CSE 571 Homework 4

Name: Student ID:

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Due: November 3., 2016

For each homework we will state here if you have to work alone or if you can team up with another student.
For this homework you are allowed to work in **teams of two students** for all questions. Each group has to submit one **handwritten** (!) copy and state their names on the front page. Use the same groups as in Blackboard. Failure to do so could result in the allegation of plagiarism! Submission is possible at the end of a lecture or during an office hour till the due date of this homework (which you can find above).
Print this homework and write all your answers in the space below the questions. If you need additional space you might want to use the backside of the pages.
Also,

- unstapled homework will result in a decrease of at least 30% of the achieved points.
- handwritten text which is not readable will be graded with zero points.

	Q1	Q2	Q3	Q4	Q5	Sum
Points	3	4	4	7	30	48
Achieved						

1. Question (3 Points) Probability

Below is a table listing the probabilities of three binary random variables. In the empty table cells, fill in the correct values for each marginal or conditional probability. Your can round your answers to 4 decimal places.

X_0	X_1	X_2	$P(X_0, X_1, X_2)$
0	0	0	0.14
1	0	0	0.14
0	1	0	0.18
1	1	0	0.08
0	0	1	0.08
1	0	1	0.16
0	1	1	0.1
1	1	1	0.12

	Value
$P(X_0 = 1, X_1 = 0, X_2 = 1)$	
$P(X_0 = 0, X_1 = 1)$	
$P(X_2 = 0)$	
$P(X_1 = 0 \mid X_0 = 1)$	
$P(X_0 = 1, X_1 = 0 \mid X_2 = 1)$	
$P(X_0 = 1 \mid X_1 = 0, X_2 = 1)$	

2. Question (4 Points) Probability

You are given the prior distribution $P(X)$, and two conditional distributions $P(Y|X)$ and $P(Z|Y)$ as below (you are also given the fact that Z is independent from X given Y). All variables are binary variables. Compute the table of their joint distribution based on the chain rule. Your might round your answers to 4 decimal places.

X	$P(X)$
0	0.1
1	0.9

Y	X	$P(Y\|X)$
0	0	0.5
1	0	0.5
0	1	0.2
1	1	0.8

Z	Y	$P(Z\|Y)$
0	0	0.8
1	0	0.2
0	1	0.8
1	1	0.2

X	Y	$P(X,Y)$
0	0	
1	0	
0	1	
1	1	

X	Y	Z	$P(X,Y,Z)$
0	0	0	
1	0	0	0.144
0	1	0	0.04
1	1	0	
0	0	1	0.01
1	0	1	
0	1	1	0.01
1	1	1	

3. Question (4 Points) Probability

For the following four subparts, you are given three joint probability distribution tables. For each distribution, please identify if the given independence / conditional independence assumption is true or false. For your convenience, we have also provided some marginal and conditional probability distribution tables that could assist you in solving this problem.

3.1) Given the probabilities

X	Y	$P(X,Y)$
0	0	0.3
1	0	0.24
0	1	0.08
1	1	0.38

X	P(X)
0	0.38
1	0.62

Y	P(Y)
0	0.54
1	0.46

X is independent from Y: ○ true ○ false

3.2) Given the probabilities

X	Y	$P(X,Y)$
0	0	0.26
1	0	0.26
0	1	0.34
1	1	0.14

X	$P(X)$
0	0.6
1	0.4

X	Y	$P(X\|Y)$
0	0	0.5
1	0	0.5
0	1	0.708
1	1	0.292

X is independent from Y: ○ true ○ false

3.3) Given the probabilities:

X	Z	$P(X\|Z)$
0	0	0.9
1	0	0.1
0	1	0.9
1	1	0.1

Y	Z	$P(Y\|Z)$
0	0	0.6
1	0	0.4
0	1	0.6
1	0	0.4

X	Y	Z	$P(X,Y\|Z)$
0	0	0	0.54
1	0	0	0.06
0	1	0	0.36
1	1	0	0.04
0	0	1	0.54
1	0	1	0.06
0	1	1	0.36
1	1	1	0.04

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.108
1	0	0	0.012
0	1	0	0.072
1	1	0	0.008
0	0	1	0.432
1	0	1	0.048
0	1	1	0.288
1	1	1	0.032

