Homework 11: Regression Inference

Reading:

- Inference for Regression (https://www.inferentialthinking.com/chapters/16/Inference_for_Regression.html)

Please complete this notebook by filling in the cells provided. Before you begin, execute the following cell to load the provided tests. Each time you start your server, you will need to execute this cell again to load the tests.

Homework 11 is due **Thursday, 4/23 at 11:59pm**. You will receive an early submission bonus point if you turn in your final submission by Wednesday, 4/22 at 11:59pm. Start early so that you can come to office hours if you're stuck. Check the website for the office hours schedule. Late work will not be accepted as per the policies (http://data8.org/sp20/policies.html) of this course.

Directly sharing answers is not okay, but discussing problems with the course staff or with other students is encouraged. Refer to the policies page to learn more about how to learn cooperatively.

For all problems that you must write our explanations and sentences for, you **must** provide your answer in the designated space. Moreover, throughout this homework and all future ones, please be sure to not re-assign variables throughout the notebook! For example, if you use `max_temperature` in your answer to one question, do not reassign it later on.

In [1]:
```
# Don't change this cell; just run it.

import numpy as np
from datascience import *

# These lines do some fancy plotting magic.
import matplotlib
%matplotlib inline
import matplotlib.pyplot as plt
plt.style.use('fivethirtyeight')
import warnings
warnings.simplefilter('ignore', FutureWarning)
from matplotlib import patches
from ipywidgets import interact, interactive, fixed
import ipywidgets as widgets

from client.api.notebook import Notebook
ok = Notebook('hw11.ok')
```

```
=====================================================================
Assignment: Homework 11: Regression Inference
OK, version v1.14.19
=====================================================================
```

```
---------------------------------------------------------------------------
LoadingException                          Traceback (most recent call last)
<ipython-input-1-6bfe8d1ff078> in <module>
     16 
     17 from client.api.notebook import Notebook
---> 18 ok = Notebook('hw11.ok')

/opt/anaconda3/lib/python3.7/site-packages/client/api/notebook.py in __init__(self, filepath, cmd_args, debug, mode)
     13         ok_logger = logging.getLogger('client')   # Get top-level ok logger
     14         ok_logger.setLevel(logging.DEBUG if debug else logging.ERROR)
---> 15         self.assignment = load_assignment(filepath, cmd_args)
     16         # Attempt a login with enviornment based tokens
     17         login_with_env(self.assignment)

/opt/anaconda3/lib/python3.7/site-packages/client/api/assignment.py in load_assignment(filepath, cmd_args)
     22     if cmd_args is None:
     23         cmd_args = Settings()
---> 24     return Assignment(cmd_args, **config)
     25 
     26 def _get_config(config):

/opt/anaconda3/lib/python3.7/site-packages/client/sources/common/core.py in __call__(cls, *args, **kargs)
    185                 raise ex.SerializeException('__init__() missing expected '
    186                                             'argument {}'.format(attr))
--> 187         obj.post_instantiation()
    188         return obj
    189 

/opt/anaconda3/lib/python3.7/site-packages/client/api/assignment.py in post_instantiation(self)
    151     def post_instantiation(self):
    152         self._print_header()
--> 153         self._load_tests()
    154         self._load_protocols()
    155         self.specified_tests = self._resolve_specified_tests(

/opt/anaconda3/lib/python3.7/site-packages/client/api/assignment.py in _load_tests(self)
    205 
    206         if not self.test_map:
--> 207             raise ex.LoadingException('No tests loaded')
    208 
    209     def dump_tests(self):

LoadingException: No tests loaded
```

Regression Inference for the NFL Draft

In this homework, we will be analyzing the relationship between draft position and success in the NFL. The NFL draft is an annual event in which every NFL team takes turns choosing players that they will add to their team. There are around 200 selections, called "picks" made every year, although this number has changed over the years.

The `nfl` table has five columns, the name of the `Player`, the `Salary` that player made for the 2019 season, the year that player was drafted (`Year Drafted`), the number of the draft pick that was used when the player was drafted (`Pick Number`), and the `Position` in football that player plays.

