$X_1, X_2, \cdots$ indep $EX_i \le 0$, $X_i \le a$ bounded where $a \ge 0$

$S_n = \sum X_i$ $\quad \sum EX_i^2 \le B_i$

① $P(S_n \ge x) \le \exp\left(-\frac{x^2}{2B_n^2 + ax}\right)$

② $P(S_n \ge x) \le \exp\left(-\frac{B_n^3}{a^2}\left[\left(1+\frac{ax}{B_n^2}\right)\ln\left(1+\frac{ax}{B_n^2}\right) - \frac{ax}{B_n^2}\right]\right)$

Prove ① & ② For normal X_i

$X_i \sim N(0, \sigma_i^2)$ $\quad B_n^2 = \sum_i \sigma_i^2$

$S_n \sim N(0, B_n^2)$

$\frac{S_n}{B_n} \sim N(0,1)$

(i) Note : $Z \sim N(0,1)$, $x > 0$ $\Rightarrow P(Z \ge x) \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$ ← when x is small, upper bound not interesting, ignore $\frac{1}{x} \to 1$

$P(Z \ge x) \ge \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^2}\right) e^{-\frac{x^2}{2}}$

$P(Z \ge x) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$ when $x \to \infty$

$ax \ll B_n^2$

$$P(S_n \ge x) = P\left(\frac{S_n}{B_n} \ge \frac{x}{B_n}\right) \le \exp\left(-\frac{x^2}{2B_n^2}\right) \le \exp\left(-\frac{x^2}{2B_n^2 + ax}\right) \quad ①//$$

(ii) $P(S_n \ge x) \le e^{-tx} E(e^{tS_n})$, $t > 0$ $\qquad e^{tx} P(S_n \ge x) \le E(e^{tS_n})$

$= e^{-tx} \prod_{i=1}^n E(e^{tX_i})$ (3)

$s \le a$, $e^s \le 1 + s + \frac{s^2}{2} C_a$

check $s \to +\infty$, a

$\Rightarrow$ need to find appropriate C_a

$\frac{e^s - (1+s)}{s^2/2} \le C_a$ for $\forall s \le a$, $C_a = \sup_{s \le a}\left[\frac{e^s - (1+s)}{s^2/2}\right] = \frac{e^a - (1+a)}{a^2/2}$ ← increasing fn

$$e^{tX_i} \le 1 + tX_i + t^2X_i^2\left(\frac{e^{ta}-1-ta}{t^2a^2}\right)$$

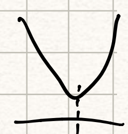
$$E(e^{tX_i}) \le 1 + t\overset{\le 0}{EX_i} + \cancel{t^2}E(X_i^2)\left(\frac{e^{ta}-1-ta}{\cancel{t^2}a^2}\right)$$

$$\le 1 + E(X_i^2)\left(\frac{e^{ta}-1-ta}{a^2}\right) \underset{1+x\le e^x}{\le} \exp\left(EX_i^2\left(\frac{e^{ta}-1-ta}{a^2}\right)\right)$$ Plug back to (3)

we have $P(S_n \ge x) \le e^{-tx}\exp\left(\sum_i EX_i^2\left(\frac{e^{ta}-1-ta}{a^2}\right)\right)$

$$\le \exp\Big(\underbrace{-tx + B_n^2\left(\frac{e^{ta}-1-ta}{a^2}\right)}_{g(t)}\Big)$$

$$g'(t) = -x + B_n^2\left(\frac{ae^{ta}-a}{a^2}\right) = -x + \frac{B_n^2}{a}(e^{ta}-1)$$



$$g'(t) = 0 \Rightarrow t = \ln\left(\frac{ax}{B_n^2}+1\right)/a$$ ② //

③ $EX_i \le 0$, $\sum EX_i^2 \le B_n^2$

then for $\forall p \ge 1$,

$$P(S_n \ge xB_n) \le P(\max X_i \ge xB_n/p) + \left(\frac{e^p}{p+x^2}\right)^p, \quad x>0$$ ④

Prove the above inequality ④: use truncation!

Let $Y_i = X_i \mathbb{1}(X_i \le xB_n/p)$ ← one-side truncation. 2-side: $|X| \le c$

$$EY_i = \underset{\le 0}{EX_i} - \underbrace{EX_i\mathbb{1}(X_i > xB_n/p)}_{\text{non-negative}} \le 0$$

Also note $Y_i \le xB_n/p$ by construction

$$P(S_n \geq x B_n) \leq P(\max X_i \geq \frac{xB_n}{p}) + P(S_n \geq xB_n,\ \max X_i < \frac{xB_n}{p})$$

(note X_i are not all positive)

