# Exact Statistics and Semi-Parametric Tests for Small Network Data

**George G. Vega Yon, MS** Andrew Slaughter, PhD Kayla de la Haye, PhD

Sunbelt 2019, Montreal
June 20, 2019

# Acknowledgements

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Computation for the work described in this paper was supported by the University of Southern California's Center for High-Performance Computing (hpc.usc.edu).

We thank members of our MURI research team, USC's Center for Applied Network Analysis, Garry Robins, Carter Butts, Johan Koskinen, Noshir Contractor, and attendees of the NASN 2018 conference for their comments.

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Two research questions

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  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

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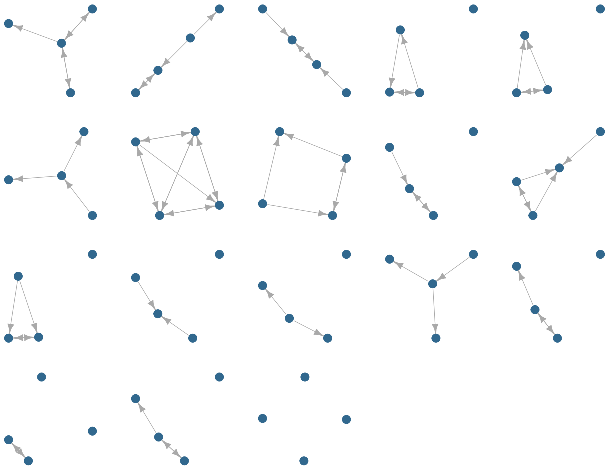

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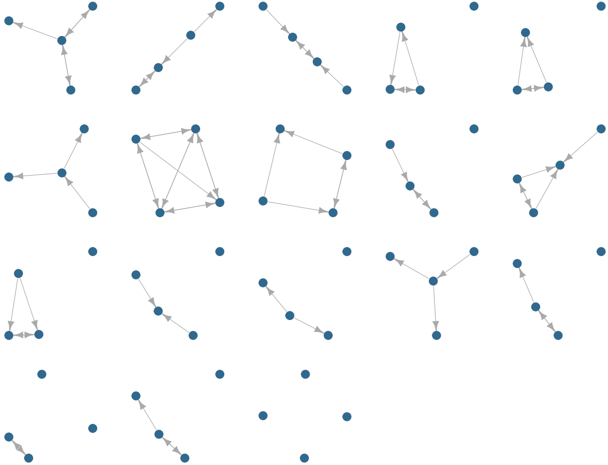

We can do a lot of simple statistics: density, prop of *[blank]*, etc. but… **how can we go beyond that?**

# Exponential random graph models

| Representation | Description |
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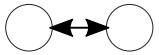
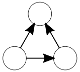
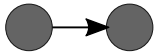
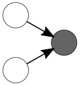
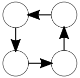
|  | Mutual Ties (Reciprocity)<br>$\sum_{i \neq j} y_{ij} y_{ji}$ |
|  | Transitive Triad (Balance)<br>$\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$ |
|  | Homophily<br>$\sum_{i \neq j} y_{ij} \mathbf{1} \left( x_i = x_j \right)$ |
|  | Covariate Effect for Incoming Ties<br>$\sum_{i \neq j} y_{ij} x_j$ |
|  | Four Cycle<br>$\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$ |

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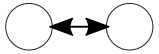
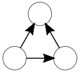
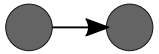
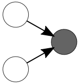
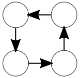
|  | Mutual Ties (Reciprocity) $\sum_{i\neq j} y_{ij}y_{ji}$ |
|  | Transitive Triad (Balance) $\sum_{i\neq j\neq k} y_{ij}y_{jk}y_{ik}$ |
|  | Homophily $\sum_{i\neq j} y_{ij}\mathbf{1}\left(x_i = x_j\right)$ |
|  | Covariate Effect for Incoming Ties $\sum_{i\neq j} y_{ij}x_j$ |
|  | Four Cycle $\sum_{i\neq j\neq k\neq l} y_{ij}y_{jk}y_{kl}y_{li}$ |

ERGMs can do the job, but the only problem is... have you tried estimating ERGMs in small networks?

