Collocation Method in American Option Pricing: A Numerical Study of Hybrid Algorithm

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December 20, 2018

Section 1

Literature Review

- **Binomial method:** Cox, Ross, and Rubinstein (1979)
- **Finite difference method:** Brennan and Schwartz (1977)
  - Artificial boundary conditions: Han and Wu (1985)
  - Projected SOR method: Wilmott (1993)
  - Front-fixing technique: Wu and Kwok (1997)
  - Penalty method: Zvan, Forsyth, and Vetzal(1998)
  - Far field boundary conditions: Kangro and Nicolaides (2000)
- **Radial basis function method:** Kansa (1990)
  - Error bounds: Franke and Schaback (1998)
  - Numerical Analysis: Hon and Mao (1999), Hon and Schaback (2001), Larsson and Fornberg (2003)
  - H2 Convergence of Lease-Square Kernel Collocation Method: Cheung, Ling, and Schaback (2017)

Radial Basis Function (RBF) Method

RBF method is a computational method to solve scattered interpolation problem by approximating the solution as a linear combination of smooth RBFs: $\phi_j : \mathbf{R}^+ \to \mathbf{R}$, which are induced from kernel functions $\Phi_j : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$.

$$u = \sum_{j=1}^{n_Z} \lambda_j \phi(||\cdot - z_j||_2) = \sum_{j=1}^{n_Z} \lambda_j \Phi(\cdot, z_j) \tag{1}$$

where $Z = \{z_1, ..., z_{n_Z}\}$ are the centers of radial basis functions. Later Kansa imposed strong-form collocation conditions (2) to identify the unknown coefficients λ_i to solve PDE (3)

$$\begin{cases} Lu(x_i) = \sum_j \lambda_i L\phi(||x_i - z_j||), & \text{for } x_i \in \Omega \\ Bu(x_i) = \sum_j \lambda_i B\phi(||x_i - z_j||), & \text{for } x_i \in \Gamma \end{cases} \tag{2}$$

$$\begin{cases} Lu = f, & \text{on } \Omega \\ Bu = g, & \text{on } \Gamma = \partial\Omega \end{cases} \tag{3}$$

Radial Basis Functions

Popular RBFs:

- Gaussian: $\phi(r) = e^{-(\varepsilon r)^2}\phi(r) = e^{-(\varepsilon r)^2}$
- Multiquadric: $\phi(r) = \sqrt{1+(\varepsilon r)^2}\phi(r) = \sqrt{1+(\varepsilon r)^2}$

Advantages of RBF method:

- Dimension independence
- Simplicity to program
- More accurate spatial derivatives

H2 Convergence of Meshless Method

Theorem (Constrained Least Square by *Cheung, Ling, and Schaback, 2017*)

Under RCs on domain, solution, differential operator and kernel functions. Let $u_* \in H_m(\Omega)$ denote the exact solution of the elliptic PDE (5). Let $u_{CLS} \in U_{Z \cup Y}$ be the constrained least-squares solution X,Y defined as

$$u_{X,Y}^{CLS} := arg\, inf_{u \in \mathcal{U}_{Z \cup Y}} ||L_u f||_X^2 \text{ subject to } u|_Y = g|_Y \tag{4}$$

Then the error estimate $\|u_{X,Y}^{CLS} - u^*\|_{2,\Omega}$ is bounded uniformly given Ω, Φ, L, and γ_X

$$Lu := \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)) + \sum_{j=1}^{d} \frac{\partial}{\partial x^j}(b^j(x)u(x)) + \sum_{i=1}^{d} c^i(x)\frac{\partial}{\partial x^i}u(x) + d(x)u(x) \tag{5}$$

Section 2

Hybrid Numerical Scheme

Difficulties:

- Bounded Domain v.s. Unbounded & Undeterministic Domain
- Elliptic v.s. Parabolic

Propose:
Hybrid numerical scheme to solve the boundedness problem. Specificly, collocation is used to search for exercising boundary after which backward time integration is applied.

Hybrid Numerical Scheme: American Call under Black-Scholes Model

B-S Model:

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial C}{\partial x} - rC \qquad \text{when } x \le x^*_\tau \tag{6}$$

$$C(\tau, x) = (K\,e^x - K)^+ \qquad \text{when } x > x^*_\tau \tag{7}$$

$$C(0, x) = (K\,e^x - K)^+ \qquad x^*_0 = 1 \tag{8}$$

$$\lim_{x \to -\infty} C(\tau, x) = 0 \tag{9}$$

$$\lim_{x \to +\infty} C(\tau, x) = K\,e^x \tag{10}$$

where

- $x_n = a + n\delta x$, $x \in [a, x_p]$, $n = 1, \ldots, N$
- $\tau_m = m\delta\tau$, $m = 0, 1, ..., M$, $M = 400$
- $C_n^m := C(\tau_m, x_n)$
- $x^{*,m} := x^*(\tau_m)$
- $\phi(x) = e^{-\epsilon\|x\|^2}$, $\epsilon = 1$

