Marginals

- What is $p_{XY}(1, i)$? $(= \frac{1}{12})$.
- $\sum_i p_{XY}(1, i) = \mathbb{P}\{\omega \in \Omega : X(\omega) = 1\} = \frac{1}{2} = p_X(x)$.
- Similarly, $p_{XY}(1, i) + p_{XY}(0, i) = \frac{1}{6} = p_Y(i)$.

> The marginal PMF's p_X and p_Y can be obtained from the joint PMF as follows:
> $p_X(x) = \sum_y p_{XY}(x, y)$ and $p_Y(y) = \sum_x p_{XY}(x, y)$.

This is true in general, and requires a proof.

Marginals

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$p_X(x) = \sum_y p_{XY}(x,y)$ and $p_Y(y) = \sum_x p_{XY}(x,y)$.

Proof:

$$
\begin{aligned}
p_X(x) &= \mathbb{P}\{\omega \in \Omega : X(\omega) = x\} \\
&= \mathbb{P}\{\bigcup_y \{\omega \in \Omega : X(\omega) = x, Y(\omega) = y\}\} \\
&= \sum_y \mathbb{P}\{\{\omega \in \Omega : X(\omega) = x, Y(\omega) = y\}\}
\end{aligned}
$$

Independence

- Back with the running example of coin and dice.
- Write down $p_{XY}(x, y)$ and $F_{XY}(x, y)$.
- Notice that $p_{XY}(1, i) = p_X(1)p_Y(i)$ and $F_{XY}(1, i) = F_X(1)F_Y(i)$.
- In general, if $p_{XY}(x, y) = p_X(x)p_Y(y)$ and $F_{XY}(x, y) = F_X(x)F_Y(y)$ we say X and Y are independent.

> Two random variables, X and Y are independent if the following is true:
> $p_{XY}(x, y) = p_X(x)p_Y(y)$ and $F_{XY}(x, y) = F_X(x)F_Y(y)$

Independence

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- How does this relate to $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$?
- $A = \{\omega \in \Omega : X(\omega) \leq x\}$ and $B = \{\omega \in \Omega : Y(\omega) \leq y\}$.
- $F_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x \text{ and } Y(\omega) \leq y\} = \mathbb{P}(A \cap B)$.

$E[XY]$

- $E[X] = \sum_x x p_X(x)$ and $E[Y] = \sum_y y p_Y(y)$
- $E[X] = \sum_x \sum_y x p_{XY}(x, y)$ and $E[Y] = \sum_x \sum_y y p_{XY}(x, y)$
- How do we define $E[XY]$?
- You want to search over all values $X \times Y$ can take ($\{1, 2, .., 6\}$) and weight it by the corresponding probabilities.
- $E[XY] = \sum_x \sum_y xy p_{XY}(x, y) = 1.75 = E[X]E[Y]$.

If X and Y are independent, $E[XY] = E[X]E[Y]$.

Example where X and Y are Dependent

- Now consider rolling a dice.
- $X = \begin{cases} 1 \text{ if outcome is odd} \\ 0 \text{ otherwise .} \end{cases}$ and $Y = \begin{cases} 1 \text{ if outcome is even} \\ 0 \text{ otherwise .} \end{cases}$
- What is $p_X(x), p_Y(y), p_{XY}(x, y)$ and $F_{XY}(x, y)$?
- What is $E[XY]$?

Consistency conditions

- $\sum_{x,y} p_{XY}(x, y) = 1.$
- $F_{XY}(\infty, \infty) = 1.$
- $F_{XY}(-\infty, -\infty) = 0.$
- $F_{XY}(-\infty, \infty) = 0.$
- $F_{XY}(\infty, -\infty) = 0$
- $F_{XY}(x, \infty) = F_X(x)$ (marginal CDF)
- $F_{XY}(\infty, y) = F_Y(y)$ (marginal CDF)

Multiple continuous random variables

- Pick a number uniformly at random from a unit square centered at $(.5, .5)$.
- Random variables X and Y represent the respective x and y coordinate of the point chosen.
- $F_{X,Y}(x, y)$ denotes the probability that the point chosen lies below and to left of point (x, y).
- In this example, $F_{X,Y}(x, y) = xy$.
- Now visualize $F_{X,Y}(x + h, y) - F_{X,Y}(x, y)$. This is the probability that the point chosen lies in the thin strip below y and between x and $x + h$.

Multiple continuous random variables

- Visualize $F_{X,Y}(x+h, y) - F_{X,Y}(x, y)$. This is the probability that the point chosen lies in the thin strip below y and between x and $x+h$.
- $\frac{\partial F_{XY}(x,y)}{\partial x} = lim_{h \to 0} \frac{F_{X,Y}(x+h,y) - F_{X,Y}(x,y)}{h}$.
- This is the rate of change of the joint CDF $F_{XY}(x, y)$ in the x direction.

Multiple continuous random variables

- $\frac{\partial F_{XY}(x,y)}{\partial y} = \lim_{h \to 0} \frac{F_{X,Y}(x,y+h) - F_{X,Y}(x,y)}{h}$ denotes the rate of change of the joint CDF in the y direction.
- $f_{X,Y}(x,y) := \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$ represents the joint probability density function.
- $f_{X,Y}(x,y)dxdy$ denotes the probability that (X, Y) are in a rectangle of area $dxdy$ around (x, y).
- In this example, $f_{X,Y}(x,y) = 1$.
- $F_{XY}(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,t)dsdt$.

Summary for Continuous random variable

- $f_{XY}(x,y)$ denotes the joint pdf for X and Y.
- $F_{XY}(x,y) := \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,t)dsdt$. $f_{X,Y}(x,y) := \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$.

> The marginal pdf's f_X and f_Y can be obtained from the joint PDF as follows:
> $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$

> Two random variables, X and Y are independent if the following is true:
> $f_{XY}(x,y) = f_X(x)f_Y(y), F_{XY}(x,y) = F_X(x)F_Y(y)$ and $E[XY] = E[X]E[Y]$.

- Rules similar for more than 2 random variables.