

# Performance Modelling for Computer Systems

## Assignment

Q1: Consider an  $M/M/K/2K$  queue. Model the system as a Markov chain and find its stationary distribution. Does the stationary distribution have a closed form? What is the blocking probability of the system?

Q2: Consider a two server system  $S_1$  and  $S_2$  with no buffer and the same exponential service rate of  $\mu$ . Arriving jobs are of two types  $C_1$  and  $C_2$ . Both arrive according to a Poisson process with rate  $\lambda$ .  $C_1$  and  $C_2$  jobs can be processed by either server at the given service rate. Additionally, if both  $S_1$  and  $S_2$  are free then they together process jobs of type  $C_2$  with rate  $\bar{\mu}$  instead. Model the described system as a Markov chain and obtain the stationary distribution for different states.

Q3: Consider a system with two identical servers of unit rate and no buffer. Let's call the two servers  $S_1$  and  $S_2$ . Arriving jobs are of two types. Low priority jobs  $C_1$  arrive according to a Poisson process with rate  $\lambda_1$  and can only use Server  $S_1$ . High priority jobs  $C_2$  arrive according to a Poisson process with rate  $\lambda_2$  and are supposed to use  $S_2$  first if it is idle. But if  $S_2$  is busy while  $S_1$  is idle, they are allowed to use server  $S_1$ . Assume both type of jobs have exponential service requirement with rate  $\mu_1$  and  $\mu_2$  respectively. Model this system as a Markov chain, obtain the stationary distribution for different states.

Q4: Find the probability that a job is declined service (i.e blocking probability) for the previous two systems (Q2 and Q3).

Q5: Consider the following queueing systems:

- A  $M/M/1$  system with arrival rate  $\lambda$  and exponential service rate  $K\mu$
- A  $M/M/K$  system with arrival rate  $\lambda$  and each server having a service rate of  $\mu$
- $n$  independent  $M/M/1$  systems with arrival rate  $\lambda/K$  and exponential service rate  $\mu$

Report the following metrics for each system:

- Expected number of customers in system
- Expected number of customers in the queue
- Expected time spent in the system
- Expected time spent in the queue

Additionally, compare these metrics for all three systems and report which system performs best (according to each metric).

Q6: As we all know, no one comes on time for a class scheduled at 8:30 AM. Fed up with this, the instructor decided that there is a uniform probability that the doors will be closed sometime between 8:30 AM and 8:45 AM. If students always come to class late and as a Poisson process with rate  $\lambda$ , what is the expected number of people who will get attendance?

Q7: Consider a discrete time Markov chain  $X_n$  with  $X_0 = i$ . Let  $N$  be the total number of visits made by the chain to a state  $j$ . Prove that

$$P(N = n) = \begin{cases} 1 - F_{ij}, & n = 0 \\ F_{ij}F_{jj}^{n-1}(1 - F_{jj}), & n \geq 1 \end{cases}$$

Here  $F_{ij}$  denotes the probability of ever coming to state  $j$  from state  $i$ .

Q8: During *Felicity*, a canteen shop is set up with two service counters to efficiently serve the attendees. The Fast Service Counter ( $S_1$ ) has a higher service rate  $\mu_1$  and is the preferred choice for all customers. If  $S_1$  is busy, incoming customers are directed to the Backup Counter ( $S_2$ ), which has a lower service rate  $\mu_2$ . If both counters are busy, the arriving customer leaves without being served, leading to job loss. Customer arrivals follow a Poisson process with rate  $\lambda$ .

As a key organizer of *Felicity*, you have been given the responsibility to manage the queuing system at the canteen. Your goal is to ensure smooth operations and minimize customer loss. To achieve this, you need to:

- Model the canteen system as a Markov Chain and construct the transition rate matrix  $Q$ .
- Find the stationary distribution of the system and compute the blocking probability (the probability that a customer is unable to buy food).
- Compare this system to an M/M/2 queue, where both counters serve at a combined rate  $\mu = \frac{\mu_1 + \mu_2}{2}$ .

Q9: Consider a Continuous-Time Markov Chain (CTMC) with a probability transition matrix  $P(t)$  given by

$$P(t) = \begin{bmatrix} 1 - e^{-t} & \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \\ \frac{e^{-t/2}}{3} & 1 - \frac{2e^{-t/2}}{3} & \frac{e^{-t/2}}{3} \\ e^{-t/4} & \frac{e^{-t/4}}{2} & 1 - \frac{3e^{-t/4}}{2} \end{bmatrix}$$

Find its stationary distribution and discuss whether the system always reaches a steady state. If not, explain why.