Big Problems for Small Networks: Statistical Analysis of Small Networks and Team Performance

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Contents

Acknowledgements

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We thank members of our MURI research team, USC's Center for Applied Network Analysis, Andrew Slaughter, and attendees of the NASN 2018 conference for their comments.

Part I: Exponential Random Graph Models for Small Networks

Exponential Random Graph Models

Figure 1: Friendship network of a UK university faculty. Source: `igraphdata` R package (Gabor Csardi, 2015)

How can we explain what we see here?

ERGMs

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$$\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\kappa\left(\theta, \mathbf{X}\right)}, \quad \forall \mathbf{g} \in \mathcal{G} \tag{1}$$

Where $\kappa\left(\theta, \mathbf{X}\right)$ is the normalizing constant and equals $\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}$. Figure 2 shows some examples of values in $s()$.

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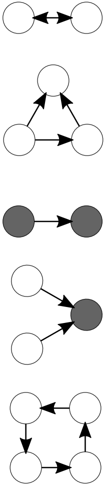
- See Wasserman, Pattison, Robins, Snijders, Handcock and others.

Structures



Mutual Ties
$\sum_{i \neq j} y_{ij} y_{ji}$

Transitive Triad
$\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$

Homophily
$\sum_{i \neq j} y_{ij} \mathbf{1}(x_i = x_j)$

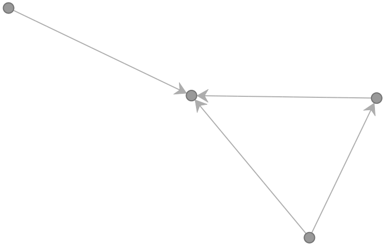
Covariate Effect for Incoming Ties
$\sum_{i \neq j} y_{ij} x_j$

Four Cycle
$\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

Figure 2: Besides of the common edge count statistic (number of ties in a graph), ERGMs allow measuring other more complex structures that can be captured as sufficient statistics.

Example of model

In this network



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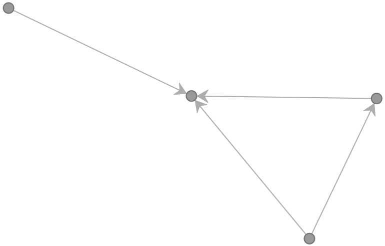

We see 4 **edges**, 1 **transitive triad** and **no mutual ties**.

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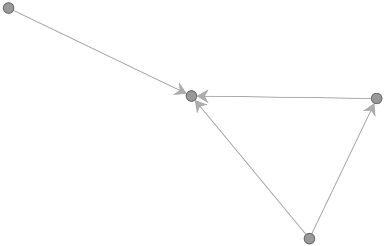

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The probability function of this model would be

$$\Pr\left(\mathbf{G}=\mathbf{g}\mid\theta\right)=\frac{\exp\left\{4\theta_{edges}+\theta_{ttriads}\right\}}{\sum_{\mathbf{g}'\in\mathcal{G}}\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g}'\right)\right\}}$$

with $\theta=\left[\theta_{edges}\quad\theta_{ttriads}\quad\theta_{mutual}\right]^{\mathbf{t}}$

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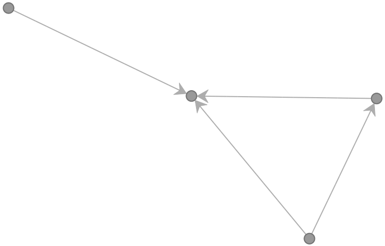

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with $\theta = [\theta_{edges} \quad \theta_{ttriads} \quad \theta_{mutual}]^{\mathbf{t}}$

This model has **MLE parameter estimates** of -0.19 (low density), 0.27 (high chance of ttriads), and -9.75 (low chance of mutuality) for the parameters `edges`, `ttriads`, and `mutual` respectively.

Estimation of ERGMs

- Calculating of the normalizing constant in (1), $\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}$, makes ERGMs difficult to estimate.

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- While significant advances have been made in the area, simulation based models can suffer from **model degeneracy**.
- Model degeneracy is particularly problematic with small networks.

Estimation of ERGMs (cont'd)

Description of the MCMC-MLE algorithm (one of the approaches)

1. Make an initial guess of the model parameters using MPL (maximum pseudo likelihood), $\hat{\theta}_0$

2. While the algorithm doesn't converge, do:

   a. Simulate B graphs from $\Pr\left(\mathbf{G} = \mathbf{g} \,\middle|\, \hat{\theta}_0, \mathbf{X}\right)$ using an important sampler

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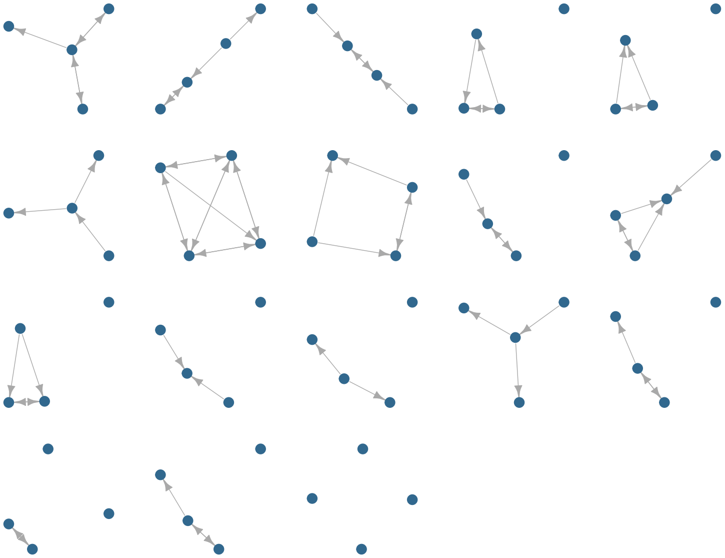
By skiping the MCMC part we are able to get MLE estiamates directly avoiding the degeneracy problem latent in MCMC, faster, and more accurately.

