CS 302.1 - Automata Theory

Lecture 10

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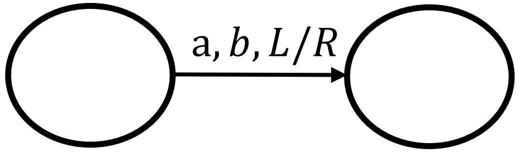
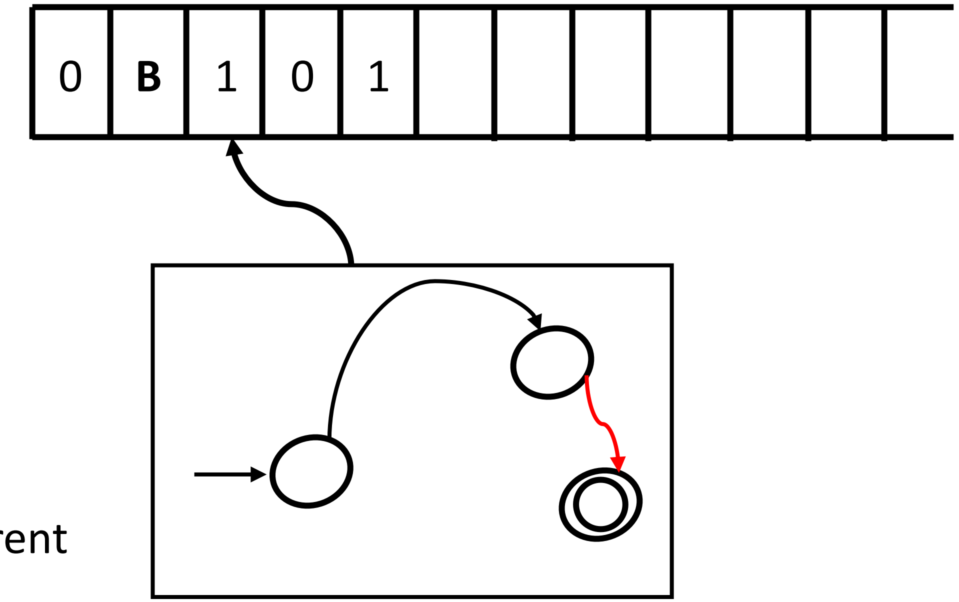
Quick Recap



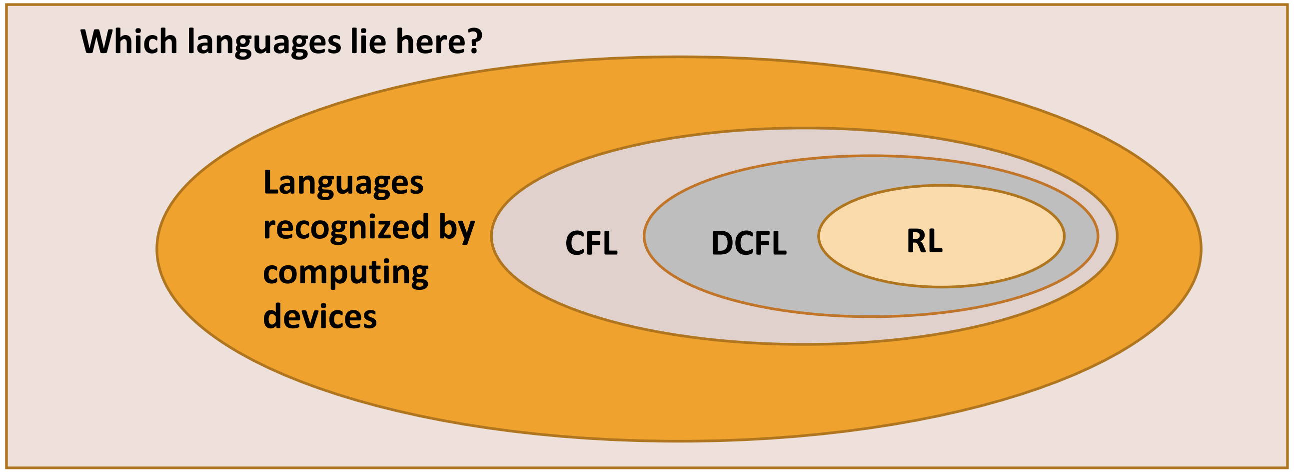


Turing Machines:

- TM **halts** and **accepts/rejects** on reaching the **accept** or **reject** states
- TM may never halt – it may loop forever.

Configuration of a TM: Combination of the current tape contents, the current state and the current head location.

$$X\ 0\ 0\ \underset{\uparrow q_1}{1}\ 1\ 1\ B\ B\ B\ B\ \dots$$



A TM ***M*** **accepts *w*** if there exists a sequence of configurations C_1 to C_k, where

- C_1 is the start configuration M on w.
- Each C_i yields C_{i+1}.
- C_k is an accepting configuration

Variants of Turing Machine models

A TM model $\mathcal{M}_1$ **is equivalent to another model** $\mathcal{M}_2$:

- $\mathcal{M}_2$ **can be simulated by** $\mathcal{M}_1$ **and vice versa**.

