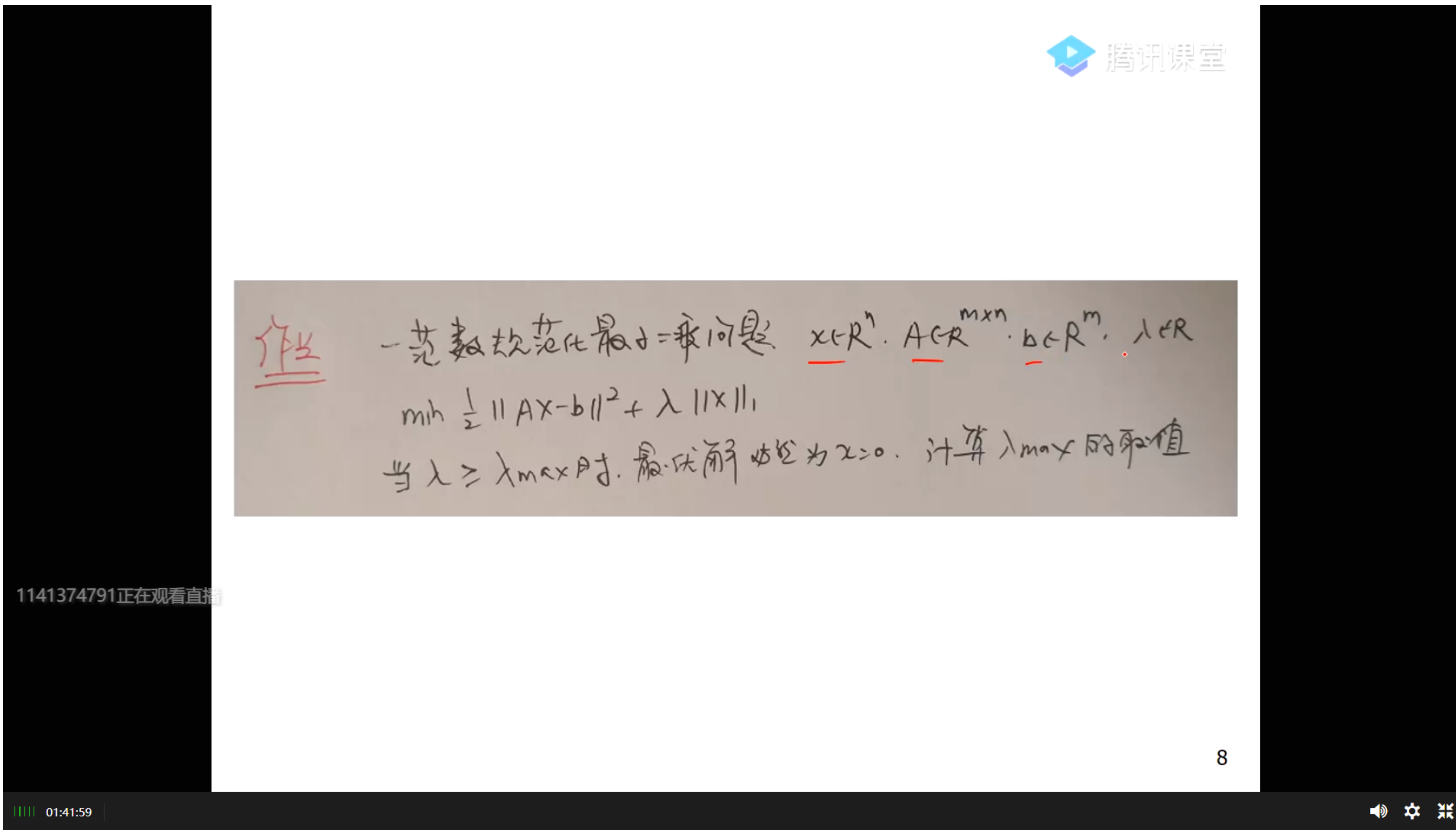