Hybrid Numerical Scheme

Consider following algorithm

1. Start from τ, solve $C(\tau+\delta\tau,x)$, i.e. first-order backward time integration scheme (BD1)
   - $x \leq x^*_{\tau+\delta\tau}$

$$\frac{C(\tau+\delta\tau,x)-C(\tau,x)}{\delta\tau} = \left[\frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial C}{\partial x} - rC\right]_{\tau+\delta\tau,x}$$

$$\frac{C(\tau+\delta\tau,x)}{\delta\tau} - \frac{C(\tau,x)}{\delta\tau} = \mathcal{L}(C(\tau+\delta\tau,x))$$

i.e.

$$\textbf{Unknown} \quad [C - \delta\tau\mathcal{L}(C)]_{\tau+\delta\tau,x} = [C]_{\tau,x} \quad \textbf{Known} \tag{11}$$

where

$$\mathcal{L}(C) = \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial C}{\partial x} - rC \tag{12}$$

   - $x \geq x^*_{\tau+\delta\tau}$

$$C(\tau+\delta\tau,x) = (K\,e^x - K)^+ \tag{13}$$

Hybrid Numerical Scheme

2. Regard $C^\tau(x) := C(x|\tau)$ as a function of x, i.e.

$$C^\tau(x) \approx \sum_j \alpha_j^\tau \phi(||x - z_j||_2) =: \sum_j \alpha_j^\tau \phi_j(x) \tag{14}$$

Plug (14) in (11) for every x, we have

$$\phi(x) = e^{-x^2} \tag{15}$$

$$\frac{\partial \phi_j(x)}{\partial x} = \phi_j(x)(-2(x - z_j)) \tag{16}$$

$$\frac{\partial^2 \phi_j(x)}{\partial x^2} = \phi_j(x)(-2 + 4(x - z_j)^2) \tag{17}$$

$$\begin{aligned}\mathcal{L}(C^{\tau+\delta\tau}(x)) = &- r \sum_j \alpha_j^{\tau+\delta\tau} \phi_j(x) + \frac{\sigma^2}{2} \sum_j \alpha_j^{\tau+\delta\tau} \frac{\partial^2 \phi_j(x)}{\partial x^2} \\ &+ (r - D_0 - \frac{\sigma^2}{2}) \sum_j \alpha_j^{\tau+\delta\tau} \frac{\partial \phi_j(x)}{\partial x}\end{aligned} \tag{18}$$

Hybrid Numerical Scheme

3. Boundary Condition:

$$\lim_{x \to -\infty} C(\tau + \delta\tau, x) = 0 \tag{19}$$

$$\lim_{x \to +\infty} C(\tau + \delta\tau, x) = K\,exp(x) \tag{20}$$

$$C(\tau + \delta\tau, x^*) = K\,exp(x^*) - K \tag{21}$$

$$\partial C(\tau + \delta\tau, x^*)/\partial x = K\,exp(x^*) \tag{22}$$

$$x^{*,0} = ln(K/K) = 0 \tag{23}$$

$$C(0, x) = (K\,exp(x) - K)^+ \tag{24}$$

where $-\infty$ can be handled using artificial boundary condition (Han & Wu, 2003).

Hybrid Numerical Scheme

4. Together with (11), we solve following linear system for $\{\alpha_j^{\tau+\delta\tau}\}_{\hat{x}}$

$$\begin{bmatrix} A_{1,\hat{x}}^{\tau+\delta\tau}(x_1) & \cdots & A_{J,\hat{x}}^{\tau+\delta\tau}(x_1) \\ \vdots & \ddots & \vdots \\ A_{1,\hat{x}}^{\tau+\delta\tau}(x_{N-1}) & \cdots & A_{J,\hat{x}}^{\tau+\delta\tau}(x_{N-1}) \\ \phi_{1,\hat{x}}(\hat{x}) & \cdots & \phi_{J,\hat{x}}(\hat{x}) \\ \phi_{1,\hat{x}}(a) & \cdots & \phi_{J,\hat{x}}(a) \end{bmatrix} \begin{bmatrix} \alpha_{1,\hat{x}}^{\tau+\delta\tau} \\ \vdots \\ \alpha_{J,\hat{x}}^{\tau+\delta\tau} \end{bmatrix} = \begin{bmatrix} C(\tau, x_1) \\ \vdots \\ C(\tau, x_{N-1}) \\ Kexp(\hat{x}) - K \\ 0 \end{bmatrix}$$

where

$$A_{j,\hat{x}}^{\tau+\delta\tau}(x) = \phi_{j,\hat{x}}(x) \left\{ 1 - \delta\tau \left[-r + \frac{\sigma^2}{2}\frac{\partial^2 \phi_{j,\hat{x}}(x)}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial \phi_{j,\hat{x}}(x)}{\partial x} \right] \right\}$$

$$z_{j,\hat{x}}(x) := a + j\,(\hat{x} - a)/N$$

$$x_i := (a + j\,(\hat{x} - a)/N)\,\rho, \text{ where } \rho = 1.1 \text{ in our case}$$

In order to maintain the comparability of comparing errors between two quadratic programming results, let the centers of radial basis functions be uniform in the region $[a, \hat{x}]$ to fix the size of linear system.

