

# Big Problems for **Small Networks**: Statistical Analysis of Small Networks and Team Performance

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## Network Science of Teams

a Multidisciplinary University Research Initiative

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# Research Problem

In the context of network science of small teams:

What characterizes the social networks that emerge from small teams?

Is there any association between how team networks are structured and their performance?

# Research Problem (cont'd)

We are trying to answer these two questions with the following experimental data:

- ▶ 42 teams typically made of 4 to 5 mixed-gender teammates.
- ▶ Each team completed 1 hour of group tasks, in particular, a set of tasks to measure collective intelligence developed by MIT (Kim et al. 2017).
- ▶ We surveyed team members to capture information regarding socio-demographics **and**:
  - ▶ Social Intelligence (SI domains): Social Perception (measured by RME), Social Accomodation, Social Gregariousness, and Social Awareness
  - ▶ Social Networks: Advice Seeking, Leadership, Influence (among others).

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**Part I: Network Structure**

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# **Part I: Network Structure**

# Exponential Random Graph Models (ERGMs)



**Figure 1:** Friendship network of a UK university faculty. Source: **igraphdata** R package (Csardi, 2015). Figure drawn using the R package **netplot** (yours truly, <https://github.com/usccana/netplot>)

# ERGMs... the *lingua franca* of SNA

- ▶ Seeks to answer the question: What local social structures gave origin to a given observed graph?
- ▶ The model is centered around a vector of **sufficient statistics**  $s()$ , and is operationalized as:




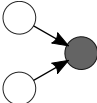

$$\Pr(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}) = \frac{\exp\{\theta^t s(\mathbf{g}, \mathbf{X})\}}{\kappa(\theta, \mathbf{X})}, \quad \forall \mathbf{g} \in \mathcal{G} \quad (1)$$

Where  $\kappa(\theta, \mathbf{X})$  is the normalizing constant and equals  $\sum_{\mathbf{g}' \in \mathcal{G}} \exp\{\theta^t s(\mathbf{g}', \mathbf{X})\}$ .

- ▶ The set of sufficient statistics reflects social and psychological mechanisms that are hypothesized to drive the network structure. Figure 2 shows some examples of values in  $s()$ .
- ▶ In the case of directed networks,  $\mathcal{G}$  has  $2^{n(n-1)}$  terms.
- ▶ See Wasserman, Pattison, Robins, Snijders, Handcock and others.



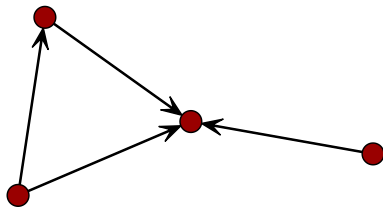
# Structures

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

**Figure 2:** Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

## Example of model

In this network



We see 4 **edges**, 1 **transitive triad**  
and **no mutual ties**.

The probability function of this model  
would be

$$\Pr(\mathbf{G} = \mathbf{g} \mid \theta) = \frac{\exp \{ 4\theta_{edges} + \theta_{ttriads} + 0\theta_{mutual} \}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp \{ \theta^t s(\mathbf{g}') \}}$$

with  $\theta = [\theta_{edges} \quad \theta_{ttriads} \quad \theta_{mutual}]^t$

This model has **MLE parameter estimates** of -0.19 (low density), 0.28 (high chance of ttriads), and -8.48 (low chance of mutuality) for the parameters edges, ttriads, and mutual respectively.

# Estimation of ERGMs

- ▶ Calculating of the normalizing constant in (1),  $\kappa = \sum_{\mathbf{g}' \in \mathcal{G}} \exp \{ \theta^t s(\mathbf{g}', \mathbf{X}) \}$ , makes ERGMs difficult to estimate.
- ▶ For this reason, statistical methods have focused on avoiding the direct calculation of  $\kappa$ ; most modern methods for estimating ERGMs rely on MCMC.

