Big Problems for Small Networks: Statistical Analysis of Small Networks and Team Performance

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Contents

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Network Science of Teams

a Multidisciplinary University Research Initiative

Part I: Exponential Random Graph Models for Small Networks

Exponential Random Graph Models

Figure 1: Friendship network of a UK university faculty. Source: `igraphdata` R package (Gabor Csardi, 2015)

How can we explain what we see here?

ERGMs

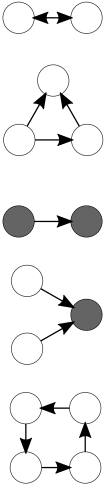
- The lingua franca of social network analysis.
- Seeks to answer the question: What local social structures gave origin to a given observed graph?
- The model is centered around a vector of **sufficient statistics** $s()$, and is operationalized as:

$$\Pr\left(\mathbf{G}=\mathbf{g}\mid\theta,\mathbf{X}\right)=\frac{\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g},\mathbf{X}\right)\right\}}{\kappa\left(\theta,\mathbf{X}\right)},\quad\forall\mathbf{g}\in\mathcal{G}\tag{1}$$

  Where $\kappa\left(\theta,\mathbf{X}\right)$ is the normalizing constant and equals $\sum_{\mathbf{g}'\in\mathcal{G}}\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g}',\mathbf{X}\right)\right\}$. Figure 2 shows some examples of values in $s()$.

- Overall, an ERGM identifies the set of parameters θ that maximize the likelihood of observing a given graph $\mathbf{g}$ over the entire set of possible networks, $\mathcal{G}$,
- In the case of directed networks, $\mathcal{G}$ has $2^{n(n-1)}$, terms.
- See Wasserman, Pattison, Robins, Snijders, Handcock and others.

Structures



Mutual Ties
$\sum_{i \neq j} y_{ij} y_{ji}$

Transitive Triad
$\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$

Homophily
$\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$

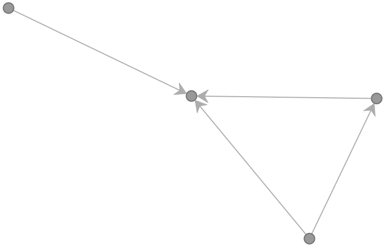
Covariate Effect for Incoming Ties
$\sum_{i \neq j} y_{ij} x_j$

Four Cycle
$\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

Figure 2: Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

Example of model

In this network



We see 4 **edges**, 1 **transitive triad** and **no mutual ties**.

The probability function of this model would be

$$\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta\right) = \frac{\exp\left\{4\theta_{edges} + \theta_{ttriads}\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}'\right)\right\}}$$

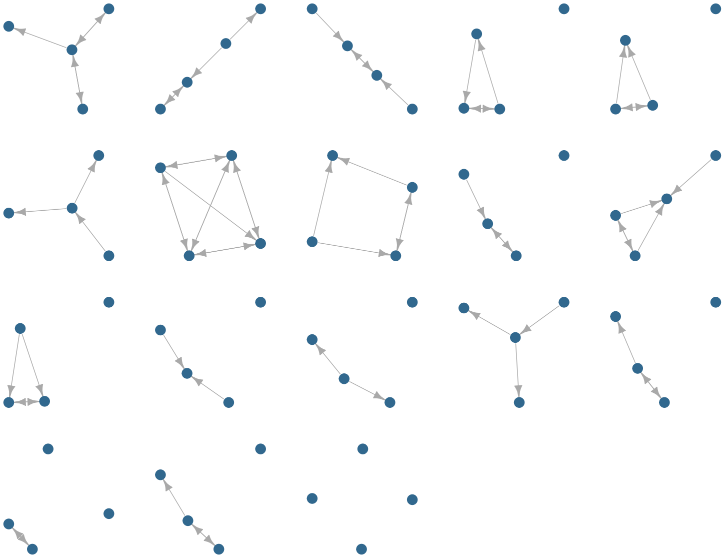
with $\theta = \left[\theta_{edges} \quad \theta_{ttriads} \quad \theta_{mutual}\right]^{\mathbf{t}}$

This model has **MLE parameter estimates** of -0.19 (low density), 0.27 (high chance of ttriads), and -9.75 (low chance of mutuality) for the parameters `edges`, `ttriads`, and `mutual` respectively.

Estimation of ERGMs

- Calculating of the normalizing constant in (1), $\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}$, makes ERGMs difficult to estimate.
- For this reason, statistical methods have focused on avoiding the direct calculation of κ; most modern methods for estimating ERGMs rely on MCMC.
- While significant advances have been made in the area, simulation based models can suffer from **model degeneracy**.
- Model degeneracy is particularly problematic with small networks.

