Homework 4 - STAT 231

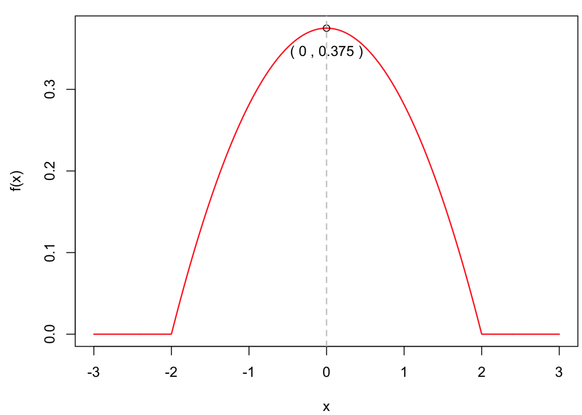
Suggested Solution [*,†]

Due in class: 24 Sep 2019

Problem 1-5 are from the Devore textbook.

1. **(Chapter 4: Problem 3)**

   (a) The sketch is as follows



   (b) $P(X > 0) = \int_0^\infty f(x)dx = \int_0^2 .09375(4 - x^2)dx = .5$

   (c) $P(-1 < X < 1) = \int_{-1}^1 f(x)dx = \int_{-1}^1 .09375(4 - x^2)dx = .6875$

   (d) $P(X < -.5 \text{ or } X > .5) = \int_{-\infty}^{-.5} f(x)dx + \int_{.5}^\infty f(x)dx \overset{(symmetry)}{=} 2\int_{.5}^2 f(x)dx = .6328$

2. **(Chapter 4: Problem 11)**

   (a) $P(X \leq 1) = F(1) = 1/4 = .25$

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†Please inform me if there is any error or possible improvement

(b) $P(.5 \le X \le 1) = P(X \le 1) - P(X < .5) = F(1) - F(.5) = 3/16 = .1875$

(c) $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 7/16 = .4375$

(d) Solve $F(\tilde{\mu}) = .5$. Note $F(x) \ne .5$ when $x < 0, x \le 2$. Therefore the solution is only possible to be in the range $[0, 2)$. Solve $\tilde{\mu}^2/4 = .5$, we have $\tilde{\mu} = \sqrt{2} = 1.414$

(e) $f(x) = \frac{dF(x)}{dx} = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x < 2 \\ 0, & 2 \le x \end{cases}$

(f) $E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x \cdot \frac{x}{2} dx = \frac{x^3}{6}|_0^2 = 4/3 = 1.333$

(g) $V(X) = E(X^2) - E^(X) = \int_0^2 x^2 f(x)dx - E^(X) = \frac{x^4}{8}|_0^2 - E^2(X) = 2 - (\frac{4}{3})^2 = 2/9 = .222$; $\sigma(X) = \sqrt{(V(X))} = \sqrt{2}/4 = .471$

(h) $Eh(X) = E(X^2) = 2$

3. **(Chapter 4: Problem 36)** Let X denote the size of the droplet, $X \sim N(\mu, \sigma^2)$ where $\mu = 1050, \sigma = 150$.

(a) $P(X < 1500) = P(\frac{X-\mu}{\sigma} < \frac{1500-1050}{150}) = P(Z < 3) = .9987$;
$P(X \ge 1000) = P(\frac{X-\mu}{\sigma} \ge \frac{1500-1000}{150}) = P(Z \ge -1/3) = .6293$

(b) $P(1000 < X < 1500) = P(X < 1500) - P(X \le 1000) = P(X < 1500) - (1 - P(X \ge 1000) = .628$

(c) Solve $P(X \le X^*) = 2\%$. $P(X < x^*) = P(\frac{X-\mu}{\sigma} \le \frac{x^*-1050}{150}) = P(Z \le \frac{x^*-1050}{150}) = 2\%$. Look up the table for .02, we have $\frac{x^*-1050}{150} = -2.05$, then $x^* = 742.5$. Therefore the size of droplet is smaller than 742.5 um for the lowest 2%.