Each row in `nfl` corresponds to one player who played in the **2019 season**.

In [2]:
```
# Just run this cell!
nfl = Table.read_table("nfl.csv")
nfl.show(5)
```

Player	Salary	Year Drafted	Pick Number	Position
Baker Mayfield	570000	2018	1	QB
Cam Newton	16200000	2011	1	QB
Eli Manning	11500000	2004	1	QB
Eric Fisher	10350000	2013	1	OT
Jadeveon Clowney	15967200	2014	1	DE

... (1157 rows omitted)

Question 1

Add a column to the table called `Career Length` that corresponds to how long a player has been in the NFL to the `nfl` table. `Career Length` is from when they were drafted to this year, 2020. So, if a player was drafted in 2015, their career length is 5:

$$2020 - 2015 = 5$$

```
BEGIN QUESTION
name: q1_1
manual: false
```

In [3]:
```
nfl = nfl.with_column("Career Length", 2020 - nfl.column("Year Drafted"
)) # SOLUTION
nfl.show(5)
```

Player	Salary	Year Drafted	Pick Number	Position	Career Length
Baker Mayfield	570000	2018	1	QB	2
Cam Newton	16200000	2011	1	QB	9
Eli Manning	11500000	2004	1	QB	16
Eric Fisher	10350000	2013	1	OT	7
Jadeveon Clowney	15967200	2014	1	DE	6

... (1157 rows omitted)

In [4]:
```
# TEST
# Did you add Career Length column?
nfl.num_columns == 6
```

Out[4]: True

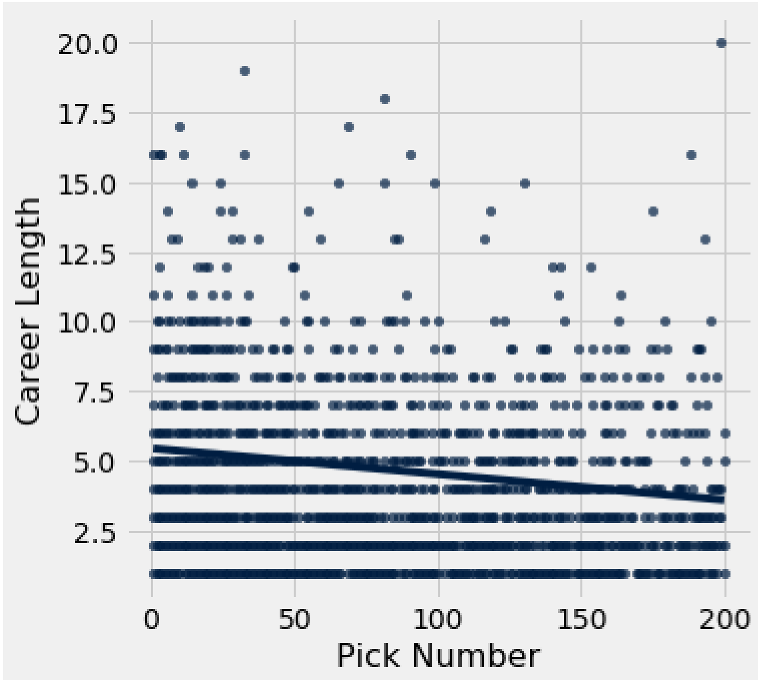
In [5]:
```
# TEST
# Checking that the first 10 rows are correct
nfl.take(np.arange(10))
```

Out[5]:

Player	Salary	Year Drafted	Pick Number	Position	Career Length
Baker Mayfield	570000	2018	1	QB	2
Cam Newton	16200000	2011	1	QB	9
Eli Manning	11500000	2004	1	QB	16
Eric Fisher	10350000	2013	1	OT	7
Jadeveon Clowney	15967200	2014	1	DE	6
Jameis Winston	20922000	2015	1	QB	5
Jared Goff	4259683	2016	1	QB	4
Kyler Murray	495000	2019	1	QB	1
Matthew Stafford	13500000	2009	1	QB	11
Myles Garrett	3229750	2017	1	DE	3

As usual, let's investigate our data visually before analyzing it numerically. The first relationship we will analyze is the relationship between a player's `Pick Number` and their `Career Length`. Run the following cell to see a scatter diagram with the line of best fit already plotted for you.

