练习* 5.2 设全集U是自然数集$\mathbb{N}$，定义集合A：

$$A = \{x + y \mid x, y \in \mathbb{N}, 1 \leq x \leq 4, 1 \leq y^2 \leq 10\}$$

(1) 使用元素枚举法给出集合A；

(2) 判断下面公式的真值（注意，个体变量的论域是全集$\mathbb{N}$）。

$$\begin{aligned}
&(1) \quad \forall x(x \in A \leftrightarrow \exists y \exists z(1 \leq y \leq 4 \ \wedge \ 1 \leq y^2 \leq 10 \ \wedge x = y + z)) \\
&(2) \quad \forall x \forall y((x + y) \in A \to (1 \leq x \leq 4 \ \wedge \ 1 \leq y^2 \leq 10)) \\
&(3) \quad \forall x \forall y((1 \leq x \leq 4 \ \wedge \ 1 \leq y^2 \leq 10) \to (x + y) \in A) \\
&(4) \quad \exists x \exists y((x + y) \in A \wedge (x > 5 \vee y^2 > 10))
\end{aligned}$$

(1)

$A = \{2, 3, 4, 5, 6, 7\}$

(2)

(1)

$\because$ 论域是$\mathbb{N}$

$\therefore \forall x(x \in A \leftrightarrow \exists y \exists z(x = y + z \wedge 1 \leq x \leq 4 \wedge 1 \leq y^2 \leq 10))$是性质概括法定义集合$A$的逻辑含义

$\therefore$ 真值为1

(2)

$\because$ 论域是$\mathbb{N}$

$\therefore$ 当$x = 0, y = 2$时，$x + y = 2 \in A$真值为1，$(1 \leq x \leq 4 \wedge 1 \leq y^2 \leq 10)$真值为0

$\therefore$ 真值为0

(3)

$\because$ 论域是$\mathbb{N}$

$\therefore$ 对$1 \leq x \leq 4 \wedge 1 \leq y^2 \leq 10$即$x = 1, 2, 3, 4$，$y = 1, 2, 3$显然有$x + y \in A$

$\therefore$ 真值为1

(4)

$\because$ 论域是$\mathbb{N}$

$\therefore$ 当$x=6,y=1$时，$x+y\in A$真值为1且$x>5\vee y^2>10$真值为1

$\therefore$ 真值为1

练习* 5.5 设全集U是自然数集，集合A和B都是全集U的子集，且$A=\{3k\mid k\in\mathbb{N}\}$，$B=\{4k\mid k\in\mathbb{N}\}$，计算$A\cap B, A\cup B$和$A-B$。

$$\begin{aligned}A\cap B&=\{3k|k\in N\}\cap\{4k|k\in N\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\}\cap\{x|\exists k\in\mathbb{N}(x=4k)\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\wedge\exists k\in\mathbb{N}(x=4k)\}\end{aligned}$$

$\because$ 对任意正整数a,b,c，有$a|c$且$b|c$当且仅当$lcm(a,b)|c$

$\therefore A\cap B=\{x|\exists k\in\mathbb{N}(x=12k)\}$

$$\begin{aligned}A\cup B&=\{3k|k\in N\}\cup\{4k|k\in N\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\}\cup\{x|\exists k\in\mathbb{N}(x=4k)\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\vee\exists k\in\mathbb{N}(x=4k)\}\\&=\{x|\exists k\in\mathbb{N}(x=3k\vee x=4k)\}\end{aligned}$$

$$\begin{aligned}A-B&=\{3k|k\in N\}-\{4k|k\in N\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\}-\{x|\exists k\in\mathbb{N}(x=4k)\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\wedge\neg(\exists k\in\mathbb{N}(x=4k))\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\wedge\forall k\in\mathbb{N}(\neg(x=4k))\}\\&=\{x|\exists k\in\mathbb{N}(x=3k)\wedge\forall k\in\mathbb{N}(x\neq4k)\}\end{aligned}$$

练习* 5.9 计算下面集合的幂集。

(1) $\{\varnothing,\{\varnothing\}\}$　　(2) $\{a,\{b\},\{\{c\}\}\}$　　(3) $\wp(\{\{a\}\})$

(1)

$\wp(\{\varnothing,\{\varnothing\}\})=\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}$

$\wp(\{a,\{b\},\{\{c\}\}\})=$
$\{\varnothing,\{a\},\{\{b\}\},\{\{\{c\}\}\},\{a,\{b\}\},\{a,\{\{c\}\}\},\{\{b\},\{\{c\}\}\},\{a,\{b\},\{\{c\}\}\}\}$

