CS 302.1 - Automata Theory

Lecture 07

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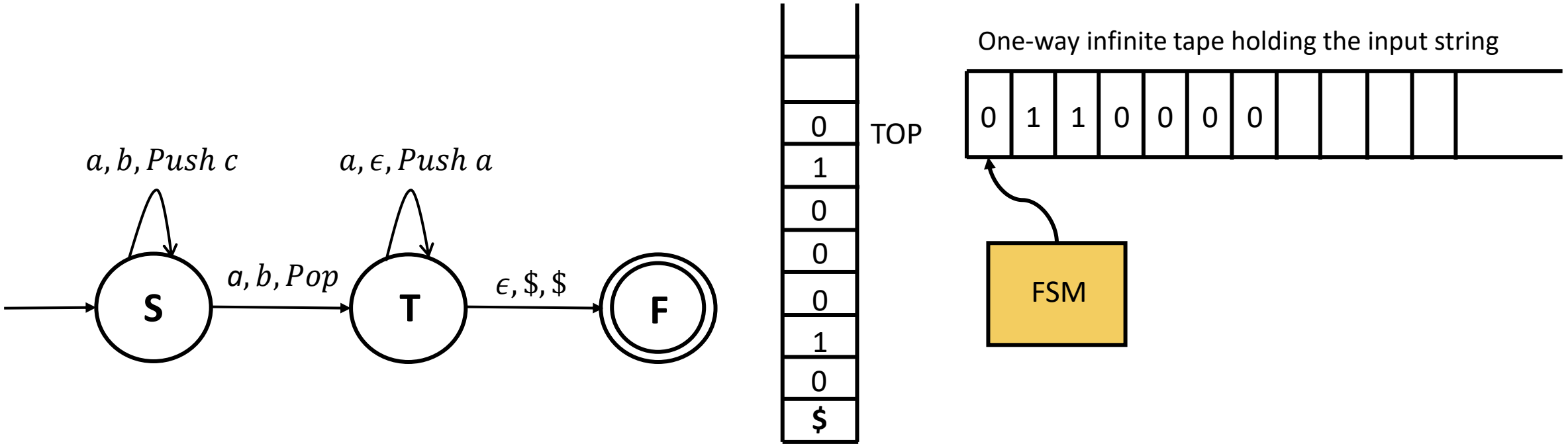
Quick Recap

Pushdown Automata

- Automata that recognizes CFLs
- FSM + stack
- FSM transitions by reading an input symbol and by interacting with the stack

- $\boldsymbol{\delta(q_i, a, b) = (q_j, c)}$: If the input symbol read is a and the stack top = b, then push c onto the stack and transition from q_i to q_j
- $\boldsymbol{\delta(q_i, a, \epsilon) = (q_j, c)}$: If the input symbol read is a, then push c onto the stack and transition from q_i to q_j
- $\boldsymbol{\delta(q_i, a, b) = (q_j, \epsilon)}$: If the input symbol read is a, and the stack top = b, then transition from q_i to q_j
- $\boldsymbol{\delta(q_i, \epsilon, \$) = (q_j, \$)}$: Transition from q_i to q_j if the stack is empty.



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  - The PDA does this non-deterministically (by taking ϵ transitions).

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- The above intuition is applicable for even length palindromes of the form ww^R.

- What about odd length palindromes?
  - Non-determinism to the rescue once again

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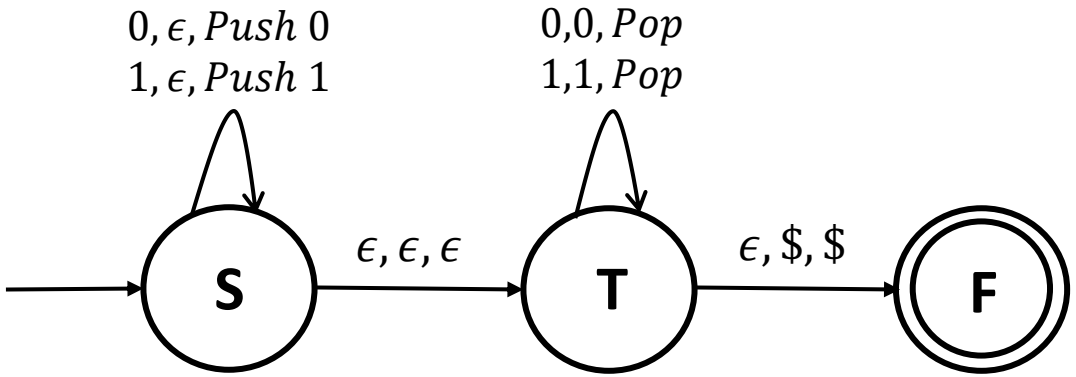

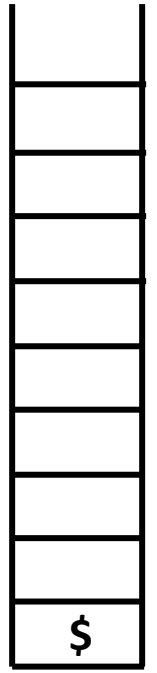

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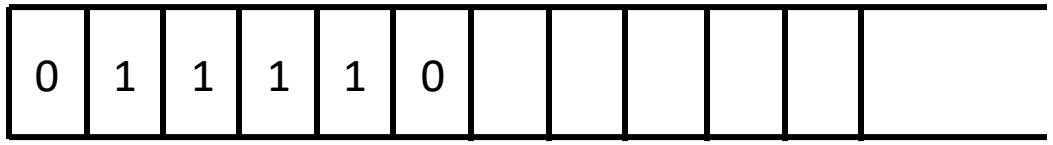

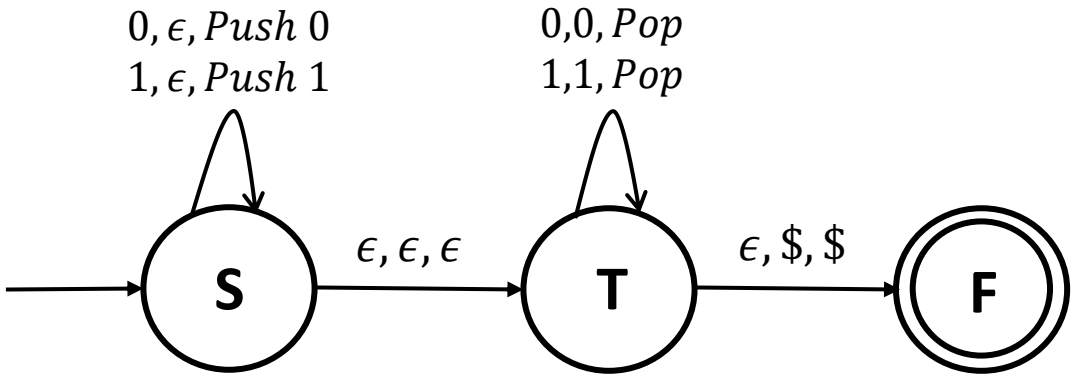

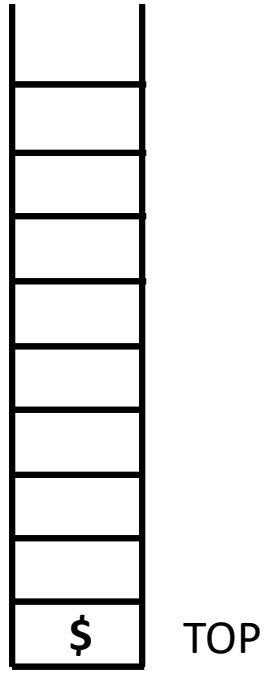

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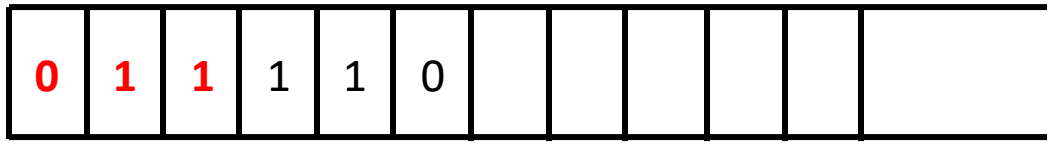

