/ /

ASSIGNMENT

Name:- Yarun Gupta
Roll No:- 2023101108
Group:- 4

Ans 1: (Source - LA by Kenneth Hoffman & Kunze)

Given:- W is a Subspace of F^n and $\dim W \leq m$ where $m, n \in \mathbb{Z}^+$

RTP:- There is precisely one $m \times n$ row reduced echelon matrix over F which has W as its row space.

Proof:- We will proove it in two parts:
(a) Existence of a rre matrix.
(b) Uniqueness of that matrix.

(a) Since W is a subspace of F^n with $\dim(W) \leq m$, there exist a basis for W consisting of at most m linearly independent vectors on F^n.
Let this basis be $\{v_1, v_2, \dots v_k\}$ $k \leq m$

Let a set $\{v_1, v_2, \dots v_k, u_1, u_2, \dots u_{m-k}\}$ Such that u_i ($\forall i$ s.t $1 \leq i \leq m-k$) behave like a basis for F^n as u_i is linearly independent as W is its subspace it also act as basis for W.

Let matrix A with u_1 to u_{m-k} as its rows has its row space as W because it spans W and M is also a row reduced echleon as its row are linearly independent & no zero rows can precede the non-zero rows. Hence there must exist a Matrix $m \times n$ with rows space as W.

Sadarvar

(Discussed with ~~[illegible]~~ Soham)

(b) Suppose there exist 2 diffrent $m \times n$ row-reduced echeleon matrix, Let R_1 & R_2 over F^n that has W as its row space.

~~Since, W is a row space for~~

Now R_1 & R_2 be two RREF's which can differ in three ways:-

① Have diffrent number of non-zero rows:

Clearly this ~~will~~ is not possible because this means the basis from $R_2 \neq k$ which means it won't span W as the dimension for a space is fixed and if a set is not of that dimension it cannot span W or would be dependent set.

② The set of first non zero entry is diffrent

Let's assume they have same no of non zero rows. Let P_1 be set of first non zero entry of RREF. R_1.

Let's assume $\ell \in P_1$ but $m \notin P_1$

$\therefore$ For matrix R_1 and R_2 situation would be like

$$
\begin{bmatrix} - & - & - & 0 \\ - & - & & \vdots \\ - & & & 1 \\ & & & 0 \\ & & & \vdots \end{bmatrix}
\qquad\qquad
\begin{bmatrix} 1 & - & - & - & a & - & - \\ - & - & 1 & - & b & - & - \\ - & - & - & & 0 & - & - \\ - & - & - & & 0 & - & - \end{bmatrix}
$$

ℓ^{th} Column $\qquad\qquad\qquad\qquad$ ℓ^{th} Column

R_1 $\qquad\qquad\qquad\qquad\qquad\qquad$ R_2

R_1's ℓ^{th} Column only one element ~~represent hence~~ is present Hence the ℓ^{th} Coordinate of vectors in W is totally independent of other coordinate while in R there are diff. values present in diff rows which will depend on their first non-zero entry (i.e if they are multiplied by c then they will also be multiplied by c).

Here it may be possible that the ℓ^{th} coordinate will not contain all possible value in F which is possible for row reduced echleon R and hence the subspace spanned by them would be diffrent.

(iii) values occuring after first non-zero element are different and rest all conditions hold.

⇒ let's assume that their u^{th} column allow entry other than 1 and rest 0. Also let us assume that it is allowed only in some rows only. (say v^{th} row)

$$\begin{array}{c c}
\begin{array}{ccccc} & w^{th} & & u^{th} & \end{array} & \\
\begin{bmatrix} \vdots & 0 & \vdots & c_1 & \vdots \\ \vdots & \vdots & \vdots & c_2 & \vdots \\ \vdots & \vdots & \vdots & c_3 & \vdots \\ \vdots & 1 & \vdots & a & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow v^{th} \text{ row} \\ \\ \end{array}
\end{array}$$

R_1

$$\begin{array}{c c}
\begin{array}{ccccc} & w^{th} & & u^{th} & \end{array} & \\
\begin{bmatrix} \vdots & 0 & \vdots & d_1 & \vdots \\ \vdots & \vdots & \vdots & d_2 & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots & b & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow v^{th} \text{ Row} \\ \\ \end{array}
\end{array}$$

R_2

Here $a \neq b$,

Let first non zero element in v^{th} row be present in w^{th} column $(w < u)$

Let's say $\alpha \in \omega$ and let ωth coordinate be p_ω; $\alpha = (p_1, p_2, \ldots$
$\ldots p_\omega, \ldots p_n)$

Now p_v from $R_1 = p_1 c_1 + p_2 c_2 + \ldots$
$\ldots p_v a + \ldots$, p_v from R_2

$= p_1 d_1 + p_2 d_2 + \ldots + p_v b + \ldots$

Now its not necessary that they would be equal.