In [6]:
```
# Just run this cell
nfl.scatter("Pick Number", "Career Length", fit_line=True)
```



Question 2

Use the functions given to assign the correlation between `Pick Number` and `Career Length` to `pick_length_correlation`. `correlation` takes in three arguments, a table `tbl` and the labels of the columns you are finding the correlation between, `col1` and `col2`.

```
BEGIN QUESTION
name: q1_2
manual: false
```

In [7]:
```
def standard_units(arr):
    return (arr- np.mean(arr)) / np.std(arr)

def correlation(tbl, col1, col2):
    r = np.mean(standard_units(tbl.column(col1)) * standard_units(tbl.co
lumn(col2)))
    return r

pick_length_correlation = correlation(nfl, "Pick Number", "Career Lengt
h") # SOLUTION
pick_length_correlation
```

Out[7]: -0.16517332737646848

In [8]:
```
# TEST
# Correlation is a number between -1 and 1
-1 <= pick_length_correlation <= 1
```

Out[8]: True

In [9]:
```
# HIDDEN TEST
np.round(pick_length_correlation, 3) == -0.165
```

Out[9]: True

We can see that there is a negative association between `Pick Number` and `Career Length` ! If in the sample, we found a linear relation between the two variables, would the same be true for the population? Would it be exactly the same linear relation? Could we predict the response of a new individual who is not in our sample?

Question 3

Evan thinks that the slope of the true line of best fit for `Pick Number` and `Career Length` is not zero: that is, there is some correlation/association between `Pick Number` and `Career Length` . To test this claim, we can run a hypothesis test! Define the null and alternative hypothesis for this test.

```
BEGIN QUESTION
name: q1_3
manual: true
```

SOLUTION:

NULL: The slope of the true line of best fit for Pick Number vs Career Length is zero.

ALT: The slope of the true line of best fit for Pick Number vs Career Length is not zero.

Question 4

Saurav says that instead of finding the slope for each resample, we can find the correlation instead, and that we will get the same result. Why is he correct? What is the relationship between slope and correlation?

```
BEGIN QUESTION
name: q1_4
manual: true
```

SOLUTION: Saurav is correct because if the correlation in a resample is zero, the slope of the best fit line will be also be zero, because slope is equal to:

$$slope = correlation * \frac{SD_Y}{SD_X}$$

Question 5

Define the function `one_resample_r` that performs a bootstrap and finds the correlation between `Pick Number` and `Career Length` in the resample. `one_resample_r` should take three arguments, a table `tbl` and the labels of the columns you are finding the correlation between, `col1` and `col2` .

```
BEGIN QUESTION
name: q1_5
manual: false
```

In [10]:
```
def one_resample_r(tbl, col1, col2):
    # BEGIN SOLUTION
    sampled_tbl = tbl.sample()
    return correlation(sampled_tbl, col1, col2)
    # END SOLUTION

# Don't change this line below!
one_resample = one_resample_r(nfl, "Pick Number", "Career Length")
one_resample
```

Out[10]:
```
-0.1881511906614756
```

In [11]:
```
# TEST
type(one_resample) in set([float, np.float32, np.float64])
```

Out[11]:
```
True
```

In [12]:
```
# HIDDEN TEST
np.random.seed(19)
np.round(one_resample, 3) == -0.129
```