# Exponential random graph models for small networks

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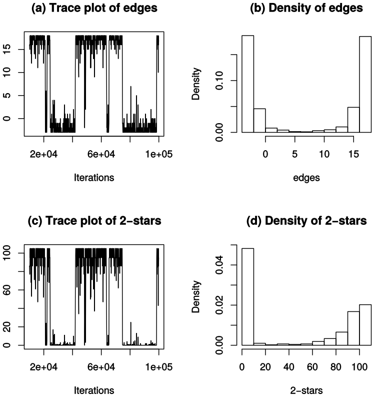
This fails too often (smaller networks = higher chance of model degeneracy).

# Revising model degeneracy

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



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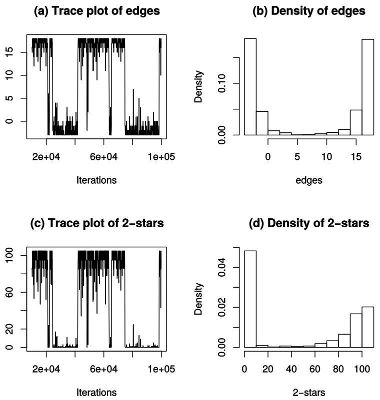


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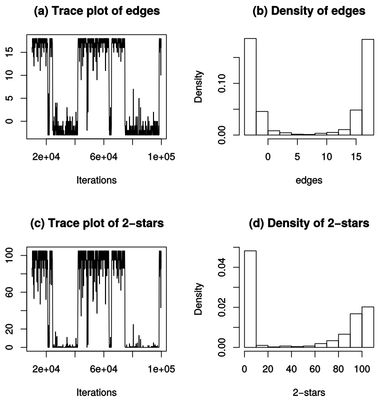


- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE
- Not in the interior: **We are in trouble**, we mostly get degenerate estimates (more with MC-MLE, but still with MLE)

# ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

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\Pr\left(\mathbf{G} = \mathbf{g} \mid \theta, \mathbf{X}\right) = \frac{\exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}, \mathbf{X}\right)\right\}}{\sum_{\mathbf{g}' \in \mathcal{G}} \exp\left\{\theta^{\mathbf{t}} s\left(\mathbf{g}', \mathbf{X}\right)\right\}}
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(built on top of Statnet's amazing `ergm` (Hunter et al. 2008; Handcock et al. 2018) R package)

Sidetrack...

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e.g.:

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**Special thanks to George Barnett who proposed the name during the 2018 NASN!**

# Quick example

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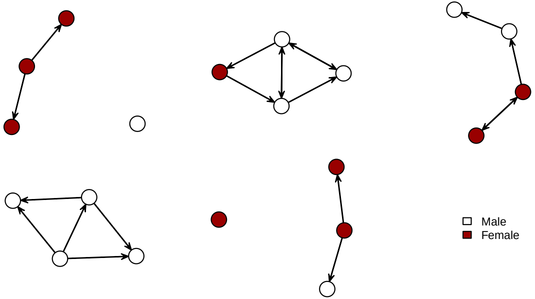


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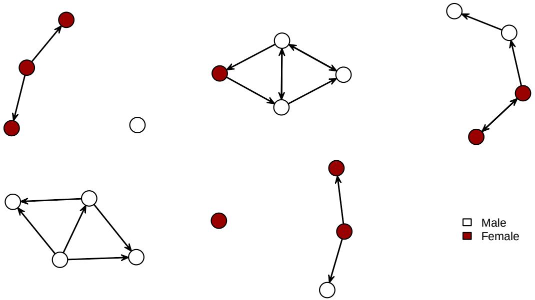


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How can we do it?

