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Assignment - 3

A1

Given :- A, B, C are matrices over a field F such that the products BC and A(BC) are well defined and so are the products AB and (AB)C.

RTP :- (Required to Proove) $A(BC) = (AB)C$

Proof :- Definition : Matrix multiplication is defined as, (for $C = AB$);

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$ ~~(assuming~~

Let $A_{m \times \theta}$, $B_{\theta \times p}$, $C_{p \times n}$ so.
for any arbitrary i, j such that $1 \le i \le m$ & $1 \le j \le n$

$$[A(BC)]_{ij} = \sum_{r=1} A_{ir} (BC)_{rj}$$

$$= \sum_{r=1}^{\theta} A_{ir} \left(\sum_{s=1}^{p} B_{rs} C_{sj} \right)$$

$$= \sum_{r=1}^{\theta} \sum_{s=1}^{p} A_{ir} B_{rs} C_{sj}$$

Because summation of are Independent of each other as they are commutative

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$\therefore$,
$$= \sum_{s=1}^{P} \sum_{r=1}^{\theta} A_{ir} B_{rs} C_{sj}$$

$$= \sum_{s=1}^{P} \left(\sum_{r=1}^{\theta} A_{ir} B_{rs} \right) \cdot C_{sj}$$

$$= \sum_{s=1}^{P} (AB)_{is} \cdot C_{sj}$$

$$= [(AB)C]_{ij}$$

As for any arbitary i, j

$(A(BC))_{ij} = ((AB)C)_{ij}$

$\therefore$ Both the matrix are same

$$\boxed{A(BC) = (AB)C}$$

Hence Proved

Ans 2 :

Given :- e be an elementary row operation and E be an elementary ~~operation~~ matrix of size $m \times m$ such that $E = e(I_{m \times m})$

RTP :- $e(A) = EA$ holds $\forall$ A of $m \times n$.

Proof :- There are total three types of elementary let named e_1, e_2, e_3 defined as :-

Let α be a scalar.

$$e_1(A)_{ij} = \begin{cases} \alpha a_{sj}, & i = s \\ a_{ij}, & i \neq s \end{cases}$$

$$e_2(A)_{ij} = \begin{cases} a_{sj} + \alpha a_{tj}, & i = s \\ a_{ij}, & i \neq s \end{cases}$$

$$e_3(A)_{ij} = \begin{cases} a_{sj}, & i = r \\ a_{rj}, & i = s \\ a_{ij}, & i \neq s, r \end{cases}$$

So if prove the equation for all these elem. operation then we can generalize statement.

for e_1: $$E = e(I)_{m \times m} = \begin{cases} \alpha I_{sj}, & i = s \\ I_{ij}, & i \neq s \end{cases}$$

$$I_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$E_{ij} = \begin{cases} \alpha, & i = j = s \\ 1, & i = j \neq s \\ 0, & i \neq j \end{cases}$$

$$e(A)_{ij} = \begin{cases} \alpha a_{sj}, & i = s \\ a_{ij}, & i \neq s \end{cases}$$

$$(EA)_{ij} = \sum_{r=1}^{m} E_{ir} A_{rj}$$

$$= \quad I_{ij} = \delta_{ij}$$

Chronical Delta Met

$$E_{ik} = \begin{cases} \alpha\, \delta_{sk} & i = s \\ \delta_{ik} & , i \neq s \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{n} E_{ik} a_{kj}$$

$$= \begin{cases} \sum_{k=1}^{m} (\alpha \delta_{sk}) a_{kj} & , i = s \\ \sum_{k=1}^{m} \delta_{ik} a_{kj} & , i \neq s \end{cases}$$

$$= \begin{cases} \alpha \sum_{k=1}^{m} \delta_{sk} a_{kj} & , i = s \\ \sum_{k=1}^{m} \delta_{ik} a_{kj} = \delta_{ii} a_{ij} & , i \neq s \end{cases}$$

