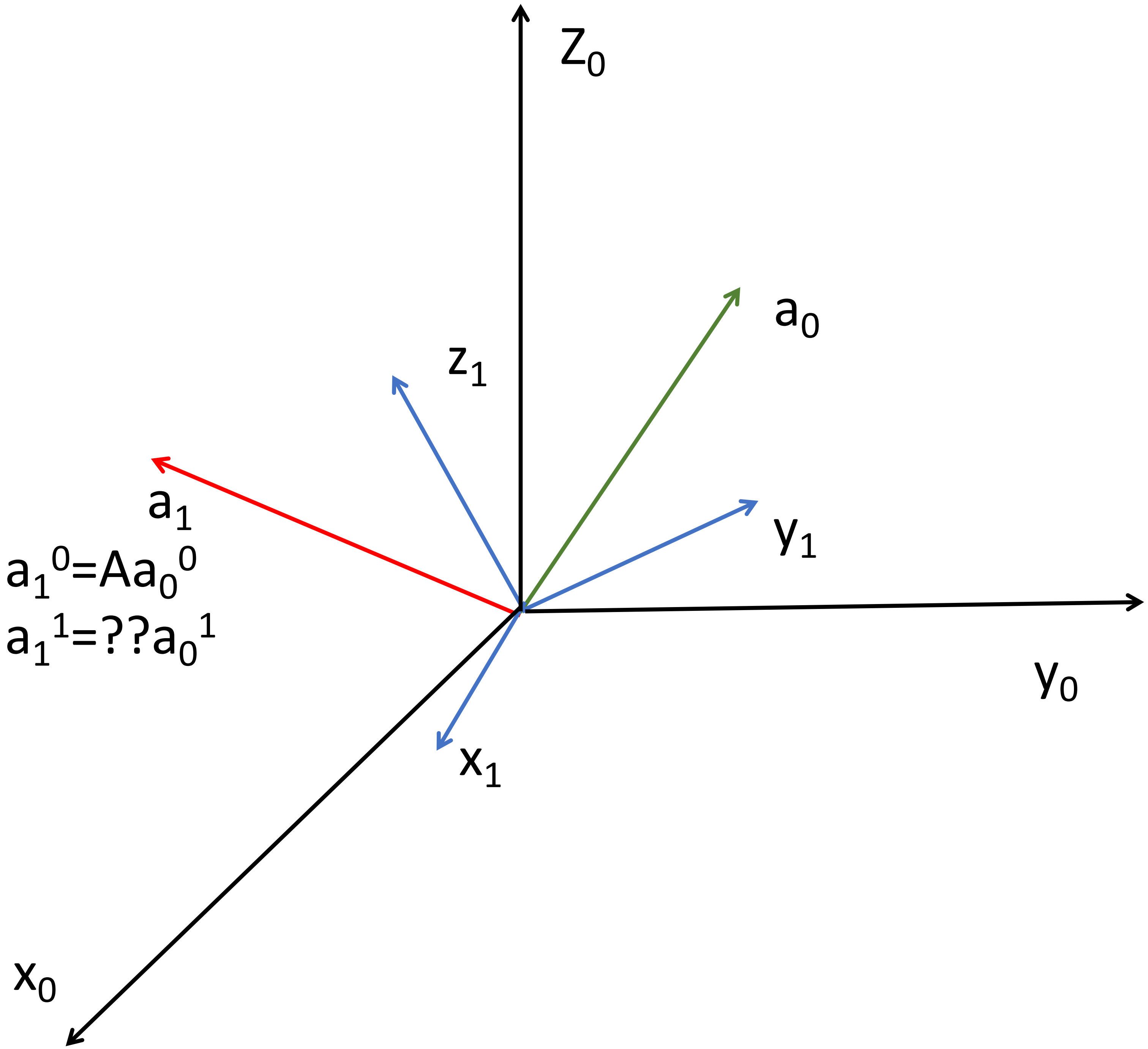
# Intro to Robotics

Lecture 3

# Similarity Transformation

- Rotation A in Frame 0
- Frame 1 to Fame 0 -- $R_1^0$
- What about the rotation A in Frame 0 relative to Frame 1?