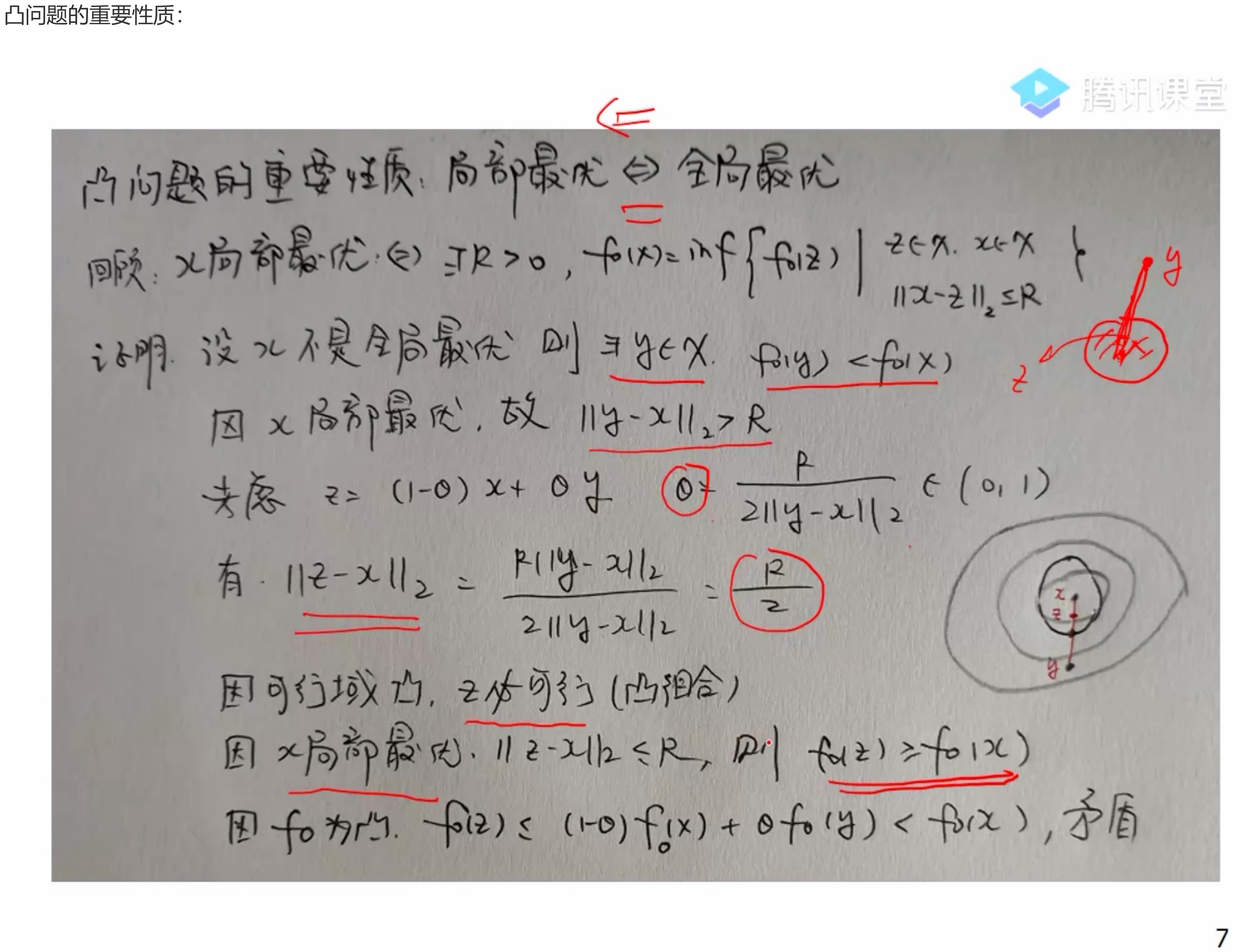
一范数规范化最小二乘问题

