# Inverse Fractional Knapsack Problem with Profits and Costs Modification

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We address in this paper the problem of modifying both profits and costs of a fractional knapsack problem optimally such that a prespectified solution becomes an optimal solution with prespect to new parameters. This problem is called the inverse fractional knapsack problem. Concerning the  $l_1$ -norm, we first prove that the problem is NP-hard. The problem can be however solved in quadratic time if we only modify profit parameters. Additionally, we develop a quadratic-time algorithm that solves the inverse fractional knapsack problem under  $l_{\infty}$ -norm.

### 1 Introduction

The  $\{0,1\}$  Knapsack Problem plays an important role in real-life decision making; for instance, finding the least wasteful way to cut raw materials, selection of investments and portfolios, cargo loading, etc. In order to solve this problem, we apply some non-polynomial but effective algorithms such as dynamic programming, greedy algorithm, branch and bound, and so on. The relaxation version of this problem is called the fractional knapsack problem, which can be solved by the greedy algorithm in  $O(n \log n)$  time or by the algorithm of Balas and Zemel [?] in linear time. Here, we denote the input size of the problem by n.

The inverse (combinatorial) optimization problem consists of changing parameters of the problem at minimum total cost such that a prespecified solution becomes optimal with respect to new parameters. The first who investigated the inverse optimization problem were Burton and Toint [5]. They developed an

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efficient algorithm that solves the inverse shortest path problem, which could be applied to predict the path of an earthquake. From here on, inverse optimization problem has increased interests from the community because of its potential applications. Ajuha et al. [2] showed that the inverse linear programming optimization problem can be reduced to a problem of the same type, based on the so-called complementary slackness condition. In 2002, Ahuja and Orlin [2] examined the inverse network flow problem with  $l_1$ - and  $l_{\infty}$ -norm. They presented combinatorial algorithms for solving this problem. Also, researchers focused on the inverse version of minimum spanning tree problem. Zhang et al. [10] was the first who investigated the problem with partition constraints in 1996 with practical applications. Then Sokkalingam et al. [11] solved the problem in  $O(n^3)$  time. Ahuja and Orlin further improved the complexity of this problem to  $O(n^2 \log n)$ . For terminology concerning the inverse optimization problem and solution methods, readers refer to the survey of Heuberger [8].

Recently, the inverse  $\{0,1\}$  knapsack problem has been investigated by Roland [9]. He first showed that this problem under  $l_{\infty}$ -norm is co-NP-complete. Hence, there exists no approach to solve the problem in polynomial time, unless P=NP. He also developed a pseudo-polynomial time algorithm based on a binary search to deal with the uniform-cost inverse  $\{0,1\}$  knapsack problem. Besides, the author proposed a bilevel programming model for the problem under  $l_1$ -norm. Computation showed that this model is efficient enough. In this paper we study the inverse fractional knapsack problem. According to the best of our knowledge, this problem has not been studied so far.

This paper is stated as follows. Section 2 includes preliminary concepts and optimality criterion of the fractional knapsack problem. In Section 3, we formulate the inverse fractional knapsack problem under  $l_1$ -norm and show the NP-hardness in general case. If the cost coefficients are fixed, a quadratic algorithm is developed. Section 4 considers the inverse problem under  $l_{\infty}$ -norm. It is shown that the problem is solvable in  $O(n^2)$  time.

### 2 Problem definition

Let us first revisit the 0-1 knapsack problem and its fractional version. The 0-1 knapsack problem can be roughly stated as follows. Given a set of items, each with a cost and a profit, we wish to determine the number of each item to include in a collection so that the total cost is less than or equal to a given budget and the total profit is as large as possible. The relaxation of 0-1 Knapsack problem is

the so-called fractional knapsack problem (FKP), which is formulated as follows.

$$\max \sum_{i=1}^{n} p_i x_i$$
s.t. 
$$\sum_{i=1}^{n} c_i x_i \le b$$

$$0 \le x_i \le 1 \quad \forall i = 1, \dots, n.$$

$$(1)$$

Here, the profits  $p_i$  and the costs  $c_i$  are positive intergers for all i = 1, ..., n. (FKP) can be solved by a simple greedy algorithm, where we takes the items with respect to the smaller ratios (profit over cost) until the budget constraint fulfills. Another solution approach with linear time complexity was proposed by Balas and Zemel [3], where we ruins a half of solution set until obtaining a stoping condition. For both of these two algorithms, an optimal solution has the following form.