$$B = (R_1^0)^{-1} A R_1^0$$




$a_1^1 = ?? a_0^1$, $B = ??$

We know

$a_0^0 = R_1^0 a_0^1$

$a_1^0 = A a_0^0$

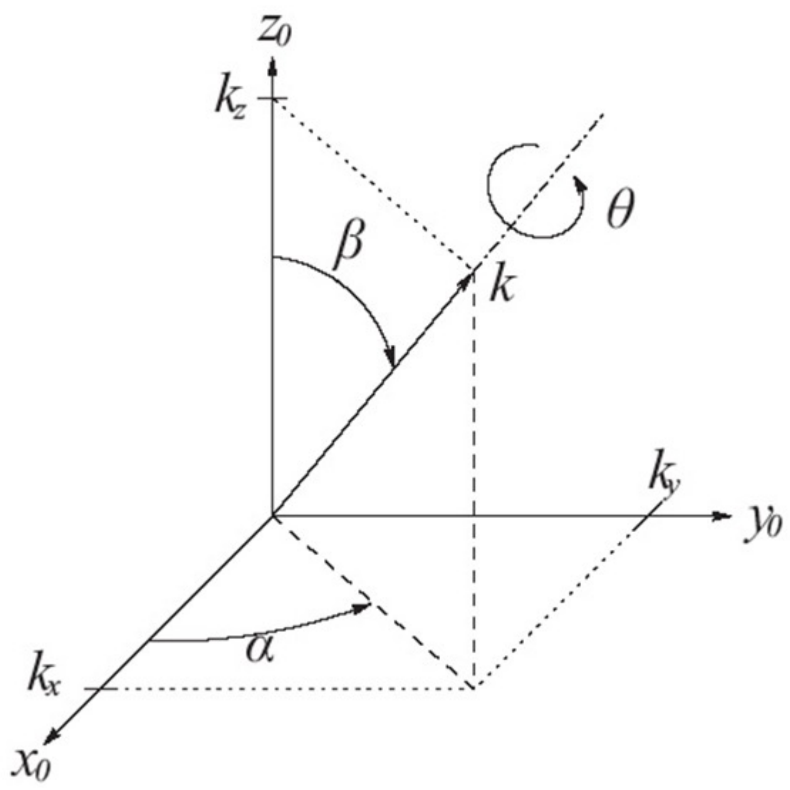
$a_1^1 = R_0^1 a_1^0$

So

$a_1^1 = R_0^1 A R_1^0 a_0^1$

$B = R_0^1 A R_1^0$

# Rotate around a Vector

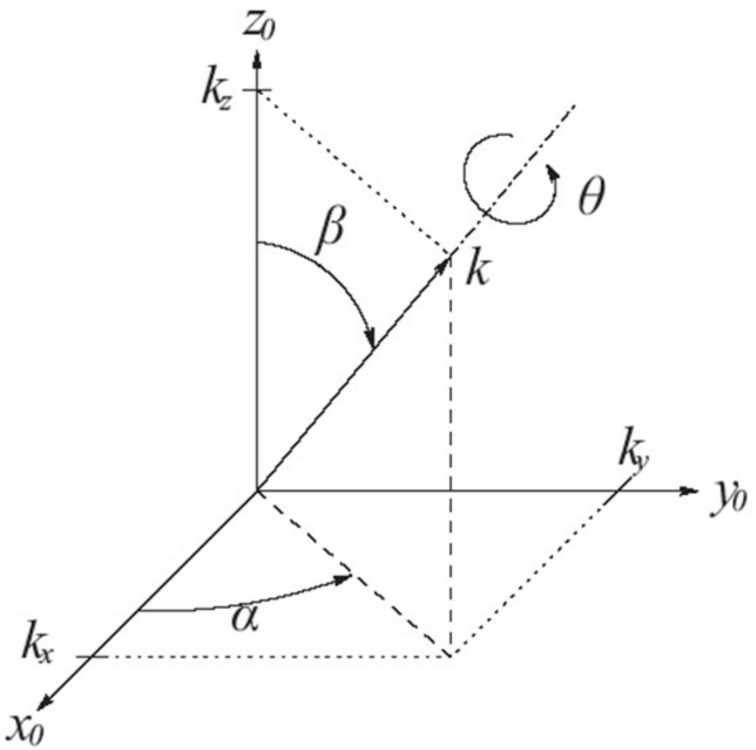



- k as $z_1$
- Rotate around k for θ
- $R_\theta$ is in frame 1, what is the rotation in frame 0

