Stein's Method

$$W \sim N(0,1) \iff Ef'(W) - E\,Wf(W) = 0$$

$$f'(w) - wf(w) = h(w) - Eh(Z), \quad Z \sim N(0,1)$$

- Aim: $W_n \xrightarrow{d} N(0,1) \iff E\,h(W_n) - Eh(Z) \to 0 \quad \forall \|h'\| \le 1$

- $Eh(W_n) - Eh(Z) = Ef'(W_n) - EW_n f(W_n)$

want to show close to each other

Example: Let ξ_i, $1 \le i \le n$ be indep. $E\xi_i = 0$

$\sum_{i=1}^n \underset{\sim}{\xi_i}^2 = 1 \qquad W_n = W = \sum_{i=1}^n \xi_i$

$\xi_{i,n}$ (for standardization purpose)

[Equivalently: $X_1, \dots, X_n$ indep $EX_i = 0$ $B_n^2 = \sum_{i=1}^n EX_i^2$

$\Rightarrow \xi_i = X_i / B_n$]

Aim: under Linderberg Condition

$$\forall \varepsilon > 0, \sum_{i=1}^n E\xi_i^2 1(|\xi_i| > \varepsilon) \to 0$$

we have $W_n \xrightarrow{d} N(0,1)$

$$\iff Eh(W_n) - Eh(Z) \to 0$$

$$\iff Ef_h'(W_n) - EW_n f_h(W_n) \to 0$$

[Proof: write $W = W_n$

$$E(Wf(W)) = E\left(\sum_{i=1}^n \xi_i f(W)\right)$$

$$= \sum_{i=1}^n E(\xi_i f(W))$$

$W^{(i)} \triangleq W - \xi_i \perp\!\!\!\perp \xi_i$

$\boxed{\text{important trick!}}$

$$= \sum_{i=1}^n E(\xi_i [f(W) - f(W - \xi_i)])$$

$f(W^{(i)} + \xi_i) - f(W^{(i)})$

$f(a+b) - f(a)$

$= \int_0^b f'(a+t)\,dt$

$= \int_{-\infty}^{\infty} f'(a+t)\big(1(0<t<b) - 1(b<t<0)\big)\,dt$

$$= \int_{-\infty}^{\infty} f'(W^{(i)}+t)\big(1(0<t<\xi_i) - 1(\xi_i<t<0)\big)\,dt$$

$$= \sum_{i=1}^n \int_{-\infty}^{\infty} E\big(f'(W^{(i)}+t)\big) \underbrace{E\big(\xi_i(1(0<t<\xi_i) - 1(\xi_i<t<0))\big)}_{K_i(t)}\,dt$$

$$= \sum_{i=1}^n E \int_{\infty}^{\infty} f'(W^{(i)}+t)\, K_i(t)\,dt$$

① non-negative

② $\int_{-\infty}^{\infty} K_i(t)\,dt = E\xi_i^2$

③ $\int_{-\infty}^{\infty} |t|\, K_i(t)\,dt = \frac{1}{2} E|\xi_i|^3$

$E f'(W) - E W f(W)$

$$= E f'(W) - \sum E \int_{-\infty}^{\infty} f'(W^{(i)}+t) K_i(t)\,dt$$

think abt $f'(W^{(i)}+t) - f'(W)$

$\int_{-\infty}^{\infty} f'(W)\, K_i(t)\,dt$

$$= f'(W) \underbrace{\int_{\infty} K_i(t)\,dt}_{E(\xi_i^2)}$$

$$= \underbrace{Ef'(W) - Ef'(W)\sum_i \xi_i^2}_{=0} - \sum E\int_{-\infty}^{\infty}\left(f'(W^{(i)}+t) - f'(W)\right)K_i(t)\,dt.$$

$$f_h'(w) - wf_h(w) = h(w) - Eh(Z)$$

$$\|h'\|<\infty, \quad \|f''\| \le 2\|h'\|, \quad \|f'\| \le \|h'\|$$

By mean-value theorem,

$$|f'(W^{(i)}+t) - f'(W)| \le \|f''\|\,(|t| + |\xi_i|)$$

$$(*) \quad E\int_{-\infty}^{\infty} |f'(W^{(i)}+t) - f'(W^{(i)})|\,K_i(t)\,dt$$

$$\le \|f''\|\,E\int_{-\infty}^{\infty}(|t| + |\xi_i|)\,K_i(t)\,dt$$

$$= \|f''\|\left(\frac{1}{2}E|\xi_i|^3 + \underbrace{E|\xi_i|\,E|\xi_i|^2}\right)$$

By Holder's Inequality

$\le E|\xi_i|^3$

$$\le \frac{3}{2}\|f''\|\,E|\xi_i|^3 \le \frac{3}{2}\cdot 2\|h'\|\,E|\xi_i|^3$$

require us to assume $E|\xi_i|^3 < \infty$

$$|Eh(W) - Eh(Z)| \le 3\|h'\|\sum_{i=1}^n E|\xi_i|^3 \quad //$$

what if we only have Lindeburg's condition?

$$\sum \frac{E|X_i|^3}{B_n^3} \quad \text{iid} \quad \frac{E|X_i|^3}{\sqrt{n}\,\sigma^3}$$

$$|f'(W^{(i)}+t) - f'(W)| \le 2\|h'\| \left(\min(|t|+|\xi_i|, 1)\right) \quad \text{(truncation)}$$

(*) $$E\int_{-\infty}^{\infty} |f'(W^{(i)}+t) - f'(W^{(i)})|\, K_i(t)\,dt$$

$$\le 8\|h'\| \left\{ E(\xi_i^2 1(|\xi_i|>1)) + E|\xi_i^3| (1(|\xi_i|\le 1) \right\}$$

then

$$|Eh(W) - Eh(Z)| \le 8\|h'\| (\beta_2+\beta_3)$$

where $$\beta_2 = \sum E\xi_i^2 1(|\xi_i|>1)$$

$$\beta_3 = \sum E\xi_i^3 1(|\xi_i|\le 1)$$

then Lindeburg Condition $\iff \beta_2+\beta_3 \to 0$ #

When h is an indicator function, f'' does not exist.

Now, look at the relationship b/w continuous function & indicator function

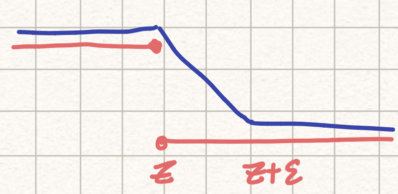
- W, if $\exists \delta$ s.t. for h $|Eh(W) - E(h(Z))| \le \|h'\|\delta$

  then $\sup_z |P(W\le z) - \Phi(z)| \le 2\sqrt{\delta}$

  ← drawback of this inequality.

[Proof: introduce an abs. cont. func. $\ge$ Indicator function

h_ε $1(W\le z)$



$$P(W \le z) - \Phi(z) \le E h_\varepsilon(W) - E h_\varepsilon(Z) + E h_\varepsilon(Z) - E 1(Z \le z)$$

$$\le \frac{1}{\varepsilon} \cdot \delta + E 1(z \le Z \le z+\varepsilon)$$

$$\le \frac{1}{\varepsilon}\delta + \varepsilon \qquad \text{let } \varepsilon = \sqrt{\delta} \qquad \#]$$

$$\sup_z |P(W \le z) - \Phi(z)| \le 4 (\sum E|\xi_i|^3)^{1/2}$$

- Berry-Esseen Theorem (use Stein's Method to prove)

$$\sup_z |P(W \le z) - \Phi(z)| \le \underline{\underline{4.1}} \sum E|\xi_i|^3$$

0.49

$$f'(w) - wf(w) = 1(w \le z) - \Phi(z)$$

$$E f'(W) - EWf(W) = \sum E \int_{-\infty}^{\infty} (f'(W) - f'(W^{(i)}+t)) K_i(t) dt$$

$$f'(W) - f'(W^{(i)}+t)$$

$$= \underbrace{Wf(W) - (W^{(i)}+t) f(W^{(i)}+t)}_{①} + \underbrace{1(W \le z) - 1(W^{(i)}+t \le z)}_{②}$$

① $(W = W^{(i)} + \xi_i)$ $\quad \swarrow \|f'\| \le 1$

$$= W^{(i)} (f(W) - f(W^{(i)}+t))$$

$$+ \xi_i f(W) - t f(W^{(i)}+t) \quad \swarrow |f| \le 1$$

$$\le |W^{(i)}| (|t| + |\xi_i|) \times 1$$

$$+ |\xi_i| \cdot 1 + |t| \cdot 1$$

$$= (|W^{(i)}| + 1)(|t| + |\xi_i|)$$

$$\sum E \int_{-\infty}^{\infty} (|W^{(i)}|+1)(|t|+|\xi_i|) K_i(t) dt$$

$$\le 2 \cdot \sum E|\xi_i|^3 \cdot \frac{3}{2}$$

$$= 3 \sum E|\xi_i|^3$$

② $\sum \int_{-\infty}^{\infty} \underbrace{\left(P(W \le z) - P(W^{(i)} + t \le z)\right)} K_i(t)\,dt$

$$= P(W^{(i)} \le z - \xi_i) - P(W^{(i)} \le z - t)$$

$$\le P(z - t \le W^{(i)} \le z - \xi_i)$$

Claim: $P(a \le W^{(i)} \le b) \le b - a + 2\sum E|\xi_i|^3$

(Proof in Lec 5)