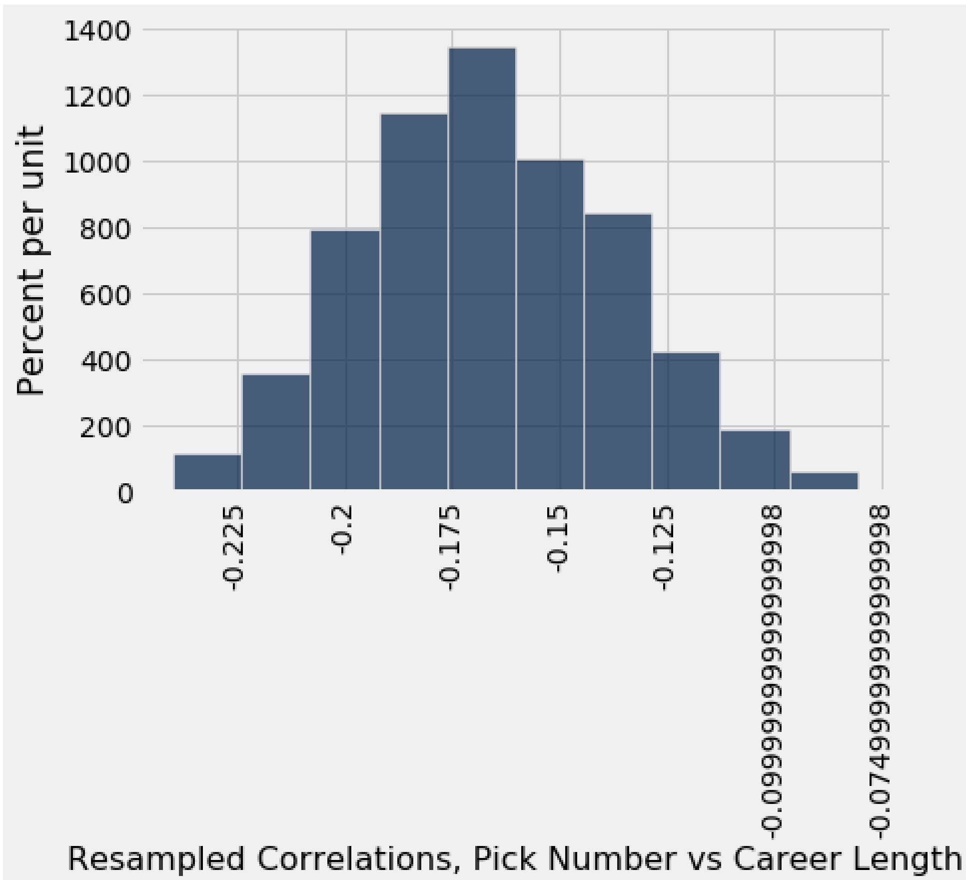
Out[12]:
```
False
```

Question 6

Generate 1000 bootstrapped correlations for `Pick Number` and `Career Length` , store your results in the array `resampled_correlations_pc` , and plot a histogram of your results.

```
BEGIN QUESTION
name: q1_6
manual: true
```

In [13]:
```
resampled_correlations_pc = make_array() # SOLUTION
# BEGIN SOLUTION
for i in np.arange(1000):
    resample_corr_pc = one_resample_r(nfl, "Pick Number", "Career Lengt
h")
    resampled_correlations_pc = np.append(resampled_correlations_pc, res
ample_corr_pc)
# END SOLUTION
Table().with_column("Resampled Correlations, Pick Number vs Career Lengt
h", resampled_correlations_pc).hist()
```



Question 7

Calculate a 95% confidence interval for the resampled correlations and assign either `True` or `False` to `reject` if we can reject the null hypothesis or if we cannot reject the null hypothesis using a 5% p-value cutoff.

```
BEGIN QUESTION
name: q1_7
manual: false
```

In [14]:
```
lower_bound_pc = percentile(2.5, resampled_correlations_pc) # SOLUTION
upper_bound_pc = percentile(97.5, resampled_correlations_pc) # SOLUTION
reject = True # SOLUTION

# Don't change this!
print(f"95% CI: [{lower_bound_pc}, {upper_bound_pc}] , Reject the null:
{reject}")
```

```
95% CI: [-0.22144880480557705, -0.10513007592087015] , Reject the null:
True
```

In [15]:
```
# TEST
all([type(lower_bound_pc) in set([float, np.float32, np.float64]),
    type(upper_bound_pc) in set([float, np.float32, np.float64]),
    type(reject) == bool])
```

