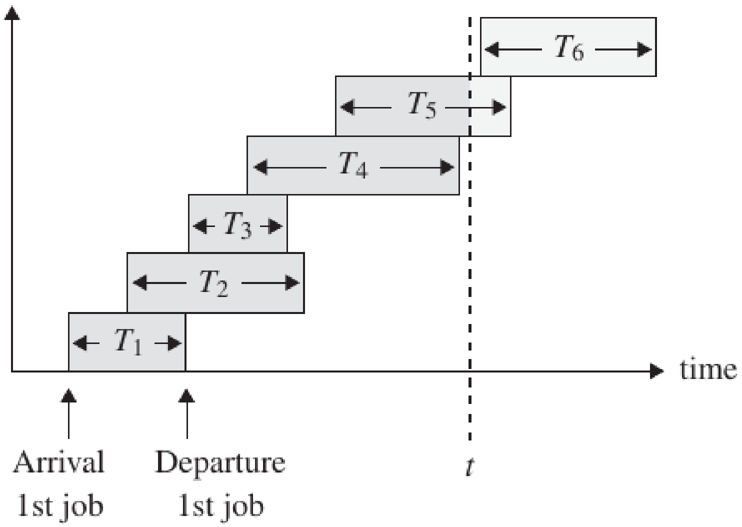
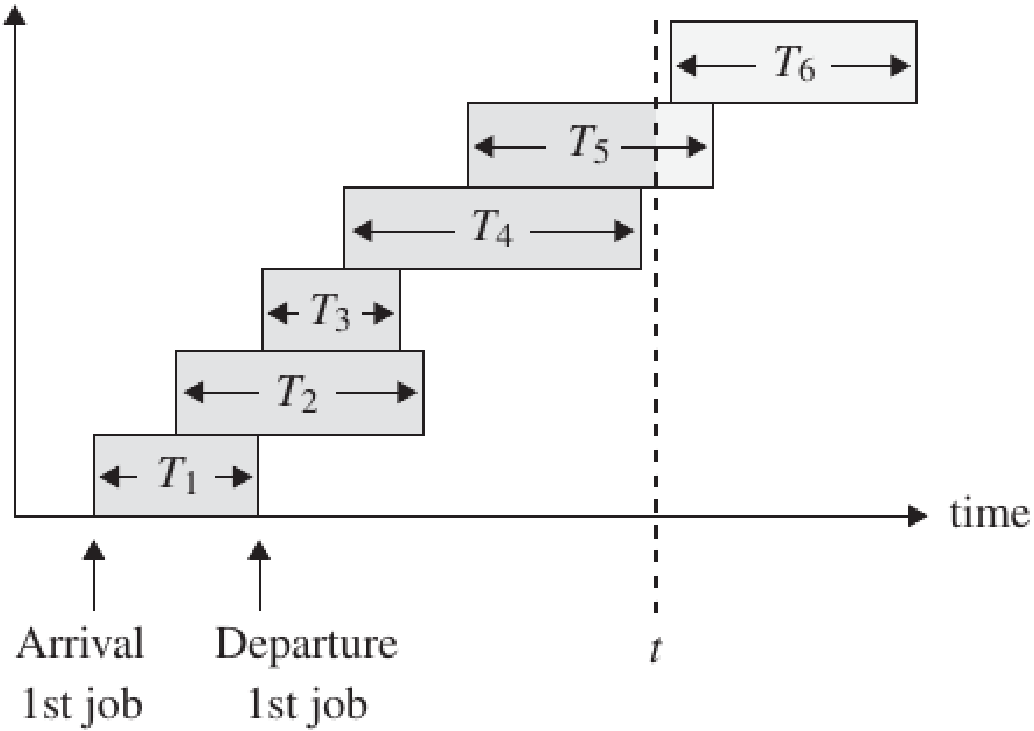
Little's law: Intuitive ensemble average proof



