MA 6.101
Probability and Statistics

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Function of continuous random variables

- Consider $Y = aX + b$ where X is a continuous random variable.
- What is $F_Y(y)$ and $f_Y(y)$?
- $F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$.
- $F_Y(y) = F_X(\frac{y-b}{a})$ if $a > 0$
- $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X(\frac{y-b}{a})$ when $a > 0$
- $F_Y(y) = 1 - F_X(\frac{y-b}{a})$ if $a < 0$
- $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{-1}{a} f_X(\frac{y-b}{a})$ when $a < 0$
- In general, $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$

Function of continuous random variables

Consider $Y = aX + b$ where X is a continuous random variable. Then $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$.

- What if $Y = g(X)$ where $g(\cdot)$ is continuous, differentiable and monotone. Any guess?
- Since $g(.)$ is monotone and continuous it is invertible. Let $h(.)$ denote the inverse function. Then $h(Y) = X$.

Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = |\frac{dh}{dy}(y)| f_X(h(y))$ where h is the inverse function of g.

Function of continuous random variables

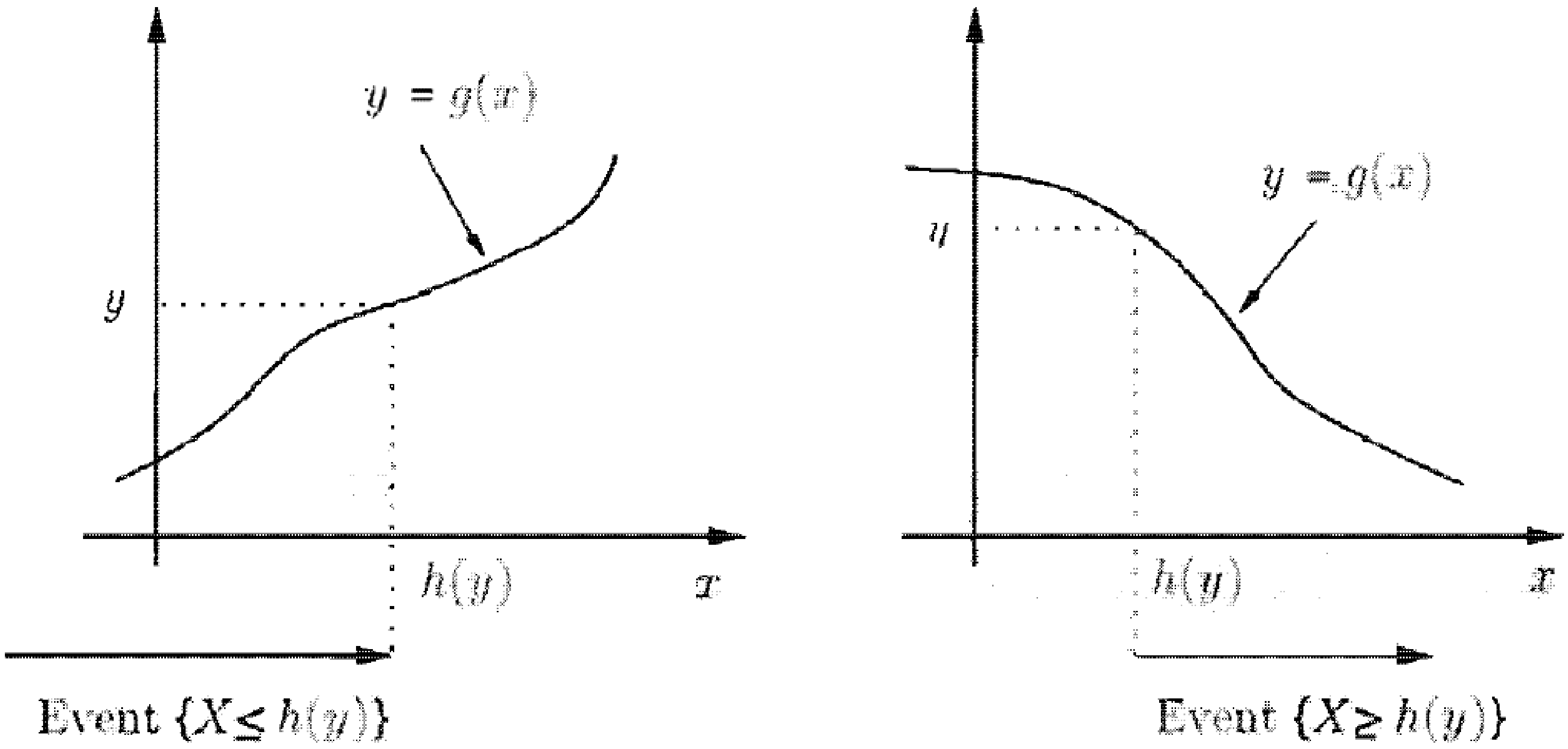
Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$ where h is the inverse function of g.