Is the standard TM M more powerful/equivalent to the following TM models where

- The head can move left, right or stay put?
- We have k read/write tapes instead of one?
- We had a two-way infinite tape, instead of one?
- TM has a printer attached?
- We introduced non-determinism?

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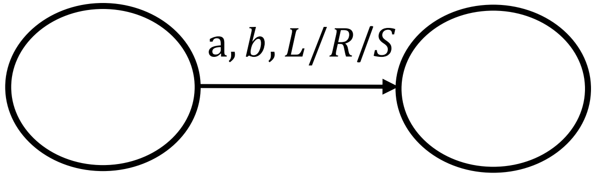
Lazy Turing Machine: The head can either move left, move right or stay put (L, R, S)



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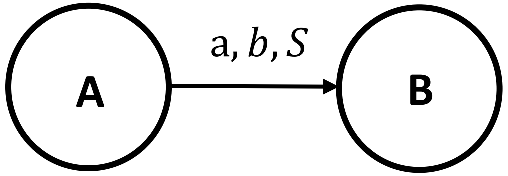
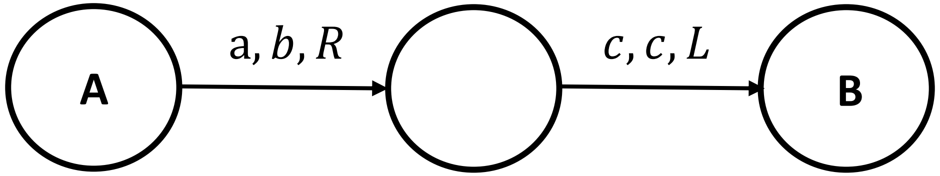


For each $c \in \Gamma$:



Hence a *lazy Turing machine* model **is equivalent** to a standard Turing Machine model.

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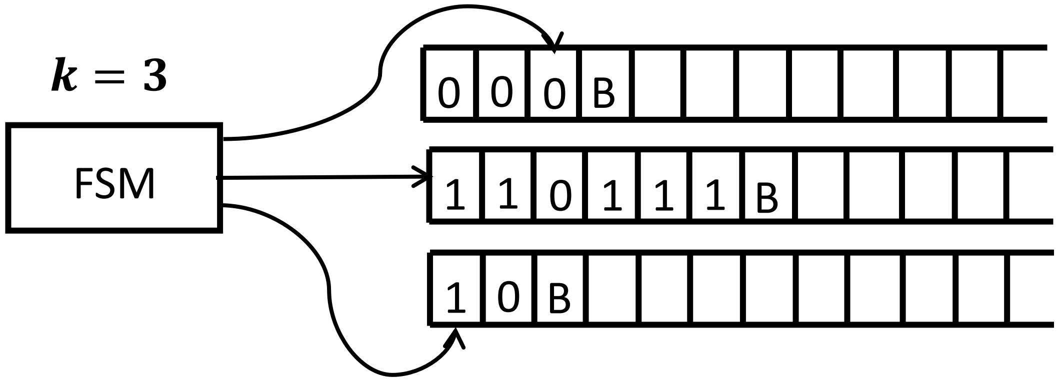
A TM model $\mathcal{M}_1$ is equivalent to another model $\mathcal{M}_2$ if $\mathcal{M}_2$ can be simulated by $\mathcal{M}_1$ and vice versa.

k-read/write tapes instead of one: What does a k-tape TM S look like? k-tape TM also has k heads, each associated with a tape. New transition function $\delta_S: Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$, i.e. $\delta_S(q_i, a_1, \cdots, a_k) = (q_j, b_1, \cdots, b_k, L, R, \cdots, L)$



- To simulate S with M, we store the entire information of the k tapes in one single tape.
- M uses \$ to separate the contents of the k tapes.
- To keep track of the locations of the k heads, M marks the symbols where the heads would be, with a '_'.