**Proposition 1.** Let  $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$  be an optimal solution of (1). Then all items in  $x^0$  obtain value 0 or 1 except at most one item, say  $x_r^0$ , s.t.  $0 < x_r^0 < 1$ .

We now consider a solution  $x^*$  where all items in  $x^*$  obtain value 0 or 1. The conditions for  $x^*$  to be an optimal solution of (1) can be straightforward derived from the greedy algorithm as below.

**Theorem 2.1.** (Optimality Criterion) A solution  $x^* = (x_1^*, x_2^*, \dots, x_n^*)$  is an optimal solution of (1) iff the following conditions hold.

(i) 
$$\sum_{i=1}^{n} c_i x_i^* = b$$
.

$$\text{(ii)} \ \min_{\{i: x_i^* \neq 0\}} \left\{ \frac{p_i}{c_i} \right\} \geq \max_{\{i: x_i^* = 0\}} \left\{ \frac{p_i}{c_i} \right\}.$$

Next we formulate the inverse setting of (FKP). Given an instance of (1) and a prespecified solution  $x^*$ , where  $x_i^* = 0$  or  $x_i^* = 1$  for i = 1, ..., n. The profits and the costs can be either increased or reduced, i.e.,  $\tilde{p}_i = p_i + u_i - v_i$  and  $\tilde{c}_i = c_i + \lambda_i - \mu_i$ . Let  $(u, v, \lambda, \mu)$  be the vector of modification and  $C(u, v, \lambda, \mu)$  is the cost function. The inverse continuous knapsack problem is stated as follows:

- The vector  $x^*$  become an optimal solution of the modified knapsack instance.
- The cost  $C(u, v, \lambda, \mu)$  is minimized.
- $u_i, v_i, \lambda_i, \mu_i$  are feasible for all i = 1, ..., n. It means

$$u_i \in [0, \bar{u}_i] \cap \mathbb{Z}, v_i \in [0, \bar{v}_i] \cap \mathbb{Z}, \lambda_i \in [0, \bar{\lambda}_i] \cap \mathbb{Z}, \mu_i \in [0, \bar{\mu}_i] \cap \mathbb{Z}$$

Note that the modifications must be intergers to guarentee that the profits and costs are intergers, too. We can formulate the problem as a programming as below.

$$\begin{aligned} & \min \quad & C(u, v, \lambda, \mu) \\ & \text{s.t.} \quad & x^* \in \mathbf{argmax} \left\{ \sum_{i=1}^n \tilde{p}_i x_i | \sum_{i=1}^n \tilde{c}_i x_i \leq b, x_i \in [0, 1] \right\} \\ & \tilde{p}_i = p_i + u_i - v_i, \forall i = 1, \dots, n \\ & \tilde{c}_i = c_i + \lambda_i - \mu_i, \forall i = 1, \dots, n \\ & u_i \in [0, \bar{u}_i] \cap \mathbb{Z}, \forall i = 1, \dots, n \\ & v_i \in [0, \bar{v}_i] \cap \mathbb{Z}, \forall i = 1, \dots, n \\ & \lambda_i \in [0, \bar{\lambda}_i] \cap \mathbb{Z}, \forall i = 1, \dots, n \\ & \mu_i \in [0, \bar{\mu}_i] \cap \mathbb{Z}, \forall i = 1, \dots, n. \end{aligned}$$

Here, objective function  $C(u, v, \lambda, \mu)$  is a nondecreasing function. Recently, one often considers the objectives w.r.t.  $l_1$ -,  $l_2$ -,  $l_\infty$ -norm, or Hamming distance for measuring the paying costs of the inverse optimization problems. In the following, we investigate properties and algorithms regarding (FIFKP) under  $l_1$ - and  $l_\infty$ -norm.

## 3 The problem under $l_1$ -norm

Let us consider (IFKP) under  $l_1$ -norm. Assume that we pay  $w_i$  ( $w'_i$ ) for modifying one unit of profit (cost), the corresponding objective function can be written as

$$C(u, v, \lambda, \mu) := \sum_{i=1}^{n} (w_i(u_i + v_i) + w'_i(\lambda_i + \mu_i)).$$

We first get the following result concerning (IFKP) under  $l_1$ -norm.

