Exact Statistics and Semi-Parametric Tests for Small Network Data

George G. Vega Yon, MS Andrew Slaughter, PhD Kayla de la Haye, PhD

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Acknowledgements

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Network Science of Teams
a Multidisciplinary University Research Initiative

UC SANTA BARBARA
USC University of Southern California
MIT Massachusetts Institute of Technology
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Context: Social abilities and team performance

Two research questions

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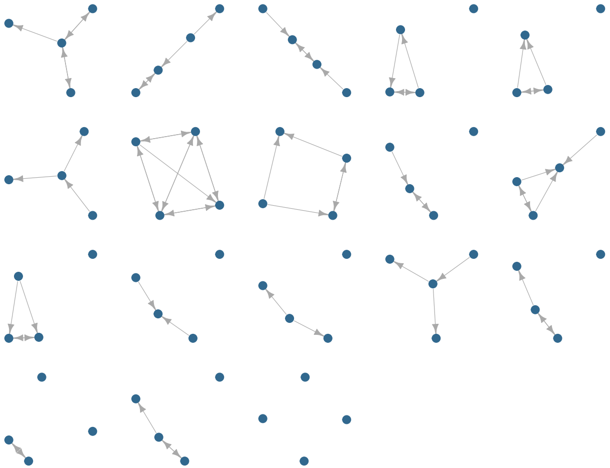
  - **Social Networks**: Advice Seeking, Leadership, Influence (among others).

Context (cont'd)



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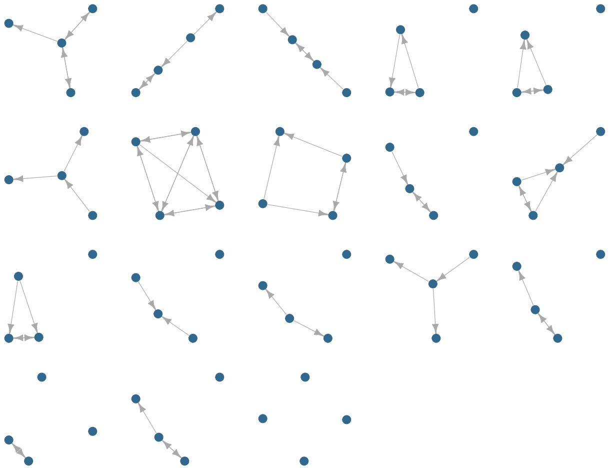

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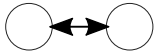
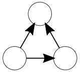
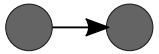
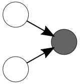
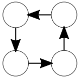
Exponential random graph models

Representation	Description
	Mutual Ties (Reciprocity) $\sum_{i \neq j} y_{ij} y_{ji}$
	Transitive Triad (Balance) $\sum_{i \neq j \neq k} y_{ij} y_{jk} y_{ik}$
	Homophily $\sum_{i \neq j} y_{ij} \mathbf{1} \left(x_i = x_j \right)$
	Covariate Effect for Incoming Ties $\sum_{i \neq j} y_{ij} x_j$
	Four Cycle $\sum_{i \neq j \neq k \neq l} y_{ij} y_{jk} y_{kl} y_{li}$

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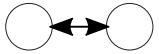
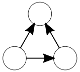
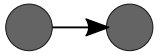
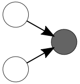
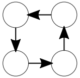
	Mutual Ties (Reciprocity) $\sum_{i\neq j} y_{ij}y_{ji}$
	Transitive Triad (Balance) $\sum_{i\neq j\neq k} y_{ij}y_{jk}y_{ik}$
	Homophily $\sum_{i\neq j} y_{ij}\mathbf{1}\left(x_i = x_j\right)$
	Covariate Effect for Incoming Ties $\sum_{i\neq j} y_{ij}x_j$
	Four Cycle $\sum_{i\neq j\neq k\neq l} y_{ij}y_{jk}y_{kl}y_{li}$

ERGMs can do the job, but the only problem is... have you tried estimating ERGMs in small networks?

