CS 3.307: Intro to Stochastic Processes

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Introduction to Stochastic processes

Introduction to Stochastic processes

- Stochastic process $\{X(t), t \in T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables defined such that for every $t \in T$ we have $X(t) : \Omega \to \mathcal{S}$.
- T is the parameter space (often resembles time) and $\mathcal{S}$ is the state space.
- When T is countable, we have a discrete time process. If T is a subset of real line, we have a continuous time process.
- State space could be integers or real numbers
- State space could even be $\mathbb{R}^n$ or $\mathbb{Z}^n$ valued

Examples of Stochastic Processes

- Sequence of i.i.d random variables.

- General random walk: If $X_1, X_2, \ldots$ is a sequence i.i.d of random variables, then $S_n = \sum_{i=1}^{n} X_i$ is a random walk.

- Weiner process: $\{X(t), t \geq 0\}$ is a Weiner process if
  1. $X(0) = 0$
  2. $\{X(t), t \geq 0\}$ has stationary and independent increments
  3. for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance t.

- $\{X(t), t \geq 0\}$ is a Markov process if for $t_1 < t_2 < \ldots t_n < t$ we have
  $P(X(t) \leq x | X(t_1) = x_1, \ldots, X(t_n) = x_n) = P(X(t) \leq x | X(t_n) = x_n)$

- Random walk and Weiner process are examples of Markov processes.

Introduction to Stochastic processes

A c.t.s.p. is called an *independent increment process* if for any choice of parameters $t_0 < t_1 < \ldots < t_n$, the n increment random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

The c.t.s.p. is said to have *stationary increments* if in addition $X(t_2+s)-X(t_1+s)$ has the same distribution as $X(t_2)-X(t_1)$ for all $t_1, t_2 \in T$ and any $s > 0$.

Counting process

Stochastic process $\{N(t), t \geq 0\}$ is a counting process if it represents the total number of events upto time t.

It satisfies the following

- $N(t) \geq 0$ and is integer valued
- For $s \leq t$, we have $N(s) \leq N(t)$. $N(t) - N(s)$ denotes the number of events in the interval (t, s)
- We will now look at some examples of couting processes that have independent and stationary increments.

Bernoulli/Binomial process

- Bernoulli(p) random variable represents an event.
- Bernoulli process is a sequence of independent r.v.'s $\{X_i, i = 1, 2, \ldots\}$ where X_i is a Bernoulli(p) random variable at time i.
- Let S_n counts the number of Bernoulli events by time n. S_n is essentailly sum of n independent Bernoulli(p) variables.
- $S_n = \sum_{i=1}^{n} X_i$ which denotes the number of event by time n and $P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$.
- $E[S_n]$? $\mathrm{Var}(S_n)$?

np

Binomial

$E[X^2] - (E[X])^2$

Bernoulli/Binomial process

First Success at n [Geom]

- $\{S_n = \sum_{i=1}^{n} X_i, n = 1, 2, \ldots\}$ is called as a Binomial process.
- Let $T := \{\text{smallest } n : S_n > 0.\}$.
- T is a geometric random variable with parameter p, i.e., $P(T = n_1) = p(1-p)^{(n_1 - 1)}$.
- Memoryless property: $P(T > m + n / T > n) = P(T > m)$.

Poisson process

A Poisson process with rate $\lambda, \lambda \geq 0$ is a counting process $\{N(t), t \geq 0\}$ with the following properties

- $N(0) = 0$
- $N(t)$ has independent increments
- Number of events in an interval of length t is a Poisson distribution with mean λt. (Hence stationary increments)
- $E[N(t+s) - N(t)] = \lambda s$

Condition 3 is difficult to verify ! Hence ...

Poisson process – Alternative definition

A function f is said to be $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

A Poisson process with rate $\lambda, \lambda \geq 0$ is a counting process $\{N(t), t \geq 0\}$ with the following properties

- $N(0) = 0$
- $N(t)$ has independent and stationary increments
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$