

CS 302.1 - Automata Theory

Lecture 06

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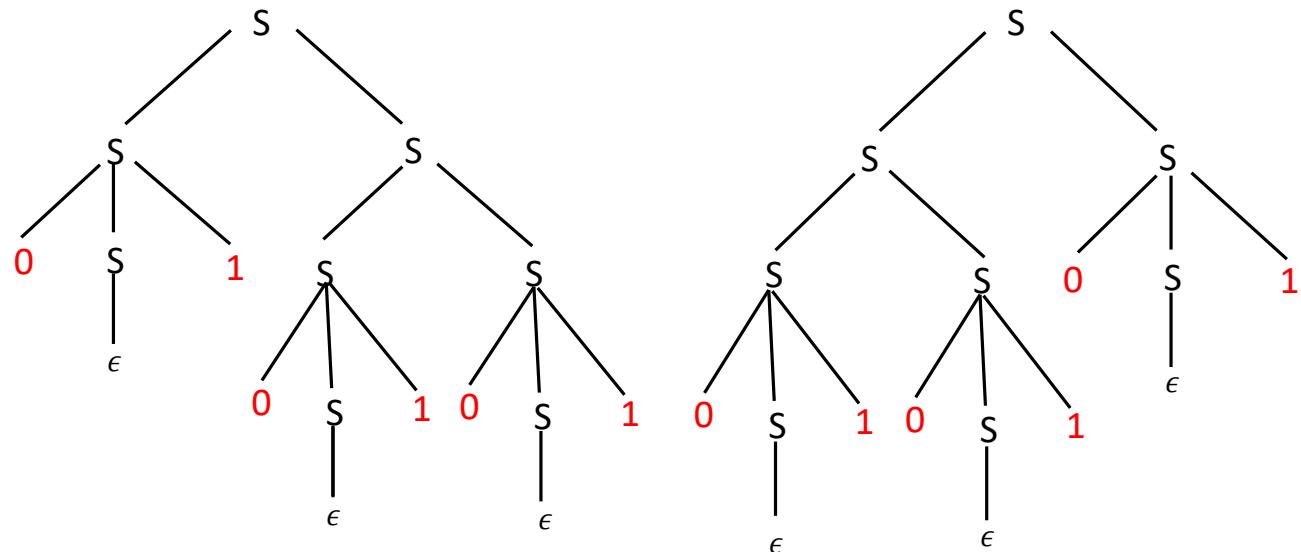
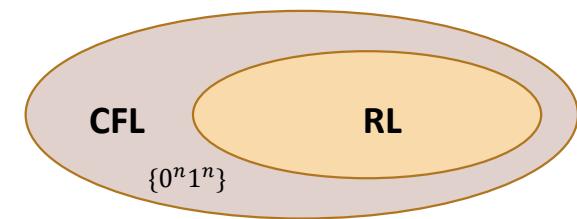
IIIT Hyderabad



Quick Recap

Context-Free Grammars: If the *rules* of the underlying grammar G are of the form
 $V \rightarrow (VUT)^*$

then such a grammar is called **Context-Free**.



Parse trees: These are ordered trees that provide alternative representations of the derivation of a grammar.

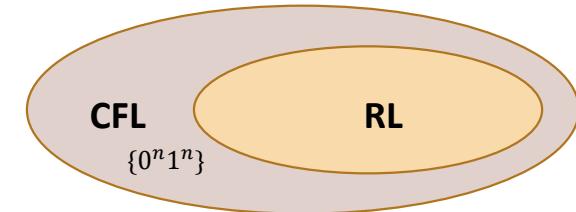
Ambiguous grammars: There exists $\omega \in L(G)$, such that there are **two or more leftmost derivations for ω** (or equivalently two or more rightmost derivations) or equivalently **two or more parse trees for ω** . Ambiguity may not be desirable

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Chomsky Normal Form: If every *rule* of the CFG is of the form

$$\begin{array}{ll} A \rightarrow BC & [B, C \text{ are not start variables}] \\ A \rightarrow a & [a \text{ is a terminal}] \\ S \rightarrow \epsilon & [S \text{ is the Start Variable}] \end{array}$$

- Any CFG can be converted to a grammar in CNF that generates the same language.
- The number of steps required to derive a string $w = 2|w| - 1$.
- Is crucial for deciding whether w is generated by a CFG G .

Chomsky Normal Form

Any CFL can be generated by a CFG written in Chomsky Normal Form.

Proof: The proof is constructive. Suppose we have a CFG G with a set of rules. To convert G into CNF, we do the following:

1. Add a new start variable $S' \rightarrow S$
2. Remove ϵ rules of the form $A \rightarrow \epsilon$
 - Remove nullable symbols/rules
3. Remove unit (short) rules of the form $A \rightarrow B$
 - Remove useless symbols/rules
4. Remove long rules of the form $A \rightarrow u_1 u_2 \cdots u_k$
 - Remove useless symbols/rules

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For each occurrence of A in the right side of the rule, we add a new rule with the occurrence of A deleted.

E.g.: Consider any rule $B \rightarrow uAvAw$ (u, v, w can be strings of variables and terminals)

Then new rules: $B \rightarrow uAvAw|uvAw|uAvw|uvw$

What if you had a rule such as $B \rightarrow A$? Then we would have needed to add a rule $B \rightarrow \epsilon$ (unless this rule has been already removed) as B is a **nullable variable**.

Repeat this procedure, until all ϵ -rules are removed.

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1. Add a new start variable $S' \rightarrow S$
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E.g.: $S \rightarrow 0|X0|XYZ$
 $X \rightarrow Y|\epsilon$
 $Y \rightarrow 1|X$

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To remove $X \rightarrow \epsilon$, we add new rules: $S \rightarrow 0|X0|XYZ$

$X \rightarrow Y$
 $Y \rightarrow 1|X|\epsilon$

To remove $Y \rightarrow \epsilon$, we add:

$S \rightarrow 0|X0|XYZ|ZZ$
 $X \rightarrow Y$
 $Y \rightarrow 1|X$

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1. Add a new start variable $S' \rightarrow S$
2. Remove ϵ rules of the form $A \rightarrow \epsilon$
3. **Remove unit rules of the form $A \rightarrow B$**

We remove the rule $A \rightarrow B$ and whenever a rule $B \rightarrow u$ appears (u is a string of terminals and variables), we add a new rule $A \rightarrow u$, unless this rule was already removed.

Repeat these steps until all unit rules are removed.

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E.g.:

$$\begin{aligned}S &\rightarrow A|11 \\A &\rightarrow B|1 \\B &\rightarrow S|0\end{aligned}$$

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| Remove $S \rightarrow A$ | Remove $A \rightarrow B$ | Remove $B \rightarrow S$ | Remove $B \rightarrow B$ | Remove $S \rightarrow B$ | Remove $A \rightarrow S$ |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $S \rightarrow 11 B 1$ | $S \rightarrow 11 0 1$ | $S \rightarrow 11 0 1$ |
| $A \rightarrow B 1$ | $A \rightarrow 1 S 0$ | $A \rightarrow 1 11 0$ |
| $B \rightarrow S 0$ | $B \rightarrow S 0$ | $B \rightarrow 0 11 1 B$ | $B \rightarrow 0 11 1$ | $B \rightarrow 0 11 1$ | $B \rightarrow 0 11 1$ |

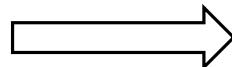
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3. Remove unit rules of the form $A \rightarrow B$
4. **Remove long rules of the form $A \rightarrow u_1u_2 \cdots u_k$**

Note that each u_i could be a variable or a terminal. We do the following:

- Replace $A \rightarrow u_1u_2 \cdots u_k$, ($k \geq 3$) with the rules $A \rightarrow u_1A_1$, $A_1 \rightarrow u_2A_2$, \cdots , $A_{k-2} \rightarrow u_{k-1}u_k$
- We replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \rightarrow u_i$

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Remove unit rules of the form $A \rightarrow B$ (Whenever a rule $B \rightarrow u$ appears, we add a new rule $A \rightarrow u$, unless this rule was already removed. Repeat these steps until all unit rules are removed.)

Remove long rules of the form $A \rightarrow u_1u_2 \cdots u_k$ (Replace $A \rightarrow u_1u_2 \cdots u_k$, ($k \geq 3$) with the rules $A \rightarrow u_1A_1$, $A_1 \rightarrow u_2A_2, \dots, A_{k-2} \rightarrow u_{k-1}u_k$; Replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \rightarrow u_i$).

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|-------------|--------------------------|-----------------------------------|
| CNF: | $A \rightarrow BC$ | [B, C are not start variables] |
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| | $S \rightarrow \epsilon$ | [S is the Start Variable] |

Convert the CFG

$$\begin{aligned} S &\rightarrow ASA|aB \\ A &\rightarrow B|S \\ B &\rightarrow b|\epsilon \end{aligned}$$

to CNF.