$$B = (R_1^0)^{-1} A R_1^0$$

$$R_{k,\theta} = R R_{z,\theta} R^{-1}$$

$$R_{k,\theta} = \underbrace{R_{z,\alpha} R_{y,\beta}}_{R} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$




$$\begin{bmatrix} k_1^2 \mathrm{v}\theta + \mathrm{c}\theta & k_1 k_2 \mathrm{v}\theta - k_3 \mathrm{s}\theta & k_1 k_3 \mathrm{v}\theta + k_2 \mathrm{s}\theta \\ k_1 k_2 \mathrm{v}\theta + k_3 \mathrm{s}\theta & k_2^2 \mathrm{v}\theta + \mathrm{c}\theta & k_2 k_3 \mathrm{v}\theta - k_1 \mathrm{s}\theta \\ k_1 k_3 \mathrm{v}\theta - k_2 \mathrm{s}\theta & k_2 k_3 \mathrm{v}\theta + k_1 \mathrm{s}\theta & k_3^2 \mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix}$$

$$\mathrm{v}\theta = 1 - \mathrm{c}\theta$$

# Inverse Problem

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} k_1^2 \mathrm{v}\theta + \mathrm{c}\theta & k_1 k_2 \mathrm{v}\theta - k_3 \mathrm{s}\theta & k_1 k_3 \mathrm{v}\theta + k_2 \mathrm{s}\theta \\ k_1 k_2 \mathrm{v}\theta + k_3 \mathrm{s}\theta & k_2^2 \mathrm{v}\theta + \mathrm{c}\theta & k_2 k_3 \mathrm{v}\theta - k_1 \mathrm{s}\theta \\ k_1 k_3 \mathrm{v}\theta - k_2 \mathrm{s}\theta & k_2 k_3 \mathrm{v}\theta + k_1 \mathrm{s}\theta & k_3^2 \mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix}$$

cθ

$$\mathrm{c}\theta = \frac{\mathrm{Tr}(\mathbf{R}) - 1}{2}$$

$r_{32}$-$r_{23}$

$r_{13}$-$r_{31}$

$r_{21}$-$r_{12}$

$$\mathrm{s}\theta = \pm\frac{1}{2}\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2}$$

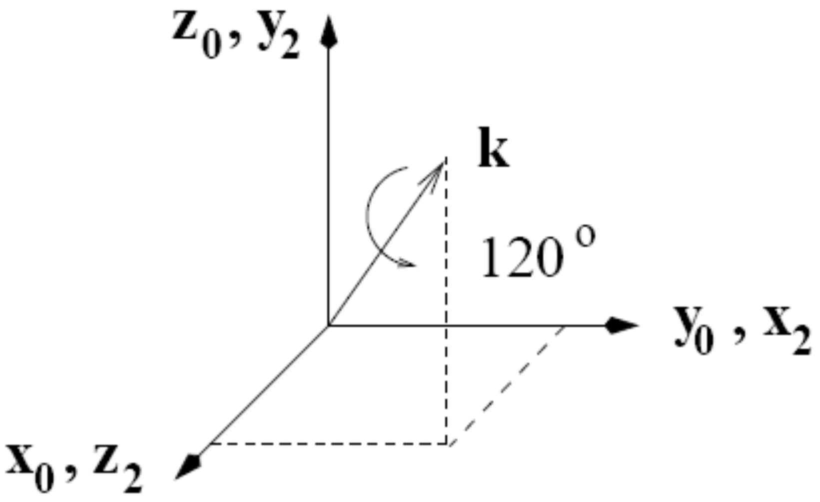
$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2\mathrm{s}\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