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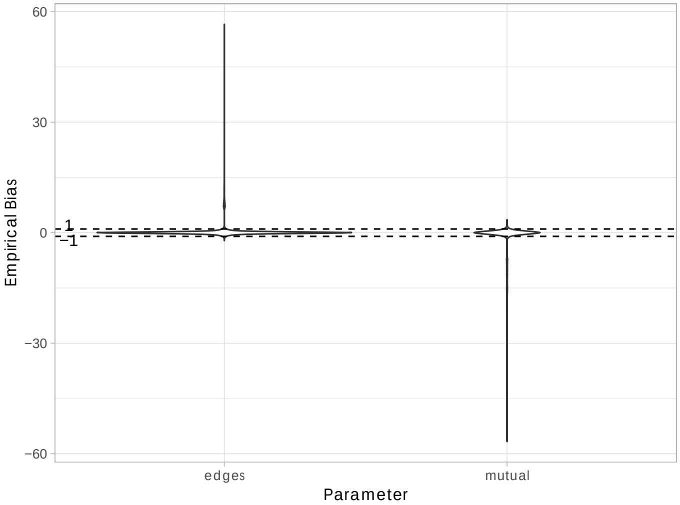
Go to https://github.com/muriteams/ergmito for more on this R package.

# How many networks?

- Thinking about power and unbiasedness, we did a simulation study
- Simulated 100,000 samples of networks using the following steps:
  1. Draw parameters for edges and mutual from a uniform(-3, 3).
  2. Draw group sizes $n_1 \sim \text{Poisson}(10), n_2 \sim \text{Poisson}(10), n_3 \sim \text{Poisson}(10)$, networks of size 3, 4, and 5 respectively.
  3. Using 1. and 2., simulate networks using ERGM
- We looked at empirical bias (sanity check), and power

# How many networks? Bias



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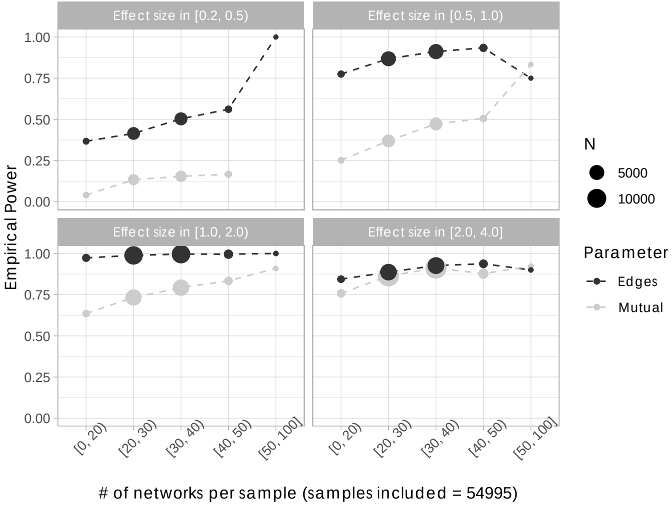

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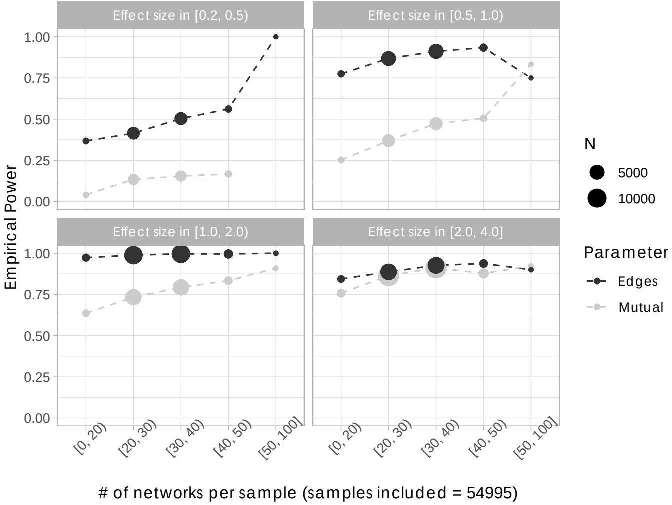


What about a real data set?

