where we have made the (fairly safe) assumption that p_k is sufficiently small for $k \gtrsim k_{\max}$ that $np_k \ll 1$ and $P_k \ll 1$.

For example, if $p_k \sim k^{-\alpha}$ in its tail, then we find that

$$k_{\max} \sim n^{1/(\alpha-1)}. \tag{13}$$

As shown by Cohen *et al.* [93], a simple rule of thumb that leads to the same result is that the maximum degree is roughly the value of k that solves $nP_k = 1$. Note however that, as shown by Dorogovtsev and Samukhin [129], the fluctuations in the tail of the degree distribution are very large for the power-law case.

Dorogovtsev *et al.* [126] have also shown that Eq. (13) holds for networks generated using the "preferential attachment" procedure of Barabási and Albert [32] described in Sec. VII.B, and a detailed numerical study of this case has been carried out by Moreira *et al.* [295].

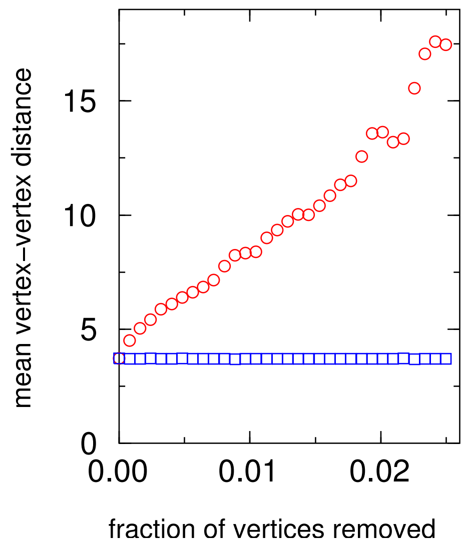

FIG. 7 Mean vertex–vertex distance on a graph representation of the Internet at the autonomous system level, as vertices are removed one by one. If vertices are removed in random order (squares), distance increases only very slightly, but if they are removed in order of their degrees, starting with the highest degree vertices (circles), then distance increases sharply. After Albert *et al.* [15].

D. Network resilience

Related to degree distributions is the property of resilience of networks to the removal of their vertices, which has been the subject of a good deal of attention in the literature. Most of the networks we have been considering rely for their function on their connectivity, i.e., the existence of paths leading between pairs of vertices. If vertices are removed from a network, the typical length of these paths will increase, and ultimately vertex pairs will become disconnected and communication between them through the network will become impossible. Networks vary in their level of resilience to such vertex removal.

There are also a variety of different ways in which vertices can be removed and different networks show varying degrees of resilience to these also. For example, one could remove vertices at random from a network, or one could target some specific class of vertices, such as those with the highest degrees. Network resilience is of particular importance in epidemiology, where "removal" of vertices in a contact network might correspond for example to vaccination of individuals against a disease. Because vaccination not only prevents the vaccinated individuals from catching the disease but may also destroy paths between other individuals by which the disease might have spread, it can have a wider reaching effect than one might at first think, and careful consideration of the efficacy of different vaccination strategies could lead to substantial advantages for public health.

Recent interest in network resilience has been sparked by the work of Albert *et al.* [15], who studied the effect of vertex deletion in two example networks, a 6000-vertex network representing the topology of the Internet at the level of autonomous systems (see Sec. II.C), and a 326 000-page subset of the World Wide Web. Both of the Internet and the Web have been observed to have degree distributions that are approximately power-law in form [14, 74, 86, 148, 401] (Sec. III.C.1). The authors measured average vertex–vertex distances as a function of number of vertices removed, both for random removal and for progressive removal of the vertices with the highest degrees.[14] In Fig. 7 we show their results for the Internet. They found for both networks that distance was almost entirely unaffected by random vertex removal, i.e., the networks studied were highly resilient to this type of removal. This is intuitively reasonable, since most of the vertices in these networks have low degree and therefore lie on few paths between others; thus their removal rarely affects communications substantially. On the other hand, when removal is targeted at the highest degree vertices, it is found to have devastating effect. Mean vertex–vertex distance increases very sharply with the fraction of vertices removed, and typically only a few percent of vertices need be removed before essentially all communication through the network is destroyed. Albert *et al.* expressed their results in terms of failure or sabotage of network nodes. The Internet (and the Web) they suggest, is highly resilient against the random failure of vertices in the network, but highly vulnerable to deliberate attack on its highest-degree vertices.

Similar results to those of Albert *et al.* were found independently by Broder *et al.* [74] for a much larger subset of the Web graph. Interestingly, however, Broder *et al.*

[14] In removing the vertices with the highest degrees, Albert *et al.* recalculated degrees following the removal of each vertex. Most other authors who have studied this issue have adopted a slightly different strategy of removing vertices in order of their *initial* degree in the network before any removal.