# Example

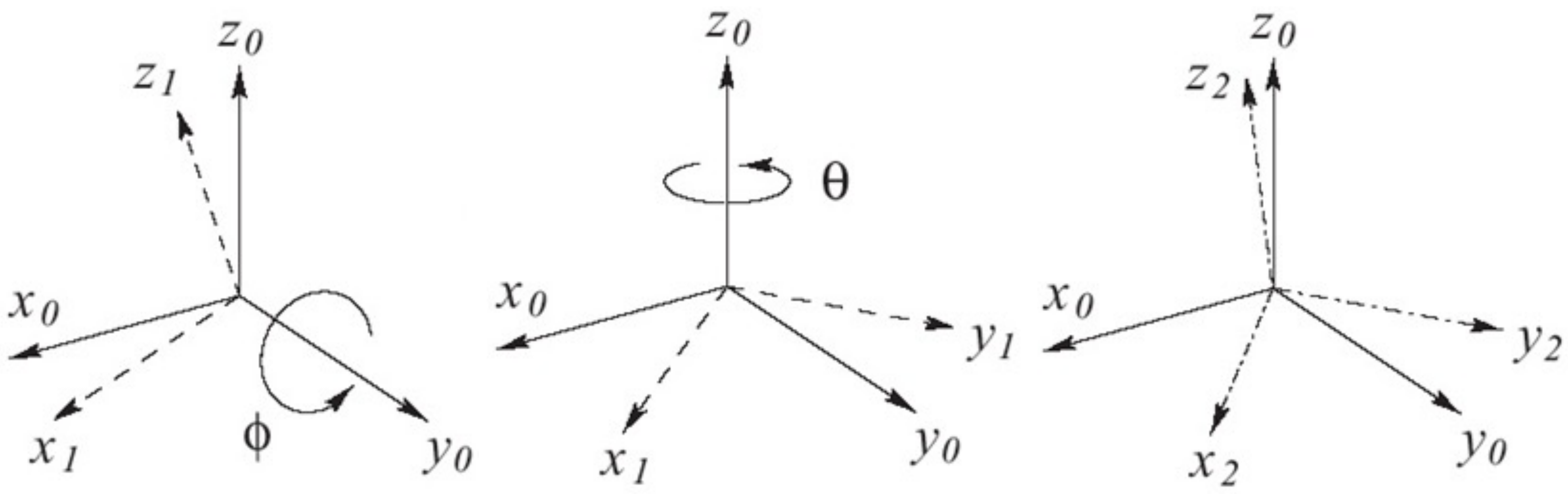
$$\mathbf{R} = \mathbf{R}_y(\pi/2)\mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathrm{c}\theta = -1/2, \quad \mathrm{s}\theta = \sqrt{3}/2, \quad \theta = \mathrm{atan2}(\sqrt{3}/2, -1/2) = 120^o$$

$$\mathbf{k} = \frac{1}{2\frac{\sqrt{3}}{2}} \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



