Properties of Q

Theorem

$P(Y_i > u|X(t) = i) := e^{q_{ii}u}$, *i.e.*, $q_{ii} = -a_i$.
$P\left(X(t + Y_t) = j|X(t) = i\right) = \frac{q_{ij}}{|q_{ii}|}$ *where* $q_{ij} \geq 0$.

- q_{ii} is negative. $q_{ii} = -\sum_{j \neq i} q_{ij}$.
- $|q_{ii}|$ is the exponential rate at which you leave state i.
- q_{ij} is the exponential rate at which you leave state i to go to state j.
- minimum of exponentials is exponential with aggregated rate.
- This justifies the rate of leaving state i to be $\sum_{j \neq i} q_{ij}$.

Equivalent definition of a CTMC using Q

- in the CTMC, you stay in state i for a random duration that has exponential($|q_{ii}|$) distribution.
- From i, you will move to state j with probability $\frac{q_{ij}}{|q_{ii}|}$.
- Equivalently, in state i, you have $M-1$ exponential(q_{ij}) clocks for $j = 1, 2, \ldots, i-1, i+1, \ldots M$.
- You move to that state whose clock rings first!

Kolmogorov forward/backward equations CTMC

$t+s$

- $\frac{dP(t)}{dt} = \lim_{s \to 0} \frac{P(t+s)-P(t)}{s}$
- $\frac{dP(t)}{dt} = P(t) \lim_{s \to 0} \frac{P(s)-I}{s}$
- $\frac{dP(t)}{dt} = P(t)Q.$
- $P(t) = e^{tQ}$ satisfies the above. (Calculus of Matrix exponentials)
- $P(t) = e^{tQ} := I + tQ + \ldots + \frac{(tQ)^n}{n!} \ldots$

Example: Poisson process $N(t)$ as a CTMC

- States $\mathcal{S} = Z_{\geq 0}$.
- Why is it a Markov process / Markov property satisfied?
- $P(N(t) = k|N(t_1) = k_1, \ldots, N(t_m) = k_m) = P(N(t) = k|N(t_m) = k_m)$?
- $P(N(t) = k|N(t_1) = k_1, \ldots, N(t_m) = k_m) = P(N(t - t_k) = k - k_m)$. Therefore the above is true.
- $p_{ij}(t) = P(N(t) = j|N(0) = i)$. $max(j - i, 0)$ arrivals in time t.
- We know that this has Possion distribution.
- How does $P(t)$ look for a Poisson process ?

Example: Poisson process $N(t)$ as a CTMC

- How does $P(t) = [[P(N(t) = j | N(0) = i)]]$ look ?
- Entries below the diagonal are zero.
- Diagonal entries have the value $e^{-\lambda t}$
- ijth entry above the diagonal has the value $e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}$

Example: Poisson process $N(t)$ as a CTMC

- How does $Q = \frac{dP(h)}{dh}|_{h=0}$ look ?
- ijth entry above the diagonal $p_{ij}(t) = e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}$
- what is $\frac{d}{dt}(e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!})|_{t=0}$?
- If $j - i = 1$, then $\frac{d}{dt}(e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}) = \lambda$.
- If $j - i > 1$, then $\frac{d}{dt}(e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}) = 0$.
- How does Q for Poisson process look like ?
- $P(t) = e^{tQ} = I + tQ + \ldots + \frac{(tQ)^n}{n!} + \ldots$

Example 3: Binomial process as a DTMC

- DO IT YOURSELF!

Limiting probabilities

- $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 \end{bmatrix}$ $P^5 = \begin{bmatrix} .06 & .3 & .64 \\ .18 & .38 & .44 \\ .38 & .44 & .18 \end{bmatrix}$ $P^{30} = \begin{bmatrix} .23 & .385 & .385 \\ .23 & .385 & .385 \\ .23 & .385 & .385 \end{bmatrix}$

- $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ $\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

- What is the interpretation of $\lim_{n\to\infty} p_{ij}^{(n)} = [\lim_{n\to\infty} P^n]_{ij}$?

- $\alpha_j = \lim_{n\to\infty} p_{ij}^{(n)}$ denotes the probability of being in state j after a large time from starting in state i.

- For an M state DTMC, $\alpha = (\alpha_1, \dots, \alpha_M)$ denotes the limiting distribution.

- How do we obtain the limiting distribution α? Does it always exist?

Stationary distribution

The **stationary distribution** of a Markov chain is defined as a solution to the equation $\pi = \pi P$.

- πP is essentially the p.m.f of X_1 having picked X_0 according to π.
- $\pi = \pi P$ says that, if the initial distribution is π, then the distribution of X_1 is also π.
- Continuing this argument, the p.m.f of X_n for any n is π and there is no dependence on the starting state.
- MCMC algorithms use this idea (at stationarity successive states of the Markov chain have p.m.f π) to sample from target distribution π.