ERGMs for Small Networks



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- More importantly, in the case that a common data generating process can be assumed, a pooled version of the ERGMs can be estimated.

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- We have implemented this and more in the `ergmito` (lifecycle experimental) R package (https://github.com/muriteams/ergmito)

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- In the same spirit of the exhaustive enumeration, includes a simulation function for small networks sampling from the true distribution.

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Simulation Study

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We simulated 100,000 samples, each one composed of an average of 30 networks.

A drawing of the simulation process

Simulation Study (cont'd)

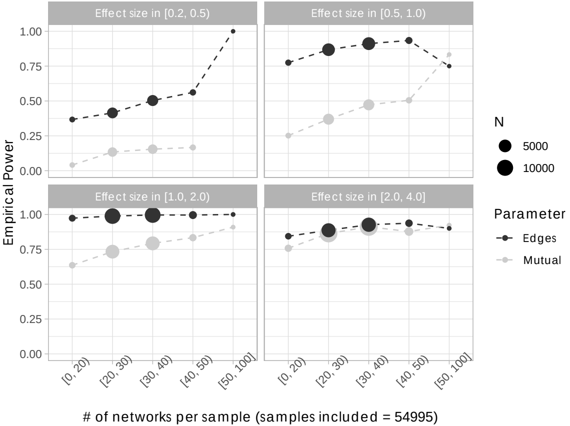

Figure 3: Empirical power of Pooled-ERGM estimates at various levels of effect size. As expected, power increases significantly with sample size (# of networks per sample). Interestingly, the discovery rate of an effect size within $[1, 2)$ is very high even with a sample size of 20-30 networks. More extreme points have higher volatility due to small number of samples included.

Part II: Association between group structure and team performance

Testing effects of social network structure on group performance

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- Permutation tests:
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  - This is oversimplifying/constraining
  - And worse, in a network of size 4, how many different networks can be observed **holding the degree sequence fixed**?

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- Sure, but what about ERGMs?Ultimately these models describe a distribution of graphs that have <u>on average</u> the same set of network statistics
- This overcomes the problem observed in permutation tests.

A semiparametric test

Notation

- $\mathbf{G} = \{\mathbf{g}_j\}$ is a sequence of J graphs that share a common data-generating-process, e.g. teams formed in a lab.
- Each network has node-level attributes $x \in \mathcal{X}$.
- A group(graph) level outcome variable, such as team performance, Y.
- Under the null, network structure and group performance are not associated, this is $Y \perp \mathbf{G}$.

Algorithm

1. Estimate an ERGM (estimates can come from a single graph or pooled estimates). We denote the data-generating-process of this model as $\mathcal{D} : \Theta \times \mathcal{X} \mapsto \mathcal{G}$.
2. Calculate the value $s_0 = s(\mathbf{G}, Y)$.
3. Now, for $b \in \{1, \dots, B\}$ do:
   - **3.1** For each group j in $\{1, \dots, J\}$, draw a new network $\mathbf{g}_j^b \sim \mathcal{D}(\hat{\theta}, X_j)$, this new sequence is denoted $\mathbf{G}^b$
   - **3.2** Using $\mathbf{G}^b$ and Y, calculate $s_b = s(\mathbf{G}^b, Y)$
   - **3.3** Next b.

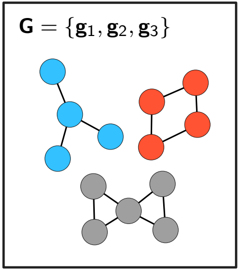
This will generate a null distribution for the statistic s, which we can use to compare against the observed statistic, s_0.

Note An important distinction to make is that structures that gave origin to the graph need not to be relevant for the team's performance <u>per se</u>.

Illustrated example

Suppose that we have a 3 networks of sizes 4, 4, and 5 respectively. The

Step 1:
Fit the ERGMito



Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

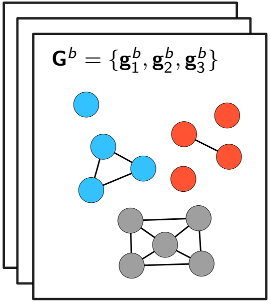
Step 2:
Calculate $s_0 =$

$$s\left(\begin{bmatrix}\mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3\end{bmatrix}, \begin{bmatrix}y_1 \\ y_2 \\ y_3\end{bmatrix}\right)$$

This is the observed statistic.

We will generate a null distribution where $\mathcal{G} \perp Y$.

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $s_b = s(\mathbf{G}^b, Y)$

We can use the distribution of the sequence $\{s_1, \dots, s_B\}$ as null to compare against s_0

Part III: An empirical example

Small teams performance

Data

Results

Concluding remarks

Thanks!

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