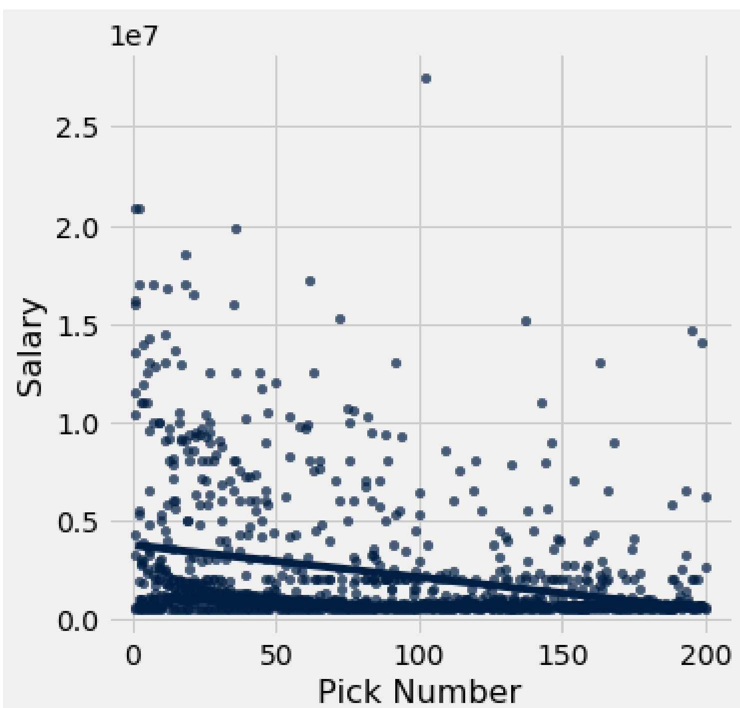
Out[15]: True

In [16]:
```
# HIDDEN TEST
all([lower_bound_pc == percentile(2.5, resampled_correlations_pc),
     upper_bound_pc == percentile(97.5, resampled_correlations_pc),
    reject == True])
```

Out[16]: True

Now let's investigate the relationship between `Pick Number` and `Salary` . As usual, let's inspect our data visually first. A line of best fit is plotted for you.

In [17]:
```
# Just run this cell!
nfl.scatter("Pick Number", "Salary", fit_line=True)
```



Question 8

Using the function `correlation`, find the correlation between `Pick Number` and `Salary` and assign it to `pick_salary_correlation`.

```
BEGIN QUESTION
name: q1_8
manual: false
```

In [18]:
```
pick_salary_correlation = correlation(nfl, "Pick Number", "Salary") # SO
LUTION
pick_salary_correlation
```

Out[18]: -0.2812388644684761

In [19]:
```
# TEST
# Correlation is a number between -1 and 1
-1 <= pick_salary_correlation <= 1
```

Out[19]: True

In [20]:
```
# HIDDEN TEST
np.round(pick_salary_correlation, 3) == -0.281
```

Out[20]: True

We can see that there is a negative association between `Pick Number` and `Salary`!

Question 9

Once again, Evan thinks that the slope of the true line of best fit for `Pick Number` and `Salary` is not zero: that is, there is some correlation/association between `Pick Number` and `Salary`. To test this claim, we can run a hypothesis test! Define the null and alternative hypothesis for this test.

```
BEGIN QUESTION
name: q1_9
manual: true
```

SOLUTION:

NULL: The slope of the true line of best fit for Pick Number vs Salary is zero.

