STAT 547 HW1 - Zhiling GU

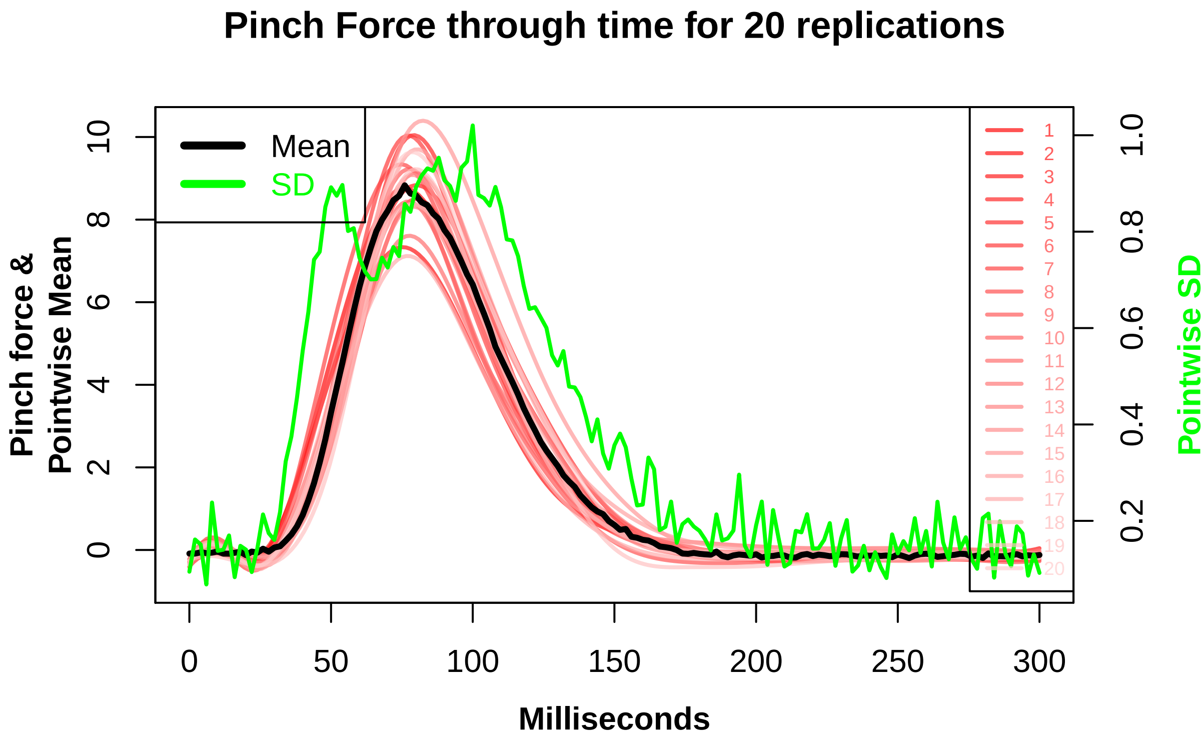
Q1

```
########################
########   Q1   ########
########################

# (a)
# Input a vector, return a fd object
nobs <- nrow(pinch)
x <- ((1:nobs)-1)*2
nrep <- ncol(pinch)

bspline4 <- function(y) {
  basis <- create.bspline.basis(rangeval =c(min(x), max(x)), nbasis = 15, norder = 4)
  fdobj <- smooth.basis (x, y,  fdParobj=basis)
}


# Store the fd objects in fd_list
fd_list <- list()
# Color the curves gradiantly
col <- sapply(1:nrep, function(i){rgb (1,0.8*i/nrep, 0.8*i/nrep, alpha =0.7)})

par(mar = c(5, 4, 4, 4) + 0.3)  # Leave space for z axis
plot(1, type="n", xlab="", ylab="", xlim=range(x), ylim=range(pinch))
for (i in 1:nrep){
  y <- pinch[,i]
  fd_i <- bspline4(y)
  fd_list <- append(fd_list, fd_i)
  lines(fd_i,lwd=2,col = col[i])
}


# (b)
y.mean <- apply(pinch, 1, mean)
# y.mean <- colMeans(pinch)
y.sd <- apply(pinch , 1, sd)
mean_col <- 'black'
sd_col <- 'green'


lines(x,y.mean, bg = mean_col, col = mean_col, cex =0.6, axes = FALSE, lwd=3)
```

