What drives social networks? A gentle introduction to exponential random graph models (with a focus on small networks)

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Social networks

Figure 1: Friendship network of a UK university faculty. Source: **igraphdata** R package (Csardi, 2015). Figure drawn using the R package **netplot** (yours truly, https://github.com/usccana/netplot)

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Why are you and I are *[blank]* ? (friends, collaborators, etc.)

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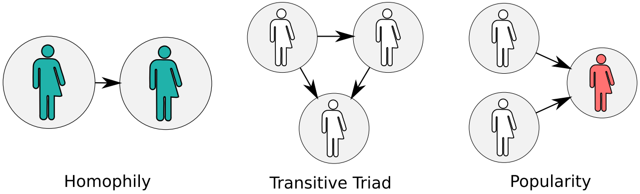

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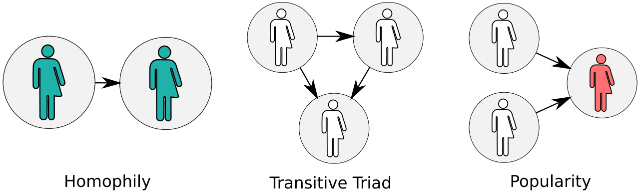
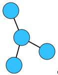


Let's build a model for this!

Exponential Family Random Graph Models (ERGMs)

We need to build a probability function for  . . .

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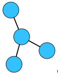
We need to build a probability function for  . . .

$\#edges, \#homophilic\ ties, \ldots$

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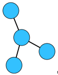
We need to build a probability function for  ...

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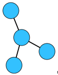
We need to build a probability function for  ...

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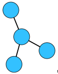
We need to build a probability function for  ...

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$$\theta_1 \times \#edges + \theta_2 \times \#homophilic\ ties + \ldots$$

$$\exp\left\{\theta_1 \times \#edges + \theta_2 \times \#homophilic\ ties + \ldots\right\}$$

$$\frac{\exp\{\theta_1 \times \#edges + \theta_2 \times \#homophilic\ ties + \ldots\}}{\sum \exp\{\ldots\}}$$

You got yourself an ERGM!

ERGMs... the *lingua franca* of SNA

A vector of model parameters

A vector of sufficient statistics

$$\Pr\left(\mathbf{Y} = \mathbf{y} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{y}' \in \mathcal{Y}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}', \mathbf{X}\right)\right\}}, \quad \forall \mathbf{y} \in \mathcal{Y}$$

Observed data

The normalizing constant

All possible networks

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$$\Pr\left(\mathbf{Y} = \underbrace{\mathbf{y} \mid \theta, \mathbf{X}}_{\text{Observed data}}\right) = \frac{\exp\left\{\overbrace{\theta^{\mathbf{t}}}^{\text{A vector of model parameters}} \overbrace{s\left(\mathbf{y}, \mathbf{X}\right)}^{\text{A vector of sufficient statistics}}\right\}}{\underbrace{\sum_{\mathbf{y}' \in \mathcal{Y}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{y}', \mathbf{X}\right)\right\}}_{\text{The normalizing constant}}}, \quad \forall \mathbf{y} \in \underbrace{\mathcal{Y}}_{\text{All possible networks}}$$

because of $\mathcal{Y}$, the **normalizing constant** is

a summation of $2^{n(n-1)}$ terms 😱!

To solve this, instead of directly computing this function, estimation is done by approximating ratios of likelihood functions instead (TL;DR we use simulations).

Let's get going

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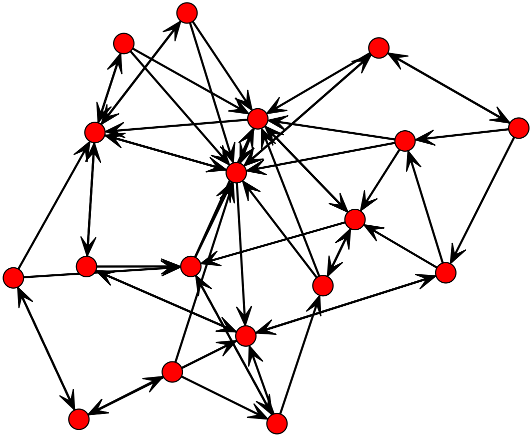
```

This is an object of class network