Limiting vs Stationary distribution

- Obtain stationary distribution for the Markov Chain with transition probability $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 \end{bmatrix}$

- The limiting distribution α need not exist for some Markov chains, but the stationary distribution π exists. For example for $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- The limiting distribution if it exists, is same as the stationary distribution, i.e. $\alpha_i = \pi_i$ for all i.

- For a CTMC, we know that $\frac{dP(t)}{dt} = P(t)Q$. When $\lim_{t\to\infty} P(t) = \Pi$, this means that at stationarity $\frac{dP(t)}{dt} = 0$. Therefore we have $\pi Q = 0$ in case of CTMC.

Embedded DTMC in a CTMC

- Consider a CTMC over state space $\mathcal{S}$.
- Let $Y_n, n \geq 0$ denote the sequence of times spent in successive states of the CTMC
- Define T_n to be the jump times of the CTMC, i.e., the times of successive state transitions.
- Then $T_n = \sum_{k=1}^{n} Y_k$.
- Define $X_n = X(T_n)$ for $n \geq 0$. X_n is the DTMC embedded in the CTMC.
- The corresponding TPM has $p_{ij} = \frac{q_{ij}}{|q_{ii}|}$.
- $\{X_n,\}$ is such that there are no one step transitions from a state to itself, i.e., $p_{ii} = 0$.

Transience and Recurrence

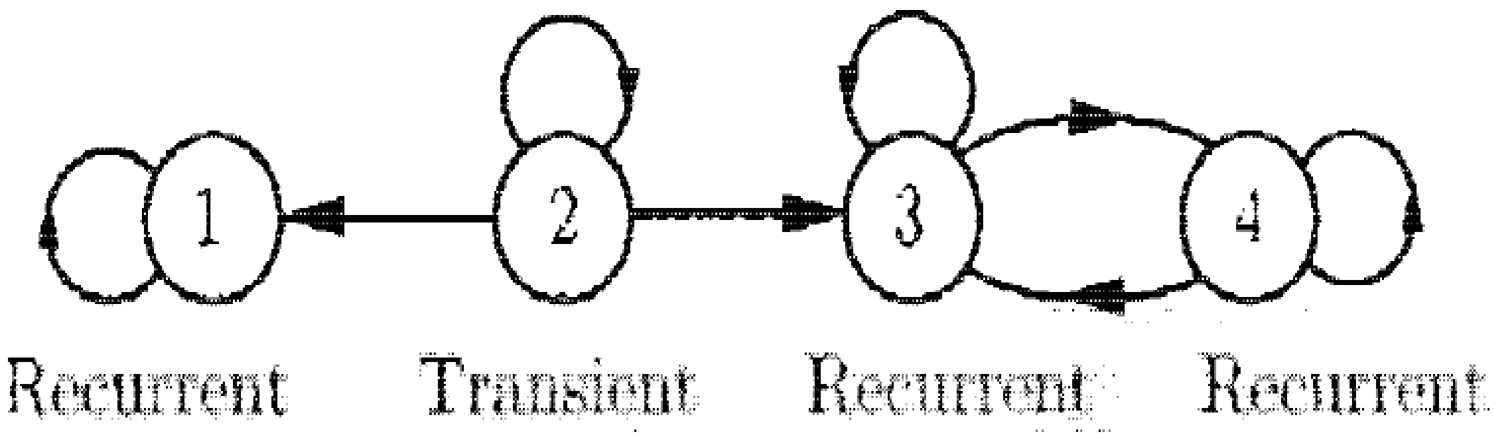
Classification of states

- Consider a Markov process with state space $\mathcal{S}$
- We say that j is accessible from i if $p_{ij}^n > 0$ for some n.
- This is denoted by $i \to j$.
- if $i \to j$ and $j \to i$ then we say that i and j communicate.
  This is denoted by $i \leftrightarrow j$.

> A chain is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$.

Recurrent and Transient states

- We say that a state i is recurrent if $F_{ii} = P($ ever returning to i having started in i $) = 1$.
- F_{ii} is not easy to calculate. (We will see this after quiz)
- If a state is not recurrent, it is transient.
- For a transient state i, $F_{ii} < 1$.
- If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.



First recurrence probabilities

- Define: $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{n}$.
- f_{ii}^{n} : probability of starting in i and returning to state i for the first time exactly after n steps.
- $f_{ii}^{n} := P(X_n = i, X_k \neq j \text{ for } 1 \leq k \leq n-1 | X_0 = i)$. $(f_{ii}^{0} = 0)$.
- F_{ii} has the interpretation of the probability of ever returning to state i.
- If $F_{ii} = p < 1$, then there is a finite probability $1 - p$ with which you may not return to state i .
- If $F_{ii} = 1$, then the from i you can certainly return to i.
- For any $i \in \mathcal{M}$, the first return time T_{ii} has the probability mass function $\{f_{ii}^{n}, n \geq 0\}$.