**Theorem 3.1.** (IFKP) under  $l_1$ -norm with both variable profits and costs is NP-hard.

**Proof.** Consider an instance of Partition problem (PP). Given a set of integers  $S = \{a_1, a_2, ..., a_n\}$  such that  $\sum_{i=1}^n a_i = 2B$ , where B is a positive integer. Does there exist a subset S' of S such that  $\sum_{a_i \in S'} a_i = B$ ? This problem is NP-complete; see Garey and Johnson [12].

The decision version of (IFKP) is stated as follows. Given an instance of the inverse fractional knapsack problem. Does there exist a modification of profits and costs such that a prespectified solution become optimal and the objective value is at most C?

Given an instance of (PP). We construct an instance of (IFKP) in polynomial time.

- The profits are  $p_i := 4a_i$  for  $i = 1, \ldots, n$  and  $p_{n+1} = 4$ .
- The costs are  $c_i := 2a_i$  for  $i = 1, \ldots, n$  and  $c_{n+1} = 1$ .
- Let  $\bar{\mu}_i := a_i$  and  $\bar{u}_i := 4a_i$  for i = 1, ..., n;  $\bar{\lambda}_i = \bar{v}_i := 0$  for i = 1, ..., n + 1;  $\bar{u}_{n+1} = \bar{v}_{n+1} := 0$ .
- Set b := 3B and  $x^* = (1, ..., 1, 0)$  with n 1's and choose C := 7B.
- The corresponding weight to modify one unit of profit  $p_i$  is  $w_i = 1$  and cost  $c_i$  is  $w'_i = 3$  for i = 1, ..., n.

Observe that, in the current state of the problem we obtain  $\frac{p_i}{c_i} < \frac{p_{n+1}}{c_{n+1}}$  for  $i = 1, \ldots, n$ . Furthermore,  $\sum_{i=1}^{n+1} c_i x_i^* = 4B$ . Hence, vector  $x^*$  is not feasible. To make it an optimal solution, we increase the profits  $p_i$  for  $i = 1, \ldots, n$  and reduce the costs  $c_i$  for  $i = 1, \ldots, n$ . In what follows we prove that the answer to (PP) is 'yes' iff the answer to (IFKP) is 'yes'.

Assume that the answer to (PP) is 'yes'. Then there exists a subset  $S' \subset S$  such that  $\sum_{a_i \in S'} = B$ . We set  $u_i = 0$  and  $\mu_i := a_i$  for  $a_i \in S'$ . Otherwise, let  $u_i := 4a_i$  and  $\mu_i := 0$  for  $a_i \notin S'$ . It is trivial to check that  $x^*$  is an optimal solution of the modified fractional knapsack problem and the objective is B.

Conversely, assume that the answer to (IFKP) is 'yes'. We prove that the answer to (PP) is 'yes'. We first prove that the modification of profit  $p_i$  can be shifted to the modification of  $c_i$  without increasing the objective value. Indeed, assume that we modify profit  $p_i$  by x units and  $c_i$  by y units  $(y \le a_i)$ . By the optimality criterion, we get  $\frac{4a_i+x}{2a_i-y}=4$  or  $x+4y=4a_i$ . Hence, x is a multiplier of 4. The modification yield an objective value  $x+3y=4a_i-y\ge 3a_i$ . Furthermore, if we shift  $\frac{x}{4}$  units from  $p_i$  to  $c_i$ , we get  $\frac{4a_i}{2a_i-\frac{x}{4}-y}=\frac{4a_i}{a_i}=4$  and the corresponding objective value  $3(\frac{x}{4}+y)=\frac{3}{4}(x+4y)=3a_i$ . Hence, it is trivial that the second option of modification reduce the objective value. We can assume that there exists at most one modification  $\lambda_{i_0}$  such that  $\lambda_{i_0}=a_{i_0}-k$  with  $0 < k < a_{i_0}$  and  $\lambda_i=0$  or  $\mu_i=a_i$  for  $i\neq i_0$ . Let us set  $I:=\{i\in\{1,\ldots,n\}: \mu_i=a_i\}$ . As  $x^*$  become an optimal solution of (FKP), we get

$$\sum_{i \in I} a_i + 2\sum_{j \notin I} a_j - k = 3B \text{ or } \sum_{i \notin I} a_i = B + k$$
 (1).