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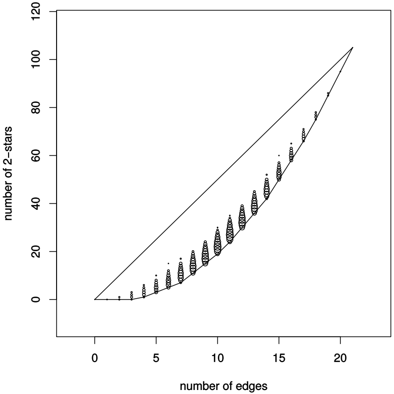
This fails too often (smaller networks = higher chance of model degeneracy).

Revising model degeneracy

Following Handcock (2003), the key question is: Where do the sufficient statistics live?



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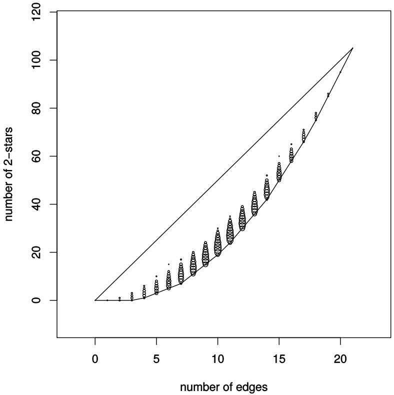

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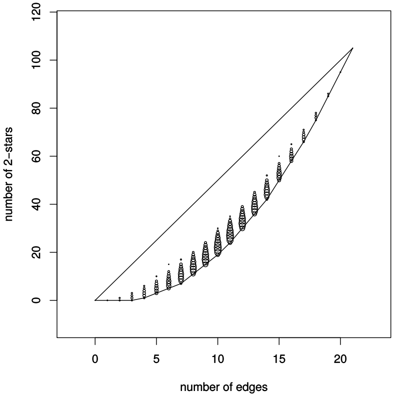

- In the interior: **Good**, we (possibly) get nice estimates in both MC-MLE and MLE
- Not in the interior: **We are in trouble**, we mostly get degenerate estimates (more with MC-MLE, but still with MLE)

ERGMs for small networks

- Calculating the likelihood function for a directed graph means (at some point) enumerating $2^{n(n-1)}$ **terms**.

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(built on top of Statnet's amazing `ergm` (Hunter et al. 2008; Handcock et al. 2018) R package)

Sidetrack...

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e.g.:

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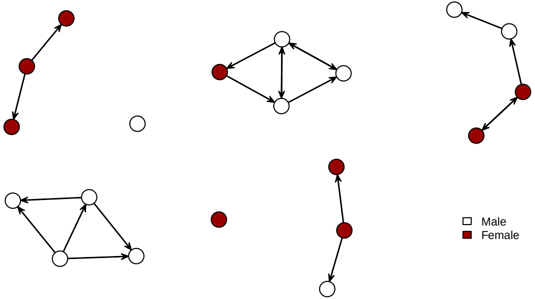
Special thanks to George Barnett who proposed the name during the 2018 NASN!

Quick example

Suppose that we have 5 `networks` (as in the R package `network`)



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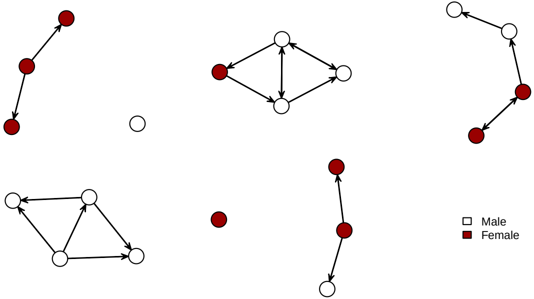

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How can we do it?

