Recap: Modes of Convergence

$\{X_n, n \geq 0\}$ converges to X pointwise or surely if for all $\omega \in \Omega$ we have $\lim_{n\to\infty} X_n(\omega) = X(\omega)$

X_n converges to X almost surely if $P\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1$.

$\{X_n, n \geq 0\}$ is a sequence of i.i.d random variables with mean μ and $S_n = \sum_{i=1}^{n} X_i$. Then $\hat{\mu}_n := \frac{S_n}{n} \to \mu$ a.s. (SLLN)

- Estimator $\hat{\mu}_n$ has mean μ and Variance $\frac{\sigma^2}{n}$.
- $\hat{\mu}_{n+1} = \hat{\mu}_n + \frac{1}{n+1}\left[X_{n+1} - \hat{\mu}_n\right]$

Borel Cantelli Lemma

Self-Study: Theorem 7.5 (probabilitycourse.com)
Consider a sequence of random variables $X_1, X_2, \ldots$. If for all ϵ we have

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty$$

then $X_n \to X$ a.s.

- This is only a sufficient condition for almost sure convergence!
- Thm 7.6 (HW) gives necessary and sufficient conditions.
- Lot of problems in probabilitycourse, practice them!

Another example of a.s. convergence

- Consider a uniform r.v. U and define $X_n = n1_{\{U \leq \frac{1}{n}\}}$.
- $X_n = n$ when $U \leq \frac{1}{n}$ and $X_n = 0$ otherwise.
- Given a realization of U, what can you say about the sequence $\{X_n\}$?
- Once an X_n is zero, all higher indexed variables are also zero!
- This happens for all realizations U other than $U = 0$. In this case since $0 \leq \frac{1}{n}$ for all n, $X_n' s$ run off to infinity and we don't see convergence to 0.
- But $P(U = 0) = 0$.
- Does $E[X_n] \to 0$?
- Almost sure convergence does not imply their means converge!

Towards convergence in probability

- Now define $X_n = n 1_{\{U_n \le \frac{1}{n}\}}$ where $\{U_n\}$ are i.i.d uniform.
- $X_n = n$ when $U_n \le \frac{1}{n}$ and $X_n = 0$ otherwise.
- What can you say about the sequence $\{X_n\}$?
- Is it true that once an X_n is zero, all higher indexed variables are also zero!? No!
- Every time (on every run of the experiment or every sample path), we will have a sequence of zero and non-zero values, where the non-zero values become rarer and rarer but will keep happening once in a while.
- On no sample path would you see convergence to zero but occurrence of non-zero values become rare.
- We now characterize this notion of convergence.

Convergence in probability (w.h.p)

X_n converges to X in probability if

$$\lim_{n\to\infty} P\left(|X_n - X| > \epsilon\right) = 0 \text{ for all } \epsilon > 0.$$

- How would you compute $P\left(|X_n - X| > \epsilon\right)$ when X_n, X are either continuous or discrete random variables ?
- Ex: $X_n = n$ with probability $\frac{1}{n}$ and $X_n = 0$ otherwise.
- $P(|X_n - X| > \epsilon) = P(X_n > \epsilon) = \frac{1}{n}$ when $n > \epsilon$.
- When $n < \epsilon$, we have $P(|X_n - X| > \epsilon) = 0$.
- Once $n > \epsilon$ we have $\lim_{n\to\infty} P(X_n > \epsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$.
- X_n converges to 0 in probability, but not almost surely.
- a.s. convergence implies convergence in probability

Convergence in r^{th} mean

X_n converges to X in r^{th} mean if

$$\lim_{n\to\infty} E[|X_n - X|^r] = 0.$$

- How will you compute $E[|X_n - X|^r]$?
- When $r = 2$, it is convergence in mean squared sense. In addition if $X = 0$, it implies that the second moments converge to 0.
- In the convergence in probability example, do we have convergence in mean or mean square?
- Convergence in r^{th} mean implies convergence in probability.