1. Add a new start variable

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB \\ A &\rightarrow B|S \\ B &\rightarrow b|\epsilon \end{aligned}$$

2a. Remove ϵ rules ($B \rightarrow \epsilon$)

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB|a \\ A &\rightarrow B|S|\epsilon \\ B &\rightarrow b \end{aligned}$$

2b. Remove ϵ rules ($A \rightarrow \epsilon$)

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB|a|AS|SA|S \\ A &\rightarrow B|S \\ B &\rightarrow b \end{aligned}$$

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3a. Remove $S \rightarrow S$

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB|a|AS|SA \\ A &\rightarrow B|S \\ B &\rightarrow b \end{aligned}$$

3b. Remove $S' \rightarrow S$

$$\begin{aligned} S' &\rightarrow ASA|aB|a|AS|SA \\ S &\rightarrow ASA|aB|a|AS|SA \\ A &\rightarrow B|S \\ B &\rightarrow b \end{aligned}$$

3c. Remove $A \rightarrow B$

$$\begin{aligned} S' &\rightarrow ASA|aB|a|AS|SA \\ S &\rightarrow ASA|aB|a|AS|SA \\ A &\rightarrow S|b \\ B &\rightarrow b \end{aligned}$$

3d. Remove $A \rightarrow S$

$$\begin{aligned} S' &\rightarrow ASA|aB|a|AS|SA \\ S &\rightarrow ASA|aB|a|AS|SA \\ A &\rightarrow b|ASA|aB|a|AS|SA \\ B &\rightarrow b \end{aligned}$$

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4a. Remove long rules

$$\begin{aligned} S' &\rightarrow ASA|aB|a|AS|SA \\ S &\rightarrow ASA|aB|a|AS|SA \\ A &\rightarrow b|ASA|aB|a|AS|SA \\ B &\rightarrow b \end{aligned}$$

There are other rules of the form: $\text{Var} \rightarrow ASA$

4b. Remove long rules

$$\begin{aligned} S' &\rightarrow AU|aB|a|AS|SA \\ S &\rightarrow AU|aB|a|AS|SA \\ A &\rightarrow b|AU|aB|a|AS|SA \\ U &\rightarrow SA \\ B &\rightarrow b \end{aligned}$$

4c. Remove long rules

$$\begin{aligned} S' &\rightarrow AU|VB|a|AS|SA \\ S &\rightarrow AU|VB|a|AS|SA \\ A &\rightarrow b|AU|VB|a|AS|SA \\ U &\rightarrow SA \\ V &\rightarrow a \\ B &\rightarrow b \end{aligned}$$

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Convert the CFG

to CNF.

$$\begin{aligned}S &\rightarrow ASA|aB \\A &\rightarrow B|S \\B &\rightarrow b|\epsilon\end{aligned}$$

$$\begin{aligned}S' &\rightarrow AU|VB|a|AS|SA \\S &\rightarrow AU|VB|a|AS|SA \\A &\rightarrow b|AU|VB|a|AS|SA \\U &\rightarrow SA \\V &\rightarrow a \\B &\rightarrow b\end{aligned}$$

Pushdown Automata

- For regular languages we had
 - Designed Finite automata (DFA, NFA) that recognize the strings by the language. Helped us decide whether a given string ω belongs to the language.

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- Any automata that recognizes **ALL** context free languages will need unbounded memory.

Pushdown Automata

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- Any automata that recognizes **ALL** context free languages will need unbounded memory.

Intuition to build an Automata for CFL

- It should be some **Finite State Machine** that has access to a memory device with infinite memory, i.e.

Automata for CFL = FSM + Memory device

- **FSM may choose to ignore the memory device** completely in which case it behaves like a DFA/NFA.
- FSM makes use of the Memory device to recognize “non-Regular” CFLs.

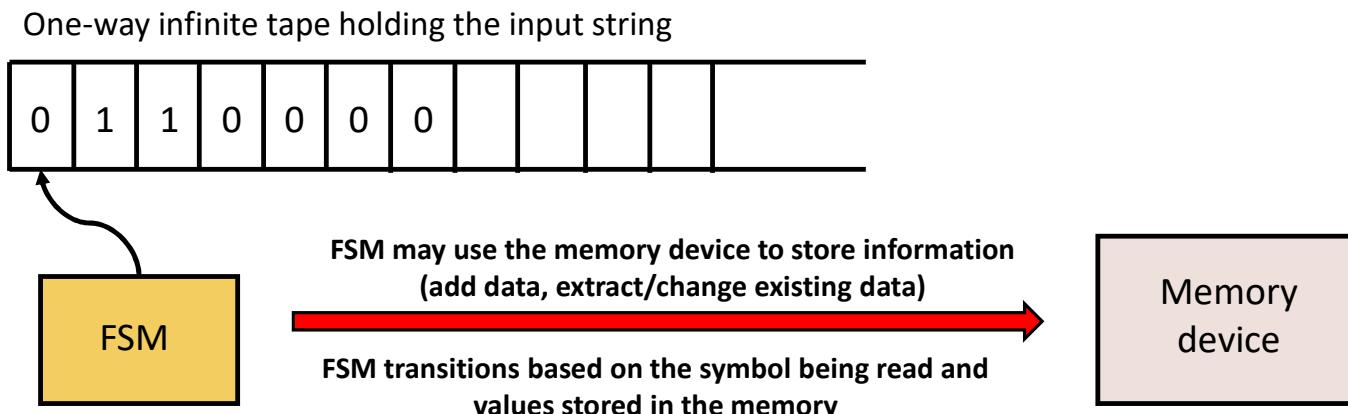
E.g.: $\{0^n 1^n, n \in \mathbb{N}\}$

Pushdown Automata

Intuition to build an Automata for CFL

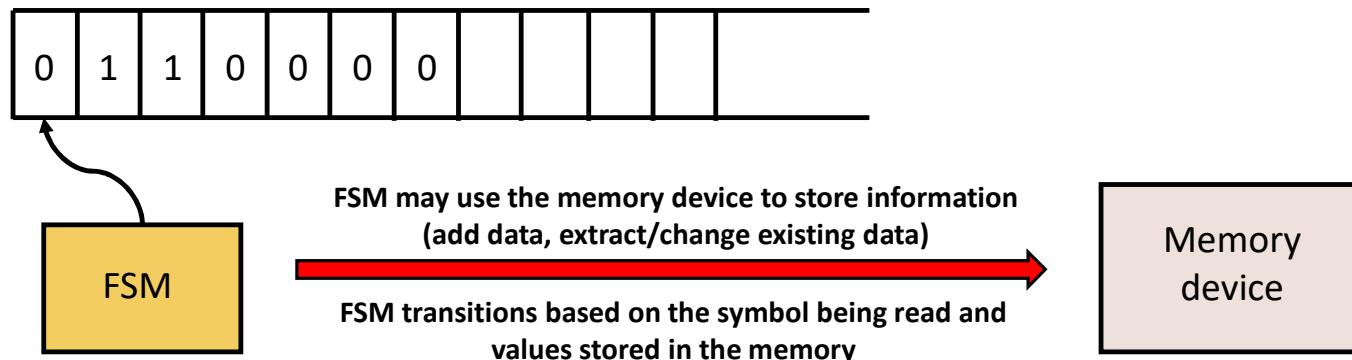
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Pushdown Automata

One-way infinite tape holding the input string



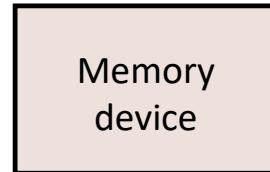
The memory device

- Simple memory device with unbounded memory.

Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, new elements can be added to the Stack (**PUSH**).
- At any stage, the element at the **top** of the STACK can be read by removing it from the stack (**POP**).



Pushdown Automata

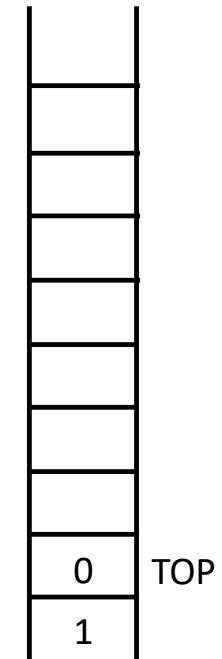
The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.

PUSH

- New symbols can be **pushed** in to the STACK.
E.g: PUSH 1
- The Top of the STACK now covers the old stack top, i.e.
 $\text{TOP} = \text{TOP} + 1$
- The size of the stack keeps growing.