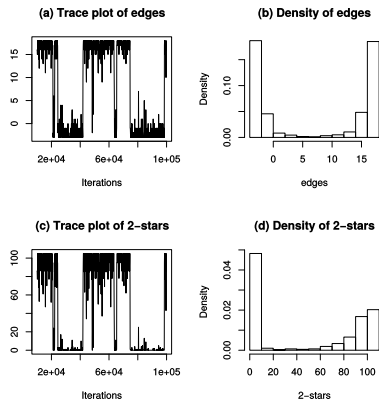
# Estimation of ERGMs (cont'd)

Description of the MCMC-MLE algorithm (Geyer and Thompson 1992)

1. Make an initial guess of the model parameters  $\hat{\theta}_0$
2. While the algorithm doesn't converge, do:
  - a. Generate a large sample of graphs from  $\Pr(\mathbf{G} = \mathbf{g} \mid \hat{\theta}_0, \mathbf{X})$  using MCMC
  - b. Use the simulated sample of graphs **to approximate the normalizing constant**, and so, the likelihood function. And with this approximation update the parameter  $\theta_0$  using a Newton-Raphson step.
  - c. next iteration

Once it converges, the last value of  $\hat{\theta}_0$  is the MCMC-MLE estimates. The variance approximation is another story.

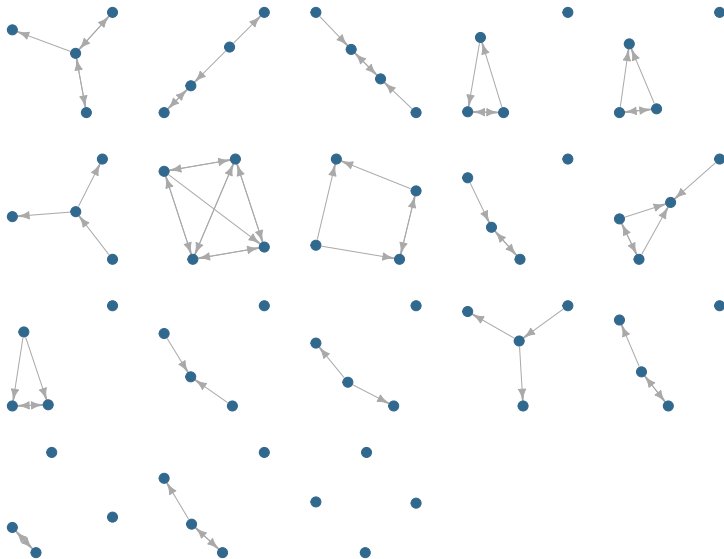
- ▶ While significant advances have been made in the area, simulation based models can suffer from **model degeneracy** (Handcock 2003).
- ▶ This occurs when the model parameters *live in a degenerate region*. This impacts directly the MCMC process yielding poor mixing



**Figure 3:** Model degeneracy. Figure 5 in Handcock (2003)

- ▶ Model degeneracy is particularly problematic with small networks. . . (says anyone who has tried to fit one).

# ERGMs for Small Networks



## ERGMs for Small Networks (cont'd)

- ▶ In the case of small networks (e.g. at most 6 nodes), the calculation of  $\kappa$  becomes computationally feasible.
- ▶ This allows direct calculation of (1), **avoiding the need for simulations** and allowing us to obtain Maximum Likelihood Estimates using *standard* optimization techniques.
- ▶ In addition, most of the time samples of small networks include multiple of them, e.g.: Families, Small teams (like our data), Ego-nets, etc.
- ▶ This makes pooled ERGM estimates a natural way of modeling the data:

$$\Pr(\mathbf{G}_1 = \mathbf{g}_1, \dots, \mathbf{G}_p = \mathbf{g}_p \mid \theta, \mathbf{X}_1, \dots, \mathbf{X}_p) = \prod_p \frac{\exp\{\theta^t s(\mathbf{g}_p, \mathbf{X}_p)\}}{\kappa_p(\theta, \mathbf{X}_p)}$$

How different is this from the “normal” way to fit ERGMs?

# Estimation of ERGMitos (cont'd)

Description of the MCMC-MLE algorithm (Geyer and Thompson, 1992)

1. Make an initial guess of the model parameters  $\hat{\theta}_0$
2. While the algorithm doesn't converge, do:
  - a. Simulate  $B$  graphs from  $\Pr(\mathbf{G} = \mathbf{g} \mid \hat{\theta}_0, \mathbf{X})$  using MCMC
  - b. Use the simulated sequence of graphs to approximate the likelihood function. And with this approximation **update the parameter  $\theta_0$  using a Newton-Raphson step.**
  - c. next iteration

By skipping the MCMC part we:

1. are able to get MLE estimates directly,
2. avoid the degeneracy problem latent in MCMC, and
3. obtain more accurate (exact) estimates faster.