As the objective function is at most B, we get

$$3(\sum_{i \in I} a_i + k) + 4\sum_{j \notin I} a_j - 2k \le 7B \text{ or } \sum_{j \notin I} a_j - \frac{k}{2} \le B$$
 (2).

From (1) and (2) we get  $\frac{k}{2} \leq 0$ . In other words, k = 0 and  $\sum_{i \in I} a_i = B$ . Set  $S' := \{a_i : i \in I\}$ , then the sum of items in S' is B.

As (IFKP) is NP-hard, there does not exist a polynomial time algorithm to solve it, unless P = NP. Therefore, approximation, heuristic approach, or special polynomially solvable cases of the problem are interesting topics.

Now let  $x^*$  be a feasible solution of (FKP), i.e.,  $\sum_{i=1}^n c_i x_i^* = b$ . We focus on the problem of modifying only profit parameters while the costs are fixed, i.e., we set  $\lambda_i = \mu_i = 0$  for i = 1, ..., n. We call this problem the fixed cost inverse fractional knapsack problem (FIFKP). It is trivial to get the following result.

**Proposition 2.** There exists an optimal modifications of (FIFKP) s.t. the profits  $\tilde{p}_i$  are increased if  $i \in \{i : x_i^* = 1\}$  and reduced if  $i \in \{i : x_i^* = 0\}$ 

**Proof.** It is straight forward as we have to increase the ratios  $\frac{\tilde{p}_i}{c_i}$  for  $i \in \{i : x_i^* \neq 0\}$  and reduce the ones for  $i \in \{i : x_i^* = 0\}$ .

We denote by  $I^0 = \{i : x_i^* = 0\}$  and  $I^1 = \{i : x_i^* = 1\}$ . We first consider the feasibility condition of (FIFKP). Let

$$L = \max_{i \in I^0} \left\{ \frac{p_i - \bar{z}_i}{c_i} \right\}, \quad U = \min_{i \in I^1} \left\{ \frac{p_i + \bar{z}_i}{c_i} \right\}.$$

Then we obtain the following result.

**Proposition 3.** (FICKP) is feasible iff  $L \leq U$ .

**Proof.** L is the maximum reduction of ratios  $\frac{\tilde{p}_i}{c_i}$  for  $i \in I^0$ , while U is the maximum augmentation of ratios  $\frac{\tilde{p}_i}{c_i}$  for  $i \in I^1$ . The result follows.

From now on, we always assume that (FIFKP) is feasible. By Proposition 2, we set  $v_i = 0$  for  $i \in I^1$  and  $u_i = 0$  for  $i \in I^0$ . We further denote by  $z_i := \begin{cases} u_i, & \text{if } i \in I^1, \\ v_i, & \text{if } i \in I^0, \end{cases}$  and  $\bar{z}_i := \begin{cases} \bar{u}_i, & \text{if } i \in I^1, \\ \bar{v}_i, & \text{if } i \in I^0. \end{cases}$  We say that a profit  $p_i$  is modified by  $z_i$  if it is reduced (increased) by  $z_i$  for  $i \in I^0$  ( $I^1$ ). Assume that modifying an amount of  $p_i$  yields a corresponding cost  $w_i$ . The objective function can be written as

$$C(z) = \sum_{i=1}^{n} w_i z_i.$$

**Presolution:** We presolve the problem by increasing the profits w.r.t. the items in  $\{i \in I^1 : \frac{p_i}{c_i} < L\}$  to L and reducing the items in  $\{i \in I^0 : \frac{p_i}{c_i} > U\}$  to U. In other words, For  $i \in I^1$  and  $\frac{p_i}{c_i} < L$ , we find the minimum value  $z_i^0$  such that  $\frac{p_i + z_i^0}{c_i} \ge L$ . As  $z_i$  is an integer, we can set  $z_i^0 := \lceil c_i L - p_i \rceil$ . By the same argument, we set  $z_i^0 := \lceil p_i - c_i U \rceil$  for  $i \in I^0$  and  $\frac{p_i}{c_i} > U$ . The corresponding cost is  $C_0 := \sum_{i \in \mathcal{P}} w_i z_i^0$ , where  $\mathcal{P} := \{i \in I^1 : \frac{p_i}{c_i} < L\} \cup \{i \in I^0 : \frac{p_i}{c_i} > U\}$ .

