RECAP

- Renewal Process $N(t)$ with arbitrary interarrival times
- Renewal equation: $m(t) = \int_0^t (1 + m(t - x))dF(x)$
- Stopping times and Wald's Equation:

$$E[\sum_{i=1}^{N} X_i] = ENEX$$

- Stopping times for renewal process is $N(t) + 1$
- $E[S_{N(t)+1}] = E[X](m(t) + 1)$

Time average versus Ensemble average

$$\bar{X}^{time-avg} = \lim_{t\to\infty} \frac{\int_0^t X(u,\omega)du}{t}$$

$$\bar{X}^{ensemble} = \lim_{t\to\infty} E(X(t))$$

For an ergodic process, $\bar{X}^{time-avg} = \bar{X}^{ensemble}$

- Consider a Markov coin (with unknown transition probabilities) and given a budget of 10,00,000 (10 lakh) tosses, how will you find the stationary probability of head?
- Exhaust all at once (time average)
- Perform 100 runs each of length 10000 and average across the last toss in each run! (ensemble average)

Renewal theorem

Lemma

- *With probability 1,* $\frac{N(t)}{t} \to \frac{1}{E[X_1]}$ *as* $t \to \infty$.
  (Proof hint:- $S_{N(t)} \leq t \leq S_{N(t)+1}$*)*

- $\frac{m(t)}{t} \to \frac{1}{E[X_1]}$ *as* $t \to \infty$.

See Sheldon Ross (Stochastic Processes, 2nd edition) Proposition 3.3.1 and Thm 3.3.4 for proof.
NOTE: $S_{N(t)+1} > t$. Taking Expectations on both sides, invoking Wald's lemma, and rearranging gives us $\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{E[X_1]}$

Renewal Reward theorem

- Consider a renewal process with interarrival times $X_i, i = 1, 2, \ldots$. Suppose a random reward Y_i is earned at the time of the ith arrival. While Y_i may depend on X_i, the pairs (X_i, Y_i) are independent and identically distributed.

- Let $Y(t)$ denote the total reward accrued till time t. Then $Y(t) = \sum_{i=1}^{N(t)} Y_i$.

Lemma

- *With probability 1,* $\frac{Y(t)}{t} \to \frac{E[Y]}{E[X]}$ *as* $t \to \infty$.

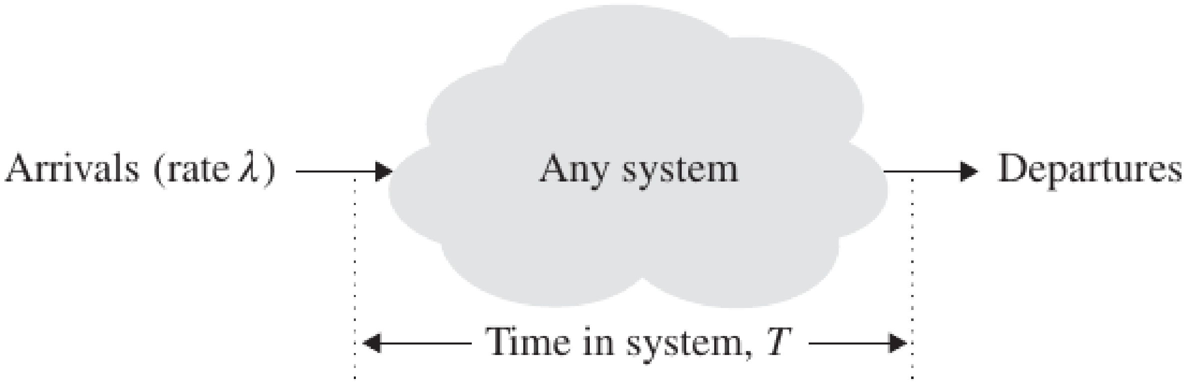
- $\frac{E[Y(t)]}{t} \to \frac{E[Y]}{E[X]}$ *as* $t \to \infty$.

See Sheldon Ross Theorem 3.6.1 for proof.

Renewal reward theorem – Application to $M/M/1/1$

On board

Little's law



$$E[N] = \lambda E[T]$$

$$\begin{bmatrix}\text{Avg. number of}\\ \text{people in Ikea}\end{bmatrix} = \lambda \begin{bmatrix}\text{Avg. time spent}\\ \text{per customer in Ikea}\end{bmatrix}$$