```
samplk1
```

```
##  Network attributes:
##   vertices = 18
##   directed = TRUE
##   hyper = FALSE
##   loops = FALSE
##   multiple = FALSE
##   bipartite = FALSE
##   total edges= 55
##     missing edges= 0
##     non-missing edges= 55
##
##  Vertex attribute names:
##     cloisterville group vertex.names
##
## No edge attributes
```

```r
library(sna) # Tools for SNA
set.seed(1)  # Graph layout is usually random-driven
gplot(samplk1)
```



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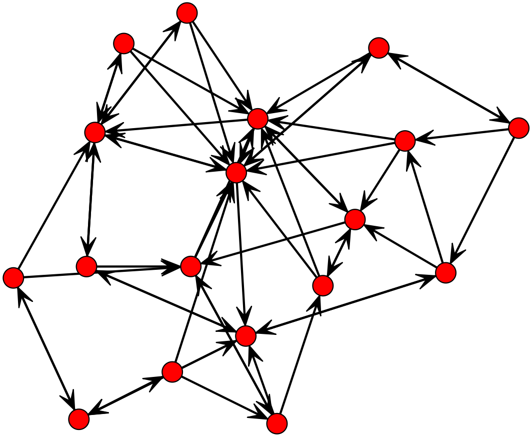
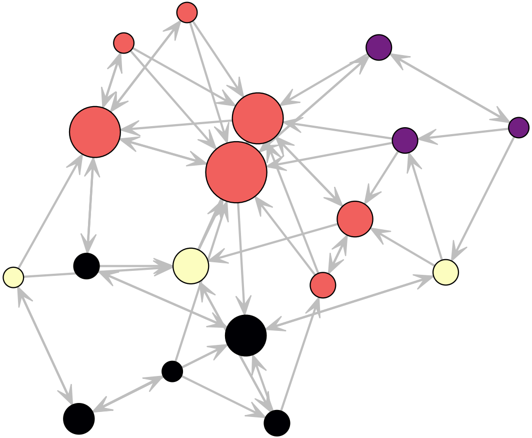


Let's add some color and other features

```r
set.seed(1)
cols <- viridisLite::magma(4)[as.factor((samplk1 %v% "group"))]
gplot(samplk1, vertex.cex = degree(samplk1)/4, vertex.col = cols, edge.col = "gray")
```



A simple ergm model

- Suppose we want to test wether homophily on *group* (individuals of the same group tend to connect with each other) and transitive triads (the friend of my friend) are driving the structure:

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summary(samplk1 ~ edges + nodematch("group") + ttriad)
```

```
##          edges nodematch.group         ttriple
##             55              30              24
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```

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##             55              30              24
```

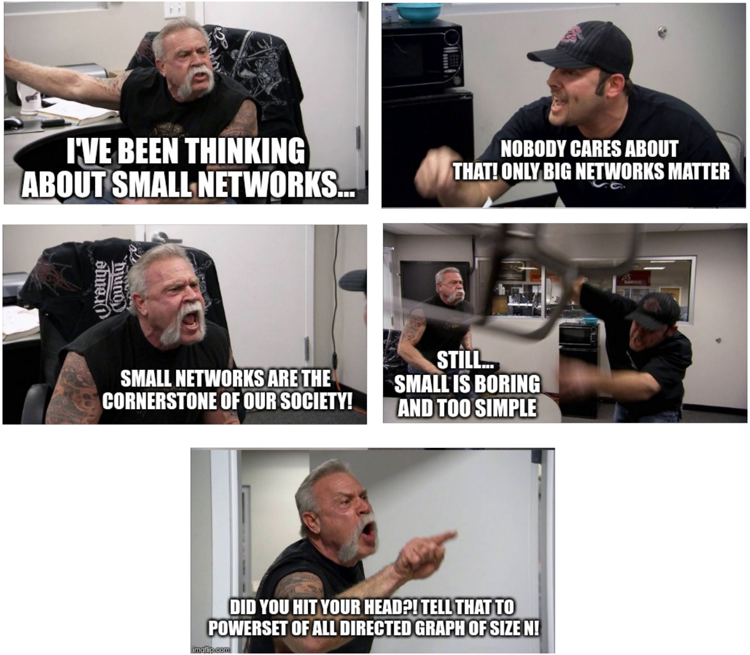
- To estimate this model we do:

```
ans <- ergm(
  samplk1 ~ edges + nodematch("group") + ttriad,
  control = control.ergm(seed = 112)
  )
```

```r
summary(ans)
```

```
##
## ==========================
## Summary of model fit
## ==========================
##
## Formula:   samplk1 ~ edges + nodematch("group") + ttriad
##
## Iterations:  2 out of 20
##
## Monte Carlo MLE Results:
##                 Estimate Std. Error MCMC % z value Pr(>|z|)
## edges            -1.7738     0.3049      0  -5.819   <1e-04 ***
## nodematch.group   1.9730     0.3906      0   5.052   <1e-04 ***
## ttriple          -0.2984     0.1954      0  -1.527    0.127
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##      Null Deviance: 424.2  on 306  degrees of freedom
##  Residual Deviance: 255.8  on 303  degrees of freedom
##
## AIC: 261.8    BIC: 272.9    (Smaller is better.)
```