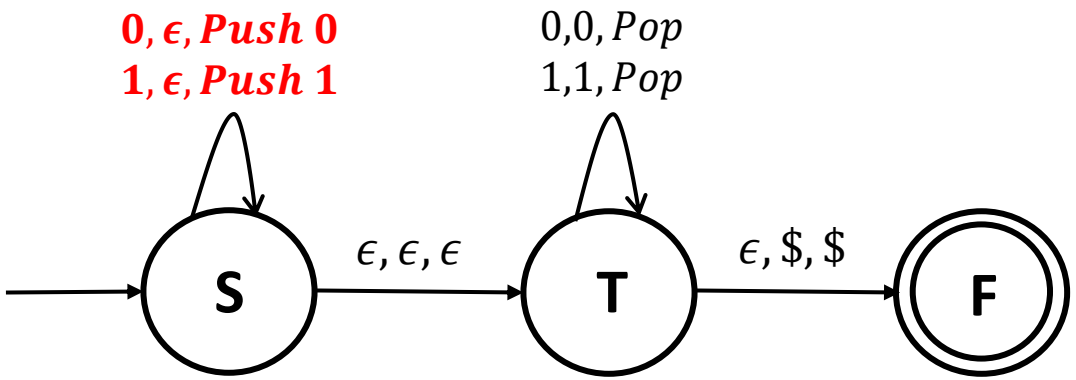

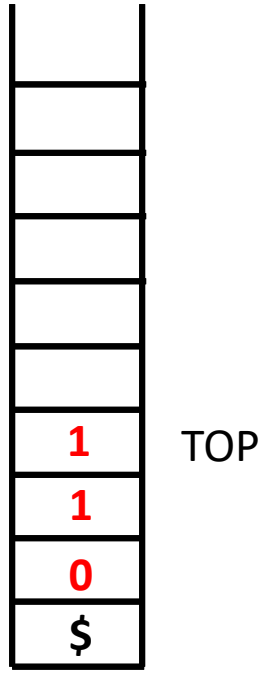

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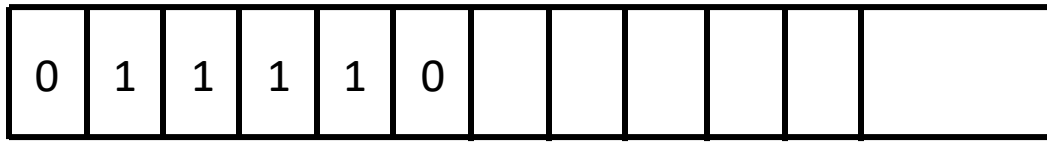

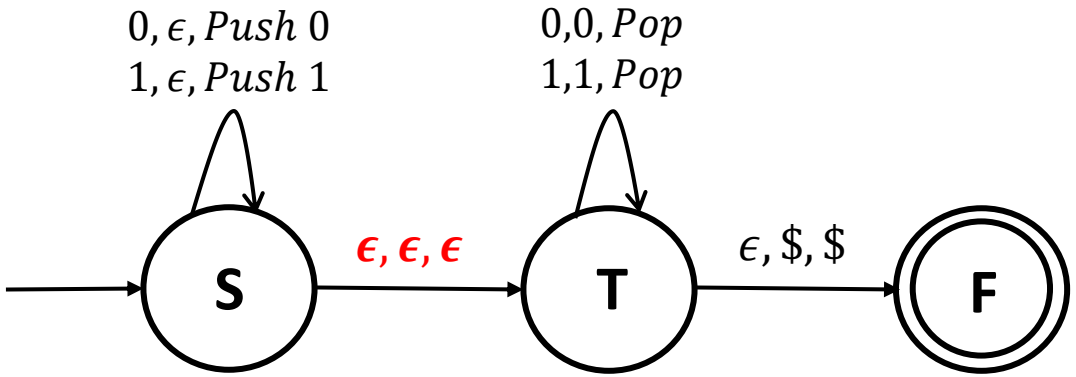

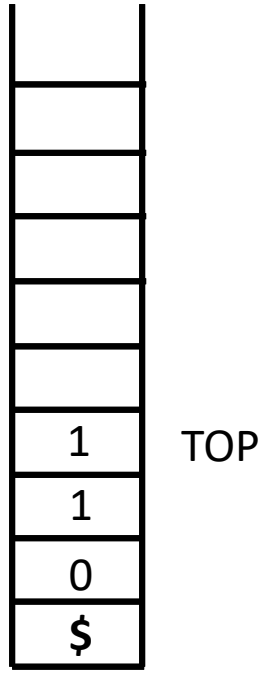

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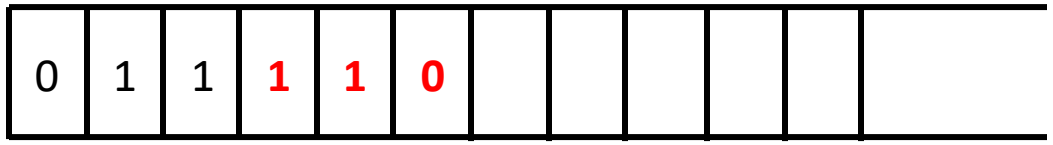

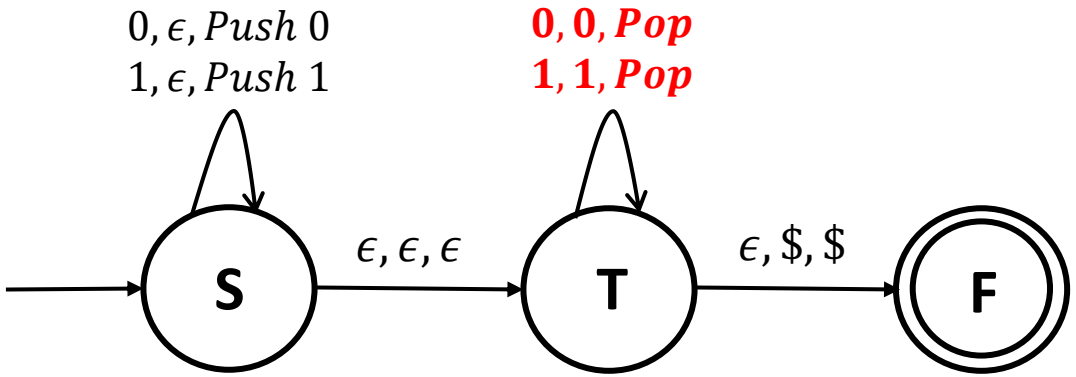

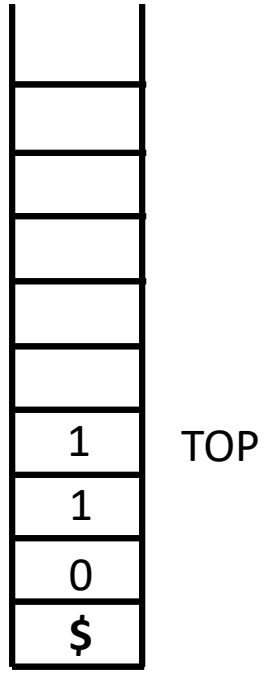

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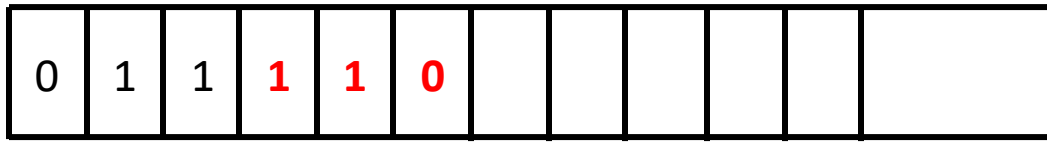

