Makeup Assignment for LA Ass-5

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Grp:- 4

Assignment-5

A1

RTP:- Supspace spanned by a non-empty Subset S of a vector space V is set of all linear combinations of Vectors in S.

Def:- The Subspace ~~spann~~ formed by intersection of all subspaces containing S is called span of a subspace.

Proof:- Let W be a span of the set S ($S \subseteq V$)

Now, since by definition W is intersection of all subspaces containing S, W is the smallest subspace which contains S.

Let $\alpha_1, \alpha_2, \ldots \alpha_n$ are elements of S ($\subseteq W$)
Now as W is a Subspace :,
if $\alpha_1, \alpha_2 \ldots \alpha_n \in W$
$c_1\alpha_1, c_2\alpha_2, \ldots c_n\alpha_n \in W$ ($c_1, c_2, \ldots c_n \in F$)
Aslo, $c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n \in W$
i.e, $\sum_{i=1}^{n} c_i\alpha_i \in W$

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Hence the Subspace W contains all the linear combinations of vectors of Subset S. (defined later to be L)

$$L \subseteq W \qquad \text{—— (1)}$$

where, L be the set of all Linear Combination of the set S.

hence, L contains S and is non-empty.

Let $\alpha, \beta \in L$

By definition of L,

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

$$\beta = y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n$$

$$(\alpha_i, \beta_i \in S) \ \& \ x_i, y_i \in F$$

For any Scalar c ($\in F$);

$$c\alpha + \beta = c\sum_{i=1}^{m} x_i\alpha_i + \sum_{j=1}^{n} y_j \beta_j$$

$$= \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{i=1}^{n} y_j \beta_j$$

Clearly $c\alpha+\beta$ is also a Linear Combination of vectors of set S

$\therefore$ $c\alpha + \beta \in L$

$\Rightarrow$ L is a subspace containing set S. (A-4 Q-3 Proof).

But by def of span, any subspace which contains S contains the span of S.

$$\therefore W \subseteq L \qquad \text{—— (2)}$$

From ① & ② we get;

$$W = L$$

∴ The Span of Subset S of V is the set of all linear combinations of all Vectors of S.

Hence, Prooved

Q2

Given :- W_1 and W_2 are finite dimensional Subspace of a Vector space V.

RTP :- (a) $W_1 + W_2$ is finite dimensional.

(b) $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

Proof :-

Assertions from Theorems / Corollaries :

(i) If W is a subspace of finite dimensional Vector space V, every linearly independent Subset of W is finite and is a part of a (finite) basis for W. or extended to basis

(ii) If W is a proper subspace of a finite dimensional vector space V then W is finite dimensional and $\dim W < \dim V$.

$$\left.\begin{array}{l} W_1 \cap W_2 \subseteq W_1 \\ W_1 \cap W_2 \subseteq W_2 \end{array}\right\} \quad W_1, W_2 \text{ be finite dimension Vector space}$$

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let $\{y^1, y^2, \ldots y^l\}$ be a basis of $W_1 \cap W_2$
Now as this is also a linearly independent subset of W_1 and W_2 so it can be extended to basis of W_1 and W_2 (From above assertions).

$\therefore$ Let $\{y^1, y^2, \ldots y^l, u^1, u^2, \ldots u^s\}$ be a basis of W_1
and $\{y^1, y^2, \ldots y^l, v^1, v^2, \ldots v^k\}$ be basis of W_2

Claim:- $B = \{y^1, y^2, \ldots y^l, u^1, u^2, \ldots u^s, v^1, v^2, \ldots v^k\}$ is a basis of $W_1 + W_2$.

Before Prooving our Claim let's see what happen if B is a basis of $W_1 + W_2$

$\Rightarrow \dim(W_1 + W_2) = \dim(B)$

$\dim(W_1 + W_2) = l + s + k$

$\Rightarrow W_1 + W_2$ is finite dimensional

$\Rightarrow$ Hence Prooved part(a)

Also $\dim(W_1) = l + s$

$\dim(W_2) = l + k$

and $\dim(W_1 \cap W_2) = l$

$\therefore \dim(W_1) + \dim(W_2) = (l + s) + (l + k)$

$= (l + s + k) + l$

$= \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$

Hence Prooved Part(B)

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Now all we to do is to proove our Claim:

RTP:- $B = \{y^1, y^2, \ldots y^e, u^1, u^2, \ldots u^s, v^1, v^2, \ldots v^k\}$ is a basis of $W_1 + W_2$

Proof:-
Now we can see the subspace $W_1 + W_2$ is spanned by set B.

because every element $w \in W_1 + W_2$ can be written as $w = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$

and w_1 & w_2 can be written as basis of W_1 and W_2 respectively
As B contain basis of both W_1 and W_2 any vector w can be expressed as linear combination of vectors of B

$\therefore$ B spans the space $W_1 + W_2$
or $Sp(B) = W_1 + W_2$ NB

Now we need to show that B is linearly Independent.

