(*) ■ $X_1, X_2, \dots$ iid $Ee^{t_0 X_1} < \infty$, $t_0 > 0$

then $\forall x > E(X_1)$

$$P\left(\frac{S_n}{n} \geq x\right)^{\frac{1}{n}} \to \inf_{t \geq 0} e^{-tx} E e^{tX_1}$$

$T_n \Leftrightarrow \frac{S_n}{V_n}$

■ $V_n^2 = \sum_1^n X_i^2$, $x > \frac{EX_1}{\sqrt{EX_1^2}}$ if $EX_1^2 = \infty$ then RHS $= 0$

(and/or $EX_1 = \infty$)

$$P\left(\underbrace{\frac{S_n}{V_n \sqrt{n}}} \geq x\right)^{\frac{1}{n}} \to ?$$

control order of denom. to be $O(n)$

■ $P\left(\frac{S_n}{V_n^2} \geq x\right) = P(S_n - xV_n^2 \geq 0)$ $(x > 0)$ assume

no stat. meaning $= P\left(\sum_{i=1}^n \underbrace{(X_i - xX_i^2)}\right)$

Y_i has upper bound, therefore $\exists$ mgf. $< \infty$

$X_i - xX_i^2 \leq C_x < \infty$

$$P\left(\sum (X_i - xX_i^2) > 0\right)^{\frac{1}{n}} \xrightarrow{(*)} \inf_{t > 0} Ee^{t(X_1 - xX_1^2)}$$ true for $\forall$ r.v. X

$\forall\, 0 > E(X_1 - xX_1^2)$, $x > \frac{E(X_1)}{E(X_1^2)}$, $x > 0$

■ Recall Cauchy Inequality: $xy \le \frac{1}{2}(x^2+y^2)$

$$xy = x\sqrt{c}\cdot\frac{y}{\sqrt{c}} \le \frac{1}{2}(x^2c + y^2/c)$$

$$xy = \frac{1}{2}\inf_{c>0}\left(\frac{c^2x^2+y^2}{c}\right)$$

write $x\sqrt{n}\cdot V_n = \frac{1}{2}\inf_{c>0}\left(\frac{c^2V_n^2+x^2n}{c}\right)$

$$\begin{aligned}
P(S_n \ge xV_n\sqrt{n}) &= P\left(S_n \ge \frac{1}{2}\inf_{c>0}\left(\frac{c^2V_n^2+x^2n}{c}\right)\right)\\
&= P\left(\bigcup_{c>0}\{2cS_n \ge c^2V_n^2+x^2n\}\right)\\
&= P\left(\bigcup_{c>0}\left\{\sum_{i=1}^{n} 2cX_i - c^2X_i^2 \ge x^2n\right\}\right)
\end{aligned}$$

$$P(S_n \ge xV_n\sqrt{n})^{\frac{1}{n}} \ge \sup_{c\ge 0} P\left(\sum_i (2cX_i - c^2X_i^2)/n \ge x^2\right)^{\frac{1}{n}}$$

by $(*)$ $\to \sup_{c\ge 0}\inf_{t\ge 0} e^{-tx^2}E\left[e^{t(2cX_1-c^2X_1^2)}\right]$

upper bound is also

$a_{n,i} \ge c_i$, $n\to\infty$

$a_{n,i}^{\frac{1}{n}} \to \max c_i$

(Shao, 1977) If $EX=0$, $EX_1^2 1(|X_1| \le x)$ slowly varying

then $\forall x_n \to \infty$, $x_n = o(\sqrt{n})$

$$\ln P\left(\frac{S_n}{V_n} \ge x\right) \sim -\frac{x_n^2}{2} \quad \text{i.e. } \forall\, 0<\varepsilon<1$$

$$e^{-(1+\varepsilon)x_n^2/2} \le P\left(\boxed{\frac{S_n}{V_n}} \ge x_n\right) \le e^{-\frac{(1-\varepsilon)x_n^2}{2}}$$

← more robust becuz no assumption on moment existence

$$P\left(\boxed{\frac{S_n}{\sigma\sqrt{n}}} \ge x_n\right) \to 0 \iff Ee^{t_0 X_1} < \infty$$

■ $S_n = \sum X_i 1(|X_i| \le z_n) + \sum X_i 1(|X_i| > z_n)$

$\triangleq \sum X_{i,1} + \sum X_{i,2}$

$P\left(\frac{S_n}{V_n} \ge x_n\right)$

$P(X+Y \ge x) \le P(X \ge (1-\varepsilon)x) + P(Y \ge \varepsilon x)$

$$\le \underbrace{P\left(\frac{\sum X_{i,1}}{V_n} \ge (1-\varepsilon)x_n\right)}_{2} + \underbrace{P\left(\frac{\sum X_{i,2}}{V_n} \ge \varepsilon x_n\right)}_{1}$$

1 $$P\left(\frac{\sum X_{i,2}}{V_n} \ge \varepsilon x_n\right) = P\left(\frac{\sum X_i 1(|X_i| > z_n)}{V_n} \ge \varepsilon x_n\right)$$

$$\le P\left(\left(\sum 1(|X_i| > z_n)\right)^{\frac{1}{2}} \ge (\varepsilon x_n)^2\right)$$