Variants of Turing Machine Models

Is the standard TM M more powerful/equivalent to the following TM models where

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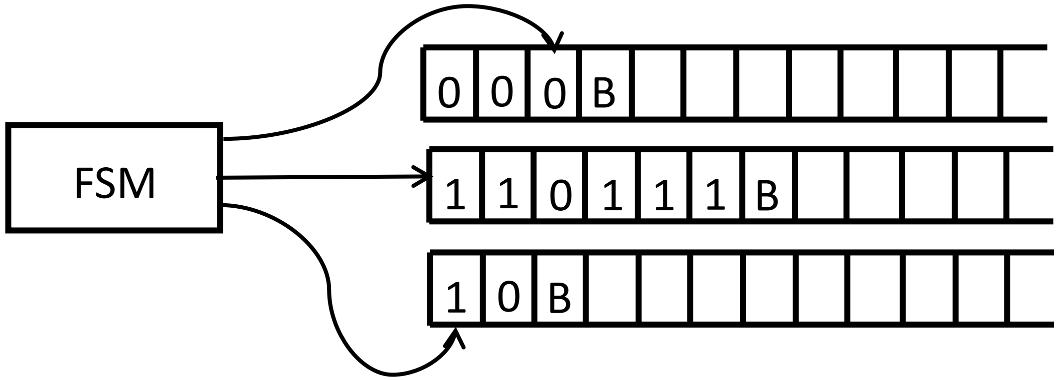
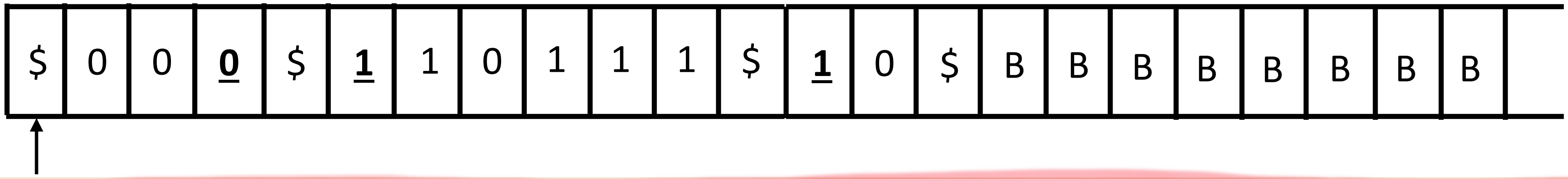


k-read/write tapes instead of one: What does a k-tape TM S look like? k-tape TM also has k heads, each associated with a tape. New transition function $\delta_S: Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$, i.e. $\delta_S(q_i, a_1, \cdots, a_k) = (q_j, b_1, \cdots, b_k, L, R, \cdots, L)$



- Single tape TM M **first scans the entire tape from leftmost** \$ **to rightmost** \$ ($k+1$ in all) to determine the symbols under the virtual heads. Then it makes a second pass to update the tape according to δ_S.

- If it so happens that ***M*'s head needs to go to the right of any of the intermediate \$** $\Rightarrow S$ has moved the head on the corresponding tape to the unread blank symbols. Starting from this cell to the rightmost \$, **shift one cell to the right to make space to write a blank on the empty tape cell** and simulate as before.

Variants of Turing Machine Models

Is the standard TM model $\mathcal{M}_1$ more powerful/equivalent to the following TM models where

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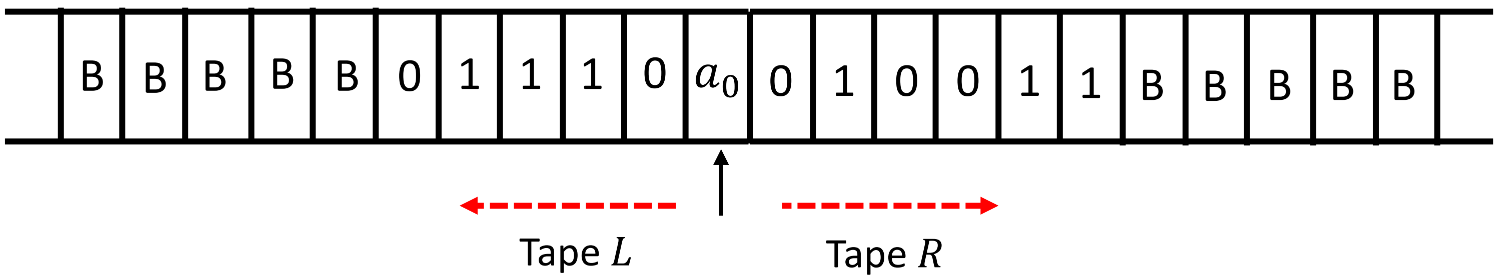
Two-way infinite Tape: Let M_D be the TM equipped with this power.



A TM model $\mathcal{M}_1$ is equivalent (as powerful as) to another model $\mathcal{M}_2$ if $\mathcal{M}_2$ can be simulated by $\mathcal{M}_1$ and vice versa.

- Cut the two tapes of M_D into Tape R and $(\text{Tape } L)^R$. We get a two-tape TM.
- Whenever M_D uses the tape to the right of the a_0, Tape R is used.
- When M_D uses the tape to the left of a_0, $(\text{Tape } L)^R$ is used.

M_D isn't any more powerful than a one way infinite tape TM.

So a TM with a two-way infinite tape is equivalent to a TM with a one-way infinite tape.

