Assignment - 4

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A1 Given: $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$

$c\,\bar{\alpha} = -c\bar{\alpha}$

RTF :- Find which axioms for the vector space are satisfied by $(R^n, \oplus, \cdot)$

Proof :-

(1) For $\oplus$;

Commutative :- $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$

$\bar{\beta} \oplus \bar{\alpha} = \bar{\beta} - \bar{\alpha}$

Not Commutative

(2) Associative :-

$\bar{\alpha} \oplus (\bar{\beta} \oplus \gamma) = \bar{\alpha} - (\bar{\beta} \oplus \bar{\gamma})$

$= \bar{\alpha} - (\bar{\beta} - \bar{\gamma})$

$= \bar{\alpha} - \bar{\beta} + \bar{\gamma}$

$(\alpha \oplus \beta) \oplus \gamma = (\bar{\alpha} - \bar{\beta}) \oplus \bar{\gamma}$

$= (\bar{\alpha} - \bar{\beta}) - \bar{\gamma} = \bar{\alpha} - \bar{\beta} - \bar{\gamma}$

Not associative

(3) Let e be

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$\oplus$

1) Commutative :-

LHS $= (0,0,\ldots n\text{tt}) \oplus (1,1,\ldots n \text{ times})$

$= (-1,-1,\ldots n \text{ times})$

RHS $= (1,1,\ldots 1) \oplus (0,\ldots 0)$

$= (1,1,\ldots 1)$

LHS $\neq$ RHS

Not Commutative

2) $\oplus$ is not associative since

$((1,\ldots 1) \oplus (1,\ldots 1)) \oplus (2,\ldots 2)$

$= (0,\ldots 0) \oplus (2,\ldots 2) = (-2,\ldots -2)$

$(1,\ldots 1) \oplus ((1,\ldots 1) \oplus (2,\ldots 2))$

$= (1,\ldots 1) \oplus (-1,\ldots -1) = (2,\ldots 2)$

3) There does ~~not~~ exist right additive identity i.e vector 0 that satisfied $\alpha + 0 = \alpha \ \forall \alpha$.

$\beta = (0,\ldots 0)$ satisfies $\alpha + \beta = \alpha \ \forall \alpha$. ~~$\beta$~~

$\beta' = (b_1,\ldots b_n)$ also satisfy

$(x_1,\ldots x_n) + \beta' = (x_1,\ldots x_n)$

$x_i - b_i = x_i \ \forall i \therefore b_i = 0 \ \forall i$

$\beta = (0,\ldots 0)$ is unique

$\hookrightarrow$ No left additive identity.

4) There do exist right additive. For vector α clearly only α itself satisfy $\bar{\alpha} \oplus \bar{\alpha} = \bar{0}$

5) The element 1 doesn't satisfy $1 \cdot \alpha = \alpha$ for any non zero α since;

$1 \cdot (x_1,\ldots x_n) = (-x_1,\ldots -x_n) = (x_1,\ldots x_n)$

only if $x_i = 0 \ \forall i$

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6.) $(c_1 c_2)\cdot\alpha = c_1\cdot(c_2\cdot\alpha)$ doesn't hold since
$(c_1 c_2)\cdot\alpha = (-c_1 c_2)\alpha$
& $(c_1(c_2\alpha)) = c_1(-c_2\alpha) = (-c_1)(-c_2\alpha)$
$= +(c_1 c_2)\alpha$

7.) It holds: $c\cdot(\alpha \oplus \beta) = c\cdot\alpha \oplus c\cdot\beta$
$c\cdot(\alpha \oplus \beta) = c\cdot(\alpha - \beta) = -c(\alpha - \beta)$
$= -c\alpha + c\beta$

$c\cdot\alpha \oplus c\cdot\beta = (-c\alpha) \oplus (-c\beta) = (-c\alpha) - (-c\beta)$
$= -c\alpha + c\beta$

8) It doesn't hold $(c_1 + c_2)\cdot\alpha = (c_1\cdot\alpha) \oplus (c_2\cdot\alpha)$
$(c_1 + c_2)\cdot\alpha = -(c_1 + c_2)\alpha = -c_1\alpha - c_2\alpha$
$c_1\cdot\alpha \oplus c_2\cdot\alpha = (-(c_1\cdot\alpha)) \oplus (-c_2\cdot\alpha)$
$= -c_1\alpha + c_2\alpha$

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A2) Given:- a) V is set of all complex valued functions f.
b) $\forall\, t \in \mathbb{R}$, $f(-t) = f^*(t)$ or $\overline{f(t)}$ ← conjugate

RTP:- (a) V with operation $(f+g)(t) = f(t) + g(t)$ and $(cf)(t) = cf(t)$ is a vector space over the field R.

(b) Give an example of a function fn in V which is not real-valued.

Proof:-

(a) ① Before we show that V satisfies the eight properties of vector space, we must show that vector addition & scalar multiplication

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are truly well defined i.e they are indeed operations on V.

