Exact Statistics and Semi-Parametric Tests for Small Network Data

George G. Vega Yon, MS[⋆] Andrew Slaughter, PhD[†] Kayla de la Haye, PhD[⋆]

[⋆]University of Southern California
Department of Preventive Medicine

[†]U.S. Army Research Institute for the
Behavioral and Social Sciences

Sunbelt 2019, Montreal
June 20, 2019

Acknowledgements

This material is based upon work support by, or in part by, the U.S. Army Research Laboratory and the U.S. Army Research Office under grant number W911NF-15-1-0577

Computation for the work described in this paper was supported by the University of Southern California's Center for High-Performance Computing (hpc.usc.edu).

The views expressed in this presentation are those of the authors, and do not represent the official policy or positions of the Department of the Army, the DoD, or the U.S. Government.

Network Science of Teams
a Multidisciplinary University Research Initiative
UC SANTA BARBARA
USC University of Southern California
MIT Massachusetts Institute of Technology
NORTHWESTERN UNIVERSITY

Context: Social abilities and team performance

Two research questions

Context: Social abilities and team performance

Two research questions

How do **social abilities** impact **network structure**?

Context: Social abilities and team performance

Two research questions

How do **social abilities** impact **network structure**?

How does **collective intelligence** affect team (network) **performance**?

Context: Social abilities and team performance

Two research questions

How do **social abilities** impact **network structure**?

How does collective intelligence affect team (network) performance?

To answer this question, we have the following experimental data:

To answer this question, we have the following experimental data:

- 42 mixed-gender teams,

To answer this question, we have the following experimental data:

- 42 mixed-gender teams,
- Which completed 1 hour of group tasks (MIT's IQ test for teams)

To answer this question, we have the following experimental data:

- 42 mixed-gender teams,
- Which completed 1 hour of group tasks (MIT's IQ test for teams)
- Individual survey capturing information regarding socio-demographics **and**:

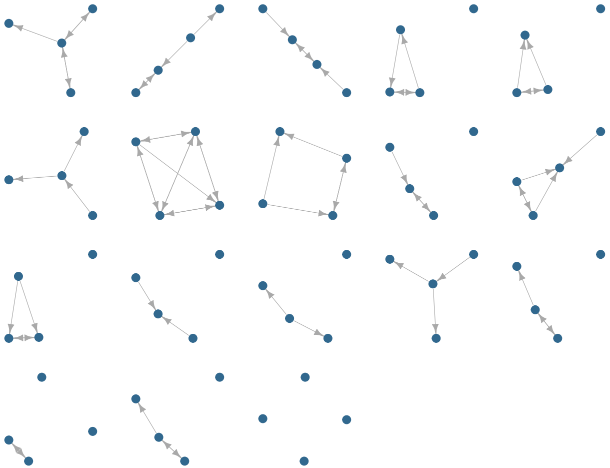
To answer this question, we have the following experimental data:

- 42 mixed-gender teams,
- Which completed 1 hour of group tasks (MIT's IQ test for teams)
- Individual survey capturing information regarding socio-demographics **and**:
  - **Social Intelligence**: Social Perception (measured by RME), Social Accomodation, Social Gregariousness, and Social Awareness

To answer this question, we have the following experimental data:

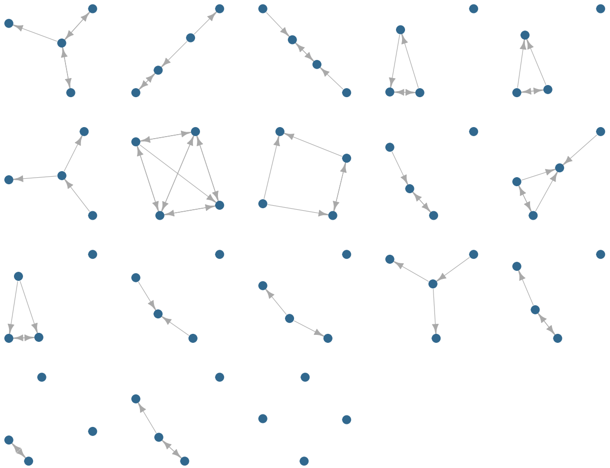
- 42 mixed-gender teams,
- Which completed 1 hour of group tasks (MIT's IQ test for teams)
- Individual survey capturing information regarding socio-demographics **and**:
  - **Social Intelligence**: Social Perception (measured by RME), Social Accomodation, Social Gregariousness, and Social Awareness
  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

Context (cont'd)



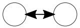
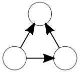
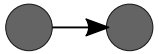
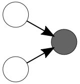
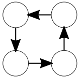
We can do a lot of simple statistics: density, % of *[blank]*, etc. but...

