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Assignment - 7

Ans 1) Given :- V and W be vector spaces over the Field F and let T be a linear Transformation from V into W. & V is finite dimensional

RTP :- rank (T) + nullity (T) = dim V

Proof :- Let r denotes the dimension of rank of T and nullity of T denoted by n is the dimension of the null space of T. $n = \dim N(T)$

Then we have to proove :-

$$\boxed{r + n = \dim V}$$

Let $\{\vec{\alpha_1}, \ldots \vec{\alpha_k}\}$ be a basis for N_T which is the null space of T.

Now there are $\vec{\alpha_{k+1}}, \ldots \vec{\alpha_n}$ vectors (in V) such that $\{\vec{\alpha_1}, \ldots \vec{\alpha_n}\}$ is a basis for V.

Now we need to proove that $\{\vec{T\alpha_{k+1}}, \ldots \vec{T\alpha_n}\}$ is a basis for the range of T, to complete our proof.

Firstly, we can observe this fact :

If $\{\vec{\alpha_1}, \ldots \vec{\alpha_n}\}$ is a basis for V then we know that any vector $\vec{\alpha} \in V$ can be

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expressed as a linear combination of these basis vectors with some scalar coordinates $x_1, \ldots x_n \in F$.

Furthermore, we know that linear transformation of vectors preserves the linear combination, that is

$$\boxed{T(x_1\vec{\alpha_1} + \ldots + x_n\vec{\alpha_n}) = x_1(T\vec{\alpha_1}) + \ldots x_n(T\vec{\alpha_n})}$$

So, $\Rightarrow T(\vec{\alpha}) = \vec{\beta}$ for any $\vec{\alpha} \in V$. Now, $\vec{\beta} \in W$ & more importantly $\vec{\beta} \in$ range(T)

Then, since linear combination is preserved, for any vector $\vec{\alpha} \in V$, if

$\vec{\alpha} = \sum_{i=1}^{n} x_i\vec{\alpha_i}$; then

$$\boxed{T\vec{\alpha} = \sum_{i=1}^{n} x_i T\vec{\alpha_i} = \vec{\beta}}$$

So, clearly $\vec{\beta}$ is a linear combination of the vectors

$\{T\vec{\alpha_1}, \ldots, T\vec{\alpha_n}\}$.

$\Rightarrow \overline{T\alpha_1}, \ldots \overline{T\alpha_n}$ span the range of T

Now as $\{\vec{\alpha_1}, \ldots \vec{\alpha_k}\}$ is a basis for N_V,

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$$\Rightarrow \boxed{T\vec{\alpha_1} = \cdots = T\vec{\alpha_k} = \vec{0_W}}$$

Since this is true, we can clearly say that

$T\vec{\alpha_{k+1}}, \cdots T\vec{\alpha_n}$ span the range of T — ①

Now, this is true, since $0(T\vec{\alpha_{k+1}}) + \cdots + 0(T\vec{\alpha_n}) = \vec{0_W}$ which is in the range as well.

Now to show that $T\vec{\alpha_{k+1}}, \cdots, T\vec{\alpha_n}$ are linearly independent vectors:

Suppose, we have the scalars c_i such that:

$$\boxed{\sum_{i=k+1}^{n} c_i (T\vec{\alpha_i}) = \vec{0_W}}$$

Clearly, this implies that $T\left(\sum_{i=k+1}^{n} c_i \vec{\alpha_i}\right) = \vec{0_W}$

since linear transformation preserves linear combinations.

From the above Statement it is clear that $\sum_{i=k+1}^{n} c_i \vec{\alpha_i}$ is in the null space of

T.

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Since $\vec{\alpha_1}, \dots \vec{\alpha_k}$ form a basis for N_V, there must be some scalars $b_1, \dots b_k$ such that:

$$\boxed{\sum_{i=k+1}^{n} c_i \vec{\alpha_i} = \sum_{j=1}^{k} b_j \vec{\alpha_j}}$$

$$\Rightarrow \boxed{\sum_{j=1}^{k} b_j \vec{\alpha_j} - \sum_{i=k+1}^{n} c_i \vec{\alpha_i} = \vec{O_w}}$$

& since $\vec{\alpha_1}, \dots \vec{\alpha_n}$ are linearly indep. (as they are basis vectors for V);

$$\Rightarrow \boxed{b_1 = \dots = b_k = c_{k+1} = \dots = c_n = 0}$$

$\Rightarrow$ Thus, $\sum_{i=k+1}^{n} c_i (\vec{T\alpha_i}) = \vec{O_w}$, only if

$c_{k+1} = \dots = c_n = 0$;

$\Rightarrow$ $\vec{T\alpha_{k+1}}, \dots, \vec{T\alpha_n}$ are linearly independent vectors.

Earlier, we also proved that they span the range of T (from ①)

$\Rightarrow \{\vec{T\alpha_{k+1}}, \dots, \vec{T\alpha_n}\}$ are the basis vectors of the range of T.