Now its time for small networks!


I'VE BEEN THINKING ABOUT SMALL NETWORKS...
NOBODY CARES ABOUT THAT! ONLY BIG NETWORKS MATTER
SMALL NETWORKS ARE THE CORNERSTONE OF OUR SOCIETY!
STILL... SMALL IS BORING AND TOO SIMPLE
DID YOU HIT YOUR HEAD?! TELL THAT TO POWERSET OF ALL DIRECTED GRAPH OF SIZE N!
imgflip.com

ERGMs for Small Networks

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- In addition, most of the time samples of small networks include multiple of them, e.g.: Families, Small teams (like our data), Ego-nets, etc.
- This makes pooled ERGM estimates a natural way of modeling the data.
- This and more can be found in the **ergmito** R package (/muriteams/ergmito)

Sidetrack...

ito, ita: From the latin -*$\bar{i}$ttus*. suffix in Spanish used to denote small or affection.

e.g.:

*¡Qué lindo ese perr**ito**! / What a beautiful little dog!*
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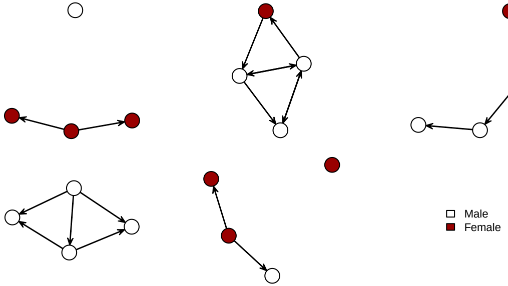
Special thanks to George Barnett who proposed the name during the 2018 NASN!

`ergmito` example

```r
library(ergmito)
data(fivenets, package = "ergmito")
```



```r
# Looking at one of the five networks
fivenets[[1]]
```

```
##  Network attributes:
##   vertices = 4
##   directed = TRUE
##   hyper = FALSE
##   loops = FALSE
##   multiple = FALSE
##   bipartite = FALSE
##   total edges= 2
##     missing edges= 0
##     non-missing edges= 2
##
##  Vertex attribute names:
##     female name
##
## No edge attributes
```

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##
##  Vertex attribute names:
##     female name
##
## No edge attributes
```

How can we fit an ERGMito to this 5 networks?

ergmito example (cont'd)

The same as you would do with the ergm package:

```r
model1 <- ergmito(fivenets ~ edges + nodematch("female"))
summary(model1)
```

```
##
## ERGMito estimates
##
## formula:  fivenets ~ edges + nodematch("female")
##
##                    Estimate Std. Error   z value    Pr(>|z|)
## edges             -1.704748  0.5435573 -3.136280 0.001711055
## nodematch.female   1.586965  0.6430475  2.467882 0.013591530
```

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And much more!

Thanks!

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https://ggvy.cl

gvegayon gvegayon

Appendix

Structures

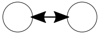
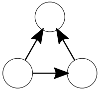
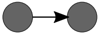
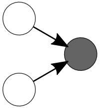
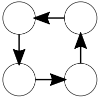
Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

Figure 2: Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

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