Proof:

- Since $g(.)$ is monotone and continuous it is invertible. Let $h(.)$ denote the inverse function. Then $X = h(Y)$.
- $F_Y(y) = P(g(X) \leq y)$.
- Is $P(g(X) \leq y) = P(X \leq h(y))$ always?
- Are the two events $\{g(X) \leq y\}$ and $X \leq h(y)$ same?
- If they are same, then the two probabilities are equal.

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?



- Same when g is increasing and compliments when g is decreasing.

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?
- Same when g is increasing and compliments when g is decreasing.
- CASE 1: $g(x)$ is non-decreasing
- $F_Y(y) = P(g(X) \leq y) = P(X \leq h(y)) = F_X(h(y))$.
- $f_Y(y) = \frac{d}{dy}(F_X(h(y))) = f_X(h(y))\frac{dh}{dy}(y)$ where $\frac{dh}{dy}(y) \geq 0$ as h is also non-decreasing.
- Rewritten therefore as $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?
- Same when g is increasing and compliments when g is decreasing.
- CASE 2: $g(x)$ is non-increasing
- $F_Y(y) = P(g(X) \leq y) = P(X > h(y)) = 1 - F_X(h(y))$.
- $f_Y(y) = -\frac{d}{dy}(F_X(h(y))) = -f_X(h(y))\frac{dh}{dy}(y)$ where $\frac{dh}{dy}(y) \leq 0$ as h is non-increasing as well.
- Rewritten therefore as $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$. $\square$

HW: What about the case when g is not monotone ?

Q: Suppose $Y = X^2$, then what is $f_Y(y)$ in terms of $f_X(x)$?

Mixed random variables

Mixed Random variables

- Random variables that are neither continuous nor discrete are called as mixed random variables.
- Their CDF is partly continuous and partly piece-wise continuous.
- Example: X is a $U[0,1]$ random variable and $Y = X$ if $X \leq 0.5$ and $Y = 0.5$ if $X > 0.5$.
- What is the CDF and PDF of Y?

Mixed Random variables

- Let $F_Y(y) = C(y) + D(y)$ where $C(y)$ corresponds to the continuous part and $D(y)$ for the discontinuous part.
- $$E[Y] = \int_{-\infty}^{\infty} x c(x) dx + \sum_{y_k} y_k P(Y = y_k)$$
  where $\{y_1, y_2, \ldots\}$ are jump points of $D(y)$ where $P(Y = y_k) > 0$.
- See section 4.3.1 from probabilitycourse.com for more examples
- Amount of workload (pending) on a server! A server on a cluster may be idle with a finite probability. If busy, the pending work is a continuous random variable.

Multiple random variables

A running example

- Consider an experiment of tossing a coin and a dice together.
- $\Omega = \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$. $\mathcal{F} = 2^{\Omega}$. $\mathbb{P}(\omega) = \frac{1}{12}$.
- Let X and Y denote the random variables depicting outcome of a coin and dice respectively.
- For $\omega = (1, 5)$ we have $X(\omega) = 1$ and $Y(\omega) = 5$.
- We are now interested in the joint PMF $p_{XY}(x, y)$ and joint CDF $F_{XY}(x, y)$ of X and Y together.