Next we solve (FIFKP). Let  $\alpha:=\min_{i\in I^1}\left\{\frac{p_i}{c_i}\right\}$  and  $\beta:=\max_{i\in I^0}\left\{\frac{p_i}{c_i}\right\}$ . Also, we denote by  $\tilde{I}^\star:=I^\star\cap\left\{i:\frac{p_i}{c_i}\in[\alpha,\beta]\right\}$  for  $\star=0,1$ . Observe that, we only modify the profits with indices in  $\tilde{I}^0\cup\tilde{I}^1$ . For a parameter  $t\in[\alpha,\beta]$ , we reduce (increase) the ratio  $\frac{\tilde{p}_i}{c_i}$  such that it is less than (greater than) t for  $j\in\tilde{I}^0$  ( $i\in\tilde{I}^0$ ). Denote by  $I^1(t):=\{i\in\tilde{I}^1:\frac{p_i}{c_i}< t\}$  and  $I^0(t):=\{i\in\tilde{I}^0:\frac{p_i}{c_i}> t\}$ . We find  $z_i$  such that  $\frac{p_i-z_i}{c_i}\leq t$  or  $z_i\geq p_i-c_it$  for  $i\in I^0(t)$ . As  $z_i$  is an integer for  $i\in I^0(t)$ , we get  $z_i:=\lceil p_i-c_it\rceil$ . Analogously, we can set  $z_i:=\lceil c_it-p_i\rceil$  for  $i\in I^1(t)$ . Therefore, the objective function w.r.t. parameter t can be written as follows.

$$C(t) = \sum_{i \in I^1(t)} w_i \lceil c_i t - p_i \rceil + \sum_{i \in I^0(t)} w_i \lceil p_i - c_i t \rceil.$$

Let  $\mathcal{B} := \{\frac{p_i}{c_i} : i \in \mathcal{P}\}$ . Assume that  $\mathcal{B} := \{t_1, t_2, \dots, t_n\}$  with  $t_1 < t_2 < \dots < t_n$ . For  $t \in (t_i, t_{i+1})$  with  $t_i$  and  $t_{i+1}$  being two consecutive members of  $\mathcal{B}$ , the set  $I^0(t)$  and  $I^1(t)$  do not change. In other words,  $I^*(t) = I^*(t')$  for \* = 0, 1 and  $t, t' \in (t_i, t_{i+1})$ . As [.] is a quasi-concave function, we get the following result.

**Proposition 4.** C(t) is a quasi-concave function for  $t \in (t_i, t_{i+1}), t_i, t_{i+1} \in \mathcal{B}$ .

As C(t) is quasi-concave for  $t \in (t_i, t_{i+1})$ , it is also quasi-concave in  $[t_i, t_{i+1}]$ . Therefore, the minimum value of C(t) on  $[t_i, t_{i+1}]$  is obtained at  $t_i$  or  $t_{i+1}$  as  $C(t) \ge \min\{C(t_i), C(t_{i+1})\}$  for  $t \in [t_i, t_{i+1}]$ .

The following example states that C(t) is however neither quasi-convex nor quasi-concave.

**Example 3.1.** Given  $x^* = (1, 1, 0, 1, 0)$  be a feasible solution. The corresponding profits and costs are given in the following table.

| i           | 1 | 2   | 3   | 4  | 5  |
|-------------|---|-----|-----|----|----|
| $p_i$       | 8 | 7   | 9   | 10 | 11 |
| $c_i$       | 5 | 10  | 10  | 10 | 10 |
| $\bar{z}_i$ | 3 | 4   | 3   | 1  | 4  |
| $w_i$       | 3 | 1/2 | 1/2 | 1  | 1  |

Table 1: An instance of (FICKP)

First of all, the set  $\mathcal{B}$  consists of  $t_1 = \frac{3}{5}; t_2 = \frac{7}{10}; t_3 = \frac{9}{10}; t_4 = \frac{10}{10}; t_5 = \frac{11}{10}.$  We compute the objective value at each break-points as follows.  $C(t_1) = \frac{3}{2} + 5 = 6, 5; C(t_2) = 3 + 1 + 4 = 8; C(t_3) = 6 + 1 + 2 = 9; C(t_4) = 6 + \frac{3}{2} + 1 = 8, 5; C(t_5) = 9 + 2 + 1 = 12.$  Hence, C(t) is neither quasi-convex nor quasi-concave for  $t \in [t_1 = \frac{3}{5}; t_1 = \frac{11}{10}].$ 

Now we know that the objective function is neither quasi-convex nor quasi-concave. To find the optimal solution of C(t), we first compute the values of C(t) at all break-points in  $\mathcal{B}$ . Then we take the best one. The value of C(t) at each break-point can be computed in linear time. Furthermore, there are at most linearly many break-points. Hence, the optimal solution of C(t) can be found in quadratic time. We get the main result of this section.