$\wp(\wp(\{\{a\}\}))=\wp(\{\varnothing,\{\{a\}\}\})=\{\varnothing,\{\varnothing\},\{\{\{a\}\}\},\{\varnothing,\{\{a\}\}\}\}$

练习* 5.10 设a是全集的某个元素，判断下面的命题是否为真。

(1) $a\in\{a\}$　　(2) $\{a\}\in\{a\}$　　(3) $\{a\}\in\{a,\{a\}\}$

(4) $\{a\}\subseteq\{a\}$　　(5) $\{a\}\subseteq\{a,\{a\}\}$　　(6) $\{\{a\}\}\subseteq\{a,\{a\}\}$

(1)

$\because a$是$\{a\}$中的元素

$\therefore$ 是

(2)

$\because \{a\}$不是$\{a\}$中的元素

$\therefore$ 否

(3)

$\because \{a\}$是$\{a, \{a\}\}$中的元素

$\therefore$ 是

(4)

$\because a$是$\{a\}$中全部元素且a是$\{a\}$中的元素

$\therefore$ 是

(5)

$\because a$是$\{a\}$中全部元素且a是$\{a, \{a\}\}$中的元素

$\therefore$ 是

(6)

$\because \{a\}$是$\{\{a\}\}$中全部元素且$\{a\}$是$\{a, \{a\}\}$中的元素

$\therefore$ 是

练习* 5.14 设A, B是任意集合，试给出下列各式成立的充分必要条件，并说明理由。

(1) $A \cap B = A$ (2) $A \cup B = A$

(3) $A \oplus B = A$ (4) $A \cap B = A \cup B$

(1)

$A \subseteq B$

$$
\begin{aligned}
\because A \cap B = A \text{当且仅当} & A \cap B \subseteq A \text{且} A \subseteq A \cap B \\
\text{当且仅当} & A \subseteq A \cap B \\
\text{当且仅当} & A \subseteq A \text{且} A \subseteq B \\
\text{当且仅当} & A \subseteq B
\end{aligned}
$$

(2)

$B \subseteq A$

$$
\begin{aligned}
\because A \cup B = A \text{当且仅当} & A \cup B \subseteq A \text{且} A \subseteq A \cup B \\
\text{当且仅当} & A \cup B \subseteq A \\
\text{当且仅当} & A \subseteq A \text{且} B \subseteq A \\
\text{当且仅当} & B \subseteq A
\end{aligned}
$$

(3)

$B = \varnothing$

$$
\begin{aligned}
\because A \oplus B = A \text{当且仅当} & A \cup B - A \cap B = A \\
\text{当且仅当} & (A \cup B) \cap (\overline{A \cap B}) = A \\
\text{当且仅当} & (A \cup B) \cap (\overline{A} \cup \overline{B}) = A // \text{德摩尔根律} \\
\text{当且仅当} & (A \cup B) \cap (\overline{A} \cup \overline{B}) \subseteq A \text{且} A \subseteq (A \cup B) \cap (\overline{A} \cup \overline{B}) \\
\because (A \cup B) \cap (\overline{A} \cup \overline{B}) \subseteq A \text{当且仅当} & (A \cap \overline{A}) \cup (A \cap \overline{B}) \cup (B \cap \overline{A}) \cup (B \cap \overline{B}) \subseteq A // \text{分配律} \\
\text{当且仅当} & (A \cap \overline{B}) \cup (B \cap \overline{A}) \subseteq A // \text{矛盾律} \\
\text{当且仅当} & A \cap \overline{B} \subseteq A \text{且} B \cap \overline{A} \subseteq A \\
\text{当且仅当} & B \cap \overline{A} \subseteq A \\
\text{当且仅当} & B \cap \overline{A} = B \cap \overline{A} \cap A = \varnothing // \text{结合律，矛盾律，同一律} \\
A \subseteq A \cup B \cap \overline{A} \cup \overline{B} \text{当且仅当} & A \subseteq A \cup B \text{且} A \subseteq \overline{A} \cup \overline{B} \\
\text{当且仅当} & A \subseteq \overline{A} \cup \overline{B} \\
\text{当且仅当} & \overline{A} \cup \overline{B} = A \cup \overline{A} \cup B = U // \text{排中律，零律} \\
\text{当且仅当} & A \cap B = \varnothing // \text{德摩尔根律} \\
\therefore A \oplus B = A \text{当且仅当} & B \cap \overline{A} = \varnothing \text{且} A \cap B = \varnothing
\end{aligned}
$$

