1. Concentration Inequality

$$W=\sum \xi_i \qquad \Delta=\Delta(\xi_1,\dots,\xi_n)$$

$$\left|P(W+\Delta \le z)-\Phi(z)\right|$$

$$\le 8\left(\sum E|\xi_i|^2+E|\Delta| + \sum_{i=1}^n E|\xi_i(\Delta-\Delta_i)|\right.$$

$\Delta_i=\Delta_i(\xi_j,\ j\neq i)$ two possible choices:

1° $\Delta_i=\Delta(\xi_1,\dots,\ 0,\xi_{i+1},\dots)$

2° $\Delta_i=\Delta(\xi_1,\dots,\xi_i^*,\dots)$

$\{\xi_i^*\}$ indep. copy of $\{\xi_i\}$

Found in chapter 10. U-Stat, multi-variate U-Stat.

2. Exchangeable Pair Approach.

$W=W_n$

- What is the limiting distribution?
- What is the error of approximation?

Let (W,W') be exchangeable.

i.e. $(W,W')\overset{d}{=}(W',W)$

trivial cases: $W \perp\!\!\!\perp W'$, $W=W'$

choice: W' close enough to W

- For asymmetric $h(x,y)$, we have

$$Eh(W,W')=0$$

joint dist.   asy.

$$= E\,h(W',W) = -E\,h(W,W')$$

$$\uparrow \; g(w',w)$$

check!

$$Eh(w',w) = \iint h(w',w)\, f(w',w)\, dw'dw$$

$$= \iint g(w',w) \left|\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right| dw'dw$$

- $EWf(W) = Ef'(W)$ Recall Stein's Method.

Assume

$$E(W-W'|W) = \lambda(W+R_1)$$

↳ Normal will satisfy this cond.

$$0 = E[(W-W')(f(W)+f(W'))] \qquad \text{define } \Delta = W-W'$$

trick

$$= 2E(W-W')f(W) - E[(W-W')(f(W)-f(W'))]$$

$$= 2E(E((W-W')f(W)|W)) - E(\Delta(f(W)-f(W-\Delta)))$$

$$= 2E(f(W)\lambda(W+R_1)) - E\left(\Delta\int_{-\Delta}^{0} f'(W+t)\,dt\right)$$

$$= 2\lambda\left[(E(Wf(W)) + E(R_1 f(W))) - E\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)\,dt\right]$$

$$\hat{K}(t) = \Delta\left(1(-\Delta \le t < 0) - 1(0 \le t \le -\Delta)\right)/2\lambda$$

$$EWf(W) = E\left(\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)\,dt\right) - Ef(W)R_1 \qquad \#$$

$$(*) \quad \int \hat{K}(t)\,dt = \frac{\Delta^2}{2\lambda}$$

• $|Eh(W) - E(h(Z))| \le 2\|h'\| \left(E\left|1 - \frac{E(\Delta^2(W))}{2\lambda}\right| + \frac{E|\Delta|^3}{\lambda} + E|R_1| \right)$

[Pf: $f'(W) - Wf(W) = h(W) - Eh(Z)$

$$E\,h(W) - Eh(Z)$$

$$= E(f'(W)) - EWf(W)$$

plug (*)

$$= Ef'(W) - E\int_{-\infty}^{\infty} f'(W+t)\hat{k}(t)\,dt \quad + ER_1 f(W)$$

$$- f'(W)$$

$$+ f(W)$$

$$= \underbrace{E\left(f'(W) - f'(W)\frac{\Delta^2}{2\lambda}\right)}_{f'(W)\left(1 - \frac{\Delta^2}{2\lambda}\right)\text{ not good enough}} - E\int_{-\infty}^{\infty}\left(\overbrace{f'(W+t) - f'(W)}^{|t|\,\|f''\|}\right)\hat{k}(t)\,dt + ER_1 f(W)$$

$$\le 2\|h\|\frac{E|\Delta|^3}{4\lambda}$$

]

$$E\left[f'(W)\left(1 - \frac{E(\Delta^2|W)}{2\lambda}\right)\right]$$

can be replaced by any σ-field generated by W (larger than W)

Recall: $\|f'\| \le \|h'\|$, $\|f''\| \le 2\|h'\|$, $\|f\| \le 2\|h'\|$

If $\lambda = 1$, or $\lambda = 0$, everything breaks down.

- $W = W(\xi_1, \dots, \xi_n)$ where ξ_i are independent

$$\boxed{W' = W(\xi_1, \dots, \xi_I', \dots, \xi_n)}$$

HOMEWORK ! VERIFY !

where I is a random index, indep. of other r.v.s

$P(I=k) = \frac{1}{n}, \quad 1 \le k \le n$

$\{\xi_I'\}$ indep. copy of $\{\xi_I\}$

then W & W' exchangeable.