Memory
device



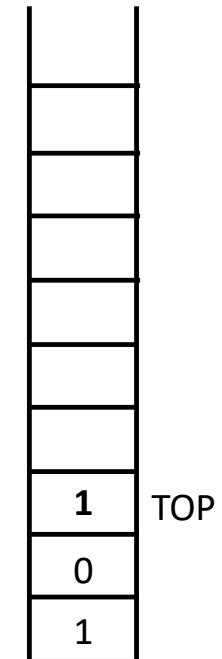
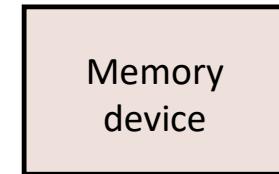
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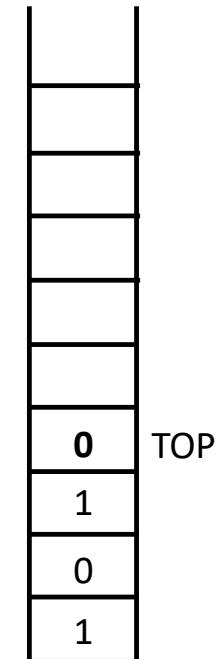
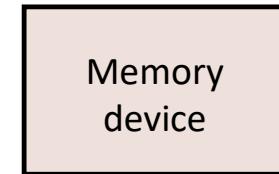
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Pushdown Automata

The memory device

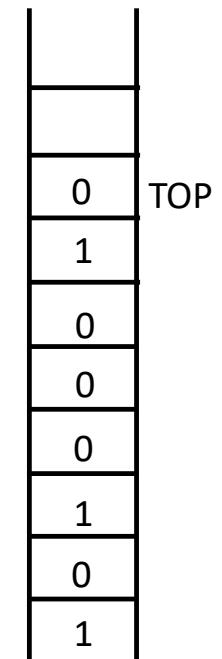
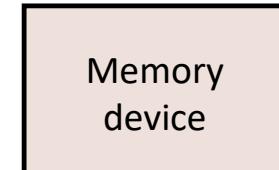
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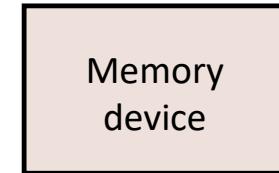
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POP

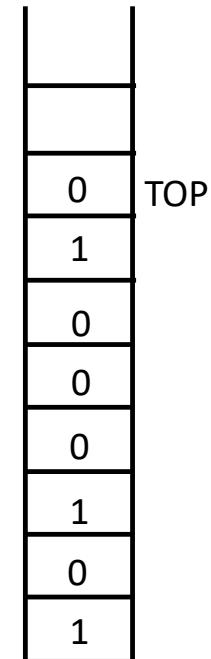
- The element from the **TOP** of the stack can be **popped** out

E.g.: **POP 0**

- The Top of the STACK moves to the element below.

$$\text{TOP} = \text{TOP} - 1$$

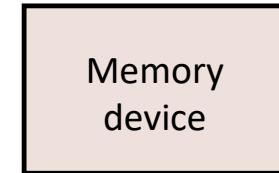
- Successive **POP** operations shrink the stack size. Elements can be popped until **EMPTY**.
- **Last In First Out (LIFO)**: The last element that was pushed is the first to be popped out



Pushdown Automata

The memory device

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- Consider a **STACK**
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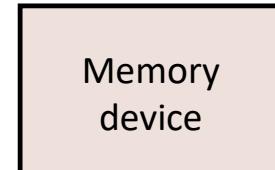
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POP

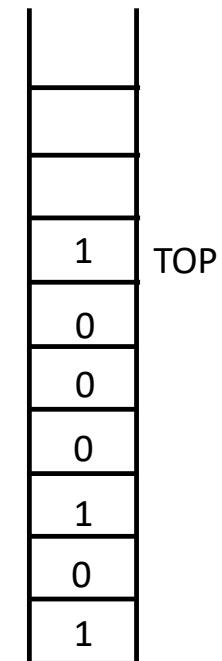
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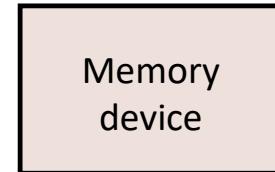
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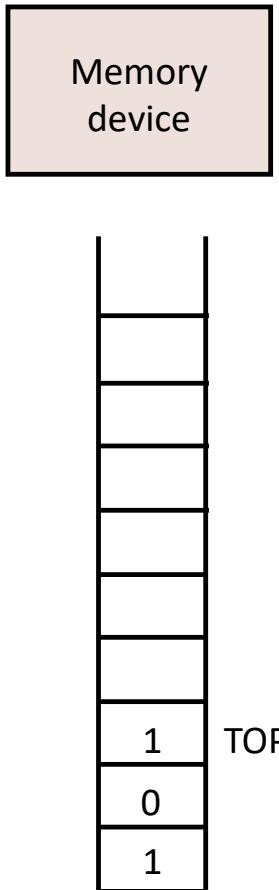
Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- Last In First Out (**LIFO**)

POP

- The element from the **TOP** of the stack can be **popped** out.
- $\text{TOP} = \text{TOP} - 1$
- Elements can be popped until STACK is EMPTY.
- How would you know that the STACK is EMPTY?



Pushdown Automata

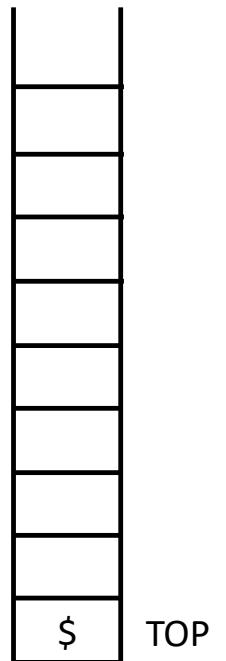
The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- Last In First Out (**LIFO**)

POP

- The element from the **TOP** of the stack can be **popped** out.
- $\text{TOP} = \text{TOP} - 1$
- Elements can be popped until STACK is EMPTY.
- How would you know that the STACK is EMPTY?
- There is generally some special symbol (say \$) that demarcates the bottom of the STACK.
- This element is Pushed at the very beginning. Whenever the popped element = \$, the STACK is EMPTY.

Memory
device



Pushdown Automata

Memory device of PDA: STACK

- STACK is a **LIFO** data structure of unbounded memory
- Only the TOP element can be read from the STACK.
- The bottom of the STACK contains a special symbol (\$)
- Characterized by two operations:

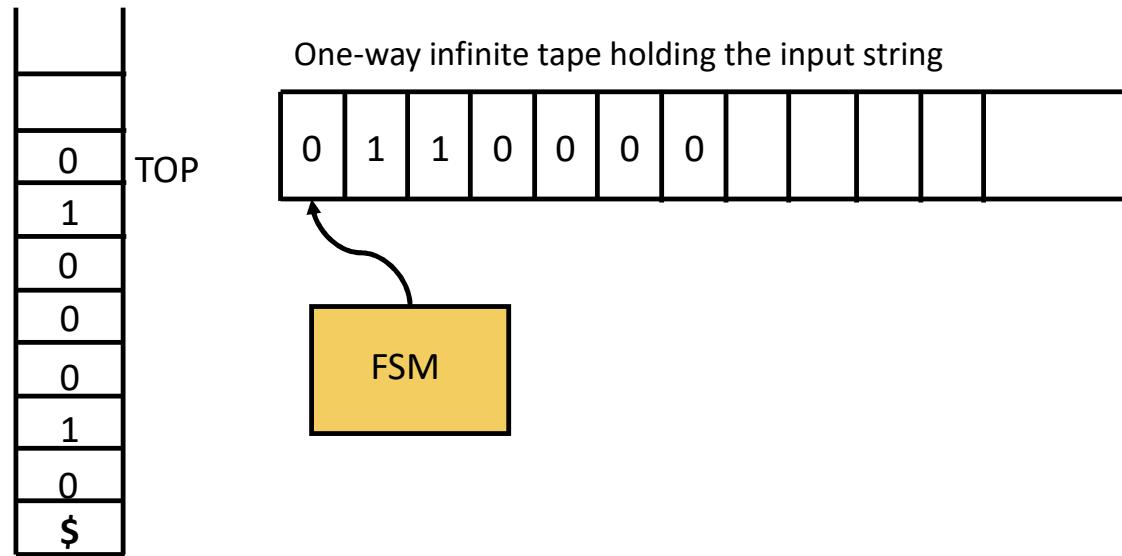
PUSH

- New symbols can be **pushed** in to the STACK.
- $\text{TOP} = \text{TOP} + 1$

POP

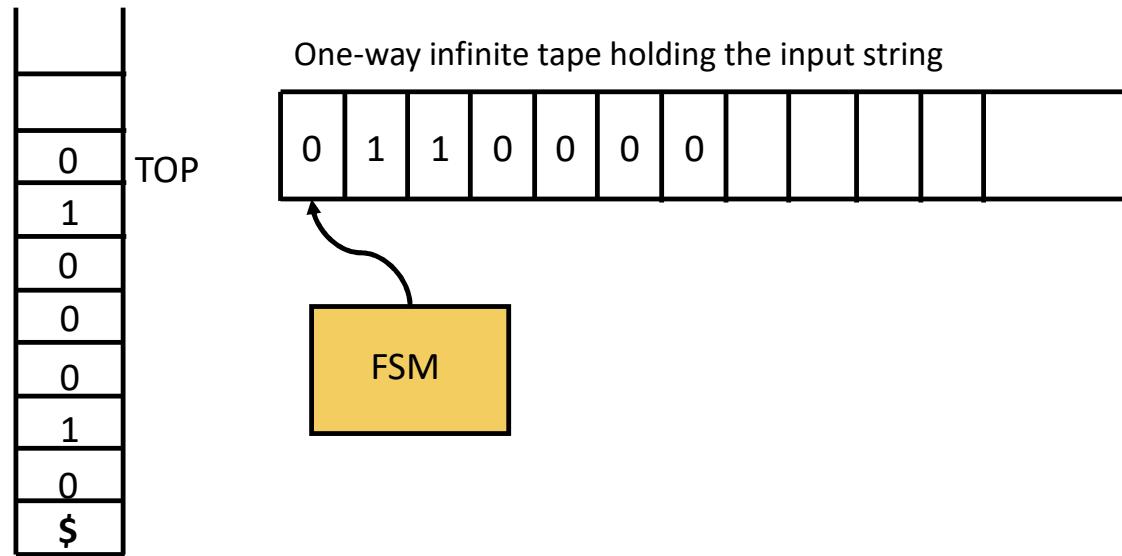
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Pushdown Automata