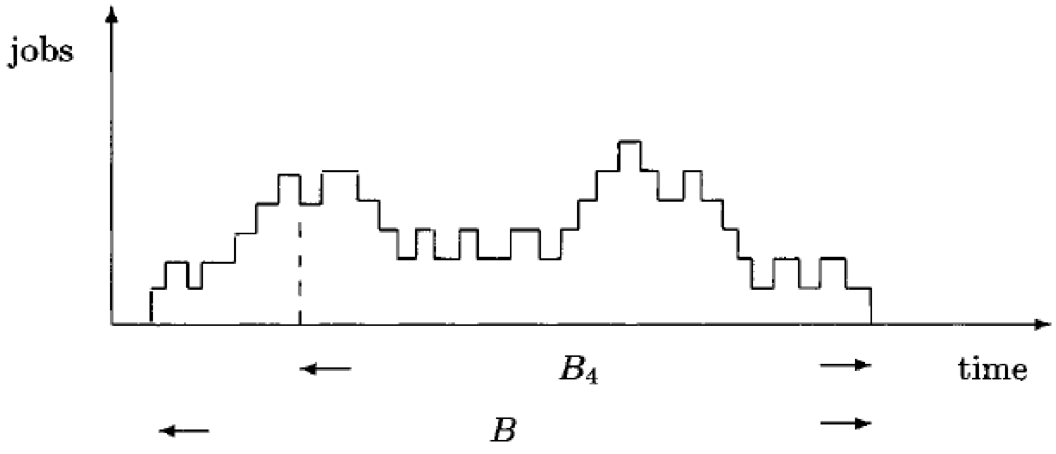
- Let $F(\cdot)$ denote the CDF of T_i.
- At time t, consider those in the system have arrived before t and are staying beyond t.
- At $(t-x, t-x+dx)$, jobs arrive with probability λdx.
- Each such job will stay beyond t with probability $1 - F(x)$
- $E[N(t)] = \int_0^t [1 - F(x)]\lambda dx$.
- $L = \lim_{t\to\infty} E[N(t)] = \int_0^\infty [1 - F(x)]\lambda dx$.

Little's law: Intuitive time average proof



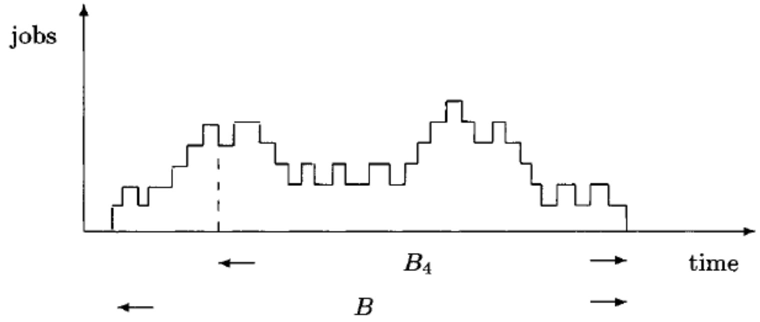
- $N(t) = A(t) - D(t)$
- $\sum_{n=1}^{D(t)} T_n \leq \int_0^t N(t) \leq \sum_{n=1}^{A(t)} T_n$
- $\frac{D(t)}{t} \frac{\sum_{n=1}^{D(t)} T_n}{D(t)} \leq \frac{\int_0^t N(t)}{t} \leq \frac{A(t)}{t} \frac{\sum_{n=1}^{A(t)} T_n}{A(t)}$
- What is $\lim_{t\to\infty} \dfrac{D(t)}{t}$?

Busy cycles of a work-conserving system



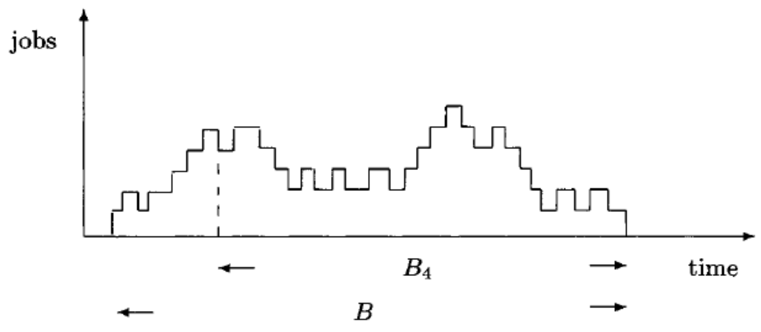
- When the system is work conserving, the system oscillates between busy periods and idle periods.
- Busy period + idle period = Busy cycle
- The start of a busy cycle constitutes a renewal point.
- The interarrival times between busy cycles are i.i.d
- We can therefore also use renewal-reward theorems to calculate L, W and even $\lim D(t)/t$.