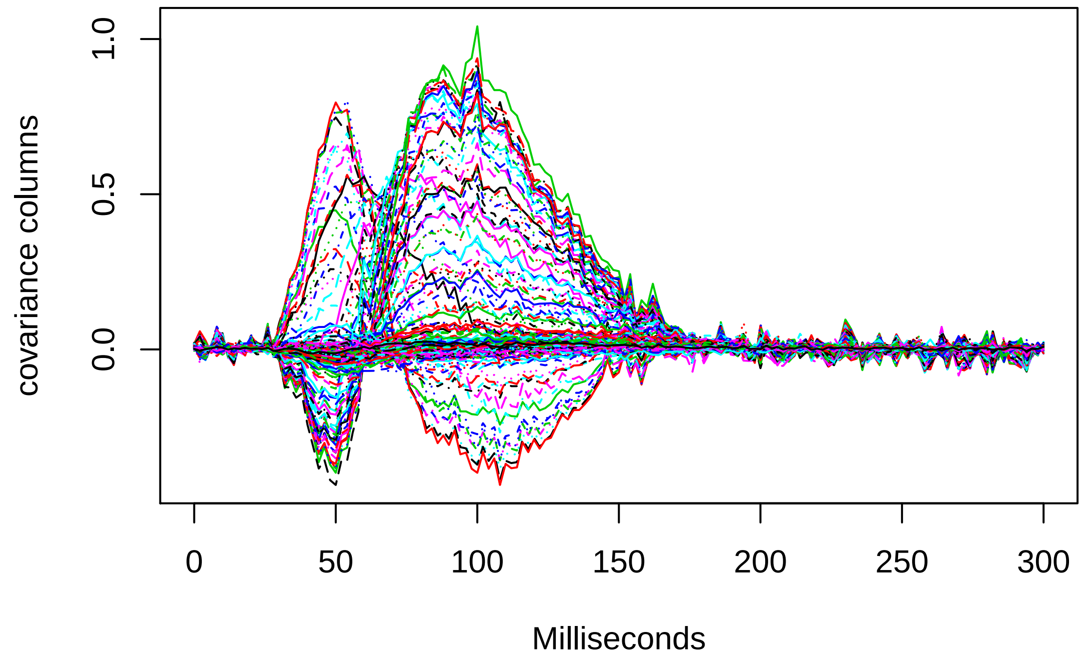
```
## Warning in plot.xy(xy.coords(x, y), type = type, ...): "axes" is not a
## graphical parameter
```

```
par(new = TRUE)
plot(x, y.sd, type = "l",  bg = sd_col, col = sd_col, cex =0.5, axes = FALSE, xlab = '', ylab = '',xlim
axis(side = 4, at = pretty(range(y.sd)))
axis(side = 2, at = pretty(range(y.mean)))
```

