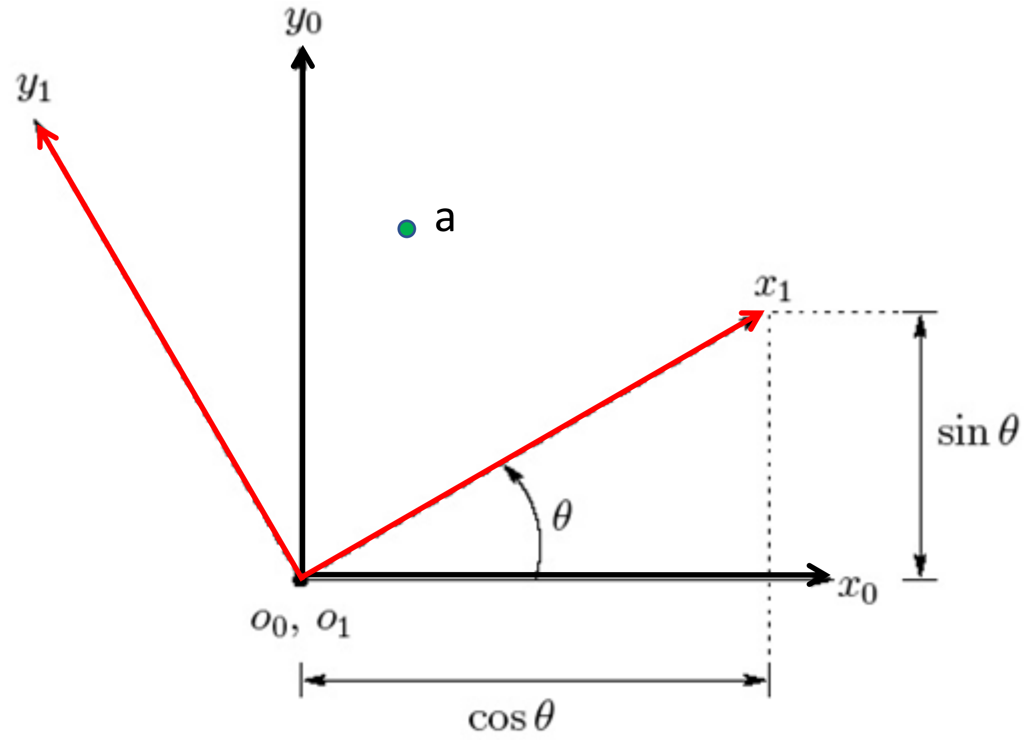
Intro to Robotics

Lecture 1

Transformation

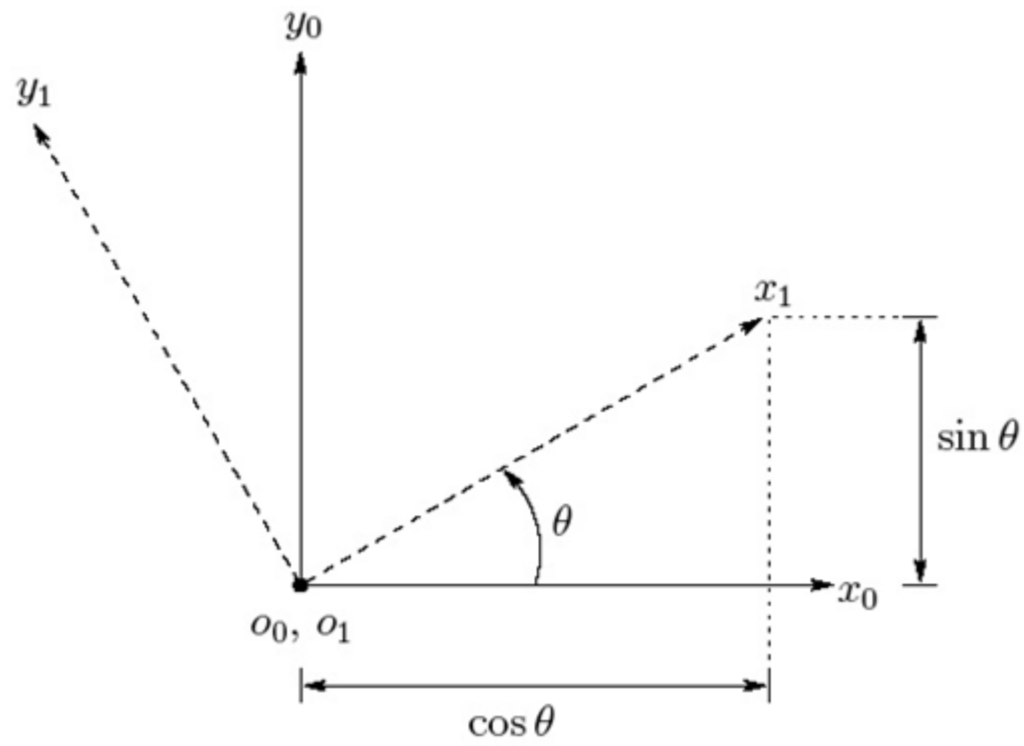


$x_1^0 \quad y_1^0$?

$a_0^1 = \left(a_{x1},\ a_{y1} \right)$

a_0^0 ?

