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LA Assignment - 1
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A1:

To Prove: Any Subfield of $(\mathbb{C}, +, \cdot)$ must contain every Rational Number.

Proof:

Let '0' be the identity element under Addition and '1' be the identity element under Multiplication.
Now Every Field must contain atleast 2 eleme
$\{0, 1\}$

Let $F \subset \mathbb{C}$ be a Subfield.

Claim:- $\mathbb{Z} \subset F$ where $\mathbb{Z}$ denotes set of Integers.

Proof: $1 \in F$ (Above Bi)

By Closure Property (Binary Op);
$1+1 \in F \Rightarrow 2 \in F$

Also $1+1+\cdots n$ times $\in F \quad \forall\, n \in \mathbb{Z}^+$
$n \in F \quad \forall\, n \in \mathbb{Z}^+$

Also by Inverse Property;
$-n \in F \quad \forall\, n \in \mathbb{Z}^+$

Also $0 \in F$ (Above)

$\therefore \mathbb{Z} \subset F$

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Let 'r' be an arbitary Rational Number s.t $r = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} - \{0\}$

$a \in F$ (Claim)

$b \in F$ (Claim)

By Inverse; $b^{-1} \in F$

By Closure, $a \cdot b^{-1} \in F$ (under Multiplication)

Or $\frac{a}{b} \in F$

$\Rightarrow r \in F$

But 'r' was an arbitary Rational No

$r \in \mathbb{Q}$

$\Rightarrow \mathbb{Q} \subset F$

$\therefore$, Every Subfield of $(\mathbb{C}, +, \cdot)$ must contain every Rational Number.

Hence Prooved

Q2

To Proove: Set of all Complex No of the form $x + \sqrt{2}y$, $x, y \in \mathbb{Q}$ is a Subfield of $\mathbb{C}$.

Proof:-

Let F be a Such Set s.t;

$F = \{ z \mid z = x + \sqrt{2}y \; ; \; x, y \in \mathbb{Q} \}$

Clearly;

$z \in \mathbb{C} \quad \forall \; z \in F$

$\therefore \quad F \subset \mathbb{C}$

Clearly, $0 \in F$ ($z = 0$ for $x = 0$ & $y = 0$)

$1 \in F$ ($z = 1$ for $x = 1$ & $y = 0$)

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Argument: If $x, y \in F$ and $x+y, -x, xy, x^{-1} \in F$ then F is a Subfield of $\mathbb{C}$ assuming $F \subset \mathbb{C}$ and $0, 1 \in F$

(Defination of Subfield Ref. Book :- Pg 2 Para 3)

Let $z_1, z_2 \in F$ s.t

$z_1 = x_1 + \sqrt{2} y_1 \quad ; \quad x_1, y_1 \in \mathbb{Q}$

$z_2 = x_2 + \sqrt{2} y_2 \quad ; \quad x_2, y_2 \in \mathbb{Q}$

$$z_1 + z_2 = (x_1 + \sqrt{2} y_1) + (x_2 + \sqrt{2} y_2)$$
$$= (x_1 + x_2) + \sqrt{2}(y_1 + y_2)$$

$x_1 + x_2 \in \mathbb{Q} = x$ and $y_1 + y_2 \in \mathbb{Q} = y$

$$= x + \sqrt{2} y \in F$$

$z_1 \cdot z_1^{-1}$: $\quad z_1 \cdot z_1^{-1} = 1$

$$z_1^{-1} = \frac{1}{z_1}$$

$$z_1^{-1} = \frac{1}{x_1 + \sqrt{2} y_1}$$

$$= \frac{1}{x_1 + \sqrt{2} y_1} \times \frac{x_1 - \sqrt{2} y_1}{x_1 - \sqrt{2} y_1}$$

$$= \frac{x_1 - \sqrt{2} y_1}{x_1^2 - 2y_1^2}$$

$$= \frac{x_1}{x_1^2 - 2y_1^2} - \frac{\sqrt{2}}{x_1^2 - 2y_1^2} y$$

$$= \left(\frac{x_1}{x_1^2 - 2y_1^2}\right) + \left(\frac{-y_1}{x_1^2 - 2y_1^2}\right) \sqrt{2}$$