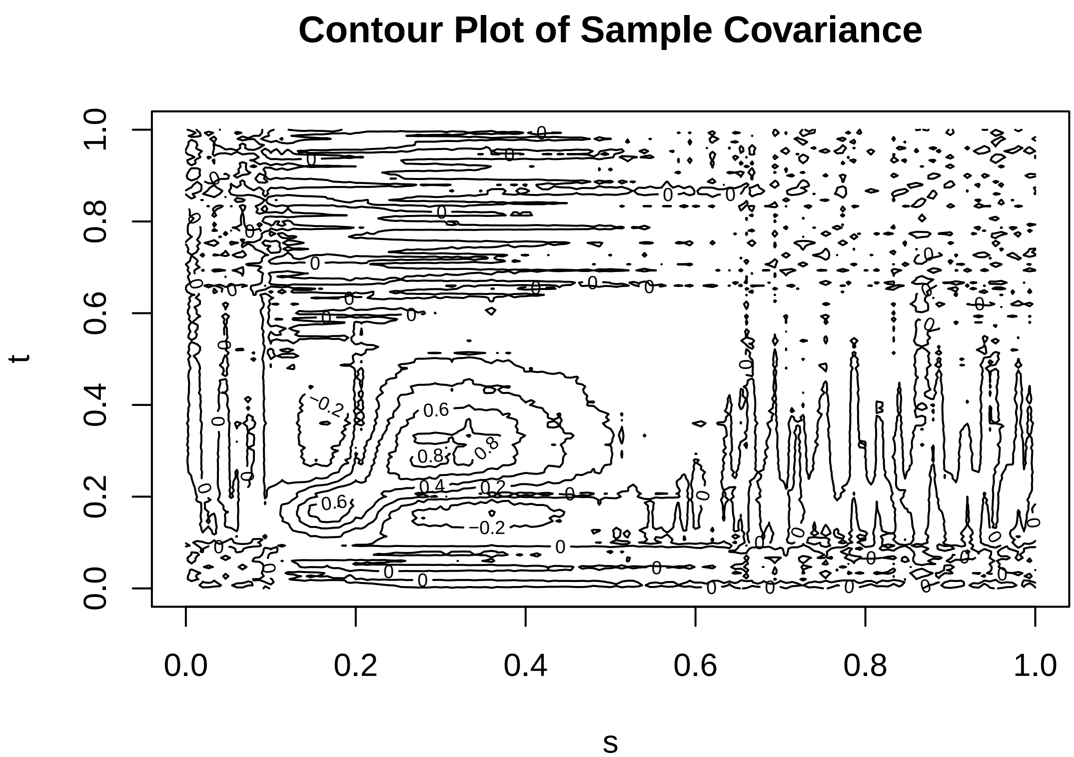
```r
# axis(side = 1, at = pretty(range(x)))
mtext("Pinch force &",  side=2, line=3, col = 'black',   font =2 )
mtext("Pointwise Mean",side=2,line=2,  col = mean_col, font =2)
mtext("Pointwise SD", side=4, line=2.5, col = sd_col,   font =2)
mtext("Milliseconds", side=1, line=2.5, col = 'black',   font =2)
legend('topright', legend = c(sapply ((1:ncol(pinch)), function(x){ paste("",x)})), col = c(col), text.
legend("topleft", legend =c("Mean","SD"),
       col=c(mean_col, sd_col),text.col =c(mean_col, sd_col), lty =1, lwd=4)
title(main = "Pinch Force through time for 20 replications")
```




```r
# (c)
t.pinch <- t(pinch)  # each row is an observation
df <- data.frame(t.pinch)
colnames(df) <- x   # change the column units to milliseconds
Ghat <- cov(df)
matplot(x = x, y = Ghat, type='l', xlab="Milliseconds", ylab= "covariance columns")
```

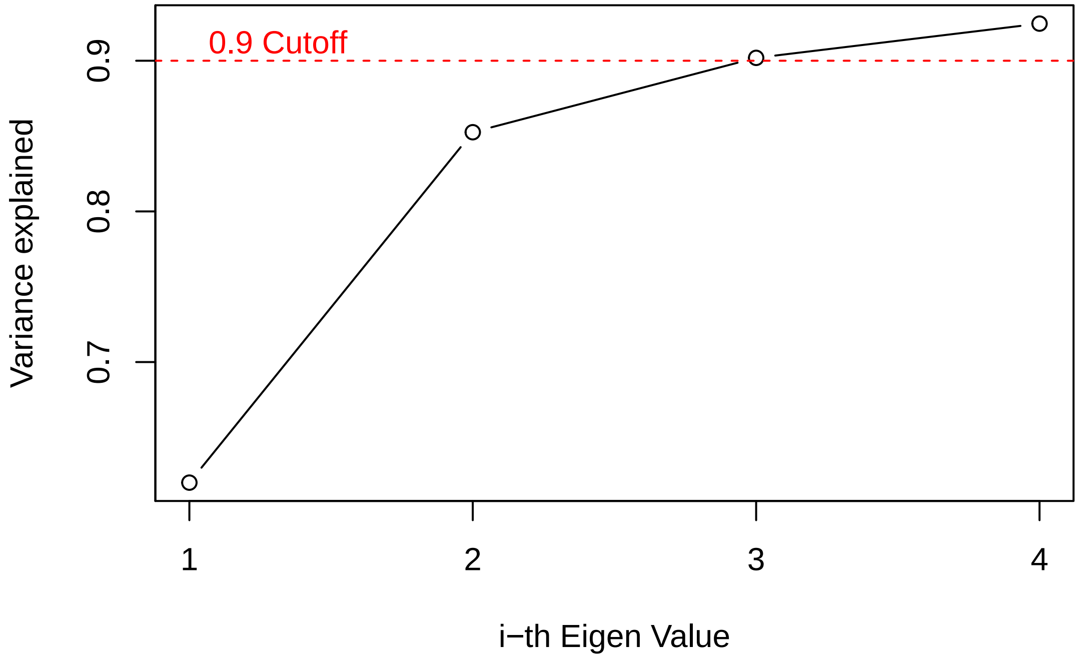



```
rho <- seq(0, 1, length.out=ncol(df))
persp3d(rho, rho, Ghat, col='white', xlab='s', ylab='t' , zlab = 'Sample Cov',main= "Perspective of Samp
contour(rho, rho, Ghat, xlab='s', ylab='t',  main= "Contour Plot of Sample Covariance")
```




```
# (d)
Sig <- Ghat
eig <- eigen(Sig)
FVE <- cumsum(eig$values) / sum(eig$values)
plot(FVE[1:4], type='b', axes = FALSE, ann = FALSE) # scree plot
```

```r
axis(side =1 , at = 1:4)
axis(side =2 , at = pretty(range(FVE)))
title (xlab = 'i-th Eigen Value')
title (ylab = 'Variance explained')
box()
abline(0.9,0,lwd =1, col = 'red', lty=2)
text(c(0.91,1),labels = "0.9 Cutoff", col = 'red', pos = 4)
```




```r
for (i in 1:length(FVE)){
  if(FVE[i]>=0.9){
    flag = i
    break
  }
}
cat(paste('The', flag, "-th eigen value explains 90% of variations" ))
```

```
## The 3 -th eigen value explains 90% of variations
```