Coordination Rotation in 2D

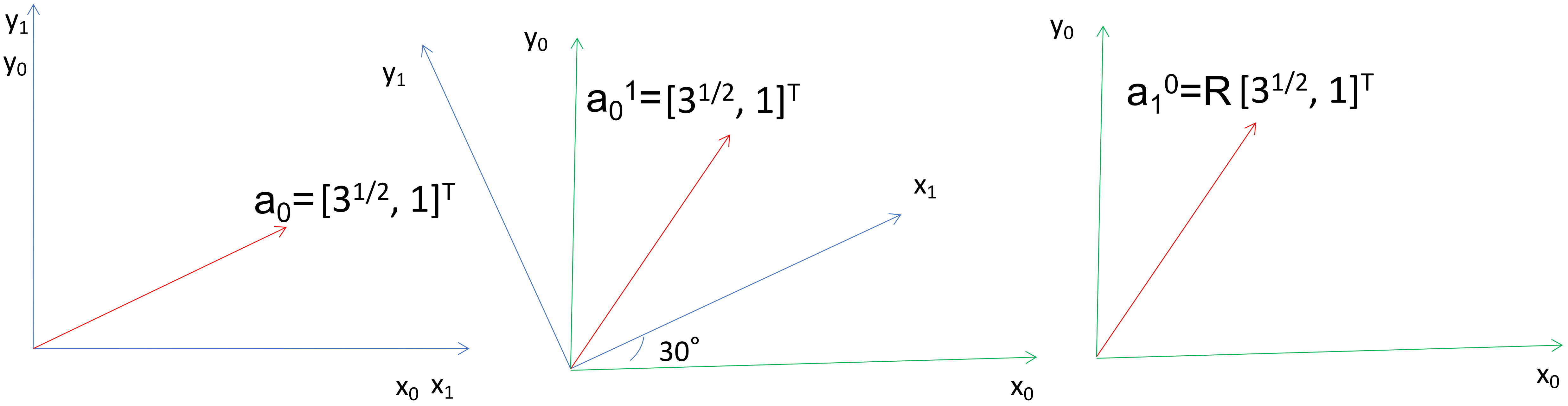


$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example



Rotation Matrix

$$R_0^1 = (R_1^0)^T$$

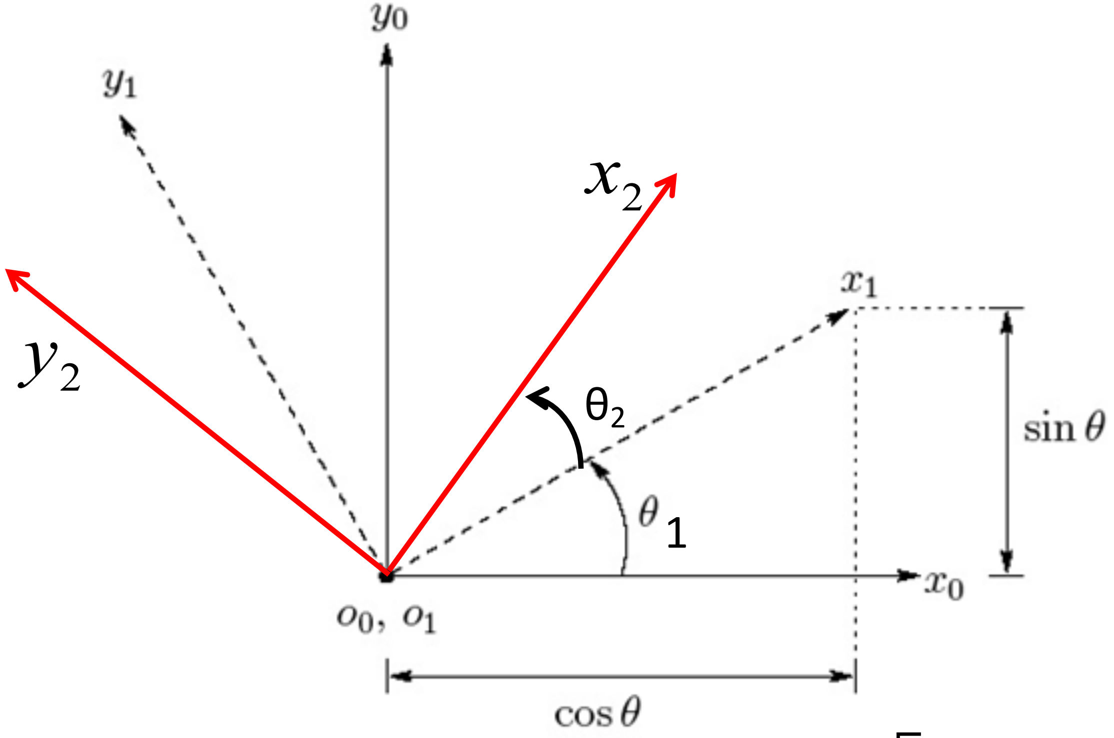
$$(R_1^0)^T = (R_1^0)^{-1}$$

