$X_1, X_2 \dots \overset{iid}{\sim}$ $EX_i = 0$, $EX_i^2 1(|X_i| \le x)$ slowly converging

$V_n^2 = \sum X_i^2$ $\quad \ln P(\frac{S_n}{V_n} \ge x_n) \sim -\frac{x_n^2}{2}$, $x_n \to \infty$, $x_n = o(\sqrt{n})$ $\forall \varepsilon > 0$

$$\exp\left(-\frac{(1+\varepsilon)x_n^2}{2}\right) \le P\left(\frac{S_n}{V_n} \ge x_n\right) \le \exp\left(-(1-\varepsilon)\frac{x_n^2}{2}\right)$$

$H_0: \mu = 0$

$H_1: \mu > 0$

t-statistic: $\quad$ p-value $= P(T_n \ge y)$

$$= P\left(\frac{S_n}{V_n} \ge x\right)$$

$$\left| P\left(\frac{S_n}{V_n} \ge x_n\right) - (1 - \Phi(x_n)) \right| \to 0$$

Question: Is $1-\Phi(x_n)$ "close to" $P(\frac{S_n}{V_n} \ge x_n)$?

$$\frac{P(\frac{S_n}{V_n} \ge x_n)}{1-\Phi(x_n)} \to 0 \; ?$$

Cramer (1938)

If $X_1, \dots, X_n$ iid $EX_i = 0$, $EX_i^2 = \sigma^2$

$E e^{t_0 |X_1|^{1/2}} < \infty$ then $\dfrac{P(\frac{S_n}{\sqrt{n}\sigma} \ge x)}{1-\Phi(x)} \to 1$ uniformly for $x \in (0, o(n^{1/6}))$

originally mgf., later loosened to $e^{t_0 |X_1|^{1/2}}$

Shao (1999)

$E|X_i|^{2+\delta} < \infty, \quad 0 < \delta \le 1$

If $EX_i = 0$, $E|X_i|^3 < \infty$, then

$n^{\frac{\delta}{2(2+\delta)}}$

$$\frac{P(\frac{S_n}{V_n} \ge x)}{1 - \Phi(x)} \to 1 \text{ uniformly for } x \in (0, o(n^{1/6}))$$

[Proof:

$P(\frac{S_n}{V_n} \ge x_n) \le \cdots$

$P(\frac{S_n}{V_n} \ge x_n) \ge \cdots$

trick: use V_n & V_n^2 relationship

Step 1: $P(\frac{S_n}{V_n} \ge x_n) \ge (1 - \Phi(x))(1 + o(1))$

$$P(\frac{S_n}{V_n} \ge x_n) = P(S_n \ge x_n V_n) \qquad EX_i^2 = 1$$

$$x_n V_n \le \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2)}{b}, \quad b = \frac{x_n}{V_n}, \quad b = \frac{x_n}{\sqrt{n}}$$

want b to be a constant, $V_n^2 = O(n)$ consider

$$\ge P(S_n \ge \frac{1}{2} \frac{(b^2 V_n^2 + x_i^2)}{b})$$

$$= P(\Sigma (bX_i - \frac{1}{2}(bX_i)^2) \ge \frac{1}{2} x^2)$$

$$= P(\Sigma \xi_i \ge \frac{1}{2} x_n^2)$$

where $\xi_i = bX_i - \frac{1}{2}(bX_i)^2$, $Ee^{t\xi_i} < \infty$, $t \ge 0$ since $\xi_i \le \frac{1}{2}$

Let η_i be independent

$$P(\eta_i \le y) = \frac{E(e^{\lambda \xi_i} \mathbb{1}(\xi_i \le y))}{E e^{\lambda \xi_i}}, \lambda > 0$$

$$P(\sum \xi_i \ge \frac{1}{2}x_n^2) = \prod E e^{\lambda \xi_i} E e^{-\lambda \sum \eta_i} \mathbb{1}(\sum \eta_i \ge \frac{1}{2}x_n^2)$$