Context (cont'd)



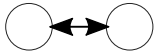
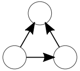
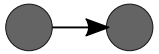
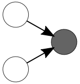
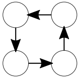
We can do a lot of simple statistics: density, % of *[blank]*, etc. but... **how can we go beyond that?**

Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}\left(x_i = x_j\right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

ERGMs can do the job, but the only problem is...

Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1} \left(x_i = x_j \right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

ERGMs can do the job, but the only problem is... have you tried estimating ERGMs on small networks?

Exponential random graph models for small networks

A lot of

Exponential random graph models for small networks

A lot of

- Playing with the MCMC control parameters to obtain sensible statistics, or

Exponential random graph models for small networks

A lot of

- Playing with the MCMC control parameters to obtain sensible statistics, or
- Sometimes we also go for using a single big (very sparse) graph

Exponential random graph models for small networks

A lot of

- Playing with the MCMC control parameters to obtain sensible statistics, or
- Sometimes we also go for using a single big (very sparse) graph
  - Block diagonal matrix
  - Constrain the sampling space puting structural zeros

Exponential random graph models for small networks

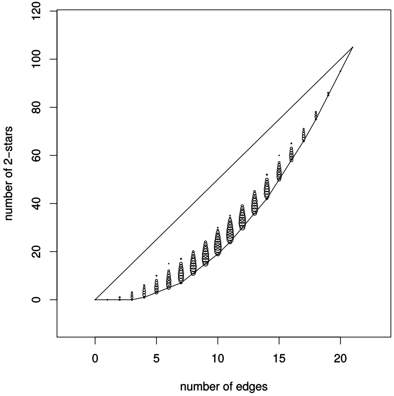
A lot of

- Playing with the MCMC control parameters to obtain sensible statistics, or
- Sometimes we also go for using a single big (very sparse) graph
  - Block diagonal matrix
  - Constrain the sampling space puting structural zeros

This fails too often (smaller networks = higher chance of model degeneracy).

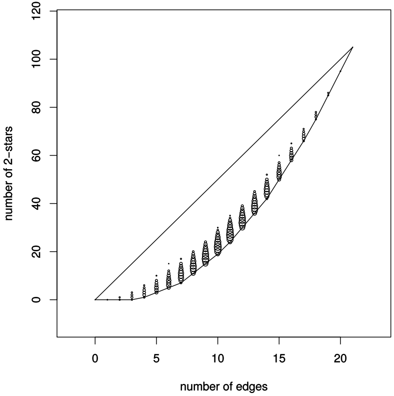
Revising model degeneracy

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



Revising model degeneracy

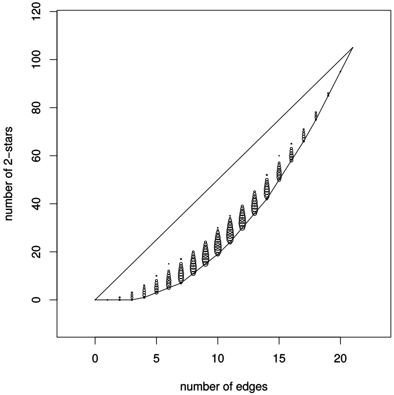
Following Handcock (2003), the key question is: Where do the sufficient statistics live?



- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE

Revising model degeneracy

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE
- Not in the interior: **We are in trouble**, we mostly get degenerate estimates (more with MC-MLE, but still with MLE)

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$\Pr\left(\mathbf{G}=\mathbf{g}\mid\theta,\mathbf{X}\right)=\frac{\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g},\mathbf{X}\right)\right\}}{\sum_{\mathbf{g}'\in\mathcal{G}}\exp\left\{\theta^{\mathbf{t}}s\left(\mathbf{g}',\mathbf{X}\right)\right\}}$$

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}$$

- So, if $n = 6$, then we have approx 1,000,000,000 terms 😱.

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$
\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}
$$

- So, if $n = 6$, then we have approx 1,000,000,000 terms 😱.
- This has lead the field to aim for (very neat) simulation based methods

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$
\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}
$$

- So, if $n = 6$, then we have approx 1,000,000,000 terms 😱.
- This has lead the field to aim for (very neat) simulation based methods
- But, if our small networks have (at most) 6 nodes...

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}$$

- So, if $n = 6$, then we have approx 1,000,000,000 terms 😱.
- This has lead the field to aim for (very neat) simulation based methods
- But, if our small networks have (at most) 6 nodes...

We can go back to the good-old-fashion MLE

Keeping $n \leq 6$ we can

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster**

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...obtain pooled estimates. Which helps with **power** *and* **degeneracy**)

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...obtain pooled estimates. Which helps with **power** *and* **degeneracy**)
- etc.

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...obtain pooled estimates. Which helps with **power** *and* **degeneracy**)
- etc.

This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

Keeping $n \leq 6$ we can

- Compute the likelihood function exactly, and hence use ``simple'' optimization to get MLEs.
- Obtain more **accurate** estimates **faster** (in most cases).
- Since (usually) small networks come in many...obtain pooled estimates. Which helps with **power** *and* **degeneracy**)
- etc.

This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

(built on top of Statnet's amazing `ergm` Hunter et al. (2008); Handcock et al. (2018) R package)

Sidetrack...

ito, ita: From the latin -*īttus*. suffix in Spanish used to denote small or affection.

e.g.:

*¡Qué lindo ese perr**ito**! / What a beautiful little dog!*
*¿Me darías una tac**ita** de azúcar? / Would you give me a small cup of sugar?*

Sidetrack...

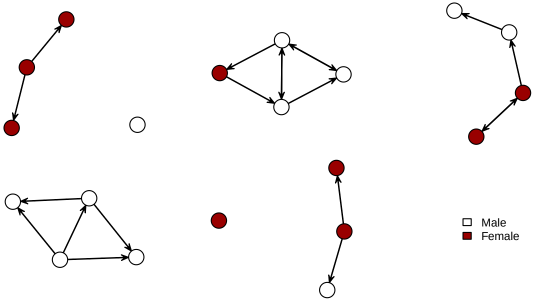
ito, ita: From the latin -*īttus*. suffix in Spanish used to denote small or affection.
e.g.:

*¡Qué lindo ese perr**ito**! / What a beautiful little dog!*
*¿Me darías una tac**ita** de azúcar? / Would you give me a small cup of sugar?*

Special thanks to George Barnett who proposed the name during the 2018 NASN!

Quick example

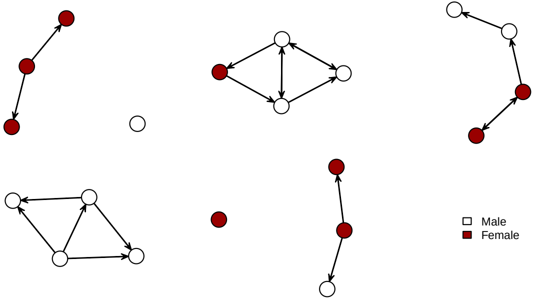
Suppose that we have 5 `networks` (as in the R package `network`)



And we would like to fit a model using the edgecount and number of gender-homophilic ties.

Quick example

Suppose that we have 5 `networks` (as in the R package `network`)



And we would like to fit a model using the edgecount and number of gender-homophilic ties.

How can we do it?

