Exact Statistics and Semi-Parametric Tests for Small Network Data

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Network Science of Teams

a Multidisciplinary University Research Initiative

UC SANTA BARBARA · USC University of Southern California · MIT Massachusetts Institute of Technology · NORTHWESTERN UNIVERSITY

Context: Social abilities and team performance

Two research questions

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How do **social abilities** impact **network structure**?

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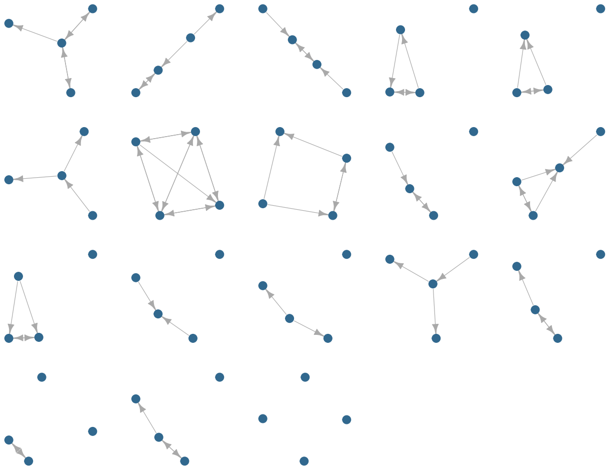
  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

Context (cont'd)



We can do a lot of simple statistics: density, % of *[blank]*, etc. but...

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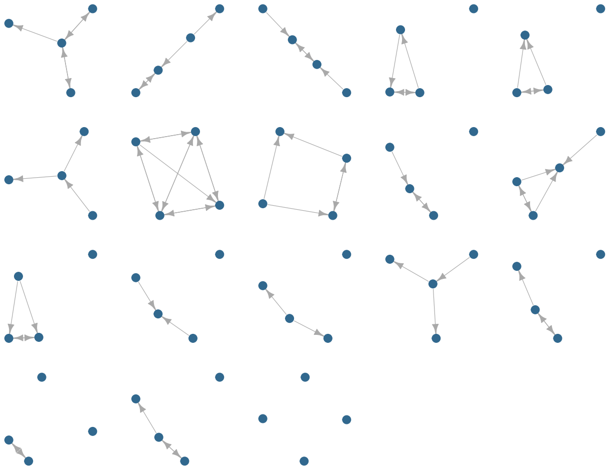

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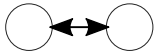
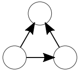
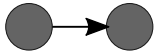
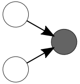
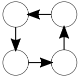
Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}\left(x_i = x_j\right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

ERGMs can do the job.

Exponential random graph models (a 1 slide crah course)

A vector of
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A vector of
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Observed data

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All possible
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ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

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We can go back to the good-old-fashion MLE

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This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

Sidetrack...

ito, ita: From the latin -*īttus*. suffix in Spanish used to denote small or affection.
e.g.:

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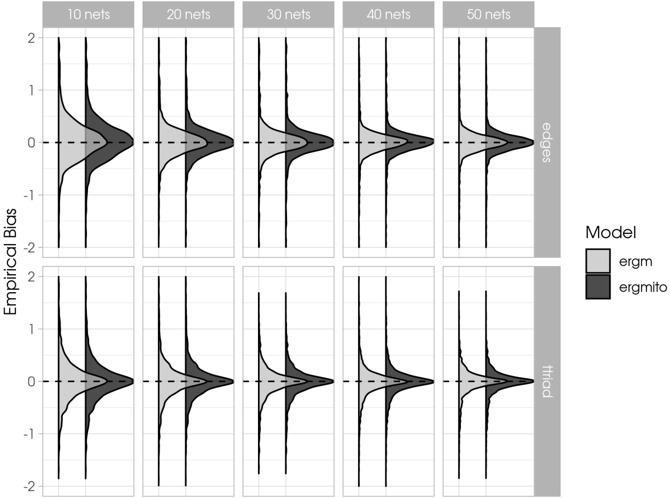
Special thanks to George Barnett who proposed the name during the 2018 NASN!

