In Exercises 1 through 5 solve the differential equations by the Euler method.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

(a)

$$y(0.2) = y(0) + hy'(0, y(0)) = 1 + 0.2(0^2 - 1) = 0.8$$
$$y(0.4) = y(0.2) + hy'(0.2, y(0.2)) = 0.8 + 0.2(0.2^2 - 0.8) = 0.648$$

$$y(0.1) = y(0) + hy'(0, y(0)) = 1 + 0.1(0^2 - 1) = 0.9$$
$$y(0.2) = y(0.1) + hy'(0.1, y(0.1)) = 0.9 + 0.1(0.1^2 - 0.9) = 0.811$$
$$y(0.3) = y(0.2) + hy'(0.2, y(0.2)) = 0.811 + 0.1(0.2^2 - 0.811) = 0.7339$$
$$y(0.4) = y(0.3) + hy'(0.3, y(0.3)) = 0.7339 + 0.1(0.3^2 - 0.7339) = 0.66951$$

(b)

t_k	y_k		$y(t_k)$ **Exact**
	$h = 0.2$	$h = 0.1$	
0.4	0.648	0.66951	0.689679953964361

(c)

h	**Number of steps,** M	**F.G.E. Error at** $t = 0.4$
0.4	2	0.041679953964361
0.2	4	0.020169953964361

$\therefore 0.020169953964361 / 0.041679953964361 = 0.483924573947649 \approx 0.5$

$\therefore$Yes

In Exercises 1 through 5, solve the differential equations by Heun's method.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

(a)

$$\begin{aligned} y^0(0.2) &= y(0) + hy'(0, y(0)) \\ &= 1 + 0.2(0^2 - 1) \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} y(0.2) &= y(0) + h(y'(0, y(0)) + y'(0.2, y^0(0.2)))/2 \\ &= 1 + 0.2((0^2 - 1) + (0.2^2 - 0.8))/2 \\ &= 0.824 \end{aligned}$$

$$\begin{aligned} y^0(0.4) &= y(0.2) + hy'(0.2, y(0.2)) \\ &= 0.824 + 0.2(0.2^2 - 0.824) \\ &= 0.6672 \end{aligned}$$

$$\begin{aligned} y(0.4) &= y(0.2) + h(y'(0.2, y(0.2)) + y'(0.4, y^0(0.4)))/2 \\ &= 0.824 + 0.2((0.2^2 - 0.824) + (0.4^2 - 0.6672))/2 \\ &= 0.69488 \end{aligned}$$

$$\begin{aligned} y^0(0.1) &= y(0) + hy'(0, y(0)) \\ &= 1 + 0.1(0^2 - 1) \\ &= 0.9 \end{aligned}$$

$$\begin{aligned} y(0.1) &= y(0) + h(y'(0, y(0)) + y'(0.1, y^0(0.1)))/2 \\ &= 1 + 0.1((0^2 - 1) + (0.1^2 - 0.9))/2 \\ &= 0.9055 \end{aligned}$$

$$\begin{aligned} y^0(0.2) &= y(0.1) + hy'(0.1, y(0.1)) \\ &= 0.9055 + 0.1(0.1^2 - 0.9055) \\ &= 0.81595 \end{aligned}$$

$$\begin{aligned} y(0.2) &= y(0.1) + h(y'(0.1, y(0.1)) + y'(0.2, y^0(0.2)))/2 \\ &= 0.9055 + 0.1((0.1^2 - 0.9055) + (0.2^2 - 0.81595))/2 \\ &= 0.8219275 \end{aligned}$$

$$\begin{aligned} y^0(0.3) &= y(0.2) + hy'(0.2, y(0.2)) \\ &= 0.8219275 + 0.1(0.2^2 - 0.8219275) \\ &= 0.74373475 \end{aligned}$$

$$\begin{aligned} y(0.3) &= y(0.2) + h(y'(0.2, y(0.2)) + y'(0.3, y^0(0.3)))/2 \\ &= 0.8219275 + 0.1((0.2^2 - 0.8219275) + (0.3^2 - 0.74373475))/2 \\ &= 0.7501443875 \end{aligned}$$

$$
\begin{aligned}
y^0(0.4) &= y(0.3) + hy'(0.3, y(0.3)) \\
&= 0.7501443875 + 0.1(0.3^2 - 0.7501443875) \\
&= 0.68412994875
\end{aligned}
$$

$$
\begin{aligned}
y(0.4) &= y(0.3) + h(y'(0.3, y(0.3)) + y(0.4, y^0(0.4))))/2 \\
&= 0.7501443875 + 0.1((0.3^2 - 0.7501443875) + (0.4^2 - 0.68412994875))/2 \\
&= 0.6909306706875
\end{aligned}
$$

(b)

t_k	y_k		$y(t_k)$ **Exact**
	$h = 0.2$	$h = 0.1$	
0.4	0.69488	0.6909306706875	0.689679953964361

(c)

h	**Number of steps, M**	**F.G.E. Error at $t = 0.4$**
0.4	2	0.005200046035639
0.2	4	0.001250716723139

$\therefore 0.001250716723139/0.005200046035639 = 0.240520317429326 \approx 0.25$

$\therefore$No

In Exercises 1 through 5, solve the differential equations by the Runge-Kutta method of order $N = 4$.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

(a)