Q2

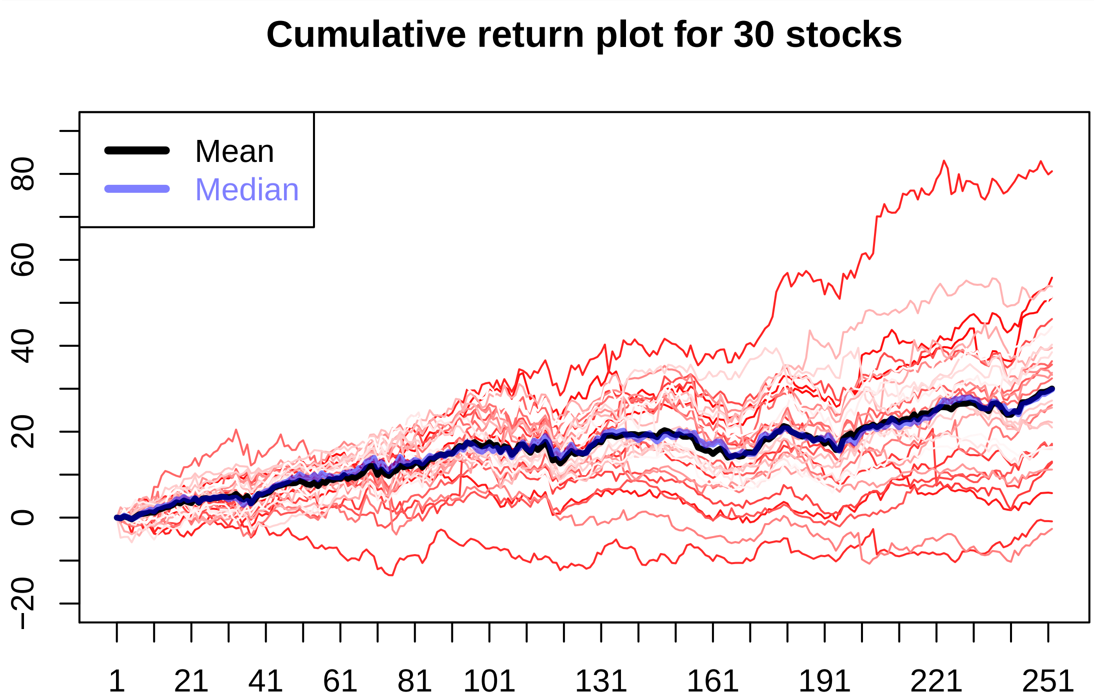
```r
########################
########   Q2   ########
########################
d <- read.csv("./DataSets/Dow_companies_data.csv")
data = d[,2:31]

# (a)
xom <- d$XOM
head(xom)
```

```
## [1] 85.75 85.59 85.99 84.99 85.52 85.19
```

```r
r <- sapply (xom, function(x){(x-xom[1])/xom[1]})*100
cat("Exxon-Mobil stock price increase in 2013 has increased by ", r[length(r)], " percent in return.")
```

```
## Exxon-Mobil stock price increase in 2013 has increased by  17.1895  percent in return.
```

```r
# (b)
# Create a 252 by 30 matrix cr that contains the cumulative returns on all stocks in DJIA.
# Plot all the cumulative return functions in one plot. Add the mean and median functions in color.

cr <- apply(d[,2:ncol(d)], 2, function(x){(x - x[1])/x[1]}) *100
rownames(cr) <-  1:252  # d[1:252,1]
colnames(cr) <-  1:30   # colnames(d)[2:31]

median <- apply (cr,1, median); mean_col = 'black'
mean <- apply(cr,1, mean); median_col = rgb(0,0,1,alpha =0.5)

plot(1, type="n", xlab="", ylab="", axes = FALSE, xlim = c(1,252), ylim =c(-20,90))
for (i in 1:ncol(cr)){
  lines(rownames(cr), cr[,i], col = rgb(1, 1*i/ncol(cr),1*i/ncol(cr)))
}
lines(rownames(cr), mean, col = mean_col,lwd =3)
lines(rownames(cr), median, col = median_col,lwd =3,lty =1)

axis(side =1, at = seq(1,252,by=10))
axis(side =2 , at = seq(-20,90,by=10))
box()
legend("topleft", legend =c("Mean","Median"),
       col=c(mean_col, median_col),text.col =c(mean_col, median_col), lty =1, lwd=4)
title(main = "Cumulative return plot for 30 stocks")
```