Busy cycles of a work-conserving system



- Consider $\lim_{t\to\infty} D(t)/t$.
- Let C_i denote the length of the i^{th} cycle.
- Let N_i denote the number of jobs served in i^{th} cycle.
- Assume that every departure earns a reward of 1 unit. Then in i^{th} cycle, the reward earned is N_i.

Busy cycles of a work-conserving system



- From renewal reward theorem, we have $\frac{D(t)}{t} \to \frac{E[N]}{E[C]}$
- Using Wald's lemma, we can show that $E[C] = \frac{E[N]}{\lambda}$.
- This implies that $\frac{D(t)}{t} \to \lambda$.

(See Thm 3.62 and Section 3.6.1 of Sheldon Ross)

Consequences of Little's law

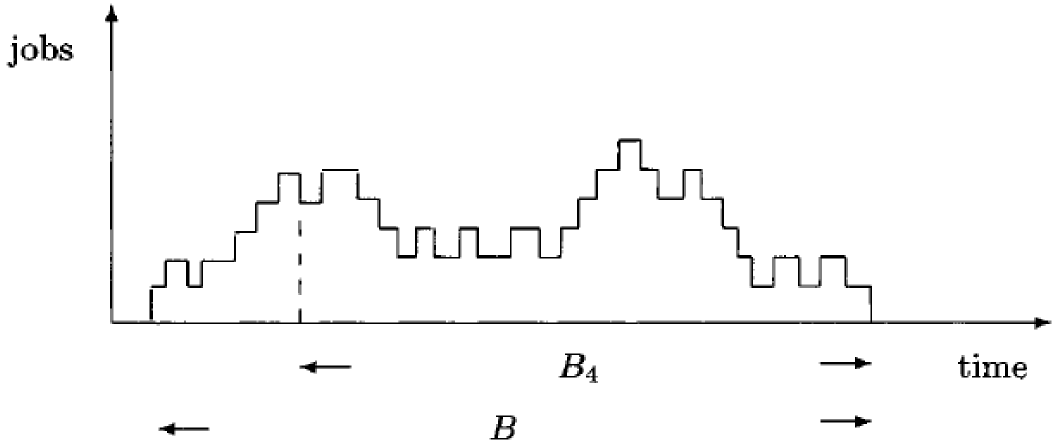
- let N_q denote the mean number of waiting jobs in a queueing system at stationarity.
- Similarly N denotes the mean number of jobs in the system.
- $N_s = N - N_q$ denotes the mean number of jobs receiving service.
- W_q denotes the mean time spent by any job waiting for service while S denotes the mean sojourn time.
- $S - W_q$ denotes the mean service time.

Consequences of Little's law – $M/M/1$

- $N_q =?$
- $N =?$
- $N_s = N - N_q =?$
- $W_q = \frac{\rho}{\mu-\lambda}$
- $S = \frac{1}{\mu-\lambda}$
- $S - W_q =?$