$$\det(R_0^1) = 1$$

$$R_1^0 = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Orthogonal}}$$

Continue Rotation

Continue rotate θ_1, then θ_2



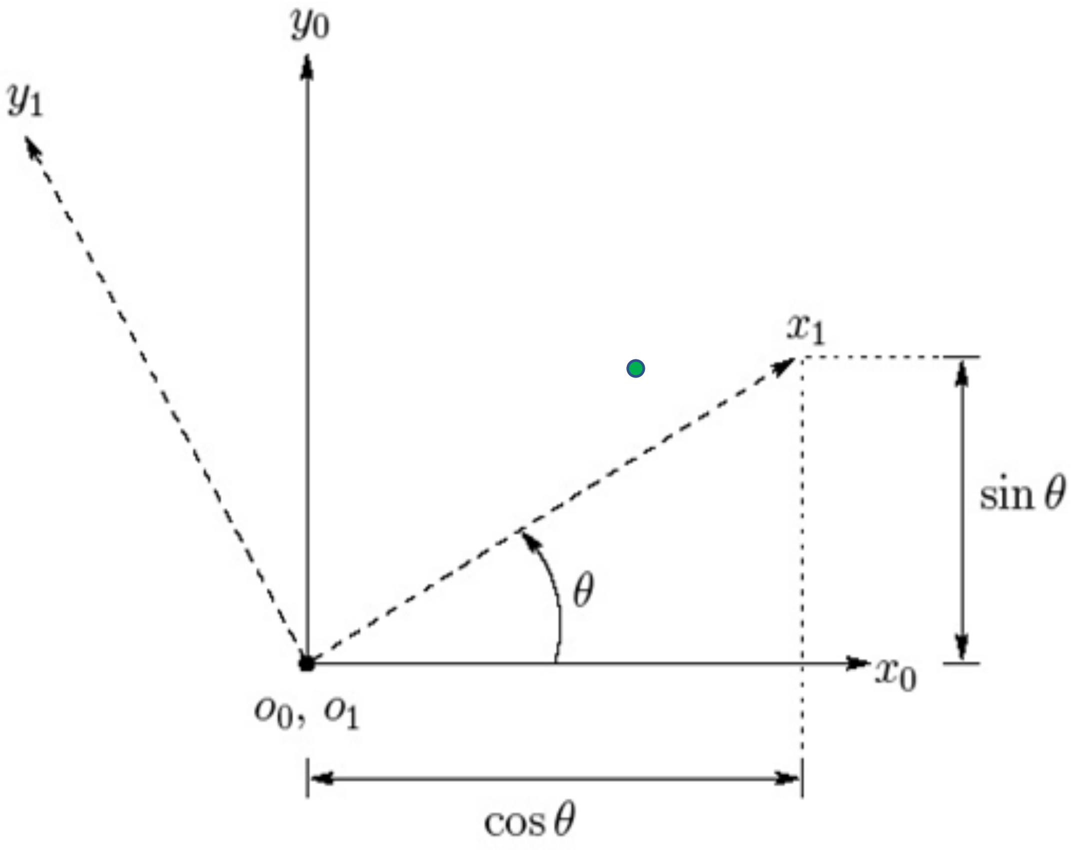
$$R_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