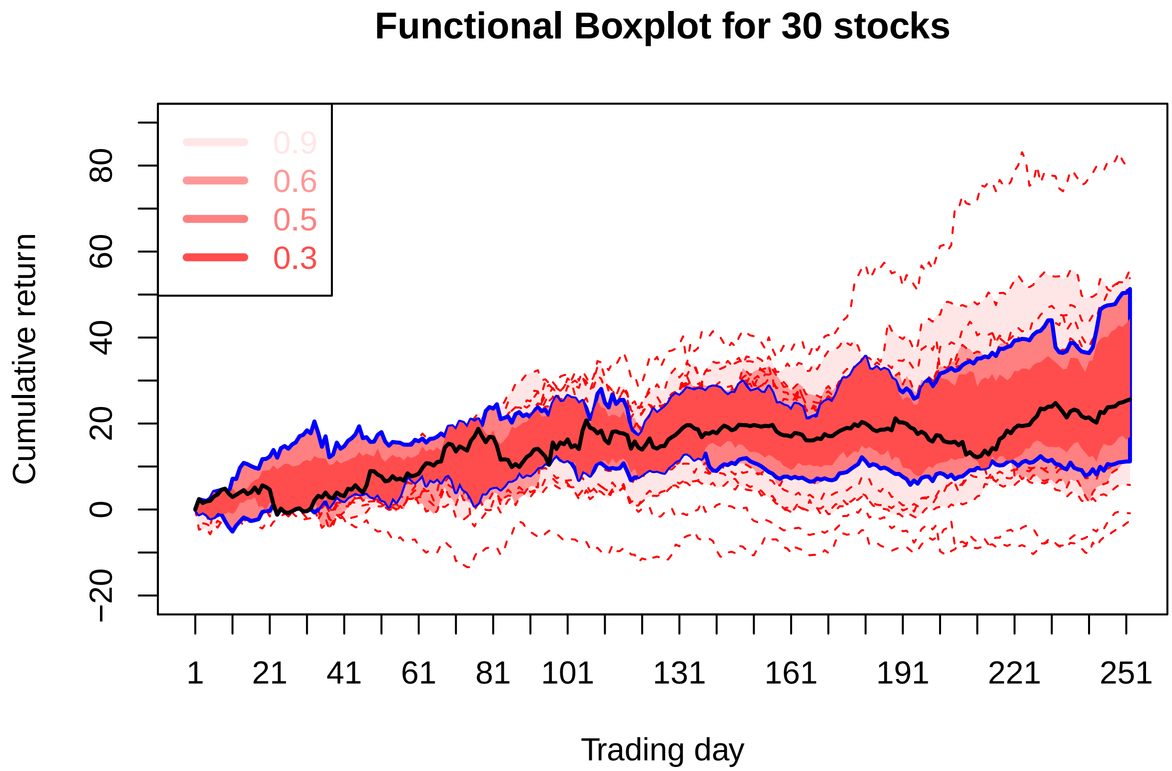



```r
# (c)
prob <- c(0.9, 0.6, 0.5, 0.3)
prob_col <- sapply(prob, function(x) {rgb(1,x,x )})
```

```r
fbplot(fit = cr, ylim = c(-20,90), xlim =c(1,252),xlab="Trading day", ylab = "Cumulative return",prob=pr
```

```
## $depth
##         1         2         3         4         5         6         7
## 0.4152915 0.3129926 0.2976852 0.1558498 0.1287926 0.4631477 0.4010787
##         8         9        10        11        12        13        14
## 0.3526204 0.3906062 0.2521096 0.4612320 0.4046730 0.4735837 0.3979771
##        15        16        17        18        19        20        21
## 0.2310368 0.3479132 0.4568167 0.4265850 0.4872856 0.3478768 0.3127919
##        22        23        24        25        26        27        28
## 0.4678366 0.4624726 0.4250160 0.3453772 0.4552842 0.4319308 0.4645525
##        29        30
## 0.4375319 0.3458881
##
## $outpoint
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30
##
## $medcurve
## 19
## 19
```

```r
axis(side =1, at = seq(1,252, by =10))
axis(side =2 , at = seq(-20,90,by=10))
title(main = "Functional Boxplot for 30 stocks")
box()
legend("topleft", legend =c(0.9, 0.6, 0.5, 0.3),
       col=prob_col,text.col = prob_col, lty =1, lwd=4)
```



