MA 6.101
Probability and Statistics

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Towards $E[g(X, Y)]$

- What about $E[aX + bY + c]$?

$$
\begin{aligned}
E[aX + bY + c] &= \sum_{x,y} (ax + by + c) p_{XY}(x, y) \\
&= a \sum_{xy} x p_{XY}(x, y) + b \sum_{xy} y p_{XY}(x, y) \\
&+ c \sum_{xy} p_{XY}(x, y) \\
&= aE[X] + bE[Y] + c.
\end{aligned}
$$

- Along similar lines, one would expect:

$$\boxed{E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{XY}(x, y)}$$

Finding $p_Z(\cdot)$ where $Z = g(X, Y)$.

- Suppose $Z = g(X)$. Then what is $p_Z(z)$?
- $p_Z(z) = \sum_{\{x:g(x)=z\}} p_X(x)$.
- Now suppose $Z = g(X, Y)$ then we have

$$p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x, y)$$

$E[g(X,Y)]$

- How do we define $E[g(X,Y)]$?
- One way is to define $Z = g(X,Y)$ and find $E[Z] = \sum_z z p_Z(z)$
- Recall $p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x,y)$
- This gives us $E[Z] = \sum_z \sum_{\{x,y:g(x,y)=z\}} z p_{XY}(x,y)$.
- This is same as $E[g(X,Y)] = \sum_{\{x,y\}} g(x,y) p_{XY}(x,y)$.

$E[g(X,Y)] = \sum_{xy} g(x,y) p_{XY}(xy)$

$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dxdy$ (for continuous r.v)

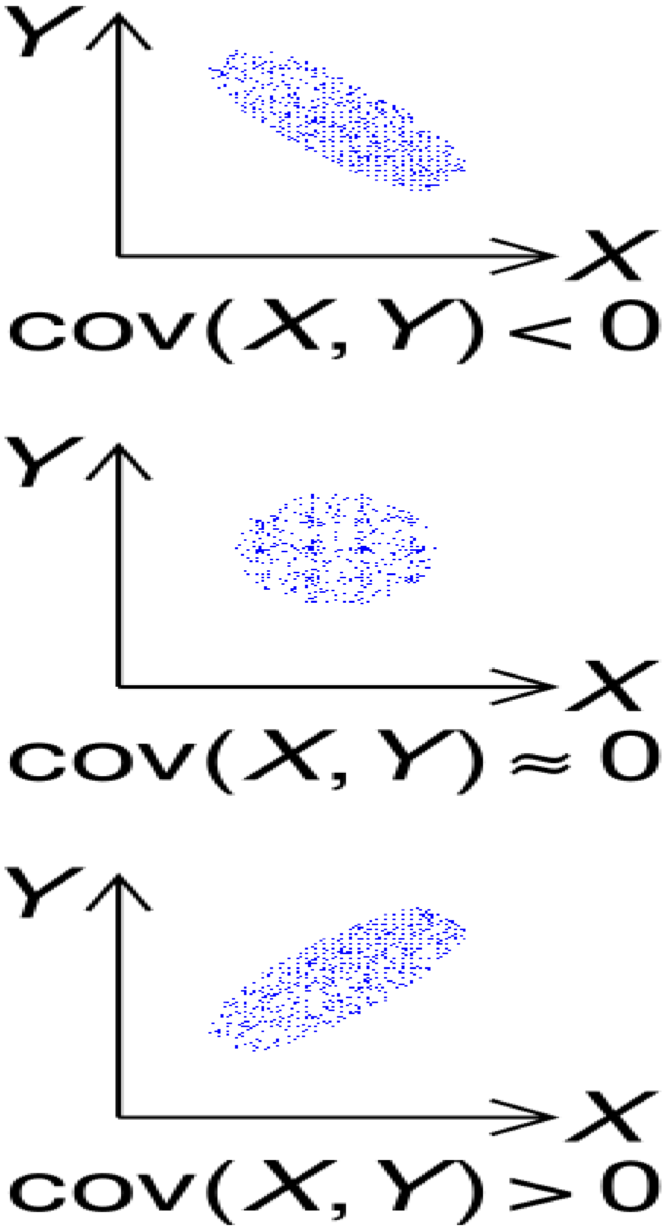
Example of function of Random variables

- Suppose X and Y are continuous independent random variables. Let $W = max(X, Y)$ and $Z = min(X, Y)$ Find the CDF and pdf of Z.
- HW: When X and Y are exponential with parameters λ_1 and λ_2 then Z is also exponential with parameter $\lambda_1 + \lambda_2$.

Covariance of X and Y

- $Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY]-E[X]E[Y]$.
- When Covariance is zero, they are said to be uncorrelated.
- (Section 4.2 Bertsekas)
- `https://en.wikipedia.org/wiki/Covariance`

Covariance of X and Y



Conditioning with random variables

Today's class

- Conditioning X on an event $A \in \mathcal{F}$.
- Conditional Expectation $E[X|A]$.
- Conditioning X with disjoint partitions $\{A_i\}$ of Ω.
- Conditioning X on an event $\{X \in A\} \in \mathcal{F}'$
- Conditioning X on another random variable Y.
- Conditional expectation $E[X|Y = y]$.

A new running example

- Pick 2 integers from $\{1,2,3\}$ without replacement.
- $\Omega = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$
- $\mathbb{P}\{\omega\} = \frac{1}{6}$ for all $\omega \in \Omega$.
- Denote them by random variables X and Y.
- For $\omega = (1,3)$ $X(\omega) = 1$ and $Y(\omega) = 3$.
- Write down their joint PMF $p_{X,Y}(x,y)$.
- Write down their marginal PMFs p_X and p_Y?
- What is $E[X], E[Y]$ and $E[XY]$?

Conditioning on an event A

- Consider a discrete r.v. X with pmf $p_X(x)$. Suppose an event A has happened where $A \in \mathcal{F}$.
- Consider event $\{\omega \in \Omega : X(\omega) = x\}$. We will use shorthand $\{X = x\}$.
- What is $\mathbb{P}(X = x|A)$? $\mathbb{P}(X = x|A) = \frac{\mathbb{P}(\{X=x\}\cap A)}{\mathbb{P}(A)}$.

$$p_{X|A}(x) := \frac{\mathbb{P}(\{X=x\}\cap A)}{\mathbb{P}(A)}.$$

- $p_{X|A}(x)$ denotes the conditional PMF of X under event A.
- In the running example say A is the event that the first number is odd and second is even. $A = \{(1,2),(3,2)\}$. Compute $p_{X|A}(\cdot)$.
- How do we know that it is consistent, i.e., $\sum_x p_{X|A}(x) = 1$?