RECAP

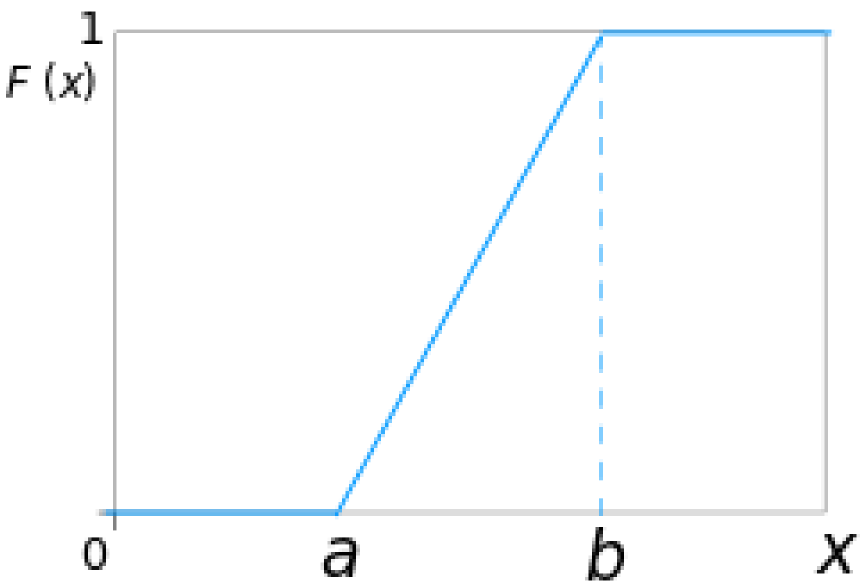
- CDF $F_X(a) := P_X(\{X \in (-\infty, a]\})$
- CDF is non-decreasing and right continuous.
- A continuous r. v.'s have continuous $F_X(\cdot)$.
- The also have a pdf $f_X(\cdot)$.
- $P_X(a \leq X \leq b) = \int_a^b f_X(u)du$. (Area under the curve)
- $F_X(x) = \int_{u=-\infty}^{x} f_X(u)du$.

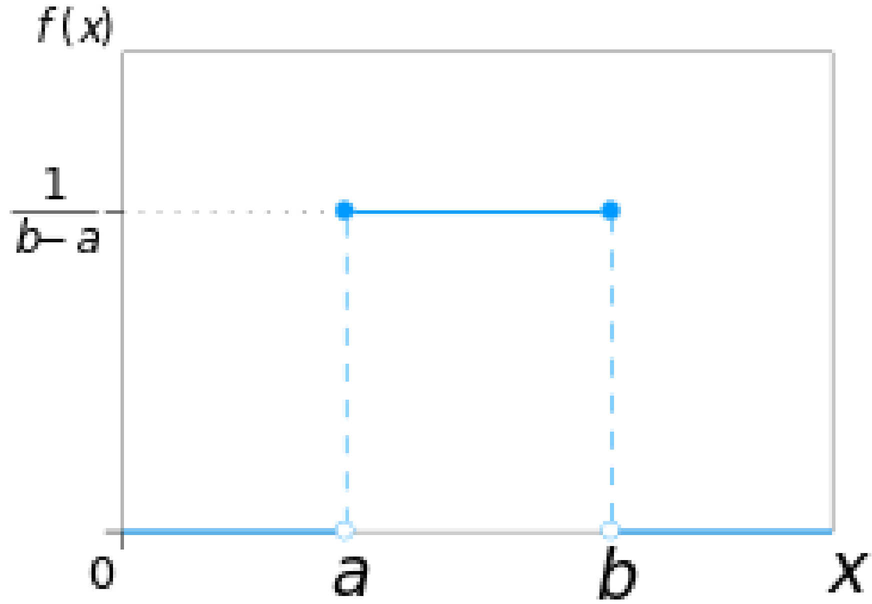
Standard Examples

Uniform random variable ($U[a, b]$)

- This is a real valued r.v.
- Its pdf $f_X(x) = \frac{1}{b-a}$ for all $x \in [a, b]$.
- Its CDF is given by $F_X(x) = \begin{cases} 0 \text{ for } x < a. \\ \frac{x-a}{b-a} \text{ for } x \in [a, b] \\ 1 \text{ otherwise.} \end{cases}$
- HW: Verify $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