**Theorem 3.2.** The inverse fractional knapsack problem with variable profits can be solved in quadratic time.

### 4 Problem under $l_{\infty}$ -norm

Now we investigate the uniform-cost inverse fractional knapsack problem under  $l_{\infty}$ -norm. The corresponding objective function can be rewritten as follows.

$$\max_{i=1}^n \{u_i, v_i, \lambda_i, \mu_i\}.$$

Let us recall that  $I^1$  and  $I^0$  are the set of items in  $x^*$  with value 1's and 0's, respectively. Moreover, a property of modifying profits and costs is given as follows.

**Proposition 5.** There exists an optimal solution such that we increase (reduce) the profits of items in  $I^1$  ( $I^0$ ) and increase the costs of items in  $I^0$ .

**Proof.** Similar to Proposition 2.

By Proposition 5, we set  $v_i := 0$  for  $i \in I^1$ ,  $u_i = \mu_i := 0$  for  $i \in I^0$ . We study the two following situations.

Case 1: If  $\sum_{i=1}^{n} c_i x_i^* = b$ , then  $x^*$  is a feasible solution. Therefore, we do not modify the cost coefficients  $c_i$  for  $i \in I^1$ . Otherwise, if we do modify the costs of items in  $I^1$ , the optimality criterion does not hold according to the infeasibility of  $x^*$ .

Let us set  $\lambda_i = \mu_i := 0, \forall i \in I^1$ . By the optimality criterion, the following condition must hold

$$\min_{i \in I^1} \left\{ \frac{\tilde{p}_i}{\tilde{c}_i} \right\} \ge \max_{j \in I^0} \left\{ \frac{\tilde{p}_j}{\tilde{c}_j} \right\}.$$

Hence, for each i, j such that  $i \in I^1$ ,  $j \in I^0$  and  $\frac{p_i}{c_i} < \frac{p_j}{c_j}$ , we calculate the minimum object value such that

$$\frac{\tilde{p}_i}{c_i} \ge \frac{\tilde{p}_j}{\tilde{c}_j}$$

Replacing  $\tilde{p}_i, \tilde{p}$ , and  $\tilde{c}_j$  by  $p_i + u_i, p_j - v_j$ , and  $c_j + \lambda_j$ , we get

$$\frac{p_i + u_i}{c_i} \ge \frac{p_j - v_i}{c_j + \lambda_j}$$

After some elementary computations, we get the inequality

$$u_i \lambda_j + p_i \lambda_j + u_i c_j + c_i v_i \ge c_i p_j - p_i c_j \tag{3}$$

We sort the corresponding upper bounds  $\bar{u}_i, \bar{\lambda}_j, \bar{u}_i, \bar{v}_i$ , then we compute the value on the left hand side of (3) w.r.t. the threholds. Then it is trivial to compute the smallest objective such that (3) holds. Let  $K_{ij}$  be the minimum value such that  $\frac{\tilde{p}_i}{c_i} \geq \frac{\tilde{p}_j}{\tilde{c}_j}$ . Then  $K = \max \min_{\substack{i \in I^1 \\ j \in I^0}} K_{ij}$  is the optimal object value.

Case 2: If  $\sum_{i=1}^{n} c_i x_i^* < b$ , the vector  $x^*$  is not feasible. Therefore, we first modify the cost optimally so that  $x^*$  become a feasible solution as follows.