$$(EA)_{ij} = \begin{cases} \alpha\, a_{sj} & , i = s \\ a_{ij} & , i \neq s \end{cases}$$

$$\boxed{(EA)_{ij} = e(A)_{ij}}$$

~~Hence Proved~~

for e_2:

$$E_{ik} = e(I) = \begin{cases} \delta_{sk} + \alpha\, \delta_{tk} & , i = s \\ \delta_{ik} & , i \neq s \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj}$$

$$= \begin{cases} \sum_{k=1}^{m} (\delta_{sk} + \alpha\, \delta_{tk}) a_{kj} & , i = s \\ \sum_{k=1}^{m} \delta_{ik} a_{kj} & ; i \neq s \end{cases}$$

$$= \begin{cases} \sum_{k=1}^{m} \delta_{sk} a_{kj} + \alpha \delta_{tk} a_{kj} & , i = s \\ \delta_{ii} a_{ik} & , i \neq s \end{cases}$$

$$= \begin{cases} \cancel{a_{kj} + \alpha} \delta_{ss} a_{sj} + \alpha \delta_{tt} a_{tj} & , i = s \\ a_{ik} & , i \neq s \end{cases}$$

$$= \begin{cases} a_{sj} + \alpha a_{tj} & , i = s \\ a_{ik} & , i \neq s \end{cases}$$

$= \cancel{e(A)}$

$= (EA) = eA$

<u>for e_3</u>

$$E_{ik} = \begin{cases} \delta_{rk} & i = s \\ \delta_{sk} & i = r \\ \delta_{ik} & i \neq s, r \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} a_{kj}$$

$$= \begin{cases} \sum_{k=1}^{m} \delta_{rk} a_{kj} & i = s \\ \sum_{k=1}^{m} \delta_{sk} a_{kj} & i = r \\ \sum_{k=1}^{m} \delta_{ik} a_{kj} & i \neq s, r \end{cases}$$

$$= \begin{cases} a_{rj} & , i = s \\ a_{sj} & , i = r \\ a_{ij} & , i \neq s, r \end{cases}$$

$\therefore \boxed{EA = e(A)}$

Hence Proved

Ans 3

Given :- If A is $n \times n$ then
1) A is Invertible
2) A is row-equivalent to the $I_{n\times n}$ identity matrix.
3) A is product of elementary matrices.

RTP :- Statements, (1); (2); (3) are equivalent.

Proof :- Consider R to be ~~an~~ row reduced echelon matrix which is row equivalent to A. (Proof - Ass 2 q5)

Means, it must be obtained by performing a finite number (say K) of elementary row transformations on A i.e,

$$R = e_K(e_{K-1}(\cdots\cdots(e_2(e_1(A))\cdots))$$

Also we Know that $e(A) = EA$
(Proof - Ass: 3 q2)

$$\therefore R = E_K E_{K-1} \cdots E_2 E_1 A \quad \text{—①}$$

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Claim: All elementary ~~row~~ matrice $E_1, --$ are invertible

Proof: $E = e(~~A~~ I)$

e^{-1} is inverse of e & exist ($\because$ Ass-2 Q2)

Let $E^{-1} = e^{-1}(I)$

$EE^{-1} = ee^{-1}(I)$

$EE^{-1} = I$

$\therefore \Rightarrow E^{-1}$ is inverse of E.

So we can modify R as;

$A = E_1^{-1} E_2^{-1} --- E_k^{-1} R$ — (2)

$A = E^{-1} R$ — (3)

This is evident that A is invertible iff R is invertible as the matrix product $E^{-1}R$ cannot be invertible without R being invertible.

(1) ⇒ (2): Given that A is invertible. So, from eq (3) R must be invertible as well.

However, R is the Row-reduced echelon matrix that is row-equivalent to A.