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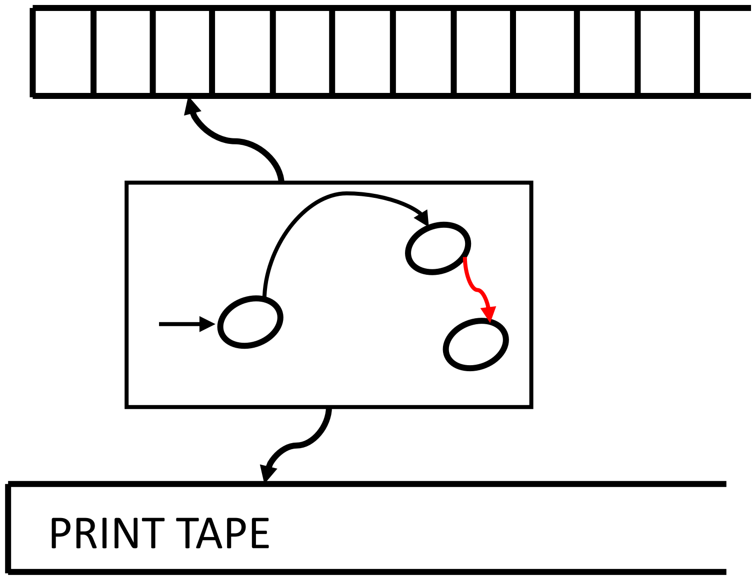
- **TM with a printer**
- We introduce non-determinism?

Enumerators: TM attached with a printer

- The Enumerator E uses the print tape to output strings
- The input tape is initially blank
- The language of E is the set of strings that it prints out
- If E does not halt, it may print infinitely many strings in some order

$$\mathcal{L}(E) = \{\text{w} \in \Sigma^* | \text{w is printed by E}\}$$



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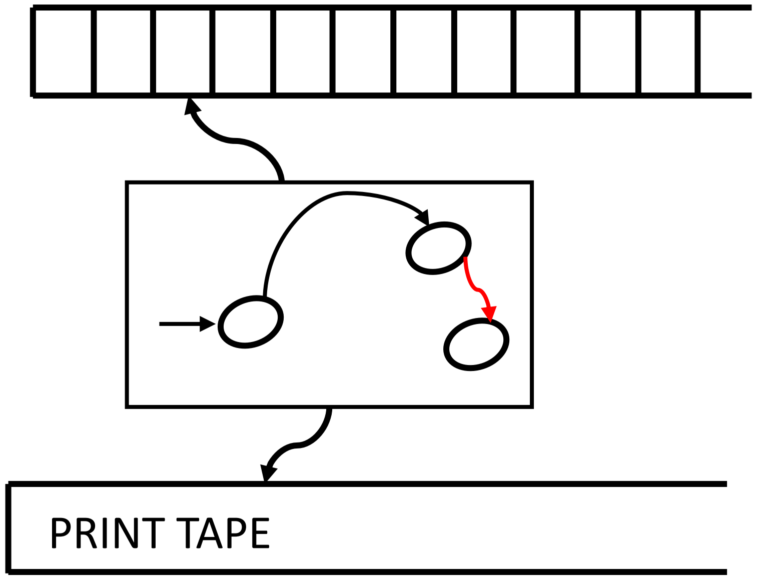

Enumerators: TM attached with a printer

If $L(M)$ is the language recognized by a Turing Machine M then there exists an enumerator E that enumerates it.

The set of all finite length (binary) strings is **countably infinite**.

- Lexicographically generate all binary strings one after the other. There exists a one-one correspondence with $\mathbb{N}$.
- We can lexicographically generate all (binary) strings and number them:

$$s_1 = 0, s_2 = 1, s_3 = 00, \cdots$$

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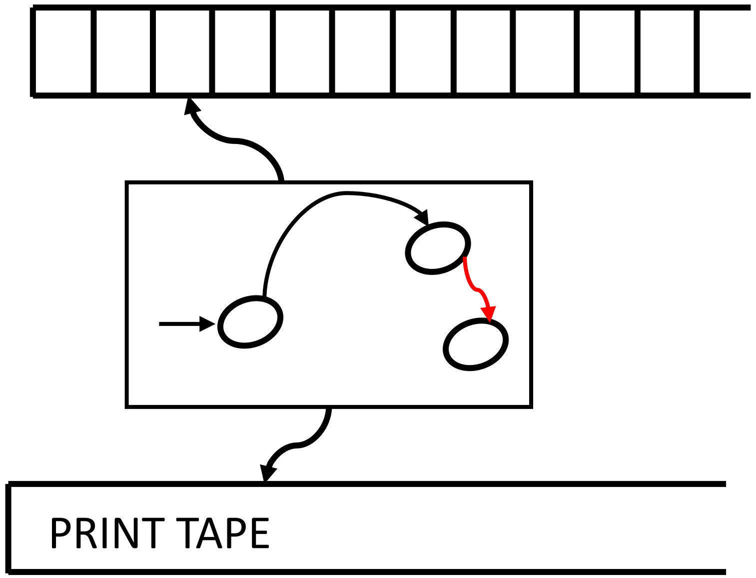

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If $L(M)$ is the language recognized by a Turing Machine M then there exists an enumerator E that enumerates it.

Proof:

For $i = 1, 2, \cdots$

For $j = 1, 2, \cdots, i$

Run M with string s_j for i steps.

If any string is accepted, then PRINT it.

