ASSIGNMENT - 2

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A1) For a matrix to be diagonalizable any of below condⁿ is sufficient :
ⓐ n linearly independent eigen vectors
ⓑ n distinct eigen values
ⓒ Sum of geom mult. is n.
ⓓ for each λ, gm = am

(a) $A = \begin{pmatrix} 1 & K & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let's first find eigen values,

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} 1-\lambda & K & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$

$(2-\lambda) \cdot \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0$

$(2-\lambda)(1-\lambda)(1-\lambda) = 0$

$\lambda = 1$ w alg. mult = 2
2 w " " = 1

for $\lambda = 1$,

$$A \cdot X = X$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 0 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$K x_2 = 0 \quad \& \quad x_2 = 0$$

$$\hookrightarrow x_2 = 0$$

$$\therefore X = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

for $\lambda = 2$

$$(A - 2I)X = 0$$

$$\begin{pmatrix} -1 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + Kx_2 = 0$$

$$-x_3 = 0$$

$$\hookrightarrow x_3 = 0, \quad x_1 = K x_2$$

$$X = \begin{bmatrix} Kx_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix}$$

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$$E_2 = \text{Span}\left(\begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix}\right)$$

Now all Eigen vector must be lin independet

$$\underbrace{V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{E_1}, \; \underbrace{V_3 = \begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix}}_{E_2}$$

$C_1 V_1 + C_2 V_2 + C_3 V_3 = 0 \Rightarrow C_1 = C_2 = C_3 = 0$

To Proove

$$\Rightarrow C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} C_1 + C_3 K \\ C_3 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow C_2 = 0 \;\&\; C_3 = 0$$

$$\Rightarrow C_1 = 0$$

$\Downarrow$

Hence linearly Indep.

$\therefore$ So $\forall K \in R$ A is diagonalizable.

(b) $B = \begin{pmatrix} 1 & 1 & K \\ 1 & 1 & K \\ 1 & 1 & K \end{pmatrix}$

$$\det(B - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 & K \\ 1 & 1-\lambda & K \\ 1 & 1 & K-\lambda \end{pmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & K \\ 1 & 1-\lambda & K \\ 1 & 1 & K-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\big((1-\lambda)(K-\lambda) - K\big) - 1\big((K-\lambda) - K\big)$$
$$+ K\big(1 - (1-\lambda)\big) = 0$$

$$(1-\lambda)(\lambda^2 - K\lambda - \lambda) + \lambda + K\lambda = 0$$
$$\lambda^2 - \cancel{K\lambda} - \cancel{\lambda} - \lambda^3 + K\lambda^2 + \lambda^2 + \cancel{\lambda} + \cancel{K\lambda} = 0$$

$$-\lambda^3 + (K+2)\lambda^2 = 0$$
$$\lambda^2(K+2-\lambda) = 0$$
$$\lambda = 0 \text{ w/ alg. mult} = 2$$
$$K+2 \text{ w/ } '' \quad '' = 1$$

<u>for $\lambda = 0$</u>

$$B \cdot X = 0$$

$$\begin{pmatrix} 1 & 1 & K \\ 1 & 1 & K \\ 1 & 1 & K \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\to$ $[B \mid 0]$

$$R_2 \leftarrow R_2 - R_1$$
$$R_3 \leftarrow R_3 - R_1$$

$$\begin{pmatrix} 1 & 1 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + Kx_3 = 0$$

$$X = \begin{bmatrix} -x_2 - Kx_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -K \\ 0 \\ 1 \end{bmatrix}$$

$$E_0 = \text{Span}\left(\underset{\uparrow \atop V_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} , \underset{\uparrow \atop V_2}{\begin{bmatrix} -K \\ 0 \\ 1 \end{bmatrix}} \right)$$

for $\lambda = K+2$

$$(B - (K+2)I)X = 0$$

$$\begin{pmatrix} -K-1 & 1 & K \\ 1 & -K-1 & K \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving,

$$x_2 + Kx_3 = (K+1)x_1$$

$$x_1 + Kx_3 = (K+1)x_2$$

$$x_1 + x_2 = 2x_3$$

$$\to x_2 = 2x_3 - x_1$$

$$\to x_1 + Kx_3 = (K+1)(2x_3 - x_1)$$

$$x_1 + Kx_3 = (2K+2)x_3 - Kx_1 - x_1$$

$$(K+2)x_1 = (K+2)x_3$$

$$x_3 = x_1$$

$$\to x_2 + Kx_1 = (K+1)x_1$$

$$x_2 = x_1$$

$$\therefore x_1 = x_2 = x_3$$

$$E_{2+K} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} \Rightarrow E_{2+K} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

v_3

By defn; v_1, v_2, v_3 should be lin. independ.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ —— Eqn 1

