Exact Statistics and Semi-Parametric Tests for Small Network Data[1]

George G. Vega Yon, MS Andrew Slaughter, PhD Kayla de la Haye, PhD

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[1] Contact: vegayon@usc.edu. We thank members of our MURI research team, USC's Center for Applied Network Analysis, Garry Robins, Carter Butts, Johan Koskinen, Noshir Contractor, and attendees of the NASN 2018 conference for their comments.

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Context: Social abilities and team performance

Two research questions

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How does **collective intelligence** affect team (network) **performance**?

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How does collective intelligence affect team (network) performance?

We will focus on the first one

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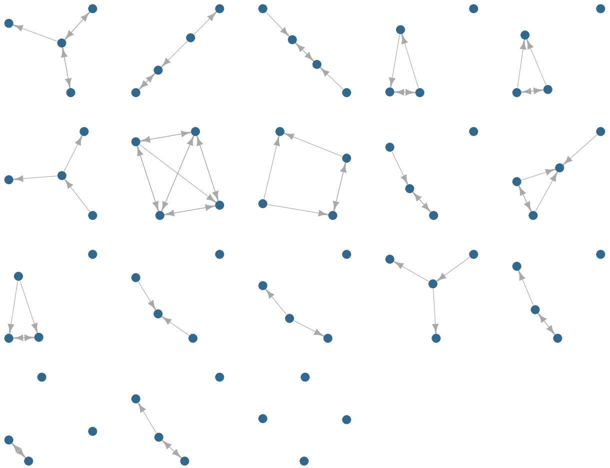
- Which completed 1 hour of group tasks (collective intelligence developed by MIT (Kim et al. 2017))
- Survey capturing information regarding socio-demographics **and**:
  - **Social Intelligence** (SI domains): Social Perception (measured by RME), Social Accomodation, Social Gregariousness, and Social Awareness
  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

Context (cont'd)



We can do a lot of simple statistics: density, prop of *[blank]*, etc. but...

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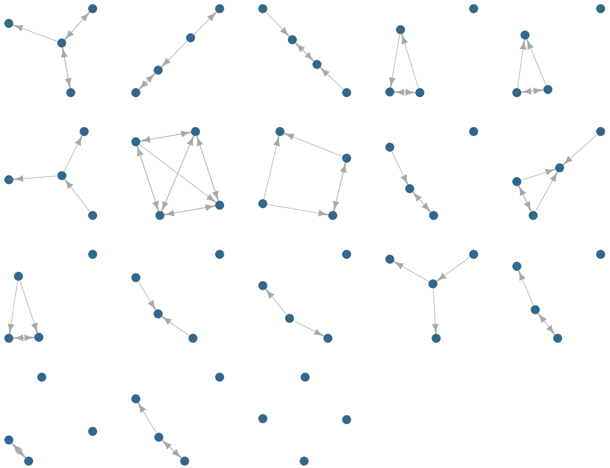
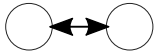
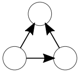
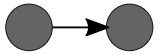
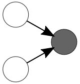
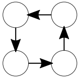


We can do a lot of simple statistics: density, prop of *[blank]*, etc. but... **how can we go beyond that?**

Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}\left(x_i = x_j\right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

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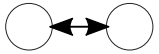
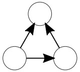
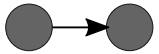
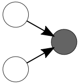
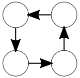
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1}\left(x_i = x_j\right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

ERGMs can do the job, but the only problem is... have you tried estimating ERGMs in small networks?

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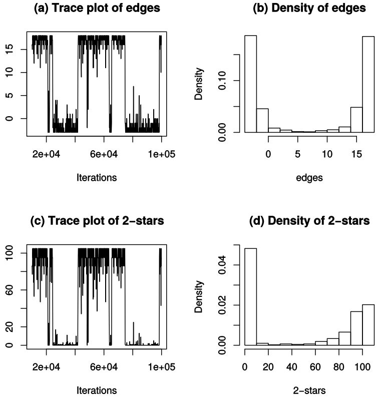
This fails too often (smaller networks = higher chance of model degeneracy).

