

Sequence of sets

- ▶ Given (Ω, \mathcal{F}) , If $A_1 \subset A_2 \dots$ is an increasing sequence of events defined on \mathcal{F} and $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say that the sequence of sets A_n are increasing to A ($A_n \uparrow A$).
- ▶ Similarly when $A_1 \supset A_2 \dots$ is a decreasing sequence of events and $\bigcap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$.
- ▶ Alternative notation: For an increasing sequence of sets A_n we often write $\lim_{n \rightarrow \infty} A_n$ for $\bigcup_{n=1}^{\infty} A_n$ and for a decreasing sequence of sets A_n that $\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

Continuity of set-function \mathbb{P}

Lemma

For sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P}(A).$$

Proof

- ▶ Consider increasing sequence first. Similar arguments follow for decreasing seq.
- ▶ Define $F_n = A_n - A_{n-1}$
- ▶ $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} F_n$.
- ▶ $\mathbb{P}(A) = \mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} F_n)$
- ▶ But $\mathbb{P}(\bigcup_{n=1}^{\infty} F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{P}(F_i) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$. □

Equivalently if $A_n \rightarrow \emptyset$, then $\mathbb{P}(A_n) \rightarrow 0$.

Conditional probability

- ▶ Given/If dice rolls odd, what is the probability that the outcome is 1?
- ▶ Given/If $\bar{\omega} \in [0, 0.5]$ what is the probability that $\bar{\omega} \in [0, 0.25]$?

The conditional probability of event B given event A is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.

Conditional probability

- ▶ Show that $P(A/B)P(B) = P(B/A)P(A)$.
- ▶ Bayes rule: $P(B/A) = \frac{P(A/B)P(B)}{P(A)}$.
- ▶ What is $P(A/(B \cap C))$? This is also denoted as $P(A/BC)$
- ▶ Prove the chain rule
$$P(A \cap B \cap C) = P(A)P(B/A)P(C/(AB)).$$

HW: Prove the chain rule for conditional probability given by

$$P(A_1 \cap A_2 \dots A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2) \dots P(A_n/A_{n-1} \dots A_1).$$

Conditional probability – Examples

- ▶ Suppose you draw 4 cards from a deck at random without replacement. What is the probability that (in order) these cards are 9 of club, 8 of diamond, king of spade and king of club?
- ▶ What if you do the above with replacement?
- ▶ Consider a finite sample space Ω where each outcome is equally likely. Then what is $P(B/A)$?
- ▶ $P(B/A) = \frac{|A \cap B|}{|A|}$.

Law of total probability

- ▶ $A = (A \cap B) \cup (A \cap B^c)$. What is $P(A)$?
- ▶ $P(A) = P(A \cap B) + P(A \cap B^c)$.
- ▶ This is same as $P(A) = P(A/B)P(B) + P(A/B^c)P(B^c)$.
- ▶ This formula is useful when $P(A)$ is not given or is difficult to find but $P(B)$ or $P(A/B)$ is readily available.

Let B_1, B_2, \dots, B_n be the partition of the sample space Ω .
Then for any event A we have

$$P(A) = \sum_{i=1}^n P(A \cap B_i) = \sum_{i=1}^n P(A/B_i)P(B_i).$$

Example 1

- ▶ I have 3 bags that contain M marbles. Bag i has R_i red and B_i blue marbles respectively (for $i = 1, 2, 3$).
- ▶ I choose a bag at random and then draw a marble. What is the probability that the chosen marble is red ?
- ▶ Solution: $P(\text{Red}) = \sum_i P(\text{Red}/B_i)P(B_i)$

Example 2

1. If an item is defective, a robot can spot it with 98% accuracy.
 2. If an item is not defective, a robot will declare it so with 99% accuracy.
 3. A total of 0.1% items are defective.
 4. If the robot says that the item you drew at random is defective, what is the probability that the robot is correct?
- ▶ $P(\text{defective}/\text{robot says defective}) = \frac{P(\text{robot says defective}/\text{defective})P(\text{defective})}{P(\text{robot says defective})}$
 - ▶ What is $P(\text{robot says defective})$?

Bayes rule revisited

Let B_1, B_2, \dots, B_n be the partition of the sample space Ω .
Then for any event A with $P(A) > 0$ we have

$$P(B_j/A) = \frac{P(A/B_j)P(B_j)}{\sum_{i=1}^n P(A/B_i)P(B_i)}.$$

In the marble example, given that the marble drawn is red, what is the probability that bag 1 was chosen ?

Independence

- ▶ Consider the experiment of tossing a coin and a dice simultaneously.
- ▶ Identify its underlying probability space.
- ▶ What is $\mathbb{P}(\{H, 6\})$?
- ▶ What is $\mathbb{P}(\{T, \text{odd}\}) = \mathbb{P}(\{\cup_{i=1,3,5} \{T, i\}\})$?
- ▶ In both cases above we have $\mathbb{P}(A \cap B) = P(A)P(B)$.
- ▶ This implies that $\mathbb{P}(A/B) = \mathbb{P}(A)$.

- ▶ Two events A, B are independent iff $P(A/B) = P(A)$ and $P(B/A) = P(B)$.
- ▶ Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.

Independence

▶ Two events A, B are independent if and only if $P(A \cap B) = P(A)P(B)$.

- ▶ If A and B are independent, then are A^c and B^c independent?
- ▶ What about A and B^c ? Are they independent?
- ▶ If A_1, A_2, \dots, A_n are independent, then prove that

$$P(\cup_{i=1}^n A_i) = 1 - \prod_{i=1}^n [1 - P(A_i)]$$

Mutual and Pairwise Independence

- ▶ A collection of events $\{A_i, i \in I\}$ are said to be **mutually independent** if the $P(\cap_{j \in J} A_j) = \prod_{j \in J} P(A_j)$ for any subset J of I .
- ▶ A collection of events $\{A_i, i \in I\}$ are said to be **pairwise independent** if any pair of events from the collection are independent.
- ▶ Mutual independence implies pairwise independence but not the other way around.
- ▶ HW: Find an example where pairwise independence does not imply mutual independence.

Independence - Example

- ▶ Pick a number randomly from the set $\{1, \dots, 10\}$.
- ▶ Event A says that the number is less than 7.
- ▶ Event B says that the number is less than 8.
- ▶ Event C says that the number is even.
- ▶ Are the events mutually independent?
- ▶ Which pair of event is independent?

Correlation between events

- ▶ Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.
- ▶ Two events A and B are positively correlated iff $P(A/B) > P(A)$.
- ▶ Two events A and B are negatively correlated iff $P(A/B) < P(A)$.
- ▶ A and B have the same correlation as A^c and B^c .
- ▶ A and B have the opposite correlation as A and B^c .