# Composition of Rotations

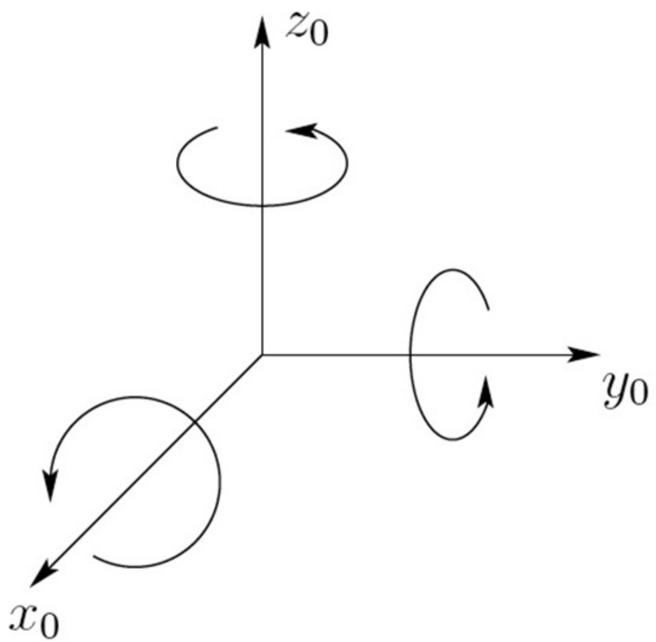



Respect to Fixed Frame

$$R_2^0 = R_1^0[(R_1^0)^{-1} R_\theta R_1^0] = R_\theta R_1^0$$

# Example

-- Roll, Pitch, Yaw Angles




Respect to Fixed Frame

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$

# What about Translation?

$$p^0 = R_1^0 p^1 + d^0$$

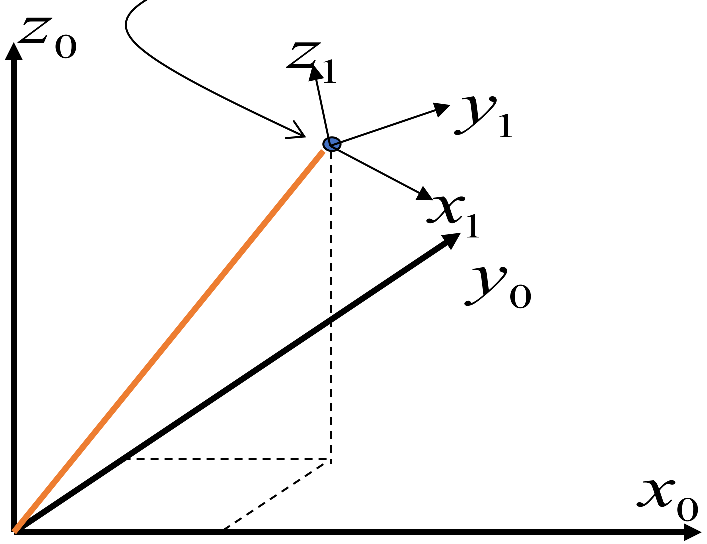
$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

Homogeneous transformation

# What about Translation?

$$d^0 = (d_x, d_y, d_z)^T$$




$$p^0 = R_1^0 p^1 + d^0$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

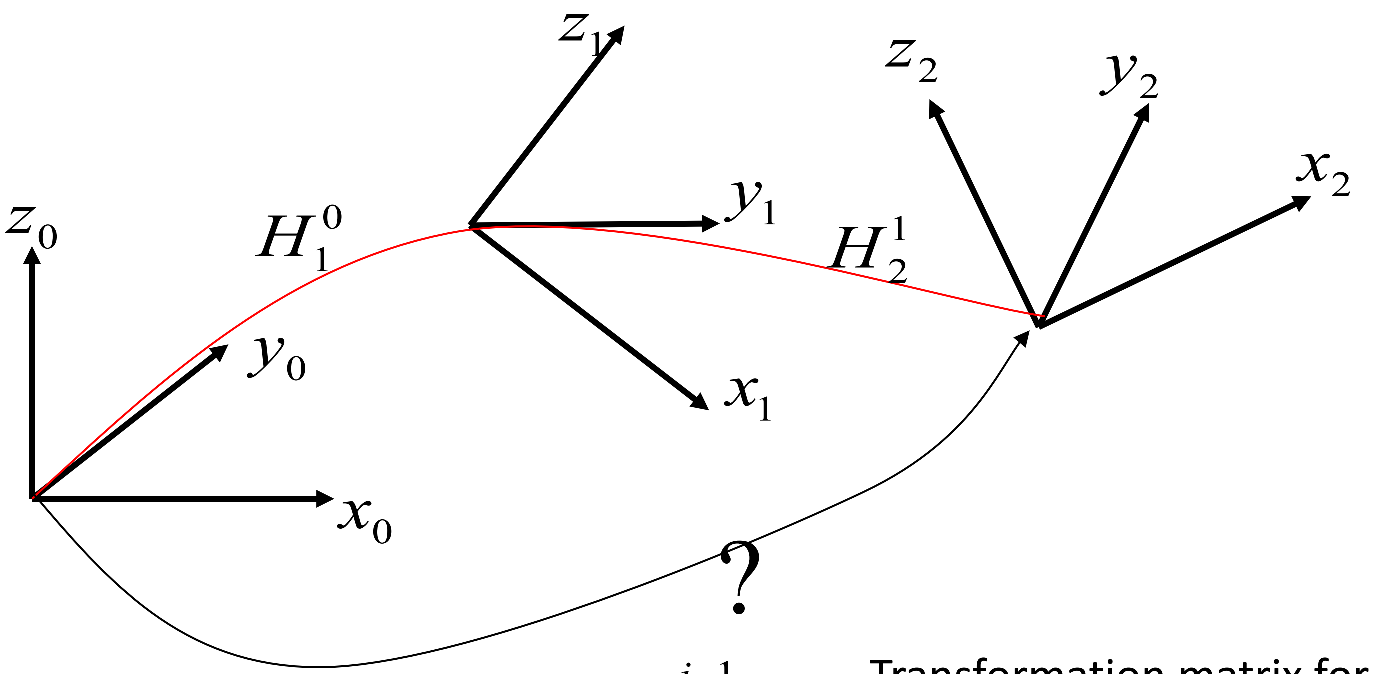
$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{R_{3\times3}} & \boxed{d_{3\times1}} \\ 0 & \boxed{1} \end{bmatrix}$$

