First recurrence probabilities

- Define: $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{n}$.
- f_{ii}^{n} : probability of starting in i and returning to state i for the first time exactly after n steps.
- $f_{ii}^{n} := P(X_n = i, X_k \neq j \text{ for } 1 \leq k \leq n-1 | X_0 = i)$. $(f_{ii}^{0} = 0)$.
- F_{ii} has the interpretation of the probability of ever returning to state i.
- If $F_{ii} = p < 1$, then there is a finite probability $1 - p$ with which you may not return to state i .
- If $F_{ii} = 1$, then the from i you can certainly return to i.
- For any $i \in \mathcal{M}$, the first return time T_{ii} has the probability mass function $\{f_{ii}^{n}, n \geq 0\}$.

Mean passage and recurrence times

- Let μ_{ii} be the mean recurrence time at state i, i.e. $E[T_{ii}]$.
- $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{n}$
- All the above definitions have an equivalent counterpart in a CTMC.
- For eg: f_{ii}^{t} has a natural interpretation. We wont go further into this.

Transient and recurrent states

- Suppose for a state i we have $F_{ii} = 1$. Then we say that state i is recurrent.
- Once in i, you are certain to come back to i.
- If $\mu_{ii} = \infty$, it is called null recurrent. The chain is bound to return to state i, but possibly after an infinite time.
- $\mu_{ii} < \infty$, it is called positive recurrent.
- If all states of the Markov chain are (null /positive) recurrent, it is called as a (null /positive) recurrent Markov chain.
- Null recurrence is possible in infinite state space models.
- State i is transient if $F_{ii} < 1$.
- You may not return back to i with a finite probability.

Transient and recurrent states

- Consider a DTMC and consider $X_0 = i$.
- Let us count the number of times the chain is in state i.
- Let I_n denote an indicator variable which is 1 if $X_n = i$ and 0 if $X_n \neq i$.
- $\sum_{n=1}^{\infty} I_n$ counts the number of times state i was visited.
- $E[I_n] = P(X_n = i | X_0 = i) = p_{ii}^{(n)}$.
- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^{n}$
- Convergence or divergence of this sum also defines transient or recurrent states.

Recurrent criteria

- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^{n}$
- Suppose the chain visits state i only exactly n times.
- The P(exactly n visits to $i) = F_{ii}^{n}(1 - F_{ii})$.
- For a recurrent state, $F_{ii} = 1$. Hence P(exactly n visits to $i) =$ 0.
- P(exactly infinite visits to $i) = 1$.
- Mean total number of visits is also infinite and hence $\sum_{n=1}^{\infty} p_{ii}^{n}$ diverges.

Transient state criteria

- The mean total number of visits to state i is given by $\sum_{n=1}^{\infty} p_{ii}^n$
- Suppose the chain visits state i only exactly n times.
- The P(exactly n visits to i) $= F_{ii}^n(1 - F_{ii})$.
- For transient state i, $F_{ii} < 1$.
- The P(exactly n visits to i) $= F_{ii}^n(1 - F_{ii})$. Compare this with geometric random variable.
- Mean total number of visits to state i is $\frac{F_{ii}}{1-F_{ii}}$ which is finite.
- Hence for transient state $\sum_{n=1}^{\infty} p_{ii}^n$ must converge.

Classification of states

- Consider a Markov process with state space $\mathcal{S}$
- We say that j is accessible from i if $p_{ij}^n > 0$ for some n.
- For a CTMC, the condition is $p_{ij}(t) > 0$ for some t.
- This is denoted by $i \to j$.
- if $i \to j$ and $j \to i$ then we say that i and j communicate.
  This is denoted by $i \leftrightarrow j$.

A chain is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$.

Communicating class

- A set $\mathcal{C} \subset \mathcal{S}$ is called a communicating class if all states in $\mathcal{C}$ communicate with each other
- These states can reach to states outside the class (hence the class is not closed)
- The communicating class is closed if states from the class cannot reach outside $\mathcal{C}$.
- The above is true for DTMC as well as CTMC.

Communicating class is an equivalence relation

- A binary relation R on $\mathcal{S}$ is a subset of the product space $\mathcal{S} \times \mathcal{S}$.

- Equivalence relation is a binary relation which is symmetric, reflexive and transitive.

- $i \leftrightarrow i$ (Reflexivity)

- If $i \leftrightarrow j$ then $j \leftrightarrow i$ (Symmetry)

- If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$. (Transitivity)

Periodicity

- Periodicity is a class property
- Period d_i of state i is defined as $d_i = gcd\{m : p_{ii}^{(m)} > 0\}$
- If $d_i = 1$ for state i, it is called aperiodic else its called periodic.
- For a CTMC, what is its periodicity ?
- A CTMC is generally considered aperiodic.
- What about the Embedded DTMC ? Can it be periodic ?

Recurrence and Transience as class property

- Recall the definition of closed communicating class.
- There is a positive probability with which you can go from one state to another within the class.
- All states in such a closed communicating class are either transient or recurrent.