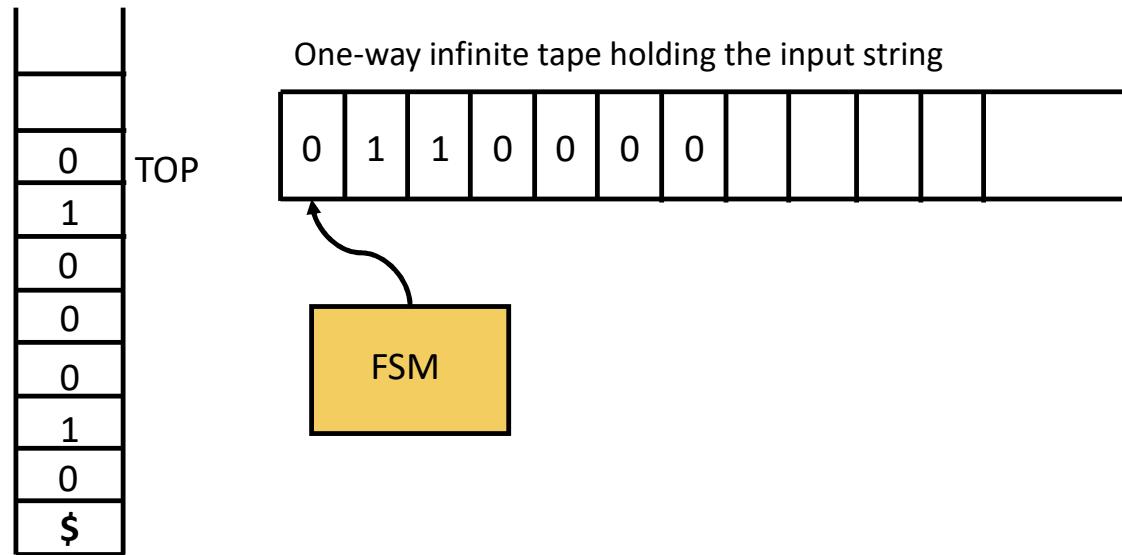
- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:

Pushdown Automata



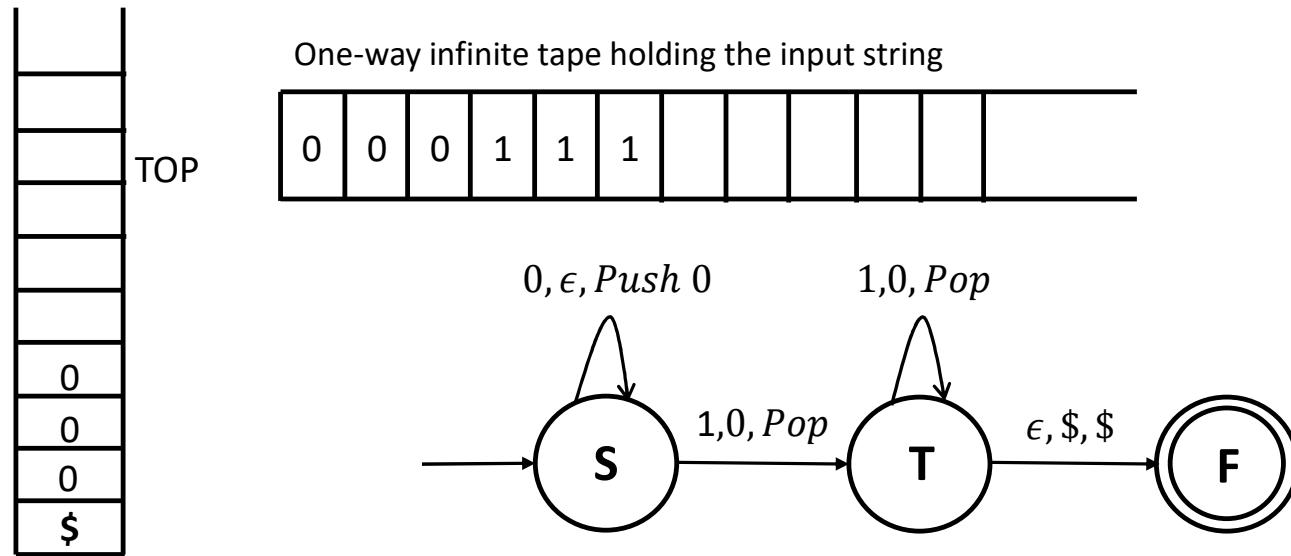
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 - Transitions based on the Input symbol and the element at the top of the stack (e.g.: If I/P symbol = 0 & 0 is popped, transition from i to j)

Pushdown Automata



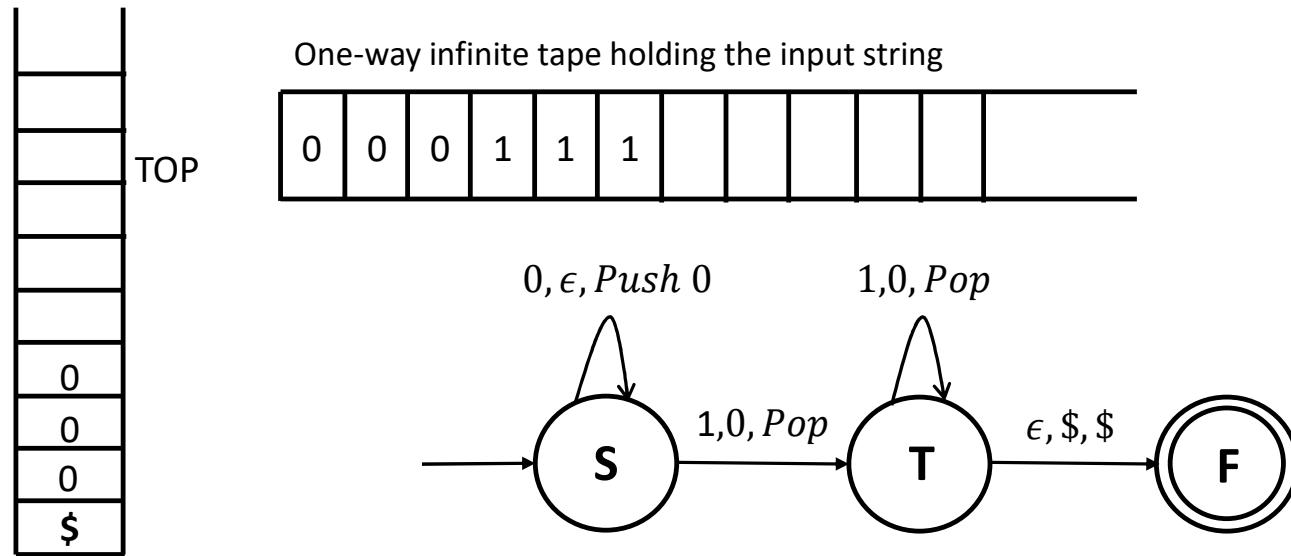
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 - Transitions based on the Input symbol and the element at the top of the stack (e.g.: If I/P symbol = 0 & POP 0, transition from i to j)
 - Pushes new elements into the Stack (e.g.: If I/P symbol = 0, PUSH 0, transition from i to j).

Pushdown Automata



- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:
 - Transitions based on the Input symbol and the element at the top of the stack
 - Pops the element at the top of the Stack.
 - Pushes new elements into the Stack.

Pushdown Automata

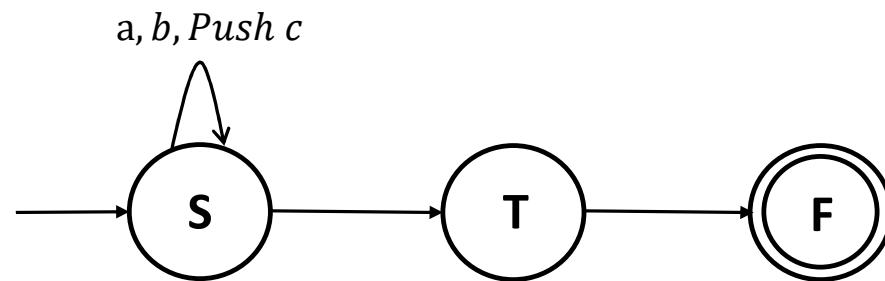


PDAs are **non-deterministic**.