Note that $\sum E\eta_i = \frac{1}{2}x_n^2$

$$E\eta_i = \frac{1}{2}\frac{x_n^2}{n}$$

$$E\eta_i = \frac{E(\xi e^{\lambda \xi_i})}{E(e^{\lambda \xi_i})}$$

$$E(e^{\lambda \xi_i}) = E e^{\lambda(bX_i - \frac{1}{2}(bX_i)^2)}$$

$$= E e^{\lambda(bX_i - \frac{1}{2}(bX_i)^2)} \mathbb{1}(|bX_i| \le 1)$$

$$+ \underbrace{E e^{\lambda(bX_i - \frac{1}{2}(bX_i)^2)}}_{\le e^{-Y}} \mathbb{1}(|bX_i| > 1)$$

taylor expansion performed up to 3rd derivative

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + O(1)t^4$$

$$E(e^{\lambda \xi_i}) = 1 + \frac{1}{2}\lambda(\lambda - 1)b^2 \underbrace{EX_i^2}_{=1} + O(1)b^3 E|X_i|^3$$

$$E(\xi_1 e^{\lambda \xi_1}) = (\lambda - \frac{1}{2})b^2 EX_i^2 + O(1)b^3 E|X_i|^3$$

let $\lambda = 1$

$$W = \sum \eta_i$$

$$E\, e^{-\lambda W} \mathbb{1}(W \geq \tfrac{1}{2}x_n^2)$$

$$= E\, e^{-\lambda(W - EW)} e^{-\lambda EW} \mathbb{1}(W - EW \geq \tfrac{1}{2}x_n^2 - EW)$$

$$= e^{-\lambda EW} E\, e^{-\lambda \sqrt{Var(W)} \frac{(W-EW)}{\sqrt{Var(W)}}} \mathbb{1}\left(\frac{W-EW}{\sqrt{Var(W)}} \geq \frac{y_n}{\sqrt{Var(W)}}\right)$$

Claim: $\lambda^* > 0$ assume

$$\left| E e^{-\lambda^* W^*} \mathbb{1}(W^* \geq y) - E(e^{-\lambda^* Z} \mathbb{1}(Z \geq y)) \right|$$

$$\leq e^{-\lambda^* \min(y,0)} \sup_z \left| P(W^* \geq z) - (1 - \Phi(z)) \right|$$

$$E g(X) = g(0) + E \int_0^X g'(t)\,dt$$

$$= g(0) + E \int_0^\infty g'(t) \mathbb{1}(t \leq X)\,dt$$

$$E e^{-\lambda^* W^*} \mathbb{1}(W^* \geq y)$$

$$= E(\lambda^* \int_{W^*}^\infty e^{-\lambda^* t}\,dt) \mathbb{1}(W^* \geq y)$$

$$= \lambda^* E \int_{-\infty}^\infty e^{-\lambda^* t} \mathbb{1}(t > W^*) \mathbb{1}(W^* \geq y)\,dt$$

$$= \lambda^* \int_y^\infty e^{-\lambda^* t} P(W^* \geq y, W^* \leq t)\,dt$$

Check! $\Leftarrow$ $E(e^{-\lambda^* Z} \mathbb{1}(Z \geq y)) = \lambda^* \int_y^\infty e^{-\lambda^* t} P(Z \geq y, Z \leq t)\,dt$

Step 2: $P(S_n \geq x_n V_n) \leq (1-\Phi(x_n))(1+o(1))$

$x_n V_n \leq \frac{1}{2} \frac{b_n^2 V_n^2 + x_n^2}{b}$

not $(x_n-\varepsilon)$, to avoid additional terms

$\{S_n \geq x_n V_n\} \subset_{1} \{S_n \geq \frac{1}{2}(\underbrace{\frac{b^2 V_n^2 + x_n^2 - \varepsilon^2}{b}}_{})\}$

close to normal dist $1-\Phi(x_n) \sim \frac{e^{-x_n^2/2}}{\sqrt{2\pi}\, x_n}$

$\cup_{2} \{S_n \geq x_n V_n, S_n < \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2 - \varepsilon^2)}{b}\}$