# Preliminary results

From our sample of 42 small networks:

| | Advice | Dislike | Influence | Leader | Trust |
|---|---|---|---|---|---|
| edges | $-0.85^{***}$ | $-2.30^{***}$ | $-0.77^{***}$ | $-0.53^{***}$ | $-0.47^{***}$ |
| | $(0.17)$ | $(0.20)$ | $(0.13)$ | $(0.14)$ | $(0.14)$ |
| ttriple | $0.24^{***}$ | | $0.21^{**}$ | | $0.20^{***}$ |
| | $(0.06)$ | | $(0.08)$ | | $(0.06)$ |
| nodeicov.RME | $0.40^{***}$ | | $0.21^{*}$ | $0.42^{***}$ | $0.25^{**}$ |
| | $(0.09)$ | | $(0.09)$ | $(0.11)$ | $(0.09)$ |
| nodeocov.Female | $0.53^{**}$ | | | | |
| | $(0.18)$ | | | | |
| nodematch.Female | | $0.56^{*}$ | | | |
| | | $(0.27)$ | | | |
| nodeicov.SI3Fac1 | | $-0.35^{*}$ | | | |
| | | $(0.15)$ | | | |
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| | | | | $(0.20)$ | |
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| nodeocov.SI3Fac1 | | | | | $0.31^{***}$ |
| | | | | | $(0.09)$ |
| AIC | 695.07 | 381.72 | 756.84 | 637.01 | 776.82 |
| BIC | 712.13 | 394.52 | 769.92 | 654.07 | 794.25 |
| Log Likelihood | -343.54 | -187.86 | -375.42 | -314.50 | -384.41 |
| Num. networks | 38 | 38 | 41 | 38 | 41 |
| Convergence | 0 | 0 | 0 | 0 | 0 |

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

**Table 1:** Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.

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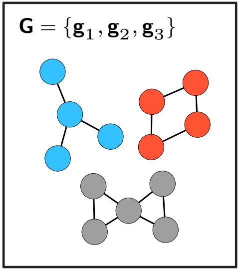
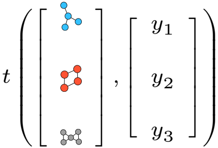
Perhaps ERGMs can help us here (to generate null distributions)

**Step 1**:
Fit the ERGMito



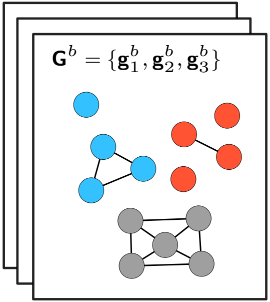

Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

**Step 2**:
Calculate $t_0 =$




Throughout the simulations the only part that changes is the networks, not $Y$

**Step 3**:
For $b \in 1, \ldots, B$ do




3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

**Step 1**:
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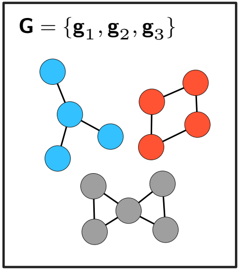
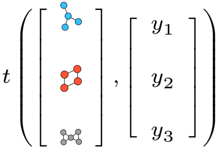


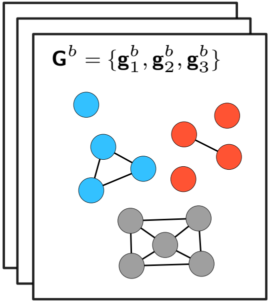

Fit the ERGMito,
This will give us $\mathcal{D}(\hat{\theta}, X_j)$

**Step 2**:
Calculate $t_0 =$




Throughout the simulations the only part that changes is the networks, not $Y$

**Step 3**:
For $b \in 1, \dots, B$ do




3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We are still working (thinking) about this...

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- Taking this approach we can improve our estimates (power) and help with degeneracy
- The tool is working(according to the simulation study...)
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- What about goodness-of-fit? Still need to better think about it

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But we are still (very) interested about the problem of identifying associations between group and structure.

# Thanks!

## Exact Statistics and Semi-Parametric Tests for Small Network Data

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