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -K \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -c_1 - Kc_2 + c_3 \\ c_1 + c_3 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad c_1 = -c_3 \;\; \& \;\; c_2 = -c_3$

$-c_1 - Kc_2 + c_3 = 0$

$c_3 + Kc_3 + c_3 = 0$

$c_3 (K+2) = 0$

Now Eqn 1 is only true for $c_1 = c_2 = c_3 = 0$

$\therefore \; K+2 \neq 0$

$K \neq -2$

So $K \in \mathbb{R} - \{-2\}$ for B to be diagonalizable

A2)

(a) Given :- A is Invertible (Assume A^{-1} is its Inv.) & A is diagonalizable

RTP :- A^{-1} is diagonalizable.

Proof :- for diagonalizable,

$A = CDC^{-1}$ where D is diagonal matrix.

Also if D is diagonal matrix then so is D^n $\forall$ $n \in \mathbb{Z}$. (~~Also prooved in Q4~~)

$A \cdot A^{-1} = I$

$(CDC^{-1})A^{-1} = I$

Pre multiply C^{-1}

$(\underbrace{C^{-1} \cdot C}_{I} DC^{-1})A^{-1} = C^{-1}$

$(DC^{-1})A^{-1} = C^{-1}$

Pre multipy by D^{-1} and then C

$A^{-1} = CD^{-1}C^{-1}$

Clearly D^{-1} is diagonalizable

$\therefore$ A^{-1} is diagonalizable.

(b) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

We have to Calculate A^{2015}.

Method-1 :-

$$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-1 & 1-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

i.e, $A^2 = I$

$\therefore A^{2015} = A^{2 \times 1007 + 1}$

$= (A^2)^{1007} \cdot A$

$= I \cdot A = A$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ Ans

~~Method 2:~~

A3)

(a) $a = (1-i, 1+2i)$
$b = (2+i, 2)$ in C^2

Assuming $\langle a, b \rangle = a_1 b_1 + a_2 b_2$

$\{a, b\}$ is orthogonal set of vectors

$\therefore \langle a, b \rangle = 0$

$(1-i)(2+i) + (1+2i)(z) = 0$

$z = \frac{(2+i)(i-1)}{(2i+1)}$

$z = \frac{2i - 2 + i^2 - i}{(2i+1)} = \frac{i-3}{(2i+1)}$

$z = \frac{(i-3)(2i-1)}{(2i+1)(2i-1)} = \frac{2i^2 - i - 6i + 3}{4i^2 - 1}$

$$\boxed{z = \frac{1-7i}{-5} = \frac{7i-1}{5}}$$

finding orthonormal set of vectors;
Let be $\{v_1, v_2\}$

$v_1 = \frac{a}{\|a\|} = \frac{(1-i, 1+2i)}{\sqrt{(1-i)^2 + (1+2i)^2}}$

$= \frac{(1-i, 1+2i)}{\sqrt{\cancel{1} + \cancel{i^2} - 2i + 1 + 4i^2 + 4i}} = \frac{(1-i, 1+2i)}{\sqrt{2i-3}}$

$v_1 = \left(\frac{1-i}{\sqrt{2i-3}}, \frac{1+2i}{\sqrt{2i-3}} \right)$

$v_2 = \frac{b}{\|b\|} = \frac{(2+i, \ 7/5 i - 1/5)}{\sqrt{(2+i)^2 + \left(\frac{7i-1}{5}\right)^2}}$