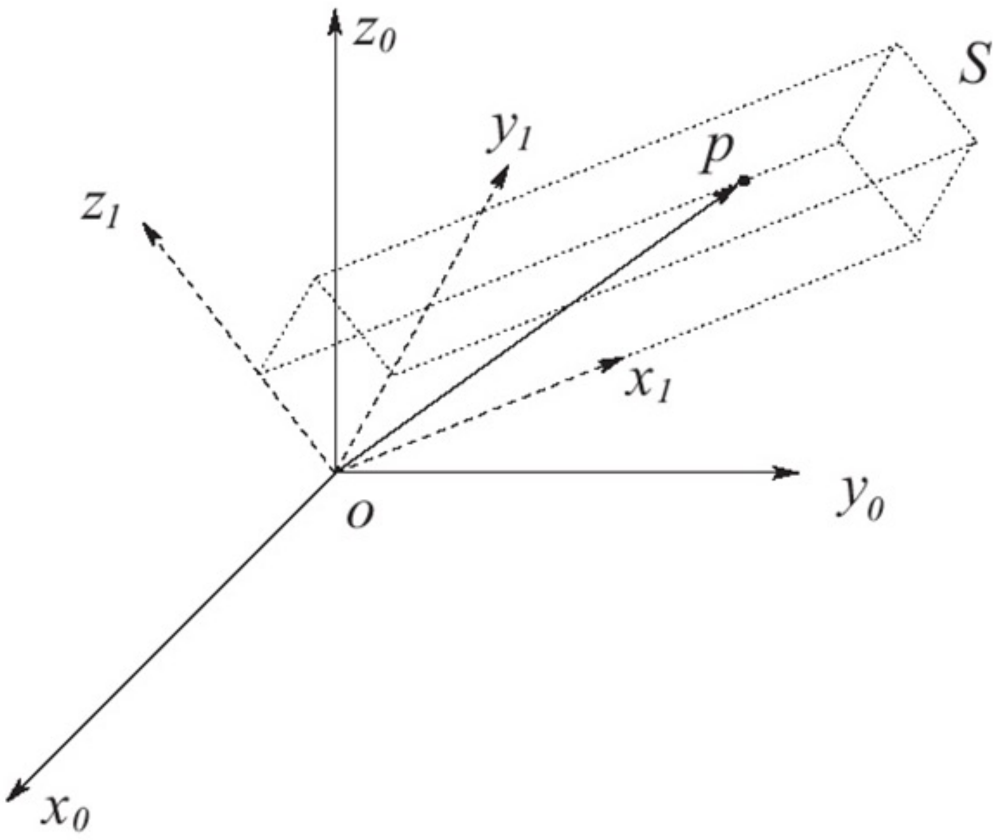
$$R_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$\Downarrow$$