Variants of Turing Machine models

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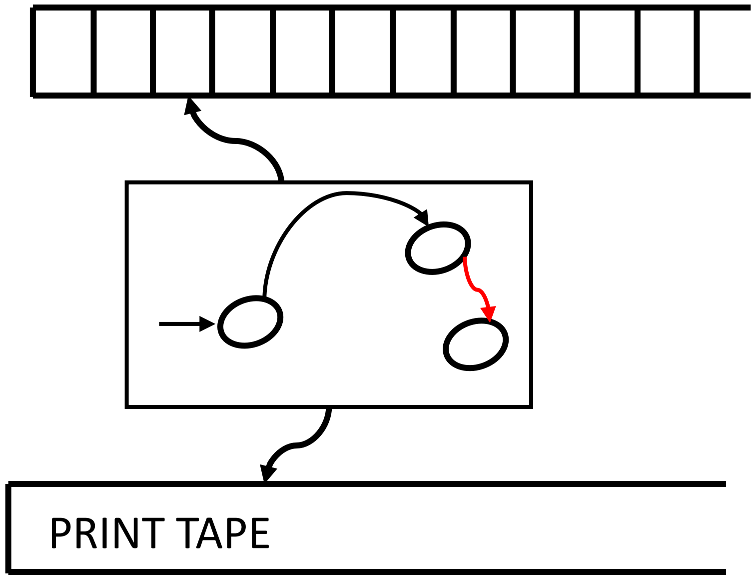

Enumerators: TM attached with a printer

If there exists an Enumerator E, then there exists a Turing Machine M such that $L(M) = L(E)$.

Proof:

$M =$ On input w:

1. Run E. Every time E prints some string, compare it with w.
2. If they match, ACCEPT.

E and M are equivalent

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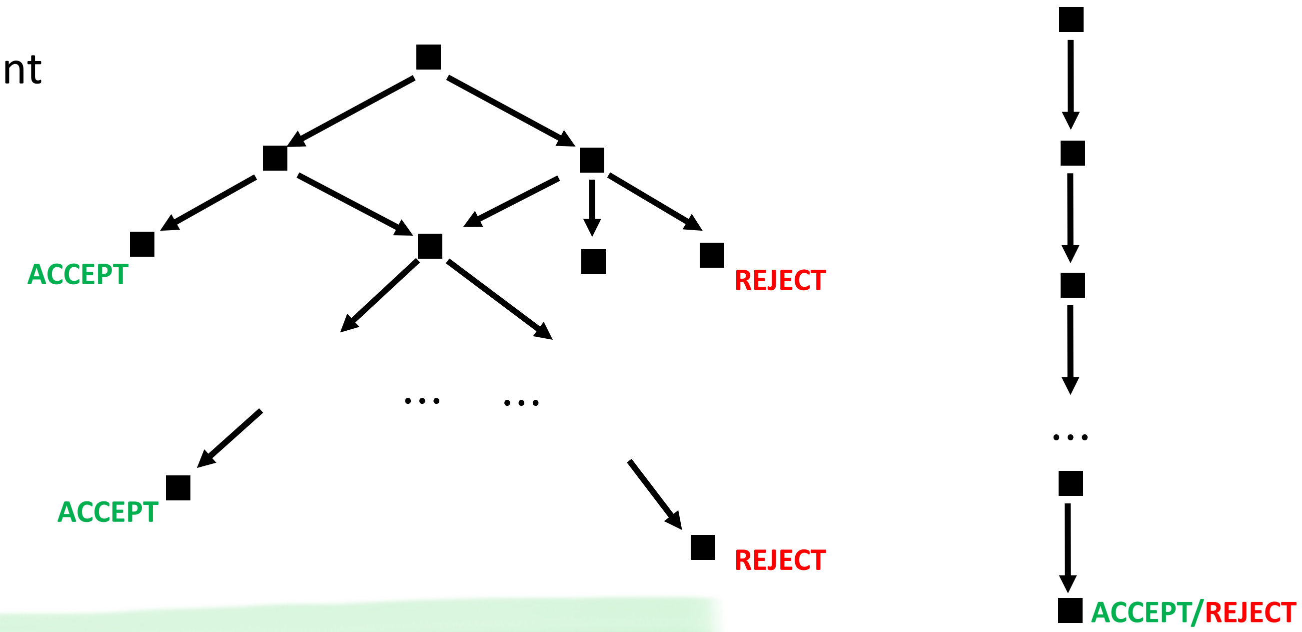

Non-deterministic Turing Machines (NTM):

- For a DTM, from a given configuration, exactly one configuration available to it at any stage.
- For an NTM, any point in the computation, several possible configurations are available.
- Its transition function is $\delta_N: Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$, i.e. $\delta(q_i, a) \rightarrow \{(q_j, b, R), (q_k, c, L) \cdots\}$
- The computation corresponds to a configuration tree: From the starting configuration, the computation has several branches, each of which leads to a different configuration.
- If any branch of the computation, leads to an accepting configuration, the NTM accepts. Immediately, DTM is a special case of NTM.

Are NTMs more powerful than DTMs? No. Any NTM can be simulated by a DTM. How?