- Missing transitions
- ϵ -transitions
- Multiple transitions/input symbol possible

Pushdown Automata

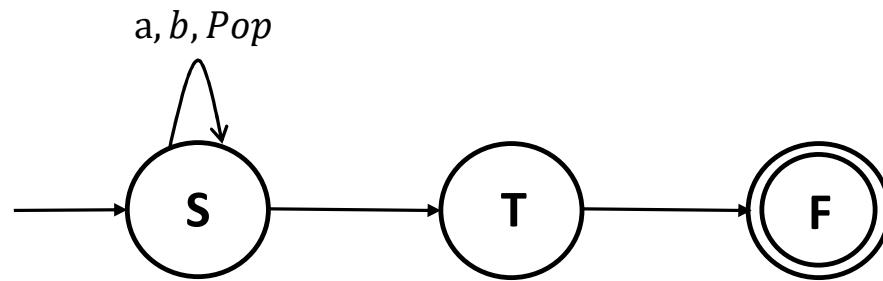
- How to represent a transition in a PDA?



If input symbol = a , Stack top = b (if b is popped), Push c onto the Stack a remain in S

Pushdown Automata

- How to represent a transition in a PDA?

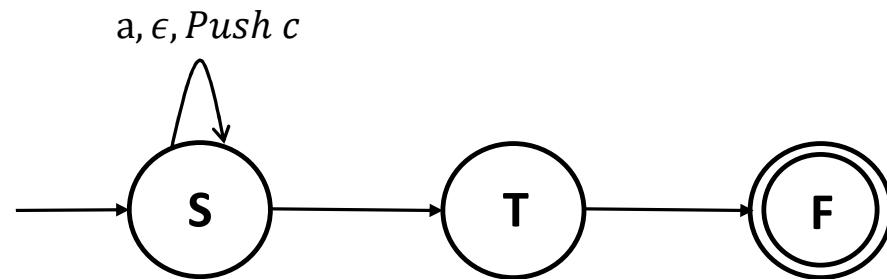


If input symbol = a , and b is popped, remain in S .

(If the symbol read is a and the element at the Stack TOP = b , then remain in S)

Pushdown Automata

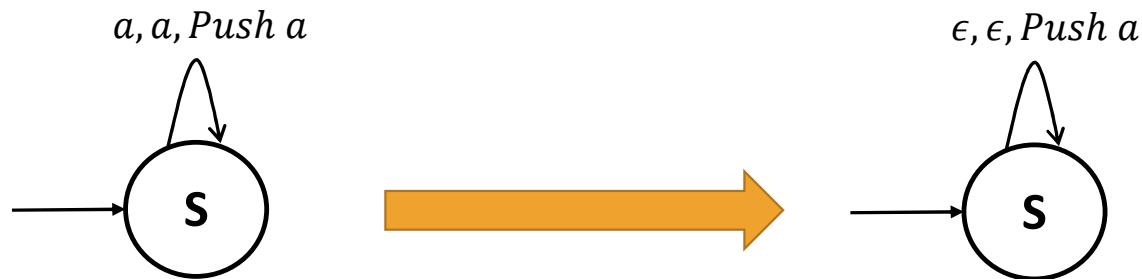
- How to represent a transition in a PDA?



If input symbol = a , then Push c

Pushdown Automata

- How to represent a transition in a PDA?

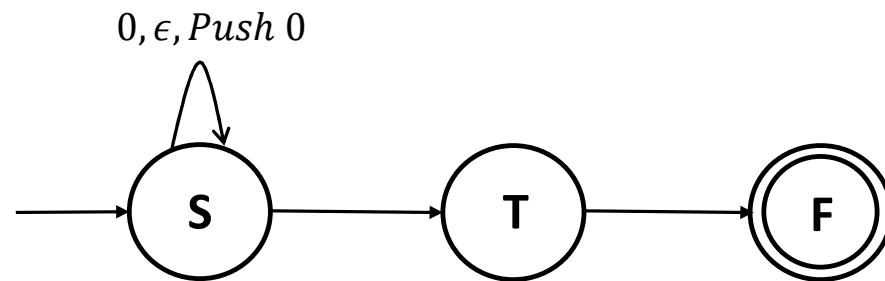


- (i) If input symbol = a , and a is popped, then Push a and remain in S .
- (ii) Push a on to the stack and remain in S .

Through Steps (i) and (ii), the PDA pushes a onto the stack if it reads a on the input tape and the element at the stack top = a .

Pushdown Automata

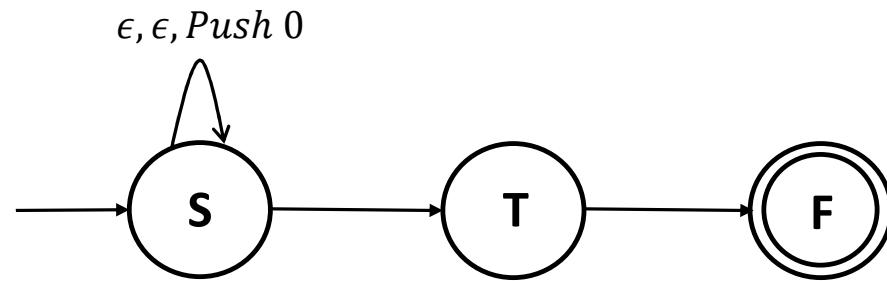
- How to represent a transition in a PDA?



If input symbol = 0, Push 0 onto the Stack irrespective of the element at the top of the stack

Pushdown Automata

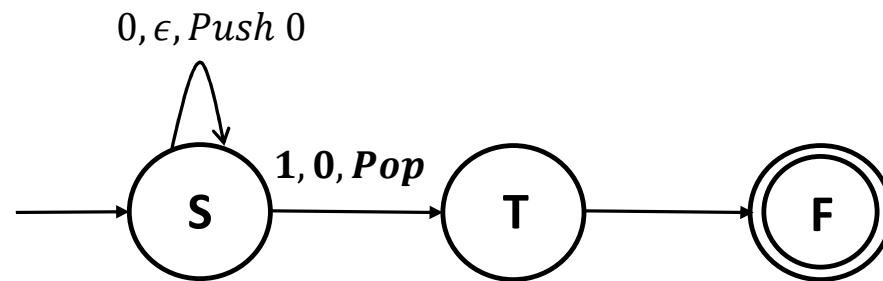
- How to represent a transition in a PDA?



Without reading the input symbol and the Stack top, Push 0 onto the Stack

Pushdown Automata

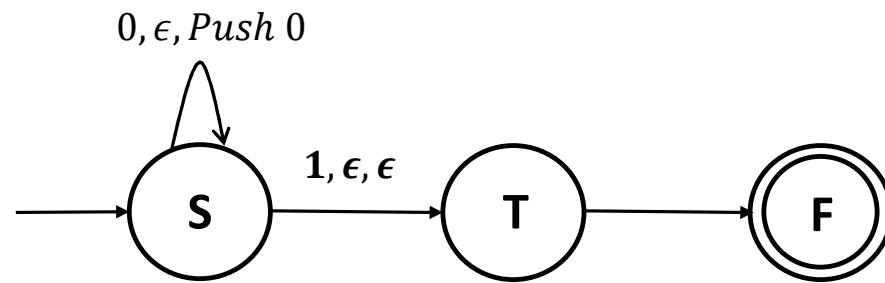
- How to represent a transition in a PDA?



If the input symbol is 1, and the element 0 is popped (**Pop 0**), then transition from *S* to *T*

Pushdown Automata

- How to represent a transition in a PDA?

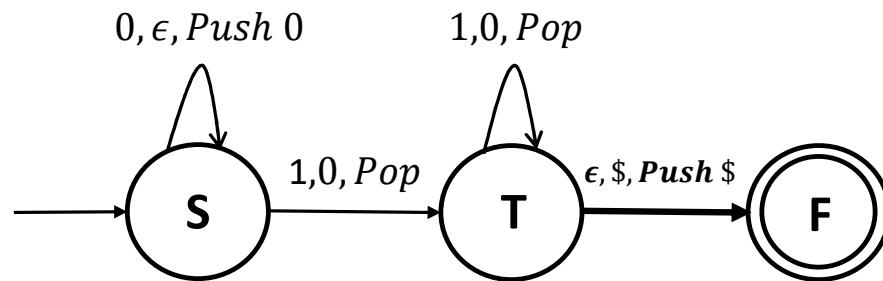


If the input symbol is 1, transition to T by ignoring the stack completely.

If this happens at every step of the execution of the PDA, then it is as powerful as an NFA.