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Now to summarise :-

i) $\{\vec{\alpha_1}, \ldots \vec{\alpha_K}\}$ are the basis vectors of N_V (null space of T)

$\Rightarrow \boxed{\dim(N_V) = \text{Nullity}(T) = \eta}$

ii) $\{\vec{\alpha_1}, \ldots, \vec{\alpha_n}\}$ are the basis vectors of V

$\Rightarrow \boxed{\dim(V) = K}$

iii) $\{\vec{\alpha_{K+1}}, \ldots, \vec{\alpha_n}\}$ are the basis vectors of range (T)

$\dim\{\text{range}(T)\} = \text{rank}(T) = \nu = k - \eta$

from these,

$\nu = K - \eta$

$\nu + \eta = K$

$\boxed{\text{rank}(T) + \text{Nullity}(T) = \dim(V)}$

Hence Prooved.

Ans 2 Given :- V be an n-dimensional vector space over the Field F and let W be an m-dimensional vectorspace over F. Then the space $L(V, W)$ is finite-dimensional and has dimension

mn.

RTP:- $\dim L(V, W) = \dim(V) \times \dim(W)$

Proof :-
Let $\beta = \{\vec{\alpha_1}, \ldots \vec{\alpha_n}\}$ be the ordered basis for the vector space V, and let $\beta' = \{\vec{\beta_1}, \ldots, \vec{\beta_m}\}$ be the ordered basis for the vector space W.
Now, since $\dim(V)$ is not necessarily equal to $\dim(W)$, let us define a linear Transformation $E^{p,q}$ from V to W as follows,

$$E^{p,q}(\vec{\alpha_i}) = \begin{cases} 0, & \text{if } i \neq q \\ \vec{\beta_p}, & \text{if } i = q \end{cases}$$

where $1 \leq p \leq m$
$1 \leq q \leq n$

Now, according to the theorem:

"If $\{\vec{\alpha_1}, \ldots, \vec{\alpha_n}\}$ is an ordered basis for vector space V & $\vec{\beta_1}, \vec{\beta_2}, \ldots \vec{\beta_n}$ are any n vectors in W, there there is precisely one linear tranformation $T: V \to W$ such that

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$$T(\vec{\alpha_j}) = \vec{\beta_j} \quad \forall j \in \{1, \ldots n\}.$$

Such a linear transformation $E^{p,q}$ is unique.

We need to show that the ~~non~~ mn linear transformations $E^{p,q}$ form a basis for $L(V, W)$.

Now, let $T: V \to W$ be a linear Transformation.
For each j; (where $1 \le j \le n$), let $A_{1j}, \ldots A_{mj}$ be the coordinates of the vector $T\vec{\alpha_j}$ in the ordered basis β', that is:

$$\boxed{T\vec{\alpha_j} = \sum_{p=1}^{m} A_{pj} \vec{\beta_p}}$$

Now, let $U: V \to W$ be the linear tranformation defined by
~~$U\vec{\alpha_j}$~~

$$U\vec{\alpha_j} = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\vec{\alpha_j})$$

$$= \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \delta_{jq} \vec{\beta_p}$$

$$= \sum_{p=1}^{m} A_{pj} \vec{\beta_p} = T\vec{\alpha_j}$$

(Since, if $j \ne q$, $\delta_{jq} = 0$, So, $\sum_{q=1}^{n}$ is effectively meaningless It is $\sum_{q=1}^{n} A_{pq} \delta_{jq} = A_{pj}$)

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where δ_{jq} is the Kronecker delta

$$\delta_{jq} = \begin{cases} 0, & \text{if } j \neq q \\ 1, & \text{if } j = q \end{cases}$$

$\therefore$ $U = T$ and thus,

$$\boxed{T\vec{\alpha_j} = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{p,q}(\vec{\alpha_j})} \quad \text{--- } (1)$$

$(\forall \vec{\alpha_j} \in V)$

Hence from eq (1) we can clearly observe that any linear transformation $T \in L(V, W)$ can be written as a linear combination of the linear transformations $E^{p,q} \in L(V, W)$.

$\therefore$ $\boxed{\text{The } E^{p,q} \text{ linear tranformations span } L(V, W)}$

Now, all we need to show is that they are linearly independent.

Observe; In the definition of $U\vec{\alpha_j}$ above, if U is the zero transformation, then

$$\Rightarrow \forall \vec{\alpha_j} \in \beta,\ U\vec{\alpha_j} = \vec{0_W}$$

$$\Rightarrow \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{p,q}(\vec{\alpha_j}) = \vec{0_W} \quad \forall\, j \in \{1, \ldots, n\}$$

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$$\Rightarrow \boxed{\sum_{p=1}^{m} A_{pj} \vec{\beta_p} = \vec{O_w}} \text{ ——— (2)}$$

from the derivation above $\forall j \in \{1, \dots n\}$

However $\beta' = \{\vec{\beta_1}, \vec{\beta_2}, \dots \vec{\beta_m}\}$ is know to be a basis vector set for the vector space $W \Rightarrow$ they are linearly independent.

Thus eq (2) holds only if $A_{pj} = 0$ $\forall p \in \{1, \dots m\}$ & $j \in \{1, \dots n\}$

So, the $E^{p,q}$ linear transformations are linearly independent

$\therefore$ The mn linear transformations E^{pq} ~~for~~ form a basis for $L(V, W)$ & thus $L(V, W)$ is a finite dim. vector space &

$$\dim(L(V,W)) = mn$$

Hence Proved