→ For R to be invertible, it must not have any zero rows. ($\because$ R is a square matrix)

So each row of the row-reduced echelon matrix R must have a non-zero element i.e $R=I$ is the only possibility) — (a)

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Addendum :- The $n \times n$ identity matrix I. regardless of however many elementary row transformation you may perform can NEVER have a zero row.

Reason: $I = \begin{bmatrix} 1 & 0 & - & - & 0 \\ 0 & 1 & - & - & 0 \\ | & | & & & | \\ 0 & 0 & - & - & 1 \end{bmatrix}$ and we have 3 elementary row ~~to~~ operation to choose from

(i) Multiply a Row with non-zero scalar α.
Can not get zero-row with this.

(ii) Add a row to a scalar contant 'c' times another row
If you do this and try to subtract that row from another to try and make a column zero you will observe that while one column in target row becomes zero the another becomes non-zero.

~~(iii) Swap Rows~~

Eg: $I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) Swap two rows
You can't get a zero-row with this since

there is no zero to swap with in first place.

$\therefore$ from statement ⓐ, If A is invertible than A is row-equivalent to the $n \times n$ identity matrix (since A is row equivalent to R & $R = I$)

③ ⟶ ①:

If A is the product of elementary matrices we have two facts to consider,

ⓐ Elementary Matrices are invertible (Prooved in Claim-1) ~~(Ass 3 Q3)~~

ⓑ Product of Invertible matrices is invertible (Thm)

So, from above we can say A is invertible.

$$\left.\begin{array}{l} A = E_1 \cdots E_k \\ A^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} \end{array}\right\}$$

② ⇒ ③: If A is row equivalent to the $n \times n$ identity matrix $\Rightarrow R = I$ where R is the row reduced echelon matrix.

from eq ②, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ i.e. A is product of elementary matrices

Now ① ⟹ ② ; ② ⟹ ③ ; ③ ⟹ ①

⇒ ① ⟹ ③

Also ① ~ ③

$\therefore$ ① ~ ③ ~ ②

Hence Prooved

Ans4.)

Method-1

Given:- A be a $n \times m$ matrix and B be a $n \times 1$ vector with real entities, ~~suppose the~~ and $AX = B$ admits a unique Solution.

RTP:- Find which is correct condition on n & m :

(i) $m >= n$

(ii) $n >= m$

(iii) $m = n$

(iv) $n > m$

Proof:- if $n < m$:

Proof by counter example

$$x + y + z = 1 \quad —①$$
$$x + y + z = 2 \quad —②$$

Here $n = 2$ and $m = 3$

But Eq ① and ② have no Solution which is trivial bcz if $x + y + z = S$ (let say) then S can not be 1 and 2 both simult.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ b_n \end{bmatrix}$$

Let A' be its augmented matrix

$$A' = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & & b_2 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Now convert A into R (R = Row Reduced echleon matrix) then A' be like;

$$A' = \left[\begin{array}{cccccc|c} 0 & 1 & - & - & - & & y_1 \\ 0 & 0 & - & - & 1 & - - & y_2 \\ 0 & - & - & - & 1 & - & \vdots \\ 0 & - & - & - & - & . & y_n \end{array}\right]$$

Let r be no of non zero rows in A'.
Clearly $r \le n$.

So now $n-r$ be no of zero rows.

$y_i = 0$ for $n \ge i > n-r$

Now for each row;

$$x_{ki} + \sum C x_j = y_i$$

Now for unique Solution

$$\sum C x_j = 0$$

$$x_{ki} = y_i$$

Now $i \leq r$

we have ~~r sotot~~ solution to r variables.

Now if $r \leq m$

then we have condition or values of r variables.

$m-r$ variable don't have any Condition

$\Rightarrow$ Equation has infinite Solution which contradicts ~~our~~ fact that ~~A~~ x Equation has a unique solution which contradicts $r < m$

$\therefore r \geq m$

Now $r \leq n$

$\therefore m \leq r \leq n$

$\therefore m \leq n$

or $n \geq m$ (ii) ✓

Hence Prooved