Exact Statistics and Semi-Parametric Tests for Small Network Data

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Context: Social abilities and team performance

Two research questions

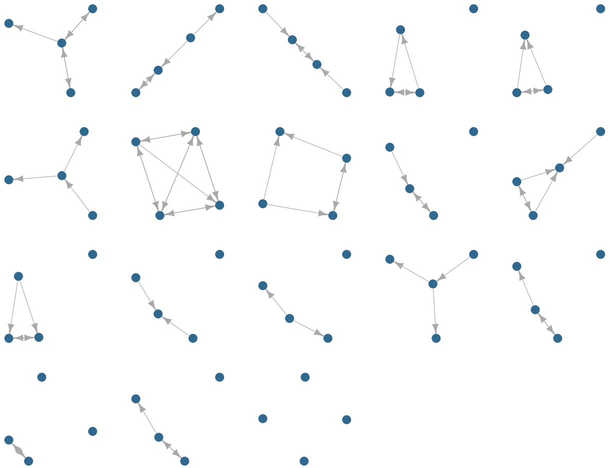
How do **social abilities** impact **network structure**?

How does **collective intelligence**collective intelligence affect team (network) **performance**performance?

To answer this question, we have the following experimental data:

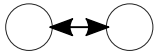
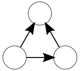
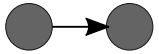
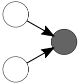
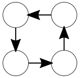
- 42 mixed-gender teams,
- Which completed 1 hour of group tasks (MIT's IQ test for teams)
- Individual survey capturing information regarding socio-demographics **and**:
  - **Social Intelligence**: Social Perception (measured by RME), Social Accomodation, Social Gregariousness, and Social Awareness
  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

Context (cont'd)



We can do a lot of simple statistics: density, % of *[blank]*, etc. but... **how can we go beyond that?**

Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1} \left(x_i = x_j \right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

ERGMs can do the job.

Exponential random graph models (a crash course)

A vector of
model parameters

A vector of
sufficient statistics

$$\Pr\left(\mathbf{Y} = \mathbf{y} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{y}' \in \mathcal{Y}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}', \mathbf{X}\right)\right\}}, \quad \forall \mathbf{y} \in \mathcal{Y}$$

Observed data

The normalizing
constant

All possible
networks

There is one problem with this model ...

A vector of model parameters

A vector of sufficient statistics

$$
\Pr\left(\mathbf{Y} = \mathbf{y} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{y}'\in\mathcal{Y}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}', \mathbf{X}\right)\right\}}, \quad \forall \mathbf{y} \in \mathcal{Y}
$$

Observed data

The normalizing constant

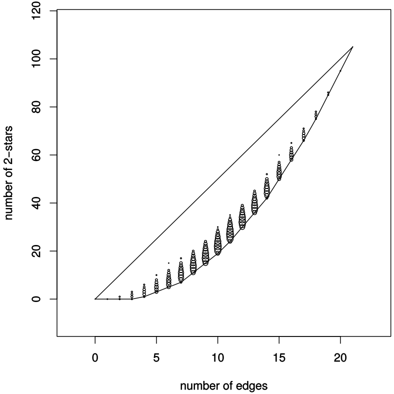
All possible networks

because of $\mathcal{Y}$, the **normalizing constant** is

a summation of $2^{n(n-1)}$ terms 😱!

Revising model degeneracy and existance of MLE

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE
- Not in the interior: **We are in trouble**, MLE may not exist

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$
\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}
$$

- So, if $n = 6$, then we have approx 1,000,000,000 terms 😱.
- This has lead the field to aim for (very neat) simulation based methods
- But, if our small networks have (at most) 6 nodes...

We can go back to the good-old-fashion MLE

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...obtain pooled estimates. Which helps with **power** *and* **degeneracy**)
- And more:
  - All MLE goodies, e.g., LRT
  - Enhanced simulation methods: resampling, cross-validation
  - Trivially extend ERGM: mixed-effects models, dependency structures across net
  - etc.

This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

(built on top of Statnet's amazing `ergm` Hunter et al. (2008); Handcock et al. (2018) R package)

Sidetrack...

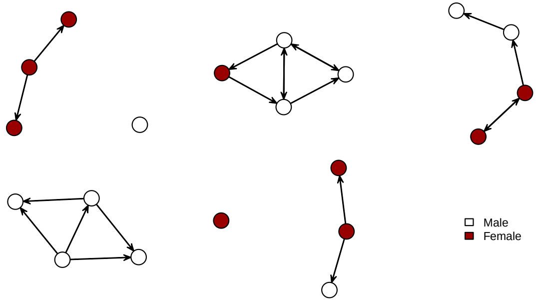
ito, ita: From the latin -*īttus*. suffix in Spanish used to denote small or affection. e.g.:

*¡Qué lindo ese perr**ito**! / What a beautiful little dog!*
*¿Me darías una tac**ita** de azúcar? / Would you give me a small cup of sugar?*

Special thanks to George Barnett who proposed the name during the 2018 NASN!