We have implemented this and more in the `ergmito` R package



Sidetrack...


**ito, ita:** From the latin *-ītus*. suffix in Spanish used to denote small or affection.  
e.g.:

*¡Qué lindo ese perrito! / What a beautiful little dog!*

*¿Me darías una tacita de azúcar? / Would you give me a small cup of sugar?*

**Special thanks to George Barnett who proposed the name during the 2018 NASN!**

# Features of ergmito

This () R package has the following features

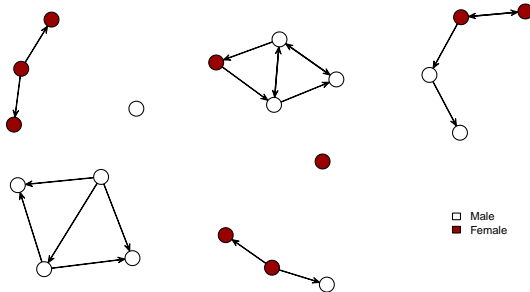
- ▶ Built on top of **statnet**'s `ergm` R package (Hunter et al. 2008; Handcock et al. 2018).
- ▶ Allows estimating ERGMs for small networks (less than 7 and perhaps 6)<sup>1</sup> via MLE.
- ▶ Implements pooled ERGM models.
- ▶ Includes a simulation function for efficiently drawing samples of small networks, and by **efficiently** we mean **fast**.

---

<sup>1</sup>A directed graph of size 6 has a support set with  $2^{6 \times (6-1)} = 1,073,741,824$  elements.

# ergmito example

```
library(ergmito)
data(fivenets, package = "ergmito")
```



```
# Looking at one of the five networks  
fivenets[[1]]
```

```
## Network attributes:  
##   vertices = 4  
##   directed = TRUE  
##   hyper = FALSE  
##   loops = FALSE  
##   multiple = FALSE  
##   bipartite = FALSE  
##   total edges= 2  
##     missing edges= 0  
##     non-missing edges= 2  
##  
## Vertex attribute names:  
##   female name  
##  
## No edge attributes
```

How can we fit an ERGMito to this 5 networks?

## ergmito example (cont'd)

The same as you would do with the ergm package:

```
(model1 <- ergmito(fivenets ~ edges + nodematch("female")))
```

```
##
## ERGMito estimates
##
## Coefficients:
##          edges  nodematch.female
##        -1.705          1.587
```

	Model 1
edges	-1.70** (0.54)
nodematch.female	1.59* (0.64)
AIC	73.34
BIC	77.53
Log Likelihood	-34.67
Num. networks	5
*** $p < 0.001$ , ** $p < 0.01$ , * $p < 0.05$	

**Table 1:** Statistical models

```
(gof1 <- gof_ergmito(model1))
```

```
##
```

```
## Goodness-of-fit for edges
```

```
##
```

```
##      obs min mean max lower upper lower prob. upper prob.
## net 1   2  0  3.7 12      0      6      0.0081      0.96
## net 2   7  0  3.7 12      0      6      0.0081      0.96
## net 3   4  0  3.1 12      0      6      0.0206      0.99
## net 4   5  0  5.6 12      2      8      0.0309      0.95
## net 5   2  0  3.7 12      0      6      0.0081      0.96
```

```
##
```

```
##
```

```
## Goodness-of-fit for nodematch.female
```

