# IIIT Hyderabad

Spring 2024 - Performance Modelling for Computer Systems
Assignment 1

## 1 Question 1

1. IIIT wants to build a new academic block which will stand for a long time by protecting it against earthquakes. If earthquakes strike Telangana as a Poisson process with rate $\lambda$ per year and we want the building to last for $Y$ years with probability $p$. What is the smallest number of earthquakes the new building should be able to endure? Calculate with values $\lambda = 0.1,\ Y = 20,\ p = 0.9$
   Solutions :-
   Let number of earthquakes that happen in Y years be N. Thus we need to find the smallest n such that $P(N > n) < p$.
   This can also be rewritten as $P(N \leqslant n) > 1 - p = 0.1$
   Now $P(N = k) = e^{-\lambda Y}(\lambda T)^k/k!$. Putting k=0, Y and $\lambda$. we get $P(N = 0) = e^{-2}$ which is around 0.13. Thus, the building does not need to survive even one earthquake to satify the requirments.

2. Before starting construction on the academic block, it was decided that simulations should be run to find the earthquake resistance of the planned project. The simulation had 2 types of earthquakes, one of magnitude 4.0 and magnitude 5.0 and rates $\lambda_1$ and $\lambda_2$ per year respectively. The building fell with probability $p_1$ if a magnitude 4.0 earthquake was applied and with probability $p_2$ if a magnitude 5.0 earthquake was applied. If the simulation ran for Y virtual years and Let total number of quakes encountered be N. Find $P\{N = n|Y = y\}$
   Solutions:- We are only concerned with the amount of earthquakes that have happened. Thus we can ignore the probabilities of the building falling down. Thus, We can combine the process of the two poisson process by the principle of merging. So earthquakes happen as a poisson process with combined rate $\lambda_1 + \lambda_2$. Using the formula for the poisson distribution, we get that $P(N = n|Y = y) = \frac{e^{-(\lambda_1+\lambda_2)y}(y(\lambda_1+\lambda_2))^n}{n!}$

3. There has been a lot of traffic on the road from the main gate to Bakul Niwas, as a result students have trouble crossing the stress. Sneha wants to cross the street but she will only cross it if there is no traffic for the next $T$ seconds on the road. Cars travel on the road as a poisson process with rate $\lambda$. What is the expected amount of time Sneha has to wait
   Solution :-
   First notice that if the first car comes after T seconds, she does not have to wait at all. Moreover, once the first car passes, the process is again at its initial state. Let W denote the total waiting time and let X denote the arrival time for the first car. Then by law of iterated expectation and conditioning on first arrival time, we have.

$$\begin{aligned}
E[W] &= E[E[W|X]] \\
&= \int_0^T (E[W] + x)\lambda e^{-\lambda x} dx \\
&= E[W] + \frac{1}{\lambda} - e^{-\lambda T}(T + E[W] + \frac{1}{\lambda}) \\
E[W] &= \frac{e^{\lambda T} - 1}{\lambda} - T
\end{aligned}$$

4. As we all know, no one comes on time for a class scheduled at 8:30 AM. Fed up with this, the instructor decided that there is a uniform probability that the doors will be closed sometime between 8:30 AM and 8:45 AM. If students always come to class late and as a poissson

process with rate $\lambda$, what is the expected number of people who will get attendance?
Solution :-
Let the time at which the doors are closed by given by $T$. Then the expected number of arrivals is $\lambda T$. Thus, the expected number of people who come to class before doors are closed is given by

$$\int_0^{15} \lambda t P(T = t) dt$$

Since $T$ is given by a uniform distribution with range from 0 to 15. We can rewrite this as

$$\begin{aligned} &\int_0^{15} \lambda t \frac{1}{15} dt \\ =&\frac{1}{15}\int_0^{15} \lambda t dt \\ =&\frac{1}{15}\frac{15^2}{2}\lambda \\ =&7.5\lambda \end{aligned} \tag{1}$$

5. $X(t)$ is known as a birth process with rate $\lambda_n,\ n \in N$ if :

$$\forall t, h > 0, X(t+h) \geqslant X(t)$$

$$P\{X(t+h) - X(t) = 1 | X(t) = n\} = \lambda_n h + o(h)$$
$$P\{X(t+h) - X(t) = 0 | X(t) = n\} = 1 - \lambda_n h + o(h)$$
$$P\{X(t+h) - X(t) > 1 | X(t) = n\} = o(h)$$

   (a) Find the conditions for this to be a poisson process.
   (b) Consider a process where $\lambda_n = \lambda n$. Define the generator matrix for this process.

   Solution:-

   (a) For this to be a Poisson process, $X(0) = 0$ and $\lambda_n = \lambda$ for all n.
   (b) When $\lambda_n = \lambda n$, then the generator matrix is defined as follows

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -2\lambda & 2\lambda & 0 & 0 & \cdots \\ 0 & 0 & -3\lambda & 3\lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

   Marks will be given if first row is defined as all zeroes as well.

