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A1)

(a) $$A = \begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \leftarrow R_1/a$
$R_2 \leftarrow R_2/b$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/a & 0 & 0 \\ 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \longleftrightarrow R_2$, $R_3 \leftarrow R_3/d$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right)$$

$R_3 \leftarrow R_3 - R_2$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & -1/a & 0 & 1/d \end{array}\right)$$

As Matrix A can not be row reduced to form an Identity Matrix as one of the rows of A is converted to zero and we already know the fact that an Identity matrix can not be row reduced to a matrix with zero row.
Therefore, Matrix A does not have an Inverse.

(b) $A = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$

$\Rightarrow \left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$

$R_1 \leftarrow R_1/a$
$R_2 \leftarrow R_2/a$
$R_3 \leftarrow R_3/a$

$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 1/a & 1 & 0 & 0 & 1/a & 0 \\ 0 & 1/a & 1 & 0 & 0 & 1/a \end{array}\right)$

$R_2 \leftarrow R_2 - \frac{1}{a} \cdot R_1$
$R_3 \leftarrow R_3 - \frac{1}{a} \cdot R_2$

$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 0 & 1 & 1/a^3 & -1/a^2 & 1/a \end{array}\right)$

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$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}$$

Hence inverse exist and A^{-1} is its Inverse.

A2

~~To~~ RTP :- If a Symmetric Matrix is invertible, then proove its inverse is also symmetric.

Definition :- If A^T is transpose of Matrix A then A is Symmetric if and only if $A = A^T$.

Proof :- Let A be a symmetric Matrix. $(A^T = A)$ and A^{-1} exist and be inverse of A.

By definition of Inverse,

$$A \cdot A^{-1} = I \quad \text{--- (1)}$$

Also $I^T = I$

Taking transpose on both sides,

$$(A \cdot A^{-1})^T = I^T \quad \text{--- (2)}$$

from (1),

$$(A \cdot A^{-1})^T = A \cdot A^{-1}$$

Also we know that, $(AB)^T = B^T \cdot A^T$

$$\therefore \quad (A^{-1})^T \cdot A^T = A \cdot A^{-1}$$

Now as A and A^{-1} are ~~commutable~~ commutative i.e,

$$A \cdot A^{-1} = A^{-1} \cdot A$$

$$(A^{-1})^T \cdot A^T = A^{-1} \cdot A$$

as $A^T = A$,

$$(A^{-1})^T \cdot A = A^{-1} \cdot A$$

Post multiplying A^{-1} on both sides,

$$(A^{-1})^T = A^{-1}$$

By definition, A^{-1} is a Symmetric Matrix.

Hence Proved.

Q3)

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^3 = -I$$

$$A^4 = A \cdot A^3 = A \cdot (-I) = -A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$A^5 = A^3 \cdot A^2 = -A^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A^6 = (A^3)^2 = (-I)^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$A^7 = A \cdot A^6 = A \cdot I = A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

By Observing we can say that matrix repeats after every 6 powers of A

$$A^{2015} = (A^6)^{350} \cdot A^5$$
$$= I^{350} \cdot A^5 = A^5$$
$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$ Ans

A4) RTP:- There are no square matrices X and Y with the property that $XY - YX = I$.

Proof:-

As we know trace of a square matrix A (represented by tr(A)) is the sum of diagonal entries of A i.e $\sum_{i=1}^{n} A_{ii}$ where A is $n \times n$.

Now ~~lets~~ there are few properties of trace which are as follows:-

(i) $tr(A+B) = tr(A) + tr(B)$

(ii) $tr(cA) = c \cdot tr(A)$

(iii) $tr(AB) = tr(BA)$

Now for the sake of contradiction, lets assume there exist matrices X and Y such that $XY - YX = I$.

Taking Trace on both sides,

$$tr(XY - YX) = tr(I)$$

$$\Rightarrow tr(XY) - tr(YX) = tr(I) \quad (I_{n\times n})$$

$$\Rightarrow \quad 0 = n$$

This contradicts our assumption that there exist matrices X and Y such that $XY - YX = I$.

Therefore Hence Prooved.