Weak convergence (in distribution)

X_n converges to X in distribution if

$$\boxed{\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ for all continuity points of } F_X(\cdot).}$$

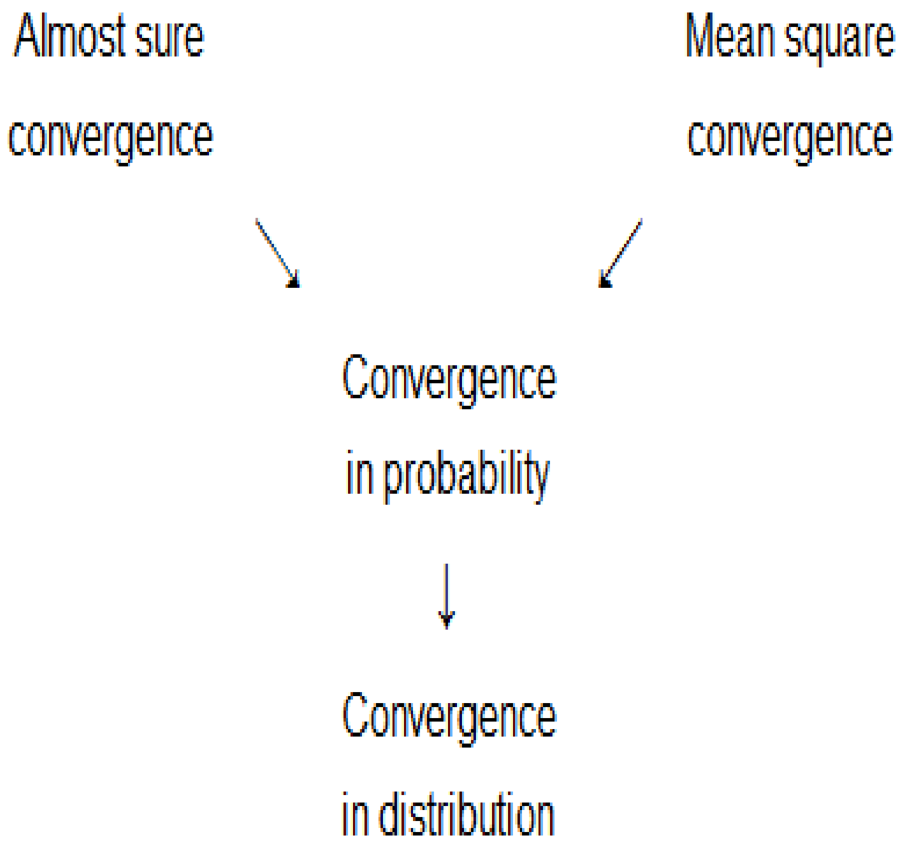
- a.s. convergence and convergence in probability imply convergence in distribution.
- Example: X_n is an exponential random variable with parameter λn.
- In this case, $F_{X_n}(x) = 1 - e^{-n\lambda x}$ and $F_X(x) = 1$ for all x.
- Note $x = 0$ is point of discontinuity as $F_X(0) = 1$ and $F_{X_n}(0) = 0$.
- HW EX2: X_n are i.i.d Binomial$(n, \frac{\lambda}{n})$. It converges in distribution to Poisson(λ).

Summary

Pointwise convergence	$\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for every ω
Almost sure convergence	$\lim_{n\to\infty} X_n(\omega) = X(\omega)$ almost surely
Convergence in probability	$\lim_{n\to\infty} \mathrm{P}(\lvert X_n - X\rvert > \epsilon) = 0$ for any $\varepsilon > 0$
Mean-square convergence	$\lim_{n\to\infty} \mathrm{E}[(X_n - X)^2] = 0$
Convergence in distribution	$\lim_{n\to\infty} F_n(x) = F(x)$ for any continuity point x

https://en.wikipedia.org/wiki/Convergence_of_random_variables

Relation between modes of convergence (no proofs)



https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables