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1. Classical Limit Theory

① Jacob Bernoulli : LLN $EX<\infty \Rightarrow \bar{X} \to \mu$ a.s.

② de Moire & Linderberg : CLT $EX<\infty$, $\mathrm{Var}X<\infty \Rightarrow \frac{S_n - n\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$

③ Khinchin & Kolmogorov : law of interated logarithm

$$\limsup_{n\to\infty} \frac{S_n - n\mu}{\sqrt{2n\log\log n}} = 1 \text{ a.s.}$$

$$\Downarrow$$

$$\liminf_{n\to\infty} \frac{S_n - n\mu}{\sqrt{2n\log\log n}} = -1 \text{ a.s.}$$

arg limsup $(\cdot)$ = arg(-lim inf)$(\cdot)$

2. Central Limit Theorem & Related Problems

assume $\mu=0$, observe CLT $\Leftrightarrow$ $\sup_x \Big|\underbrace{P(\frac{S_n}{\sqrt{n}\sigma} \geq x)}_{\triangleq A} - \underbrace{(1-\Phi(x))}_{\triangleq B}\Big| \to 0$

$\Uparrow$ usually unknown

estimate

① Berry-Esseen Inequality: abt the abs error of the estimation

$$|A-B| \leq \frac{0.4748}{\sqrt{n}} \frac{E|X|^3}{\sigma^3}$$

$\Uparrow$ also called Kolmogorov distance.

② Cramér Moderate Deviation Theorem : Relative error.

$$A/B = \exp\left(x^2 \lambda\left(\tfrac{x}{\sqrt{n}}\right)\right) \left|1+O\left(\tfrac{1+x}{\sqrt{n}}\right)\right| \quad \text{if } Ee^{t_0|X|} < \infty \text{ for } t_0 > 0$$

where $\lambda(t)$ is Cramér's series. $x \geq 0$ and $x = o(n^{1/2})$

afterwards Yuri Linnick relaxed to $E(e^{t_0\sqrt{|X|}})<\infty$

$o(n^{1/6})$

☆ Depend on how many moments of X can match those of standard norma

③ Comparison b/w Abs err & Rel err

p-value $= P(T \geq x_n)$

$T \xrightarrow{d} N(0,1)$

$\hat{p} = 1 - \Phi(x_n)$

References:

Petrov. (1972). Sum of Ind. R.V.
Petrov. (1995) Limit Theorems of Prob. Theory

$\hat{p} - p \to 0$

$\uparrow$

unknown, but always very small

$\Downarrow$

$p / \hat{p} \to 1$

Singularity prob. of Random Bernoulli matrices

$A_n = (\varepsilon_{ij})_{n \times n}$

Textbook: "Self-Normalized Process"

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"Normal Approx. by Stein's Method"

$S_n = \sum_{i=1}^n X_i$, X_i indep. $EX_i = 0$

$P(\max_{1 \le k \le n} |S_k| \ge x) \le ES_n^2/x^2$ (maximum inequality)

Chapter 1 Inequalities.

1. Chebyshev's Inequalities

(a) $\forall x \ge 0$, $P(X \ge x) \le E|X|/x$

[Pf: $1(X \ge x) \le X^+/x$]

$$P(X \ge x) \le \frac{E(X1(X \ge x))}{x}$$

(b) $P(X \ge x) = P(e^{tX} \ge e^{tx}) \le E(e^{tX})/e^{tx}$

transformation using increasing non-neg. fn

useful for exponential ineq. for r.v.