Quick example

Suppose that we have 5 `networks` (as in the R package `network`)



And we would like to fit a model using the edgecount and number of gender-homophilic ties.

How can we do it?

ergmito example (cont'd)

The same as you would do with the ergm package

```r
model1 <- ergmito(fivenets ~ edges + nodematch("female"))
```

```r
summary(model1) #
```

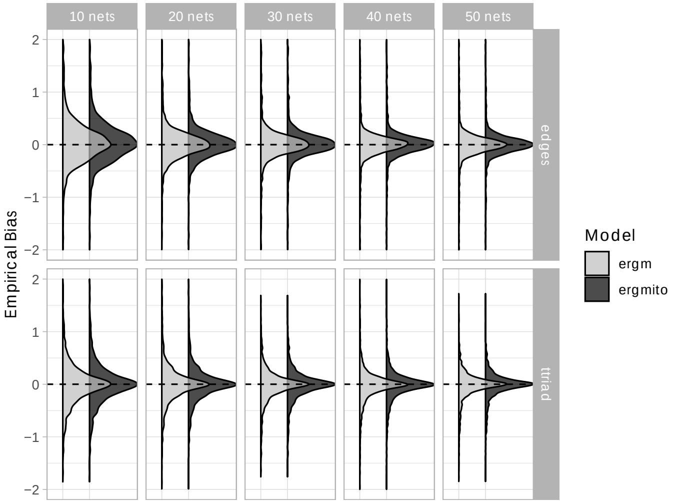
```
##
## ERGMito estimates
##
## formula:  fivenets ~ edges + nodematch("female")
##
##                  Estimate Std. Error z value Pr(>|z|)
## edges            -1.70475    0.54356 -3.1363 0.001711 **
## nodematch.female  1.58697    0.64305  2.4679 0.013592 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## AIC: 73.34109    BIC: 77.52978    (Smaller is better.)
```

Go to https://github.com/muriteams/ergmito for more on this R package.

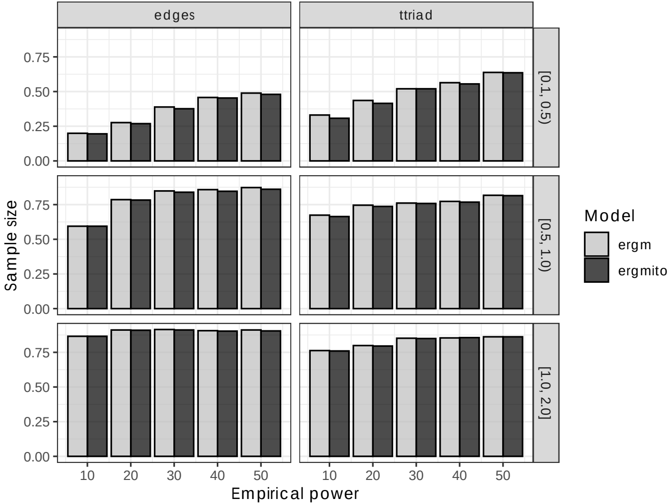
How many networks?

- Thinking about power and unbiasedness, we did a simulation study
- Simulated 20,000 samples of networks using the following steps:
  1. Draw parameters for the model based on the terms edges and ttriad (transitive triples) from a uniform(-2, 2).
  2. Draw group sizes by randomly selecting the number of networks of size 4 and size 5. Each sample has any of $\{10, 20, \dots, 50\}$ networks.
  3. Using 1. and 2., simulate networks using an ERGM model
- We fitted the models using both MC-MLE (block-diagonal ergm) and MLE (ergmito).
- We looked (are looking) at empirical bias (sanity check), power and elapsed time.