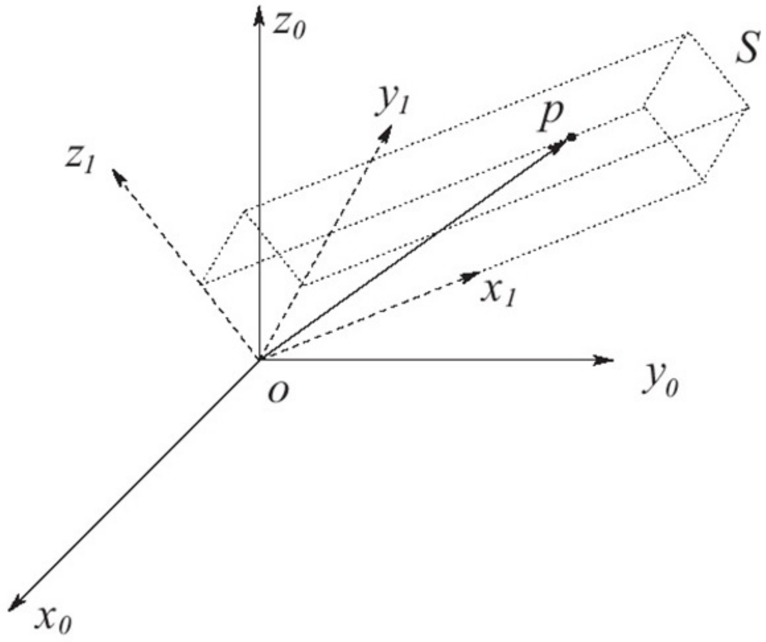
$$R_2^0 = R_1^0 R_2^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Rotation in 3D





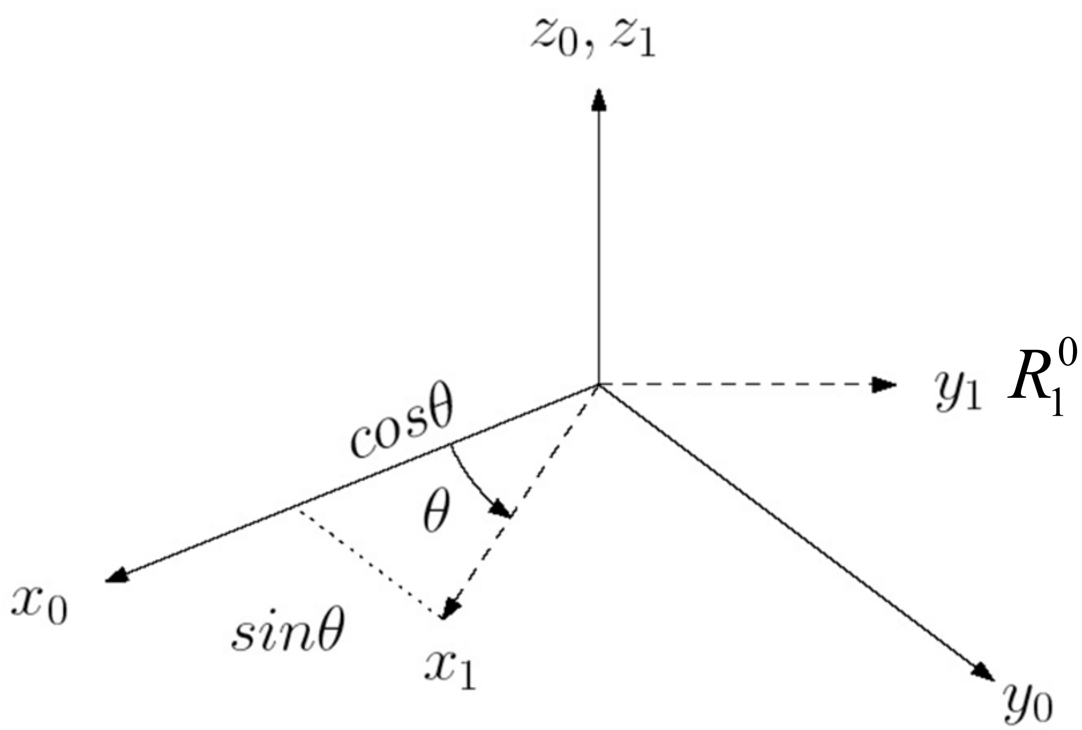
Rotation with Dot Product



$$
\begin{array}{cccc}
 & x_1^0 & y_1^0 & z_1^0 \\
R = & \multicolumn{3}{c}{\begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}}
\end{array}
$$

