Performance Modelling for Computer Systems

Quiz 1 Solutions

Q1: Let $Y \sim \text{Exp}(\mu)$ be an exponentially distributed random variable with rate μ. Suppose that jobs arrive according to a Poisson process with rate λ over a random interval Y. Define $N(Y)$ as the number of arrivals in this random time interval Y. Find the distribution of $N(Y)$. Hint:

$$\int_0^\infty x^k e^{-\beta x} dx = \frac{k!}{\beta^{k+1}}$$

A:

$$\begin{aligned}
P(N(Y) = k) &= \int_0^\infty P(N(Y) = k | Y = y) f_Y(y)\, dy \\
&= \int_0^\infty \frac{e^{-\lambda y}(\lambda y)^k}{k!} \cdot \mu e^{-\mu y}\, dy \\
&= \frac{\mu \lambda^k}{k!} \int_0^\infty y^k e^{-(\lambda+\mu)y}\, dy
\end{aligned}$$

Using the hint,

$$\begin{aligned}
P(N(Y) = k) &= \frac{\mu \lambda^k}{k!} \frac{k!}{(\lambda+\mu)^{k+1}} \\
&= \frac{\mu \lambda^k}{(\lambda+\mu)^{k+1}} \\
&= \frac{\mu}{(\lambda+\mu)} \left(\frac{\lambda}{\lambda+\mu}\right)^k
\end{aligned}$$

Thus we have that $N(Y) \sim Geometric(\frac{\mu}{\mu+\lambda})$

$$P(N(Y) = k) = \frac{\mu}{\lambda+\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^k, \quad k = 0, 1, 2, ...$$

Q2: Consider a Continuous Time Markov Chain (CTMC) with the following infinitesimal generator matrix Q:

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

Obtain its stationary distribution. Now consider the corresponding embedded Discrete Time Markov Chain (DTMC) and derive its transition probability matrix. What is the stationary distribution of the embedded DTMC? Are the two stationary distributions the same? Why?

A:

1. Stationary Distribution of the CTMC

Let π_0 and π_1 be the stationary probabilities of states 0 and 1 respectively. The stationary distribution satisfies the equation $\pi Q = 0$ and the normalization condition $\pi_0 + \pi_1 = 1$.

$$\begin{aligned}
\pi Q &= 0 \\
\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} &= 0 \\
\pi_0(-\lambda) + \pi_1 \mu &= 0 \\
\pi_0 \lambda &= \pi_1 \mu.
\end{aligned}$$

Using the normalization condition $\pi_0 + \pi_1 = 1$, we get:

$$\pi_0 = \frac{\mu}{\mu + \lambda}, \qquad \pi_1 = \frac{\lambda}{\mu + \lambda}.$$

2. Transition Probability Matrix of the Embedded DTMC

The embedded DTMC has transition probabilities given by:

$$p_{ij} = \frac{q_{ij}}{|q_{ii}|}, \quad p_{ii} = 0.$$

Thus, the transition matrix P becomes:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3. Stationary Distribution of the Embedded DTMC

Let $\pi_0^{'}$ and $\pi_1^{'}$ be the stationary probabilities for the embedded DTMC. The stationary distribution satisfies $\pi P = \pi$ and $\pi_0^{'} + \pi_1^{'} = 1$.

$$\begin{aligned}
\pi_0^{'} &= \pi_1^{'} \\
\pi_0^{'} + \pi_1^{'} &= 1.
\end{aligned}$$

From this, we get:

$$\pi_0^{'} = \pi_1^{'} = \frac{1}{2}.$$

4. Comparison of Stationary Distributions

The stationary distributions of the CTMC and the embedded DTMC are different. The CTMC distribution depends on the rates λ and μ, while the embedded DTMC always has equal stationary probabilities for both states, regardless of these rates.

The difference arises because the CTMC captures the time spent in each state, while the embedded DTMC only considers the sequence of transitions without accounting for the duration of stays.

Q3: State the criteria for classifying state i of a DTMC as recurrent (null and positive) and transient based on F_{ii}, μ_{ii}, f_{ii}^n and $\sum_n p_{ii}^n$. Justify your formulas and give proofs/explanations wherever applicable.

A:

- $f_{ij}^n = P(X_n = j, X_k \neq j \text{ for } 1 \leq k < n | X_0 = i)$ is the probability that, starting from state i, the first visit to state j occurs at step n.
- $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$ is the probability that, starting from state i, state j is ever visited. F_{ii} is the probability that, starting from state i, we ever return to state i.
- $\sum_{n=1}^{\infty} p_{ii}^n$ is the mean total number of visits to state i.
- $\mu_{ij} = \mathbb{E}[\text{time of the first visit to j}|X_0 = i] = \sum_{n=1}^{\infty} n f_{ij}^n$ is the expected time of the first visit to state j, starting from state i. μ_{ii} is the expected time to return to state i, starting from state i.