`ergmito` example (cont'd)

The same as you would do with the `ergm` package

ergmito example (cont'd)

The same as you would do with the ergm package

```r
model1 <- ergmito(fivenets ~ edges + nodematch("female"))
```

ergmito example (cont'd)

The same as you would do with the ergm package

```r
model1 <- ergmito(fivenets ~ edges + nodematch("female"))
```

```r
summary(model1) #
```

```
##
## ERGMito estimates
##
## formula:  fivenets ~ edges + nodematch("female")
##
##                   Estimate Std. Error   z value    Pr(>|z|)
## edges            -1.704748  0.5435573 -3.136280 0.001711055
## nodematch.female  1.586965  0.6430475  2.467882 0.013591530
```

ergmito example (cont'd)

The same as you would do with the ergm package

```r
model1 <- ergmito(fivenets ~ edges + nodematch("female"))
```

```r
summary(model1) #
```

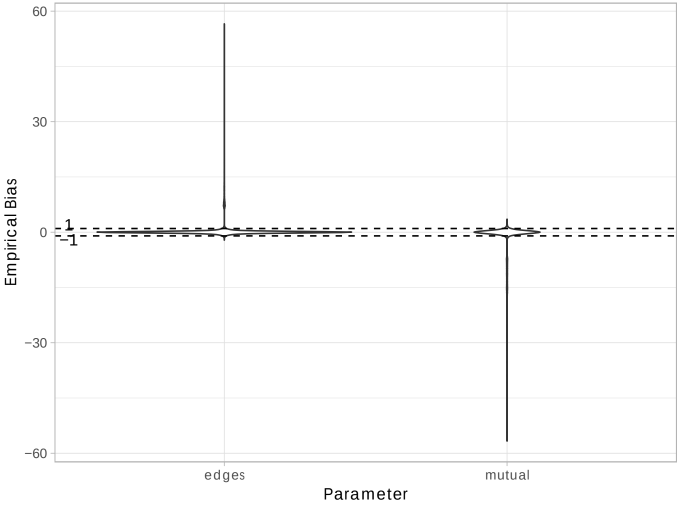
```
##
## ERGMito estimates
##
## formula:  fivenets ~ edges + nodematch("female")
##
##                   Estimate Std. Error   z value    Pr(>|z|)
## edges            -1.704748  0.5435573 -3.136280 0.001711055
## nodematch.female  1.586965  0.6430475  2.467882 0.013591530
```

Go to https://github.com/muriteams/ergmito for more on this R package.

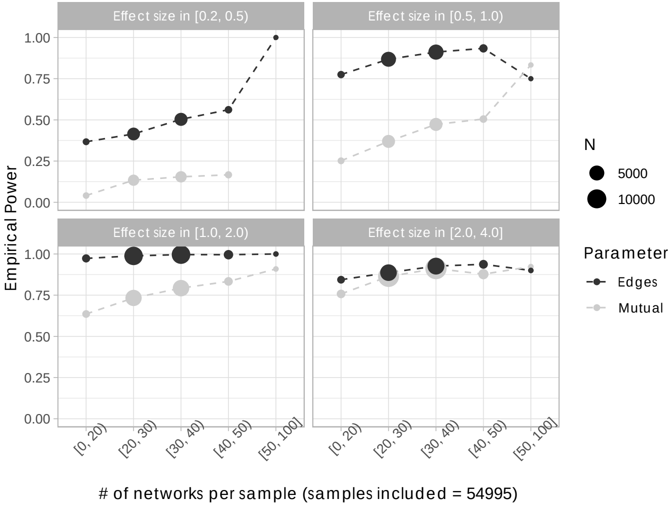
How many networks?

- Thinking about power and unbiasedness, we did a simulation study
- Simulated 100,000 samples of networks using the following steps:
  1. Draw parameters for edges and mutual from a uniform(-3, 3).
  2. Draw group sizes $n_1 \sim \text{Poisson}(10), n_2 \sim \text{Poisson}(10), n_3 \sim \text{Poisson}(10)$, networks of size 3, 4, and 5 respectively.
  3. Using 1. and 2., simulate networks using ERGM
- We looked at empirical bias (sanity check), and power

