Exact Statistics and Semi-Parametric Tests for Small Network Data[1]

George G. Vega Yon, MS   Andrew Slaughter   Kayla de la Haye, PhD

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[1] Contact: vegayon@usc.edu. We thank members of our MURI research team, USC's Center for Applied Network Analysis, Garry Robins, Carter Butts, Johan Koskinen, Noshir Contractor, and attendees of the NASN 2018 conference for their comments.

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Network Science of Teams

a Multidisciplinary University Research Initiative

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Context: A tale about social abilities and team performance

Recruited

- 42 mixed gender groups of 3 to 5 participants (unknown)
- Eligibility: (1) 18+ years, (2) Native English speaker

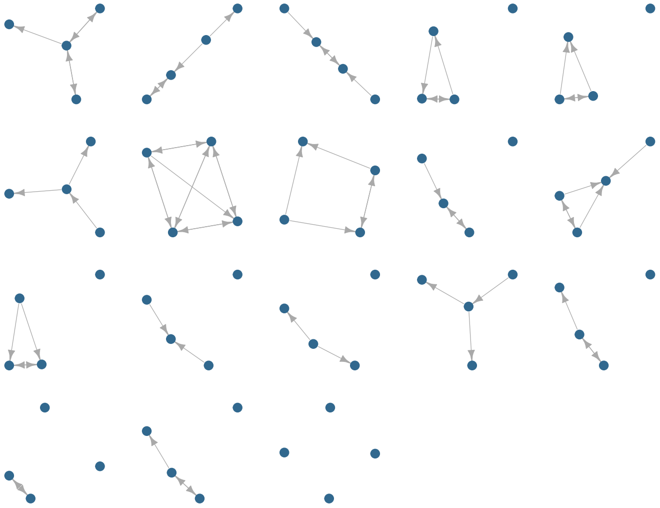
2 hour group session

- Group tasks (2 sets of tasks x 30 minutes each)
- Measure group social networks and individual social intelligence (SI)

Study motivation

- Overall, a very limited set of SI domains have been tested as predictors of social networks
- Very little research on the emergence of networks in teams.

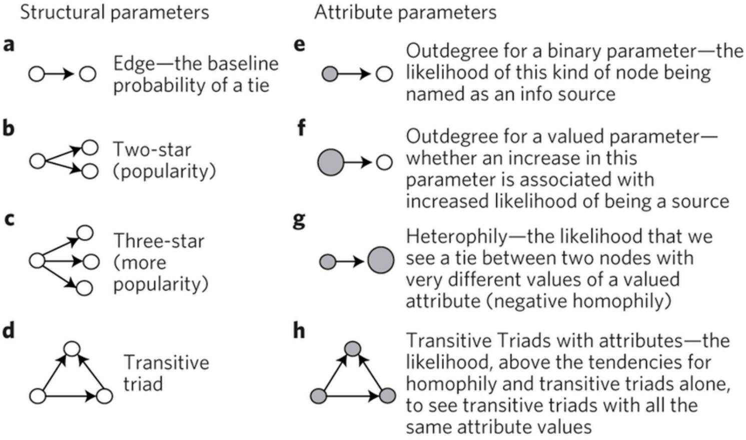
Context (cont'd)



How can we go beyond descriptive statistics?

Small networks and Exponential Random Graph Models

Exponential Random Graph Models: What are the structures that give origin to a given observed graph?



Structural parameters

a Edge—the baseline probability of a tie

b Two-star (popularity)

c Three-star (more popularity)

d Transitive triad

Attribute parameters

e Outdegree for a binary parameter—the likelihood of this kind of node being named as an info source

f Outdegree for a valued parameter—whether an increase in this parameter is associated with increased likelihood of being a source

g Heterophily—the likelihood that we see a tie between two nodes with very different values of a valued attribute (negative homophily)

h Transitive Triads with attributes—the likelihood, above the tendencies for homophily and transitive triads alone, to see transitive triads with all the same attribute values

(In general, ties are not IID, moreover, the entire graph is a single observation.)

Small networks and Exponential Random Graph Models (Cont'd)

- Estimating ERGMs is a complex problem. The likelihood function has $2^{n(n-1)}$ terms, which means that for a graph of size 6 we have about 1 billion terms to compute.
- Current approaches to estimate ERGMs rely on simulation methods, for example MCMC-MLE

When trying to estimate ERGMs in little networks

- MCMC fails to converge when trying to estimate a block diagonal (structural zeros) model,
- Same happens when trying to estimate an ERGM for a single (little) graph,
- Even if it converges, model degeneracy, i.e. bad fit, arises too often.