Rotation matrix

Position vector

# Homogeneous Transformation

Composite Homogeneous Transformation Matrix

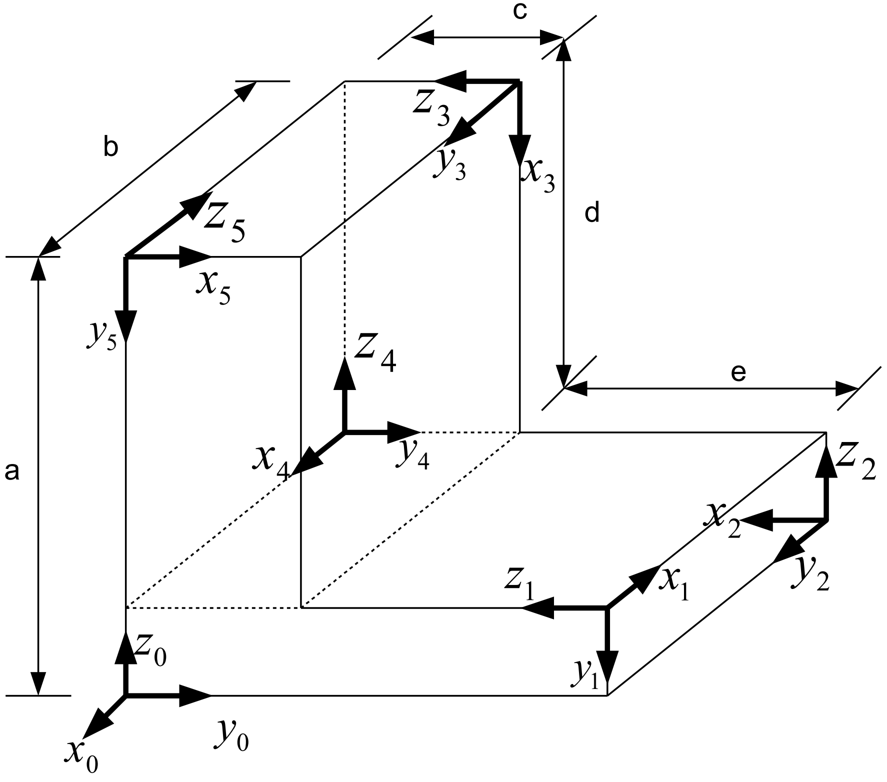



$H_i^{i-1}$ Transformation matrix for adjacent coordinate frames

$$H_2^0 = H_1^0 H_2^1$$

Chain product of successive coordinate transformation matrices

# Example




$$H_1^0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & e+c \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & e+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$