RECAP

- We looked at n-length random vectors $\mathbf{X}$ which are essentially multivariate random variables.
- Its CDF/pdf is simply joint CDF/pdf of components
- $E[\mathbf{X}] = [E[X_1], \ldots E[X_n]]$. $C_{\mathbf{X}}$ is the covariance matrix.
- Let $\mathbf{Y} = G(\mathbf{X})$ where $G : \mathbb{R}^n \to \mathbb{R}^n$, continuous invertible with continuous partial derivatives. Let H denote its inverse. Then $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(H(\mathbf{y}))|J|$ where J is the determinant of the Jacobian matrix.
- If $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, then $C_{\mathbf{Y}} = AC_{\mathbf{X}}A^T$ and

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|det(A)|} f_{\mathbf{X}}(A^{-1}(\mathbf{y} - \mathbf{b}))$$

Standard Normal Vector

- An n length random vector $\mathbf{Z}$ is called as a standard normal vector if its components Z_i are independent and standard normal.
- What is $E[\mathbf{Z}]$ and $C_{\mathbf{Z}}$?
- Show that the pdf is given by

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} e^{\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\}}$$

Standard Normal Vector

- Now suppose $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$. What is $E[\mathbf{X}]$ and $C_{\mathbf{X}}$?
- $E[\mathbf{X}] = \boldsymbol{\mu}$ and $C_{\mathbf{X}} = AA^T$.
- Note that A can have dimension $n \times l$ in which case $\mathbf{Z}$ is an l length standard normal.
- What is $f_{\mathbf{X}}(\mathbf{x})$?

Multivariate Gaussian Vector

- What is $f_{\mathbf{X}}(\mathbf{x})$?

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{|det(A)|} f_{\mathbf{Z}}(A^{-1}(\mathbf{x}-\boldsymbol{\mu})) \\
&= \frac{1}{(2\pi)^{n/2}\sqrt{det(C_{\mathbf{X}})}} e^{\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T C_X^{-1}(\mathbf{x}-\boldsymbol{\mu})\}}
\end{aligned}
$$

- Henceforth we will use the notation Σ to represent $C_{\mathbf{X}}$.

Definition 1: $\mathbf{X}$ is multivariate Gaussian with mean vector μ and covariance matrix Σ (denoted by $\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$) if for some A and $\boldsymbol{\mu}$, it can be written as $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$ where $\mathbf{Z}$ is a standard normal vector and $\Sigma = AA^T$.

Equivalent definitions of a Gaussian vector

The following are equivalent definitions (without proof)

$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$ if for some A and $\boldsymbol{\mu}$, it can be written as $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$

$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$ if it has the pdf

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{det(\Sigma)}} e^{\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\}}$$

$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ iff for all vectors $\mathbf{a} \in \mathbb{R}^n$, it turns out that $\mathbf{a}^\mathbf{T}\mathbf{X}$ is univariate Gaussian $\mathcal{N}(a^T\boldsymbol{\mu}, a^T\Sigma a)$.

For equivalent definitions see `https://en.wikipedia.org/wiki/Multivariate_normal_distribution`

Affine transformations preserve Gaussianity

- Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, then what can we say about $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$? Is it a Gaussian vector ?
- Easy to see that $E[\mathbf{Y}] = AE[\mathbf{X}] + b$ and $C_{\mathbf{Y}} = A\Sigma A^T$.
- Like in the univariate case, we can use MGF (for multivariate MGF see Bertsekas) to show the following

> Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Now consider $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$. Then we have $\mathbf{Y} \sim \mathcal{N}(AE[\mathbf{X}] + b, A\Sigma A^T)$.