X is independent from Y given Z : ○ true ○ false

3.4) Given the probabilities

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.07
1	0	0	0.14
0	1	0	0.42
1	1	0	0.07
0	0	1	0.03
1	0	1	0.06
0	1	1	0.09
1	1	1	0.12

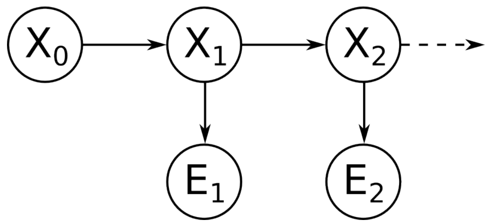
X	Y	Z	$P(X\|Y,Z)$
0	0	0	0.333
1	0	0	0.667
0	1	0	0.857
1	1	0	0.143
0	0	1	0.333
1	0	1	0.667
0	1	1	0.429
1	1	1	0.571

X	Z	$P(X\|Z)$
0	0	0.7
1	0	0.3
0	1	0.4
1	1	0.6

X is independent from Y given Z: ○ true ○ false

4. Question (7 Points) Hidden Markov Models

Consider the HMM shown below.



The prior probability $P(X_0)$, dynamics model $P(X_{t+1}|X_t)$, and sensor model $P(E_t|X_t)$ are as follows:

X_0	$P(X_0)$
0	0.7
1	0.3

X_{t+1}	X_t	$P(X_{t+1}\|X_t)$
0	0	0.4
1	0	0.6
0	1	0.45
1	1	0.55

E_t	X_t	$P(E_t\|X_t)$
a	0	0.1
b	0	0.05
c	0	0.85
a	1	0.5
b	1	0.05
c	1	0.45

We perform a first dynamics update and fil in the resulting belief distribution $B'(X_1)$:

X_1	$B'(X_1)$
0	0.414
1	0.585

.

We incorporate the evidence $E_1 = c$ and fill in the evidence-weighted distribution $P(E_1 = c|X_1)B'(X_1)$ and the (normalized) belief distribution $B(X_1)$:

X_1	$P(E_1 = c\|X_1)B'(X_1)$
0	0.35275
1	0.26325

X_1	$B(X_1)$
0	0.572646103896
1	0.427353896104

4.1) (3 Points) Your task is now to perform the second dynamics update. Fill in the resulting belief distribution $B'(X_2)$. You might want to round to four decimal places.

X_2	$B'(X_2)$
0	
1	

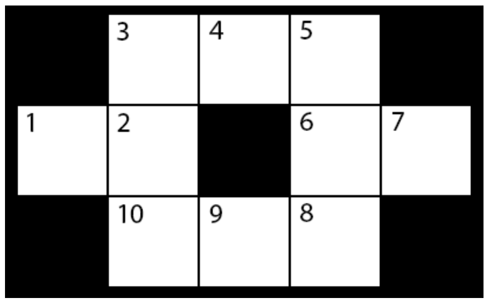
4.2) (4 Points) Now incorporate the evidence $E_2 = a$. Fill in the evidence-weighted distribution $P(E2 = a|X_2)B'(X_2)$ and the (normalized) belief distribution $B(X_2)$. You might want to round to four decimal places.

X_2	$P(E_2 = a|X_2)B'(X_2)$
0	
1	

X_2	$B(X_2)$
0	
1	

5. Question (30 Points) Particle Filter

In this question, we will use a particle filter to track the state of a robot that is lost in the small map below:



The robot's state is represented by an integer $1 \leq X_t \leq 10$ corresponding to its location in the map at time t. We will approximate our belief over this state with $N = 8$ particles.

You have no control over the robot's actions. At each timestep, the robot either stays in place, or moves to any one of its neighboring locations, all with equal probability. For example, if the robot starts in state $X_t = 7$, it will move to state $X_{t+1} = 6$ with probability $\frac{1}{2}$ or $X_{t+1} = 7$ with probability $\frac{1}{2}$. Similarly, if the robot starts in state $X_t = 2$, the next state X_{t+1} can be any element of $\{1, 2, 3, 10\}$ and each occurs with probability $\frac{1}{4}$.