Pushdown Automata

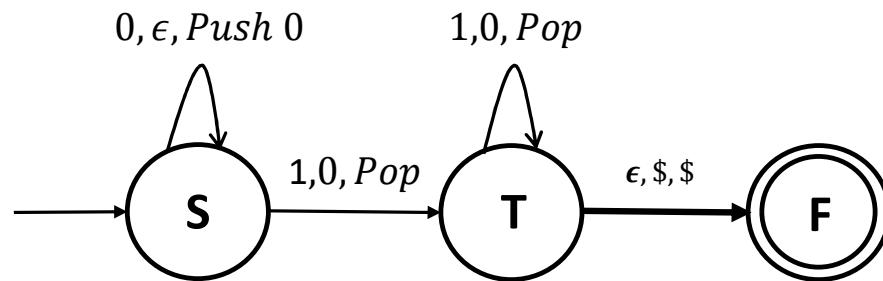
- How to represent a transition in a PDA?



Empty stack: If $\$$ is popped, push the $\$$ back onto the stack and transition to F from T , without reading the input

Pushdown Automata

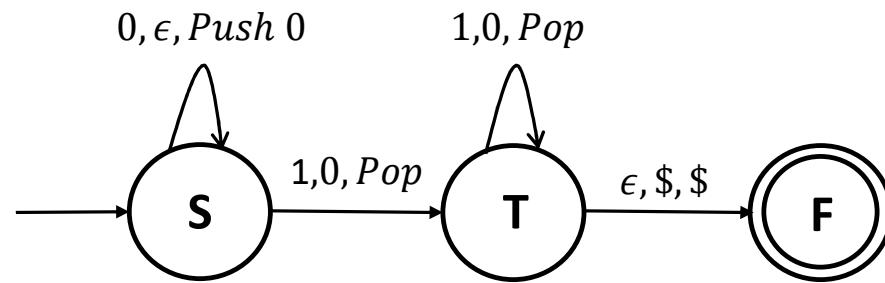
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Pushdown Automata

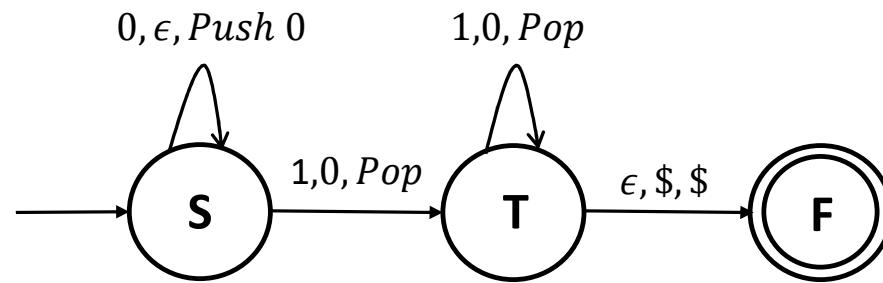
- How to represent a transition in a PDA?



What is the language recognized by this PDA?

Pushdown Automata

- How to represent a transition in a PDA?

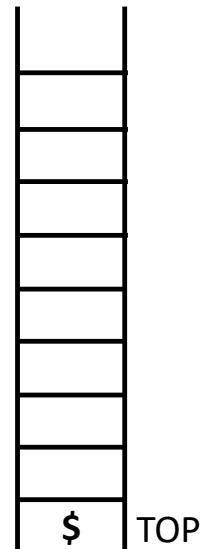
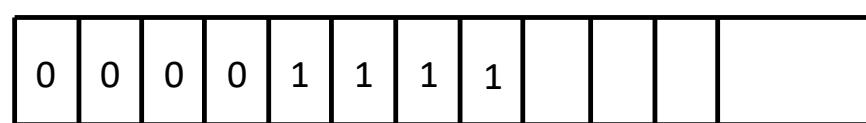
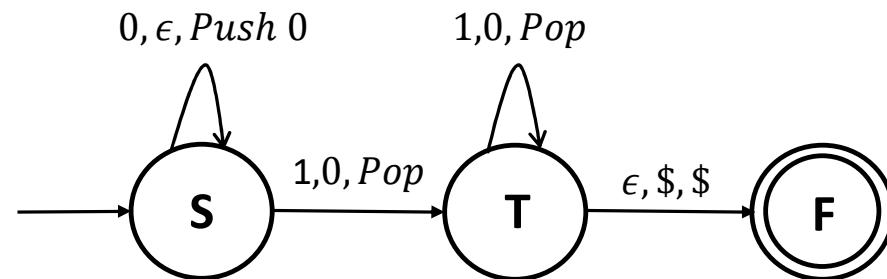


What is the language recognized by this PDA?

Verify that it is $L = \{0^n 1^n, n \geq 1\}$

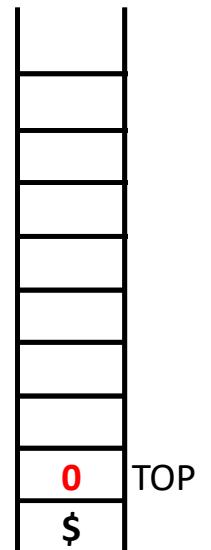
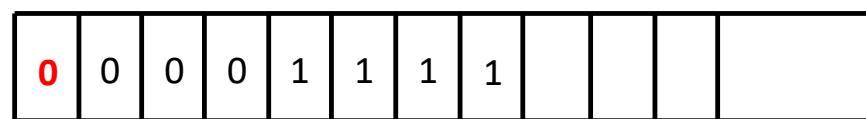
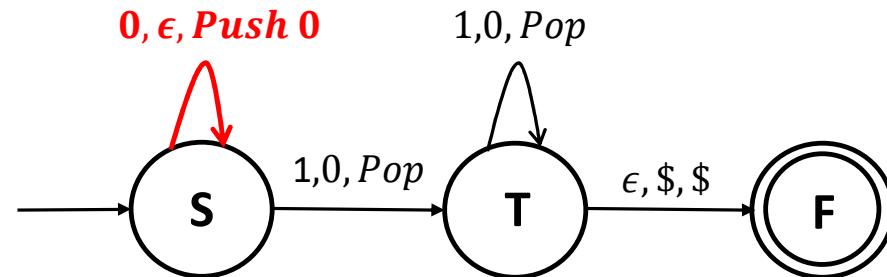
Pushdown Automata