Q3: W.L.O.G Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ be the eigen values of G, G be the covariance matrix of r.v. X. Let $u_1, \ldots, u_d$ be the corresponding eigen vectors. Also let $u_1, \ldots, u_d$ be normalized vectors.

1° Recall eigen-decomposition:

since $G u_i = \lambda_i u_i$, $i = 1, \cdots,$, we have

$$G(u_1 \cdots u_d) = (u_1 u_2 \cdots u_d)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$$

Denote $Q = (u_1 \cdots, u_d)$, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$

$$GQ = Q\Lambda \quad \Rightarrow G = Q\Lambda Q^{-1}$$

<u>Lemma</u> For any symmetric matrix, eigen vectors must be orthogonal

$$\lambda_i \langle u_i, u_j \rangle = \langle \lambda_i u_i, u_j \rangle = \langle G u_i, u_j \rangle$$

$$= \langle u_i, G^T u_j \rangle \overset{\text{Sym.}}{=} \langle u_i, G u_j \rangle$$

$$= \langle u_i, \lambda_j u_j \rangle = \lambda_j \langle u_i, u_j \rangle$$

For $\lambda_i \neq \lambda_j$, equality must hold due to $\langle u_i, u_j \rangle = 0$ #

use eigen decomposition and the lemma above, we have

$$QQ^T = I = Q^T Q \quad (Q^T = Q^{-1})$$

$$GQQ^T = G = Q\Lambda Q^T$$

Note $\operatorname{var}(l_1^T X) = E\left(l_1^T (X-\mu)(X^T - \mu^T) l_1\right)$

$$= l_1^T E\left((X-\mu)(X-\mu)^T\right) l_1$$

$$= l_1^T G l_1$$

$$= l_1^T Q \Lambda Q^T l_1$$

$$= (l_1^T u_1 \cdots l_1^T u_d)\, \Lambda \begin{pmatrix} u_1^T l_1 \\ \vdots \\ u_d^T l_1 \end{pmatrix}$$

$$= \sum_{i=1}^d l_1^T u_i \lambda_i u_i^T l_1$$

Case I: when $l_1 = u_i$, $Var(l_1^T X) = \overset{=1}{(l_1^T u_i)} \lambda_i \overset{=1}{(u_i^T l_1)} = \lambda_i \le \lambda_1$

Case II: when $l_1 = \sum_{i=1}^d c_i u_i$, $Var(l_1^T X) = \sum_{i=1}^d c_i^2 \lambda_i \le \sum_{i=1}^d c_i^2 \lambda_1 = \lambda_1$

$\boxed{\|l_1\| = 1}$

Therefore

$$v_1 = \underset{l_1}{\arg\max}\ Var(l_1^T X) = u_1 \ \#$$

2° $$v_k = \underset{\substack{\|l_k\|=1 \\ l_k^T l_j = 0,\ j=1,\dots,k-1}}{\arg\max}\ Var(l_k^T X)$$

Case I: when $l_k = u_i$, $Var(l_k^T X) = \lambda_i$, where $i \ge k$ $\leftarrow$ $\boxed{l_k^T l_j = 0,\ j=1,\dots,k-1}$

Case II: when $l_k = \sum_{i=1}^d c_i u_i$, $Var(l_k^T X) = \sum_{i=1}^d c_i^2 \lambda_i = \sum_{i=k}^d c_i^2 \lambda_i$