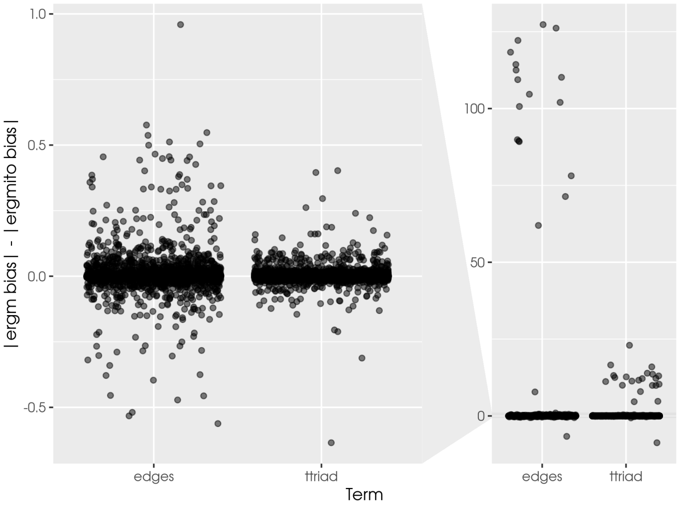
How many networks?

- Thinking about power and unbiasedness, we did a simulation study
- Simulated 20,000 samples of networks using the following steps:
  1. Draw parameters for the model based on the terms edges and ttriad (transitive triples) from a uniform(-2, 2).
  2. Draw group sizes by randomly selecting the number of networks of size 4 and size 5. Each sample has any of $\{10, 20, \dots, 50\}$ networks.
  3. Using 1. and 2., simulate networks using an ERGM model
- We fitted the models using both MC-MLE (block-diagonal ergm) and MLE (ergmito).
- We looked (are looking) at empirical bias (sanity check), power and elapsed time.

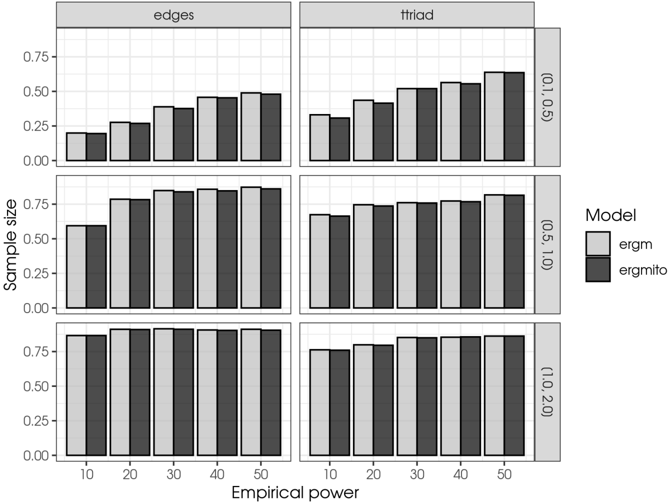
Simulation study: Bias



Simulation study: Bias (contd')



Simulation study: Power



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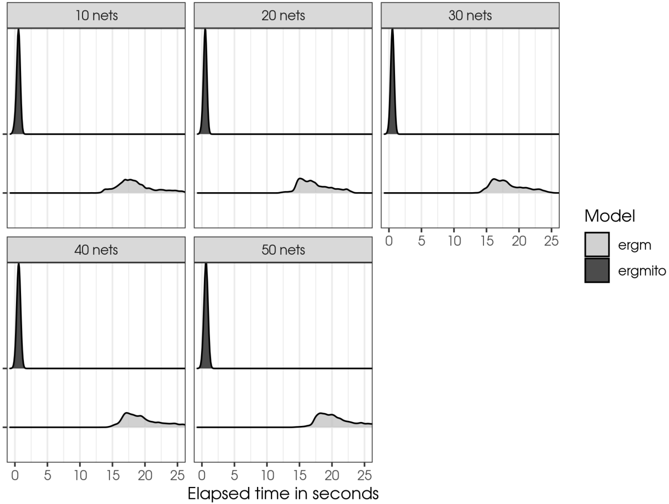