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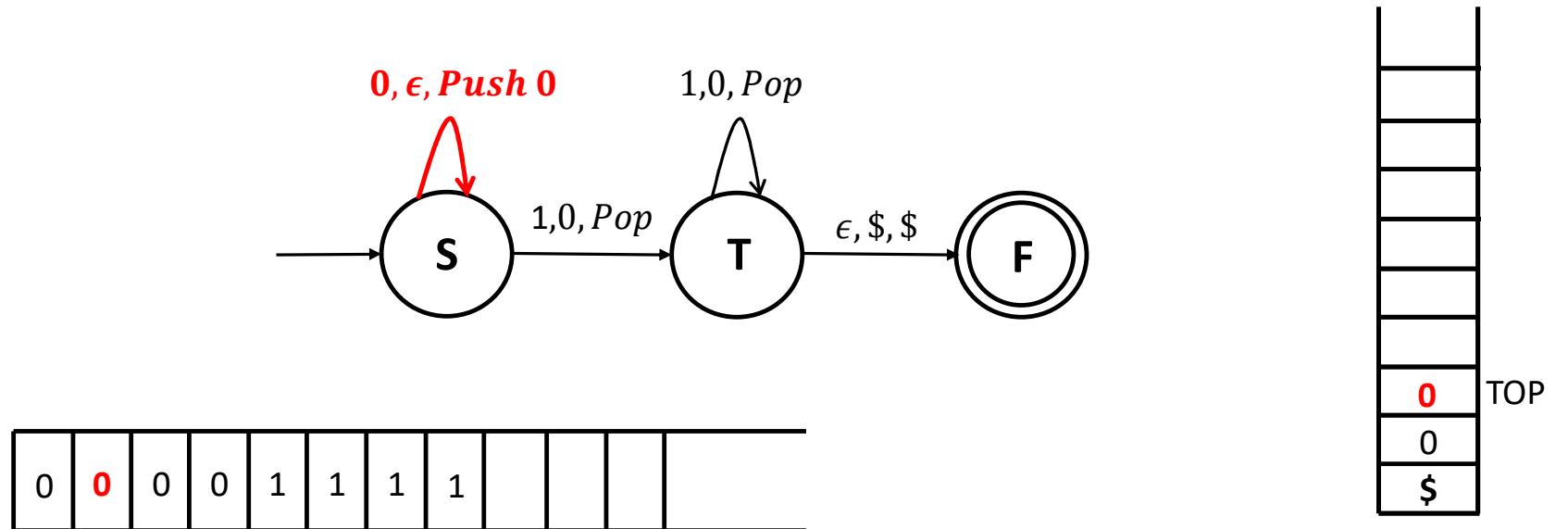
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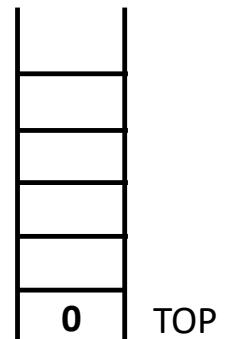
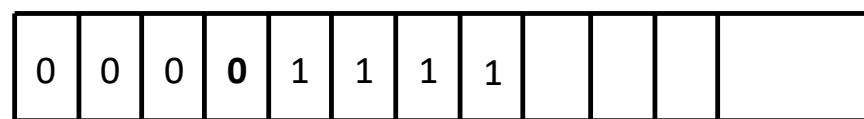
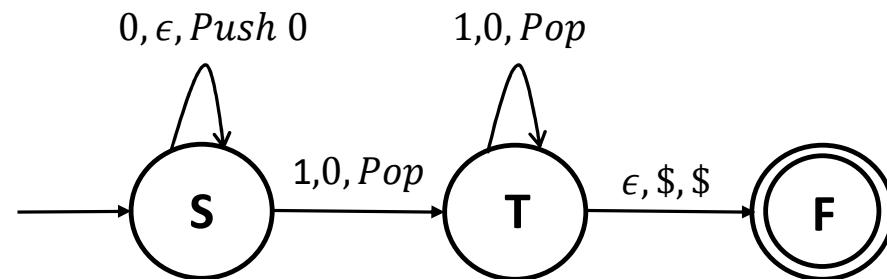
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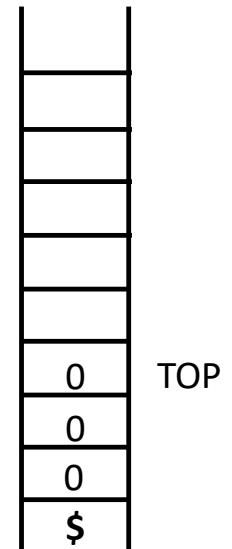
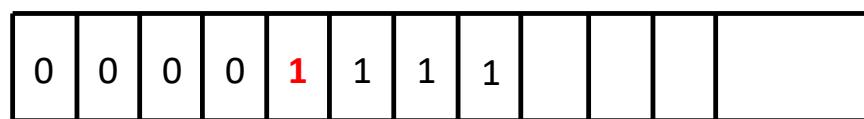
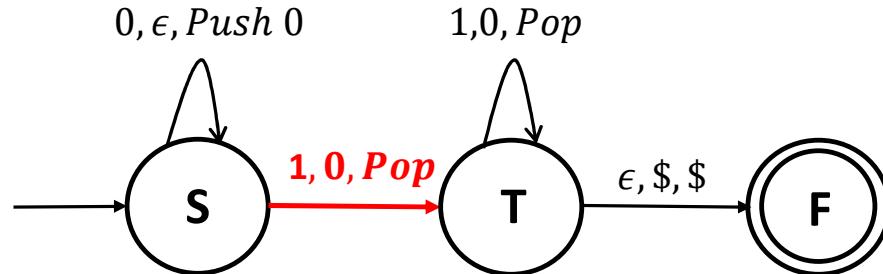
Pushdown Automata

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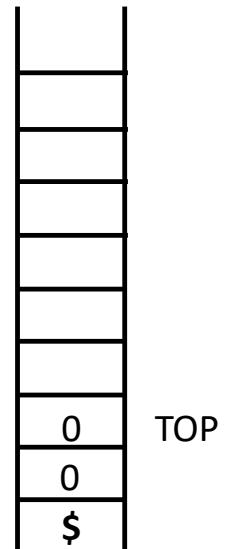
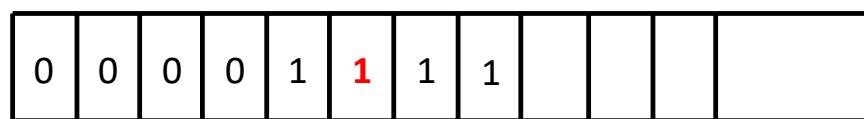
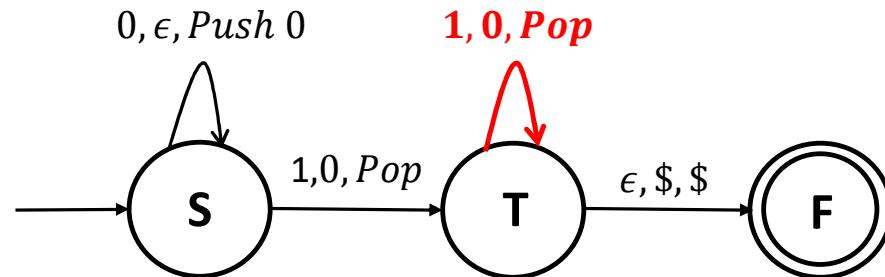
Pushdown Automata

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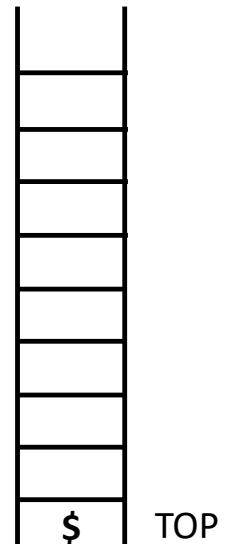
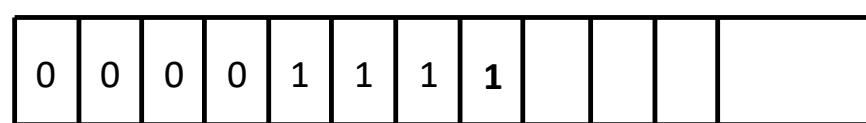
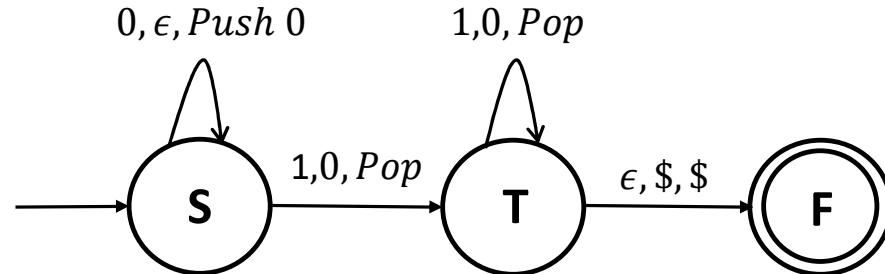
Pushdown Automata

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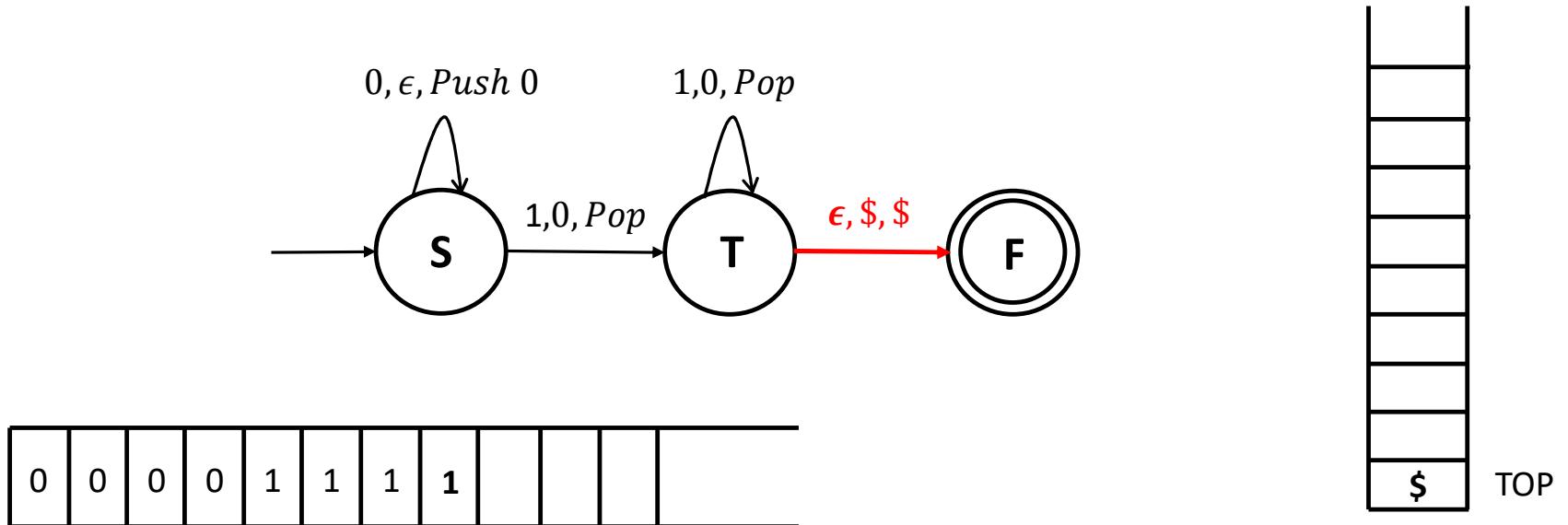
Pushdown Automata

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Pushdown Automata

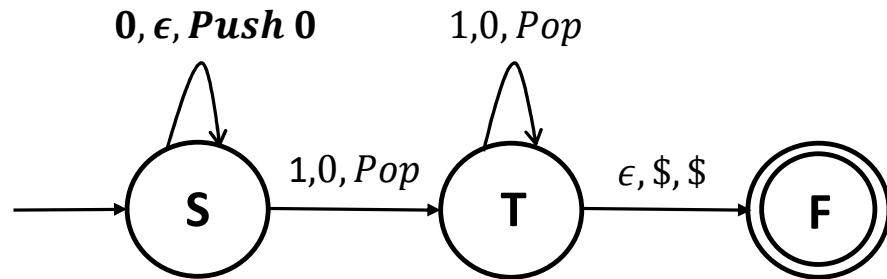
What is the language recognized by this PDA?