$\boxed{\text{with } c_1 = \dots = c_{k-1} = 0}$

$$\le \sum_{i=k}^d c_i^2 \lambda_k = \lambda_k$$

$\boxed{\|l_k\| = 1}$

therefore

$$v_k = \underset{\substack{\|l_k\|=1 \\ l_k^T l_j = 0,\ j=1,\dots,k-1}}{\arg\max}\ Var(l_k^T X) = \lambda_k \ \#$$

Q4.

a) $E(\xi_K) = v_K^T(EX-\mu) = 0$ #

b) $Var(\xi_K) = E(v_K^T(X-\mu)(X-\mu)^T v_K)$

$= v_K^T E((X-\mu)(X-\mu)^T) v_K$

$= v_K^T(G v_K)$

$= v_K^T(\lambda_K v_K) = \lambda_K$ #

c) $Cov(\xi_j, \xi_K) = E(v_j^T(X-\mu)(X-\mu)^T v_K)$

$= v_j^T G v_K$

$= v_j^T \lambda_K v_K$

$= \lambda_K v_j^T v_K$

$= \lambda_K \xi_{jK}$ # $\leftarrow$ $v_j^T v_K = \begin{cases} 0, & j \neq K \\ 1, & j = K \end{cases}$ by orthonormal.

d) $Corr(X_j, \xi_K) = Cov(X_j, \xi_K) / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= Cov(X_j, v_K^T(X-\mu)) / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= Cov(X_j, \Sigma_{i=1}^K v_{Ki}(X_i - \mu_i)) / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= \Sigma_{i=1}^K v_{Ki} Cov(X_j, X_i) / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= \Sigma_{i=1}^K v_{Ki} \lambda_K \xi_{ij} / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= v_{Kj} \lambda_K / \sqrt{\sigma_{jj}}\sqrt{\lambda_K}$

$= v_{Kj}\sqrt{\lambda_K} / \sqrt{\sigma_{jj}}$ #

Q5 (EX 10.1 K&R 2017) Prove prop 10.1.1.

a) $\|ax\| = \sqrt{\langle ax, ax\rangle} = \sqrt{a^2\langle x, x\rangle} = |a|\sqrt{\langle x, x\rangle} = |a|\,\|x\|$ #

b) $|\langle x,y\rangle| \le \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$

<Pf by projecting x on $\vec{y}$ and $\vec{y}^{\perp}$>

Consider $z = x - \dfrac{\langle x,y\rangle}{\langle y,y\rangle}\cdot y$

$$\langle z, y\rangle = \langle x, y\rangle - \left\langle \frac{\langle x,y\rangle}{\langle y,y\rangle}\cdot y, y\right\rangle$$

$\dfrac{\langle x,y\rangle}{\langle y,y\rangle}$ constant

$$= \langle x, y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\cdot\langle y,y\rangle$$

$$= 0$$

therefore z is perpendicular to y

Apply pythagorean theorem to $x = z + \dfrac{\langle x,y\rangle}{\langle y,y\rangle}\cdot y$

we have $\|x\|^2 = \|z\|^2 + \left(\dfrac{\langle x,y\rangle}{\langle y,y\rangle}\right)^2\|y\|^2$

$$= \|z\|^2 + \frac{\langle x,y\rangle^2}{(\|y\|^2)^2}\|y\|^2$$

$$\ge 0 + \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$$|\langle x,y\rangle| \le \sqrt{\|x\|^2}\sqrt{\|y\|^2}$$

$$= \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \quad \#$$

c) $\|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{\langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle}$

$$\overset{b)}{\le} \sqrt{\|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\|}$$

$$= \|x\| + \|y\| \quad \#$$

d) ① $d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle y-x, y-x\rangle}$

$= \|y-x\| = d(y-x) \geq 0$, equal when $x=y$

② $d(x,x) = \|x-x\| = \|0\| \overset{a)}{=} 0$

③ $d(x,y) = \|x-y\| = \|(x-z)+(z-y)\|$

$\overset{b)}{\leq} \|x-z\| + \|z-y\|$

$= d(x,z) + d(z,y)$ #

Q6 (EX 10.3 K&M 2017) verify ℓ^2 defined in 10.1.1 is Hibert

Recall the definitions

A complete inner-product space is Hilbert Space

A complete space : $\forall$ Cauchy sequence has limit

A Cauchy sequence: $d(x_m, x_n) \to 0$ as $m,n \to \infty$

ℓ^2: a collection of sequences $x = \{x_1, x_2, \cdots\}$ s.t. $\sum_{i=1}^{\infty} |x_i|^2 < \infty$

