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Assignment-2

Ans2)

TP:- For each elementary row op. e, there corresponds an elementary row operation e_1 such that

$$e_1(e(A)) = e(e_1(A)) = A \quad \forall A.$$

Proof:- ~~e : R~~

We have total three types of elementary row operations let be e_1, e_2, e_3 defined as

$$e_1 (A)_{ij} = \begin{cases} \alpha a_{sj} & , i = s \\ a_{ij} & , i \neq s \end{cases}$$

$$e_2 (A)_{ij} = \begin{cases} a_{sj} + \alpha a_{tj} & , i = s \\ a_{ij} & , i \neq s \end{cases}$$

$$e_3 (A)_{ij} = \begin{cases} a_{sj} & , i = r \\ a_{rj} & , i = s \\ a_{ij} & , i \neq s, r \end{cases}$$

Now we defined three elementary row operaration named e_1', e_2', e_3' as:

$$e_1'(A)_{ij} = \begin{cases} \frac{1}{\alpha} a_{sj} & , i = s \\ a_{ij} & , i \neq s \end{cases}$$

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$$e_2'(A)_{ij} = \begin{cases} a_{sj} - \alpha a_{tj} & , i = s \\ a_{sj} & , i \neq s \end{cases}$$

$$e_3'(A)_{ij} = \begin{cases} a_{rj} & , i = r \\ a_{sj} & , i = s \\ a_{ij} & , i \neq s, r \end{cases}$$

Let $e_1'(A) = B_{ij} = \begin{cases} \frac{1}{\alpha} a_{sj} & , i = s \\ a_{ij} & , i \neq s \end{cases}$

Consider;

$$(e_1 \cdot e_1')(A)$$
$$= e_1(e_1'(A))$$
$$= e_1(B)$$

$$e_1(B)_{ij} = \begin{cases} \alpha \cdot (b_{sj}) & , i = s \\ b_{ij} & , i \neq s \end{cases}$$

$$= \begin{cases} \alpha \left(\frac{1}{\alpha} a_{si}\right) & , i = s \\ a_{ij} & , i \neq s \end{cases}$$

$$= A_{ij}$$

$\Rightarrow (e_1 \circ e_1')(A) = A$

So $e_1 \circ e_1'$ = ~~the dm~~ Identity function

$e_1' \circ e_1$ = " "

Similarly,

$$(e_2 \circ e_2')(A) = A$$

And; $(e_3 \circ e_3')(A) = A$

$\Rightarrow$ For each elementary row opeeation 'e' there corresponds an elementary row op. e_1 such that $e_1(e(A)) = e(e_1(A)) = A$

Hence Prooved.

Ans 3)

(a) I : (a) $x + 2y = 18$ $\qquad$ II : (a) $-2x + 9y = 5$

(b) $-x + 11y = 23$ $\qquad$ (b) $8x - 10y = 62$

II is linear combinations of I

$\cancel{\text{II}\cdot a = 2x + 9y}$ $\quad \text{II}\cdot a = -(\text{I}\cdot a) + (\text{I}\cdot b)$

$$-2x + 9y - 5 = -(x + 2y - 18) + (-x + 11y - 23)$$

$$\text{II}\cdot b = 6(\text{I}\cdot a) - 2(\text{I}\cdot b)$$

Also, I is linear Combinations of II

$$\text{I}\cdot a = \frac{1}{2}(\text{II}\cdot a) + \frac{1}{4}(\text{II}\cdot b)$$

$$\text{I}\cdot b = \frac{3}{2}(\text{II}\cdot a) + \frac{1}{4}(\text{II}\cdot b)$$

$\therefore$ Both the Systems are equivalent.

~~(b) 1: $\cancel{x + 2y = 18}$ $x + y + z = 6$~~

~~$y + 2z = 5$~~

~~$x + 3z = 6$~~

~~2: $\cancel{3x + 2y = 7}$~~

Ans4) TP :- Every matrix has a row Reduced form.

Definition of Row-reduced matrix :

a) The first non-zero entry in each non-zero row of R is 1.

b) Each column of R which contains the leading non-zero entry of some row has all it's other entries zero.

Proof:-

→ If to proove a matrix has a row reduced form then we can show that it can be created through finite row operations or Every matrix is row equivalent to its row reduced form.

(i) If every entry in the first row of A is zero then condition <a> (mentioned above) is satisfied.

(ii) If row 1 has a non-zero entry, say K, then multiply row 1 by K^{-1} & then cond <a> is satisfied wrt row 1.