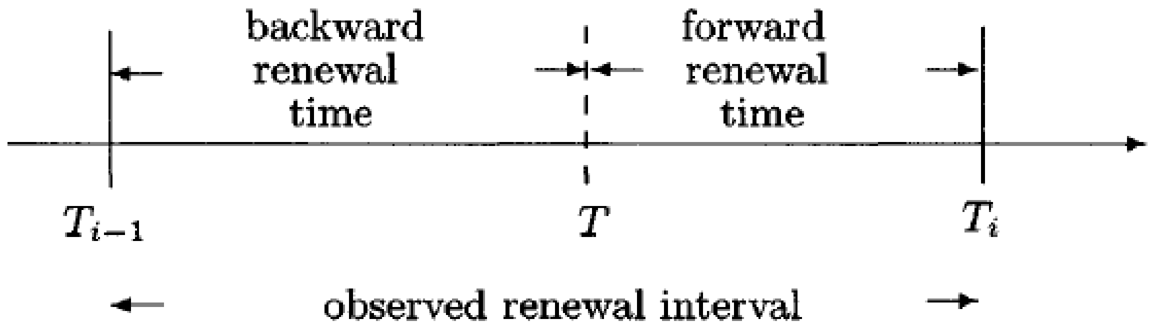
Exercise:- Identify these for $M/M/1/K, M/M/K/K$ and $M/M/K/\infty$

Busy period analysis for $M/M/1$



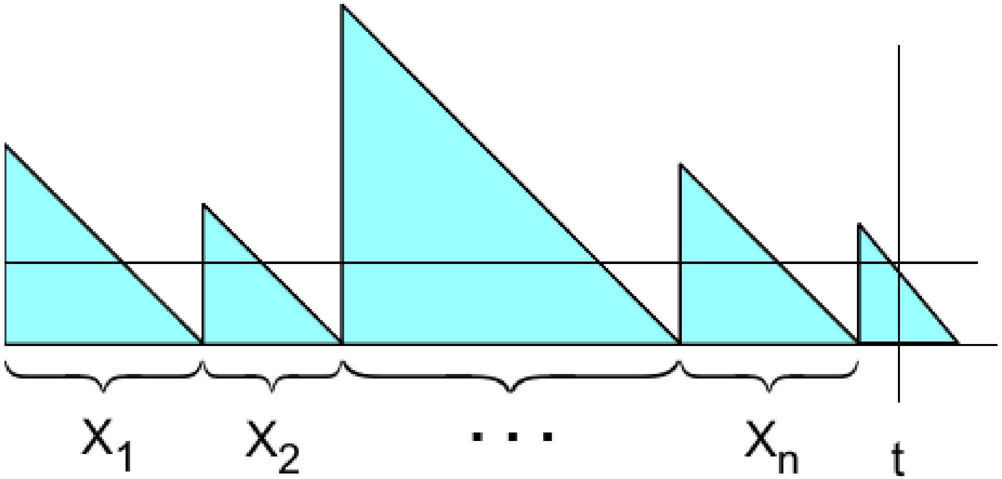
- What is the mean length of busy period, i.e., $E[B]$?
- What is the probability that the server is busy? $(1 - \pi_0 = \frac{\lambda}{\mu})$
- The time average that the server is busy is $\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{N(t)>0\}} dt$.
- Using RR theorem, this is equal to $\frac{E[B]}{E[B]+\frac{1}{\lambda}}$
- Equating the two averages give us $E[B] = \frac{1}{\mu-\lambda}$.
- Mean number of jobs served in a busy period $n_B = E[B]/\frac{1}{\mu}$.