$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$

$x+y = \{x_1+y_1, x_2+y_2, \cdots\}$

$ax = \{ax_1, ax_2, \cdots\}$

Hint : $\left|\sum_{i=1}^{\infty} x_i \overline{y_i}\right| \leq \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i|^2\right)^{1/2}$

$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle} \sqrt{\langle y, y\rangle}$

[Construct $e_k(j) = \delta_{kj}$ such that $\langle x^m, e_k \rangle = \sum_{j=1}^{\infty} x_j^m e_k(j) = x_k^m$

Consider a Cauchy Sequence x^i, $i=1, \cdots$, $x^i \triangleq \{x_1^i, x_2^i, \cdots\}$

then $|x^m(k) - x^n(k)| = |\langle x^m - x^n, e_k \rangle|$

$\leq \|x^m - x^n\|^2 \|e_k\|^2 = \|x^m - x^n\|$ holds for $j=1,2,\cdots$

$= d(x^m - x^n) \to 0$ as $n, m \to \infty$

therefore $\{x^n(k)\}_{n\in\mathbb{N}}$ is Cauchy for each k, therefore converges.

Let $\tilde{x} = (\tilde{x}(k))_{k\in\mathbb{N}}$ be the limit of $\{x^n(k)\}_{n\in\mathbb{N}}$

only need to show $\tilde{x} \in l^2$.

Convergence uniform over k

$\sum_{i=1}^{\infty} |\tilde{x}(k)|^2 = \sum_{i=1}^{\infty} |\lim_{n\to\infty} x^n(k)|^2 = \lim_{n\to\infty} \sum_{i=1}^{\infty} |x^n(k)|^2 = \lim_{n\to\infty} \|x^n\|^2$

$< \infty$ by def.

Therefore $\tilde{x} \in l^2$ #

Q7. Let $X(t)$, $t \in [0,1]$ be a stochastic process for which the sample paths lie in $L^2[0,1]$. Show that the solution to following problem coincides with the projections in the functional PCA.

$$\underset{e_1,\cdots,e_K,\,K\geq 1}{\arg\min}\ \mathbb{E}\,\|X-\sum_{k=1}^{K}\langle X, e_k\rangle e_k\|^2$$

$$\Big[\ E\,\|X-\sum_{k=1}^{K}\langle X, e_k\rangle e_k\|^2 = \mathbb{E}\,\langle X-\sum_{k=1}^{K}\langle X, e_k\rangle e_k,\ X-\sum_{k=1}^{K}\langle X, e_k\rangle e_k\rangle$$

$$= \mathbb{E}\Big(\langle X, X\rangle - 2\langle X, \sum_{k=1}^{K}\langle X, e_k\rangle e_k\rangle + \langle \sum_{k=1}^{K}\langle X, e_k\rangle e_k, \sum_{k=1}^{K}\langle X, e_k\rangle e_k\rangle\Big)$$

$$= \mathbb{E}\Big(\langle X, X\rangle - 2\sum_{k=1}^{K}\langle X, e_k\rangle^2 + \sum_{i=1}^{k}\sum_{j=1}^{k}\langle\langle X, e_i\rangle e_i, \langle X, e_j\rangle e_j\rangle\Big)$$

$$= \mathbb{E}\Big(\langle X, X\rangle - 2\sum_{k=1}^{K}\langle X, e_k\rangle^2 + \sum_{i=1}^{K}\sum_{j=1}^{K}\langle X, e_i\rangle\cdot\langle X, e_j\rangle\cdot\langle e_i, e_j\rangle\Big)$$

$$= \mathbb{E}\Big(\langle X, X\rangle - 2\sum_{k=1}^{K}\langle X, e_k\rangle^2 + \sum_{k=1}^{K}\langle X, e_k\rangle^2\cdot 1\Big)$$

$$= \mathbb{E}\Big(\langle X, X\rangle - \sum_{k=1}^{K}\langle X, e_k\rangle^2\Big)$$

$$\underset{\{e_k\}}{\arg\min}\ \mathbb{E}\,\|\langle X, X\rangle-\sum_{k=1}^{K}\langle X, e_k\rangle e_k\|^2$$

$$= \underset{\{e_k\}}{\arg\min}\ \mathbb{E}\Big(\underset{\substack{\uparrow\\ \text{does not depend on } \{e_k\}}}{\langle X, X\rangle} - \sum_{k=1}^{K}\langle X, e_k\rangle^2\Big) = \underset{\{e_k\}}{\arg\max}\ \mathbb{E}\Big(\sum_{k=1}^{K}\langle X, e_k\rangle^2\Big)$$

#