At each time step, a sensor on the robot gives a reading $E_t \in \{H, C, T, D\}$ corresponding to the type of state the robot is in. The possible types are:

- **Hallway (H)** for states bordered by two parallel walls (4,6).
- **Corner (C)** for states bordered by two orthogonal walls (3,5,8,10).
- **Tee (T)** for states bordered by one wall (2,6).
- **Dead End (D)** for states bordered by three walls (1,7).

The sensor is not very reliable: it reports the correct type with probability $\frac{1}{2}$ but gives erroneous readings the rest of the time, with probability $\frac{1}{6}$ for each of the three other possible readings.

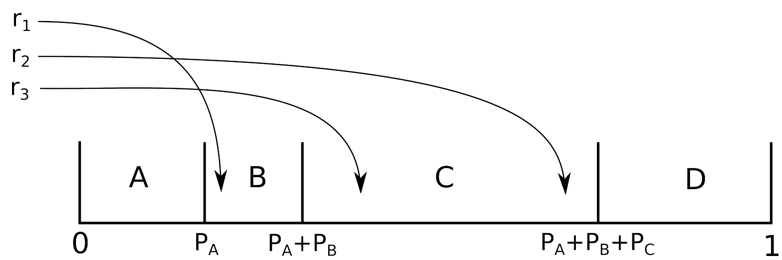
For all questions below, you can round your answers to four decimal places! Also make sure to check your answers for errors, since you might have to reuse results in later questions.

5.1) (8 Points) **Sensor Model:**
Fill in the sensor model below:

Sensor Reading	State Type	P(Sensor|State Type)
H	H	
C	H	
T	H	
D	H	
H	C	
C	C	
T	C	
D	C	
H	T	
C	T	
T	T	
D	T	
H	D	
C	D	
T	D	
D	D	

5.2) Sampling Review:

Suppose that we want to sample from a set of 4 events, $\{A, B, C, D\}$, which occur with corresponding probabilities P_A, P_B, P_X, P_C. First, we form the set of cumulative weights, given by $\{0, P_A, P_A + P_B, P_A + P_B + P_C, 1\}$. These weights partition the $[0, 1)$ interval into bins, as shown below. We then draw a number r uniformly at random from $[0, 1)$ and pick A, B, C or D based on which bin r lands in. The process is illustrated in the diagram below. If r_1, uniformly chosen from $[0, 1)$, lands in the interval $[P_A, P_A + P_B)$, then the resulting sample would be B. Similarly, if r_2 lands in $[P_A + P_B, P_A + P_B + P_C)$ the sample would be C.



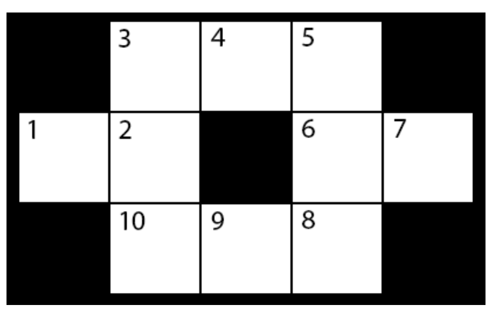
5.3) (8 Points) **Belief State for $t = 0$:**

Now we will sample the starting positions for our particles. For each particle p_i, we have generated a random number r_i sampled uniformly from $[0, 1)$. Your job is to use these numbers to sample a starting location for each particle. As a reminder, locations are integers from the rang $[1, 10]$, as shown in the map above. You should assume that the locations go in ascending order and that each location has equal probability. The random number generated for particle i, denoted by r_i, is provided in the table below. Please fill in the locations of the eight particles.

Particle	r_i	Location x_0
p_1	0.914	
p_2	0.473	
p_3	0.679	
p_4	0.879	
p_5	0.212	
p_6	0.024	
p_7	0.458	
p_8	0.154	

5.4) (4 Points) **Time update from $t = 0 \rightarrow t = 1$:**
Now we'll perform a time update using the transition model. Stated again, the transition model is as follows: At each timestep, the robot either stays in place, or moves to any one of its neighboring locations, all with equal probability.