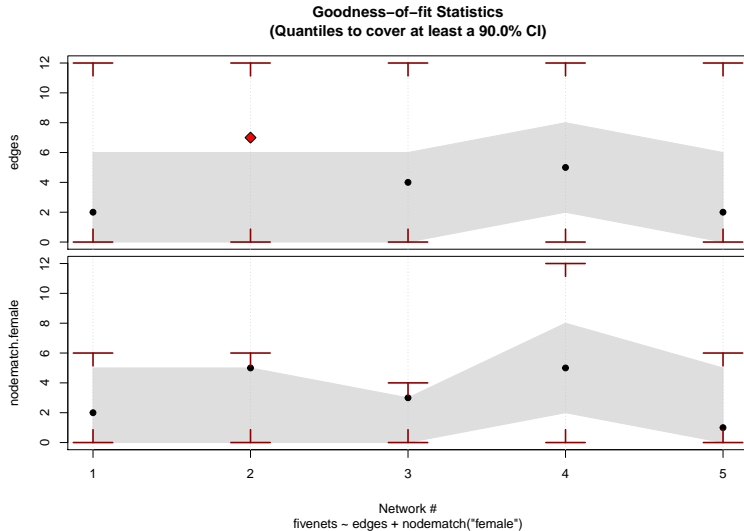
```
##
```

```
##      obs min mean max lower upper lower prob. upper prob.
## net 1   2  0  2.8  6      0      5      0.022      0.99
## net 2   5  0  2.8  6      0      5      0.022      0.99
## net 3   3  0  1.9  4      0      3      0.079      0.95
## net 4   5  0  5.6 12      2      8      0.031      0.95
## net 5   1  0  2.8  6      0      5      0.022      0.99
```

```
##
```

```
## Note: Exact confidence intervals where used. This implies that the requested CI may differ from the one used (se
```

```
plot(gof1)
```



# Simulation Study

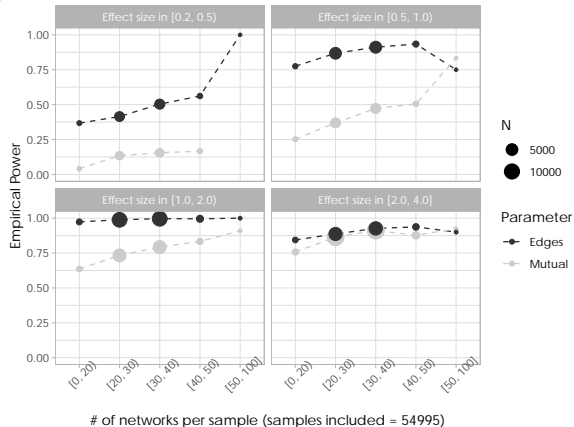
We conducted a simulation study to explore the properties of MLE for small networks (a.k.a. ERGMito). To generate each sample of teams:

1. Draw the **population parameters** from a piecewise Uniform with values in  $[-4, -.1] \cup [.1, 4]$
2. We will draw **groups of sizes 3 to 5**. The number of networks per group size are drawn from a Poisson distribution with parameter 10 (hence, an expected size of 30 networks per sample).
3. Use the **drawn parameters** and **group sizes** to **generate random graphs** using an ERGM data generating process.

We simulated 100,000 samples, each one composed of an average of 30 networks.



# Simulation Study (cont'd)



**Figure 4:** Empirical power of Pooled-ERGM estimates at various levels of effect size. As expected, power increases significantly with sample size (# of networks per sample). Interestingly, the discovery rate of an effect size within [1, 2) is very high even with a sample size of 20-30 networks. More extreme points have higher volatility due to small number of samples included.

So now that we can estimate ERGMs for small networks (cool!)...

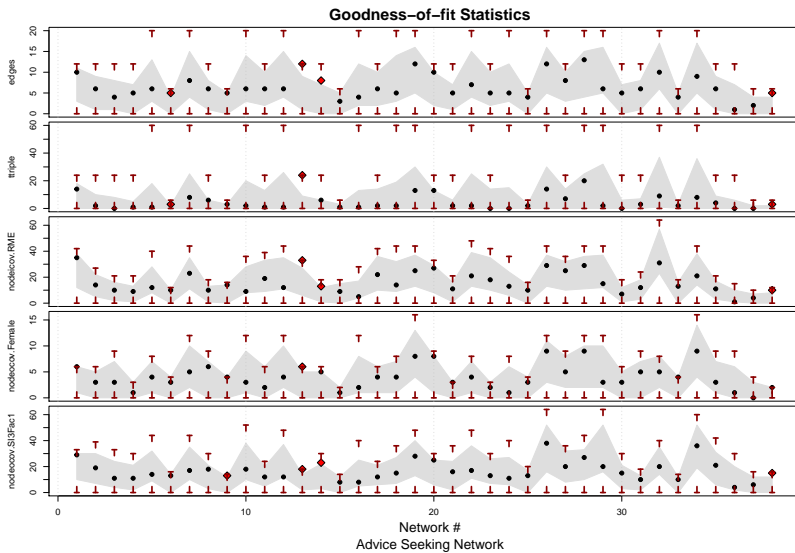
... what can this tell us about our 42 teams?

