Here we consider a vector $\mathbf{P}$ of $n+1$ points (P_0, P_1, …, P_n), where $n \geq 1$. We explain below how to choose Bezier control points $\mathbf{P}_1$ and $\mathbf{P}_2$ so that the resulting set of Bezier curves produces a cubic interpolating spline of the data $\mathbf{P}$.

1 Single Segment

If $n = 1$, we only have a single interval to consider, and the associated Bezier curve is

$$B(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_{1,1} + 3(1-t)t^2 P_{2,1} + t^3 P_1 , \tag{1}$$

where $t \in [0,1]$, and P_0 is the first knot point, $P_{1,1}$ is the first control point (close to P_0), $P_{2,1}$ is the second control point (close to P_1) and P_1 is the second knot point.

If we have just two knot points, our "smooth" Bezier curve should be a straight line, i.e. in (1) the coefficients of t^2 and t^3 should be zero. It's easy to deduce that the control points satisfy

$$3P_{1,1} = 2P_0 + P_1 \quad \text{and} \quad P_{2,1} = 2P_{1,1} - P_0 .$$

2 Multiple Segments, open case

In the case where we have more than two points, our aim is to make a sequence of individual Bezier curves that produce a cubic interpolating spline, so we need to calculate Bezier control points so that the spline curve has two continuous derivatives at knot points.

Considering a set of piecewise Bezier curves with $n+1$ points and n subintervals. We will denote the points as follows: P_0 is the starting point, and then, for $i = 1, \ldots, n$, we have

- P_i is the i-th knot point
- $P_{1,i}$ is the "first" control point (close to P_{i-1})
- $P_{2,i}$ is the "second" control point (close to P_i)

The set of Bezier curves will thus be:

$$B_i(t) = (1-t)^3 P_{i-1} + 3(1-t)^2 t P_{1,i} + 3(1-t)t^2 P_{2,i} + t^3 P_i$$

for $i = 1, \ldots, n$.

The first and second derivatives of B_i are

$$B_i'(t) = -3(1-t)^2 P_{i-1} + 3(3t^2 - 4t + 1)P_{1,i} + 3(2t - 3t^2)P_{2,i} + 3t^2 P_i ,$$
$$B_i''(t) = 6(1-t)P_{i-1} + 6(3t-2)P_{1,i} + 6(1-3t)P_{2,i} + 6tP_i$$

for $i = 1, \ldots, n$. The continuity conditions $B_{i-1}'(1) = B_i'(0)$ and $B_{i-1}''(1) = B_i''(0)$ thus give

$$P_{2,i-1} = 2P_{i-1} - P_{1,i} \quad \text{and} \quad P_{1,i-1} + 2P_{1,i} = P_{2,i} + 2P_{2,i-1} \tag{2}$$

for $i = 2, \ldots, n$. We add two more conditions at the ends of the total interval. These are the "natural" conditions $B_1''(0) = 0$ and $B_n''(1) = 0$, which give

$$P_{2,1} = 2P_{1,1} - P_0 \quad \text{and} \quad P_{2,n} = \frac{P_{1,n} + P_n}{2} . \tag{3}$$

Now, we have $2n$ conditions (2)-(3) for the two vectors of n control points $\mathbf{P}_1$ and $\mathbf{P}_2$. Solving for $P_{2,i}$ and substituting reduces the system to a system of n equations for the n unknowns $\mathbf{P}_1$, namely

$$\begin{aligned}
2P_{1,1} + P_{1,2} &= P_0 + 2P_1 \\
P_{1,1} + 4P_{1,2} + P_{1,3} &= 4P_1 + 2P_2 \\
\vdots \quad & \quad \vdots \\
P_{1,i-1} + 4P_{1,i} + P_{1,i+1} &= 4P_{i-1} + 2P_i \\
\vdots \quad & \quad \vdots \\
P_{1,n-2} + 4P_{1,n-1} + P_{1,n} &= 4P_{n-2} + 2P_{n-1} \\
2P_{1,n-1} + 7P_{1,n} &= 8P_{n-1} + P_n \,.
\end{aligned}$$

Define $\mathbf{R}$ to be the right hand side vector: $R_1 = P_0 + 2P_1$, $R_i = 4P_{i-1} + 2P_i$ for $i = 2, \ldots, n-1$ and $R_n = 8P_{n-1} + P_n$. Then the above system is thus $A \cdot \mathbf{P}_1 = \mathbf{R}$, where

$$A = \begin{pmatrix}
2 & 1 & & & & & & \\
1 & 4 & 1 & & & & & \\
 & 1 & 4 & 1 & & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & 1 & 4 & 1 & \\
 & & & & & 1 & 4 & 1 \\
 & & & & & & 2 & 7
\end{pmatrix}$$