- $W = W(\xi_1, \dots, \xi_n)$

$W' = W(\xi_1, \dots, \xi_I', \dots, \xi_n)$

where I is a random index indep of other r.v.s

$$\xi_i' \mid \xi_j, j \neq i \overset{d}{\sim} \xi_i \mid \xi_j, j \neq i$$

Then (W, W') is exchangeable.

- $W = \sum_{i=1}^{n} \xi_i$, ξ_i indep

  $W' = W - \xi_I + \xi_I'$

  $W - W' = \xi_I - \xi_I'$

Recall the assumption $E(W-W'|W) = E(\xi_I - \xi_I' \quad)$

$$E(W-W'|W) = E(\xi_I - \xi_I' \,|W)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(\xi_i - \xi_i' |W)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(E(\xi_i - \xi_i' |\sigma(\xi_j)_{1\le j\le N})|W)$$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} E(\xi_i')\right)$$

$$= \frac{1}{n} W$$

$\lambda = \frac{1}{n}$ very small

$$\frac{E|\delta|^3}{\lambda} = n\, E|\xi_I - \xi_I'|^3 = n \frac{1}{n}\sum_{i=1}^{n} E|\xi_i - \xi_i'|^3 \le 8\sum_{i=1}^{n} E|\xi_i|^3$$

$$E\left(\Delta^2 \mid \xi_j, 1\le j\le n\right) = \frac{1}{n}\sum_{i=1}^n E\left((\xi_i-\xi_i')^2 \mid \xi_j, 1\le j\le n\right)$$

$$= \frac{1}{n}\sum_{i=1}^n \left(\xi_i^2 + E(\xi_i'^2)\right)$$

$$= \frac{1}{n}\sum_{i=1}^n \left(\xi_i^2 + E(\xi_i^2)\right)$$

- If $X \in \mathcal{A}$, then $E(XY\mid\mathcal{A}) = X\,E(Y\mid\mathcal{A})$

- If X and $\mathcal{A}$ indep, then $E(X\mid\mathcal{A}) = EX$

Assume $\sum E\xi_i^2 = 1$

$$1 - \frac{E(\Delta^2 \mid \xi_j, 1\le j\le n)}{2\lambda} = 1 - \frac{1}{2}\sum(\xi_i^2 + E\xi_i^2) = \frac{1}{2}\left(1 - \sum \xi_i^2\right)$$

$$= \frac{1}{2}\sum_{i=1}^n \underbrace{\left(E\xi_i^2 - \xi_i^2\right)}_{\eta_i \text{ are indep. } E(\eta_i)=0}$$

$$E\left|\sum \eta_i\right| \le \sqrt{\operatorname{Var}\left(\sum \eta_i\right)} = \sqrt{\sum \operatorname{Var}(\eta_i)}$$

$$\le \sqrt{\sum E\eta_i^2} \le \sqrt{\sum E\xi_i^4}$$

need to assume $E\xi_i^4 < \infty$

Exchangeable pair can give us optimal rate, but the condition may not be optimal.

- Assume $E(W-W' \mid W) = \lambda(W+R_1)$ $\quad \Delta \triangleq W-W'$

$$E((W-W')^2 \mid W) = 2\lambda(1+R_2)$$

then $$|P(W \le z) - \Phi(z)| \le E|R_1| + E|R_2| + \boxed{\frac{E\left|E(\Delta|\Delta| \mid W)\right|}{\lambda}}$$

$$\frac{E\left|E(\Delta\Delta^* \mid W)\right|}{\lambda}$$ can be replaced

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where $\Delta^* \ge |\Delta|$

$$\Delta^*(W,W') = \Delta^*(W',W)$$

term may be bounded by $E|R_2|$, idea:

$\Delta = \Delta^+ - \Delta^-$ $\quad E(\Delta^2)/2\lambda = 1$

$$E(\Delta^2 \mid W) = E(\Delta^{+2} \mid W) + E(\Delta^{-2} \mid W)$$

$$E|R_2| = \frac{1}{2\lambda} E\left|E(\Delta^{+2} \mid W) - E(\Delta^{+2}) + E(\Delta^{-2} \mid W) - E(\Delta^{-2})\right|$$

$$\frac{E\left|E(\Delta|\Delta| \mid W)\right|}{\lambda} = \frac{1}{\lambda} E\left|E(\Delta^{+2} \mid W) - E(\Delta^{+2}) - \left(E(\Delta^{-2} \mid W) - E(\Delta^{-2})\right)\right|$$

$$f'(w) - w f(w) = \mathbb{1}(w \le z) - \Phi(z)$$

(Cont. at Lec 7)