$U[a, b]$

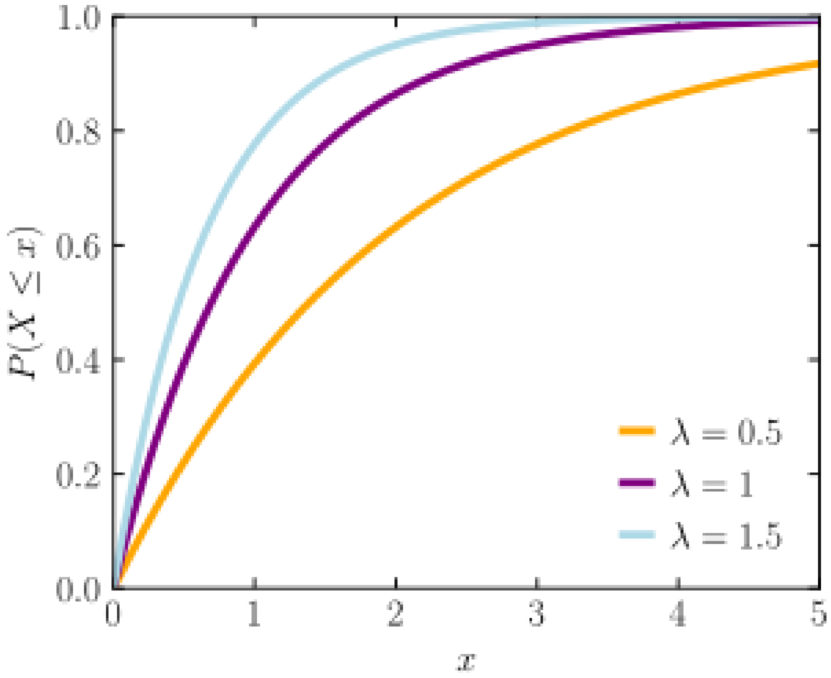


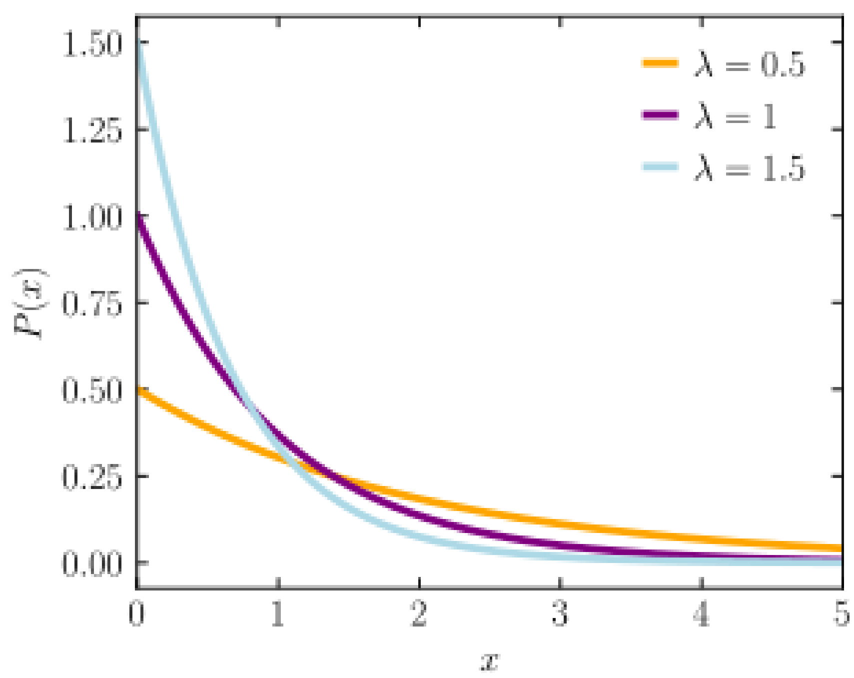


Exponential random variable ($Exp(\lambda)$)

- This is a non-negative r.v. with parameter λ.
- Its pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
- Its CDF is given by $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
- $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$
- $E[X^n] = \frac{n!}{\lambda^n}$

$Exp(\lambda)$


1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
x
$P(X \leq x)$
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

