Recall Exchangeable Pair (W, W')

$$\Delta = W - W'$$

$$E(W - W' \mid W) = \lambda (W + R)$$

Then

$$|P(W \le z) - \Phi(z)|$$

$$\le E\left|1 - \frac{1}{2\lambda} E(\Delta^2 \mid W)\right| + E(R) + \frac{1}{\lambda} E\left|E(\Delta\Delta^* \mid W)\right|$$

$$\Delta^* \ge |\Delta|$$

Ref: Ann. Prob. 47 (2019) 61–108 Shao, Zhang.

[Proof:

$$0 = E(W - W')\left(f(W) + f(W')\right)$$

$$= 2\lambda \left[E(Wf(W)) + E(Rf(W)) - E\int_{\infty}^{\infty} f'(W+t)\hat{K}(t)\,dt \right]$$

$$\hat{K}(t) = \frac{\Delta\left(1(-\Delta \le t \le 0) - 1(0 \le t \le -\Delta)\right)}{2\lambda}$$

$\leftarrow \int \hat{K}(t)\,dt = \frac{\lambda}{2\sigma^2}$

Let $f'(w) - wf(w) = 1(w \le z) - \Phi(z)$

$$P(W \le z) - \Phi(z) = E(f'(W)) - E(Wf(W))$$

$|f(w)| \le 1$

$$= E\, f'(W) - E\int f'(W+t)\hat{K}(t)\,dt + \underbrace{E(Rf(W))}_{\le E|R|}$$

$|f'(w)| < 1$

$-f'(W) + f'(W)$

↑ trick

$$= E\Big[\underbrace{f'(W)\big(1-\tfrac{\Delta^2}{2\lambda}\big)\Big]}_{\substack{= E\left\{f'(W)\left(1-\frac{E(\Delta^2 \mid W)}{2\lambda}\right)\right\} \\ \leq E\left|1-\frac{E(\Delta^2 \mid W)}{2\lambda}\right|}} - E\underbrace{\int \big(f'(W+t)-f'(W)\big)\hat{K}(t)\,dt}_{\text{Main Problem} \star}$$

$$E\int (f'(W+t)-f'(W))\hat{K}(t)\,dt$$

by def.

$$= E\int \big[(W+t)f(W+t) - Wf(W)\big]\hat{K}(t)\,dt$$

$$+ E\int \big(1(W+t\leq z) - 1(W\leq z)\big)\hat{K}(t)\,dt \quad \text{— ①}$$

$$① = E\Big(\underbrace{\Delta\int_{-\Delta}^{0}}_{2\lambda}\big(\underbrace{1(W+t\leq z) - 1(W\leq z)}_{\text{given } W,\ \text{is decreasing function of } t}\big)\,dt\Big)$$

→ when $\Delta > 0$
max. taken at $t = -\Delta$

→ when $\Delta < 0$
min. taken at $t = -\Delta$

$$0 \leq \int_{-\Delta}^{0}\big(1(W+t)\leq z - 1(W\leq z)\big)\,dt$$

$$\leq \big(1(\underbrace{W-\Delta}_{W' \text{ by def.}} \leq z) - 1(W\leq z)\big)\Delta$$

$$\leq \frac{E\left(|\Delta| \, \Delta \left(1(W' \leq z) - 1(W \leq z)\right)\right)}{2\lambda} \qquad (W, W') \overset{d}{=} (W', W)$$

$$= \frac{1}{2\lambda} \left[E|\Delta|(-\Delta)\, 1(W \leq z) - E\left(|\Delta| \, \Delta \, 1(W \leq z)\right)\right]$$

$$= -\frac{1}{\lambda} E\left(|\Delta| \Delta \, 1(W \leq z)\right)$$

$$= -\frac{1}{\lambda} E\left(1(W \leq z) E\left(|\Delta| \Delta \mid W\right)\right]$$

$$\leq \frac{1}{\lambda} E\left| E\left(|\Delta| \Delta \mid W\right)\right| \qquad \star$$

Recall $\quad Wf(W) \uparrow \qquad |Wf(W)| \leq 1$

$W = W_n$ Statistics.

- What is the limiting distribution?

If $Y \overset{pdf}{\sim} p(y)$, $p(y) > 0$

we have $E\left(\frac{(p(y) f(y))'}{p(y)}\right) = p(y) f(y) \Big|_{-\infty}^{\infty} = 0$

true for $\forall f$?

