commanded impedances which may be used for (*a*) target acquisition (*b*) obstacle avoidance and (*c*) simultaneous target acquisition and obstacle avoidance. A plan view of the isopotential contours are shown in (*a*) and (*b*).

The target-acquisition command and the obstacle-avoidance command could be combined in a number of ways, but remember that the admittance sums the impedances. The inevitable inertial behavior of the end-point guarantees the superposition of the components of the impedance-controller action independent of the linearity of the components. It is always possible to command obstacle-avoidance and target-acquisition (or any other aspect of the complete task) independently and then combine all commands by simply adding the impedances, in this case the corresponding potential fields (see Fig. 3(*c*)) [7, 8]. Furthermore, a number of obstacles and a target may be specified simultaneously. Each task component may be represented as a generalized Norton equivalent network and the combination of all the task components represented by the equivalent network of Fig. 4.

It is important to note that the combined potential field of Fig. 3(*c*) represents a *single command* to the manipulator. Of course, neither targets nor obstacles need stay fixed in the workspace and a typical task will require multiple impedance commands (just as locating the spot welds on an automobile requires multiple position commands to a conventional robot controller) and by updating the impedance commands repeatedly this approach may be used to make a manipulator avoid "invaders," objects which may move about the workspace in an unpredictable (or merely unpredicted) manner [2, 3].

The use of potential functions as commands to a robot is

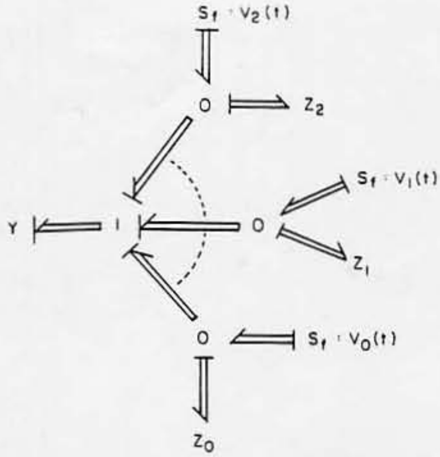

Fig. 4 A bond graph equivalent network representation of commands to an impedance-controlled manipulator specifying simultaneous target acquisition and avoidance of multiple obstacles. Each task component is represented by a generalized Norton equivalent network.

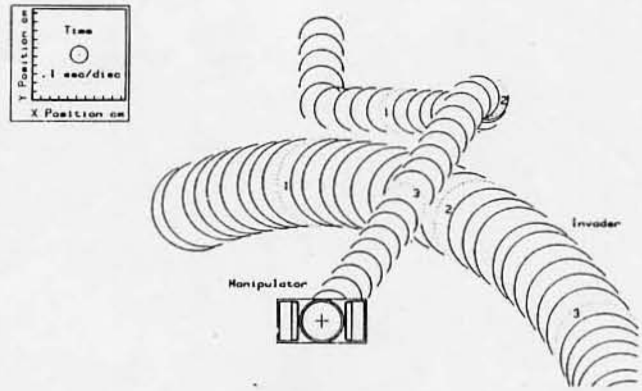

Fig. 5 Avoidance of an unpredictably moving "invader" by a spherical-coordinate manipulator controlled by 8-bit, 2MHz, Z80 microprocessors. The half circles show successive positions of the manipulator end-effector and the invader in the vertical plane at 100 millisecond intervals. All of the behavior shown here is the robot's response to a single impedance command from the supervising computer, a PDP 11/44.

similar to the approach used by Khatib and LeMaitre [12] to navigate a manipulator through a complicated environment. The distinguishing feature (and advantage) of impedance control is that the same controller used to deal with free motions can also be used to deal with real mechanical interaction. The success of impedance control as a unifying framework for dealing with both kinematically constrained manipulations and free motions (including avoiding moving "invaders") has been demonstrated by performing both of these tasks in real time using a spherical coordinate manipulator [1, 2]. The same controller was used for both tasks and the algorithm was simple enough to be implemented using 8-bit 2 MHz microprocessors (Z-80, one for each axis) for the real-time controller. One example of the obstacle-avoidance behavior achieved is shown in Fig. 5.