一、作业要求



二、作业原理

1. 凸问题的重要性质：



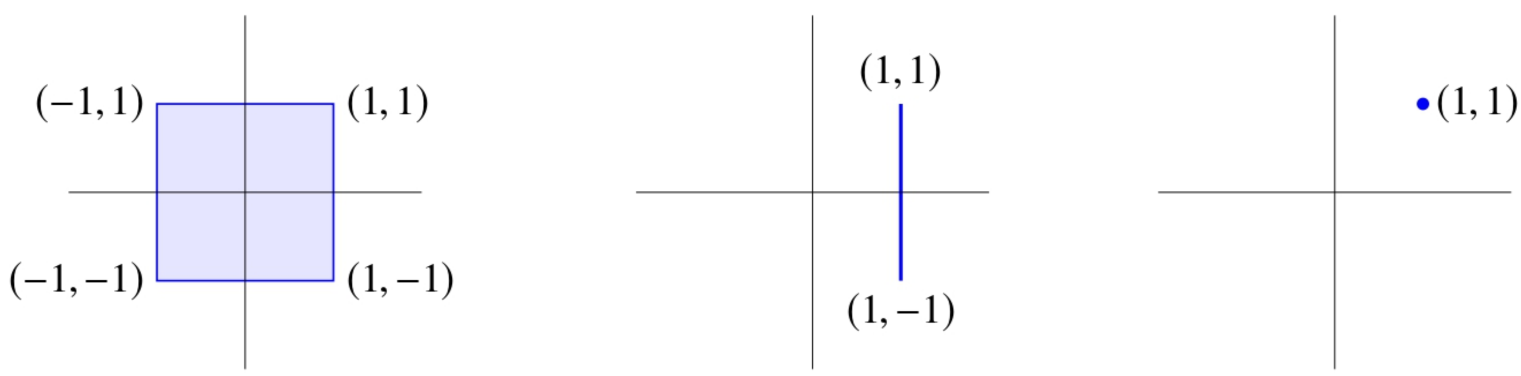
2. 1-范数的次梯度：

Example: ℓ_1-norm

$$f(x) = \|x\|_1 = \max_{s\in\{-1,1\}^n} s^T x$$

the subdifferential is a product of intervals

$$\partial f(x) = J_1 \times \cdots \times J_n, \qquad J_k = \begin{cases} [-1,1] & x_k = 0 \\ \{1\} & x_k > 0 \\ \{-1\} & x_k < 0 \end{cases}$$



$\partial f(0,0) = [-1,1]\times[-1,1]$ $\quad\partial f(1,0) = \{1\}\times[-1,1]$ $\quad\partial f(1,1) = \{(1,1)\}$

Subgradients 2.16

三、作业内容

因为1-范数，2-范数是凸函数

所以f是凸函数

所以该问题是无约束的图问题，局部最优解即全局最优解

所以$\mathbf{0}$是该问题最优解当且仅当$\exists R > 0, \forall z \in \mathbb{R}^n$，若$\| z - \mathbf{0} \| \leq R$，$f(z) > f(\mathbf{0})$即$f(z) - f(\mathbf{0}) > \mathbf{0} \cdot (z - \mathbf{0})$即$\mathbf{0} \in \partial f(\mathbf{0})$

因为$\nabla \frac{1}{2} \parallel Ax - b \parallel^2 = A^T(Ax - b)$（分子布局）。

又因为$\partial \parallel x \parallel_1 = J_1 \times \cdots \times J_n$，其中$J_k = \begin{cases} [-1,1] & x_k = 0 \\ \{1\} & x_k > 0 \\ \{-1\} & x_k < 0 \end{cases}$

所以$\partial f(\mathbf{0}) = \{A^T(-b) + \lambda C | \forall i \in \mathbb{N}^+ \wedge i \leq n, C_i \in [-1,1]\}$

所以$\mathbf{0} \in \partial f(\mathbf{0})$当且仅当$\forall i \in \mathbb{N}^+ \wedge i \leq n, 0 \in [(A^T(-b))_i - \lambda, (A^T(-b))_i + \lambda]$

当且仅当$\lambda \geq \max\limits_{1\leq i \leq n}\{|(A^T b)_i|\}$

所以$\lambda_{max} = \max\limits_{1\leq i \leq n}\{|(A^T b)_i|\}$