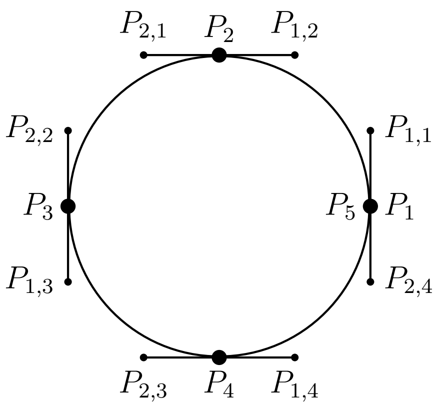
This last system is tridiagonal with diagonal dominance, hence we can solve it by (Gaussian) elimination. When P_1 is found, P_2 can be calculated from (2) and (5) through

$$P_{2,i} = 2P_i - P_{1,i+1} \quad \text{for } i = 1, \ldots, n-1 \quad \text{and} \quad P_{2,n} = \frac{P_n + P_{1,n}}{2} \,.$$

3 Multiple Segments, closed case

Here we consider points $\mathbf{P} = (P_1, P_2, \ldots, P_{n+1})$ with $P_{n+1} = P_1$. We will also consider that

- P_i is the i-th knot point
- $P_{1,i}$ is the "first" control point (close to P_i)
- $P_{2,i}$ is the "second" control point (close to P_{i+1})



for $i = 1, \ldots, n$ (the picture on the right considers the $n = 4$ case).

The set of Bezier curves will thus be:

$$B_i(t) = (1-t)^3 P_i + 3(1-t)^2 t P_{1,i} + 3(1-t)t^2 P_{2,i} + t^3 P_{i+1}$$

for $i = 1, \ldots, n$.

The first and second derivatives of B_i are now

$$\begin{aligned} B_i'(t) &= -3(1-t)^2 P_i + 3(3t^2 - 4t + 1)P_{1,i} + 3(2t - 3t^2)P_{2,i} + 3t^2 P_{i+1} \ , \\ B_i''(t) &= 6(1-t)P_i + 6(3t-2)P_{1,i} + 6(1-3t)P_{2,i} + 6tP_{i+1} \end{aligned}$$

for $i = 1, \ldots, n$. The continuity conditions are now

$$\begin{aligned} B_i'(1) = B_{i+1}'(0) \quad &\text{and} \quad B_i''(1) = B_{i+1}''(0) \quad \text{for} \quad i = 1, \ldots, n-1 \ , \\ B_n'(1) = B_1'(0) \quad &\text{and} \quad B_n''(1) = B_1''(0) \ . \end{aligned}$$

These conditions read

$$\begin{aligned} P_{2,i} &= 2P_{i+1} - P_{1,i+1} \quad \text{for} \quad i = 1, \ldots, n-1 \ , & (4) \\ P_{1,i} + 2P_{1,i+1} &= 2P_{2,i} + P_{2,i+1} \quad \text{for} \quad i = 1, \ldots, n-1 \ , & (5) \\ P_{2,n} &= 2P_1 - P_{1,1} \ , & (6) \\ P_{1,n} + 2P_{1,1} &= 2P_{2,n} + P_{2,1} \ . & (7) \end{aligned}$$

Inserting (4) with $i = 1$ and (6) into (7) gives

$$P_{1,n} + 4P_{1,1} + P_{1,2} = 4P_1 + 2P_2 \ ,$$

In the bulk ($i = 1, \ldots, n-2$), we have from (4) that

$$P_{2,i} = 2P_{i+1} - P_{1,i+1} \quad \text{and} \quad P_{2,i+1} = 2P_{i+2} - P_{1,i+2} \ ,$$

and so from (5), we get

$$P_{1,i} + 4P_{1,i+1} + P_{1,i+2} = 4P_{i+1} + 2P_{i+2} \quad \text{for} \quad i = 1, \ldots, n-2$$

or

$$P_{1,i-1} + 4P_{1,i} + P_{1,i+1} = 4P_i + 2P_{i+1} \quad \text{for} \quad i = 2, \ldots, n-1$$

Finally, for $i = n-1$, we have

$$P_{1,n-1} + 2P_{1,n} = 2P_{2,n-1} + P_{2,n} \ ,$$

which transforms into

$$P_{1,n-1} + 4P_{1,n} + P_{1,1} = 4P_n + 2P_1 \ .$$

To summarize, we now have

$$\begin{aligned} P_{1,n} + 4P_{1,1} + P_{1,2} &= 4P_1 + 2P_2 \\ P_{1,i-1} + 4P_{1,i} + P_{1,i+1} &= 4P_i + 2P_{i+1} \quad \text{for} \quad i = 2, \ldots, n-1 \\ P_{1,n-1} + 4P_{1,n} + P_{1,1} &= 4P_n + 2P_1 \ , \end{aligned}$$

and we can recover $\mathbf{P}_2$ by

$$\begin{aligned} P_{2,i} &= 2P_{i+1} - P_{1,i+1} \quad \text{for} \quad i = 1, \ldots, n-1 \ , \\ P_{2,n} &= 2P_1 - P_{1,1} \ . \end{aligned}$$