ERGMs for Small Networks



ERGMs for Small Networks (cont'd)

- In the case of small networks (e.g. at most 6 nodes), the calculation of κ becomes computationally feasible.
- This allows direct calculation of (1), **avoiding the need for simulations** and allowing us to obtain Maximum Likelihood Estimates using *standard* optimizations techniques.
- More importantly, in the case that a common data generating process can be assumed, a pooled version of the ERGMs can be estimated.

$$
\Pr\left(\mathbf{G}_1 = \mathbf{g}_1, \dots, \mathbf{G}_p = \mathbf{g}_p \mid \theta, \mathbf{X}_1, \dots, \mathbf{X}_p\right) = \prod_p \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}_p, \mathbf{X}_p\right)\right\}}{\kappa_p\left(\theta, \mathbf{X}_p\right)}
$$

How different is this from the "normal" way to fit ERGMs?

Estimation of ERGMitos (cont'd)

Description of the MCMC-MLE algorithm (one of the approaches)

1. Make an initial guess of the model parameters using MPL (maximum pseudo likelihood), $\hat{\theta}_0$
2. While the algorithm doesn't converge, do:
   a. Simulate B graphs from $\Pr\left(\mathbf{G}=\mathbf{g}\,\middle|\,\hat{\theta}_0, \mathbf{X}\right)$ using an important sampler
   b. Use the simulated sequence of graphs to approximate the likelihood function. And with this approximation update the parameter θ_0 using a Newton-Raphson step.
   c. next iteration

By skiping the MCMC part we:

1. are able to get MLE estiamates directly,
2. avoiding the degeneracy problem latent in MCMC, and
3. obtain more accurate estimates faster.

We have implemented this and more in the `ergmito` R package

An important pause...

ito, ita: Suffix in spanish used to denote small. e.g.:

*¡Que lindo ese perr**ito**! / What a beautiful little dog!*
*¿Me darias una tac**ita** de azúcar? / Would you give me a small cup of sugar?*

Special thanks to George Barnett who proposed the name during the 2018 NASN!

Features of `ergmito`

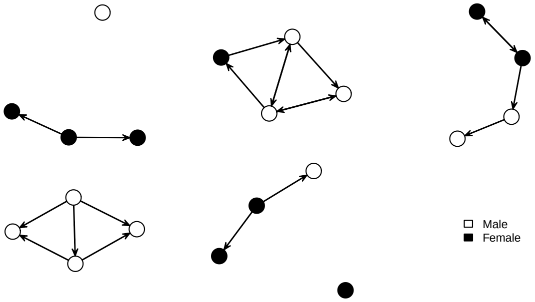
This (lifecycle experimental) R package has the following features

- Built on top of **statnet**'s `ergm` R package.
- Allows estimating ERGMs for small networks (less than 7 and perhaps 6)[1] via MLE.
- Implements pooled ERGM models.
- In the same spirit of the exhaustive enumeration, includes a simulation function for small networks sampling from the true distribution.

[1] A directed graph of size 6 has a support set with $2^{6\times(6-1)} = 1,073,741,824$ elements.

`ergmito` example

```r
library(ergmito)
data(fivenets, package = "ergmito")
```



```r
# Looking at one of the five networks
fivenets[[1]]
```

```
##  Network attributes:
##   vertices = 4
##   directed = TRUE
##   hyper = FALSE
##   loops = FALSE
##   multiple = FALSE
##   bipartite = FALSE
##   total edges= 2
##     missing edges= 0
##     non-missing edges= 2
##
##  Vertex attribute names:
##     female name
##
## No edge attributes
```

So how can we fit this model?

ergmito example (cont'd)

The same as you would do with the ergm package:

```
(model1 <- ergmito(fivenets ~ edges + nodematch("female")))
```

```
##
## ERGMito estimates
##
##  Coefficients:
##            edges  nodematch.female
##           -1.705             1.587
```

	Model 1
edges	-1.70^{**}
	(0.54)
nodematch.female	1.59^{*}
	(0.64)
AIC	73.34
BIC	77.53
Log Likelihood	-34.67
# Networks	5

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Statistical models

Simulation Study

We conducted a simulation study to explore the properties of MLE for small networks (a.k.a. ERGMito). To generate each sample of teams:

1. Draw the **population parameters** from a piecewise Uniform with values in $[-4, -.1] \cup [.1, 4]$
2. We will draw **groups of sizes 3 to 5**. The number of networks per group size are drawn from a Poisson distribution with parameter 10 (hence, an expected size of 30 networks per sample).
3. Use the **drawn parameters** and **group sizes** to **generate random graphs** using an ERGM data generating process.

We simulated 100,000 samples, each one composed of an average of 30 networks.