由

```
function [y] = my_rk4(f, t1, y1, h, M)
    y = y1;
    t = t1;
    for i = 1:M
        y = RK41step(f, t, y, h);
        t = t + h;
    end
end
```

```
function [y2] = RK41step(f, t1, y1, h)
    k1 = f(t1, y1)
    k2 = f(t1 + 0.5 * h, y1 + 0.5 * k1 * h)
    k3 = f(t1 + 0.5 * h, y1 + 0.5 * k2 * h)
    k4 = f(t1 + h, y1 + k3 * h)
    y2 = y1 + (k1 + 2 * k2 + 2 * k3 + k4) * h / 6
end
```

```
function [d] = f(t, y)
    d = t^2 - y;
end
```

$k_1 = -1$
$k_2 = -0.89$
$k_3 = -0.901$
$k_4 = -0.7798$
$y(0.2) = 0.8213$

$k_1 = -0.7813$
$k_2 = -0.6531$
$k_3 = -0.6660$
$k_4 = -0.5281$
$y(0.4) = 0.689687853777778$

$k_1 = -1$
$k_2 = -0.9475$
$k_3 = -0.9501$
$k_4 = -0.8950$
$y(0.1) = 0.9052$

$k_1 = -0.8952$
$k_2 = -0.8379$
$k_3 = 0.8408$
$k_4 = -0.7811$
$y(0.2) = 0.8213$

$k_1 = -0.7813$
$k_2 = -0.7197$
$k_3 = -0.7228$
$k_4 = -0.6590$
$y(0.3) = 0.7492$

$k_1 = -0.6592$
$k_2 = -0.5937$

$k_3 = -0.5970$
$k_4 = -0.5295$
$y(0.4) = 0.689680432829764$

(b)

t_k	y_k		$y(t_k)$ Exact
	$h = 0.2$	$h = 0.1$	
0.4	0.689687853777778	0.689680432829764	0.689679953964361

(c)

h	Number of steps, M	F.G.E. Error at $t = 0.4$
0.4	2	$7.899813416756274 \times 10^{-6}$
0.2	4	$4.788654031084860 \times 10^{-7}$

$\therefore 4.788654031084860 \times 10^{-7} / 7.899813416756274 \times 10^{-6} = 0.0606 \approx 0.0625$

$\therefore$No

In Problems 1 through 5, solve the differential equations by the Runge-Kutta method of order $N = 4$.

(a) Let $h = 0.1$ and do 20 steps with Program 9.4. Then let $h = 0.05$ and do 40 steps with Program 9.4.

(b) Compare the exact solution $y(2)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

(d) Plot the two approximations and the exact solution on the same coordinate system. *Hint.* The output matrix `R` from Program 9.4 contains the x- and y-coordinates of the approximations. The command `plot(R(:,1),R(:,2))` will produce a graph analogous to Figure 9.6.

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

```
>> rk4(@f,0,0+0.1*20,1,20)

ans =

         0    1.0000
    0.1000    0.9052
    0.2000    0.8213
    0.3000    0.7492
    0.4000    0.6897
    0.5000    0.6435
    0.6000    0.6112
    0.7000    0.5934
    0.8000    0.5907
    0.9000    0.6034
    1.0000    0.6321
    1.1000    0.6771
    1.2000    0.7388
    1.3000    0.8175
    1.4000    0.9134
    1.5000    1.0269
    1.6000    1.1581
    1.7000    1.3073
    1.8000    1.4747
    1.9000    1.6604
    2.0000    1.8647
```

```
>> rk4(@f,0,0+0.05*40,1,40)

ans =

         0    1.0000
    0.0500    0.9513
    0.1000    0.9052
    0.1500    0.8618
    0.2000    0.8213
    0.2500    0.7837
    0.3000    0.7492
    0.3500    0.7178
    0.4000    0.6897
    0.4500    0.6649
    0.5000    0.6435
    0.5500    0.6256
    0.6000    0.6112
    0.6500    0.6005
    0.7000    0.5934
    0.7500    0.5901
    0.8000    0.5907
    0.8500    0.5951
    0.9000    0.6034
    0.9500    0.6158
    1.0000    0.6321
    1.0500    0.6526
    1.1000    0.6771
    1.1500    0.7059
    1.2000    0.7388
    1.2500    0.7760
    1.3000    0.8175
    1.3500    0.8633
    1.4000    0.9134
    1.4500    0.9679
    1.5000    1.0269
    1.5500    1.0903
    1.6000    1.1581
    1.6500    1.2305
    1.7000    1.3073
    1.7500    1.3887
    1.8000    1.4747
    1.8500    1.5653
    1.9000    1.6604
    1.9500    1.7602
    2.0000    1.8647
```

(b)

t_k	y_k		$y(t_k)$ **Exact**

t_k	y_k		$y(t_k)$ **Exact**
	$h=0.1$	$h=0.05$	
2	1.864666364534275	1.864664817490470	1.864664716763387

(c)

h	**Number of steps,** M	**F.G.E. Error at** $t=0.4$
0.1	20	$1.647770887736044\times10^{-6}$
0.05	40	$1.007270828967677\times10^{-7}$

$\therefore 1.007270828967677\times10^{-7}/1.647770887736044\times10^{-6}=0.0611\approx0.0625$

∴No

(d)

由

```
%p503t1
y=@(t)-exp(-t)+t^2-2*t+2;
plot(2, y(2), 'o')
hold on
R1 = rk4(@f,0,0+0.1*20,1,20)
plot(R1(:, 1), R1(:, 2), 'b')
R2 = rk4(@f,0,0+0.05*40,1,40)
plot(R2(:, 1), R2(:, 2), '-.')
xlabel('t')
ylabel('y')
legend('Exact', 'h=0.1', 'h=0.05')
```