Our aim: Obtain samples from a continuous random variable

- Suppose you have access to samples from a uniform random variable U over support $[0, 1]$.(We will not study how to generate such samples.)
- Consider a continuous random variable X with support set $\mathcal{X}$ and let $F_X(x)$ denotes its cdf.
- Support set of X could be arbitrary.
- Our aim: Create i.i.d. samples of r.v. X using i.i.d. samples of U.
- We shall again see the inverse transform method to do this.

Sampling from continuous random variables

Lemma

Let U be uniform random variable over $[0, 1]$. Consider continuous r.v. X with cdf $F_X(.)$. Consider a random variable $\hat{X}$ defined as follows

$$\hat{X} := F_X^{-1}(U)$$

Then the cdf of $\hat{X}$ is $F_X(.)$.

Proof:

- Consider the cdf of $\hat{X}$, i.e., $F_{\hat{X}}(x) := \mathbb{P}[\hat{X} \leq x]$. Then

$$\begin{aligned} F_{\hat{X}}(x) &= \mathbb{P}[F_X^{-1}(U) \leq x] \\ &= \mathbb{P}[U \leq F_X(x)] \\ &= F_X(x) \end{aligned}$$

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Then the cdf of $\hat{X}$ is $F_X(.)$.

- Using this lemma, how to generate samples of a continuous random variable X using samples U?
- **Answer:** Draw $u \sim U$ and obtain $F_X^{-1}(u)$. This is a sample from $\hat{X}$ which has same distribution as X.
- `https://en.wikipedia.org/wiki/Inverse_transform_sampling`
- Do you observe anything "special" about this lemma?

Application in data analysis

- Lemma: $\hat{X} = F_X^{-1}(U)$ has distribution $F_X(.)$.
- What will be cdf of a random variable $Y = F_X(\hat{X})$? **Uniform!**
- A consequence of this lemma is that $F_X(X)$ is a uniform distribution.
- This property is known as "probability integral transform or universality of uniform".
- This property is used to test whether a set of observations can be modelled as arising from a specified distribution $G(.)$ or not.

Evaluating Integrals via Monte Carlo approach

- Suppose you want to compute $\theta = \int_0^1 g(x)dx$ using only samples from $U[0,1]$. How will you do it ?
- $\theta = E[g(U)]$.
- Use iid samples of U and invoke strong law of large numbers (SLLN).

> Suppose X_i are iid, and $S_n = \sum_{i=1}^n X_i$. Then $\frac{S_n}{n} \to E[X]$.

- as $n \to \infty$ we have

$$\sum_{i=1}^{n} \frac{g(U_i)}{n} \to E[g(U)] = \theta$$

.

- HW: How will you compute $\int_a^b g(x)dx$ or $\int_0^\infty g(x)dx$?

Importance Sampling

- Suppose you want to compute $E[h(X)]$ where X has pdf $f(\cdot)$.
- Assume you do not have samples from X but know $f(\cdot)$.
- Now suppose you have access to samples from random variable Y with pdf $g(\cdot)$.
- How will you use i.i.d samples of Y to compute $E[h(X)]$?