6. A drunk man tosses a coin at a rate given by $\lambda$. If the coin comes up heads, he takes one step towards his home and one step back towards the bar if the coin comes up tails. The main does not do anything if the coin comes up tails and he is already at the bar. Construct the generator matrix for this process and also define the discrete time process embedded in this continuous time process.
   Solution:-
   By the splitting process, the tails and heads can be thought of as two Poisson processes with rates $\lambda/2$ each. Thus the man has rates of $\lambda/2$ of going forward and backward in all states except the first state. In the first state, the tails results is always discarded. Using the property that sum of non diagonal elements is equal to the magnitude of the diagonal element, we can construct the matrix as follows :-

$$\begin{bmatrix} -\lambda/2 & \lambda/2 & 0 & 0 & 0 & \cdots \\ \lambda/2 & -\lambda & -\lambda/2 & 0 & 0 & \cdots \\ 0 & \lambda/2 & -\lambda & -\lambda/2 & 0 & \cdots \\ 0 & 0 & \lambda/2 & -\lambda & -\lambda/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

Marks will be given if any assumption for the terminal state is done as well.
For the embedded discrete time process, clearly there is a probability of 1 from going to state 1 from state 0. For the rest of the states, there is a 1/2 chance of going to the next state and the previous state.

7. For the previous question, consider that his home is only 2 steps away. What is the expected time for him to get home?
Solution:-
Let $t_{ij}$ denote the expected time to go from state i to state j and let $R_{ij}$ denote the rate of transition from state i to j. Now we are interested to find $t_{02}$.
We know that $t_{02} = t_{01} + t_{12}$. This is because from state 0, we can only go to state 1. So expected time would just be the sum of expected time from 0 to 1 and then from 1 to 2. However,

$$t_{12} = P(\text{Transition to 0 from 1})(\frac{1}{R_{10}} + t_{02}) + P(\text{Transition to 2 from 1})(\frac{1}{R_{12}})$$

The probabilities of transitions are 1/2 and the rates are $\lambda/2$. Thus, we get

$$t_{12} = \frac{1}{2}(\frac{2}{\lambda} + t_{02}) + \frac{1}{2}\frac{2}{\lambda} = \frac{2}{\lambda} + t_{02}/2$$

Now, $t_{01}$ is also just $\frac{2}{\lambda}$. Thus, $t_{02} = \frac{2}{\lambda} + t_{12}$. Using this value of $t_{02}$, we get

$$t_{12} = \frac{3}{\lambda} + t_{12}/2$$

. Thus, $t_{12} = \frac{6}{\lambda}$. Thus, we get $t_{02} = \frac{8}{\lambda}$

8. For the previous question, once the man is at home, he has a 0.8 probability of staying at home and a probability 0.2 for going back one step if the coin comes up heads and going back one step if the coin comes up tails. Find the stationary distribution for this process.
Solution :-
When he is at home, he has a probability of 0.6 of going back and 0.4 for staying aftr every coin flip. Hence, his rate of leaving is $0.6\lambda$. Thus, the Q matrix is :-

$$\begin{bmatrix} -\lambda/2 & \lambda/2 & 0 \\ \lambda/2 & -\lambda & \lambda/2 \\ 0 & 0.6\lambda & -0.6\lambda \end{bmatrix}$$

On solving $\pi Q = 0$, we get $\pi_0 = \pi_1 = \frac{6}{17}$ and $\pi_2 = \frac{5}{17}$

9. Consider a CTMC with a probability transition matrix $P(t)$ given by

$$\begin{bmatrix} 1-e^t & e^t & 0 \\ e^{\frac{t}{2}} & 1-2e^{\frac{t}{2}} & e^{\frac{t}{2}} \\ e^{\frac{t}{4}} & 0 & 1-e^{\frac{t}{4}} \end{bmatrix}$$

Find its stationary distribution
Solution:-
We get the Q matrix by differentiating the given transition matrix at $t = 0$. We get

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

Solving $\pi Q = 0$, we get $\pi_0 = \pi_1 = \frac{1}{6}$ and $\pi_2 = \frac{2}{3}$

10. Consider a discrete time Markov chain $X_n$ with $X_0 = i$. Let $N$ be the total number of visits made by the chain to a state $j$. Prove that

$$P(N = n) = \begin{cases} 1 - F_{ij} & n = 0 \\ F_{ij}F_{jj}^{n-1}(1 - F_{jj}) & n \geqslant 1 \end{cases}$$

Here $F_{ij}$ denotes the probability of ever coming to state $j$ from state $i$.
Solution:-

**5.** Clearly $\mathbb{P}(N = 0) = 1 - f_{ij}$, while, by conditioning on the time of the $n$th visit to $j$, we have that $\mathbb{P}(N \geq n+1 \mid N \geq n) = f_{jj}$ for $n \geq 1$, whence the answer is immediate. Now $\mathbb{P}(N = \infty) = 1 - \sum_{n=0}^{\infty} \mathbb{P}(N = n)$ which equals 1 if and only if $f_{ij} = f_{jj} = 1$.