Now for each row i ($i \geq 2$) that occurs susbsequently after row 1, we shall make the column K in row 1 ~~where~~ where the first non-zero entry appears have all elements in that column be zero, by adding $(-A_{ik})$ time row 1 for row i. The way the leading non zero entry of row 1 occurs in column K & that entry is 1 & other entry in column are zero.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \cdots & 0 & a_{1K} & \cdots \\ & * & & & * \\ & * & & & * \end{pmatrix}$$

$$R_1 \leftarrow \frac{1}{a_{1K}} R_1$$

$$A \sim \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & * & \cdots & * \\ & * & & & & * \end{array}\right)$$

$$R_2 \leftarrow -a_{2K} R_1 + R_2$$

$$\cdots$$

$$R_m \leftarrow -a_{mK} R_1 + R_m$$

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$$A \sim \begin{array}{c} \text{Column-}K \\ \left(\begin{array}{ccccc|c|ccccc} 0 & \cdots & & & 0 & 1 & * & \cdots & & & * \\ * & * & \cdots & * & * & 0 & * & \cdots & * & * \cdots & * \\ & & & & & 0 & & & & & \\ & * & & & & \vdots & & & * & & \\ & & & & & 0 & & & & & \end{array}\right) \end{array}$$

Right of Col K
Col q (q > K)

~~The~~ the left of Col K
Col P (p < K)

(iii) In Resultant matrix from above, if every entry in row 2 is zero then <a> is trivially true. But if row 2 has atleast on non-zero entry,

Case: I) Leading Non-Zero entry in Right at q.

$$A \sim \left(\begin{array}{ccc|c|ccccc} 0 & \cdots & 0 & 1 & * & & \cdots & & * \\ 0 & \cdots & 0 & 0 & 0 & * & \cdots & & * \\ & & & \vdots & & & & & \\ - & \cdots & - & 0 & - & & - & & - \end{array}\right)$$

$a_{2q} \neq 0$, q least subscript

$$R_2 \leftarrow \frac{1}{a_{2q}} R_2$$

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$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad a_{1q}/a_{1k} \\ A \sim \begin{pmatrix} 0 & - & - & - & 0 & 1 & * & * & - & * \\ 0 & - & - & - & 0 & 0 & 0 & 1 & * & - \\ & & & & & \vdots & & & & \\ - & - & & - & & 0 & - & - & - & - \end{pmatrix} \end{array}$$

$$R_1 \longleftarrow -\frac{a_{1q}}{a_{1k}} R_2 + R_1$$

$$\vdots \quad - \quad - \quad - \quad - \quad - \quad -$$

$$R_m \longleftarrow -a_{mq} R_m + R_1$$

$$A \sim \begin{pmatrix} 0 & - & - & - & 0 & 1 & * & 0 & * \\ 0 & - & - & - & - & 0 & 0 & 1 & - \\ & * & & & & \vdots & & \vdots & \\ - & & - & - & & 0 & & 0 & - \end{pmatrix}$$

<u>Case:</u> II) Leading Non-zero entry in left at p

$$A \sim \begin{pmatrix} 0 & - & - & 0 & & 1 & * & - & - \\ 0 & - & - & * & * & 0 & * & - & - \\ & & & & & \vdots & & & \\ - & - & & - & & 0 & - & - & - \end{pmatrix}$$

$a_{2p} \neq 0$, $p \neq k$, p least

$$R_2 \longleftarrow \frac{1}{a_{2p}} R_2$$

$$A \sim \begin{pmatrix} 0 & - & - & 0 & & 1 & * & - & - \\ 0 & - & - & 1 & * & 0 & * & - & - \\ & & & & & \vdots & & & \\ & - & - & & & 0 & - & & - \end{pmatrix}$$

$R_3 \longleftarrow -a_{3p}R_2 + R_3$

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$R_m \longleftarrow -a_{mp}R_2 + R_m$

$$A \sim \begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots \\ 0 & \cdots & 1 & 0 & \cdots & \cdots \\ & & \vdots & \vdots & & \\ \cdots & \cdots & 0 & 0 & \cdots & \cdots \end{pmatrix}$$

Notice the fact that the columns of row are not affected by making these changes on row 2.

So, in this way we can work with one row at a time & in a finite number of steps, we will arrive at a row-reduced matrix R.

$\therefore$ Hence Prooved

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Ans 5) TP :- Every $m \times n$ matrix A is row-equivalent to a a row reduced echlelon matrix.

Definition of Row-reduced echlelon matrix :-

(a) Row-Reduced.

(b) every row of A which has all its

entries 0 occurs below every row which has a non-zero entry.

(c) If rows $1, \dots r$ are non zero rows of R and leading non zero entry of row i occurs in column k_i, $i = 1, \dots r$ then $k_1 < k_2 < \dots < k_r$.

Proof :-

Claim: Every $m \times n$ matrix is equivalent to row reduced form.

Proof : [Ans 4]

Now after getting row-reduced matrix it is trivial to observe that by performing finite row interchange on row-reduced matrix we can satisfy condition <b> & <c>

Ex: $A \tilde{\approx} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ← Row Reduced form

$R_1 \longleftrightarrow R_3$

p

$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Hence we can bring it to row reduced echelon form.

$\therefore$ Hence Prooved.

Ans 3 (B)

System A :- $x + y + z = 6$
$y + 2z = 5$
$x + 3z = 6$

System B :- $3x + 2y - z = 12$
$3x + y - z = 10$
$y + z = 3$

System C :- $x + y - z = 4$
$2x + y = 8$
$x - z = 2$

~~G~~ For ~~all~~ all systems to be equivalent $A \sim B \sim C$. i.e $A \sim B$, $A \sim C$, $B \sim C$.