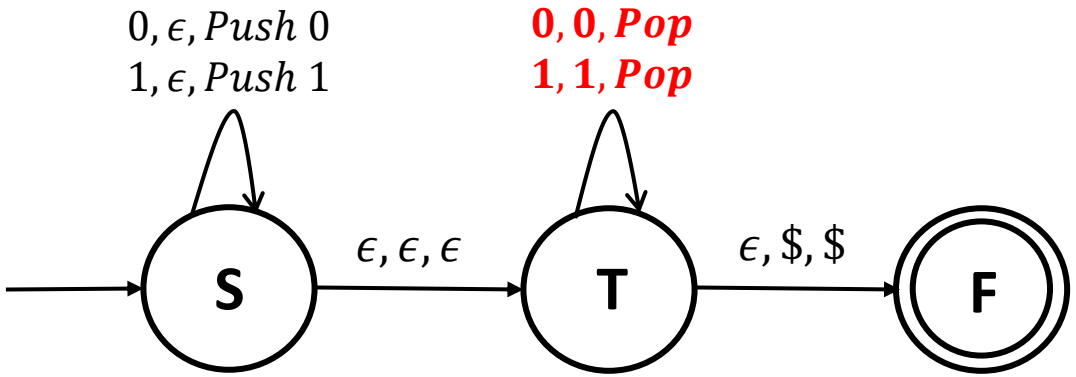

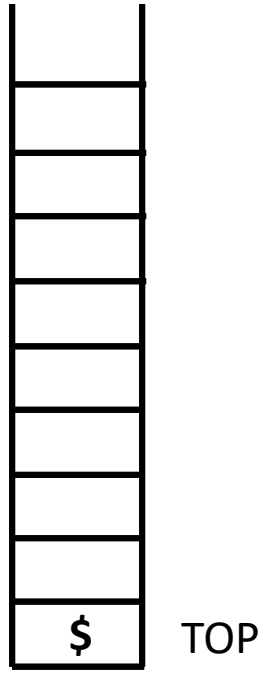

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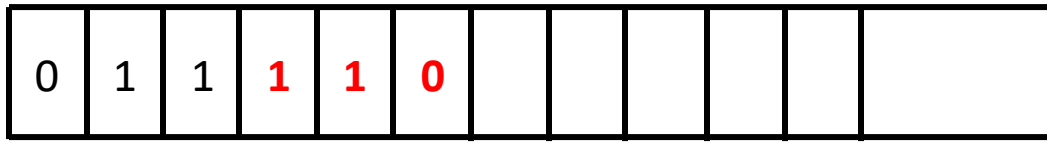

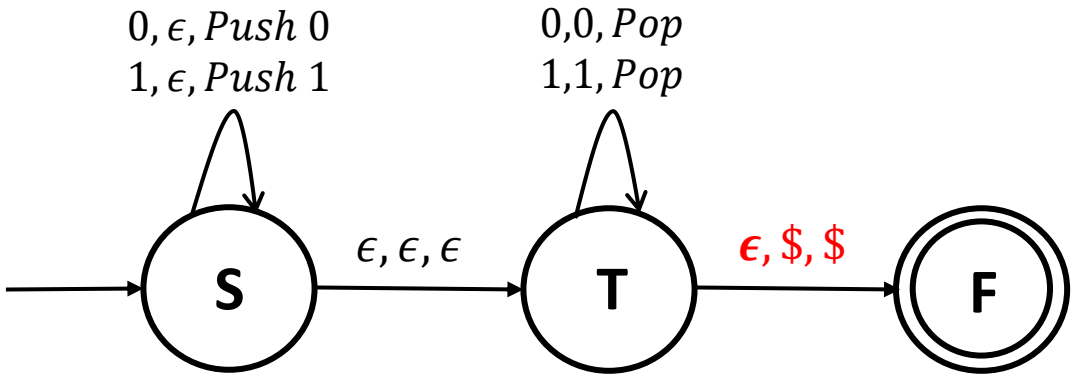

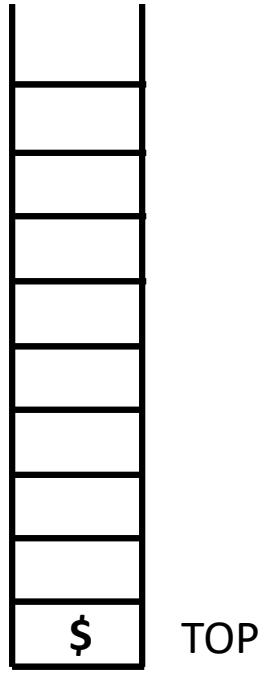

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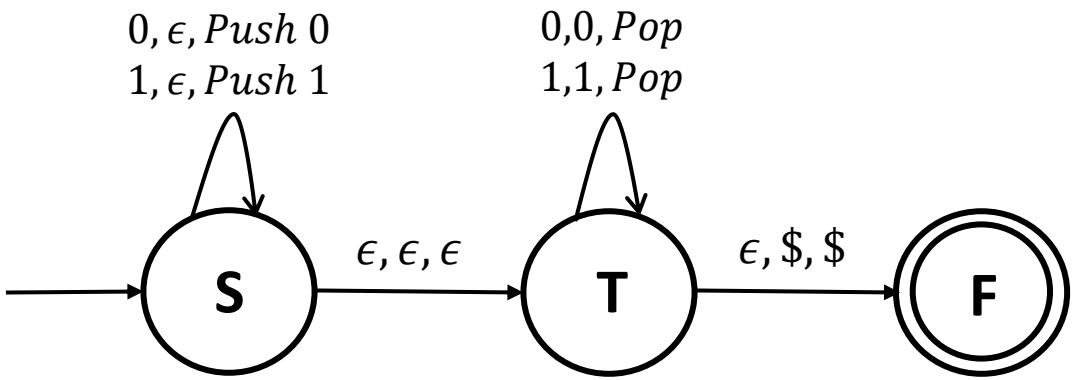

Recognizes even length palindromes of the form: ww^R

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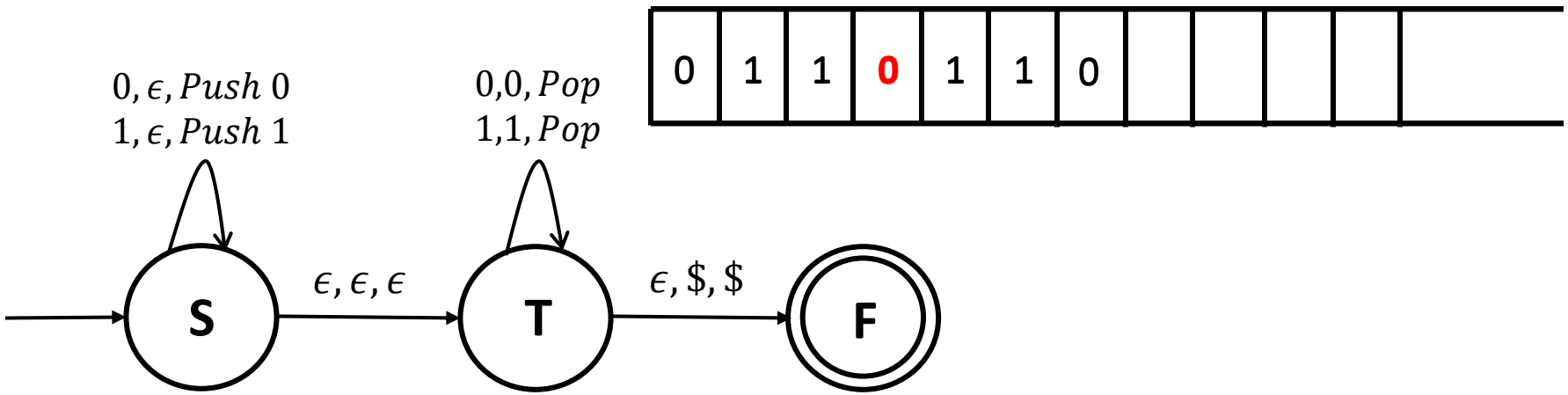