Setting now the right hand side vector $\mathbf{R} = (R_1, \ldots, R_n)$ as $R_i = 4P_i + 2P_{i+1}$ for $i = 1, \ldots, n-1$ and $R_n = 4P_n + 2P_1$, we get the system $A \cdot \mathbf{P}_1 = \mathbf{R}$, where

$$A = \begin{pmatrix} 4 & 1 & & & & & & 1 \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ 1 & & & & & & 1 & 4 \end{pmatrix} . \tag{8}$$

Consider now $\tilde{\mathbf{R}} = (R_1, \ldots, R_{n-1})$, $\tilde{\mathbf{P}}_1 = (P_{1,1}, \ldots, P_{1,n-1})$ and $\tilde{\mathbf{S}} = (-1, 0 \ldots, 0, -1)$ and[1]

$$\tilde{A} = \begin{pmatrix} 4 & 1 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 4 \end{pmatrix} .$$

The system $A \cdot \mathbf{P}_1 = \mathbf{R}$ can now be rewritten as

$$\tilde{A} \cdot \tilde{\mathbf{P}}_1 = \tilde{\mathbf{R}} + \tilde{\mathbf{S}} \cdot P_{1,n} \ , \tag{9}$$

$$P_{1,1} + P_{1,n-1} + 4P_{1,n} = R_n \ . \tag{10}$$

Owing to the linearity of (9), we can write $\tilde{\mathbf{P}}_1 = \tilde{\mathbf{B}} + \tilde{\mathbf{C}} \cdot P_{1,n}$, where $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ satisfy

$$\tilde{A} \cdot \tilde{\mathbf{B}} = \tilde{\mathbf{R}} \quad \text{and} \quad \tilde{A} \cdot \tilde{\mathbf{C}} = \tilde{\mathbf{S}} \ ,$$

two tridiagonal problems that can be solved by Gaussian elimination. We then substitute into (10), and get

$$B_1 + C_1 \cdot P_{1,n} + B_{n-1} + C_{n-1} \cdot P_{1,n} + 4P_{1,n} = R_n \ ,$$

which gives

$$P_{1,n} = \frac{R_n - B_1 - B_{n-1}}{4 + C_1 + C_{n-1}} \ ,$$

which can then be used with $\tilde{\mathbf{P}}_1 = \tilde{\mathbf{B}} + \tilde{\mathbf{C}} \cdot P_{1,n}$ to compute the other $P_{1,i}$ for $i = 1, \ldots, n-1$.

[1] here, $\tilde{A}$ is an $(n-1) \times (n-1)$ matrix and not $n \times n$ as A

4 LU decomposition

The $(n-1)\times(n-1)$ matrix $\tilde{A}$ can be decomposed as follows:

$$\tilde{A} = \underbrace{\begin{pmatrix} 1 & & & & & & \\ l_2 & 1 & & & & & \\ & l_3 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & & 1 & & \\ & & & & l_{n-2} & 1 & \\ & & & & & l_{n-1} & 1 \end{pmatrix}}_{=L} \cdot \underbrace{\begin{pmatrix} d_1 & 1 & & & & & \\ & d_2 & 1 & & & & \\ & & d_3 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & \\ & & & & & d_{n-2} & 1 \\ & & & & & & d_{n-1} \end{pmatrix}}_{=U} .$$

The coefficients l_i and d_i are determined through the forward recurrence relations

$$\begin{aligned} d_1 &= 4 \\ l_i &= \frac{1}{d_{i-1}} \quad \text{for } i = 2, \ldots, n-1 \\ d_i &= 4 - l_i \quad \text{for } i = 2, \ldots, n-1 \,. \end{aligned}$$

We then solve $\tilde{A} \cdot \tilde{\mathbf{P}} = \tilde{\mathbf{R}}$ through solving $L \cdot \mathbf{Y} = \tilde{\mathbf{R}}$ then $U \cdot \tilde{\mathbf{P}} = \mathbf{Y}$. The equation $L \cdot \mathbf{Y} = \tilde{\mathbf{R}}$ is solved by forward recurrence as

$$Y_1 = R_1 \quad \text{and} \quad Y_i = R_i - l_i \cdot Y_{i-1} \quad \text{for} \quad i = 2, \ldots, n-1 \,,$$

while the equation $U \cdot \tilde{\mathbf{P}} = \mathbf{Y}$ is solved by backward recurrence as

$$P_{n-1} = \frac{Y_{n-1}}{d_{n-1}} \quad \text{and} \quad P_i = \frac{Y_i - P_{i+1}}{d_i} \quad \text{for} \quad i = n-2, \ldots, 1 \,.$$

5 Other decomposition

In the cyclic symmetric case (8), to solve $Ax = b$, write $A = LDL^T$, solve $Lz = b$ for z (forward substitution), then solve $Dc = z$ for c then solve $L^T x = c$ for x (backward substitution). The matrix D is diagonal, the matrix L has the form

$$\begin{pmatrix} 1 & & & & & & \\ l_1 & 1 & & & & & \\ & l_2 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & & 1 & & \\ & & & & l_{n-2} & 1 & \\ c_1 & & \cdots & & c_{n-2} & l_{n-1} & 1 \end{pmatrix}$$