Poisson Approximation (1995)

$$W \sim \text{Poisson}(\lambda) \xrightarrow{\text{then}} EWf(W) = \lambda Ef(W+1)$$

$$Ef'(W) - EWf(W) = 0$$

$$E\left(\frac{(f(Y)P(Y))'}{P(Y)}\right) = 0$$

$$0 = E\left(\frac{f(Y+1)P(Y+1) - f(Y)P(Y)}{P(Y)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{f(k+1)P(k+1) - f(k)P(k)}{\cancel{P(k)}} \cdot \cancel{P(Y=k)}$$

$$= f(\infty)P(\infty) - f(0)P(0)$$

Stein's Equation

$$\lambda f(w+1) - wf(w) = h(w) - Eh(Y), \quad w = 0, 1, \cdots$$

L.H.Y. Chen (1975) "Stein-Chan / Chan-Stein Method"

$\{T_\alpha, \alpha \in I\}$ $\qquad \max_{\alpha \in I} T_\alpha \xrightarrow{d} ?$

$$P(\max_{\alpha \in I} T_\alpha \le x) = P(\bigcap_{\alpha \in I} \{T_\alpha \le x\})$$

Observe that

$$\left\{\max_{\alpha\in I} T_\alpha \le x\right\} = \Big\{\sum_{\alpha\in I} \underbrace{1_{\{T_\alpha > x\}}}_{\xi_\alpha} = 0\Big\}.$$

- Y_α are independent, $P(Y_\alpha = 1) = p_\alpha = 1 - P(Y_\alpha = 0)$.

If $\sum_{\alpha\in I} p_\alpha \to \lambda$, $\max_{\alpha\in I} p_\alpha \to 0$.

then $\sum_{\alpha\in I} Y_\alpha \xrightarrow{d} \text{Poisson}(\lambda)$

- $\lambda = \sum_{\alpha\in I} P(T_\alpha > x)$

$\sum_{\alpha\in I} \xi_\alpha \xrightarrow{d} \text{Poisson}(\lambda)$?

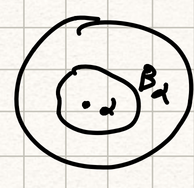
Ann. Probab. (1989) "Two moments are sufficient" L. Goldstein.

$W = \sum_{\alpha\in I} \xi_\alpha$, $P(\xi_\alpha = 1) = p_\alpha$, $P(\xi_\alpha = 0) = 1 - p_\alpha$

$\lambda = \sum_{\alpha\in I} p_\alpha$, then

$$|P(W\in A) - P(Y\in A)| \le 4(b_1 + b_2 + b_3)$$

where $Y \sim \text{Poisson}(\lambda)$ $\alpha \in B_\alpha$.



$$b_1 = \sum_{\alpha \in I,\ \beta \in B_\alpha} p_\alpha p_\beta$$

$$b_2 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha \setminus \alpha} E(\xi_\alpha \xi_\beta)$$

$$b_3 = \sum_{\alpha \in I} E \left| E\left((\xi_\alpha - p_\alpha) \mid \sigma(\xi_\beta, \beta \notin B_\alpha) \right) \right|$$

If independent then $b_3 = 0$

Q: Can you find a "suff" condition or a computable estimator for

$$| p(W \in A) - p(Y \in A) | \ ?$$ Not easy.

- Self-normalizing Limit Theorem

$X_1, X_2, \cdots \overset{iid}{\sim} \; EX_1 = \mu$

① $\frac{S_n}{n} \xrightarrow{a.s.} \mu.$

② $E|X_1|^p < \infty, \quad 1 < p < 2$

$$n^{1-\frac{1}{p}} \left(\frac{S_n}{n} - \mu\right) \xrightarrow{a.s.} 0$$

$\forall x > \mu$

$$P\left(\frac{S_n}{n} \geq x\right) \longrightarrow 0$$

Large Deviation (Cramer (1938), Chernoff (1952))

If $Ee^{t_0 X_1} < \infty$, $t_0 > 0$, then $\forall x > E(X_1)$

$$P\left(\frac{S_n}{n} \geq x\right)^{\frac{1}{n}} \longrightarrow \inf_{t \geq 0} e^{-tx} E(e^{tX_1})$$

[Pf. $P\left(\frac{S_n}{n} \geq x\right) = P\left(e^{tS_n} \geq e^{txn}\right) \leq \frac{1}{e^{txn}} E(e^{tS_n})$

$$= \frac{\left(E(e^{tX_1})\right)^n}{e^{txn}} = \left(e^{-tx} Ee^{tX_1}\right)^n$$

$$\Rightarrow P\left(\frac{S_n}{n} \geq x\right)^{\frac{1}{n}} \leq \inf_{t \geq 0} e^{-tx} Ee^{tX_1}$$

$$g(t) = e^{-tx} E e^{tX_1}$$

$$\frac{\partial g(t)}{\partial t} = -x e^{-tx} E(e^{tX_1}) + e^{-tx} E(t e^{tX_1})$$

$$= E[t e^{tX_1}(X_1 - x)) = 0$$

$$\Rightarrow E(X_1 e^{tX_1}) = x E e^{tX_1}$$

Conjugated Method (Change Measure)