Simulation Study (cont'd)

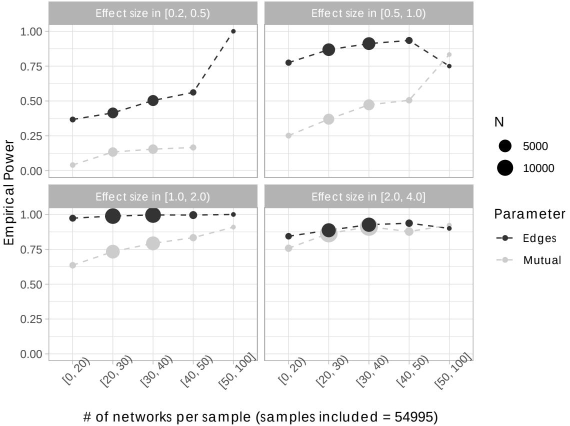


Figure 3: Empirical power of Pooled-ERGM estimates at various levels of effect size. As expected, power increases significantly with sample size (# of networks per sample). Interestingly, the discovery rate of an effect size within $[1, 2)$ is very high even with a sample size of 20-30 networks. More extreme points have higher volatility due to small number of samples included.

Part II: Association between group structure and team performance

Testing effects of social network structure on group performance

Two common approaches: Generalized Linear Models (GLMs), or Mantel-like tests (a.k.a. permutation tests). Both have limitations:

- GLMs:
  - Sample size is problematic: How costly is getting enough teams to run get a desired level of power?
- Permutation tests:
  - Common approach: sample from graphs with the same degree sequence–the observed sequence of in/out degree
  - This is oversimplifying/constraining
  - And worse, in a network of size 4, how many different networks can be observed **holding the degree sequence fixed**?

An idea

- Lack of power:What if we just simulate them?OK, but aren't we doing this with permutation tests?...
- Sure, but what about ERGMs?Ultimately these models describe a distribution of graphs that have on average the same set of network statistics
- This overcomes the problem observed in permutation tests.

A semiparametric test

Notation

- $\mathbf{G} = \{\mathbf{g}_j\}$ is a sequence of J graphs that share a common data-generating-process, e.g. teams formed in a lab.
- Each network has node-level attributes $x \in \mathcal{X}$.
- A group(graph) level outcome variable, such as team performance, Y.
- Under the null, network structure and group performance are not associated, this is $Y \perp \mathbf{G}$.

Algorithm

1. Estimate an ERGM (estimates can come from a single graph or pooled estimates). We denote the data-generating-process of this model as $\mathcal{D} : \Theta \times \mathcal{X} \mapsto \mathcal{G}$.

2. Calculate the value $s_0 = s(\mathbf{G}, Y)$.

3. Now, for $b \in \{1, \dots, B\}$ do:

  3.1 For each group j in $\{1, \dots, J\}$, draw a new network $\mathbf{g}_j^b \sim \mathcal{D}(\hat{\theta}, X_j)$, this new sequence is denoted $\mathbf{G}^b$

  3.2 Using $\mathbf{G}^b$ and Y, calculate $s_b = s(\mathbf{G}^b, Y)$

  3.3 Next b.

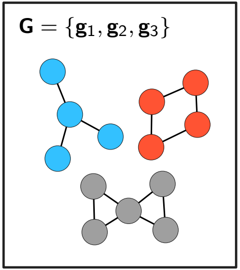
This will generate a null distribution for the statistic s, which we can use to compare against the observed statistic, s_0.

Note An important distinction to make is that structures that gave origin to the graph need not to be relevant for the team's performance per se.

Illustrated example

Suppose that we have a 3 networks of sizes 4, 4, and 5 respectively. The

Step 1:
Fit the ERGMito



Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

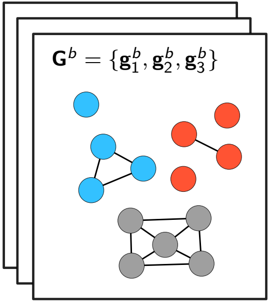
Step 2:
Calculate $s_0 =$

$$s\left(\begin{bmatrix}\mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3\end{bmatrix}, \begin{bmatrix}y_1 \\ y_2 \\ y_3\end{bmatrix}\right)$$

This is the observed statistic.

We will generate a null distribution where $\mathcal{G} \perp Y$.

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $s_b = s(\mathbf{G}^b, Y)$

We can use the distribution of the sequence $\{s_1, \dots, s_B\}$ as null to compare against s_0

Part III: An empirical example

Small teams performance

Data

Results

Concluding remarks

Exponential Random Graph Models for Small Networks:

1. Not a new thing,what's new is the tool to do so in a smooth way
2. Still work to do (on the development side of things): Goodness of fit tests, better algorithms for drawing random graphs, bayesian model
3. Can be extended to other types of ERGMs... our next target: TERGMs (Separable Exponential Random Graph Models)

Test for Association between graph level outcomes and graph structures:

1. Still need to run simulation studies
2. Key question 1: What happens when the graph structure is in both the ERGM and the stat
3. What about other properties such as type I error?

Thanks!

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Let's chat!

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