Let  $S := \{\bar{\mu}_i : i \in I^1\}$ . We aim to find the smallest value, say  $\bar{\mu}_{i^*}$ , in S such that

$$\frac{b-\sum_{j\in I^1}c_j}{|I^1|}\leq \bar{\mu}_{i^*}$$

for  $i \in I^1$ . It can be done by applying a binary search algorithm. Indeed, let m be the median of S and  $\bar{\mu}_{i_0}$  be the largest element in S which is less than or equal m. If  $\frac{b-\sum_{j\in I^1}c_j}{|I^1|} < \bar{\mu}_{i_0}$ , we know that  $\mu_{i^*} \leq \mu_{i_0}$  and one thus has to find  $\mu_{i^*}$  in  $S := S \setminus \{\mu_i > \mu_{i_0}\}$ . Otherwise, we know that  $\mu_{i^*} > \mu_{i_0}$  and consider  $S := S \setminus \{\mu_i \leq \mu_{i_0}\}$ . This algorithm find  $\bar{\mu}_{i^*}$  in linear time.

After finding  $\bar{\mu}_{i^*}$ , we set  $\mu_i := \bar{\mu}_i$  for  $i \in I^1$  and  $\bar{\mu}_i < \bar{\mu}_{i^*}$ . Then set  $\mu_i := \max\{\mu_i : \mu_i \in S \text{ and } \bar{\mu}_i \geq \bar{\mu}_{i^*}\}$  and  $J^1 := \{i \in I^1 : \bar{\mu}_i \geq \bar{\mu}_{i^*}\}$  and  $J^2 := \{i \in I^1 : \bar{\mu}_i < \bar{\mu}_{i^*}\}$ . We further find  $\mu^{\min} = \left\lceil \frac{b - \sum_{i \in I^1} c_i - \sum_{j \in J^2} \bar{\mu}_j}{|J^1|} \right\rceil$ . Then it is easy to verify that

$$\sum_{j \in J^2} (c_j + \bar{\mu}_j) + \sum_{j \in J^1} (c_j + \mu^{\min}) \ge b.$$

Next we study which variables in  $J^1$  should takes value  $\mu^{\min}$  and  $\mu^{\min} - 1$ . We first consider how many variables in  $J^1$  take value  $\mu^{\min}$ . This number equals

$$N := b - \sum_{j \in J^2} (c_j + \bar{\mu}_j) - \sum_{j \in J^1} (c_j + \mu^{\min} - 1).$$

Hence, there are N variables in  $J^1$  obtaining value  $\lambda^{\min}$  and  $|J^1| - N$  variables in  $J^1$  obtaining value  $\lambda^{\min} - 1$ .

The objective function is  $\lambda^{\min}$  to modify the costs so that  $x^*$  becomes feasible. In order to reduce the ratios  $\frac{\tilde{p}_i}{\tilde{c}_i}$  for  $i \in I^0$  and augment the the ratios  $\frac{\tilde{p}_i}{\tilde{c}_i}$  for  $i \in I^1$ , we set  $\tilde{p}_i := \begin{cases} p_i + \min\{\bar{u}_i, \lambda^{\min}\}, & \text{if } i \in I^1, \\ p_i - \min\{\bar{v}_i, \lambda^{\min}\}, & \text{if } i \in I^0, \end{cases}$  We also update  $\bar{\star}_i := \begin{cases} 0, & \text{if } \bar{\star}_i \leq \lambda^{\min}, \\ \bar{\star}_i - \lambda^{\min}, & \text{if } \bar{\star}_i > \lambda^{\min}, \end{cases}$  for  $\star = u, v$  and  $i \in I^0$  or  $i \in I^1$ , accordingly.

Let us now consider the set the current rations  $\frac{\tilde{p}_i}{\tilde{c}_i}$ . For each  $i \in I^1$  we compute the largest cost, say  $K_i$ , such that  $\frac{p'_i}{c'_i} \geq \frac{p'_j}{c'_j}$  for all  $j \in J^0$  by the similar approach in Case 1. We then consider the chance to reduce the cost as follows. We first take  $|J^1| - N$  items with respect to the  $|J^1| - N$  largest costs in  $\{K_i\}_{i \in I^1}$  and set  $\tilde{p}_i = \tilde{p}_i + \lambda^{\min} - 1$ . Then we reevaluate the costs  $K'_i$  with respect to new items.

Case 3: If 
$$\sum_{i=1}^{n} c_i x_i^* > b$$
, we can solve the problem as in Case 2.

In summary, we first check the feasibility of  $x^*$ . If it is not feasible, we can justify the cost coefficients in linear time to make it feasible. Then, it costs quadratic time to compute  $K_{ij}$  for  $i \in I^1$  and  $j \in I^0$ . The final step costs also quadratic time in order to find  $\tilde{K}_{ij}$ . Hence, the total computation complexity is quadratic.

