CS 3.307

Performance Modeling for Computer Systems

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RECAP

- Bernoulli/Binomial Process
- 3 Definitions of Poisson Process
- Relation between S_n and $N(t)$
- Splitting and Merging property
- Conditional distribution of Arrival times

First Queueing Example: Infinite server Queues

- Imagine a system with infinite servers and jobs arrive to this system according to $PP(\lambda)$.
- Every arriving job has a independent service requirement with distribution G and is immediately assigned a server for service.
- When the job receives service, he leaves the system.
- Let $N(t)$ denote the number of arrivals till time t.
- Let $X(t)$ denote the number of customers present in this system at time t.
- Example of such systems: Malls, Tourist spots, Gardens, number of active phone calls, etc

First Queueing Example: Infinite server Queues

- What is the pmf of $X(t)$, i.e., $P(X(t) = k)$?
- First condition on $N(t)$. What is $P(X(t) = k | N(t) = n)$?
- Of the n jobs that arrived (uniformly placed in the interval $[0, t]$), k are yet to complete service.
- Let p denote the probability that an arbitrary of these customers is still receiving service at time t.
- Then $P(X(t) = k | N(t) = n) = \binom{n}{k} p^k (1 - p)^{n-k}$.
- Now unconditioning on $N(t)$, we get

$$
\begin{aligned}
P(X(t) = k) &= \sum_{n=k}^{\infty} P(X(t) = k | N(t) = n) P(N(t) = n) \\
&= e^{-\lambda t p} \frac{(\lambda t p)^j}{j!}
\end{aligned}
$$

where $p = \int_0^t (1 - G(t - x)) \frac{dx}{t}$.

Non-homogeneous Poisson process

A non-homogeneous Poisson process with rate function $\lambda(t), \lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ with the following properties

- $N(0) = 0$
- $N(t)$ has independent and stationary increments
- $P\{N(h) = 1\} = \lambda(t)h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$
- Define mean function $m(t) := \int_0^t \lambda(s)ds \geq 0$
- It can be shown that $N(t+s) - N(t) \sim Poisson(m(t+s) - m(t))$.

Markov Process

- There are two versions of Markov chains- Discrete time and Continuous time.
- A stochastic process $\{X_n, n \in \mathbb{Z}_+\}$ is a discrete time Markov chain if for any $n_1 < n_2 < \ldots < n_k < n$,

$$P(X_n = j | X_{n_1} = x_1, .., X_{n_k} = i) = P(X_n = j | X_{n_k} = i)$$

- $\{X(t), t \geq 0\}$ is a Markov process (ctmc) if for $t_1 < t_2 < \ldots t_n < t$,

$$P(X(t) = j | X(t_1) = x_1, .., X(t_n) = i) = P(X(t) = j | X(t_n) = i)$$

- This is known as the Markov property.
- State space in both cases can be integers or general $(\mathbb{R}^d)$
- We will stick with integer or finite state space

Example: Coin with memory!

- In a Markovian coin with memory, the outcome of the next toss depends on the current toss.

- $X_n = 1$ for heads and $X_n = -1$ otherwise. $\mathcal{S} = \{+1, -1\}$.

- Sticky coin : $P(X_{n+1} = 1 | X_n = 1) = 0.9$ and $P(X_{n+1} = -1 | X_n = -1) = 0.8$ for all n.

- Flippy Coin: $P(X_{n+1} = 1 | X_n = 1) = 0.1$ while $P(X_{n+1} = -1 | X_n = -1) = 0.3$ for all n.

- This can be represented by a transition diagram (see board)

- The transition probability matrix P for the two cases is
$$P_s = \begin{bmatrix} 0.9 & .1 \\ 0.2 & 0.8 \end{bmatrix} \text{ and } P_f = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$$

- The row corresponds to present state and the column corresponds to next state.

Running example: Dice with memory!

- In a markovian dice with memory, the outcome of the next roll depends on the current roll.
- $X_n = i$ for $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \ldots, 6\}$.
- Example transition probability matrix

$$P = \begin{bmatrix} 0.9 & .1 & 0 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

- In the ctmc counterpart for these examples, imagine the coin tosses itself/ dice rolls itself after waiting in the state for a random time that is exponentially distributed. (more later)

Time-homogenous Markov Process

- A DTMC is said to be time homogeneous if the one step transition probabilities are same at all time.

- $P(X_{n+1} = j | X_n = i) = P(X_{n+1+s} = j | X_{n+s} = i) := p_{ij}$

- One step transition probability matrix $P = [[p_{ij}]]$

- $i, j \in \mathcal{S}$ which is countable and $|\mathcal{S}| \leq \infty$

For a CTMC ...

- For a time homogeneous CTMC, we have

$$\begin{aligned} P(X(t)=j|X(t_n)=i) &= P(X(t+s)=j|X(t_n+s)=i) \\ &= P(X(t-t_n)=j|X(0)=i). \end{aligned}$$

- We have a transition probability matrix with entries $p_{ij}(t)$, i.e., $P(t)=[[p_{ij}(t)]]$.

DTMC – Time spent in a state

- For a time homogeneous DTMC, we have a transition probability matrix with entries p_{ij}, i.e., $P = [[p_{ij}]]$.
- Let $Y_n = inf\{s > 0 : X_{n+s} \neq X_n\}$
- Y_n is the remaining time that the process spends in whichever state it is in, at time n.
- Consider a Markov coin, its state transition matrix and diagram
- Y_n is geometric random variable.
- What would be the time spent in a state for a continuous time Markov chain ?

CTMC – Time spent in a state

- For a time homogeneous CTMC, we have a transition probability matrix with entries $p_{ij}(t)$, i.e., $P(t) = [[p_{ij}(t)]]$.
- Let $Y_t = inf\{s > 0 : X(t+s) \neq X(t)\}$
- Y_t is the remaining time that the process spends in whichever state it is in, at time t.
- Intuitively, the time spent in a state should depend only on what state it is in, and not on the previous state.

Theorem

$$P(Y_t > u | X(t) = i) := \bar{G}_i(u) = e^{-a_i u}$$

for all $i \in \mathcal{S}$ *and* $t \geq 0, u \geq 0$ *and for some real number* $a_i \in [0, \infty]$.