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Now $\frac{x_1}{x_1^2-2y_1^2}$ and $\frac{-y_1}{x_1^2-2y_1^2} \in \mathbb{Q}$

only when $x_1^2-2y_1^2 \neq 0$

We Know z^{-1} is defined for $z \neq 0$

$\therefore \quad z_1 \neq 0$

~~$x_1+\sqrt{2}y_1 \neq 0$~~

$x_1 \neq -\sqrt{2}y_1$

$\Rightarrow x_1^2-2y_1^2 \neq 0$

$\therefore \quad z_1^{-1} = x+\sqrt{2}y \in F$

$$\begin{aligned} z_1 \cdot z_2 &= (x_1+\sqrt{2}y_1)(x_2+\sqrt{2}y_2) \\ &= x_1x_2+\sqrt{2}x_1y_2+\sqrt{2}x_2y_1+2y_1y_2 \\ &= (x_1x_2+2y_1y_2)+\sqrt{2}(x_1y_2+x_2y_1) \\ &= x+\sqrt{2}y \in F \end{aligned}$$

$$\begin{aligned} -z_1 &= -(x_1+\sqrt{2}y_1) \\ &= (-x_1)+\sqrt{2}(-y_1) \\ &= x+\sqrt{2}y \in F \end{aligned}$$

$\therefore$ F is a Subfield of $\mathbb{C}$

Hence Prooved

Extra:- We didn't prooved associativity and commutative property under '+' and '×' operation due to definition but the reason is if two elements x, y

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belong to a Subset of a Field then they also belong to Field then they must follow associative & commutative property.

If $z_1, z_2 \in F \Rightarrow z_1, z_2 \in \mathbb{C}$

$\Rightarrow z_1, z_2$ follow associative and Commutative and Distributive Property

Still Proof :-

~~Associativity~~: Commutative

(1) Under Addition: $z_1 + z_2 = (x_1 + \sqrt{2}y_1) + (x_2 + \sqrt{2}y_2)$

$= (x_1 + x_2) + \sqrt{2}(y_1 + y_2)$

$= (x_2 + x_1) + \sqrt{2}(y_2 + y_1)$

$= (x_2 + \sqrt{2}y_2) + (x_1 + \sqrt{2}y_1)$

$= z_2 + z_1$

(2) Under Multiplication: $z_1 \cdot z_2 = (x_1 + \sqrt{2}y_1) \cdot (x_2 + \sqrt{2}y_2)$

$= (x_1 x_2 + 2y_1 y_2) + \sqrt{2}(x_1 y_2 + x_2 y_1)$

$= (2y_1 y_2 + x_1 x_2) + \sqrt{2}(x_2 y_1 + x_1 y_2)$

$= (x_2 + \sqrt{2}y_2) \cdot (x_1 + \sqrt{2}y_1)$

Associative :

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(1) Under Addition :

$(z_1 + z_2) + z_3 = ((x_1 + x_2) + \sqrt{2}(y_1 + y_2)) + x_3 + \sqrt{2}y_3$

$= (x_1 + x_2 + x_3) + \sqrt{2}(y_1 + y_2 + y_3)$

$= (x_1 + \sqrt{2}y_1) + ((x_2 + x_3) + \sqrt{2}(y_2 + y_3))$

$= z_1 + (z_2 + z_3)$

(2) Under Multiplication :

$(z_1 \cdot z_2) \cdot z_3 = ((x_1 x_2 + 2y_1 y_2) + \sqrt{2}(x_1 y_2 + x_2 y_1)) \cdot (x_3 + \sqrt{2}y_3)$

$= ((x_1 x_2 x_3 + 2x_3 y_1 y_2 + 2(x_1 y_2 y_3 + x_2 y_1 y_3))$
$+ \sqrt{2}(x_1 y_2 x_3 + x_2 y_1 x_3 + x_1 x_2 y_3 + 2y_1 y_2 y_3)$

$= (x_1 + \sqrt{2}y_1) \cdot ((x_2 x_3 + 2y_2 y_3) + \sqrt{2}(x_2 y_3 + x_3 y_2))$

$= (x_1 + \sqrt{2}y_1) \cdot ((x_2 + \sqrt{2}y_2) \cdot (x_3 + \sqrt{2}y_3))$

$= z_1 \cdot (z_2 \cdot z_3)$

Distributive

$z_1 \cdot (z_2 + z_3) = (x_1 + \sqrt{2}y_1) \cdot ((x_2 + x_3) + \sqrt{2}(y_2 + y_3))$

$= (x_1 x_2 + x_1 x_3 + 2y_1 y_2 + 2y_1 y_3) + \sqrt{2}(x_1 y_2 + x_1 y_3 + x_2 y_1 + x_3 y_1)$

$= ((x_1 x_2 + 2y_1 y_2) + \sqrt{2}(x_1 y_2 + x_2 y_1)) + ((x_1 x_3 + 2y_1 y_3) + \sqrt{2}(x_1 y_3 + x_3 y_1))$

$= z_1 \cdot z_2 + z_1 \cdot z_3$ H