Claim 1 :- if $A \sim B$ and $B \sim C$ the $A \sim C$

Proof: Because Row equivalence is a Equivalence Relation. (Transitive)

Claim :- Prooving two ~~statements~~ systems to be equivalent is same as prooving their augmented matrix are row equivalent.

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Justification: It is so because if their augmented matrices are row-equivalent it means one can be transformed into other using elementary row Operation which is same as saying one can be written in form of other as linear Combination which is the definition of Equivalent Systems

Let A', B', C' be respective Augmented Matrix of System A, B and C

$$A' = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 1 & 0 & 3 & 6 \end{array}\right]$$

$R_3 \leftarrow \cancel{R_3}(-R_1) + R_3$

$$A' \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_2 + R_3$

$R_1 \leftarrow (-R_2) + R_1$

$$A' \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 5 \end{array}\right]$$

$R_3 \leftarrow R_3 / 4$

$R_{\cancel{2}1} \leftarrow R_3 + R_1$

$R_2 \leftarrow (-2 R_3) + R_2$

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$$A' \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 5/4 \end{array}\right]$$

$$B' = \left[\begin{array}{ccc|c} 3 & 2 & -1 & 12 \\ 3 & 1 & -1 & 10 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$R_2 \leftarrow R_2 + R_3$$

$$B' \sim \left[\begin{array}{ccc|c} 3 & 2 & -1 & 12 \\ 3 & 2 & 0 & 13 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$R_2 \leftarrow (-R_1) + R_2$$

$$B' \sim \left[\begin{array}{ccc|c} 3 & 2 & -1 & 12 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$B' \sim \left[\begin{array}{ccc|c} 3 & 2 & -1 & 12 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$R_1 \leftarrow R_1/3$$

$$B' \sim \left[\begin{array}{ccc|c} 1 & 2/3 & -1/3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$R_1 \leftarrow R_1 - \tfrac{2}{3} R_3$$
$$R_1 \leftarrow R_1 + \tfrac{1}{3} R_2$$
$$R_2 \longleftrightarrow R_3$$

$$\boxed{B' \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]}$$

$$C' = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 2 & 1 & 0 & 8 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

$$R_1 \leftarrow (-R_3) + R_1$$

$$C' \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 8 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

$$R_2 \leftarrow (-R_1) + R_2$$
$$R_2 \leftarrow R_2 / 2$$

$$C' \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

$$R_3 \leftarrow (-R_2) + R_3$$
$$R_3 \leftarrow -R_3$$

$$C' \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\boxed{C' \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]}$$

Now in their row reduced echoleon matrix form we can see that

$B' \sim C'$

$A' \nsim B'$

$A' \nsim C'$

So we can't say that

$A' \sim B' \sim C'$

$\therefore$ System B and C are equivalent but not equivalent to System A.

$\therefore$ given system of linear equation are not equivalent Ans

(a) M-2 :- (Using Claim-2 from Upper Qu)

$$A :- \begin{bmatrix} 1 & 2 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 23 \end{bmatrix}$$

$$B :- \begin{bmatrix} -2 & 9 \\ 8 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 62 \end{bmatrix}$$

Let A' & B' be their respective Augmented Matrix

$$A' = \left[\begin{array}{cc|c} 1 & 2 & 18 \\ -1 & 11 & 23 \end{array}\right]$$

$$R_2 \leftarrow \cancel{8} R_1 + R_2$$

$$A' \sim \left[\begin{array}{cc|c} 1 & 2 & 18 \\ 0 & 13 & 41 \end{array}\right]$$

$$R_2 \leftarrow R_2 / 13$$

$$A' \sim \left[\begin{array}{cc|c} 1 & 2 & 18 \\ 0 & 1 & 41/13 \end{array}\right]$$

$$R_1 \leftarrow R_1 - 2R_2$$

$$\boxed{A' \sim \left[\begin{array}{cc|c} 1 & 0 & 152/13 \\ 0 & 1 & 41/13 \end{array}\right]}$$

$$B' = \left[\begin{array}{cc|c} -2 & 9 & 5 \\ 8 & -10 & 62 \end{array}\right]$$

$$R_1 \leftarrow -R_1/2$$

$$B' \sim \left[\begin{array}{cc|c} 1 & -9/2 & -5/2 \\ 8 & -10 & 62 \end{array}\right]$$

$$R_2 \leftarrow R_2 - 8R_1$$

$$B' \sim \left[\begin{array}{cc|c} 1 & -9/2 & -5/2 \\ 0 & 26 & 82 \end{array}\right]$$

$R_2 \leftarrow R_2/26$

$$B' \sim \left[\begin{array}{cc|c} 1 & -9/2 & -5/2 \\ 0 & 1 & 41/13 \end{array}\right]$$

$R_1 \leftarrow R_1 + 9/2\, R_2$

$$\boxed{B' \sim \left[\begin{array}{cc|c} 1 & 0 & 152/13 \\ 0 & 1 & 41/13 \end{array}\right]}$$

Clearly $A' \sim B'$

$\therefore$ Systems A and B are Equivalent