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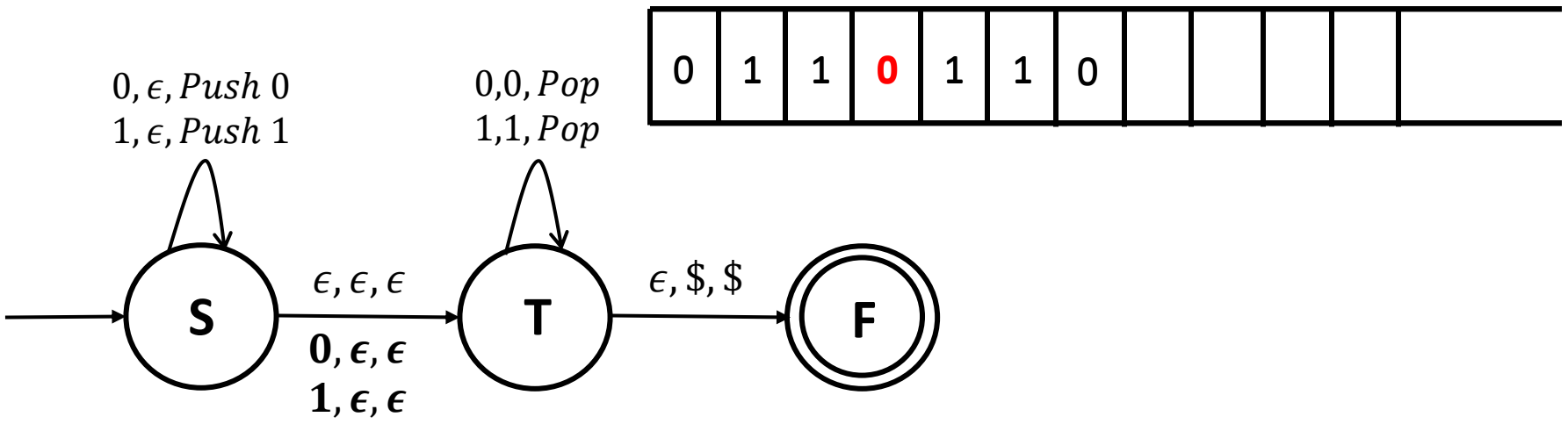

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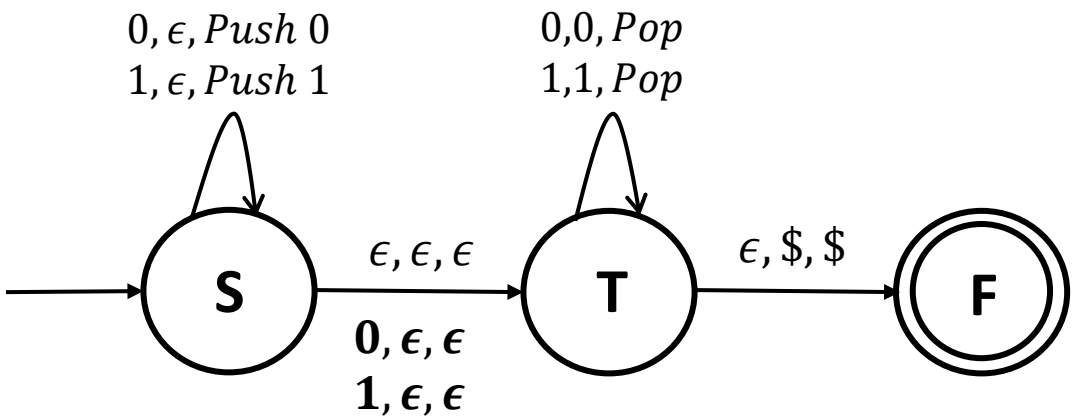

The transitions $\boldsymbol{0, \epsilon, \epsilon}$ **and** $\boldsymbol{1, \epsilon, \epsilon}$ **allow the PDA to consume one symbol** and then begin matching what it has encountered thus far.

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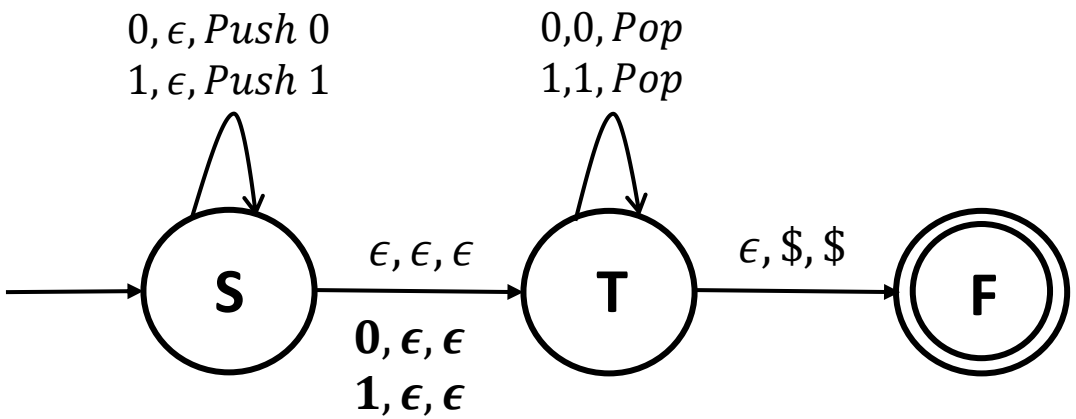

The transitions $\boldsymbol{0, \epsilon, \epsilon}$ **and** $\boldsymbol{1, \epsilon, \epsilon}$ **allow the PDA to consume one symbol** and then begin matching what it has encountered thus far.

This allows the PDA to **recognize strings of the form:** $\boldsymbol{\omega c w^R}$, where the aforementioned transitions non-deterministically guessed $c \in \{0,1\}$

Equivalence between PDA and CFL

- We already know that a language is Context-Free if and only if there exists a CFG that generates all the strings belonging to the CFL.
- It can be shown that a language is context free if and only if a PDA recognizes it.
    - If L is context free then there exists a PDA that recognizes L. (We'll prove this next)
    - If there exists a PDA for L, then L is context-free. (Won't prove this in class. Look up a standard text book)

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

- Before formally proving this, we will use some examples in order to provide some intuition.
- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.

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- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.
- The proof consists of using the rules of the CFG to build a PDA so that it can simulate any derivation $S \overset{*}{\Rightarrow} w$.
    - The PDA accepts an input w if the CFG G generates w
    - It determines whether ∃ a derivation for w.
    - Takes advantage of non determinism

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Intuitions

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Example: Consider the grammar G with the rules: $S \to aTb|b$

$T \to Ta|\epsilon$

The string $\boldsymbol{w = aaab}$ can be generated by G. Derivation:

$$\boldsymbol{S \to aTb \to aTab \to aTaab \to aaab}$$

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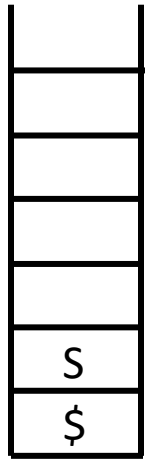
Example: $S \rightarrow aTb|b$
$T \rightarrow Ta|\epsilon$
Input to PDA: w = $aaab$

Derivation for input string $w = aaab$ can be generated by G**:**

$$S \rightarrow aTb \rightarrow aTab \rightarrow aTaab \rightarrow aaab$$

1. Push S onto the Stack.



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Derivation for input string $w = aaab$ can be generated by G**:**

$$S \rightarrow aTb \rightarrow aTab \rightarrow aTaab \rightarrow aaab$$

1. Push S onto the Stack.
2. Pop S and
   a. Push b
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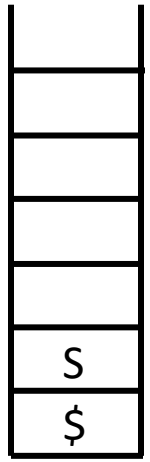