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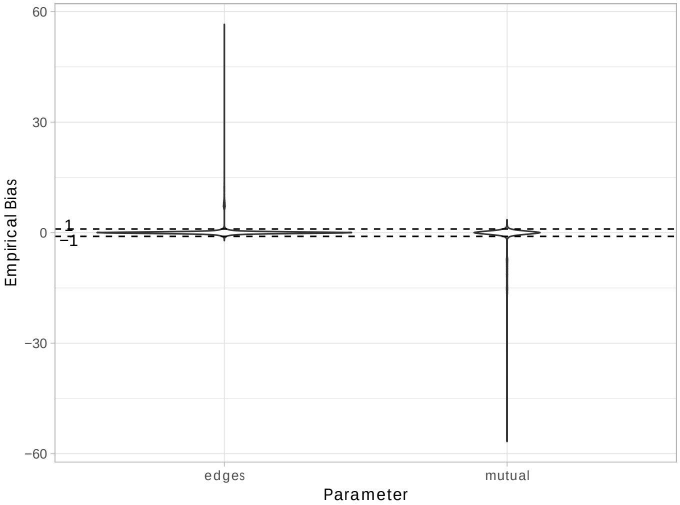
Go to https://github.com/muriteams/ergmito for more on this R package.

How many networks?

- Thinking about power and unbiasedness, we did a simulation study
- Simulated 100,000 samples of networks using the following steps:
  1. Draw parameters for edges and mutual from a uniform(-3, 3).
  2. Draw group sizes $n_1 \sim \text{Poisson}(10), n_2 \sim \text{Poisson}(10), n_3 \sim \text{Poisson}(10)$, networks of size 3, 4, and 5 respectively.
  3. Using 1. and 2., simulate networks using ERGM
- We looked at empirical bias (sanity check), and power

How many networks? Bias



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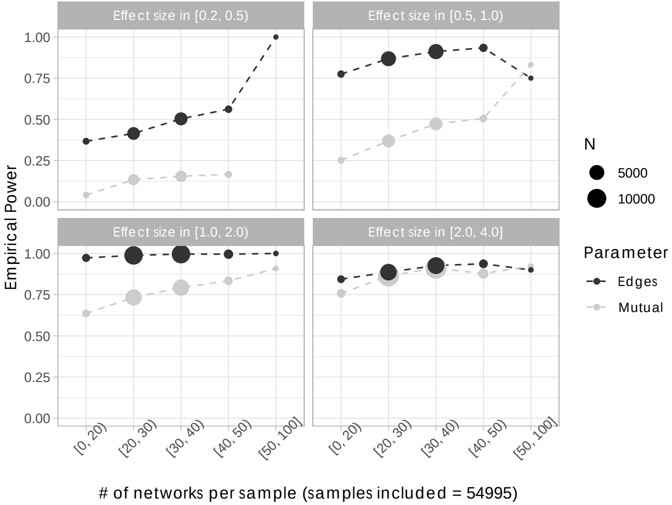

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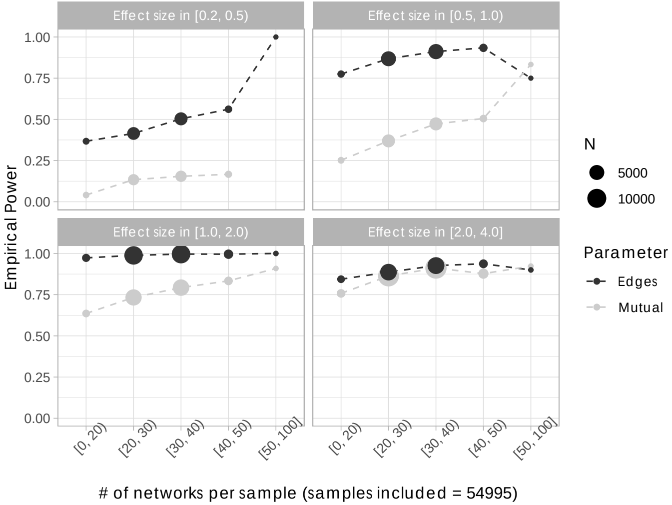

What about a real data set?

