Big Problems for Small Networks: Statistical Analysis of Small Networks and Team Performance

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We thank members of our MURI research team, USC's Center for Applied Network Analysis, Andrew Slaughter, and attendees of the NASN 2018 conference for their comments.

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  - Social Networks: Advice Seeking, Leadership, Influence (among others).

Contents

Part I: Network Structure

Exponential Random Graph Models (ERGMs)

Figure 1: Friendship network of a UK university faculty. Source: **igraphdata** R package (Csardi, 2015). Figure drawn using the R package **netplot** (yours truly, https://github.com/usccana/netplot)

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How can we explain what we see here?

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Where $\kappa\left(\theta,\mathbf{X}\right)$ is the normalizing constant and equals $\sum_{\mathbf{g}'\in\mathcal{G}}\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g}',\mathbf{X}\right)\right\}$.

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- The set of sufficient statistics reflects social and psychological mechanisms that are hypothesized to drive the network structure. Figure 2 shows some examples of values in $s\left(\right)$.
- In the case of directed networks, $\mathcal{G}$ has $2^{n(n-1)}$ terms.
- See Wasserman, Pattison, Robins, Snijders, Handcock and others.

Structures

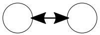
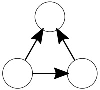
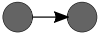
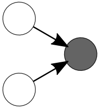
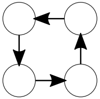
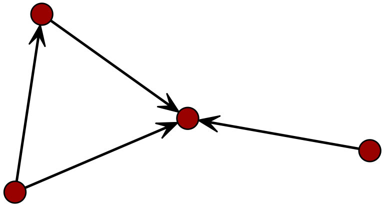
Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

Figure 2: Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

Example of model

In this network



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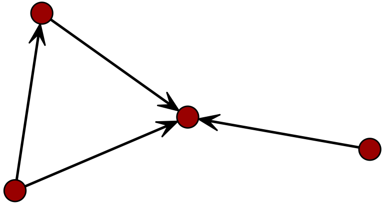

We see 4 **edges**, 1 **transitive triad** and **no mutual ties**.

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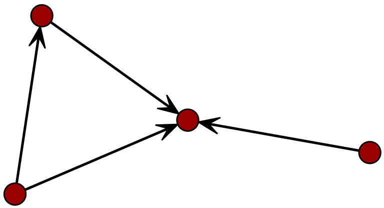

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The probability function of this model would be

$$\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta\right) = \frac{\exp\left\{4\theta_{edges} + \theta_{ttriads} + 0\theta_{mutual}\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}'\right)\right\}}$$

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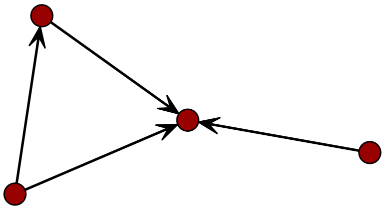

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This model has **MLE parameter estimates** of -0.19 (low density), 0.27 (high chance of ttriads), and -9.75 (low chance of mutuality) for the parameters `edges`, `ttriads`, and `mutual` respectively.

Estimation of ERGMs

- Calculating of the normalizing constant in (1), $\kappa = \sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}$, makes ERGMs difficult to estimate.

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- For this reason, statistical methods have focused on avoiding the direct calculation of κ; most modern methods for estimating ERGMs rely on MCMC.

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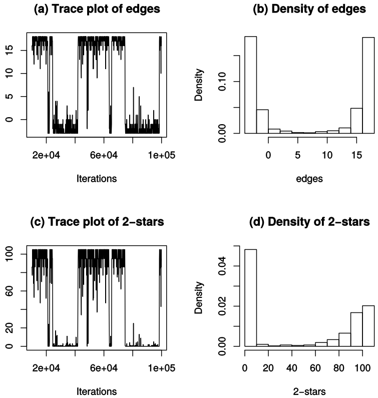

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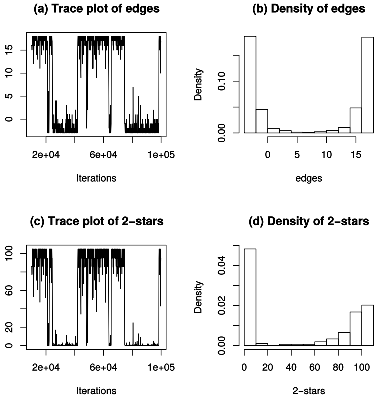