Pushdown Automata and CFL

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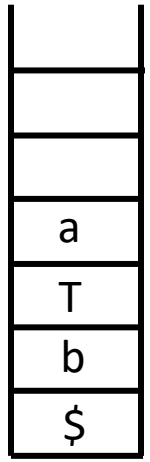

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Derivation for input string $w = \boldsymbol{a}aab$ can be generated by G**:**

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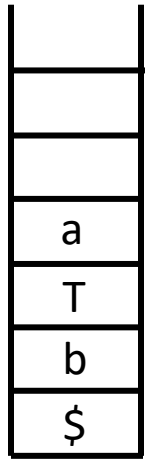

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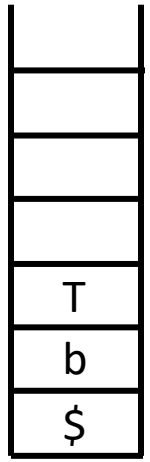

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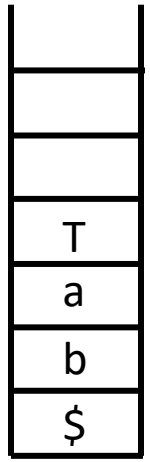

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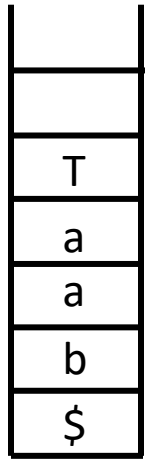

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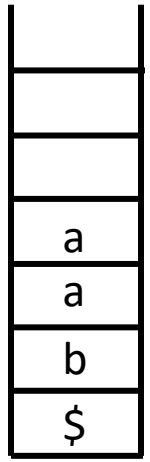

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Example: $S \rightarrow aTb|b$
$T \rightarrow Ta|\epsilon$
Input to PDA: w $= \boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaa}b$ can be generated by G**:**

$$S \rightarrow aTb \rightarrow aTab \rightarrow aTaab \rightarrow aaab$$

1. Push S onto the Stack.
2. Pop S and push aTb (Shorthand).
3. Read the input (a) (Pop a).
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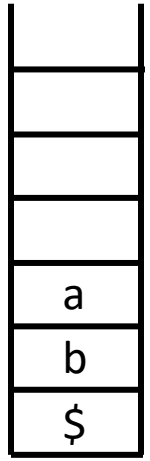

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Example: $S \rightarrow aTb|b$
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Input to PDA: w $= \boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G**:**

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1. Push S onto the Stack.
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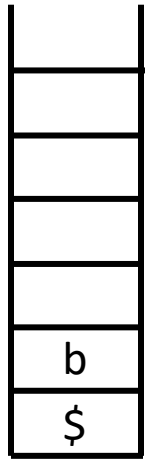

Pushdown Automata and CFL

- The PDA begins by pushing the start variable S onto the stack.
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- **Read the input symbol** if the top of the stack is some terminal a.

Example: $S \to aTb|b$
$T \to Ta|\epsilon$
Input to PDA: w = $\boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G**:**

$$S \to aTb \to aTab \to aTaab \to aaab$$

1. Push S onto the Stack.
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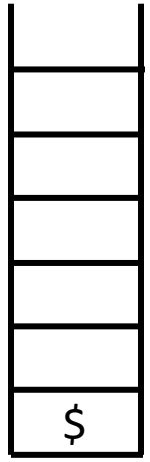
3. Read the input (a) and pop a.
4. Pop T and push Ta
5. Pop T and push Ta
6. Pop T (for the rule $T \to \epsilon$)
7. Read the input (a) (Pop a).
8. Read the input (a) (Pop a).
9. Read the input (b) (Pop b).
10. Since the stack is empty exactly when the input has been read, accept w.



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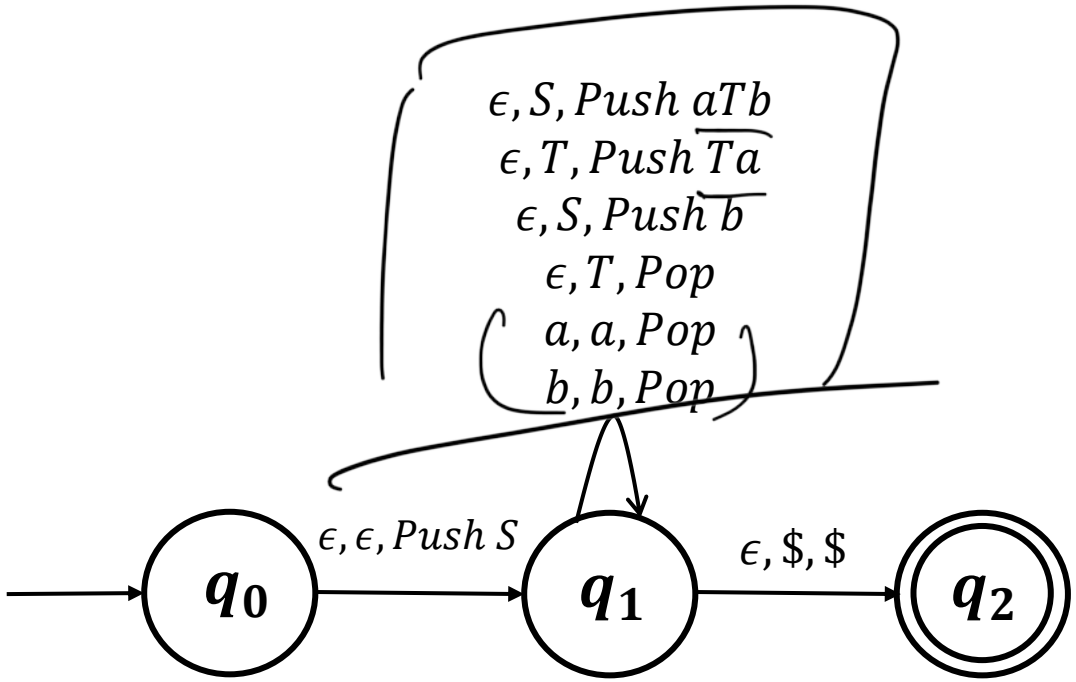

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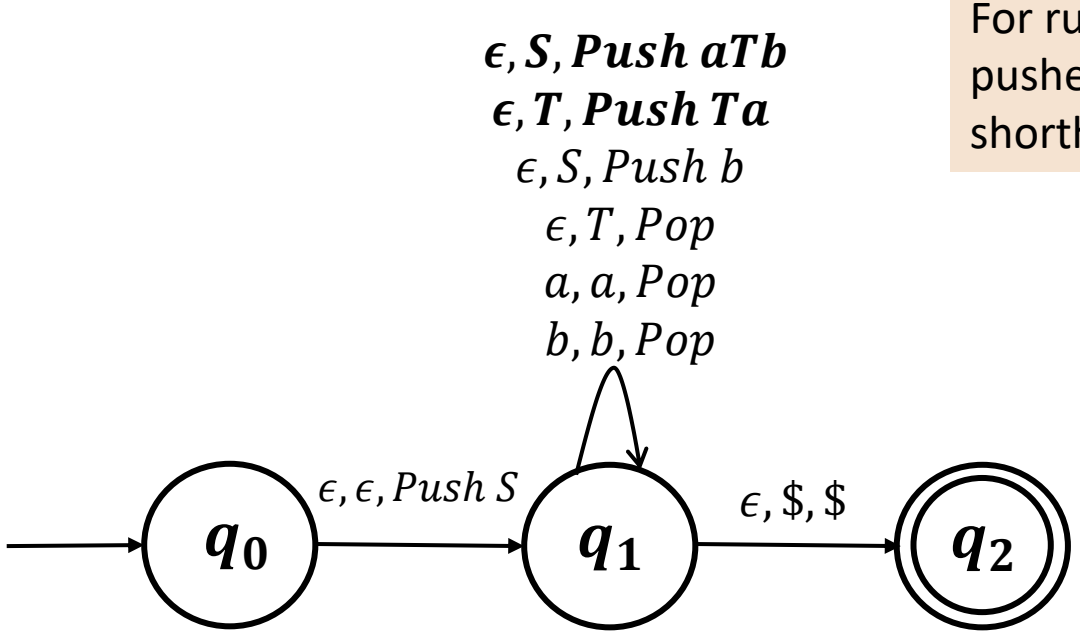

For rules where several elements need to be pushed, new states are introduced. This is only a shorthand for that.