The language recognized by the PDA: $L = \{0^n 1^n, n \geq 1\}$

Pushdown Automata

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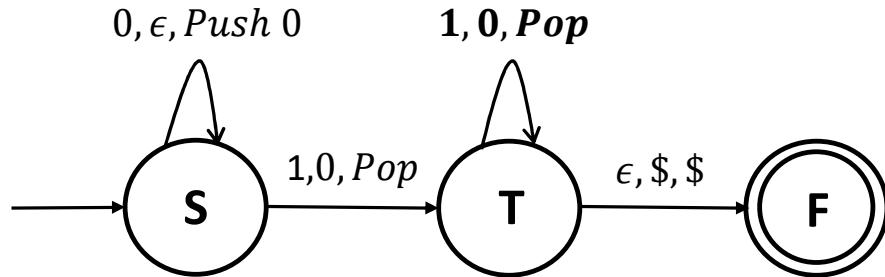


In some references (such as Sipser):

- The transitions of the PDA are labelled as " $a, b \rightarrow c$ ", implying: If the input symbol read is a , and the element at the top of the stack is b (b is popped), then push c on to the Stack.

Pushdown Automata

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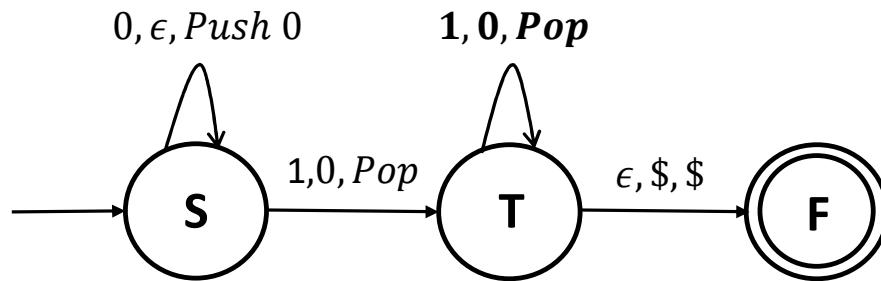


In some references (such as Sipser):

- The transitions of the PDA are labelled as " $a, b \rightarrow c$ ", implying: If the input symbol read is a , then pop b (the element at the top of the stack is b) and push c on to the Stack.
- The label " $a, b \rightarrow \epsilon$ " implies that if the input symbol is a then **pop b**.

Pushdown Automata

What is the language recognized by this PDA?



In some references (such as Sipser):

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- The label " $a, b \rightarrow \epsilon$ " implies that if the input symbol is a and b is popped.
- The symbol signifying the bottom of the Stack $\$$ is pushed at the very beginning.

Pushdown Automata

Formally, a PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set called the **states**.
- Σ is the set of input **alphabets**.
- Γ is the set of **Stack alphabets**
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the **transition function**
- $q_0 \in Q$ is the **start state**.
- $F \subseteq Q$ is the set of **accepting states**.

$$[\Sigma_\epsilon = \Sigma \cup \{\epsilon\} \text{ and } \Gamma_\epsilon = \Gamma \cup \{\epsilon\}]$$

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A PDA accepts a string $w \in L$, if there exists a run such that

- It reaches a **final state** when the entire string is read.

OR

- The **stack is empty** when the entire string is read.

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These two notions of acceptance are equivalent

Pushdown Automata

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Transition function:

- $\delta(q_i, a, b) = (q_j, c)$: If the input symbol read is a and b is popped, then push c onto the stack and transition from q_i to q_j

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- $\delta(q_i, \epsilon, \$) = (q_j, \$)$:

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- $\delta(q_i, \epsilon, \$) = (q_j, \$)$: Transition from q_i to q_j if the stack is empty.

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- If the input symbol read is a and a is popped, then Push a and remain at q_i : ?

Pushdown Automata

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Transition function:

- $\delta(q_i, a, b) = (q_j, c)$: If the input symbol read is a and b is popped, then push c onto the stack and transition from q_i to q_j
- If the input symbol read is a and a is popped, then Push a and remain at q_i : $\delta(q_i, a, a) = (q_i, a)$

Pushdown Automata

Formally, a PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set called the **states**.
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$$L = \{w \mid P \text{ accepts } w\}$$

There exists an accepting run for w on P

- If $\mathcal{L}(P) = L$, then the PDA P recognizes L

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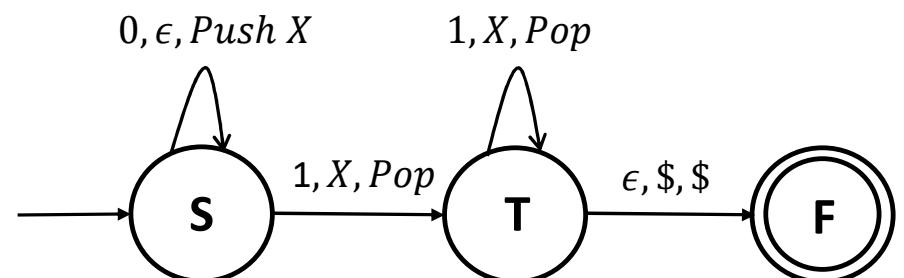
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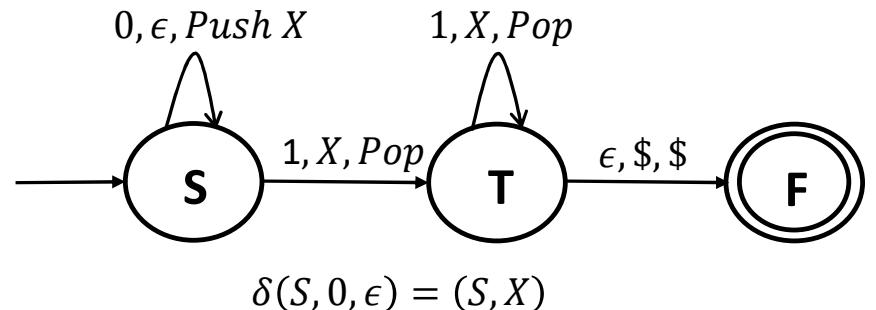
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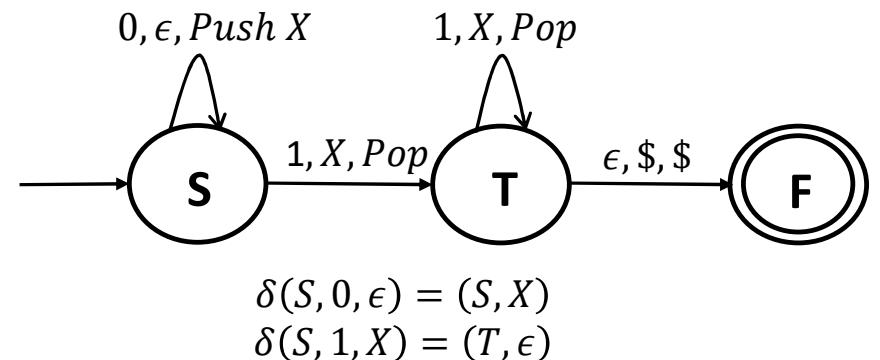
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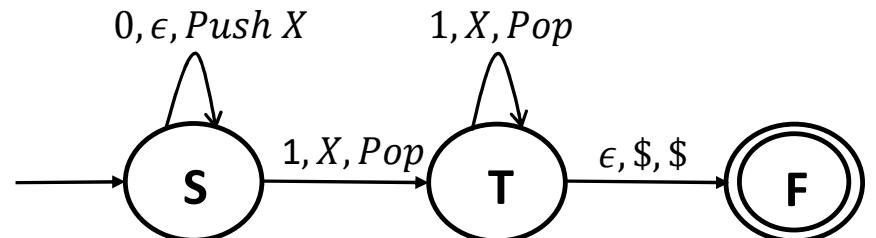
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$$\begin{aligned}\delta(S, 0, \epsilon) &= (S, X) \\ \delta(S, 1, X) &= (T, \epsilon) \\ \delta(T, 1, X) &= (T, \epsilon) \\ \delta(T, \epsilon, \$) &= (F, \$)\end{aligned}$$

Thank You!