$\Downarrow$

in this case $X_i = Y_i$

$$\leq \quad (\quad) \quad + P(\sum Y_i \geq xB_n)$$

use inequality ② , let

$\tilde{a} = \frac{xB_n}{p}$ $\quad \tilde{x} = xB_n$

$$\leq \exp\left(-\frac{p^2}{x^2}\left[\underbrace{\left(1+\frac{x^2}{p}\right)}_{\geq 1}\ln\left(1+\frac{x^2}{p}\right) - \frac{x^2}{p}\right]\right)$$

$$\leq \exp\left(-p\ln\left(1+\frac{x^2}{p}\right)+p\right)$$

$$= e^p \cdot \left[\frac{1}{\left(1+\frac{x^2}{p}\right)}\right]^p = e^p \cdot \left(\frac{p}{p+x^2}\right)^p$$

$$= \left[\frac{1}{e}\left(1+\frac{x^2}{p}\right)\right]^{-p} \leq \left(\frac{3p}{p+x^2}\right)^p \quad // ③$$

④ $EX_i = 0$

$$P(|S_n| \geq xB_n) \leq \sum P(|X_i| \geq xB_n/p) + 2\left(\frac{3p}{p+x^2}\right)^p$$

⑤ Rosential's Inequality

X_i independent $\quad EX_i = 0, \quad E|X_i|^p < \infty, \quad p \geq 2$

then $E|S_n|^p \leq C_p\left((ES_n^2)^{p/2} + \sum_i E|X_i|^p\right)$

$E|S_n|^p \geq D_p\left((ES_n^2)^{p/2} + \sum_i E|X_i|^p\right)$ HW

Relationship b/w moment & dist.

Note $E|X_i|^p = \int_0^\infty p x^{p-1} P(|X| \geq x)\, dx$

$$g(x) = g(0) + \int_0^x g'(t)dt$$
$$= g(0) + \int_0^\infty g'(t)\, 1(t \leq x)\, dt$$

$$E\, g(x) = g(0) + \int_0^\infty g'(t)\, P(X \geq t) dt$$

What if x is negative?

Change integral range from random to fixed
- Introduce indicator function

$$\star \quad \int_0^x g'(t) dx = \int_{-\infty}^\infty g'(t) \left(1(0<t<x) - 1(x<t<0) \right) dt$$

$$E|S_n|^p = \int_0^\infty p x^{p-1} P(|S_n| \geq x) dx$$

use inequality ④

$$= B_n^p \int_0^\infty p x^{p-1} P(|S_n| \geq x B_n)\, dx$$

$\begin{cases} x \to x B_n \\ B_n^2 = E S_n^2 \end{cases}$

$$\leq B_n^p \sum \int_0^\infty p x^{p-1} P(|X_i| \geq x B_n / p)\, dx$$

$P(x B_n \leq p|X_i|)$

$$+ \underbrace{B_n^p \int_0^\infty p x^{p-1} \cdot 2 \left(\frac{3p}{p + x^2} \right)^p dx.}_{\text{finite}}$$

$$\leq \sum E\left((p|X_i|)^p\right) + C_p B_n^p$$

⑥ X_i : Martingale Difference

$E(X_i|\mathcal{F}_{i-1})=0 \qquad S_n=\sum_{i=1}^n X_i$

- $E|S_n|^p \le C_p \left(\sum E|X_i|^p + E\left(\sum_{i=1}^n E(X_i^2|\mathcal{F}_{i-1})\right)^{p/2}\right)$
- $E|S_n|^p \le C_p^* E\left[\left(\sum X_i^2\right)^{p/2}\right]$ also lower-bound exists

⑦ Non-negative indep $X_i \ge 0$

$\mu_n=\sum EX_i \qquad B_n^2=\sum EX_i^2$

then $\forall x>0$

$P(S_n \le \mu_n - x) \le \exp\left(-\frac{x^2}{2B_n^2}\right)$

$P(S_n \ge \mu_n + x)$ no inequality

Prove ⑦ by Markov's Inequality

$P(S_n \le \mu_n - x) = P(-S_n \ge -\mu_n + x) \le Ee^{-tS_n} / e^{t(-\mu_n+x)}$

$= \prod_i Ee^{-tX_i} / e^{-t(-\mu_n+x)}$

$e^{-tX_i} \le 1 - tX_i + \frac{t^2X_i^2}{2}$ note $e^s \le 1+s+\frac{s^2}{2}$, $s\le 0$

$Ee^{-tX_i} \le 1 - tEX_i + \frac{t^2EX_i^2}{2}$

$\le \exp\left(-tEX_i + \frac{t^2EX_i^2}{2}\right)$

$e^{-t(-\mu_n+x)}Ee^{-tS_n}$

$\le \exp\left(t\mu_n - tx - t\sum EX_i + t^2\frac{\sum EX_i^2}{2}\right)$

$= \exp\left(-tx + \frac{t^2B_n^2}{2}\right)$, let $t=\frac{x}{B_n^2}$. // ⑦

8 0-1 random variables

$P(X_i=1)=p_i$

$P(X_i=0)=1-p_i$

then

$$P(\sum X_i \geq x) \leq \left(\frac{\mu e}{x}\right)^x \qquad \mu = \sum_{i=1}^n p_i$$

Prove by mgf, minimize using markov inequality.