Revising model degeneracy

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



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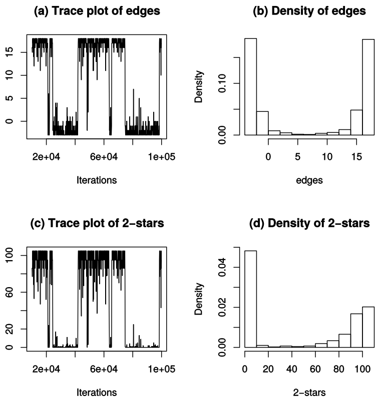

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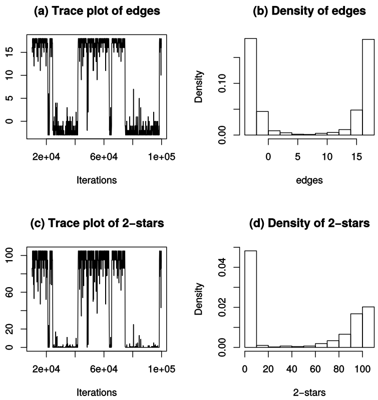

- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE
- Not in the interior: **We are in trouble**, we mostly get degenerate estimates (more with MC-MLE, but still with MLE)

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

$$
\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}
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We can go back to the good-old-fashion MLE!

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This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

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This and more has been implemented in the `ergmito` (lifecycle experimental) R package (available at https://github.com/muriteams/ergmito)

(built on top of Statnet's amazing `ergm` (Hunter et al. 2008; Handcock et al. 2018) R package)

Sidetrack...

ito, ita: From the latin -*īttus*. suffix in Spanish used to denote small or affection.

e.g.:

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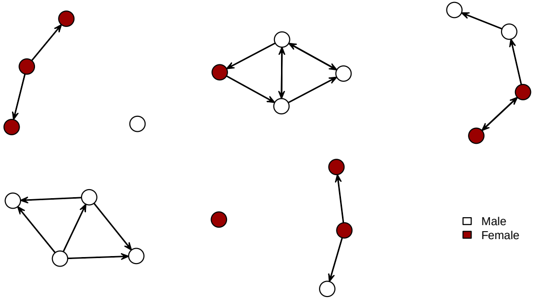
Special thanks to George Barnett who proposed the name during the 2018 NASN!

`ergmito` example

```r
library(ergmito)
data(fivenets, package = "ergmito")
```



```r
# Looking at one of the five networks
fivenets[[1]]
```

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##  Network attributes:
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How can we fit an ERGMito to this 5 networks?

`ergmito` example (cont'd)

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summary(model1) #
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## ERGMito estimates
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## nodematch.female  1.586965  0.6430475  2.467882 0.013591530
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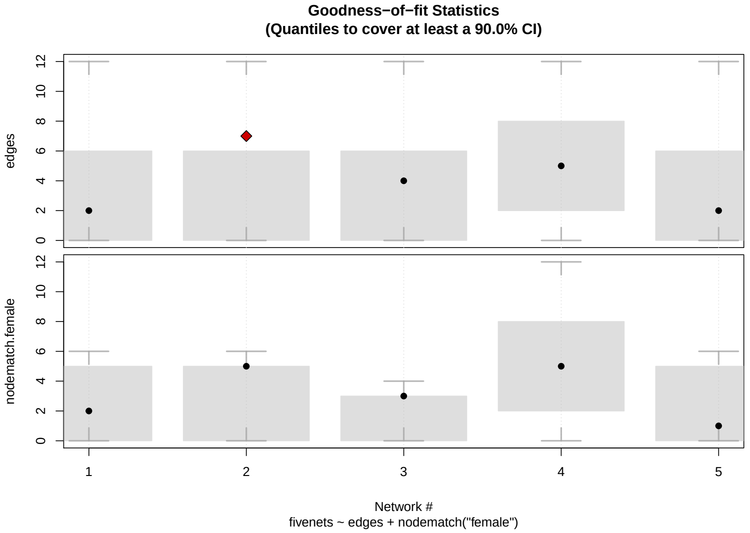
By skipping the MCMC part we:

- Obtain estimates **faster**, and
- obtain **more accurate** estimates.

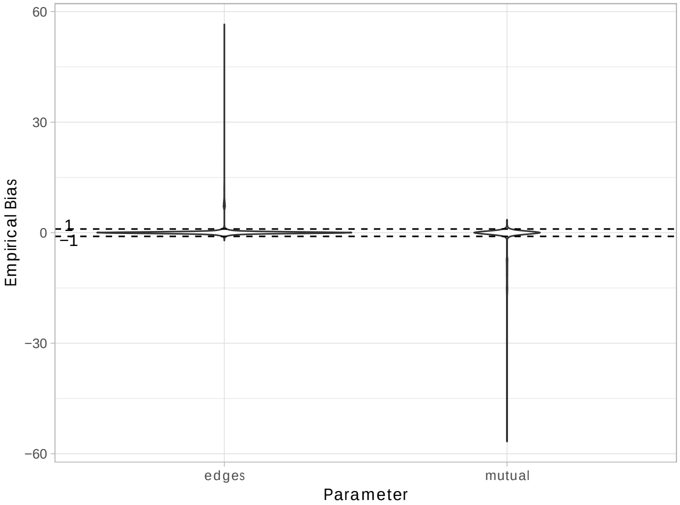
```r
gof1 <- gof_ergmito(model1)
plot(gof1)
```