$$
p^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} p^1
$$

Rotation around Z



$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around X

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Rotation around Y

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

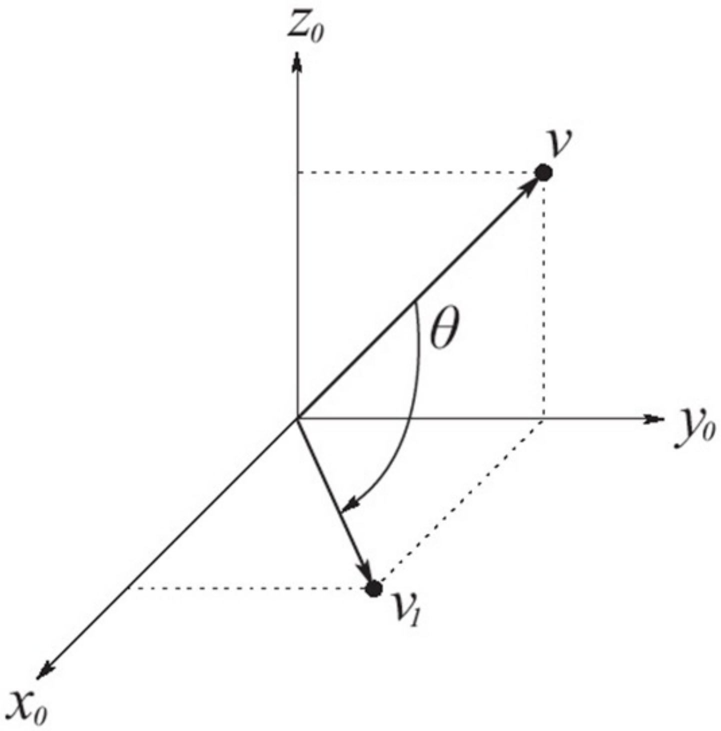
Example

A point $a = (4,3,2)$ is attached to a rotating frame 1, the frame rotates 60 degree about the OZ axis of the reference frame 0. Find the coordinates of the point relative to the reference frame 0 after the rotation.

$$a^0 = Rot(z,60)a^1$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

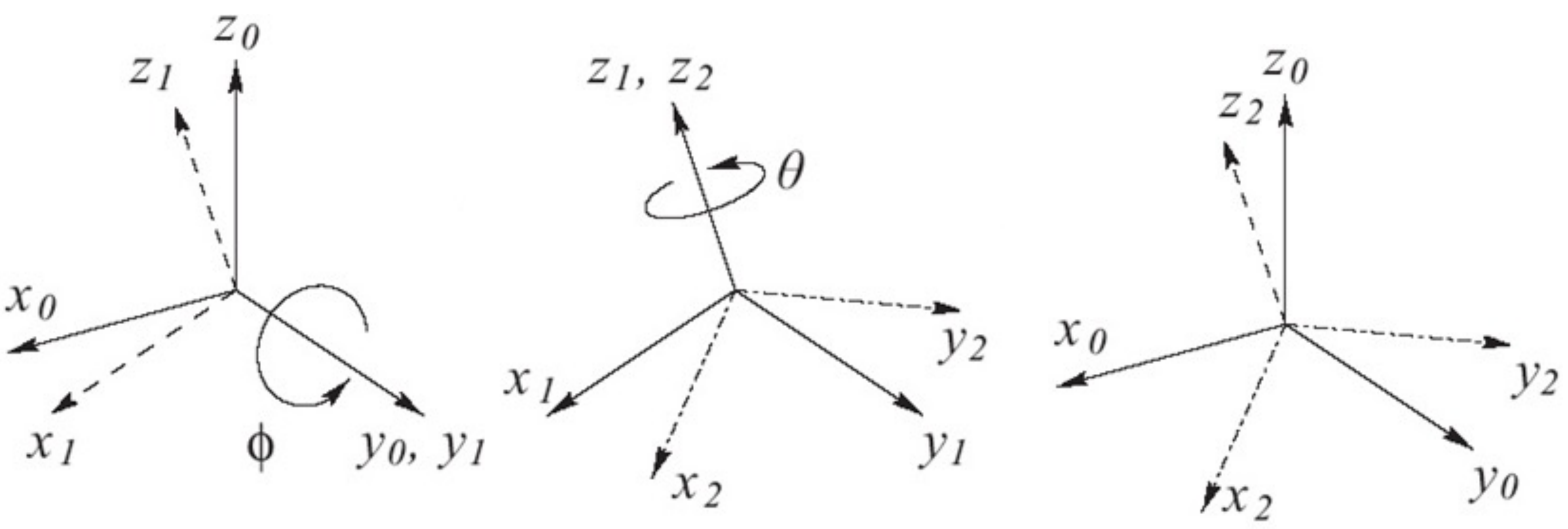
Rotate Vector



$$v_1^0 = R_{y,\frac{\pi}{2}} v^0$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Composition of Rotations