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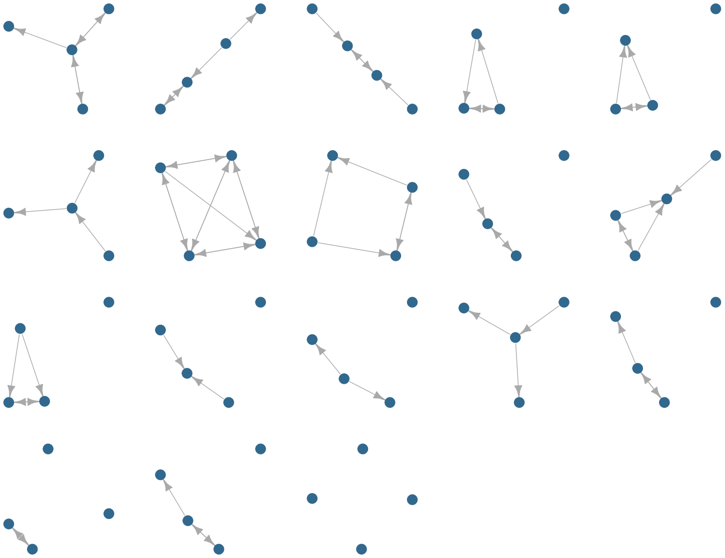
- Model degeneracy is particularly problematic with small networks... (says anyone who has tried to fit one).

ERGMs for Small Networks



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How different is this from the "normal" way to fit ERGMs?

Estimation of ERGMs (cont'd)

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We have implemented this and more in the `ergmito` R package

Sidetrack...

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e.g.:

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Special thanks to George Barnett who proposed the name during the 2018 NASN!

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- Includes a simulation function for efficiently drawing samples of small networks, and by **efficiently** we mean **fast**.

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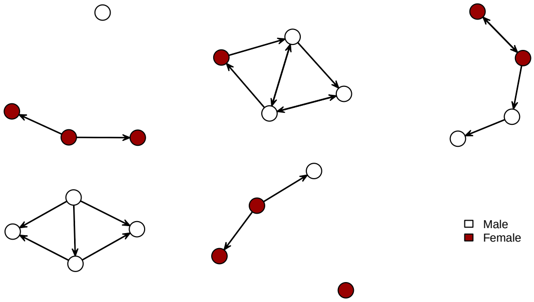
ergmito example

```r
library(ergmito)
data(fivenets, package = "ergmito")
```



```r
# Looking at one of the five networks
fivenets[[1]]
```

```
##  Network attributes:
##   vertices = 4
##   directed = TRUE
##   hyper = FALSE
##   loops = FALSE
##   multiple = FALSE
##   bipartite = FALSE
##   total edges= 2
##     missing edges= 0
##     non-missing edges= 2
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##  Vertex attribute names:
##     female name
##
## No edge attributes
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How can we fit an ERGMito to this 5 networks?

`ergmito` example (cont'd)

The same as you would do with the `ergm` package:

```
(model1 <- ergmito(fivenets ~ edges + nodematch("female")))
```

```
##
## ERGMito estimates
##
##  Coefficients:
##            edges  nodematch.female
##           -1.705             1.587
```

	Model 1
edges	-1.70^{**}
	(0.54)
nodematch.female	1.59^{*}
	(0.64)
AIC	73.34
BIC	77.53
Log Likelihood	-34.67
Num. networks	5

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Statistical models

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We simulated 100,000 samples, each one composed of an average of 30 networks.

Simulation Study (cont'd)

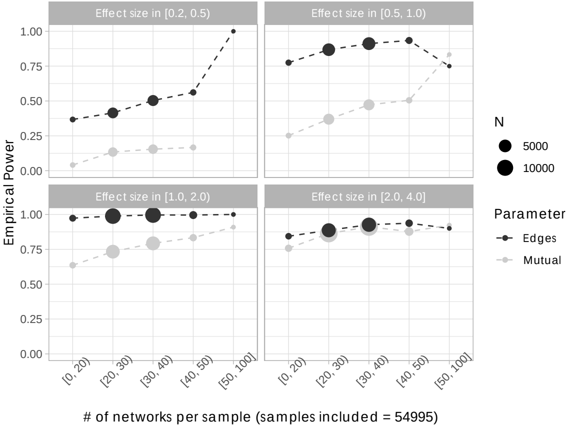

Figure 4: Empirical power of Pooled-ERGM estimates at various levels of effect size. As expected, power increases significantly with sample size (# of networks per sample). Interestingly, the discovery rate of an effect size within $[1, 2)$ is very high even with a sample size of 20-30 networks. More extreme points have higher volatility due to small number of samples included.

So now that we can estimate ERGMs for small networks (cool!)...

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... what can this tell us about our 42 teams?

Preliminary results

	Advice	Influence	Leadership
Edges	-1.87^{***}	-0.78^{***}	-0.57^{***}
	(0.30)	(0.13)	(0.14)
Transitive Triads	0.24^{***}	0.21^{**}	
	(0.06)	(0.08)	
Indeg. RME	0.35^{***}		
	(0.08)		
Outdeg. Female	0.43^{*}		
	(0.19)		
Outdeg. Social Accomodation	0.11		
	(0.08)		
Indeg. Female			-0.38^{*}
			(0.19)
AIC	693.18	760.40	655.78
BIC	714.50	769.12	664.32
Log Likelihood	-341.59	-378.20	-325.89
Num. networks	38	41	38

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 2: The two statistics that showed to be the most robust were **Indeg. RME** and **Outdeg. Female**. These two effects can be described as (1) individuals with high levels of RME receive more ties, and (2) female subjects were more likely of seeking advice than male. Other statistics such as GPA, religiousness, age, and ethnicity were not significant.

Part II: Association between network structure and team performance

Testing effects of social network structure on group performance

Two common approaches: Generalized Linear Models (GLMs), or permutation-like tests. Both have limitations:

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BTW Degree sequence $\mapsto$ Scale-free networks

> “*The **structural diversity of real-world networks** uncovered here presents both a puzzle and an opportunity. The strong focus in the scientific literature on **explaining and exploiting scale-free patterns has meant relatively less is known about mechanisms that produce non-scale-free structural patterns**, e.g., those with degree distributions better fitted by a log-normal. Two important directions of future work will be the **development and validation of novel mechanisms for generating more realistic degree structure in networks**, and novel statistical techniques for identifying or untangling them given empirical data*”
>
> **– p. 8, Broido and Clauset (2019)**

See Holme (2019) for a recent reference on the Scale-free issue.

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In principle, this would be equivalent to a revised rewiring test. . .

Algorithm

1. Estimate an ERGM (estimates can come from a single graph or pooled estimates). We denote the data-generating-process of this model as $\mathcal{D} : \Theta \times \mathcal{X} \mapsto \mathcal{G}$.

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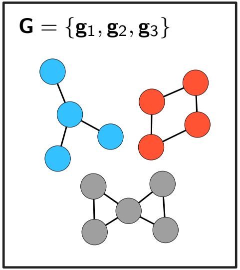
Note An important distinction to make is that structures that gave origin to the graph need not to be relevant for the team's performance per se.

Illustrated example

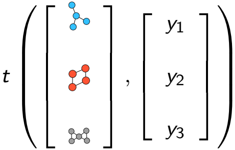
Suppose that we have a 3 networks of sizes 4, 4, and 5 respectively. The

Step 1:
Fit the ERGMito



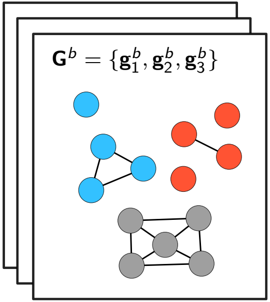
Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

Step 2:
Calculate $t_0 =$



Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We can use the distribution of the sequence $\{t_1, \dots, t_B\}$ as null to compare against t_0

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y	nodeicov("female")
