(20) ① $\left(\frac{17}{37}\right)$； ② $\left(\frac{151}{373}\right)$； ③ $\left(\frac{191}{397}\right)$； ④ $\left(\frac{911}{2003}\right)$； ⑤ $\left(\frac{37}{200\,723}\right)$； ⑥ $\left(\frac{7}{20\,040\,803}\right)$.

③ ∵ 191 和 397 是奇素数

$$\therefore \left(\frac{191}{397}\right) = (-1)^{\frac{190}{2}\cdot\frac{396}{2}}\cdot\left(\frac{397}{191}\right)$$

$$= \left(\frac{397}{191}\right)$$

$$= \left(\frac{15}{191}\right)$$

$$= \left(\frac{3}{191}\right)\cdot\left(\frac{5}{191}\right)$$

∵ 3 和 191 是奇素数

$$\therefore \left(\frac{3}{191}\right) = (-1)^{\frac{2}{2}\cdot\frac{190}{2}}\cdot\left(\frac{191}{3}\right)$$

$$= -\left(\frac{191}{3}\right)$$

$$= -\left(\frac{-1}{3}\right)$$

$$\because (-1)^{\frac{3-1}{2}} \equiv -1 \pmod 3$$

$$\therefore \left(\frac{3}{191}\right) = 1$$

∵ 5 和 191 是素数

$$\therefore \left(\frac{5}{191}\right) = (-1)^{\frac{4}{2}\cdot\frac{190}{2}}\cdot\left(\frac{191}{5}\right)$$

$$= \left(\frac{191}{5}\right)$$

$$= \left(\frac{1}{5}\right)$$

$$\because 1^{\frac{5-1}{2}} \equiv 1 \pmod 5$$

$$\therefore \left(\frac{5}{191}\right) = 1$$

$$\therefore \left(\frac{191}{397}\right) = 1$$

④ ∵ 911 和 2003 是奇素数

$$\therefore \left(\frac{911}{2003}\right) = (-1)^{\frac{910}{2}\cdot\frac{2002}{2}}\cdot\left(\frac{2003}{911}\right)$$

$$= -\left(\frac{181}{911}\right)$$

∵ 181 和 911 是素数

$$\therefore \left(\frac{181}{911}\right) = (-1)^{\frac{180}{2}\cdot\frac{910}{2}}\cdot\left(\frac{911}{181}\right)$$

$$= \left(\frac{6}{181}\right)$$

$= \left(\frac{2}{181}\right)\cdot\left(\frac{3}{181}\right)$

$\because 181 \equiv -3 \pmod 8$

$\therefore \left(\frac{2}{181}\right) = -1$

$\because 3 \nmid 181$ 是奇素数

$\therefore \left(\frac{3}{181}\right) = (-1)^{\frac{2}{2}\cdot\frac{180}{2}}\cdot\left(\frac{181}{3}\right)$

$= \left(\frac{1}{3}\right)$

$= 1$

$\therefore \left(\frac{911}{2003}\right) = 1$

(22) 求下列同余方程的解数:

① $x^2 \equiv -2 \pmod{67}$; ② $x^2 \equiv 2 \pmod{67}$;

③ $x^2 \equiv -2 \pmod{37}$; ④ $x^2 \equiv 2 \pmod{37}$.

① $\left(\frac{-2}{67}\right) = \left(\frac{-1}{67}\right)\cdot\left(\frac{2}{67}\right)$

$\because (-1)^{\frac{67-1}{2}} \equiv -1 \pmod{67}$

$\therefore \left(\frac{-1}{67}\right) = -1$

$\because 67 \equiv 3 \pmod 8$

$\therefore \left(\frac{2}{67}\right) = -1$

$\therefore \left(\frac{-2}{67}\right) = 1$

$\therefore$ 解数 2

② $\because \left(\frac{2}{67}\right) = -1$

$\therefore$ 解数 0

③ $\because \left(\frac{-2}{37}\right) = \left(\frac{-1}{37}\right)\left(\frac{2}{37}\right)$

$\because (-1)^{\frac{37-1}{2}} \equiv 1 \pmod{37}$

$\therefore \left(\frac{-1}{37}\right) = 1$

$\because 37 \equiv -3 \bmod 8$

$\therefore \left(\frac{2}{37}\right) = -1$

$\therefore \left(\frac{-2}{37}\right) = -1$

$\therefore$ 解数 0

④ $\because \left(\frac{2}{37}\right) = -1$

$\therefore$ 解数 0

(26) 判断下列同余方程是否有解:

① $x^2 \equiv 7 \pmod{227}$; ② $x^2 \equiv 11 \pmod{511}$;

③ $11x^2 \equiv -6 \pmod{91}$; ④ $5x^2 \equiv -14 \pmod{6193}$.

③ $\because 91 = 7 \times 13$

$\therefore \begin{cases} (11x)^2 \equiv -66 \pmod{7} \\ (11x)^2 \equiv -66 \pmod{13} \end{cases}$

$\because \left(\frac{-66}{7}\right) = \left(\frac{-3}{7}\right)$

$\because (-3)^{\frac{7-1}{2}} \equiv 1 \pmod{7}$

$\therefore \left(\frac{-66}{7}\right) = 1$

$\because \left(\frac{-66}{13}\right) = \left(\frac{-1}{13}\right)$

$\therefore (-1)^{\frac{13-1}{2}} \equiv 1 \pmod{13}$

$\therefore \left(\frac{-66}{13}\right) = 1$

$\therefore$ 解数 4

④ $\because 6193 = 11 \times 563$

$\therefore \begin{cases} (5x)^2 \equiv -70 \pmod{11} \\ (5x)^2 \equiv -70 \pmod{563} \end{cases}$

$\because \left(\frac{-70}{11}\right) = \left(\frac{-4}{11}\right)$

$\because (-4)^{\frac{11-1}{2}} \equiv -1024 \equiv -1 \pmod{11}$

$\therefore \left(\frac{-70}{11}\right) = -1$

$\therefore$ 解数 0

(29) 设素数 $p > 2$. 证明: $x^4 \equiv -4 \pmod p$ 有解的充要条件是 $p \equiv 1 \pmod 4$.

证 $\Rightarrow$

若 $x^4 \equiv -4 \pmod p$ 有解

$\therefore (x^2)^2 \equiv -4 \pmod p$ 有解

$\therefore \left(\frac{-4}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^2 = \left(\frac{-1}{p}\right) = 1$

$\therefore (-1)^{\frac{p-1}{2}} \equiv 1 \pmod p$

$\therefore p \equiv 1 \pmod 4$

证 $\Leftarrow$

若 $p \equiv 1 \pmod 4$

$\therefore \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = 1$

$\therefore (x+1)^2 \equiv -1 \pmod p$ 有解

$\therefore x^2 \equiv -2x-2 \pmod p$ 有解

$\therefore x^4 \equiv 4(x+1)^2 \equiv -4 \pmod p$ 有解

(34) 证明: 对任意素数 p, 同余式

$$(x^2-2)(x^2-17)(x^2-34) \equiv 0 \pmod p$$

有解.

① 若 $\left(\frac{2}{p}\right) = 1$

$\therefore$ 原式有解

② 若 $\left(\frac{17}{p}\right) = 1$

$\therefore$ 原式有解

③ 若 $\left(\frac{2}{p}\right) = \left(\frac{17}{p}\right) = -1$

$\therefore \left(\frac{34}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{17}{p}\right) = 1$

$\therefore$ 原式有解

综上, 得证。