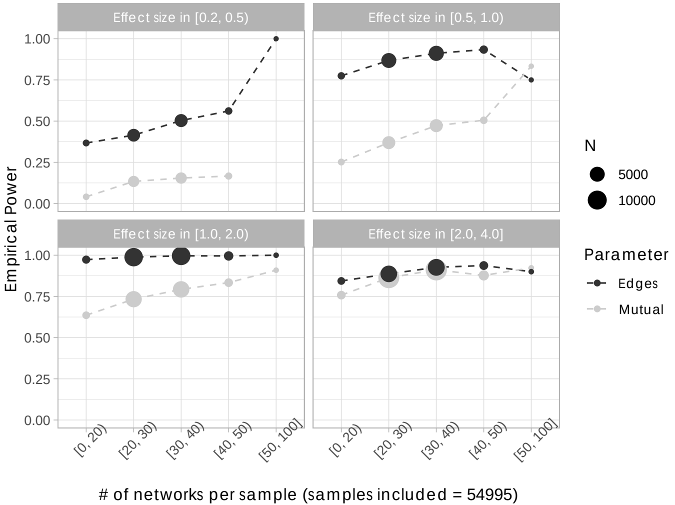
Simulation study

1. Draw parameters for edges and mutual from a uniform(-3, 3).
2. Using those parameters, sampled $n_1 \sim \text{Poisson}(10), n_2 \sim \text{Poisson}(10), n_3 \sim \text{Poisson}(10)$ networks of size 3, 4, and 5 respectively.
3. Estimated the pooled ERGMs using both the MLE and the bootstrap version.

Simulation study: Empirical Bias



Simulation study: Power



Preliminary results

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.

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- Taking this approach we can improve our estimates (power) and help with degeneracy (all by pooling)
- The tool is working (that's what the simulation say)
- Need to conduct more simulations using nodal attributes
- What about goodness-of-fit? Still need to better think about it

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- More flexible formulas (e.g. interactions between terms)

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- Separable Temporal ERGMitos, a.k.a. TERGMitos
- Mixture models and Bayesian inference (if you are into that kind of stuff)
- More flexible formulas (e.g. interactions between terms)
- Better odds ratios (not simply exponentiating the coefficients)

Thanks!

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https://ggvy.cl

gvegayon gvegayon

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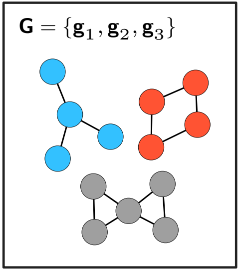
- Use fitted ERGM to our benefit, e.g. sample from ERGM distribution to do a simulation based test.

Pseudo-bonus track (contd')

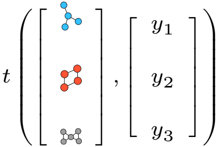
(sorry the heavy notation, but) Suppose that we have a 3 networks of sizes 4, 4, and 5 respectively.

Step 1:
Fit the ERGMito



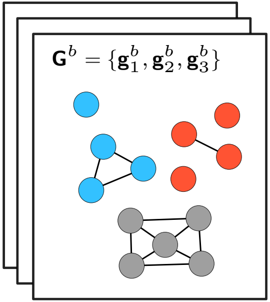
Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

Step 2:
Calculate $t_0 =$



Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We can use the distribution of the sequence $\{t_1, \dots, t_B\}$ as null to compare against t_0

Thanks (again)!

Exact Statistics and Semi-Parametric Tests for Small Network Data

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References

Handcock, Mark S. 2003. ``Assessing Degeneracy in Statistical Models of Social Networks.'' Working Paper No. 39 76 (39): 33--50. https://doi.org/10.1.1.81.5086.

Handcock, Mark S., David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, and Martina Morris. 2018. Ergm: Fit, Simulate and Diagnose Exponential-Family Models for Networks. The Statnet Project (http://www.statnet.org). https://CRAN.R-project.org/package=ergm.

Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. ``Ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks.'' Journal of Statistical Software 24 (3): 1--29.

Kim, Young Ji, David Engel, Anita Williams Woolley, Jeffrey Yu-Ting Lin, Naomi McArthur, and Thomas W. Malone. 2017. ``What Makes a Strong Team?: Using Collective Intelligence to Predict Team Performance in League of Legends.'' In Proceedings of the 2017 Acm Conference on Computer Supported Cooperative Work and Social Computing, 2316--29. CSCW '17. New York, NY, USA: ACM. https://doi.org/10.1145/2998181.2998185.