Pushdown Automata and CFL

Example: $S \rightarrow aTb|b$

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Input to PDA: w = $aaab$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G**:**

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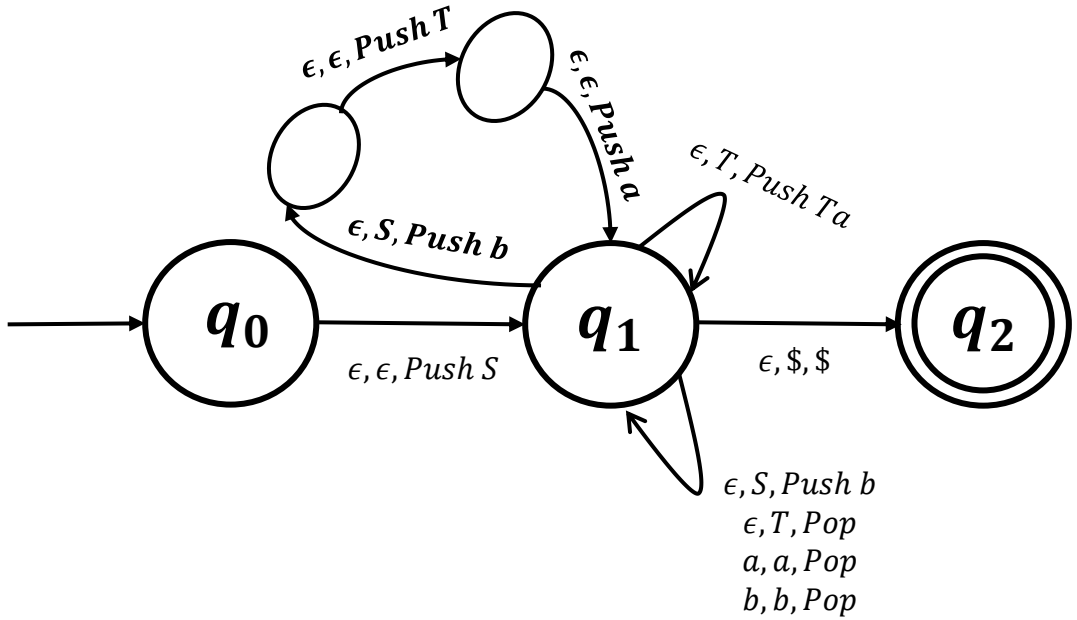
bTa



1S0

11S0

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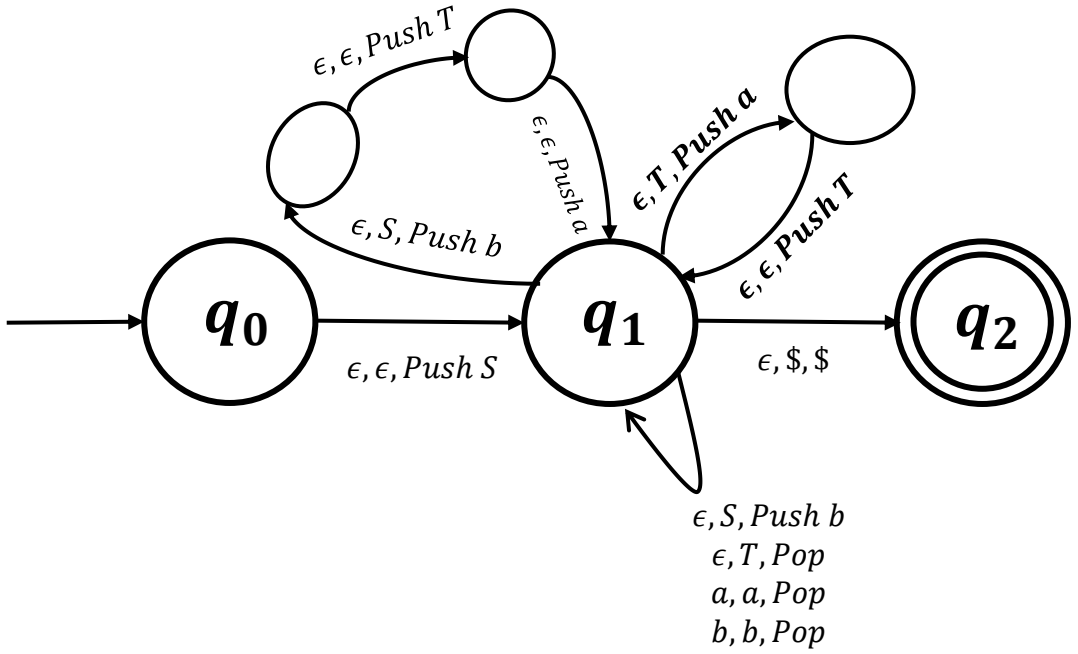


Summary

Given the rules of a CFG G, the equivalent PDA either non deterministically chooses which rule to use or matches part of the input symbol.

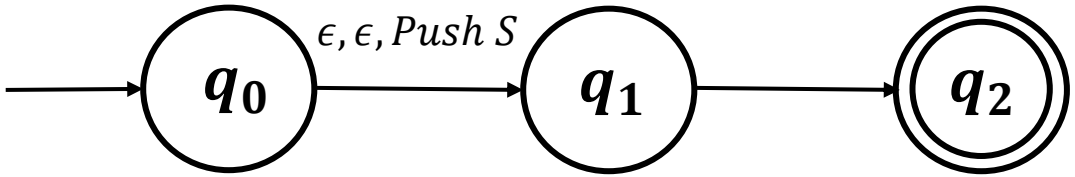
Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: For convenience, we shall be using the shorthand notation.

Let G be a CFG with a set of rules R, then the equivalent PDA P will have three states $\{q_0, q_1, q_2\}$.

The PDA P first pushes the start symbol S into the stack, irrespective of the input symbol and transitions from the initial state q_0 to q_1, i.e. $\delta(q_0, \epsilon, \epsilon) = (q_1, S)$.



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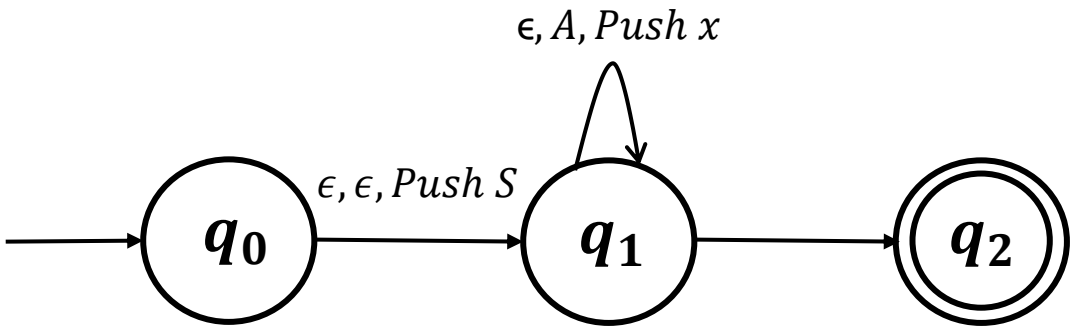
At q_1, the PDA P implements the rules R of G.

- Pop A and push x onto the stack, where $A \to x$ is a rule in R and return back to q_1, i.e. let $\delta(q_1, \epsilon, A) = (q_1, x)$.