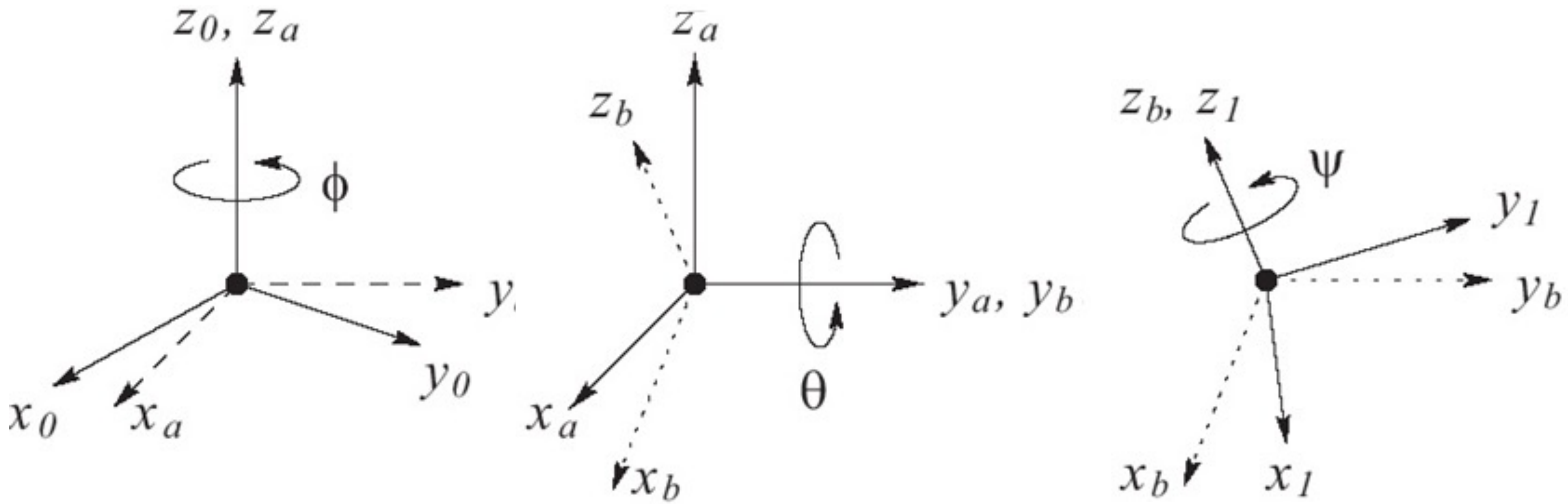
Respect to Current Frame

$$R_2^0 = R_1^0 R_2^1$$

- Any rotation can be described by three successive rotations about linearly independent axes

Example

--Euler Angles



Respect to Current Frame

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix}$$

Inverse Problem

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix}$$

Inverse Problem

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

$$r_{33} = c\theta \qquad \tan\phi = \frac{r_{23}}{r_{13}}$$

$$\begin{aligned} r_{13}c\phi + r_{23}s\phi &= (c\phi s\theta)c\phi + (s\phi s\theta)s\phi \\ &= (c^2\phi + s^2\phi)s\theta \\ &= s\theta \end{aligned}$$

Read textbook pp 54 - 56

Two sets of Euler angles for every R
for almost all R’s
- Multiple conventions
- Singular cases