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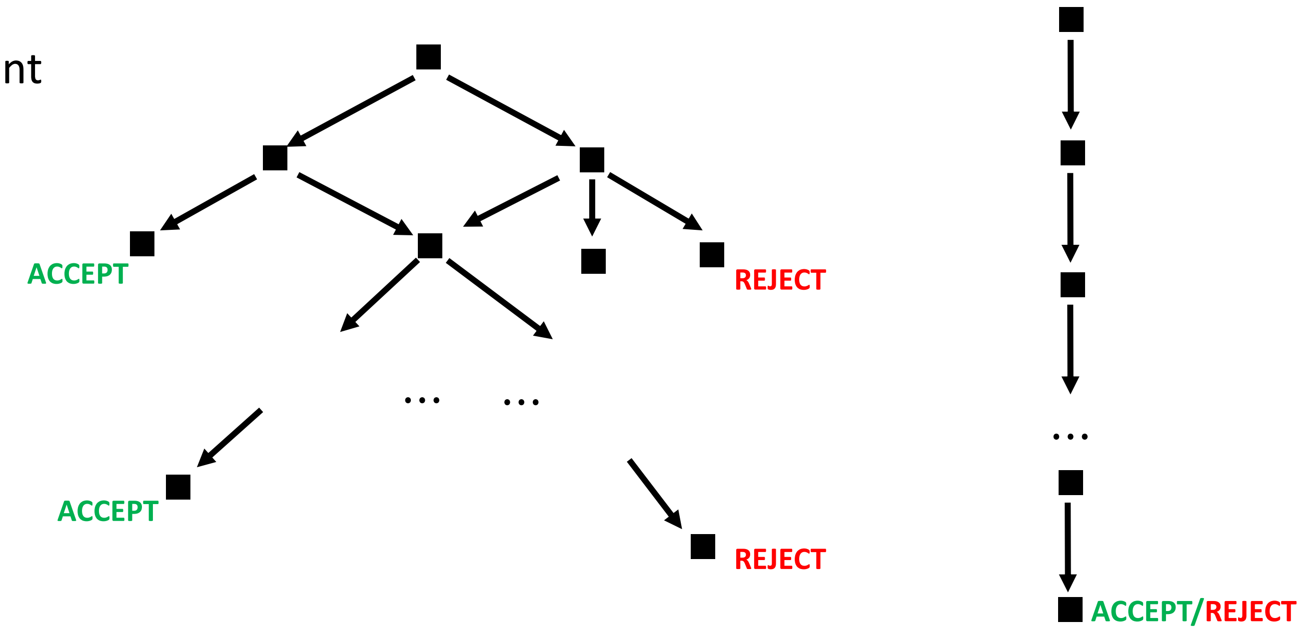
Any NTM can be simulated by a DTM



- The DTM searches from among the configurations of the NTM for an accepting configuration.
- Clearly an ordinary Depth First Search shouldn't work
- A branch of the configuration tree can be of infinite depth (when the TM loops forever for that sequence of configuration) and hence the DTM can miss a much shorter accepting configuration.

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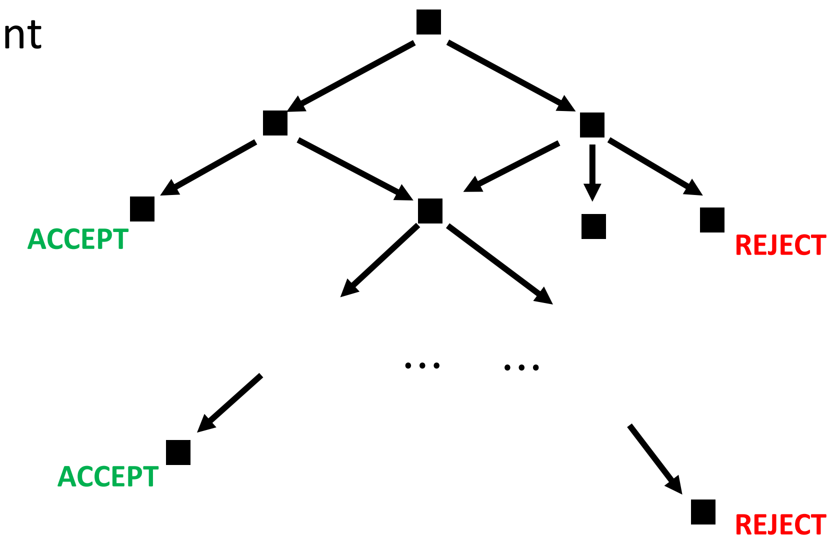
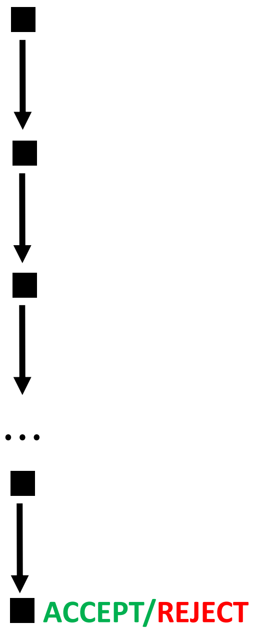
Any NTM can be simulated by a 3-tape DTM





Input string w

Generate runs lexicographically

Simulate the run for i/p w

- Tape 1 holds the input string w.
- As for the content of Tape 2, note that we can always obtain a bound for the maximum number of nodes at any level of the configuration tree (say b).
- Let $C = \{1,2,\cdots,b\}$, then we can define a run by a string $s \in C$. E.g: 122: choose the first node from level 1, second node from level 2, third node from level 3.