Hybrid Numerical Scheme

5. Searching all $\hat{x} \in [0, x_p]$ for the exercising boundary by finding the $\hat{x}$ with least absolute error in first order smoothness, i.e.

$$x^* = arg\, min_{\hat{x}} \left\{ \left| \frac{\partial C^{\tau+\delta\tau}(\hat{x})}{\partial x} - Kexp(\hat{x}) \right| \right\} \qquad (25)$$

where

$$C^{\tau+\delta\tau}(x) = \sum_j \alpha_{j,x^*}^{\tau+\delta\tau} \phi_{j,x^*}(x)$$

$$x^* := x^{*,\tau+\delta\tau}$$

6. Repeat step 4 and step 5 until $\tau = T$.

Numerical Results

a	Asset Price	ABF (M=400)	Hybrid (N=30)	Hybrid (N=58)	Hybrid (N=142)	True Value
-1	40	0.0028	-0.01241	-0.02706	-0.03612	0.002792
	50	0.0456	0.05621	0.02952	0.01192	0.045594
	60	0.3013	0.3605	0.3308	0.3101	0.301387
	70	1.1459	1.1615	1.1426	1.1345	1.145799
	80	3.0435	2.9837	2.9911	3.0056	3.041536
	90	6.3643	6.2275	6.2592	6.2963	6.328677
	100	11.1267	11.0007	11.0399	11.0917	11.108407
	110	17.2772	17.1729	17.2000	17.2564	17.266726
	120	24.5710	24.4931	24.4962	24.5479	24.565972
CPU		6.2900	0.302	2.158	70.435	

- Good choice of N (size of radial basis functions) dramatically reduces computing time.
- However, increment of dimension drives computating complexity to be very high with limited accurary improvement.

Revelation

- The boundary seaching algorithm based on smoothness performs quite well even exercising boundary points of several τ deviate from neighboring boundaries.
- Hybrid numerical scheme still requires careful choice of mesh size and collocation points, due to the finite difference mechanism.
- Naturally we think of methods make use of convergence in elliptic 2nd order PDE: if we can roughly estimate exercising boundary by evaluating the smoothness of boundary, we can achieve truly meshless algorithm.

Section 3

Numerical Scheme

Recall that when $x \le x^*_\tau =: p(\tau)$,

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial C}{\partial x} - rC$$

1. Perform Laplace transform respect to τ,

$$\hat{C}_\lambda(x) := \mathscr{L}_\lambda\left\{C(\tau, x)\right\} = \int_0^\infty C(\tau, x)e^{-\lambda\tau}\, d\tau \tag{26}$$

$$\hat{p}_\lambda := \mathscr{L}_\lambda\left\{p(\tau)\right\} = \int_0^\infty p(\tau)e^{-\lambda\tau}\, d\tau \tag{27}$$

We have: when $x \le \hat{p}_\lambda$

$$(r - D_0 - \sigma^2/2)\frac{\partial\, \hat{C}_\lambda}{\partial\, x} + \frac{1}{2}\sigma^2\,\frac{\partial^2\, \hat{C}_\lambda}{\partial x^2} + (\lambda - r)\hat{C}_\lambda = rC(0, x) \tag{28}$$

Unknown $\mathcal{B}\left(\hat{C}_\lambda(x)\right) = C(0, x)$ **Known** (29)

where

$$\mathcal{B}(\cdot) = (r - D_0 - \sigma^2/2)\frac{\partial(\cdot)}{\partial\, x} + \frac{1}{2}\sigma^2\frac{\partial^2(\cdot)}{\partial x^2} + (\lambda - r) \tag{30}$$

Numerical Scheme

2. Regard $\hat{C}_\lambda(x)$ as a function of x, i.e.

$$\hat{C}_\lambda(x) \approx \sum_j \beta_j \phi(||x - z_j||_2) =: \sum_j \beta_j \phi_j(x) \tag{31}$$