(d) For each droplet, $P(X \ge 1500) = 1 - .9987 = .0013$. Define Y to be the number of droplets exceeding 1500 um. $Y \sim Bin(n, p)$, where $n = 5, p = .0013$.
$P(Y = 2) = \binom{5}{2} p^2(1-p)^3 = 10 \times .0013^2 \times .9987^3 = 1.68 \times 10^{-5}$

4. **(Chapter 4: Problem 40)**$X \sim N(\mu, \sigma)$, where $\mu = 43, \sigma = 4.5$

(a) $P(X \le 40) = P(\frac{X-\mu}{\sigma} \le \frac{40-\mu}{\sigma}) = P(Z \le \frac{-3}{4.5}) = .2514$
$P(X \ge 60) = P(\frac{X-\mu}{\sigma} \ge \frac{60-\mu}{\sigma}) = P(Z \ge \frac{17}{4.5}) = P(Z < -3.78) < .0002(\approx 0)$

(b) $P(X \ge x^*) = P(\frac{X-\mu}{\sigma} \ge \frac{x^*-\mu}{\sigma}) = P(Z \ge \frac{x^*-\mu}{\sigma}) = .75 \iff P(Z \le \frac{x^*-\mu}{\sigma}) = .25$. Look up .25 in the table, we have $\frac{x^*-\mu}{\sigma} = -.67 \implies x^* = \mu - .67\sigma = 39.99$

5. **(Chapter 4: Problem 42)** $X \sim N(\mu, \sigma^2)$. We need to solve $95\% = P(\mu - .1 \leq X \leq \mu + .1) = P(\frac{-.1}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{.1}{\sigma}) = P(Z \leq \frac{.1}{\sigma}) - P(Z \leq \frac{-.1}{\sigma}) = 2(P(Z \leq \frac{.1}{\sigma}) - 1/2)$ $\implies P(Z \leq \frac{.1}{\sigma}) = .975 \implies \frac{.1}{\sigma} = 1.96 \implies \sigma = .05102$

6. (a) $P(X < 1) = \int_{-\infty}^{1} f(x)dx = \int_0^1 \frac{x}{4}dx = \frac{x^2}{8}\Big|_0^1 = \frac{1}{8}$

   (b) $P(X < 2.5) = \int_{-\infty}^{2.5} f(x)dx = \int_0^2 \frac{x}{4}dx + \int_2^{2.5} \frac{1}{2}dx = \frac{x^2}{8}\Big|_0^2 + \frac{x}{2}\Big|_2^{2.5} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

   (c) $F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{8}, & 0 \leq x < 2 \\ -\frac{1}{2} + \frac{x}{2}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$

   (d)

$$\begin{aligned} EX = \int_{-\infty}^{\infty} xf(x)dx &= \int_0^2 \frac{x^2}{4}dx + \int_2^3 \frac{x}{2}dx \\ &= \frac{x^3}{12}\Big|_0^2 + \frac{x^2}{4}\Big|_2^3 \\ &= \frac{2}{3} + \frac{5}{4} = 23/12 = 1.9167 \end{aligned}$$

7. (a) $X \sim N(\mu, \sigma^2), \mu = 18000, \sigma = \sqrt{2500} = 50$. $P(|X - \mu| > 100) = P(\frac{X-\mu}{\sigma} > \frac{100}{\sigma}) + P(\frac{X-\mu}{\sigma} < -\frac{100}{\sigma}) = P(Z > 2) + P(Z < -2) = 1 - P(Z \leq 2) + P(Z < -2) = .0456$

   (b) $\tilde{X} \sim N(\mu, \tilde{\sigma}^2)$. $P(|\tilde{X} - \mu| > 100) = P(\frac{\tilde{X}-\mu}{\tilde{\sigma}} > \frac{100}{\tilde{\sigma}}) + P(\frac{\tilde{X}-\mu}{\tilde{\sigma}} < -\frac{100}{\tilde{\sigma}}) = P(Z > \frac{100}{\tilde{\sigma}}) + P(Z < -\frac{100}{\tilde{\sigma}}) = 1 - P(Z \leq \frac{100}{\tilde{\sigma}}) + P(Z < -\frac{100}{\tilde{\sigma}}) = 2(1 - P(Z \leq \frac{100}{\tilde{\sigma}})) = .01 \implies P(Z \leq \frac{100}{\tilde{\sigma}}) = 1 - .01/2 = .995 \implies \frac{100}{\tilde{\sigma}} = 2.575 \implies \tilde{\sigma} = 38.83$, and the variance $\tilde{\sigma}^2 = 1508.15$

8. (a) $Y \sim Bin(1000, 30\%)$
   $P(Y \geq 310) = 1 - P(Y \leq 309) = 1 - .7448 = .2552$

   (b) $X \sim N(\mu, \sigma^2), \mu = np = 300, \sigma^2 = np(1-p) = 210, \sigma = \sqrt{210}$
   $P(X \geq 310) = P(\frac{X-\mu}{\sigma} \geq \frac{210-\mu}{\sigma}) = P(Z \geq .69) = 1 - .7549 = .2451$

   (c) In our case, the approximation is not bad. Check the criteria $np = 300 \geq 10, nq = 700 \geq 10$ both satisfied.