As an aside, note that to be of practical value, the "repulsive" force fields used to implement collision avoidance must be nonlinear; the repulsive force must drop to zero for sufficiently large separations between the end-effector and objects in the environment (see Fig. 3(*b*)). This is precisely the type of noninvertible, nonlinear force/displacement behavior for which no inverse compliance form exists. The concept of tuning the end-point stiffness and damping of a manipulator has been discussed in the literature under the general heading of "compliance," "compliant motion control," "fine motion control," or "force control" [5, 11, 17, 18, 21-24, 28]. In most of this prior work, the manipulator has been given the behavior of a linear compliance (a special case of an admittance). The control strategy presented here is considerably more general; If the end-point dynamic behavior is expressed as an impedance, the above obstacle-avoidance behavior is included as a special case; If it were expressed as a compliance this useful behavior would be excluded. In addition, the superposition property of impedances coupled to an admittance would not be preserved.

Summary and Conclusion

This paper has presented a method for controlling a manipulator which may interact dynamically with its environment. The approach is solidly based on the mechanics of interaction and was developed in Part I from some reasonable physical assumptions about manipulation: that the controlled manipulator may be represented as an equivalent physical system; that manipulation is a fundamentally nonlinear problem (therefore impedance and admittance must be distinguished); and that the environment contains kinematically constrained inertial objects and is an admittance (therefore the manipulator must have the causality of an impedance). Two theoretical consequences of these assumptions are that a broad class of nonlinear manipulators may be represented by a generalization of the familiar Norton equivalent network, and that impedances may be superimposed even when they are nonlinear.

Impedance control is an extension of conventional position control strategies. A time-varying position (the virtual position) is commanded; in addition an impedance is commanded, a relation (possibly dynamic, nonlinear, discontinuous and time-varying) between interface forces and displacements from that position. This simple strategy of commanding a relation rather than just a position (or a velocity) has a profound impact on the problems of manipulator control. In Part II it was shown that it leads to the elimination of the "inverse kinematic problem" [21] (that of determining a joint trajectory from an end-point trajectory).

Impedance control focuses on the interaction port and describes the required behavior in terms of the mechanical properties of the manipulator (e.g., its impedance) independent of the way this behavior is to be achieved. This sets the stage for considering alternatives to feedback control. These are important for high-speed manipulation; at sufficiently high frequencies the behavior of any controlled system is dominated by its open loop behavior. In Part II it was shown that multiple actuators and "excess" linkage degrees of freedom may be used to modulate end-point impedance. It is suggested that the primate central nervous system uses these non-feedback strategies and that the apparent redundancies in the primate musculoskeletal system may in fact play an essential functional role in controlling interactive behavior.

In this third part of the paper it was shown that in general, the impedance appropriate to a given task may be deduced from the task objective, and a method which uses optimization theory to do this was presented. Although the examples presented were extremely simple, they retained the structure of the basic manipulation problem, represented by the generalized Norton equivalent network coupled to an admittance. The static example led to an instructive result: while power transmission requires machine impedance to match environmental impedance, manipulation (trading off movement errors against interface forces) requires a machine impedance proportional to environmental admittance; power transmission and manipulation are, in a sense, "orthogonal" tasks. The dynamic example showed that the appropriate impedance can be expressed in terms of force and motion tolerances independent of the way the impedance is implemented e.g., without assuming feedback control. The method used is general and has been applied to a nonlinear system.

The concept of tuning the dynamic behavior of a manipulator has been explored by a number of researchers. However, most of this prior work considered only linear dynamic behavior and implemented it as an admittance (force in, motion out). The restriction to linearity is not necessary and as shown in the collision-avoidance example, nonlinear behavior has its uses. The restriction to admittance causality is not consistent with the physical constraints of interacting with a (possibly constrained) inertial environment. That approach might be justified by arguing that the environment could be modelled as an impedance, (e.g., a spring [18, 28]); Unfortunately, admittances coupled to an impedance at a common point (the end-effector of the robot) do not enjoy the superposition properties of impedances coupled to an admittance at a common point. Impedance control offers a significant advantage over this alternative.

The practical value of the additive property of nonlinear impedances was shown in this third part of the paper by using it to develop a feedback control law for avoiding unpredictably moving objects. By taking advantage of the superposition of impedances, target acquisition and obstacle avoidance could be considered separately and implemented as different components of a total commanded impedance which were combined by simple addition. This approach does not require explicit path planning and the control law was simple enough to be implemented using 8-bit MHz microprocessors. Note, however, that impedance control does not preclude a preplanning or navigational approach and the two methods may usefully complement one another; path-planning is appropriate for the predictable aspects of the environment, impedance control offers a method for dealing with its less predictable aspects.

