Poisson process

Lemma

Definition 1 $\implies$ *Definition 2*

Proof on board.

Lemma

Definition 2 $\implies$ *Definition 1*

Self Study: Refer Sheldon Ross, Stochastic processes, Theorem 2.1.1

Poisson Processes Definition 3

A ctsp $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

- $N(0) = 0$
- $N(t)$ is a counting process with stationary and independent increments
- X_i, the time interval between $i - 1$th and ith event is exponentially distributed with parameter λ.

Lemma

Definition 1/2 $\Longrightarrow$ *Definition 3*

Proof:

- What is $P(X_1 > t) =?$

$$P(X_1 > t) = P(N(0,t) = 0) = e^{-\lambda t}$$

- This implies $F_{X_1}(t) = P(X_1 \leq t) = 1 - e^{-\lambda t}$ and hence X_1 has exponential distribution.
- What is $P(X_2 > t | X_1 = s)$?

$$\begin{aligned} P(X_2 > t | X_1 = s) &= P(N(s, t+s] = 0 | X_1 = s) \\ &= P(N(s, t+s] = 0) \text{ (indep. increments)} \\ &= e^{-\lambda t} \text{ (stat. increments)} \end{aligned}$$

- This implies X_2 is exponential.Repeating the arguments yields the lemma.

Definition 3 $\implies$ Definition 1

Lemma

i.i.d exponential interarrival time implies $N(0,t)$ has Poisson distribution with rate λt.

- Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$
- If $S_n = t$, we say that the nth renewal happened at time t.
- $f_{S_n}(t) = \lambda \left[\frac{(\lambda t)^{n-1} e^{-\lambda t}}{n-1!} \right]$ and $F_{S_n}(t) = \int_{x=0}^{t} \lambda \left[\frac{(\lambda x)^{n-1} e^{-\lambda x}}{n-1!} \right] dx$

More on $F_{S_n}(t)$

- $F_{S_n}(t) = \int_0^t \lambda \left[\frac{(\lambda x)^{n-1} e^{-\lambda x}}{n-1!}\right] dx$
- Integration by parts $(u(x) = e^{-\lambda x},\ v'(x) = \lambda \left[\frac{(\lambda x)^{n-1}}{n-1!}\right])$

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

- $F_{S_n}(t) = \left[\frac{(\lambda x)^n e^{-\lambda x}}{n!}\right]_0^t - \int_0^t \left[\frac{-\lambda e^{-\lambda x}(\lambda x)^n}{n!}\right] dx$
- $F_{S_n}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} + F_{S_{n+1}}(t)$

$$F_{S_n}(t) - F_{S_{n+1}}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Relation between S_n and $N(t)$

$$N(t) = sup\{n : S_n \leq t\}$$

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

- $P\{N(t) \geq n\} = P\{S_n \leq t\}$
- $P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}.$
- $P\{N(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\}.$
- $P\{N(t) = n\} = Poisson(\lambda t).$

Lemma

Exponential interarrival times imply $N(t)$ has Poisson distribution with rate λt

Properties of Poisson Process (Self Study)

Merging: Merging two independent Poisson processes with rate λ_1 and λ_2 leads to a Poisson process with rate $\lambda_1 + \lambda_2$.

Splitting: If you label each event point of a Poisson(λ) process as type A or type B with probability p or $1 - p$ respectively, then Events of type A form a Poisson ($p\lambda$) process. Similarly Events of type B form a Poisson ($(1 - p)\lambda$) process.

Conditional distribution of Arrival times

Lemma

Given that 1 event of $P.P.(\lambda)$ has happened by time t, it is equally likely to have happened anywhere in $[0, t]$ i.e., $P\{X_1 < s | N(t) = 1\} = \frac{s}{t}$.

Proof.

$$\begin{aligned}
P\{X_1 < s | N(t) = 1\} &= \frac{P\{X_1 < s, N(t) = 1\}}{P(N(t) = 1)} \\
&= \frac{P\{N[0, s) = 1, N[s, t] = 0\}}{P(N(t) = 1)} \\
&= \frac{P\{N[0, s) = 1\} P\{N[s, t] = 0\}}{P(N(t) = 1)} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}
\end{aligned}$$

□

Conditional distribution of Arrival times

Doubt

Theorem 2.3.1 from Sheldon Ross

Lemma

Given that $N(t) = n$, the joint distribution of the arrival times of these n jobs is $\frac{n!}{t^n}$.

Proof: Self Study