# Preliminary results

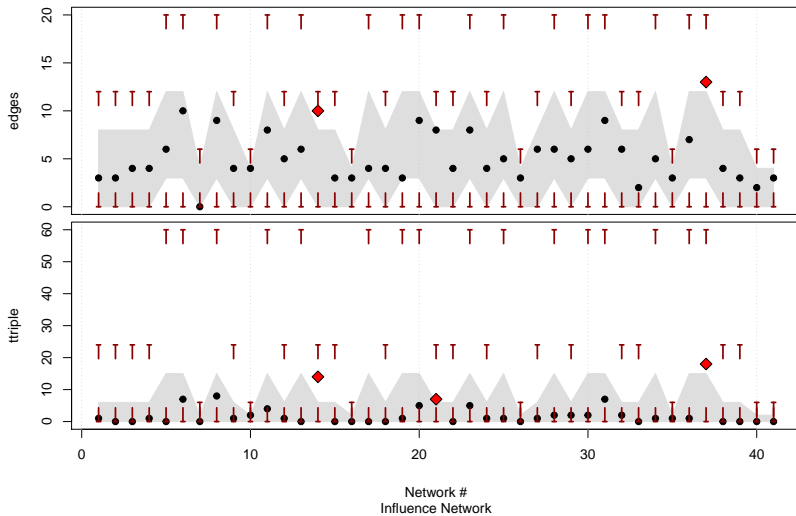
	Advice	Influence	Leadership
Edges	-1.87*** (0.30)	-0.78*** (0.13)	-0.57*** (0.14)
Transitive Triads	0.24*** (0.06)	0.21** (0.08)	
Indeg. RME	0.35*** (0.08)		
Outdeg. Female	0.43* (0.19)		
Outdeg. Social Accomodation	0.11 (0.08)		
Indeg. Female			-0.38* (0.19)
AIC	693.18	760.40	655.78
BIC	714.50	769.12	664.32
Log Likelihood	-341.59	-378.20	-325.89
Num. networks	38	41	38

\*\*\*  $p < 0.001$ , \*\*  $p < 0.01$ , \*  $p < 0.05$

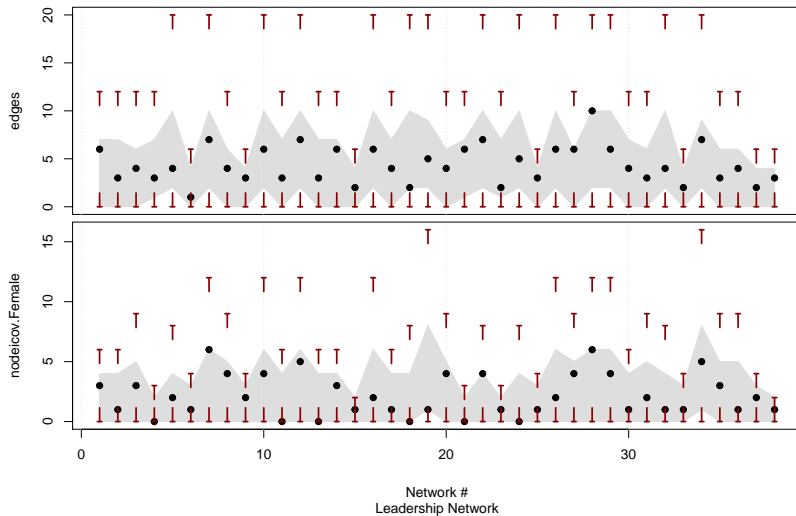
**Table 2:** The two statistics that showed to be the most robust were **Indeg. RME** and **Outdeg. Female**. These two effects can be described as (1) individuals with high levels of RME receive more ties, and (2) female subjects were more likely of seeking advice than male. Other statistics such as GPA, religiousness, age, and ethnicity were not significant.