$\sum a_i b_i \le \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$

ε_i indep. $P(\varepsilon_i = 1) = p_i$ $\quad P(\varepsilon_i = 0) = 1-p_i$

$P(\sum \varepsilon_i \ge x) \le \left(\frac{3\sum_{i=1}^n p_i}{x}\right)^x$

$$\le \left(\frac{3n\, P(|X_1| > z_n)}{\varepsilon^2 x_n^2}\right)^{\varepsilon^2 x_n^2} \le e^{-2x_n^2} \quad 1 \quad \frac{n}{x_n^2 z_n^2} \le C$$

Here we first assume Second moment exists $EX_i^2 < \infty$

$$P(|X_i| > \xi_n) \le \frac{EX_i^2 1(|X_i| > \xi_n)}{\xi_n^2}$$

$$P(|X_i| > \xi_n) = o\left(\frac{1}{\xi_n}\right)$$

Then when $l(x) = EX_i^2 1(|X_i| \le x)$ slowly varying,

$$P(|X_i| \ge x) = \frac{o(l(x))}{x^2}$$

$$P(V_n^2 < EV_n^2 - \varepsilon) \to 0 \quad ☆$$

② $P(\Sigma X_{i,1} \ge (1-\varepsilon) x_n V_n)$

$$\le P\left(\Sigma X_{i,1} \ge (1-\varepsilon) x_n V_n,\ V_n \ge (1-\varepsilon)^{\frac{1}{2}} \sqrt{n}\right)$$
$$+ P\left(\Sigma X_{i,1} \ge (1-\varepsilon) x_n V_n,\ V_n < (1-\varepsilon)^{\frac{1}{2}} \sqrt{n}\right)$$

Benette Hoeffding's Inequality)

$X_i \le a \quad EX_i \le 0$

$$P(\Sigma X_i \ge x) \le \exp\left(-\frac{x^2}{2\Sigma EX_i^2 + 2ax}\right)$$

$$\le P\left(\Sigma X_{i,1} \ge (1-\varepsilon)^{3/2} x_n \sqrt{n}\right) \le \exp\left(-\frac{x_n^2 \cdot n(1-\varepsilon)^3}{2n + x_n \sqrt{n} \xi_n}\right)$$
$$+ P(V_n^2 < n - n\varepsilon)$$

② $x_n \sqrt{n} \xi_n \le \varepsilon n$

$$\le P\left(\Sigma X_i^2 1(|X_i| \le \xi_n) \le n - \varepsilon n\right)$$

$$\le P\left(\Sigma X_i^2 1(|X_i| \le \xi_n) \le \Sigma E X_i^2 1(|X_i| \le \xi_n) - \frac{\varepsilon n}{2}\right)$$

$$\le \exp\left(-\frac{\varepsilon^2 n^2}{\boxed{8 n E X_i^4 1(|X_i| \le \xi_n)}}\right) \le \exp(-2x_n^2) \quad \#$$

$EX_i^4 1(|X_i| \le x) = o(x^2)$ as $EX_i^2 < \infty$.

$$\geqslant \frac{n}{z_n^2 o(1)} \geq \frac{x_n^2}{O(1)}$$ choose $z_n = \varepsilon\sqrt{n}/x_n$

[redacted] $$xy \le \frac{1}{2}\left(\frac{x^2 b^2 + y^2}{b}\right), \; b>0, \; b = \frac{y}{x}$$

$$x_n V_n \le \frac{1}{2}\left(\frac{b^2 V_n^2 + x_n^2}{b}\right), \text{ let } b = \frac{x_n}{V_n}$$

when $EX_i^2 = 1$

$b_n = \frac{x_n}{\sqrt{n}}, \; b_n \to 0$

$$P(S_n \geq x_n V_n)$$

$$\geqslant P\left(S_n \geq \frac{1}{2}\left(\frac{b^2 V_n^2 + x_n^2}{b}\right)\right)$$

$$= P\left(bS_n - \frac{1}{2} b^2 V_n^2 \geq \frac{x_n^2}{2}\right)$$

$$= P\Big(\sum \underbrace{\left(bX_i - \tfrac{1}{2}(bX_i)^2\right)}_{\xi_i} \geq \frac{x_n^2}{2}\Big)$$

essential that $b \to 0$

can estimate mgf then

$\{\eta_i\}$ indep.

$$P(\eta_i \le y) = \frac{E(e^{\lambda \xi_i} 1(\xi_i < y))}{E e^{\lambda \xi_i}}$$

$$P\left(\sum \xi_i \geq \frac{x_n^2}{2}\right) = E\left(e^{\lambda \xi_i}\right)^n E\left(e^{-\lambda \sum y_i} \mathbb{1}\left(\sum y_i \geq \frac{x_n^2}{2}\right)\right)$$

want to control $E(\sum y_i)$ not too far away from $x_n^2/2$

$$2 \geq \frac{\sum y_i - E y_i}{\sqrt{\sum \operatorname{Var} y_i}} \geq \boxed{\frac{\frac{x_n^2}{2} - E y_i}{\sqrt{n \operatorname{Var}(y_i)}}} \; (**)$$

$$\geq E\left(e^{\lambda \xi_i}\right)^n e^{-\lambda\left(2\sqrt{\sum \operatorname{Var} y_i} + \sum E y_i\right)} \cdot P(2 \geq \cdots \geq -2)$$

$EY = 0 \quad \operatorname{Var} Y = 1$

$P(2 \geq Y \geq -2) = 1 - P(|Y| > 2) \geq 1 - \frac{1}{4} = \frac{3}{4}$

$\lambda = 1 \pm \varepsilon$, choose λ such that $(**) < -2$

then $\dfrac{\sum y_i - E y_i}{\sqrt{\sum \operatorname{Var} y_i}}$ bounded.