* Observe, $(f+g)(t) = f(t) + g(t)$ &
$(cf)(t) = cf(t)$

we know; $\overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta}$ & $\overline{c\alpha} = c\cdot\bar{\alpha}$ $(c \in \mathbb{R})$
and $(\overline{\alpha\cdot\beta}) = \bar{\alpha}\cdot\bar{\beta}$ (bar is conjugation)

$(f+g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)}$
$\Rightarrow (f+g)(-t) = \overline{f(t)} + \overline{g(t)}$

&&

$(cf)(-t) = cf(-t) = c\,\overline{f(t)} = \overline{c\,f(t)}$
$\Rightarrow (cf)(-t) = \overline{cf(t)}$

So the operations are indeed, well-defined

(2) <u>Commutative of a vector addition :-</u>
Here, f is a complex function on real field $\mathbb{R}$. So if $t \in \mathbb{R}$ then $f(t) \in \mathbb{C}$.
Given addition on functions of V would be a addition of number in $\mathbb{C}$.
So $\mathbb{C}$ is Commutative $\Rightarrow$ V is Commutative

(3) <u>Associative of Vector Addition:</u>
Just like (2) $\mathbb{C}$ is associative $\Rightarrow$ V is associat

(4) <u>Additive Identity :-</u>
* The zero function $f(t) = 0$ is in V as $\bar{0} = -0$ holds
* Now if $f(t) = 0$ then

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$(f+g)t = f(t) + g(t) = 0 + g(t) = g(t)$

&& $(g+f)(t) = g(t) + f(t) = g(t)$

So if $f(t) = 0$ then $(f+g)(t) = (g+f)(t) = g(t)$

$\therefore$ V has an additive Identity.

⑤ Additive Inverse :-

Let $g(t) = -f(t)$, then $g(-t) = -f(-t) =$
$-\overline{f(t)} = \overline{g(t)} \Rightarrow g(t) = \overline{f(t)}$

Thus for $g \in V$;
$(f+g)(t) = f(t) + g(t) = f(t) - f(t) = 0$
$(g+f)(t) = g(t) + f(t) = -f(t) + f(t) = 0$
$\therefore$ V has an additive Inverse

⑥ Multiplicative Inverse :-
Clearly, $1 \cdot f = f$ holds, since 1 is the multiplicative identity in $\mathbb{R}$ & $\mathbb{C}$ as well.

⑦ Associative of Scalar Multiplication :-
Since $\mathbb{C}$ is associative & $f \in \mathbb{C}$ thus $(c_1 c_2)f = c_1(c_2 f)$ holds.

⑧ Distributivity of Scalars over vectors :-
As $\mathbb{C}$ is distributive $\Rightarrow c(f+g) = cf + cg$ holds

⑨ Distributivity of vector over scalars :-
As $\mathbb{C}$ is distributive $\Rightarrow (c_1+c_2)f = c_1 f + c_2 f$ holds

We can say; $\therefore$ V with the given operations is a vector field.

(b) a function in V which isn't real valued is $f(t) = it$.

Claim :- $it \in V$

Proof :- $f(-t) = (-i)t = t(-i) = t(\bar{i})$
$= \overline{it} = \overline{f(t)}$
$\Rightarrow f(-t) = \overline{f(t)} \Rightarrow f \in V$

Claim : it is not real valued

Consider; $f(1)$; $f(1) = i(1) = i$
$\Rightarrow f$ is not real valued.

Ans 3) RTP :- A non-empty subset W of vector space V is a subspace of V if and only if for each pair of vectors $\bar{\alpha}, \bar{\beta} \in W$ and each scalar $c \in F$, the vector $c\bar{\alpha} + \bar{\beta} \in W$.

Proof :-

$\Rightarrow$: If W is a subspace then, $c\bar{\alpha} + \bar{\beta} \in W$ $\forall\ \alpha, \beta \in W$ & $c \in F$

for any $\alpha, \beta \in W$ and $c \in F$.

As W is subspace;

$\bar{\alpha} \in W$

$c\bar{\alpha} \in W$

$\bar{\beta} \in W$

$\therefore \Rightarrow c\bar{\alpha} + \bar{\beta} \in W$

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$\Leftarrow$: If $c\bar{\alpha}+\bar{\beta} \in W$ $\forall \alpha, \beta \in W$ & $c \in F$ then W is a Subspace.

(Reference: LA by Hoffan and Kunze Page - 42)

To show W is a subspace we need to show ~~three~~ 4 conditions:- (1) Zero Vector

2) Closed under addition

(3) Additive Inverse

(4) Closed under Addition.

if $\alpha, \beta \in W$

$\Rightarrow c\bar{\alpha} + \bar{\beta} \in W$

$c = 1$

$\bar{\alpha} + \bar{\beta} \in W$ Closed under addition 2✓

if $\bar{\alpha} \in W$ & $c = -1$

$(-1)\bar{\alpha} + \bar{\alpha} \in W$

$\vec{0} \in W$ Zero Vector 1✓

$\bar{\alpha} \in W$ & $\bar{0} \in W$

$\Rightarrow c\bar{\alpha} + \bar{0} \in W \Rightarrow c\bar{\alpha} \in W$ Closed under Mult 4✓

$c = -1 \Rightarrow -\bar{\alpha} \in W$ Additive Inv 3✓

$\therefore$ W is subspace of V

Hence Prooved