Discrete time Markov Chains (DTMC)

- A stochastic process $\{X_n, n \in \mathbb{Z}_+\}$ is a discrete time Markov chain if for any n we have

$$P(X_n = j | X_1 = x_1, .., X_{n-1} = x_{n-1}) = P(X_n = j | X_{n-1} = x_{n-1})$$

- This is called as the Markov property.
- $P(next\ state | past\ states,\ present\ state) = P(next\ state |\ present\ state)$
- Why Chain? You can view the successive random variables as a chain of states being visited in a sequence and where the next state visited depends only on the current state.
- We will throughout assume that the state space $\mathcal{S}$ is countable.

Running example: Coin with memory!

- In a Markovian coin with memory, the outcome of the next toss depends on the current toss.
- $X_n = 1$ for heads and $X_n = -1$ otherwise. $\mathcal{S} = \{+1, -1\}$.
- Sticky coin : $P(X_{n+1} = 1 | X_n = 1) = 0.9$ and $P(X_{n+1} = -1 | X_n = -1) = 0.8$ for all n.
- Flippy Coin: $P(X_{n+1} = 1 | X_n = 1) = 0.1$ while $P(X_{n+1} = -1 | X_n = -1) = 0.3$ for all n.
- This can be represented by a transition diagram (see board)
- The one step transition probability matrix P for the two cases is $P_s = \begin{bmatrix} 0.9 & .1 \\ 0.2 & 0.8 \end{bmatrix}$ and $P_f = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$
- The row corresponds to present state and the column corresponds to next state.

Running example: Dice with memory!

- In a markovian dice with memory, the outcome of the next roll depends on the current roll.
- $X_n = i$ for $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \dots, 6\}$.
- Example one-step transition probability matrix

$$P = \begin{bmatrix} 0.9 & .1 & 0 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

- State transition diagram on board
- Consider $S_n = \sum_{i=1}^{n} X_i$ and $\hat{\mu}_n = \frac{S_n}{n}$. What is $\lim_{n\to\infty} \hat{\mu}_n$?
- Cannot invoke SLLN as $\{X_i\}$ are not i.i.d.
- We will see later SLLN for Markov chains!

Finite dimensional distributions

- Consider a Markov dice with transition probability P.
- What is $P(X_0 = 4, X_1 = 5, X_2 = 6)$?
- $= P(X_2 = 6|X_1 = 5, X_0 = 4)P(X_1 = 5|X_0 = 4)P(X_0 = 4)$
- $= p_{65}p_{54}P(X_0 = 4)$.
- What is $P(X_0 = 4)$?
- This probability of starting in a particular state is called initial distribution of the markov chain.

Finite dimensional distributions

- Consider a DTMC $\{X_n, n \geq 0\}$ with transition matrix P.
- We assume M states and X_0 denotes the initial state.
- You can start in any starting state or may pick your starting state randomly.
- Let $\bar{\mu} = (\mu_1, \ldots, \mu_M)$ denote the initial distribution, i.e., $P(X_0 = x_0) = \mu_{x_0}$.
- How does one obtain the finite dimensional distribution $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2)$?
- $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2) = p_{x_1,x_2} p_{x_0,x_1} \mu_{x_0}$.
- In general, $P(X_0 = x_0, X_1 = x_1, \ldots X_k = x_k) = p_{x_{k-1},x_k} \times \ldots \times p_{x_0,x_1} \mu_{x_0}$

Chapman Kolmogorov Equations for DTMC

- Consider a Markov coin and its transition probability matrix $P = \begin{bmatrix} p_{1,1} & p_{1,-1} \\ p_{-1,1} & p_{-1,-1} \end{bmatrix}$.

- Given $X_0 = 1$, what is $P(X_2 = 1)$?

$$\begin{aligned} P(X_2 = 1 | X_0 = 1) &= P(X_2 = 1 | X_1 = 1, X_0 = 1) P(X_1 = 1 | X_0 = 1) \\ &+ P(X_2 = 1 | X_1 = -1, X_0 = 1) P(X_1 = -1 | X_0 = 1) \\ &= p_{1,1}^2 + p_{-1,1} p_{1,-1} \end{aligned}$$

- Here the first inequality follow from the fact that

$$P(C|A) = P(C|BA)P(B|A) + P(C|B^cA)P(B^c|A)$$ HW: Verify

- Similarly, $P(X_2 = -1 | X_0 = 1), P(X_2 = 1 | X_0 = -1), P(X_2 = -1 | X_0 = -1)$ can be obtained and these are elements of a two-step transition matrix $P^{(2)}$.

Chapman Kolmogorov Equations for DTMC

- The two step transition probability matrix $P^{(2)}$ is given by
$$P^{(2)} = \begin{bmatrix} p_{1,1}^2 + p_{1,-1}p_{-1,1} & p_{1,1}p_{1,-1} + p_{1,-1}p_{-1,-1} \\ p_{-1,1}p_{1,1} + p_{-1,-1}p_{-1,1} & p_{-1,1}p_{1,-1} + p_{-1,-1}^2 \end{bmatrix}.$$

- This implies that $P^{(2)} = P \times P = P^2$.

- In general, $P^{(n)} = P^n$.

- Chapman-Kolmogorov equations are a further generalization of this.

$$\boxed{P^{(n+l)} = P^{(n)}P^{(l)}}$$

- We wont see the proof of this.