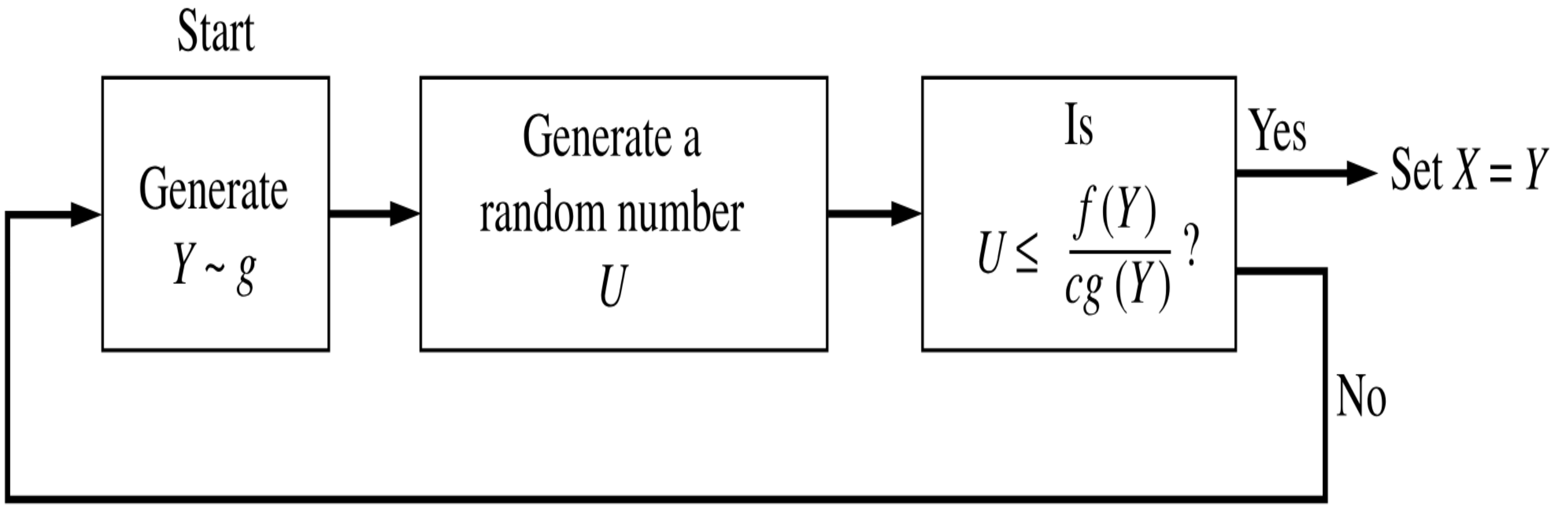
$$
\begin{aligned}
E[h(X)] &= \int h(x)f(x)dx \\
&= \int \frac{h(y)f(y)}{g(y)} g(y)dy \\
&= E_Y\left[\frac{h(Y)f(Y)}{g(Y)}\right]
\end{aligned}
$$

- Now use LLN and samples of Y to estimate $E[h(X)]$.

Accept Reject method

- Suppose you want to generate samples from X with pmf $p(\cdot)$ using samples from Y with pmf $q(\cdot)$.
- Suppose that $\frac{p(y)}{q(y)} \leq c$ for all y.
- The accept reject method is as follows:
- Step 1: Generate a sample $y \sim q(\cdot)$.
- Step 2: Generate $u \sim \mathcal{U}(0,1)$.
- Step 3: If $u \leq \frac{p(y)}{cq(y)}$, accept y as a sample from X.
- Step 4: If not, reject y and go back to Step 1.

Accept Reject method



- Why does the method work ?
- What is $P(y/accept)$?   is it $p(y)$?

Proof of Accept-Reject Method

- To prove that the method produces samples from $p(\cdot)$, we will compute the probability of accepting a sample y from $q(\cdot)$.

- The probability of accepting y is given by:

$$P(\text{accept} \mid y) = P\left(u \leq \frac{p(y)}{cq(y)}\right) = \frac{p(y)}{cq(y)}$$

  since $u \sim \mathcal{U}(0, 1)$.

- Thus, the joint probability of sampling $y \sim q(\cdot)$ and accepting it is:

$$P(\text{sample } y \text{ and accept}) = q(y) \cdot \frac{p(y)}{cq(y)} = \frac{p(y)}{c}$$

Proof (cont'd)

- The marginal probability of accepting any sample (i.e., normalizing constant) is:

$$P(\text{accept}) = \sum_y P(\text{sample } y \text{ and accept}) = \sum_y \frac{p(y)}{c} = \frac{1}{c}$$

- The conditional probability of accepting a particular sample y given that the sample was accepted is:

$$P(y \mid \text{accept}) = \frac{P(\text{sample } y \text{ and accept})}{P(\text{accept})} = \frac{\frac{p(y)}{c}}{\frac{1}{c}} = p(y)$$

- Therefore, the accepted samples are distributed according to $p(\cdot)$, proving that the method works.

Stochastic Simulation

- This was a brief introduction to this area of stochastic simulation.
- Refer the book Simulation by Sheldon Ross!
- Some popular techniques in simulation are:
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    - The inverse transform method
    - Accept-Reject method (rejection sampling)
    - Importance sampling
    - Markov Chain Monte Carlo (MCMC) methods
        - Hasting-Metropolis algorithm
        - Gibbs sampling
        - Slice sampling