Note

$$\frac{P(S_n \geq \frac{1}{2} \frac{b V_n^2 + x_n^2}{b})}{1-\Phi(x_n)} \to 1 \qquad b_n = \frac{x_n}{\sqrt{n}}.$$

$$\frac{P(S_n \geq \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2 - \varepsilon^2)}{b})}{1-\Phi(x_n)} \to 1 \quad \text{as } \varepsilon \to 0$$

2: $\{S_n \geq x_n V_n, S_n \leq \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2 - \varepsilon^2)}{b}\}$

$\subset \{S_n \geq x_n V_n, x_n V_n \leq \frac{1}{2} (\frac{b^2 V_n^2 + x_n^2 - \varepsilon^2}{b})\}$

$b^2 V_n^2 + x_n^2 - 2b x_n V_n \geq \varepsilon^2$

$(b V_n - x_n)^2 \geq \varepsilon^2$

$|b V_n - x_n| \geq \varepsilon$

$$|b^2V_n^2 - x_n^2| \geq \varepsilon(bV_n + x_n)$$

$$\subset \underbrace{\{b^2V_n^2 - x_n^2 \geq \varepsilon x_n\}}_{(*)}$$

$$\cup \{b^2V_n^2 - x_n^2 \leq -\varepsilon x_n\}$$

$$(*) = \{b^2V_n^2 - x_n^2 \geq 2x_n^2\} \cup$$

$$\{2x_n^2 \geq b^2V_n^2 - x_n^2 \geq \varepsilon x_n\}$$

$$P(bS_n \geq x(V_n^2 b^2)^{1/2},\ b^2V_n^2 \geq x_n^2 + \varepsilon n)$$

$$= P((bS_n, b^2V_n^2) \in A)$$

where $A = \{(s,t),\ s \geq x\sqrt{t},\ t \geq x_n^2 + \varepsilon x_n\}$

Recall Chebyshev's Inequality

$$P((bS_n, b^2V_n^2) \in A)$$

$$\leq E\exp(\lambda_1 bS_n - \lambda_2 b^2V_n^2)\, e^{-\inf_{(s,t)\in A}(\lambda_1 s - \lambda_2 t)}$$

where

$$\inf_{s \geq x\sqrt{t}} (\lambda_1 s - \lambda_2 t) = \inf(\lambda_1 x\sqrt{t} - \lambda_2 t) \to -\infty$$

$2x_n^2 \geq t \geq x_n^2 + \varepsilon x_n$ $\quad$ $2x_n^2 \geq t \geq x_n^2 + \varepsilon x_n$

in order to solve $-\infty$

trick truncate $(*)$

$X_1, \dots, X_n$ indep $EX_i = 0$ $EX_i^2 < \infty$, $S_n = \sum_i X_i$ $V_n^2 = \sum_i X_i^2$

$B_n^2 = \sum E X_i^2$ $\frac{S_n}{V_n} \xrightarrow{d} N(0,1)$

Interested in the conditions under which

$$\frac{P(\frac{S_n}{V_n} \geq x_n)}{1 - \Phi(x_n)} \to 1$$ holds.