Age and Residual life of a Renewal process



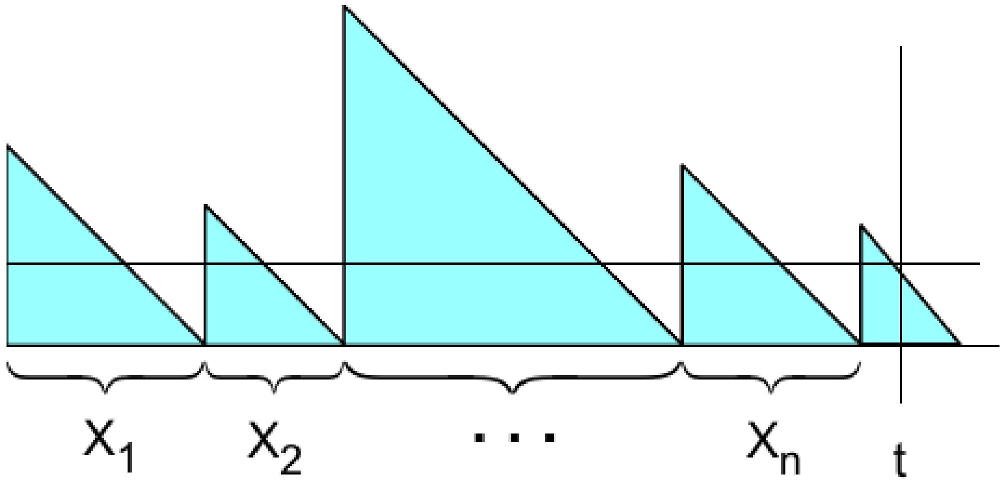
- Let $A(t)$ and $R(t)$ denote the age and the residual life of the renewal process at time t.
- Assume you arrive at a Metro station at time t.
- $A(t)$ is the time since the last metro departed.
- $R(t)$ is the time till the next Metro arrives.
- Assume that you arrive uniformly at random to the Metro.
- What is your average waiting time $\bar{R}$? $\bar{R} = E[X]/2$?

Hitchhiker's Paradox!



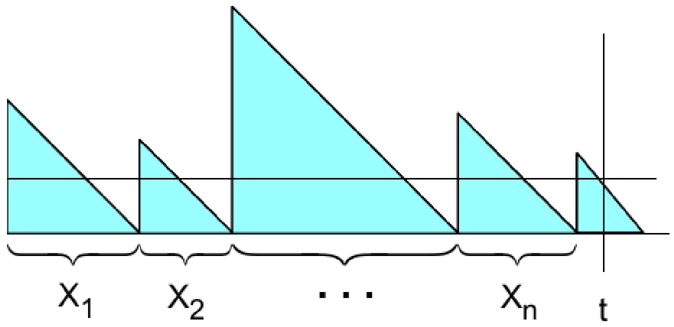
- Consider $\bar{R} = \lim_{t\to\infty} \frac{Y(t)}{t}$ where $Y(t) = \int_0^t R(t)$.
- Using Renewal reward theorem, $\bar{R} = \frac{E[Y]}{E(X)} = \frac{E[X^2]}{2E[X]} \neq E[X]/2$.
- $\frac{E[X^2]}{2E[X]} = E[X]/2$ only when interarrival times are deterministic.
- Consider $\bar{A} = \lim_{t\to\infty} \frac{Y(t)}{t}$ where $Y(t) = \int_0^t A(t)$. $\bar{A}$?.
- What is $\bar{R}$ or $\bar{A}$ when $X_i \sim exp(\lambda)$?

PASTA



- The key assumption to Hitchhikers paradox was that you arrive uniformly at random at the busy/metro stop.
- Now suppose there is a signboard at the metro that tells you the residual time till the next metro.
- Suppose that you note the residual time after every 5 min interval and compute an empirical average of the residual times.
- Will this be $\bar{R}$? No! You do not sample $(0, t)$ uniformly.

PASTA



- What if you make the residual time readings after a random time which is $exp(\lambda)$ distributed.
- Your observation process is a Poisson(λ) process.
- In this case, the empirical average will equal $\bar{R}$.

For a Poisson process, given $N(t) = n$, the arrival times $S_1, \ldots S_n$ have the same distribution as the order statistics of n i.i.d uniform points over $(0, t)$. (Thm 2.3.2 Sheldon ross)

Poisson arrivals see time average! (PASTA)

For those interested in Honors/DD

1. Stochastic Optimization
   - Bayesian Optimization (Gaussain processes for ML)
   - Reinforcement learning (Markov Decision Process under uneratainty)
   - Multi-arm bandit optmization (UCB, Thompson, Gittins index)
   - Probabilistic Machine learning (GenAI)
2. Operations Research
   - Performance modeling (this course)
   - Pricing (Data driven approaches, estimating distributions)
   - Inventory control and pricing
3. Financial Engineering
   - Porfolio Optimization, Option pricing
   - Brownian motion, Black Sholes formula, Stochastic Differential Equations

Resources: `https://sites.google.com/view/orfs/resources?authuser=0`