For rule $A \to x$ in R

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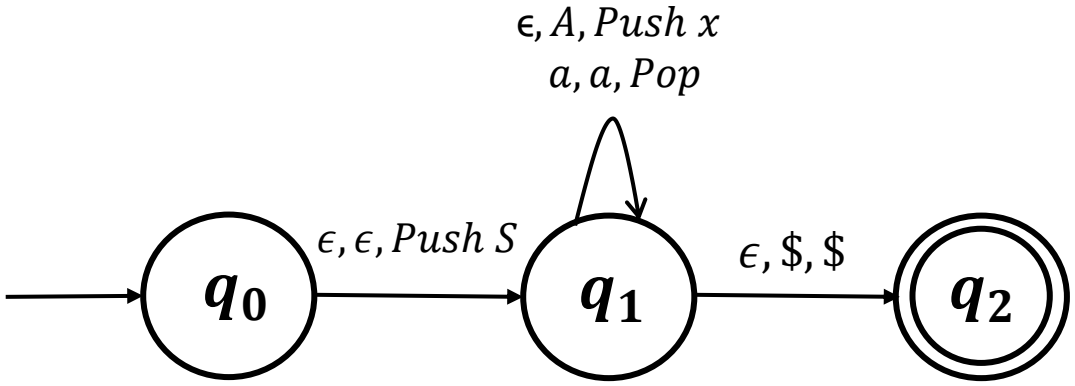

For terminal a

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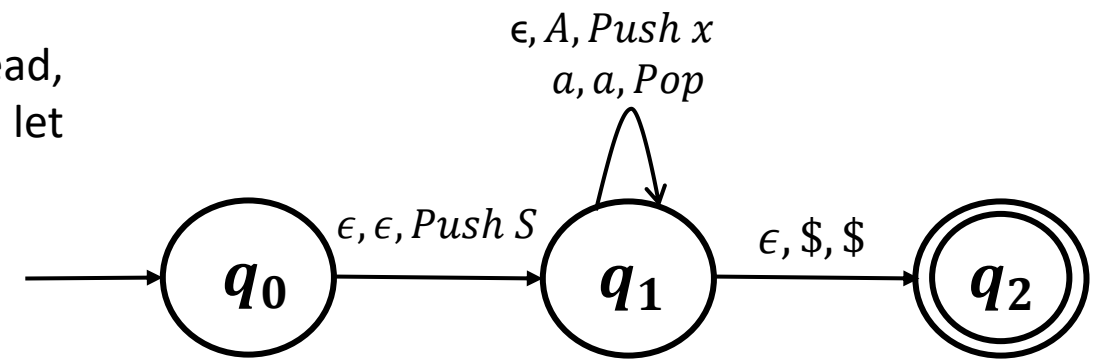
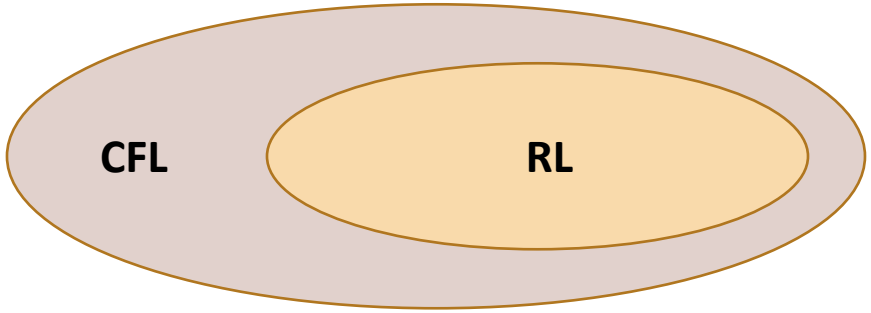
- If the stack is empty, when all the input symbols are read, transition from q_1 to the accepting state q_2, i.e. let $\delta(q_1, \epsilon, \$) = (q_2, \$)$



Equivalence between PDA and CFL

- It can be shown that a language is context free **if and only if** a PDA recognizes it.
    - If L is context free then there exists a PDA that recognizes L. (We proved this)
    - The proof for the other direction (Constructing a CFG that generates L given a PDA that recognizes L) is quite elaborate
    - We won't be covering it in class. But the proof itself is quite easy to understand.
    - Refer to a standard text book (e.g. Sipser)

$$(\boldsymbol{RL} \equiv \boldsymbol{Regular\ Grammar} \equiv \boldsymbol{Regular\ Expressions} \equiv \boldsymbol{NFA} \equiv \boldsymbol{DFA}) \subseteq (\boldsymbol{CFL} \equiv \boldsymbol{CFG} \equiv \boldsymbol{PDA})$$



Deterministic Pushdown Automata

- So far we have considered Non-deterministic PDAs (which are referred to as just PDAs)
- Multiple transitions per input symbol/stack symbol is allowed
- Recall that for regular languages, introducing non-determinism added no extra power to finite automata: NFAs and DFAs were equivalent
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Deterministic Pushdown Automata (DPDA)

DPDAs can be defined in a similar manner to PDAs with the following restriction:

- For each state in the PDA, for any combination of the current input symbol and the current stack symbol **at most** one transition is possible. (At most one valid transition is allowed for any input)

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- In fact, for a DPDA, the transition rule δ must satisfy the following conditions:

  - For any $q \in Q, a \in \Sigma_\epsilon$ and $x \in \Gamma_\epsilon$, the set $\boldsymbol{\delta(q, a, x)}$ **has at most one element.**
  - For any $q \in Q$ and $x \in \Gamma_\epsilon$, if $\delta(q, \epsilon, x) \neq \Phi$, then $\delta(q, a, x) = \Phi$ for every $a \in \Sigma$**.**

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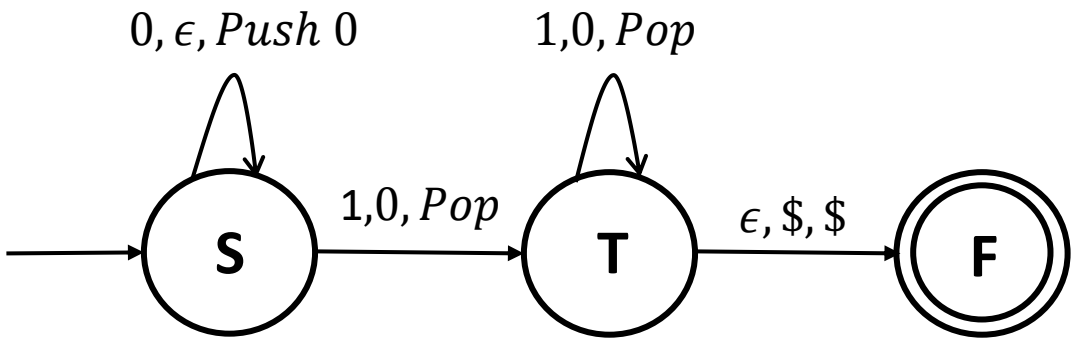
- This enforces deterministic behaviour by ruling out scenarios such as: $\delta(q, a, x) \neq \Phi$ and $\delta(q, a, \epsilon) \neq \Phi$.
- If there is an ϵ-transition for some configuration, no other input consuming move is possible.

Deterministic Pushdown Automata

DPDAs can be defined in a similar manner to PDAs with the following restriction:

- For each state in the PDA, for any combination of the current input symbol and the current stack symbol **at most** one transition is possible. (At most one valid transition is allowed for any input)
- So determinism is defined slightly differently here. For example: ϵ transitions are allowed, some transitions may be absent (leading to CRASH runs) unlike in the case of DFAs.
- In fact, for a DPDA, the transition rule δ must satisfy the following condition:
  - For any $q \in Q, a \in \Sigma_\epsilon$ and $x \in \Gamma_\epsilon$, the set $\boldsymbol{\delta(q, a, x)}$ **has at most one element.**
  - For any $q \in Q$ and $x \in \Gamma_\epsilon$, if $\delta(q, \epsilon, x) \neq \Phi$, then $\delta(q, a, x) = \Phi$ for every $a \in \Sigma$.