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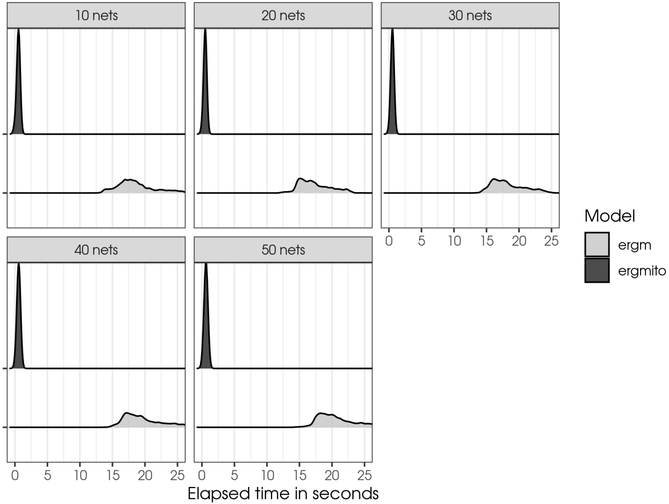


What about a real data set?

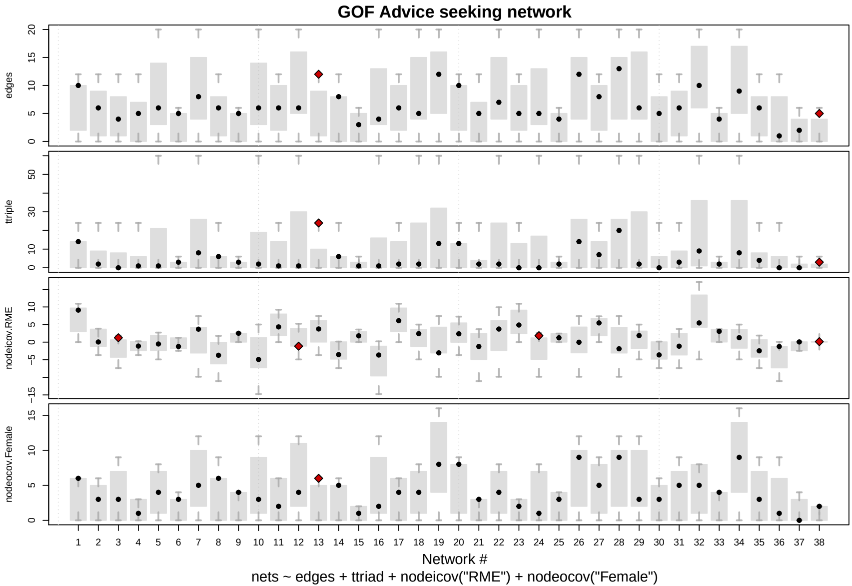
Preliminary results

From our sample of 42 small networks:

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.


GOF Advice seeking network
edges
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Network #
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b. **Simulation based methods**

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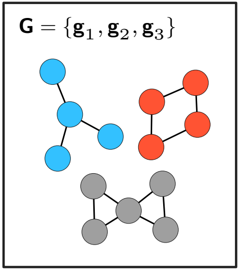

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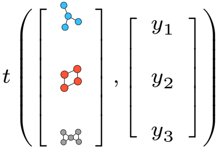

Throughout the simulations the only part that changes is the networks, not Y

Step 3:
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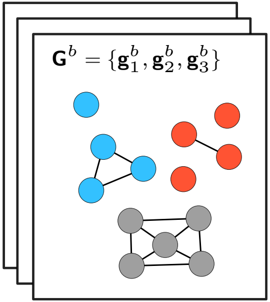

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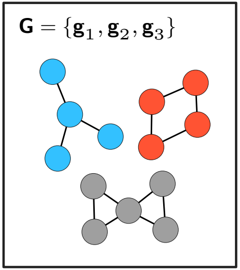

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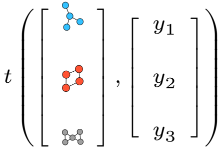

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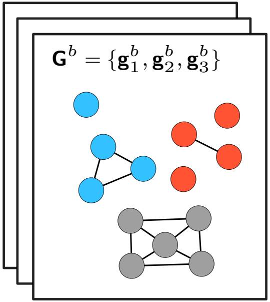

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- What about goodness-of-fit? Still need to better think about it

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  - Mixture models and Bayesian inference (if you are into that kind of stuff)

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- Still thinking about how to test for association between network structure and group outcome

Thanks!

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