**Theorem 4.1.** The inverse fractional knapsack problem under  $l_{\infty}$ -norm can be solved in quadratic time.

#### 5 Conclusions

We considered the inverse fractional knapsack problem with profit and cost modifications. It is shown that the problem under  $l_1$ -norm is, in general, NP-hard. Especially, if we can only justify the profit parameters, this problem is solvable in  $O(n^2)$  time. Moreover, we solve the problem under  $l_{\infty}$ -norm in quadratic time based on greedy type algorithm.

It is promising to study the inverse fractional knapsack problem under various of objective functions. Furthermore, it is also worthwhile to consider the inverse mixed integer knapsack problem by combining the techniques in this paper and in Roland [9].

### References

- [1] B. Alizadeh and R.E. Burkard, Combinatorial algorithms for inverse absolute and vertex 1-center location problems on trees, Networks, vol. 58 pp. 190-200, 2011.
- [2] Ahuja, R.K., Magnanti, T.L., Orlin, J.B.: Network flows: theory, algorithms, and applications (1993)
- [3] Balas, E, Zemel, E.: An algorithm for large zero-one knapsack problems, operations Research 28(5), 1130-1154 (1980)
- [4] S. Bespamyatnikh, B. Bhattacharya, M. Keil, D. Kirkpatrick and M. Segal, Efficient algorithms for centers and medians in interval and circular-arc graphs. Networks vol. 39, 144-152, 2002.

- [5] Burton, D., Toint, P.L.: On an instance of the inverse shortest paths problem. Mathematical Programming 53(1), 45-61 (1992)
- [6] Ahuja, Ravindra K and Ergun, Özlem and Orlin, James B and Punnen, Abraham P, A survey of very large-scale neighborhood search techniques, Discrete Applied Mathematics, Elsevier, vol. 123, 75-102, 2002.
- [7] R.E. Burkard, C. Pleschiutschnig, and J.Z. Zhang, Inverse median problems, Discrete optimization, vol. 1. pp. 23-39, 2004.
- [8] Heuberger, C.: Inverse combinatorial optimization: A survey on problems, methods, and results. Journal of Combinatorial Optimization 8(3), 329-361 (2004)
- [9] Roland, Julien and Figueira, The inverse {0, 1}-knapsack problem: theory, algorithms and computational experiments, Discrete Optimization, Elsevier, vol. 10, 181-192 (2013)
- [10] Zhang, J., Xu, S., Ma. Z.: An algorithm for inverse minimum spanning tree problem. Optimization Methods and Software 8(1), 69-84 (1997)
- [11] Sokkalingam, PT and Ahuja, Ravindra K and Orlin, James B, Solving inverse spanning tree problems through network flow techniques, Operations Research, vol.47, 291-298 (1999)
- [12] Garey, Michael R and Johnson, David S, A Guide to the Theory of NP-Completeness, WH Freemann, New York, vol. 70 (1979)
- [13] M. Galavii, The inverse 1-median problem on a tree and on a path, Electronic Notes in Disrete Mathematics, vol. 36, pp. 1241-1248, 2010.
- [14] E. Gassner, An inverse approach to convex ordered median problems in trees. Journal of Combinatorial Optimization, vol. 23, pp. 261-273, 2012.
- [15] K.T. Nguyen, Inverse 1-median problem on block graphs with variable vertex weights, Journal of Optimization Theory and Applications, DOI: 10.1007/s10957-015-0829-2, 2015.
- [16] K.T. Nguyen and A. Chassein, Inverse eccentric vertex problem on networks, Cent. Eur. J. Oper. Res., vol. 23, pp. 687-698, 2015.
- [17] K.T. Nguyen and L.Q. Anh, Inverse k-centrum problem on trees with variable vertex weights, Math. Meth. Oper. Res., vol. 82, pp 19-30.
- [18] K.T. Nguyen and A. Chassein, The inverse convex ordered 1-median problem on trees under Chebyshev norm and Hamming distance, European Journal of Operational Research vol. 247, pp. 774-781, 2015.
- [19] K.T. Nguyen and A.R. Sepasian, The inverse 1-center problem on trees with variable edge lengths under Chebyshev norm and Hamming distance, Journal of Combinatorial Optimization, vol. 32, pp. 872-884, 2016.