Is this a DPDA? **YES!**



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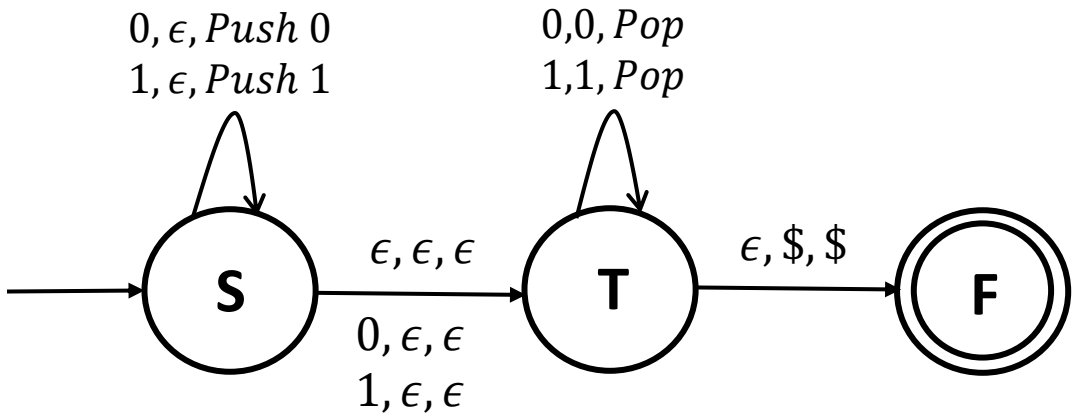

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- Interestingly, the power of DPDAs and PDAs are not the same. It turns out that PDAs are more powerful.
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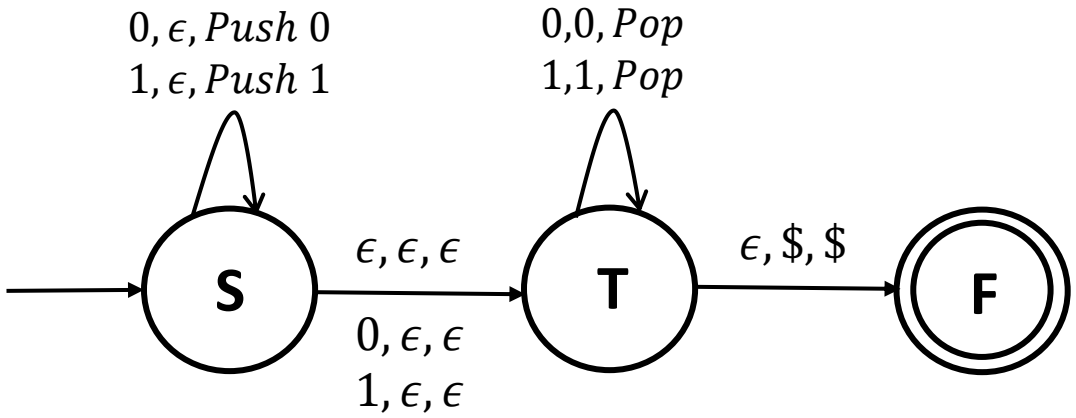
Example: $\boldsymbol{L = \{w | w}$ is a Palindrome$\}$

The PDA had to non-deterministically guess when half the string has been read and make a transition.

So although $\boldsymbol{L \in CFL, L \notin DCFL}$.



Deterministic Pushdown Automata

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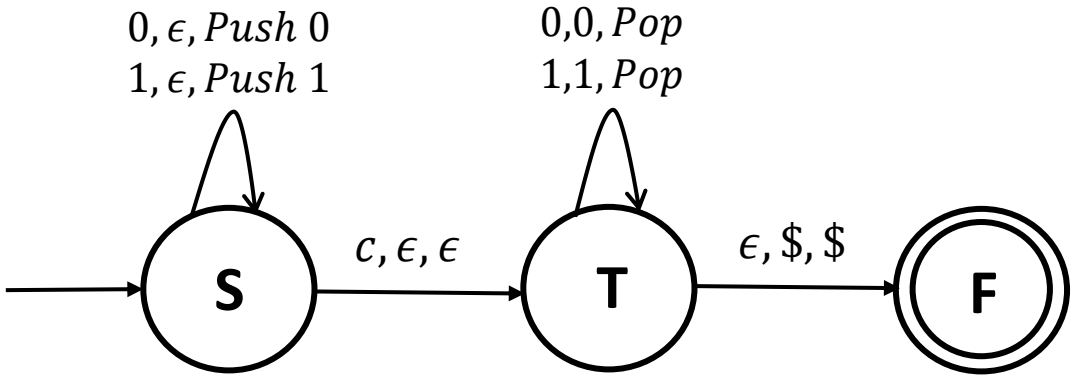
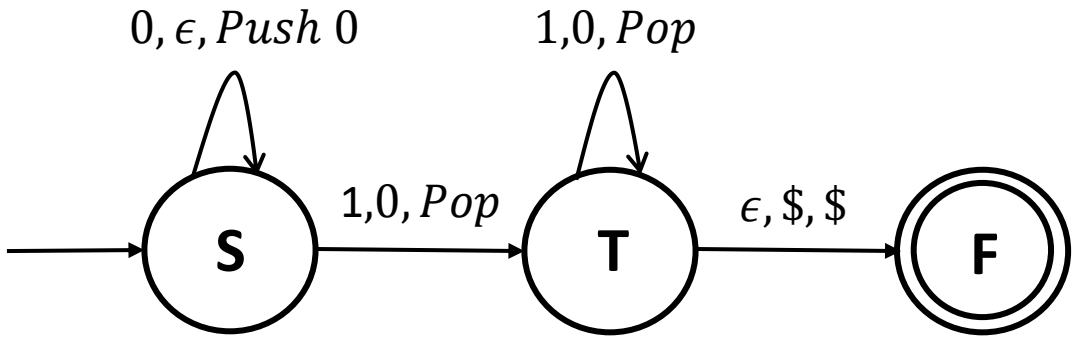
- $\boldsymbol{\Sigma = \{0, 1, c\}}$
- $\boldsymbol{L_1 = \{wcw^R | w \in \{0, 1\}^+\}}$
- $\boldsymbol{L_1 \in DCFL}$.



- $\boldsymbol{L_2 = \{0^n 1^n, n \geq 1\}}$
- $\boldsymbol{L_2 \in DCFL}$.



Deterministic Pushdown Automata

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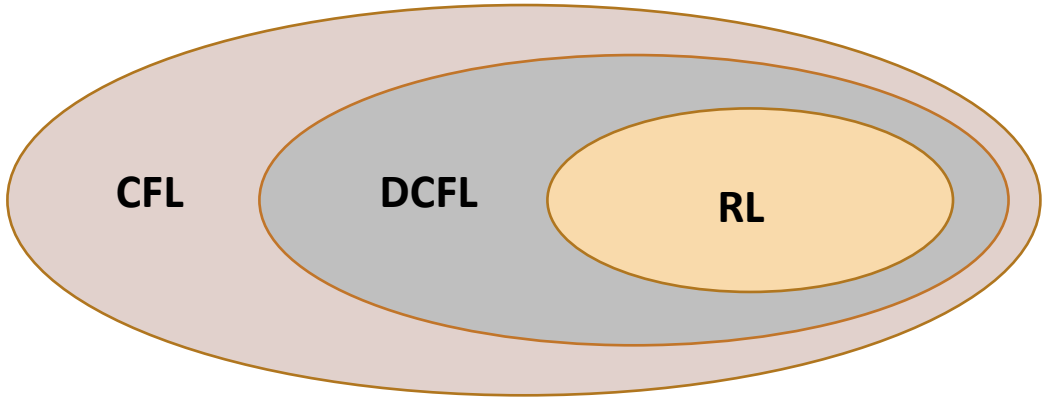
$$\boldsymbol{(RL \equiv Regular\ Grammar \equiv Regular\ Expressions \equiv NFA \equiv DFA) \subseteq (DCFL \equiv DPDA) \subseteq (CFL \equiv CFG \equiv PDA)}$$

Next lecture:

- Pumping lemma for CFLs
- Closure properties of CFLs



Thank You!