Jing, S., Wang (2003)

$$\frac{P(\frac{S_n}{V_n} \geq x_n)}{1 - \Phi(x_n)} = 1 + O(1) \frac{(1+x_n^3) \sum_{i=1}^n E|X_i|^3}{B_n^3}$$ for $0 \leq x_n \leq \frac{B_n}{(\sum E|X_i|^3)^{1/3}}$

$|O(1)| \leq C$

$$x_n V_n \leq \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2)}{b}, \quad b = \frac{x_n}{V_n} \qquad b = \frac{x_n}{B_n}$$

$$P\left(S_n \geq \frac{1}{2} \frac{(b^2 V_n^2 + x_n^2)}{b}\right)$$

$$= P\left(\sum (b X_i - \tfrac{1}{2}(b^2 X_i^2)) \geq \frac{x_n^2}{2}\right)$$

[Proof: $P(S_n \geq x_n V_n) = P(S_n \geq x_n V_n, \max|X_i| \leq a)$ ← use truncate r.v.

$+ P(S_n \geq x_n V_n, \max|X_i| > a)$

$$\sum_i P(S_n \geq x_n V_n, |X_i| > a)$$

Note $\cdot \{X + Y \geq x\sqrt{Y^2 + Z^2}\} \subset \{X \geq \sqrt{x^2-1}\sqrt{Z^2}\}$ for $\forall x > 1$

$\cdot \{S_n \geq x_n V_n, |X_i| > a\}$

$$\subset \left\{ \frac{\sum_{j \neq i} X_j}{\sqrt{\sum_{j \neq i} X_j^2}} \geq (x_n^2 - 1)^{1/2}, |X_i| > a \right\}$$

$\cdot P(S_n \geq x_n V_n, |X_i| > a)$

$$\leq \underbrace{P\left(\frac{\sum_{j \neq i} X_j}{\sqrt{\sum_{j \neq i} X_j^2}} \geq \sqrt{x_n^2 - 1} \right)}_{\frac{e^{-\frac{x_n^2-1}{2}}}{x_n}} \underbrace{P(|X_i| > a)}_{\frac{E|X_i|^3}{a^3}}$$

$$, \quad a = \frac{\beta_n}{x_n + 1}$$

S, Zhou, W.X. (2016) (Journal: Bernoulli)

$\{\xi_i\}$ indep $E\xi_i = 0$, $\sum E\xi_i^2 = 1$

$$\frac{P\left(\frac{\sum_{i=1}^n \xi_i + D_1}{\sqrt{\sum_{i=1}^n \xi_i^2 (1+D_2)}} \geq x_n\right)}{1 - \Phi(x_n)} \to 1$$

useful for P-value

"Don't care too much abt where you publish, as long as the results is good."

"Don't care too much abt the order of authors, in prob. it always follows alphabetic order."

* Self-Normalizing

$X_1, \dots, X_n$ indep. $EX_i = 0$ $\quad S_n = \sum_{i=1}^n X_i \quad V_n = \sum_{i=1}^n X_i^2$

$$\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1) \quad \checkmark$$

$$\frac{S_n}{V_n} \xrightarrow{d} N(0,1) \quad ?$$

* Previous works

① iid

$$\frac{S_n}{V_n} \xrightarrow{d} N(0,1)$$

$\Leftrightarrow$ $EX_1^2 1\{|X_i| \le x\}$ slowly varying

$$\Leftrightarrow \quad \frac{\max_{1\le i\le n} |X_i|}{V_n} \xrightarrow{p} 0$$

② General independent symmetric

$$\Leftrightarrow \max \frac{|X_i|}{V_n} \xrightarrow{p} 0$$

* S. (2018)

If (i) $\max \frac{|X_i|}{V_n} \xrightarrow{p} 0$

(ii) $\sum_{i=1}^n (E(\frac{X_i}{V_n}))^2 \to 0$

(iii) $E\left(\frac{S_n}{\max(V_n, a_n)}\right) \to 0$

where $\sum_{i=1}^{n} E\left(\frac{X_i^2}{X_i^2 + a_n^2}\right) = 1$

then $\frac{S_n}{V_n} \xrightarrow{d} N(0,1)$

If (i) is satisfied, then (ii) & (iii) are necessary for self-normalized CLT

Conjecture: (i) (ii) (iii) are nece. & suff. for self-norm CLT