What drives social networks?
A gentle introduction to exponential random graph models (with a focus on small networks)

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Social networks

Figure 1: Friendship network of a UK university faculty. Source: **igraphdata** R package (Csardi, 2015). Figure drawn using the R package **netplot** (yours truly, https://github.com/usccana/netplot)

Exponential Family Random Graph Models (ERGMs)

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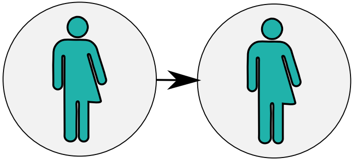
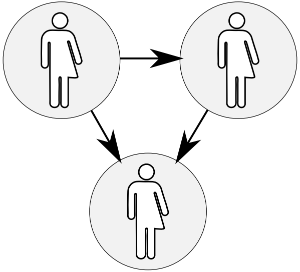
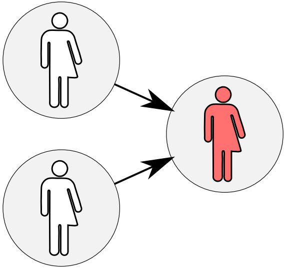


Homophily



Transitive Triad



Popularity

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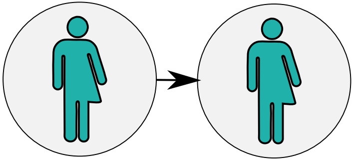

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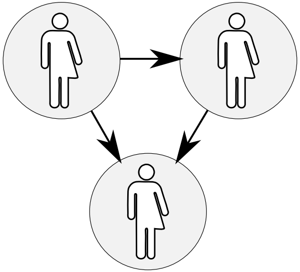

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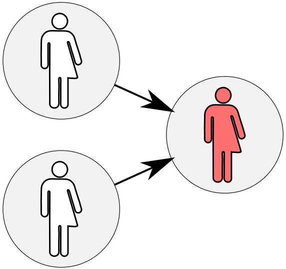

Popularity

Let's build a model for this!

ERGMs from scratch

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You got yourself an ERGM!

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Where $\kappa\left(\theta, \mathbf{X}\right)$ is the normalizing constant and equals $\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}$.

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- The set of sufficient statistics reflects social and psychological mechanisms that are hypothesized to drive the network structure. Figure 2 shows some examples of values in $s\left(\right)$.

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- In the case of directed networks, $\mathcal{G}$ has $2^{n(n-1)}$ terms.
- See Wasserman, Pattison, Robins, Snijders, Handcock, Butts, and others.

Structures

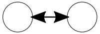
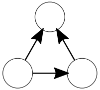
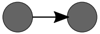
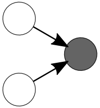
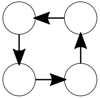
Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

Figure 2: Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

References

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