Ex :

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2$$

Now if say $(3, 3, 6, 0) \in \omega$, it can be obtained from R_1 but not from R_2.

Therefore they are different

Hence two row reduced echelon matrix span same subspace then they must be equal.

_/_/_

A2

(a) Let $p(x) \in W$ and write

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

for some coeff $a_0, a_1, a \ldots a_4 \in \mathbb{R}$

So,
$$p(1) = a_0 + a_1 + a_2 + a_3 + a_4$$
$$p(-1) = a_0 - a_1 + a_2 - a_3 + a_4$$
$$p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4$$
$$p(-2) = a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4$$

Also $p(x) \in W$, so it satisfies
$$p(1) + p(-1) = 0 \quad \& \quad p(2) + p(-2) = 0$$

$$\Rightarrow 2a_0 + 2a_2 + 2a_4 = 0 \quad \& \quad 2a_0 + 8a_2 + 32a_4 = 0$$

$$\Rightarrow a_0 + a_2 + a_4 = 0 \quad \& \quad a_0 + 4a_2 + 16a_4 = 0$$

Reducing the augmented matrix of the system formed by elementary row operation as follows :-

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 4 & 16 & 0 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - R_1]{\cancel{R_2 \rightleftharpoons R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 15 & \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

$\Rightarrow \quad a_0 - 4a_4 = 0 \Rightarrow a_0 = 4a_4$

$\& \quad a_2 - 5a_4 = 0 \Rightarrow a_2 = 5a_4$

(a_4 is free variable)

Substituting them in $p(x)$ we get

$$p(x) = \underbrace{4a_4}_{a_0} + a_1 x - 5a_4 x^2 + a_3 x^3 + a_4 x^4$$

$$= a_1 x + a_3 x^3 + a_4 (4 - 5x^2 + x^4)$$

Let $\left.\begin{aligned} p_1(x) &= x \\ p_2(x) &= x^3 \\ p_3(x) &= 4 - 5x^2 + x^4 \end{aligned}\right\}$ vectors in W.

So any vector $p(x)$ in W is linear combination;

$$\Rightarrow p(x) = a_1 p_1(x) + a_3 p_2(x) + a_4 p_3(x)$$

Thus, $\{p_1(x), p_2(x), p_3(x)\}$ is a spanning set of W.

(b) Also, the vectors $p_1(x), p_2(x), p_3(x)$ are linearly independent.

Proof:- If $c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0$ then we have,

$$0 = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x)$$

$$= c_1 x + c_2 x^3 + c_3 (4 - 5x^2 + x^4)$$

$$= 4c_3 + c_1 x - 5c_3 x^2 + c_2 x^3 + c_3 x^4$$

__/__/__

$\Rightarrow$ Coefficients must be zero, so $c_1 = c_2 = c_3 = 0$

This prooves that $p_1(x), p_2(x), p_3(x)$ are linearly independent.

Therefore set $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent spanning set of subspace W, ~~here it~~ hence it is basis for W Dimension of subspace W is 3.

A3) Suppose that a set of vectors $S_1 = \{v_1, v_2, v_3\}$ is a spanning set of a subspace V in R^5. If v_4 is another vector in V, then is the set $S_2 = \{v_1, v_2, v_3, v_4\}$ still a spanning set for V? If so proove it Otherwise give a canter example.

A Given: $S_1 = \{v_1, v_2, v_3\}$ as a spanning set for subspace V in R^5.

Also v_4 is another vector in V.
Set $S_2 = \{v_1, v_2, v_3, v_4\}$
As v_4 is also in V then it can be written as a linear combination of vectors in set S because S_1 spans subspace V.