# Goodness-of-fit Statistics



# Goodness-of-fit Statistics



## **Part II: Association between network structure and team performance**

# Testing effects of social network structure on group performance

Two common approaches: Generalized Linear Models (GLMs), or permutation-like tests. Both have limitations:

- ▶ GLMs:
  - ▶ Sample size is problematic: How costly is getting enough teams to run get a desired level of power?
- ▶ Permutation-like tests:
  - ▶ Permutation: shuffle the dependent variable and compute, for example, correlations. **Which does not control for other factors in the system.**
  - ▶ Rewiring (see Milo et al. 2004): Sample networks by generating new graphs using a rewiring algorithm, usually preserving some property such as degree sequence—the observed sequence of in/out degree. **This assumes an oversimplifying data generating process.** And worse, in a network of size 4, how many different networks can be observed **holding the degree sequence fixed?**

BTW: Talking about Degree sequence leads directly to the now controversial Scale-free networks.



“Scale-free networks are rare”

“

The **structural diversity of real-world networks** uncovered here presents both a puzzle and an opportunity. The strong focus in the scientific literature on **explaining and exploiting scale-free patterns** has meant **relatively less is known about mechanisms that produce non-scale-free structural patterns**, e.g., those with degree distributions better fitted by a log-normal. Two important directions of future work will be the **development and validation of novel mechanisms for generating more realistic degree structure in networks**, and novel statistical

”

*techniques for identifying or untangling them given empirical data*

– p. 8, Broido and Clauset (2019)

See Holme (2019) for a recent reference on the Scale-free issue.

# An Idea

What about using ERGMs to generate null distributions?

- ▶ Even for small networks, we can generate thousands of unique graphs, so sample size is not a problem.
- ▶ As a difference from permutation tests:
  - ▶ ERGMs provide a more meaningful/realistic (or comprehensive if you like) data generating process
  - ▶ On average, sampled networks will **look like the original** graph (addressing Broido and Clauset (2019)'s comments).

In principle, this would be equivalent to a revised rewiring test. . .

# Algorithm

1. Estimate an ERGM (estimates can come from a single graph or pooled estimates). We denote the data-generating-process of this model as  $\mathcal{D} : \Theta \times \mathcal{X} \mapsto \mathcal{G}$ .
2. Calculate the value  $t_0 = t(\mathbf{G}, Y)$ , the observed test statistic (e.g. correlation between triads and  $y$ ).
3. Now, for  $b \in \{1, \dots, B\}$  do:
  - 3.1 For each group  $j$  in  $\{1, \dots, J\}$ , draw a new network  $\mathbf{g}_j^b \sim \mathcal{D}(\hat{\theta}, X_j)$ , this new sequence is denoted  $\mathbf{G}^b$
  - 3.2 Using  $\mathbf{G}^b$  and  $Y$ , calculate  $t_b = t(\mathbf{G}^b, Y)$
  - 3.3 Next  $b$ .

This will generate a null distribution for the statistic  $t$ , which we can use to compare against the observed statistic,  $t_0$ .

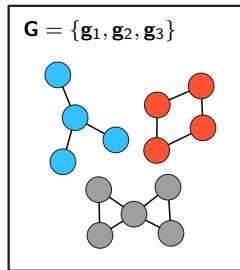
**Note** An important distinction to make is that structures that gave origin to the graph need not to be relevant for the team's performance per se.