$$\underbrace{\alpha_1 y^1 + \ldots \alpha_l y^e}_{y} + \underbrace{\beta_1 u^1 + \ldots + \beta_s u^s}_{u} + \underbrace{\gamma_1 v^1 + \ldots + \gamma_k v^k}_{v} = 0$$

$$y + u + v = 0$$

Clearly, $y \in W_1 \cap W_2$

$u \in W_1$

$v \in W_2$

Re writing same equation,

$$\underbrace{y}_{(\in W_1 \cap W_2} + \underbrace{u}_{\in W_1)} = \underbrace{-v}_{(\in W_2)}$$

$$(\in W_1) \qquad (\in W_2)$$

(W_2 is a subspace $v \in W_2 \longleftrightarrow -v \in W_2$)

$\Rightarrow v \in W_1 \cap W_2$

$\Rightarrow v = \delta_1 y^1 + \delta_2 y^2 + \cdots + \delta_l y^l$

($\because y^1, y^2, \ldots y^l$ is basis of $W_1 \cap W_2$)

But, $\gamma_1 v^1 + \gamma_2 v^2 + \cdots + \gamma_k v^k = \delta_1 y^1 + \cdots + \delta_l y^l$

$\Rightarrow \delta_1 y^1 + \cdots + \delta_l y^l - \gamma_1 v^1 - \cdots - \gamma_k v^k = 0$

But as $\{y^1, \ldots y^l, v^1, \ldots v^k\}$ is a basis of W_2 and $\therefore$ linearly independent.

$\therefore \delta_1 = \delta_2 = \cdots = \delta_l = \gamma_1 = \cdots = \gamma_k = 0$

In particular,

$v = 0$

Also, $y + u = 0$

i.e, $\underbrace{y}_{\in W_1 \cap W_2} = \underbrace{-u}_{\in W_1}$

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So, i.e,

$$\alpha_1 y^1 + \alpha_2 y^2 + \dots + \alpha_l y^l + \beta_1 u^1 + \dots + \beta_s u^s = 0$$

So, as $\{y^1, \dots y^l, u^1, \dots u^s\}$ is linearly Ind.

$\therefore$, $\alpha_1 = \dots \alpha_l = \beta_1 = \dots \beta_s = 0$

In particular,

$$\alpha_1 = \dots = \alpha_l = \beta_1 = \dots \beta_s = \gamma_1 = \dots = \gamma_k = 0$$

$\Rightarrow$ B is linearly Independent.

$\Rightarrow$ B is a basis of $W_1 + W_2$

$\Rightarrow$ Hence our claim is true.

Hence Prooved

Ans 3: Given :- R be a non zero row reduced echelon matrix.

RTP :- Non zero row vectors of R form a basis for the row space of R

Proof :-

Let us say that the row reduced echelon matrix $R_{m\times n}$ has r non zero rows

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and $n-r$ zero rows. $(r \leq m)$

Also assume that the leading non zero entry of the rows occurs in Column $k_1, k_2, \dots k_r$ where

$$1 \leq k_1, k_2, k_3, \dots k_r \leq n$$

We have to show the set of r non zero row form basis of row space R.

Let us first show that the set of the non zero rows form a linearly independent set.

We know from definition of RREF of a matrix that the first non-zero entry (i.e 1) for each or r rows occurs in a diffrent Column and all other entries in those column are zero.
Choosing r scalars $c_1, c_2, \dots c_r$ where we we multiply each non zero row with corresponding scalar and add the rows.
$\Rightarrow$ we get a row matrix $(Y)_{1 \times n}$ in which the k_ith entry is the scalar c_i.

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Formally;

$$c_1R_1 + c_2R_2 + \dots + c_rR_r = Y$$

where R_i is the i^{th} non zero row of RRE matrix.

$\Rightarrow$ we also notice that $Y = 0$ row only if each of $c_i = 0$ (as c_i occupy the k_i^{th} column of Y)

Hence, $c_1R_1 + c_2R_2 + \dots + c_rR_r = 0$
$\Rightarrow c_1 = c_2 = \dots = c_r = 0$

Hence, the set of non-zero rows of R is a linearly independent set.

Now let us proove that this set of non-zero rows of R spans the entire row space of R.

The row space of R include all rows of R and all rows which can be produced by linear combinations of R.

Note :- we can trivially produce all rows of R by a linear combination of non-zero rows of R.

$$R_i = 1 \cdot R_i \quad (i \le r)$$

and $\quad R_i = 0 \cdot R_1 + 0 \cdot R_2 + \cdots + 0 \cdot R_r$ for $i > r$

$\rightarrow$ Zero Rows

Thus the row space of R includes the rows of all matrices which are row-equivalent to R.

A finite series of elementary row operations on the rows of R will produce any ~~se~~ matrix row-equivalent to R.
(rows of a row-equivalent matrix are linear combinations of the given matrix)

Let Y be a row of any such row-eq matrix:-

$$y = x_1 R_1 + x_2 R_2 + \cdots + x_r R_r + x_{r+1} R_{r+1} + \cdots + x_m R_m$$

But as, $R_{r+1}, \cdots R_m$ are zero rows,

$$\therefore y = x_1 R_1 + x_2 R_2 + \cdots + x_r R_r$$

Hence we can say that all rows of R and any matrix row-eq to R can be represented as a linear combn of non-zero rows of R.

Set of non-zero rows of R spans the

row space of R.

Hence, the set of non-zero rows of R forms a basis of row space of R.