Rethinking the problem

- 1st Step: Forget about MCMC-MLE estimation, take advantage of small sample and use exact statistic for MLEs:

$$\Pr\left(\mathbf{Y}=\mathbf{y}|\theta,\mathcal{Y}\right)=\frac{\exp\theta^{\mathsf{T}}\mathbf{g}(\mathbf{y})}{\kappa\left(\theta,\mathcal{Y}\right)},\quad \mathbf{y}\in\mathcal{Y}$$

  Where $\mathbf{g}(\mathbf{y})$ is a vector of sufficient statistics, $\theta\in\Theta$ a vector of model parameters, and $\kappa\left(\theta,\mathcal{Y}\right)$ is the normalizing constant (a summation with $2^{n(n-1)}$ terms)

- This solves the problem of been able to estimate a small ergm.
- For this we started working on the `ergmito` R package (available at https://github.com/muriteams/ergmito):

Example 1

Let's start by trying to estimate an ERGM for a single graph of size 4

```r
library(ergmito)
set.seed(12)
x <- sna::rgraph(4)
ergmito(x ~ edges + balance + mutual)
```

```
##
## ERGMito estimates
##
##  Coefficients:
##    edges  balance   mutual
## -1.9443  -0.2417   3.4961
```

- Cool, we are able to estimate ERGMs for little networks! (we actually call them ~~ergmitos~~ ERGMitos[2]),
- Going directly to MLE, we avoid the degeneracy problem.
- Moreover, due to the size of the networks, we can actually go further and estimate pooled ERGMs

[2] Thanks to George Barnett for suggesting the name!

Solution

- When estimating a block diagonal ERGM we were essentially assuming independence across networks.
- This means that we can actually do the same with exact statistics approach to calculate a joint likelihood:

$$\Pr\left(\mathbf{Y} = \{\mathbf{y}_i\} | \theta, \{\mathcal{Y}_i\}\right) = \prod_i \frac{\exp \theta^{\mathsf{T}} \mathbf{g}(\mathbf{y}_i)}{\kappa_i\left(\theta, \mathcal{Y}_i\right)}$$

- By estimating a pooled version of the ERGM we can increase the power of our MLEs.
- We implemented this in the `ergmito` package

Example 2

Suppose that we have 3 little graphs of sizes 4, 5, and 5:

```r
library(ergmito)
set.seed(12)
x1 <- sna::rgraph(4)
x2 <- sna::rgraph(5)
x3 <- sna::rgraph(5)

ergmito(list(x1, x2, x3) ~ edges + balance + mutual)
```

```
##
## ERGMito estimates
##
##  Coefficients:
##   edges  balance   mutual
## -0.3941  -0.2085   1.4156
```

Simulation study

Scenario A

1. Draw parameters for edges and mutual from a uniform(-3, 3).
2. Using those parameters, sampled $n \sim \text{Poisson}(30)$ networks of size 4
3. Estimated the pooled ERGMs using both the MLE and the bootstrap version.

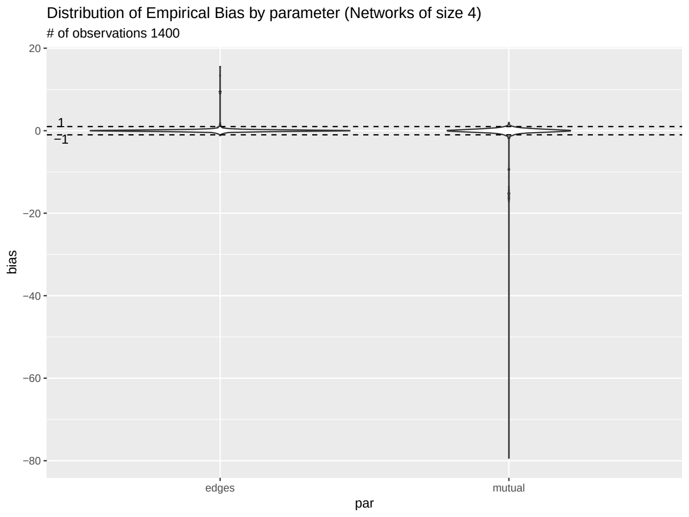
Scenario B

1. Idem.
2. Using those parameters, sampled $n_1 \sim \text{Poisson}(15), n_2 \sim \text{Poisson}(15)$ networks of size 3 and 4 respectively.
3. Idem.