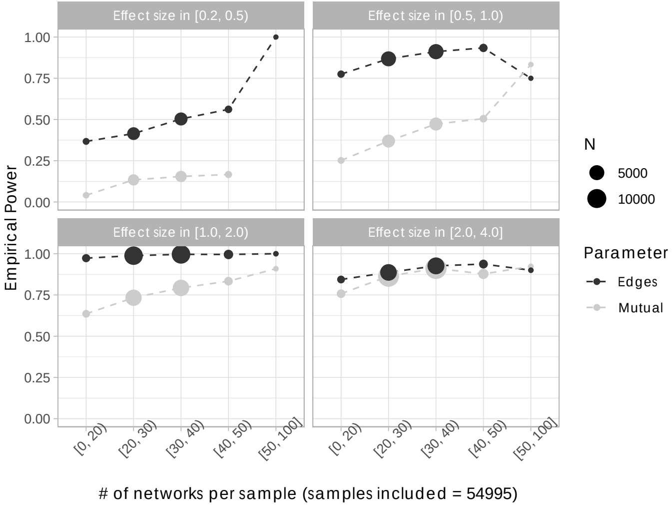
How many networks? Bias



How many networks? Power



How many networks? Power



What about a real data set?

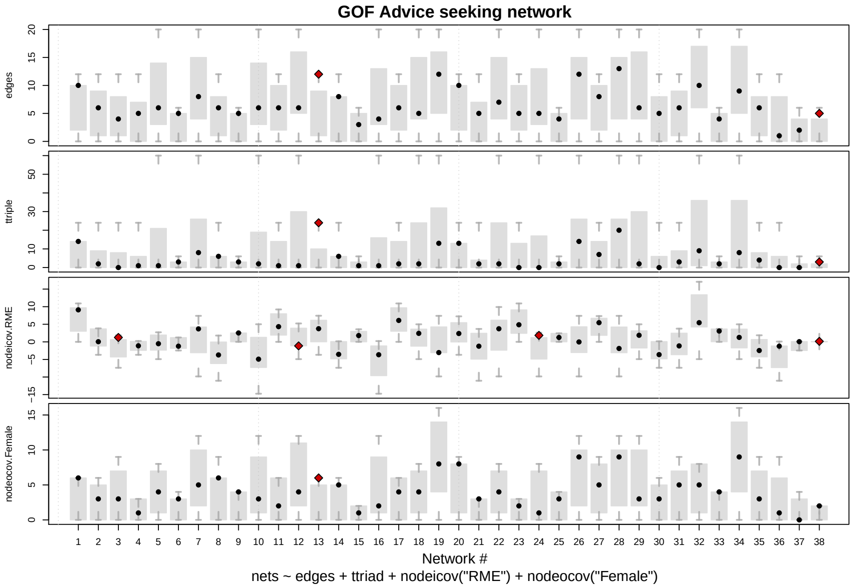
Preliminary results

From our sample of 42 small networks:

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.


GOF Advice seeking network
edges
ttriple
nodeicov.RME
nodeocov.Female
Network #
nets ~ edges + ttriad + nodeicov("RME") + nodeocov("Female")

Context: Social abilities and team performance

Two research questions

Context: Social abilities and team performance

Two research questions

~~How do **social abilities** impact **network structure**?~~

How does **collective intelligence** affect team (network) **performance**?

Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$

Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$

Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$
- Ways to solve this: parametrically (e.g. GLMs) and non-parametrically (permutation tests):

Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$
- Ways to solve this: parametrically (e.g. GLMs) and non-parametrically (permutation tests):
- For parametric approaches: Sample size?

Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$
- Ways to solve this: parametrically (e.g. GLMs) and non-parametrically (permutation tests):
- For parametric approaches: Sample size?
- Non-parametrically: Control for confounders ($\mathbf{x} \to y, \mathbf{x} \to \mathbf{g}$)?

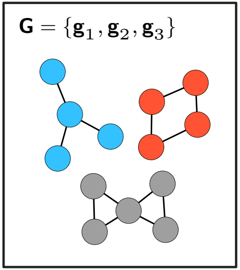
Networks and team performance

Suppose we have the following:

- Data on structure, nodes, and an outcome: $(\mathbf{g}, \mathbf{x}, y)$
- In general, we are interested on assessing the following: $\mathbf{g} \perp y$
- Ways to solve this: parametrically (e.g. GLMs) and non-parametrically (permutation tests):
- For parametric approaches: Sample size?
- Non-parametrically: Control for confounders ($\mathbf{x} \to y, \mathbf{x} \to \mathbf{g}$)?