The choice of a realistic but appropriately simple form for the impedance to be imposed leads to a dramatic simplification of the problems of controlling the complete system (manipulator and environment). Restricting attention to impedances with exact differentials (force fields with zero curl) permits the definition of potential functions for the position- and velocity-dependent behavior. Because of the simple form of the imposed dynamic equations the (elastic) potential function and the external forces are sufficient to define static stability. Asada [4] has shown how elastic fields may be used as the basis of an approach to planning stable grasp. Stable equilibrium configurations of end-effector and workpiece are defined by finding minima of the potential energy function. Gravitational forces are readily included by expressing them as a potential function and combining it with the potential function of the manipulator by simple addition. Note, however, that the dynamic stability of the end-effector is not guaranteed (that is, in principle, sustained oscillations are possible). To ensure dynamic stability the dissipative field must be chosen appropriately; the complete impedance must be controlled, not just the elastic behavior.

The use of potential functions in effect maps the end-point dynamics into a set of static functions and the visualization, prediction and planning of the behavior of the complete system is simplified. For example, in the absence of external active sources the total energy of the system, kinetic plus potential, may never increase. This permits easy prediction of the maximum velocities which may result from a given set of commands without computing the detailed trajectories. Conversely, as the potential energy function is one of the commands, it is readily chosen so that a desired maximum velocity is never exceeded. If the impedance command is given when the system is at zero velocity (e.g., a workpiece has just been grasped) then it is not even necessary to know the mass of the grasped object.

A feature of impedance control is that it permits a unified treatment of many aspects of manipulator control. The actions of both controller software and manipulator hardware may be described through an equivalent physical system. As a result powerful methods (such as bond graphs) for network analysis of nonlinear systems may profitably be applied. Real mechanical interaction may be treated in the same framework as free (unconstrained) motions. The impedance controller used to avoid unpredictably moving objects was also capable of coping with kinematically constrained motions [1, 2]. Targets to be acquired are treated in the same way as obstacles to be avoided as different components of a total task, where each component is described by a generalized Norton equivalent network. Path control [20, 25], rate control [26, 27], and acceleration control [16], could be considered in a single framework as important special cases of impedance control (e.g., position control: maximize impedance; rate control: no static impedance component). Pure force control [11] (force commanded as a function of time only) could also be considered in the same framework by regarding it as a special case in which the impedance is purely elastic. A potential function with a constant gradient defines the magnitude of the commanded force, and the virtual position (which may go outside the workspace) defines the direction of the commanded force. The hybrid combination of force and position control in orthogonal directions [17, 23] proposed for dealing with pure kinematic constraints is also included under impedance control.

Most important, the applicability of impedance control extends beyond the workless conditions imposed by free motions or pure kinematic constraints to include the control of energetic interactions such as are encountered when using a power tool. It promises to be particularly useful for understanding, controlling and coordinating the actions of mutually interacting manipulators, such as the fingers of a hand, the hand and the arm, or two arms. Using this approach each subsystem presents a simple behavior to the other subsystems; This will facilitate the prediction and control of the combined behavior of the entire system.

An alternative approach to manipulator control in the presence of significant dynamic interaction is to change the structure and/or parameters of a feedback controller as the conditions imposed by the environment change. This would require the controller to monitor the environment continuously, identify changes, and adapt its own behavior accordingly – a far-from-trivial task. Changes in the structure and parameters of the environment may take place very rapidly (consider the transition from free motion to constrained motion as an object comes in contact with a surface) and there may not be sufficient time for the usually lengthy process of system identification. On the other hand, if the controller is structured so that the manipulator always impresses a force on the environment in relation to its motion (that is, it behaves as an impedance) there are no practical situations in which its behavior is inappropriate, no practical task has been excluded, and the need to identify the structure of the environment has been reduced.

Of course, impedance control does not preclude the application of adaptive strategies, and indeed the two approaches may complement each other, controlled impedance taking care of the transitions and allowing time for identification and adaptation to optimize performance. Strictly speaking, impedance control is a subset of parameter-adaptive control; the primary distinctions are that the parameters to be modulated are expressed in terms of a physically meaningful quantity, mechanical impedance, and unlike other work on parameter adaptation, no assumption is made that the implementation of the impedance will be through feedback control strategies. An impedance may be implemented in a number of ways, using to advantage the resources of a specific manipulator.