若$B = \varnothing$

$\therefore B \cap \overline{A} = \varnothing$且$A \cap B = \varnothing$//零律

若$B \cap \overline{A} = \varnothing$且$A \cap B = \varnothing$

$\therefore B = B \cap (\overline{A} \cup A) = (B \cap \overline{A}) \cup (B \cap A) = \varnothing$//同一律，排中律，分配律，幂等律

$\therefore A \oplus B$当且仅当$B = \varnothing$

(4)

$A = B$

$$\begin{aligned}A \cap B = A \cup B&\text{当且仅当}A \cap B \subseteq A \cup B\text{且}A \cup B \subseteq A \cap B\\&\text{当且仅当}A \cup B \subseteq A \cap B\\&\text{当且仅当}A \cup B \subseteq A\text{且}A \cup B \subseteq B\\&\text{当且仅当}A \subseteq A\text{且}B \subseteq A\text{且}A \subseteq B\text{且}B \subseteq B\\&\text{当且仅当}B \subseteq A\text{且}A \subseteq B\\&\text{当且仅当}A = B\end{aligned}$$

练习* 5.16 设A, B, C是集合，证明：若$A \cap B = A \cap C$且$A \cup B = A \cup C$，则$B = C$。

$\because A \cup B = A \cup C$

$\therefore \overline{A} \cap (A \cup B) = \overline{A} \cap (A \cup C)$

$\therefore \overline{A} \cap B = \overline{A} \cap C$//分配律，矛盾律，同一律

$\therefore (A \cap B) \cup (\overline{A} \cap B) = (A \cap C) \cup (\overline{A} \cap C)$

$\therefore (A \cup \overline{A}) \cap B = (A \cup \overline{A}) \cap C$//分配律

$\therefore B = C$//排中律，同一律

练习* 5.24 设A, B, C是集合，证明：

(1) $A \cap (B \cup C) \subseteq (A \cap B) \cup C$；

(2) $(A \cap B) \cup C = A \cap (B \cup C)$当且仅当$C \subseteq A$。

(1)

$$\begin{aligned}A \cap (B \cup C) &= (A \cap B) \cup (A \cap C)//\text{分配律}\\&= ((A \cap B) \cup A) \cap (A \cap B \cup C)//\text{分配律}\end{aligned}$$

$\because ((A \cap B) \cup A) \cap (A \cap B \cup C) \subseteq (A \cap B) \cup C$

$\therefore$ 得证

(2)

$(A\cap B)\cup C=A\cap(B\cup C)$当且仅当$(A\cap B)\cup C=(A\cap B)\cup(A\cap C)$//分配律

当且仅当$(A\cap B)\cup C=((A\cap B)\cup A)\cap((A\cap B)\cup C)$//分配律

当且仅当$(A\cap B)\cup C\subseteq(A\cap B)\cup A$

当且仅当$(A\cap B)\cup C\subseteq A\cap(B\cup A)$//分配律，幂等律

当且仅当$(A\cap B)\cup C\subseteq A$且$(A\cap B)\cup C\subseteq B\cup A$

当且仅当$A\cap B\subseteq A$且$C\subseteq A$且$A\cap B\subseteq B\cup A$且$C\subseteq B\cup A$

当且仅当$C\subseteq A$且$C\subseteq B\cup A$

$\because$ 当$C\subseteq A$时，$C\subseteq B\cup A$

$\therefore$ 当$C\subseteq A$时，$C\subseteq A$且$C\subseteq B\cup A$

$\therefore$ 当$C\subseteq A$时，$(A\cap B)\cup C=A\cap(B\cup C)$

$\because$ 当$C\subseteq A$且$C\subseteq B\cup A$时，$C\subseteq A$

$\therefore$ 当$(A\cap B)\cup C=A\cap(B\cup C)$时，$C\subseteq A$

$\therefore$ 得证

练习* 5.30 设A,B是集合，证明$A=|B$当且仅当$\wp(A)=\wp(B)$。

若$A=B$

$\therefore \forall x\in\wp(A)$，有$x\subseteq A$即$x\subseteq B$即$x\in\wp(B)$

$\therefore \wp(A)\subseteq\wp(B)$

$\because \forall x\in\wp(B)$，有$x\subseteq B$即$x\subseteq A$即$x\in\wp(A)$

$\therefore \wp(B)\subseteq\wp(A)$

$\therefore \wp(A)=\wp(B)$

若$\wp(A)=\wp(B)$

$\forall x\subseteq A$，$x\subseteq B$且$\forall x\subseteq B$，$x\subseteq A$

$\because \exists A\subseteq A$且$\exists B\subseteq B$

$\therefore A\subseteq B$且$B\subseteq A$

$\therefore A=B$

$\therefore$ 得证