__/__/__

$$V_4 = c_1 V_1 + c_2 V_2 + c_3 V_3 \quad \text{---} (1)$$

$c_i \in$ Field.

Now consider a vector α, which is made from linear combination of vectors in S_2.

$$\alpha = p_1 V_1 + p_2 V_2 + p_3 V_3 + p_4 V_4$$

$p_i \in$ Field

$$\alpha = p_1 V_1 + p_2 V_2 + p_3 V_3 + p_4 (c_1 V_1 + c_2 V_2 + c_3 V_3)$$

[From (1)]

$$\alpha = (p_1 + c_1 p_4) V_1 + (p_2 + c_2 p_4) V_2 + (p_3 + c_3 p_4) V_3$$

$$\alpha = x_1 V_1 + x_2 V_2 + x_3 V_3 \quad \text{---} (2)$$

such that $x_i = p_i + c_i p_4 \in$ Field

Also the subspace V can be written as linear combination of vectors in set S_1

$$\beta = a_1 V_1 + a_2 V_2 + a_3 V_3 \quad \text{-} (3) \text{ s.t}$$

$a_i \in$ Field

As α in (2) is of the same form as β in (3)

$\Rightarrow$ The subspace V can also be expressed in terms of α vector and α is linear combination of vectors in set S_2.

_/_/_

$\Rightarrow$ Set S_2 spans V

Ans 4

Given :- V and W be subspaces of R^n such that $V \cap W = \{0\}$ & $\dim(V) + \dim(W) = n$

RTP :- (a) If $v + w = 0$, where $v \in V$ and $w \in W$ then show that $v = 0$ and $w = 0$

(b) If B_1 is a basis for the subspace V and B_2 is a basis for subspace W then Show that union $B_1 \cup B_2$ is a basis for R^n

(c) If x is in R^n then show that x can be written in the form $x = v + w$ where $v \in V$ & $w \in W$.

(d) Show that the representation obtained in Part (c) is unique.

Proof :-

(a) If $v + w = 0$ where $v \in V$ and $w \in W$ then $v = -w$. But since $V \cap W = \{0\}$ the only vector that is common in both V & W is $\vec{0}$

$\therefore$ $v = -w = \vec{0}$

$\Rightarrow$ $v = 0$ & $w = 0$

(b.) We know that if W_1 & W_2 are two subspace (finite dimensional) then

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

[Ass-5]

Put $W_1 = V$ & $W_2 = W$

$$\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V+W)$$

As $V \cap W = \{0\}$ so $\dim(V \cap W) = 0$

$$\dim(V) + \dim(W) = \dim(V+W)$$

We also know that $\dim(\mathbb{R}^n) = n$ because basis of $\mathbb{R}^n$ will be $\{(1,0,----0), (0,1,--0), ---- ----(0,0,---1)\}$ which has n elements.

Let B be a basis of $V+W$. As $V+W$ is subspace of $\mathbb{R}^n$, we can add vectors in B to make it a basis of $\mathbb{R}^n$. But this is not possible because $\dim(V+W) = \dim(\mathbb{R}^n)$. Hence both have same basis B.

$\therefore$, $V+W = \mathbb{R}^n$

B_1 is basis of V & B_2 is basis of W. So, basis of $V+W$ is $B_1 \cup B_2$

_/_/__

As $V+W=\mathbb{R}^n$ So basis of $V+W$ is basis of $\mathbb{R}^n$ Hence $B_1 \cup B_2$ is basis of $\mathbb{R}^n$.

(c) If x is in $\mathbb{R}^n$, then x can be written in the form $x = v + w$ where $v \in V$ and $w \in W$. This is because V and W span $\mathbb{R}^n$ (From part (b)) so any vector in $\mathbb{R}^n$ can be written as a linear combination of vectors in V and W.

(d) The representation obtained in part (c) is unique. Suppose $x = v_1 + w_1 = v_2 + w_2$ For some $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

Then $v_1 - v_2 = w_1 - w_2$ But since $v_1 - v_2 \in V$ & $w_1 - w_2 \in W$ and $V \cap W = \{0\}$ we must have $v_1 - v_2 = w_1 - w_2 = 0$

$\Rightarrow v_1 = v_2$ & $w_1 = w_2$

$\therefore$ the Representation is unique.