# Illustrated example

Suppose that we have a 3 networks of sizes 4, 4, and 5 respectively. The

**Step 1:**

Fit the ERGMito



Fit the ERGMito,  
This will give us  $\mathcal{D}(\hat{\theta}, X_j)$

**Step 2:**

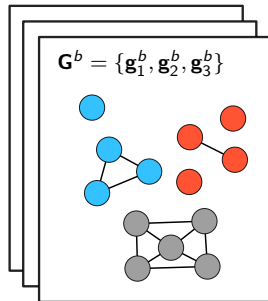
Calculate  $t_0 =$

$$t \left( \begin{bmatrix} \text{blue network} \\ \text{red network} \\ \text{gray network} \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right)$$

Throughout the simulations  
the only part that changes is  
the networks, not  $Y$

**Step 3:**

For  $b \in 1, \dots, B$  do



3.1) For  $j \in \{1, 2, 3\}$  draw a  
new network from  $\mathcal{D}$

3.2) Use the new sample to  
calculate  $t_b = t(\mathbf{G}^b, Y)$

We can use the distribution of the sequence  $\{t_1, \dots, t_B\}$  as null to compare against  $t_0$

# Extended example with fivenets

Going back to our fivenets example:

- ▶ Recall that our data-generating process for  $\mathbf{G}$  was an ERGMito with parameters  $\left(\theta_{edges}, \theta_{nodematch("female")}\right)$ .
- ▶ Imagine that  $y_{\mathbf{g}} = s(\mathbf{g}) + \varepsilon$ , with  $\varepsilon \sim N(0, 1)$  and  $s(\mathbf{g})$  as the number of ties sent to women (`nodeicov("female")` in `ergm`)
- ▶ Our test statistic  $t$  is the correlation between  $y$  and  $s(\mathbf{g})$
- ▶ The data looks something like this

$y$	$s(\mathbf{g})$
1.0138091	2
0.6051448	1
4.3085153	2
0.9547600	0
-0.1330788	1

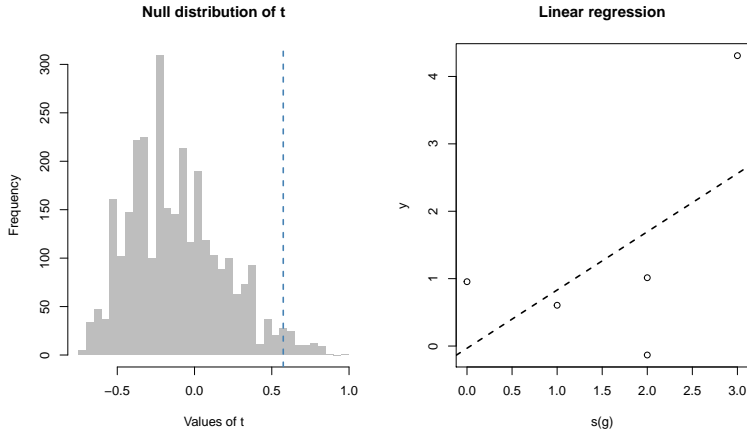
We will try to assess the presence/absence of an association between the outcome  $y$  and the number of ties received by women, `nodeicov("female")`, using both a linear regression model and our test

- ▶ Our approach Is the the observed correlation statistic  $t_0 = \text{cor}(y, s(\mathbf{g}))$  different than as expected by chance (assuming  $\mathbf{g}$  follows an ERGM distribution)?
- ▶ OLS approach In the following model:

$$y = \alpha + \theta^{OLS} s(\mathbf{g}) + \varepsilon, \quad \varepsilon \sim N(0, 1)$$

is the  $\theta_{OLS}$  parameter significantly different from zero?

## Extended example with fivenets (cont'd)



**Figure 5:** Comparing our method against a linear regression. Our proposed method returned a two sided p-value of 0.045, while the pvalue for the OLS coefficient was 0.311.

# Concluding remarks

Exponential Random Graph Models for Small Networks:

1. Not a new thing, what's new is the tool to do so in a smooth way.
2. Preliminary results (both from simulations and applications) look encouraging.
3. Still work to do (on the development side of things): Goodness of fit tests, better algorithms for drawing random graphs, Bayesian models (because is great fun. . . ), etc.
4. Can be extended to other types of ERGMs. . . our next target: TERGMs (Separable Exponential Random Graph Models)

Test for Association between graph level outcomes and graph structures:

1. Still need to run simulation studies to explore **power** and **false discovery** rates.
2. Also on the development side of things, need to make things a bit faster and lightweight.
3. Working on a more formal statistical framework (when is it a good/bad idea to use this kind of method).



# Thanks!



**George G. Vega Yon**

Let's chat!

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