Example

$$\mathbf{R} = \mathbf{R}_z(0)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix}$$

$$\phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

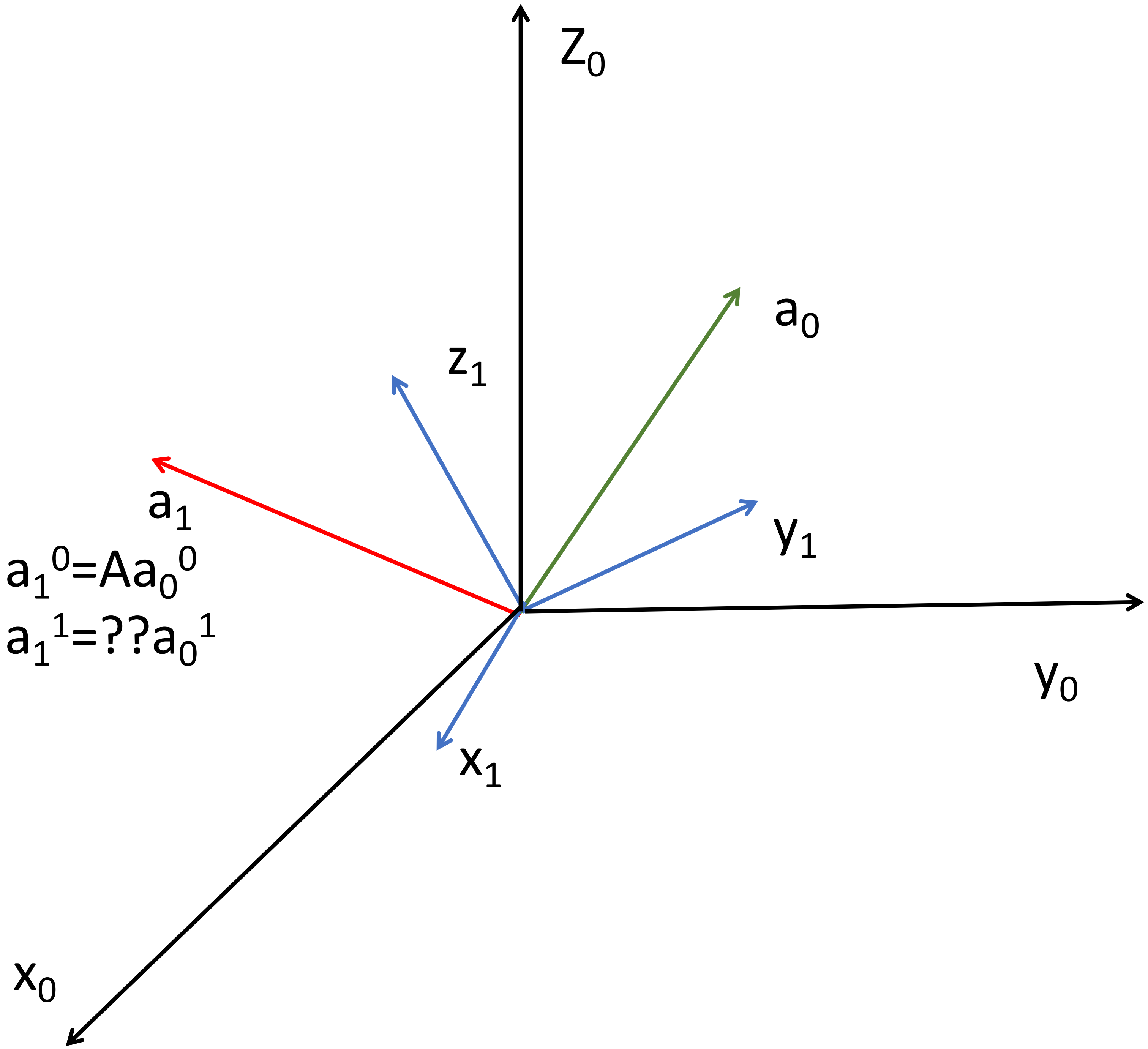
$$\theta = \mathrm{ATAN2}(1, 0) = \pi/2$$

$$\psi = \mathrm{ATAN2}(1, 0) = \pi/2$$

Similarity Transformation

- Rotation A in Frame 0
- Frame 1 to Fame 0 -- R_1^0
- What about the rotation A in Frame 0 relative to Frame 1?

$$B = (R_1^0)^{-1} A R_1^0$$



$a_1^1=??a_0^1$, B = ??

We know

$a_0^0=R_1^0a_0^1$

$a_1^0=Aa_0^0$

$a_1^1=R_0^1a_1^0$

So

$a_1^1=R_0^1AR_1^0a_0^1$

$B = R_0^1AR_1^0$