For each particle, take the starting position you found in Part 2, and perform the time update for that particle. You should again sample from the range $[0, 1)$, where the **bins are the possible locations sorted in ascending numerical order**. As an example, if $X_t = 2$, the next state can be one of $\{1, 2, 3, 10\}$, each with equal probability, so the $[0, 0.25)$ bin would be for $X_{t+1} = 1$, the $[0.25, 0.5)$ bin would be for $X_{t+1} = 2$, the $[0.5, 0.75)$ bin would be for $X_{t+1} = 3$ and the $[0.75, 1)$ bin would be for $X_{t+1} = 10$. The map is shown again below:



Particle	r_i	Location x_1
p_1	0.674	
p_2	0.119	
p_3	0.748	7
p_4	0.802	
p_5	0.357	
p_6	0.736	
p_7	0.425	
p_8	0.058	

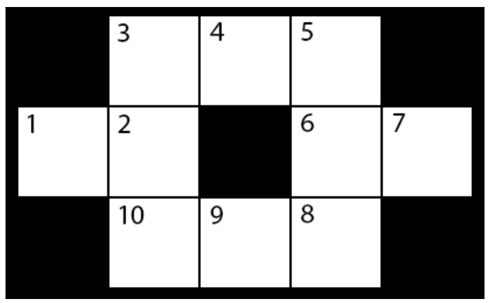
5.5) (4 Points) **Probability Distribution Induced by the Particles:** Recall that a particle filter just keeps track of a list of particles, but at any given time, we can compute a probability distribution from these particles. Using the current newly updated set of particles (that you found in the previous question) , give the estimated probability that the robot is in each location.

x_1	$P(x_1)$
1	
2	
3	
4	
5	

x_1	$P(x_1)$
6	
7	
8	
9	
10	

5.6) (2 Points) **Incorporating Evidence at** $t = 1$

The sensor reading at $t = 1$ is $E_1 = D$. Using the sensor model you specified in Part 1, incorporate the evidence by reweighting the particles. Also enter the normalized and cumulative weights for each particle. The normalized weight for a specific particle can be calculated by taking that particle's weight and dividing by the sum of all the particle weights. The cumulative weight keeps track of a running sum of all the weights of the particles seen so far (meaning, particle i at will have a cumulative weight equal to the sum of the weights of all particles j such that $j \leq i$). Refer back to question **5.4**) to get the positions of your particles. The map is shown again below:



Particle	p_1	p_2	p_3	p_4
Weight	0.1667			
Normalized Weight				0.0833
Cumulative Weight				

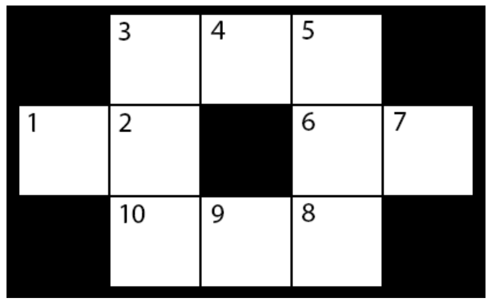
Particle	p_5	p_6	p_7	p_8
Weight				
Normalized Weight				
Cumulative Weight	0.5831			

5.7) (2 Points) **Resampling** Finally, we'll resample the particles. This reallocates resources to the most relevant parts of the state space in the next time update step.

Notice that your cumulative weights effectively tell you where the bins used in resampling the particles lie. For example, for particle 1, you calculated the cumulative weight to be some value, p. Then, on a random value draw, if a value between 0 and p was chosen, you would generate a new particle where particle 1 is. Use these bounds to resample the eight particles. In the "New Particle" row, enter the particle corresponding to the bin that the random value chose. In the "New Location" row, enter the location corresponding to this new particle. You may need to look back at Part 3 to get the locations of the particles.

Particle	r_1	New Particle	New Location
p_1	0.403		
p_2	0.218		
p_3	0.217	3	
p_4	0.826		
p_5	0.717		5
p_6	0.460		
p_7	0.794		
p_8	0.016		

5.8) (2 Points) **Analysis**
The sensor provided a reading $E_1 = D$. What fraction of the particles are now on a dead end? The map is shown again below:



Fraction of particles on a dead end: __________

This completes everything for the first time step, $t = 0 \rightarrow t = 1$. Of course, we would now continue by repeating the time update, evidence incorporation by reweighting, and resampling. We'll leave that to the computers, though. :)