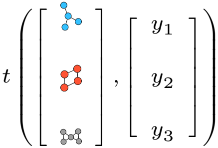
Perhaps ERGMs can help us here (to generate null distributions)

Step 1:
Fit the ERGMito



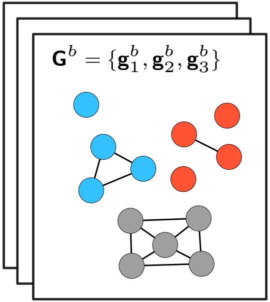
Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

Step 2:
Calculate $t_0 =$



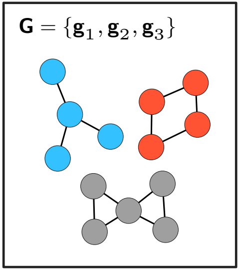
Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \dots, B$ do



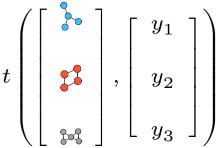
3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

Step 1:
Fit the ERGMito



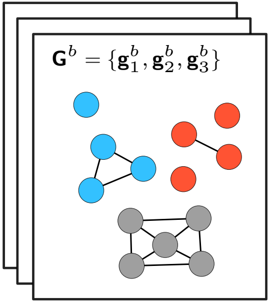
Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

Step 2:
Calculate $t_0 =$



Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We are still working (thinking) about this...

Discussion

- ERGMItos... This is not new.

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working (according to the simulation study...)

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working (according to the simulation study...)
- Need to conduct more simulations using nodal attributes and compare with ERGM block diagonal models.

Discussion

- ERGMItos... This is not new. What's new is the set of tools to apply it
- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working (according to the simulation study...)
- Need to conduct more simulations using <u>nodal</u> attributes and compare with ERGM block diagonal models.
- What about goodness-of-fit? Still need to better think about it

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)
  - Better odds ratios (not simply exponentiating the coefficients)

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)
  - Better odds ratios (not simply exponentiating the coefficients)
  - Simulation based methods (small size $\implies$ sampling from in-memory data, and exact tests)

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)
  - Better odds ratios (not simply exponentiating the coefficients)
  - Simulation based methods (small size $\implies$ sampling from in-memory data, and exact tests)
  - Cross-validation/model selection in ERGMs (thank you, Nolan !)

Discussion (contd')

- The simplicity of the estimation procedure allows us to think of:
  - Separable Temporal ERGMitos, a.k.a. TERGMitos
  - Mixture models and Bayesian inference (if you are into that kind of stuff)
  - More flexible formulas (e.g. interactions between terms and graph-level attributes)
  - Better odds ratios (not simply exponentiating the coefficients)
  - Simulation based methods (small size $\implies$ sampling from in-memory data, and exact tests)
  - Cross-validation/model selection in ERGMs (thank you, Nolan !)
- Still thinking about how to test for association between network structure and group outcome

Thanks!

We thank members of our MURI research team, USC's Center for Applied Network Analysis, Garry Robins, Carter Butts, Johan Koskinen, Noshir Contractor, and attendees of the NASN 2018 conference for their comments.

George G. Vega Yon, MS Andrew Slaughter, PhD Kayla de la Haye, PhD

vegayon@usc.edu

https://ggvy.cl

gvegayon gvegayon

References

Handcock, Mark S. 2003. ``Assessing Degeneracy in Statistical Models of Social Networks.'' Working Paper No. 39 76 (39): 33--50. https://doi.org/10.1.1.81.5086.

Handcock, Mark S., David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, and Martina Morris. 2018. Ergm: Fit, Simulate and Diagnose Exponential-Family Models for Networks. The Statnet Project (http://www.statnet.org). https://CRAN.R-project.org/package=ergm.

Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. ``Ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks.'' Journal of Statistical Software 24 (3): 1--29.