Preliminary results

From our sample of 42 small networks:

	Advice	Dislike	Influence	Leader	Trust
edges	-0.85^{***}	-2.30^{***}	-0.77^{***}	-0.53^{***}	-0.47^{***}
	(0.17)	(0.20)	(0.13)	(0.14)	(0.14)
ttriple	0.24^{***}		0.21^{**}		0.20^{***}
	(0.06)		(0.08)		(0.06)
nodeicov.RME	0.40^{***}		0.21^{*}	0.42^{***}	0.25^{**}
	(0.09)		(0.09)	(0.11)	(0.09)
nodeocov.Female	0.53^{**}				
	(0.18)				
nodematch.Female		0.56^{*}			
		(0.27)			
nodeicov.SI3Fac1		-0.35^{*}			
		(0.15)			
nodeicov.Female				-0.52^{**}	
				(0.20)	
nodeocov.RME				-0.32^{**}	
				(0.11)	
nodeocov.SI3Fac1					0.31^{***}
					(0.09)
AIC	695.07	381.72	756.84	637.01	776.82
BIC	712.13	394.52	769.92	654.07	794.25
Log Likelihood	-343.54	-187.86	-375.42	-314.50	-384.41
Num. networks	38	38	41	38	41
Convergence	0	0	0	0	0

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Selected models for each one of the studied networks. Results presented here correspond to a forward selection process.

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Perhaps ERGMs can help us here (to generate null distributions)

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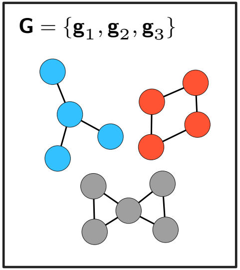
Fit the ERGMito



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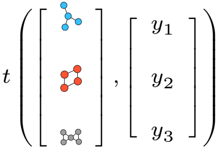
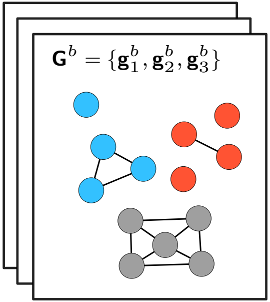
Step 2:
Calculate $t_0 =$



Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \ldots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

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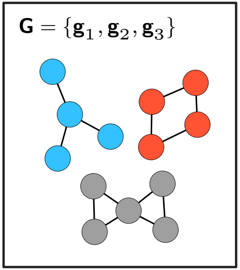

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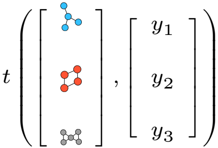
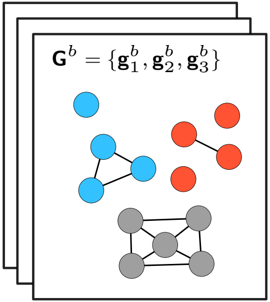


Throughout the simulations the only part that changes is the networks, not Y

Step 3:
For $b \in 1, \dots, B$ do



3.1) For $j \in \{1, 2, 3\}$ draw a new network from $\mathcal{D}$
3.2) Use the new sample to calculate $t_b = t(\mathbf{G}^b, Y)$

We are still working (thinking) about this...

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- What about goodness-of-fit? Still need to better think about it

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  - Simulation based methods (small size $\implies$ sampling from in-memory data)
- Still thinking about how to test for association between network structure and group outcome

Thanks!

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References

Handcock, Mark S. 2003. ``Assessing Degeneracy in Statistical Models of Social Networks.'' Working Paper No. 39 76 (39): 33--50. https://doi.org/10.1.1.81.5086.

Handcock, Mark S., David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, and Martina Morris. 2018. Ergm: Fit, Simulate and Diagnose Exponential-Family Models for Networks. The Statnet Project (http://www.statnet.org). https://CRAN.R-project.org/package=ergm.

Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. ``Ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks.'' Journal of Statistical Software 24 (3): 1--29.