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$$= \frac{(2+i,\ 7i-1/5)}{\sqrt{4+i^2+4i + \frac{49i^2+1-14i}{25}}}$$

$$= \frac{5(2+i,\ (7i-1)/5)}{\sqrt{100i+75+ \cancel{4} -48 -14i}}$$

$$= \frac{(5i+10,\ 7i-1)}{\sqrt{86i+27}}$$

$$v_2 = \left(\frac{5i+10}{\sqrt{86i+27}},\ \frac{7i-1}{\sqrt{86i+27}}\right)$$

$\therefore \{v_1, v_2\}$ is the required orthonormal set

Method -2

Considering inner product as complex dot product.

$$\langle u, v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n$$

$\{v_1, v_2\}$ be orthonormal Set

$$v_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \frac{(1-i,\ 1+2i)}{\sqrt{(1^2+1^2)+(1^2+4)}} = \frac{(1-i,\ 1+2i)}{\sqrt{7}}$$

$$= \left(\frac{1-i}{\sqrt{7}},\ \frac{1+2i}{\sqrt{7}}\right)$$

$$v_2 = \frac{b}{\|b\|} = \frac{(2+i,\ \frac{1}{5}(7i-1))}{\sqrt{(4+1)+\frac{1}{25}(\cancel{49+1})2}} = \left(\frac{2+i}{\sqrt{7}},\ \frac{7i-1}{5\sqrt{7}}\right)$$

Ay

(b) $V_1 = \text{Span}(a_1, a_2, a_3) = \text{Span}(v_1, v_2, v_3)$

S_1

$= \text{Span}(a_3, a_2, a_1) = \text{Span}(u_1, u_2, u_3)$

⇒ ~~One theoretical proof can be for same vector Subspace orthogonal basis would also be same as a just ordering was changed.~~

⇒ Let see :-

$v_1' = \dfrac{\vec{a_1}}{}$

$\cancel{v_2} = a_2 - (a_2 \cdot v_1)v_1$

$= a_2 - \dfrac{(a_2 \cdot a_1)\, a_1}{(\|a_1\|)^2} \langle v_1, v_2 \rangle$

$v_3 = a_3 - (a_3 \cdot v_1)v_1 - (a_3 \cdot v_2)\, v_2$

$= a_3 - \dfrac{(a_3 \cdot a_1)\, a_1}{(\|a_1\|)^2} - \dfrac{(a_3 \cdot a_2)\, a_2}{(\|a_2\|)^2}$

$u_1 = \dfrac{a_3}{\|a_3\|}$

$v_2 = a_2 - \dfrac{(a_2 \cdot v_1)\, v_1}{v_1 \cdot v_1} = a_2 - \dfrac{(a_2 \cdot a_1)\, a_1}{a_1 \cdot a_1}$

$v_3 = a_3 - \dfrac{(a_3 \cdot a_1)\, a_1}{a_1 \cdot a_1} - \dfrac{(a_3 \cdot a_2)\, a_2}{a_2 \cdot a_2}$

$$v_1 = a_3$$

$$v_2 = a_2 - \frac{(a_2 \cdot a_3) a_3}{a_3 \cdot a_3}$$

$$v_3 = a_1 - \frac{(a_1 \cdot a_3) a_3}{a_3 \cdot a_3} - \frac{(a_1 \cdot a_2) a_2}{a_2 \cdot a_2}$$

Cleary Both are not the same set of Vectore

Hence Disprooved

A4)

(a) RTP:- If M is diagonalizable then so is M^2

Proof:-

$M \sim D$ (D be diagonal Matrix)

$M = PDP^{-1}$ (P be invertible)

Pre multiply with M;

$M^2 = M \cdot (PDP^{-1})$.

$M^2 = (PDP^{-1})(PDP^{-1})$

$M^2 = PD^2P^{-1}$

D^2 is also diagonal

$M^2 \sim D^2$

M^2 is also diagonalizable

Hence Prooved

(b) RTP :- If M^2 is diagonalizable, then so is M.

Let disproove this with the help of a counterxamle.

$$M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}$$

Clearly M^2 is diagonalizable as it is a diagonal matrix itself.

Let M be $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Find eigenvalues of M;

$$\det(M - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$\lambda = 0$ w/ alg. multiplicity 2.

$\lambda = 0$

$$(A - 0I)X = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = 0$$

$$X = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

But M eigen space should have 2 linearly independent vector.

$\therefore$ M is not diagonalizable

Hence disprooved