1.0138091	2
0.6051448	1
4.3085153	2
0.9547600	0
-0.1330788	1

We will try to assess the presence/absence of an association between the outcome y and the number of ties received by women, `nodeicov("female")`, using both a linear regression model and our test

- Our approach Is the the observed correlation statistic $t_0 = \mathrm{cor}\left(y, \texttt{nodeicov("female")}\right)$ different than as expected by chance (assuming $\mathbf{g}$ follows an ERGM distribution)?
- OLS approach In the following model:

$$y = \alpha + \theta^{OLS}\texttt{nodeicov("female")}$$

is the θ_{OLS} parameter significantly different from zero?

Extended example with `fivenets` (cont'd)

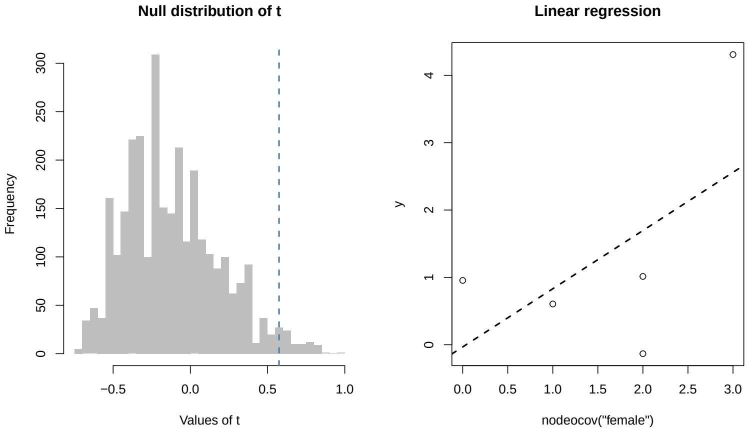

Figure 5: Comparing our method against a linear regression. Our proposed method returned a two sided p-value of 0.045, while the pvalue for the OLS coefficient was 0.311.

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3. Working on a more formal statistical framework (when is it a good/bad idea to use this kind of method).

Thanks!

George G. Vega Yon

Let's chat!

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@gvegayon

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References I

Allaire, JJ, Yihui Xie, Jonathan McPherson, Javier Luraschi, Kevin Ushey, Aron Atkins, Hadley Wickham, Joe Cheng, Winston Chang, and Richard Iannone. 2018. Rmarkdown: Dynamic Documents for R. https://rmarkdown.rstudio.com.

Broido, Anna D, and Aaron Clauset. 2019. "Scale-Free Networks Are Rare." Nature Communications 10 (1): 1017.

Csardi, Gabor. 2015. Igraphdata: A Collection of Network Data Sets for the 'Igraph' Package. https://CRAN.R-project.org/package=igraphdata.

Geyer, Charles J., and Elizabeth A. Thompson. 1992. "Constrained Monte Carlo Maximum Likelihood for Dependent Data." Journal of the Royal Statistical Society. Series B (Methodological) 54 (3): 657–99. http://www.jstor.org/stable/2345852.

Handcock, Mark S. 2003. "Assessing Degeneracy in Statistical Models of Social Networks." Working Paper No. 39 76 (39): 33–50. https://doi.org/10.1.1.81.5086.

References II

Handcock, Mark S., David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, and Martina Morris. 2018. Ergm: Fit, Simulate and Diagnose Exponential-Family Models for Networks. The Statnet Project (http://www.statnet.org). https://CRAN.R-project.org/package=ergm.

Handcock, Mark, Peng Wang, Garry Robins, Tom Snijders, and Philippa Pattison. 2006. "Recent developments in exponential random graph (p*) models for social networks." Social Networks 29 (2): 192–215. https://doi.org/10.1016/j.socnet.2006.08.003.

Holme, Petter. 2019. "Rare and everywhere: Perspectives on scale-free networks." Nature Communications 10 (1): 1016. https://doi.org/10.1038/s41467-019-09038-8.

Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. "Ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks." Journal of Statistical Software 24 (3): 1–29.

References III

Kim, Young Ji, David Engel, Anita Williams Woolley, Jeffrey Yu-Ting Lin, Naomi McArthur, and Thomas W. Malone. 2017. "What Makes a Strong Team?: Using Collective Intelligence to Predict Team Performance in League of Legends." In Proceedings of the 2017 Acm Conference on Computer Supported Cooperative Work and Social Computing, 2316–29. CSCW '17. New York, NY, USA: ACM. https://doi.org/10.1145/2998181.2998185.

Milo, R, N Kashtan, S Itzkovitz, M E J Newman, and U Alon. 2004. "On the uniform generation of random graphs with prescribed degree sequences." Arxiv Preprint Condmat0312028 cond-mat/0: 1–4. http://arxiv.org/abs/cond-mat/0312028.

R Core Team. 2018. R: A Language and Environment for Statistical Computing. Vienna, Austria: R Foundation for Statistical Computing. https://www.R-project.org/.

Wasserman, Stanley, and Philippa Pattison. 1996. "Logit models and logistic regressions for social networks: I. An introduction to Markov graphs andp." Psychometrika 61 (3): 401–25. https://doi.org/10.1007/BF02294547.

Xie, Yihui. 2018. Knitr: A General-Purpose Package for Dynamic Report Generation in R. https://yihui.name/knitr/.