Stein's identity:

$$E\left(\frac{(f(y)p(y))'}{p(y)}\right) = 0, \qquad E\left(f'(y)\right) + E\left(\frac{f(y)p'(y)}{p(y)}\right) = 0$$

Stein's Equation: Given h assume p is given

$$\frac{(f(y)p(y))'}{p(y)} = h(y) - E(h(Y))$$

integral

$$\Rightarrow f(y) = \frac{1}{p(y)} \int_{-\infty}^{y} p(t)\left(h(t) - E(h(Y))\right) dt$$

$$= -\frac{1}{p(y)} \int_{y}^{-\infty} p(t)\left(h(t) - E(h(Y))\right) dt$$

Properties: under certain conditions.

- $\|f\| \le c_1 \|h\|$, $\|f'\| \le c_2 \|h\|$
- $\|f\| \le c_3 \|h'\|$, $\|f'\| \le c_4 \|h'\|$, $\|f''\| \le c_5 \|h'\|$

Theorem: Let (W, W') exchangeable.

Assume: ① $E(W-W'|W) = \lambda(g(W) + R(W))$

② $\Delta = W - W'$

$$\frac{E(\Delta^2|W)}{2\lambda} \to 1 \text{ in prob.}$$

What if

$E(\Delta^2|W)/2\lambda = v(W)$

③ $\frac{E|\Delta|^3}{\lambda} \to 0$, $E|R| \to 0$

Then: $W \xrightarrow{d} Y$, where Y has pdf.

$$p(y) = C \cdot e^{-G(y)}, \quad G(y) = \int_0^y g(t)\,dt$$

$p(y) = C \frac{1}{v(y)} e^{-G(y)}$

$G(y) = \int_0^y \frac{g(t)}{v(t)}\,dt$

[Pf: $0 = E(W-W')(f(W)-f(W'))$

$= 2Ef(W)(W-W') - E(\Delta(f(W)-f(W')))$

$= 2E[f(W)E(W-W'|W)] - E(\Delta(f(W)-f(W-\Delta)))$

$$\lambda g(w) \qquad = E\int \delta \int_{-\delta}^{0} f'(w+t)\,dt$$

$$= 2\lambda\Big[E\big(f(w)\,g(w)\big) - \frac{E}{2\lambda}\Big(\delta\int_{-\infty}^{\infty} f'(w+t)\big(\mathbb{1}(-\delta\le t\le 0) - \mathbb{1}(0\le t\le -\delta)\big)\,dt\Big]$$

$$\frac{\big(f(y)\,g(y)\big)'}{p(y)} = h(y) - Eh(y) = f'(y) + \frac{p'(y)}{p(y)}\,f(y)$$

guess: $g(y) = \frac{p(y)}{p'(y)} \Leftarrow \big(\ln(p(y))\big)'$

$$Eh(w) - Eh(y) = E\Big(f'(w) + \frac{p'(w)}{p(w)}\cdot f(w)\Big)$$

Clerie-Weiss Model

$\sigma_1, \sigma_2, \cdots, \sigma_n, \pm 1$

the joint pdf of $(\sigma_1, \dots, \sigma_n) = \sigma$

$$p(\sigma) = C_{n,\beta} \exp\left(\frac{\beta}{n} \sum_{1 \le i \le j \le n} \sigma_i \sigma_j\right)$$

- $0 < \beta < 1$ $\quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_i \Rightarrow N(0, \frac{1}{1-\beta})$
- $\beta = 1$ $\quad \frac{1}{n^{3/4}} \sum_{i=1}^{n} \sigma_i \xrightarrow{d} Y$
- $Y \overset{pdf}{\sim} Ce^{-y^4/12}$

Define $W \triangleq \sum_{i=1}^{n} \sigma_i \cdot \frac{1}{n^{3/4}}$

construct W' to a ex. pair

$$W' = W - \frac{1}{n^{3/4}} \sigma_I + \frac{1}{n^{3/4}} \sigma_I'$$

$$\sigma_i' | \sigma_j, j \ne i \overset{d}{\sim} \sigma_i | \sigma_j, j \ne i$$

$$|W - W'| \le \frac{1}{n^{3/4}} \text{ since } |\sigma_i| \le 1$$

- $E(W - W' | W) = \frac{1}{3} n^{-3/2} \left(W^3 + \frac{O(1)}{\sqrt{n}}\right)$

- $E\left|1-\frac{n^{3/2}}{2}E(\Delta^2|W)\right| \le \delta n^{-1/2}$

$\lambda = n^{-3/2}$, $g(w) = \frac{1}{3}w^3$ (use the theorem above #)