<u>Proof of the propertig $tr(AB) = tr(BA)$</u>

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik}$$

$$= \sum_{k=1}^{n} (BA)_{kk}$$

$$= tr(BA).$$

<u>A5)</u> <u>Given</u> :- $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

also $\langle a, b\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$ for

$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

RTP:- (a) $\{u_1, u_2, u_3\}$ forms orthogonal basis for R^3

(b) Find Coordinate of vector w with respect to basis $\{u_1, u_2, u_3\}$

Claim:- If the standard basis of R^3 (let be $\{a, b, c\}$) can be expressed in terms of u_1, u_2, u_3, then it is not enough to proove that u_1, u_2, u_3 ~~forms orthogonal basis for~~ spans R^3.

Proof:- ~~Clearly~~

Let, $a = x_1 u_1 + x_2 u_2 + x_3 u_3$

$b = y_1 u_1 + y_2 u_2 + y_3 u_3$

$c = z_1 u_1 + z_2 u_2 + z_3 u_3$

for any vector v in R^3,

$$v = \alpha a + \beta b + \gamma c.$$

$$V = \alpha (x_1 u_1 + x_2 u_2 + x_3 u_3) + \beta (y_1 u_1 + y_2 u_2 + y_3 u_3) + \gamma (z_1 u_1 + z_2 u_2 + z_3 u_3)$$

$$V = (\alpha x_1 + \beta y_1 + \gamma z_1) u_1 + (\alpha x_2 + \beta y_2 + \gamma z_2) u_2 + (\alpha x_3 + \beta y_3 + \gamma z_3) u_3$$

$\Rightarrow \{u_1, u_2, u_3\}$ forms basis of R^3

Hence Prooved.

~~Now be~~ Now as we know

$a = [1,0,0]$, $b = [0,1,0]$, $c = [0,0,1]$

By Observation,

~~$[1,0,0] = \frac{1}{3}$~~ $a = \frac{1}{3} u_1 + \frac{1}{2} u_2 + \frac{1}{6} u_3$

$b = \frac{1}{3} u_1 - \frac{1}{2} u_2 + \frac{1}{6} u_3$

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and;

$$c = \frac{1}{3}u_1 + 0u_2 - \frac{1}{3}u_3$$

Therefore, $\{u_1, u_2, u_3\}$ spans ~~forms orthogonal basis for~~ R^3.

Now,

$\langle u_1, u_2\rangle = 1\cdot1 + 1(-1) + 1\cdot0 = 0$

$\langle u_2, u_3\rangle = 1\cdot1 + (-1)\cdot1 + 0\cdot(-2) = 0$

$\langle u_3, u_1\rangle = 1\cdot1 + 1\cdot1 + 1\cdot(-2) = 0$

~~Now~~ This implies, vectors u_1, u_2, u_3 are mutually perpendicular to each other.

$\Rightarrow$ Also, u_1, u_2, u_3 would be linearly independent.

Therefore, $\{u_1, u_2, u_3\}$ forms orthogonal basis for R^3. Hence Prooved (a).

(b) As $w \in R^3$,

$$w = \alpha u_1 + \beta u_2 + \gamma u_3$$

~~Now by definition~~, Now by definitions,

$$\alpha = \frac{w\cdot u_1}{u_1\cdot u_1}, \quad \beta = \frac{w\cdot u_2}{u_2\cdot u_2}, \quad \gamma = \frac{w\cdot u_3}{u_3\cdot u_3}$$

$$\alpha = \frac{4+5+6}{3} = 5$$

$$\beta = \frac{4-5+0}{2} = -1/2$$

$$\gamma = \frac{4+5-12}{6} = -1/2$$

$\therefore$ Coordinates are $\{5, -\frac{1}{2}, -\frac{1}{2}\}$

= Ae

Prooving it true

A6)

RTP:- ~~Prove~~ Product of two upper triangular matrices is upper triangular.

Proof:- Let A and B be two upper triangular matrices.

By definition, A matrix is called upper triangular if all the elements below the main diagonal are zero.

i.e $A_{ij} = 0 \quad \forall \; i > j$.

Now, $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

Now by def., for AB to Upper Triangular

$(AB)_{ij} = 0 \quad \forall \; i > j$.

Since A and B are upper triangular
$A_{ik} = 0 \;\forall\; k < i \;\;\&\;\; B_{kj} = 0 \;\forall\; j < k$.

Now, ~~[illegible]~~ $k < i \;\&\; j < k$
$\Rightarrow i > j$

i.e for $i > j$ any one of A_{ik} or B_{kj} must be true and,

$(AB)_{ij} = 0 \quad \forall \; i > j$

$\therefore$ AB is upper triangular Matrix.

Hence Prooved