Plug (31) in (29) for every x, we have

$$\phi(x) = e^{-x^2} \tag{32}$$

$$\frac{\partial \phi_j(x)}{\partial x} = \phi_j(x)(-2(x - z_j)) \tag{33}$$

$$\frac{\partial^2 \phi_j(x)}{\partial x^2} = \phi_j(x)(-2 + 4(x - z_j)^2) \tag{34}$$

$$\begin{aligned}\mathcal{B}(\hat{C}_\lambda(x)) =& (\lambda - r)\sum_j \beta_j \phi_j(x) + \frac{\sigma^2}{2}\sum_j \beta_j \frac{\partial^2 \phi_j(x)}{\partial x^2} \\ &+ (r - D_0 - \frac{\sigma^2}{2})\sum_j \beta_j \frac{\partial \phi_j(x)}{\partial x}\end{aligned} \tag{35}$$

Numerical Scheme

3. Boundary Conditions:

$$\lim_{x \to -\infty} \hat{C}_\lambda(x) = 0 \tag{36}$$

$$\lim_{x \to +\infty} \hat{C}_\lambda(x) = \mathscr{L}_\lambda(K\,exp(x)) = K\,exp(x)/\lambda \tag{37}$$

$$\hat{C}_\lambda(x^*) = \mathscr{L}_\lambda(K\,exp(x^*) - K) = (K\,exp(x^*) - K)/\lambda \tag{38}$$

$$\partial \hat{C}_\lambda(x^*)/\partial x = \mathscr{L}_\lambda(K\,exp(x^*)) = K\,exp(x^*)/\lambda \tag{39}$$

Numerical Scheme

4. Together with (11), we solve following linear system for $\{\beta_{j,\lambda}\}$

$$\begin{bmatrix} B^{\lambda}_{1,\hat{x}}(x_1) & \cdots & B^{\lambda}_{J,\hat{x}}(x_1) \\ \vdots & \ddots & \vdots \\ B^{\lambda}_{1,\hat{x}}(x_{N-1}) & \cdots & B^{\lambda}_{J,\hat{x}}(x_{N-1}) \\ \phi^{\lambda}_{1,\hat{x}}(\hat{x}) & \cdots & \phi^{\lambda}_{J,\hat{x}}(\hat{x}) \\ \phi^{\lambda}_{1,\hat{x}}(a) & \cdots & \phi^{\lambda}_{J,\hat{x}}(a) \end{bmatrix} \begin{bmatrix} \beta^{\lambda}_{1,\hat{x}} \\ \vdots \\ \beta^{\lambda}_{J,\hat{x}} \end{bmatrix} = \begin{bmatrix} C(0,x_1) \\ \vdots \\ C(0,x_{N-1}) \\ (Kexp(\hat{x})-K)/\lambda \\ 0 \end{bmatrix}$$

where

$$B^{\lambda}_{j,\hat{x}}(x) = \phi_j(x)\left[(\lambda - r) + \frac{\sigma^2}{2}\frac{\partial^2\phi_j(x)}{\partial x^2} + (r - D_0 - \frac{\sigma^2}{2})\frac{\partial \phi_j(x)}{\partial x}\right]$$

$$z_{j,\hat{x}}(x) := a + j\,(\hat{x} - a)/N$$

$$x_i := (a + j\,(\hat{x} - a)/N)\,\rho, \text{ where } \rho = 1.1 \text{ in our case}$$

Note: the linear system above is very similar to the system before the Laplace transform. However, unlike $\tau \in \mathbb{R}$, here $\lambda \in \mathbb{C}$

Numerical Scheme

5. For each λ, search all $\hat{x} \in [0, x_p]$ for the exercising boundary by finding the x with least absolute error in first order smoothness, i.e.

$$\hat{p}_\lambda = arg\,min_{\hat{x}} \left\{ \left| \frac{\partial \hat{C}_\lambda(\hat{x})}{\partial x} - Kexp(\hat{x})/\lambda \right| \right\} \tag{40}$$

where

$$\hat{C}_\lambda(x) = \sum_j \beta^\lambda_{j,\hat{x}} \phi_{j,\hat{x}}(x)$$

6. Repeat step 4 and step 5 for $\lambda_l = \lambda_1, \cdots, \lambda_L$, find $\{\beta_{j,\lambda_l}\}$ and $\hat{p}_{\lambda_l}$.
7. Perform numerical Laplace inverse transform on $\{\beta_{j,\lambda_l}\}$ and $\hat{p}_{\lambda_l}$ to calculate the original option price:

$$C(\tau, x) = \mathscr{L}_\gamma^{-1}\left\{\hat{C}_\lambda(x)\right\}(\tau) := \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{\lambda\tau} \hat{C}_\lambda(x)\, d\lambda$$

Section 4

Conclusion

- The meaning of the study lies in the generalization opportunities to higher dimensional option pricing
- The huristic hybrid algorithm is connected to penalty method, yet does not require forward and backward substitution
- Balance between efficiency and accuracy
- Further study: Extension to stochastic volatility models and higher dimensional option pricing

Thank You!