(If anyone asks, I just ran about 3 million ERGMs... :))

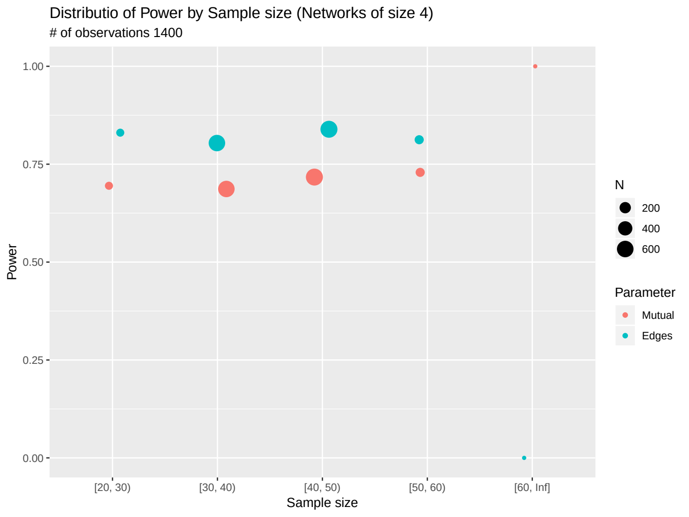
Simulation study: Scenario A

Empirical Bias



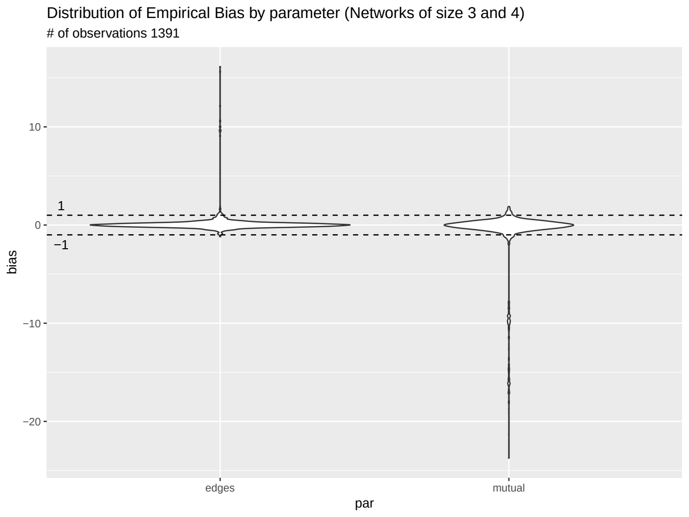
Simulation study: Scenario A

Power



Simulation study: Scenario B

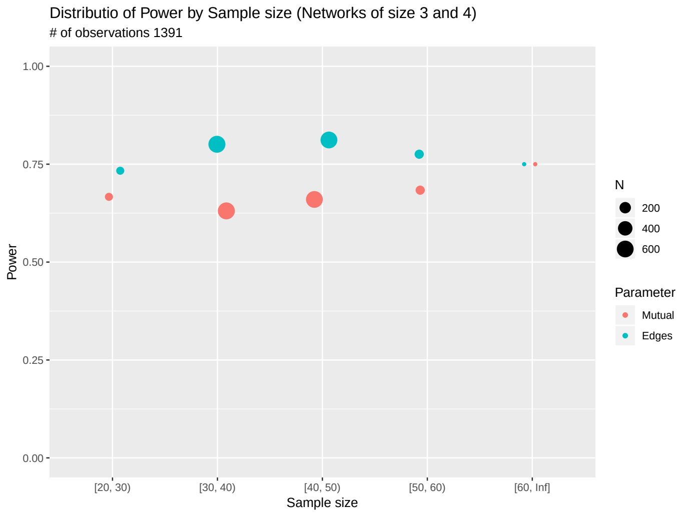
Empirical Bias


Distribution of Empirical Bias by parameter (Networks of size 3 and 4)

# of observations 1391

Simulation study: Scenario B

Power



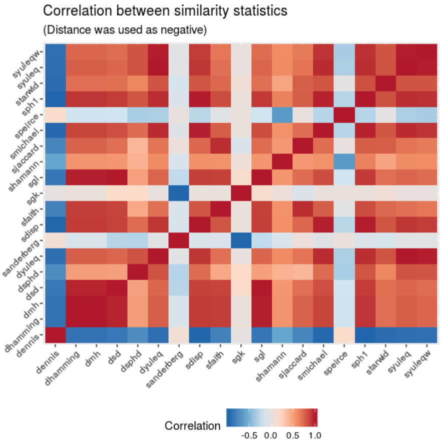
Preliminary results

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.

Other approaches


Correlation between similarity statistics

(Distance was used as negative)

Discussion

- First set of results from the simulation study are encouraging
- Need to conduct more simulations using nodal attributes and networks of size 5 (right now having problems when building the DGP).
- Small structures imply a smaller pool of parameters (which is OK), but can be more useful when including nodal attributes.
- When estimating the pooled version, we are essentially hand-waving the fact that parameter estimates implicitly encode size of the graph, i.e.

  > *Does a the estimate of `edge = 0.1` has the same meaning for a network of size 3 to a size 5? (but perhaps is not such a big deal)*

- Finally, this work can be extended to other types of small networks, including: families, ego-nets, etc. And other methods, such as TERGMs.

Thank you!

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