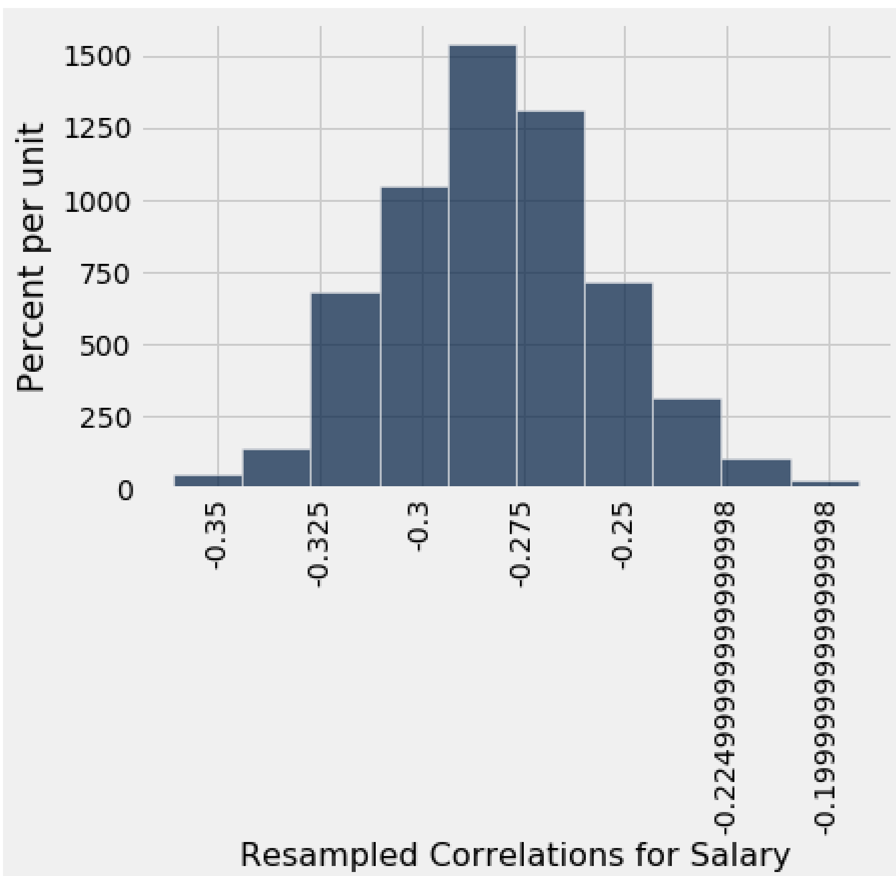
ALT: The slope of the true line of best fit for Pick Number vs Salary is not zero.

Question 10

Generate 1000 bootstrapped correlations for `Pick Number` and `Salary` , append them to the array `resampled_correlations_salary` , and then plot a histogram of your results.

```
BEGIN QUESTION
name: q1_10
manual: true
```

In [21]:
```
resampled_correlations_salary = make_array() # SOLUTION
# BEGIN SOLUTION
for i in np.arange(1000):
    resampled_corr_sal = one_resample_r(nfl, "Pick Number", "Salary")
    resampled_correlations_salary = np.append(resampled_correlations_sal
ary, resampled_corr_sal)
# END SOLUTION
Table().with_column("Resampled Correlations for Salary", resampled_corre
lations_salary).hist()
```



Question 11

Calculate a 95% confidence interval for the resampled correlations and assign either `True` or `False` to `reject_sal` if we can reject the null hypothesis or if we cannot reject the null hypothesis using a 5% p-value cutoff.

```
BEGIN QUESTION
name: q1_11
manual: false
```

In [22]:
```
lower_bound_sal = percentile(2.5, resampled_correlations_salary)
upper_bound_sal = percentile(97.5, resampled_correlations_salary)
reject_sal = True

# Don't change this!
print(f"95% CI: [{lower_bound_sal}, {upper_bound_sal}], Reject the null:
{reject_sal}")
```

```
95% CI: [-0.3298805472230466, -0.22857801897705676], Reject the null: T
rue
```

In [23]:
```
# TEST
all([type(lower_bound_sal) in set([float, np.float32, np.float64]),
    type(upper_bound_sal) in set([float, np.float32, np.float64]),
    type(reject_sal) == bool])
```

Out[23]: `True`

In [24]:
```
# HIDDEN TEST
all([lower_bound_sal == percentile(2.5, resampled_correlations_salary),
     upper_bound_sal == percentile(97.5, resampled_correlations_salary),
    reject_sal == True])
```

Out[24]: `True`

Analyzing Residuals

Next, Evan wants to predict his Career Length and Salary based on his Pick Number. To understand what his Career Length and Salary might be, Evan wants to generate confidence intervals of possible values for both career length and salary. First, let's investigate how effective our predictions for career length and salary based on pick number are.