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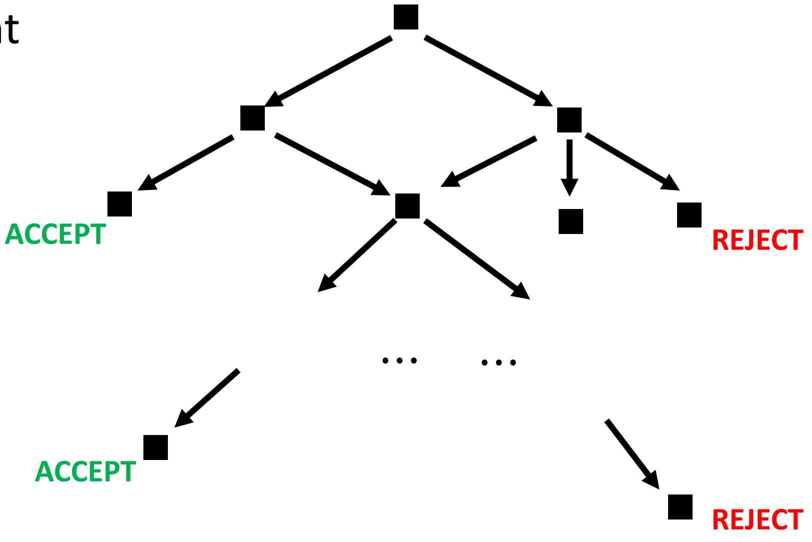

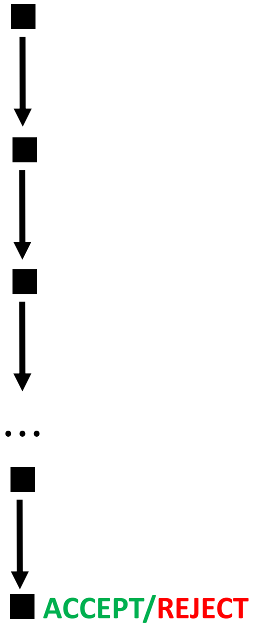

Any NTM can be simulated by a 3-tape DTM

$1100011BB$

121

Simulate the run for i/p w

- Tape 1 holds the input string w.
- Tape 2 generates a string in $C = \{1,2,\cdots,b\}$ lexicographically: Generate all strings of length 1, then strings of length 2 and so on, i.e. $\{1,2,\ldots,b,11,12,21,22,\ldots\}$.
- Some of these runs may be invalid or too short to lead to any accept/reject state.

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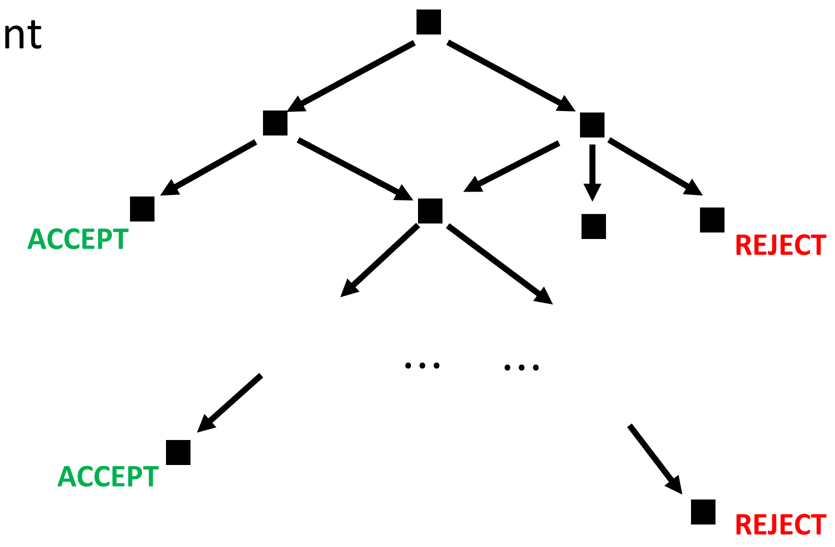
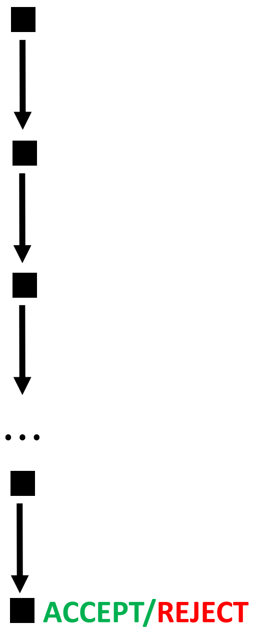
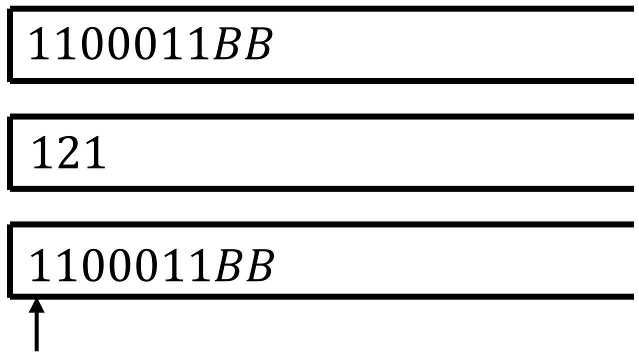