Essentially, impedance control is an attempt to combine the control of "transport" tasks (which are the philosophical underpinning of conventional robot control) with the control of "interactive" tasks such as the use of a tools. The ultimate goal of this work is to understand the subleties of adaptive tool-use, one of the distinguishing features of primate behavior. Impedance control may provide the basis for understanding tool-using behavior in primates, restoring this capability to an amputee using an artificial limb, and implementing it on an industrial robot.

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APPENDIX I

Optimal Impedance for a One-Dimensional Dynamic System

The system equations in phase variable form are:

$$\begin{bmatrix} \dot{X} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} X \\ V \end{bmatrix} + \begin{bmatrix} 0 \\ k/m \end{bmatrix} X_0 + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F\text{ext}$$

The interface force is: $F = k(X_o - X) + bV$

The objective function to be minimized is:

$$Q = \int_0^\infty \{F/F\text{tol})^2 + [(X_o - X)/X\text{tol}]^2\} dt$$

The external force Fext is a zero-mean, Gaussian, purely random process of strength S. Thus:

$$E[F\text{ext}(t)] = 0 \quad E[F\text{ext}(t)F\text{ext}(t+\tau)] = S\delta(\tau)$$

In steady state $\bar{X} = X_o$, $\bar{V} = 0$ thus without loss of generality assume $X_o = 0$. The covariance propagation equations are:

$$\dot{\overline{X^2}} = 2\overline{XV}$$

$$\dot{\overline{XV}} = \overline{V^2} - \frac{b}{m}\overline{XV} - \frac{k}{m}\overline{X^2}$$

$$\dot{\overline{V^2}} = \frac{S}{m^2} - 2\frac{b}{m}\overline{V^2} - 2\frac{k}{m}\overline{XV}$$

Because of the random forcing, the optimum impedance is obtained by minimizing the expectation of the objective function subject to the constraints imposed by the covariance propagation equations. Writing $p^2 = F\text{tol}/X\text{tol}$

$$E[Q] = \frac{1}{F\text{tol}^2}\int_0^\infty \{b^2\overline{V^2} + 2k\, b\overline{XV} + (k^2 + p^2)\overline{X^2}\} dt$$

The Pontryagin function is:

$$H = b^2\overline{V^2} + 2kb\overline{XV} + (k^2+p^2)\overline{X^2} + 2\lambda_1\overline{XV}$$
$$+ \lambda_2\left(\overline{V^2} - \frac{b}{m}\overline{XV} - \frac{k}{m}\overline{X^2}\right)$$
$$+ \lambda_3\left(\frac{s}{m^2} - 2\frac{b}{m}\overline{V^2} - 2\frac{k}{m}\overline{XV}\right)$$

The minimizing conditions are:

$$\frac{\partial H}{\partial k} = 0 = 2b\overline{XV} + 2k\overline{X^2} - \frac{\lambda_2}{m}\overline{X^2} - \frac{2\lambda_3}{m}\overline{XV}$$

$$\frac{\partial H}{\partial b} = 0 = 2b\overline{V^2} + 2k\overline{XV} - \frac{\lambda_2}{m}\overline{XV} - \frac{2\lambda_3}{m}\overline{V^2}$$

The LaGrange multipliers are determined from the costate equations:

$$\frac{\partial H}{\partial \overline{XV}} = -\dot{\lambda}_1 = (k^2+p^2) - \lambda_2\frac{k}{m}$$

$$\frac{\partial H}{\partial \overline{XV}} = -\dot{\lambda}_2 = 2kb + 2\lambda_1 - \lambda_2\frac{b}{m} - 2\lambda_2\frac{k}{m}$$

$$\frac{\partial H}{\partial \overline{V^2}} = -\dot{\lambda}_3 = b^2 + \lambda_2 - 2\lambda_3\frac{b}{m}$$

Assuming a steady-state solution exists, it may be obtained by setting all rates of change to zero. Manipulating the resulting equations yields:

$$\overline{XV} = 0 \qquad \overline{V^2} = \frac{S}{2bm} \qquad \overline{X^2} = \frac{S}{2bk}$$

$$k^2_{\text{opt}} = p^2 \qquad k_{\text{opt}} = F\text{tol}/X\text{tol}$$

$$b^2_{\text{opt}} = 2k_{\text{opt}}m \qquad b_{\text{opt}} = \sqrt{2k_{opt}m}$$