(c) $P((X,Y) \in A) = P(X \in A^1, Y \in A^2)$

$= E(1(X \in A^1, Y \in A^2))$

$x \ge a_1$ $y \ge a_2$

$\le E(\underbrace{e^{sX+tY - \inf_{(x,y)\in A}(sx+ty)}}_{\text{when } (x,y)\in A,\ \exp(\cdot) \text{ should} \ge 1}))$

2. Lyapunov Inequalities

Recall Hölder's Inequality:

$$E|XY| \le (E|X|^p)^{1/p} (E|X|^q)^{1/q} \quad , \quad E|X| \le (E|X|^p)^{1/p}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$

use higher moment for a upper bound

$0 \le r \le s \le t$

$$E|X|^s \le (E|X|^r)^{\frac{t-s}{t-r}} (E|X|^t)^{\frac{s-r}{t-r}}$$

$r=1,\ s=2,\ t=4$

$$E(X^2) \le (E|X|)^{2/3} (EX^4)^{\frac{1}{3}} \Rightarrow (E|X|)^{\frac{2}{3}} \ge \frac{E(X^2)}{(E(X^4))^{1/3}}$$

gives a lower bound using higher moments

Proof: (Make Use of Hölder's Inequality)

Let $Y=1 \in L^q$, $Z=|X|^r$

then $E(|X|^r) \le E(|X|^{rp})^{\frac{1}{p}}$

since $p \ge 1$, we have $rp \ge r$, let $rp=s$,

then $E(|X|^r) \le E(|X|^s)^{\frac{r}{s}}$ i.e. $E(|X|^r)^{\frac{1}{r}} \le E(|X|^s)^{\frac{1}{s}}$

$0 < r < s < t$

3. Kinball's Inequality

(a) $g(x)\uparrow$, $h(x)\uparrow$ $\quad E(g(x)h(x)) \geq E(g(x))E(h(x))$ 单调性相同

[Introduce $Y \stackrel{d}{=} X$ and $Y \perp\!\!\!\perp X$

then $E(g(X))E(h(X)) = E(g(X))E(h(Y)) = \frac{1}{2}\left(E(g(X)h(Y)) + E(g(Y)h(X))\right)$

$$2E\,g(X)\,E\,h(X)$$

$$= E(g(X)\,h(Y)) + E(g(Y)\,h(X))$$

$$\leq E(g(X)\,h(X) + g(Y)\,h(Y))$$

$$g(X)h(Y) + g(Y)h(X) - (g(X)h(X) + g(Y)h(Y))$$

$$= -(g(X)-g(Y))(h(X)-h(Y)) \leq 0 \quad \#\ ]$$

(b) $g(x)\uparrow$ $h(x)\downarrow$ $\quad E(g(x)h(x)) \leq E(g(x))E(h(x))$ 单调性相反

4. Bernette-Hoeffding's Inequality

$E(X_i) \leq 0$, $X_i \leq a$, $a \geq 0$. $\sum_{i=1}^{n} EX_i^2 \leq B_n^2$

then ① $P(S_n \geq x) \leq \exp\left(-\frac{x^2}{2B_n^2 + ax}\right)$

② $P(S_n \geq x) \leq \exp\left(-\frac{B_n^2}{a^2}\right)\left[\left(1+\frac{ax}{B_n^2}\right)\ln\left(1+\frac{ax}{B_n^2}\right) - \frac{ax}{B_n^2}\right]$

$$\underbrace{(1+y)\ln(1+y) - y}_{\Downarrow}$$

$$0 + \frac{1}{1+y}y - \frac{1}{(1+y)^2}y^2/2 + \frac{1}{(1+y)^3}y^3/6 \cdots$$

when $y \to 0$

$$= \cancel{y} - \frac{y^2}{2} \frac{1}{\cancel{(1+y)}}^{\nearrow 1} + \frac{y^3}{6} \frac{1}{\cancel{(1+y)^2}}^{\nearrow 1} - \cdots - \cancel{y}$$

$$\geq \frac{y^2}{2(1+y)}$$

$$\forall t > 0, \quad P(S_n \geq x) \leq e^{-tx} E e^{tS_n}$$

$$E e^{tS_n} = \prod_{i=1}^{n} E e^{tX_i}$$

$$e^x \leq 1 + x + \frac{x^2}{2} C$$