A state i of a DTMC can be classified as follows:

Transient State:

- A state i is transient if $F_{ii} < 1$. This means there is a positive probability that, starting from state i, the DTMC will never return to state i.
- if $\mu_{ii} = \infty$, then the state is possibly transient or null recurrent. This is because the $\mu_{ii} = \infty$ could imply that you never return to state i again or after infinite time.
- f_{ii}^n alone cannot be used to determine transience or recurrence.
- If $\sum_{n=1}^{\infty} p_{ii}^n < \infty$, the state is transient. This is because the total number of visits to state i is finite and you will never visit state i after a certain time.

Recurrent State:

- A state i is recurrent if $F_{ii} = 1$. This means that, starting from state i, the DTMC is certain to return to state i eventually (though possibly after a very long time).
- if $\mu_{ii} < \infty$, then the state is postive recurrent. You return to state i after a finite amount of time.
- f_{ii}^n alone cannot be used to determine transience or recurrence.
- If $\sum_{n=1}^{\infty} p_{ii}^n = \infty$, the state is recurrent. This is because the total number of visits to state i is infinite and you will always keep visiting state i.

Q4: Consider an M/G/∞ system. Find the distribution (pmf or cdf) of $X(t)$, where $X(t)$ denotes the number of jobs present in the system at time t and where the service time distribution follows 1) Shifted-Exponential distribution with parameter μ, shifted by r. 2) Uniform distribution over interval $[a, b]$.

A:

1. Shifted-Exponential Distribution

If the service time follows a Shifted-Exponential distribution with parameter μ and shift r, the cumulative distribution function (CDF) is given by:

$$G(x) = \begin{cases} 0, & x < r, \\ 1 - e^{-\mu(x-r)}, & x \geq r. \end{cases}$$

Thus, the probability that a job is still in the system at time t is:

$$1 - G(t-x) = \begin{cases} 1, & t - x < r, \\ e^{-\mu(t-x-r)}, & t - x \geq r. \end{cases}$$

Now, we compute p:

$$p = \frac{1}{t} \int_0^t (1 - G(t-x)) dx.$$

Splitting the integral into two parts:

$$p = \frac{1}{t} \left[\int_0^{t-r} e^{-\mu(t-x-r)} dx + \int_{t-r}^{t} 1 dx \right].$$

Evaluating both integrals:

$$p = \frac{1}{t} \left[e^{-\mu(t-r)} \int_0^{t-r} e^{\mu x} dx + (t - (t-r)) \right].$$

$$p = \frac{1}{t} \left[e^{-\mu(t-r)} \frac{e^{\mu(t-r)} - 1}{\mu} + r \right].$$

$$p = \frac{1}{t} \left[\frac{1 - e^{-\mu(t-r)}}{\mu} + r \right].$$

Since $X(t) \sim \text{Poisson}(\lambda t p)$, the PMF is given by:

$$P(X(t) = k) = \frac{e^{-\lambda t p} (\lambda t p)^k}{k!},$$

where p is given by:

$$p = \frac{1}{t} \left[\frac{1 - e^{-\mu(t-r)}}{\mu} + r \right].$$

Thus, $X(t)$ follows a Poisson distribution with mean:

$$\lambda t p = \lambda \left[\frac{1 - e^{-\mu(t-r)}}{\mu} + r \right].$$

2. Uniform Distribution

If the service time follows a Uniform distribution over $[a, b]$, the cumulative distribution function (CDF) is given by:

$$G(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$$

Thus, the probability that a job is still in the system at time t is:

$$1 - G(t-x) = \begin{cases} 1, & t-x < a, \\ \frac{b-(t-x)}{b-a}, & a \le t-x \le b, \\ 0, & t-x > b. \end{cases}$$

Now, we compute p:

$$p = \frac{1}{t}\int_0^t (1 - G(t-x))dx.$$

Splitting the integral into two parts:

$$p = \frac{1}{t}\left[\int_0^{t-a} \frac{b-(t-x)}{b-a}dx + \int_{t-a}^{t} 1dx\right].$$

Evaluating the first integral:

$$\int_0^{t-a} \frac{b-(t-x)}{b-a}dx = \frac{1}{b-a}\int_0^{t-a}(b-t+x)dx.$$

Expanding and solving:

$$\frac{1}{b-a}\left[(b-t)x + \frac{x^2}{2}\right]_0^{t-a}.$$

Substituting $x = t - a$:

$$\frac{1}{b-a}\left[(b-t)(t-a) + \frac{(t-a)^2}{2}\right].$$

Evaluating the second integral:

$$\int_{t-a}^{t} 1dx = a.$$

Thus, we obtain:

$$p = \frac{1}{t}\left[\frac{(b-t)(t-a) + \frac{(t-a)^2}{2}}{b-a} + a\right].$$

Since $X(t) \sim \text{Poisson}(\lambda tp)$, the PMF is given by:

$$P(X(t) = k) = \frac{e^{-\lambda tp}(\lambda tp)^k}{k!},$$

where p is given by:

$$p = \frac{1}{t}\left[\frac{(b-t)(t-a) + \frac{(t-a)^2}{2}}{b-a} + a\right].$$

Thus, $X(t)$ follows a Poisson distribution with mean:

$$\lambda t p = \lambda\left[\frac{(b-t)(t-a) + \frac{(t-a)^2}{2}}{b-a} + a\right].$$

Note: 2 marks are reserved for deriving the expression for p and making an argument for $X(t)$ being Poisson.