Let $\{X_i\}$ independent $\boxed{E(e^{\lambda X_i}) < \infty}$, $\lambda > 0$ note that it could be $-\infty$ for higher moments

Introduce $\{Y_i\}$ independent

$$P(Y_i \le y) = \frac{E[e^{\lambda X_i} 1(X_i \le y)]}{E e^{\lambda X_i}}$$

then

$$P(\sum X_i \ge y) = \left(\prod_{i=1}^{n} E e^{\lambda X_i}\right) E(e^{-\lambda \sum X_i} 1(\sum Y_i \ge y))$$

☆ HW

$$P((X_1, \dots, X_n) \in A) = \left(\prod_{i=1}^{n} E e^{\lambda X_i}\right) E(e^{-\lambda \sum X_i} 1((Y_1, \dots, Y_n) \in A))$$

Properties:

- $E(g(Y_i)) = \dfrac{E(g(X_i) e^{\lambda X_i})}{E e^{\lambda X_i}}$
- $E(Y_i) = \dfrac{E(X_i e^{\lambda X_i})}{E e^{\lambda X_i}} = x$ when $g(t)$ is minimized.

$$P\left(\frac{S_n}{n} \geq x\right) = E(e^{tX_i})^n \, E\left(e^{-\lambda \sum Y_i} \mathbb{1}\left(\frac{\sum Y_i}{n} \geq x\right)\right)$$

$$\geq (E\, e^{tX_i})^n \, E\left(e^{-\lambda \sum Y_i} \mathbb{1}\left(x \leq \frac{\sum Y_i}{n} \leq x+\varepsilon\right)\right)$$

$$\geq (E\, e^{tX_i})^n \, E\left(e^{-\lambda n(x+\varepsilon)} \mathbb{1}\left(x \leq \frac{\sum Y_i}{n} \leq x+\varepsilon\right)\right)$$

$$= (E\, e^{tX_i})^n \, e^{-\lambda n(x+\varepsilon)} \, P\left(x \leq \frac{\sum Y_i}{n} \leq x+\varepsilon\right)$$

$$\frac{\sum Y_i}{n} \sim \underline{N\left(x, \frac{\gamma^2}{n}\right)}$$

when $n \to \infty$

$P(\cdot) \to \frac{1}{2}$

$$\Rightarrow \liminf P\left(\frac{S_n}{n} \geq x\right)^{\frac{1}{n}} \geq E(e^{\lambda X_1})\, e^{-\lambda(x+\varepsilon)}$$

$$\Rightarrow \liminf P\left(\frac{S_n}{n} \geq x\right)^{\frac{1}{n}} \geq E(e^{\lambda X_1})\, e^{-\lambda x} = \inf_{t \geq 0} e^{-tx} E e^{tX_1}$$

- Self-normalized

$X_1, \cdots$ iid

$S_n = \sum_{i=1}^n X_i$

$V_n^2 = \sum_{i=1}^n X_i^2$

$S_n / V_n \to$ self-normalized, t-statistics

(1989) Griffin, Kuelbs, "Self-normalized law of the iterated logarithm"

① If $EX_1 = 0$, $\underbrace{EX_1^2 \mathbb{1}(|X_1| \le x)}_{\triangleq \, l(x)}$ "slowly varying"

$\forall t > 0$, $\dfrac{l(tx)}{l(x)} \to 1$

then $\limsup \dfrac{S_n}{V_n \sqrt{2 \log\log n}} = 1$ a.s.

② X_1 symmetric

If $P(X_i \ge x) = \dfrac{l(x)}{x^\alpha}$, $0 < \alpha < 2$, $l(x)$ slowly varying

then $\limsup \dfrac{S_n}{V_n \sqrt{\log\log n}} = C_\alpha$ a.s. $\quad 0 < C_\alpha < \infty$

- $\limsup \frac{S_n}{a_n} = \begin{cases} 0 & \text{a.s} \\ \infty & \text{a.s.} \end{cases}$ for $\forall a_n$

- $EX_1 = 0$, $EX_1^2 = 1$; $m_n \to \infty$, $\frac{m_n}{\log n} \to \infty$; $E(e^{t_0|X_1|}) < \infty$

$$\limsup_{n\to\infty} \max_{1\le k\le n} \frac{S_{k+m} - S_k}{\sqrt{m_n}\sqrt{\cdots}} = 1 \text{ a.s}$$

- $EX_1^2 < \infty$

$$\limsup_{n\to\infty} \max_{1\le k\le n} \frac{S_{k+m_n} - S_k}{\sqrt{\sum_{i=k+1}^{k+m} X_i^2}\sqrt{\cdots}} = 1 \text{ a.s.}$$

- Self-normalized Large Deviation

$$P\left(\frac{S_n}{V_n\sqrt{n}} \ge x\right)^{\frac{1}{n}} \qquad x > \frac{E(X_1)}{\sqrt{E(X_1^2)}}$$