How many networks? Bias

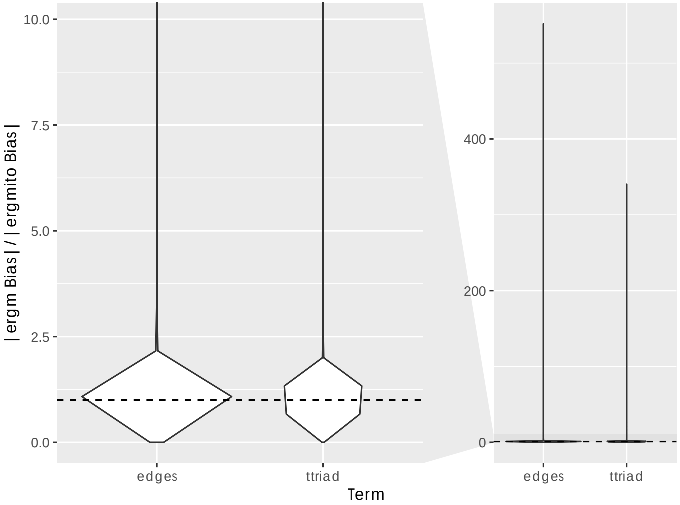


How many networks? Power

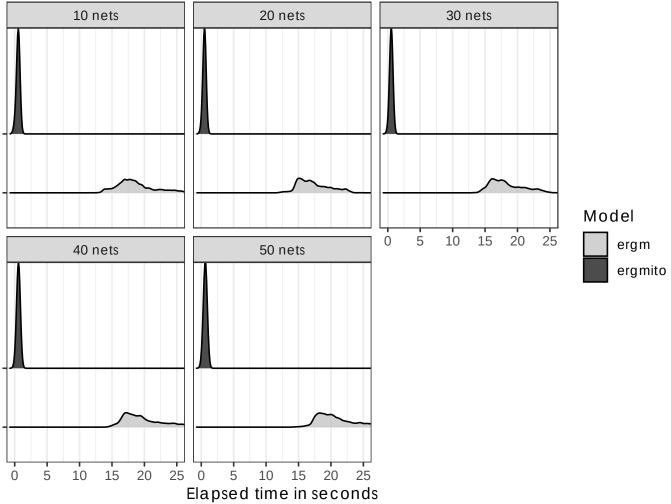


What about a real data set?

How many networks? improvements?



How many networks? improvements? (contd')



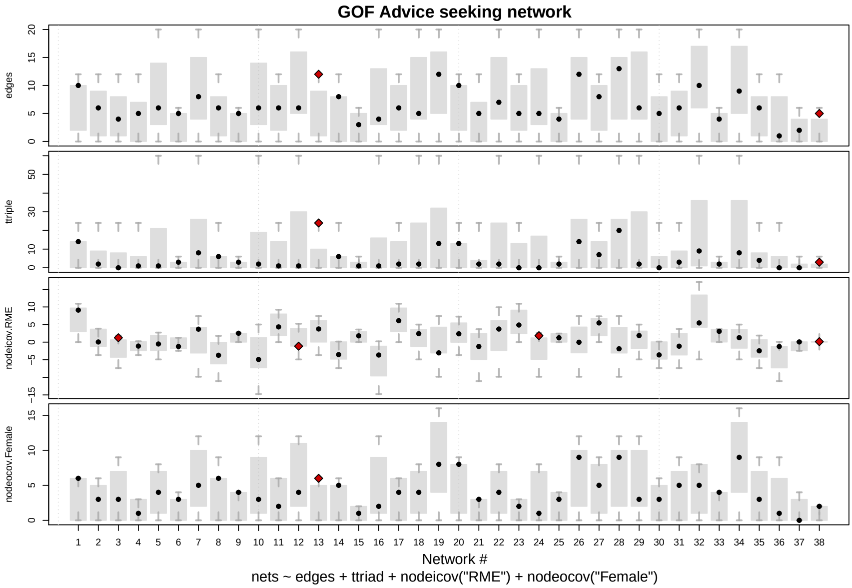
Preliminary results

From our sample of 42 small networks:

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.


GOF Advice seeking network
edges
ttriple
nodeicov.RME
nodeocov.Female
Network #
nets ~ edges + ttriad + nodeicov("RME") + nodeocov("Female")

Context: Social abilities and team performance

Two research questions

~~How do **social abilities** impact **network structure**?~~

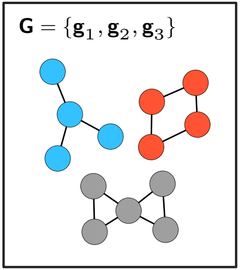
How does **collective intelligence** affect team (network) **performance**?

Networks and team performance

Suppose we have the following:

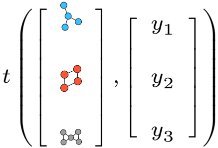
- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$
- Three scenarios:
  - **a.** Direct association
  - **b.** Known (hypothesized) mediated association (a knwon confounder)
  - **c.** Uknown (hypothesized) mediated association (an unknown confounder)

Step 1:
Fit the ERGMito



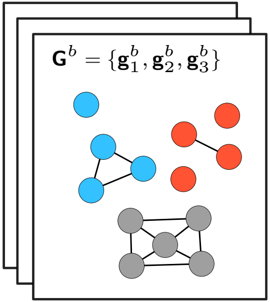
Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

Step 2:
Calculate $t_0 =$



Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \ldots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We are still working (thinking) about this...

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working (according to the simulation study...)
- Need to conduct more simulations using <u>nodal</u> attributes and compare with ERGM block diagonal models.
- What about goodness-of-fit? Still need to better think about it

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)
  - Better odds ratios (not simply exponentiating the coefficients)
  - Simulation based methods (small size $\implies$ sampling from in-memory data, and exact tests)
  - Cross-validation/model selection in ERGMs (thank you, Nolan !)
- Still thinking about how to test for association between network structure and group outcome

Thanks!

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Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. ``Ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks.'' Journal of Statistical Software 24 (3): 1--29.