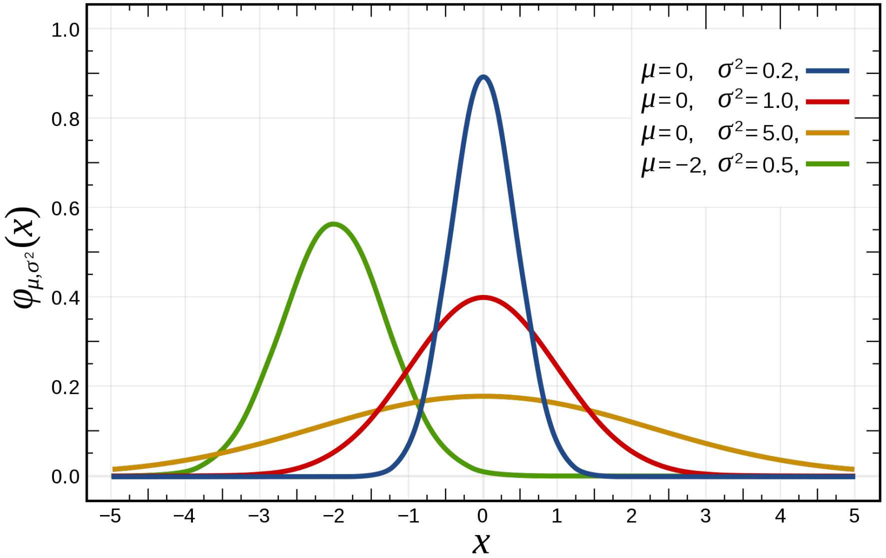

1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
x
$P(x)$
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

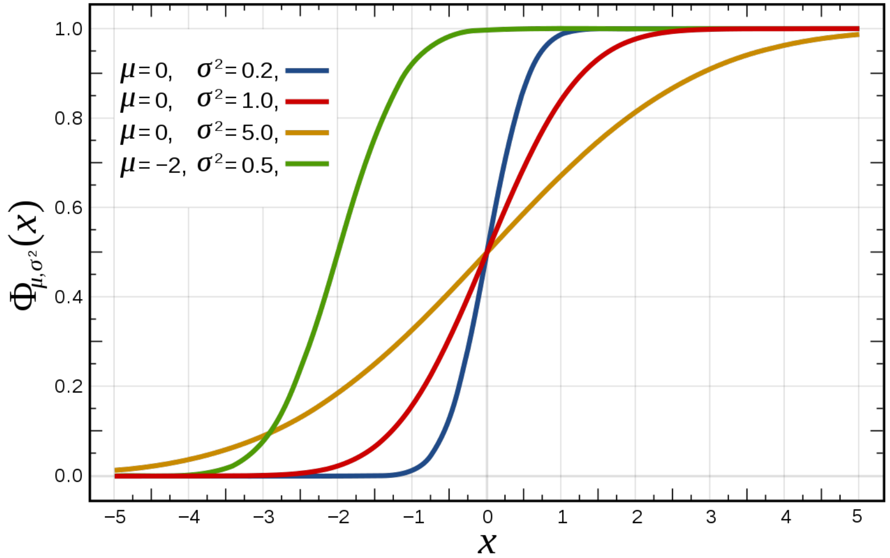
Significance of Exponential r.v.

- Building blocks for Continuous time Markov Chains.
- Demonstrate memory-less property (to be seen formally soon).
- $P(X > a + h | X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda(h)} = P(X > h)$.
- Used extensively in Queueing theory to model inter-arrival time and service time of jobs.

Gaussian random variable ($\mathcal{N}(\mu, \sigma^2)$)

- This is a real valued r.v. with two paramerters, μ and σ.
- Its pdf $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for all $x \in \mathbb{R}$.
- Verify: $\int_{-\infty}^{\infty} f_X(x)dx = 1$, $E[X] = \mu$ and $Var(X) = \sigma^2$.





Standard Normal random variable ($\mathcal{N}(0,1)$)

- When $\mu = 0$ and $\sigma = 1$, it is called as a standard normal.
- In this case $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.
- What is $\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$? How do you even solve this? $(= \sqrt{2\pi})$
- The CDF of standard normal, denoted by $\Phi(x)$ is given by
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$
- $Q(x) := 1 - \Phi(x)$ is the Complimentary CDF ($P(X > x)$). A closely related cousin in the error function $erf(x) = \frac{2}{\sqrt{pi}} \int_0^x e^{t^2} dt$.
- $\Phi =$These values are recorded in a table. (Fig. 3.10 in Bertsekas)
- https://en.wikipedia.org/wiki/Gaussian_function

Normality preserved under Linear Transformations

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b$ is also a normal variable with $E[Y] = a\mu + b$ and variance $a^2\sigma^2$. (To be proved later)

- Suppose X is standard normal, then find a and b such that $Y \sim \mathcal{N}(\mu, \sigma^2)$
- In this case, the CDF of Y in terms of X is given by $\Phi(\frac{x-\mu}{\sigma})$.

Significance of Gaussian r.v.

- Key role in Central limit theorem.
- $\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ where X_i is any random variable with mean μ and variance σ^2.
- Building block for multinomial Gaussian vector and Gaussian processes (GP).
- Gaussian process are used in Bayesian Optimization (black-box optimization).
- Brownian motion is a type of GP and is used in Finance.
- GP Regression, Gaussian mixture models, used widely in ML.

List of Probability distributions ...

https://en.wikipedia.org/wiki/List_of_probability_distributions

Important ones are Beta, Gamma, Erlang, Logistic, Weibull ....