Any NTM can be simulated by a 3-tape DTM



- Tape 1 holds the input string w.
- Tape 2 generates runs lexicographically
- Tape 3 simulates one branch of the configuration tree corresponding to the run generated in Tape 2. At each step of the simulation consult Tape 2 to decide the next configuration.
- If the simulation leads to an accept state – accept the computation
- During the simulation, if the run in Tape 2 is found to be invalid, abort and generate the next lexicographic string

3-tape DTM $\equiv$ 1-tape DTM $\Rightarrow$ NTM $\equiv$ DTM

Variants of Turing Machine Models

Variants of TM:

- **The head can move left, right or stay put?**
- **We have k read/write tapes instead of one?**
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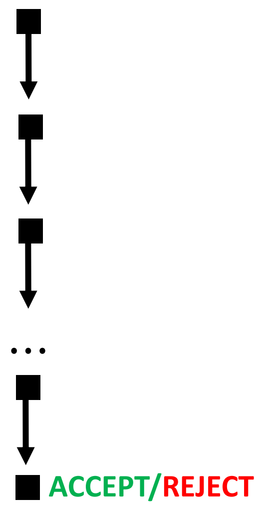
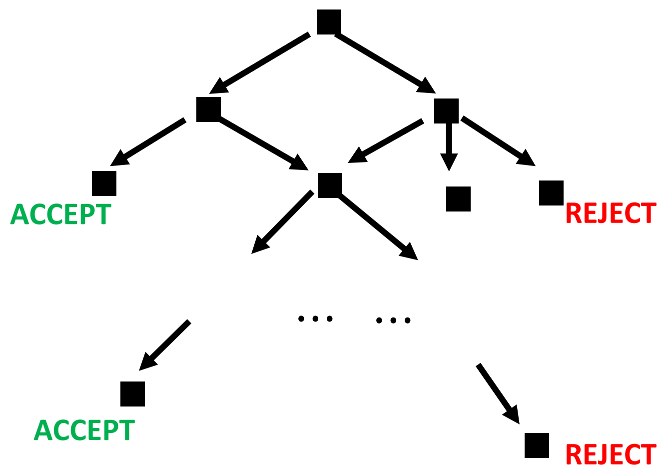
- **TM with a printer attached?**
- **We introduce non-determinism?**

Key takeaway: A standard TM is quite robust. Adding extra power seems to make no difference in computing power

Variants of Turing Machine Models

- As an aside, although $NTM \equiv DTM$, a DTM may require several more steps to perform the same computation.
- For a moment, consider problems that are computable (TM halts on all inputs).
- For a given decision problem L, let for all input strings $|w| = n$, suppose $\exists$
- **DTM that halts within $\boldsymbol{DTIME}$ steps and outputs YES if $\boldsymbol{w \in L}$ and NO if $\boldsymbol{w \notin L}$.**





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- **DTM that halts within $\boldsymbol{DTIME}$ steps and outputs YES if $\boldsymbol{w} \in \boldsymbol{L}$ and NO if $\boldsymbol{w} \notin \boldsymbol{L}$.**

$\boldsymbol{P}$ = Set of all problems for which $DTIME$ is a polynomial in n
(e.g.: $n^{1000} + 3n^2$)

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$\boldsymbol{NP} =$ Set of all problems for which $NTIME$ is a polynomial in n.

Variants of Turing Machine Models

- For a given decision problem L, let for all input strings w such that $|w| = n$, suppose $\exists$
- **DTM that halts within $\boldsymbol{DTIME}$ steps and outputs YES if $\boldsymbol{w \in L}$ and NO if $\boldsymbol{w \notin L}$.**
- **NTM that halts within $\boldsymbol{NTIME}$ steps and outputs YES if $\boldsymbol{w \in L}$ and NO if $\boldsymbol{w \notin L}$.**

$\boldsymbol{P}$ = Set of all problems for which $DTIME$ is a polynomial in n.

(e.g.: $n^{1000} + 3n^2$)

$\boldsymbol{NP}$ = Set of all problems for which $NTIME$ is a polynomial in n.

Clearly, $\boldsymbol{P \subseteq NP}$. However, is $\boldsymbol{P = NP}$?

A million dollar problem

Thank You!