Question 12

Calculate the slope and intercept for the line of best fit for `Pick Number` vs `Career Length` and for `Pick Number` vs `Salary`. Assign these values to `career_length_slope`, `career_length_intercept`, `salary_slope`, and `salary_intercept` respectively. The function `parameters` returns a two-item array containing the slope and intercept of a linear regression line.

Hint 1: Use the `parameters` function with the arguments specified!

*Hint 2: Remember we're predicting career length and salary **based off** a pick number. That should tell you what the `colx` and `coly` arguments you should specify when calling `parameters`.*

```
BEGIN QUESTION
name: q1_12
manual: false
```

In [25]:

```
# DON'T EDIT THE PARAMETERS FUNCTION
def parameters(tbl, colx, coly):
    x = tbl.column(colx)
    y = tbl.column(coly)

    r = correlation(tbl, colx, coly)

    x_mean = np.mean(x)
    y_mean = np.mean(y)
    x_sd = np.std(x)
    y_sd = np.std(y)

    slope = (y_sd / x_sd) * r
    intercept = y_mean - (slope * x_mean)
    return make_array(slope, intercept)

career_length_slope = parameters(nfl, "Pick Number", "Career Length").it
em(0) # SOLUTION
career_length_intercept = parameters(nfl, "Pick Number", "Career Length"
).item(1) # SOLUTION

salary_slope = parameters(nfl, "Pick Number", "Salary").item(0) # SOLUTI
ON
salary_intercept = parameters(nfl, "Pick Number", "Salary").item(1) # SO
LUTION
```

In [26]:
```
# TEST
all([type(career_length_slope) in set([float, np.float32, np.float64]),
    type(career_length_intercept) in set([float, np.float32, np.float64
]),
    type(salary_slope) in set([float, np.float32, np.float64]),
    type(salary_intercept) in set([float, np.float32, np.float64])])
```

Out[26]: True

In [27]:
```
# HIDDEN TEST
all([np.round(career_length_slope, 3) == -0.009,
    np.round(career_length_intercept, 3) == 5.454,
    np.round(salary_slope, 3) == -16229.882,
    np.round(salary_intercept, 3) == 3732861.986])
```

Out[27]: True

Question 13

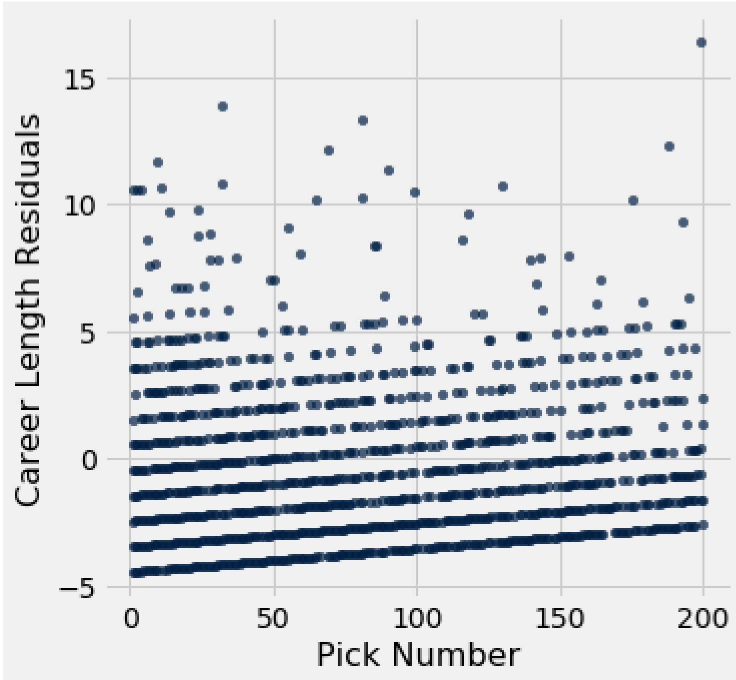
Draw a scatter plot of the residuals for each line of best fit for `Pick Number vs Career Length` and for `Pick Number vs Salary`.

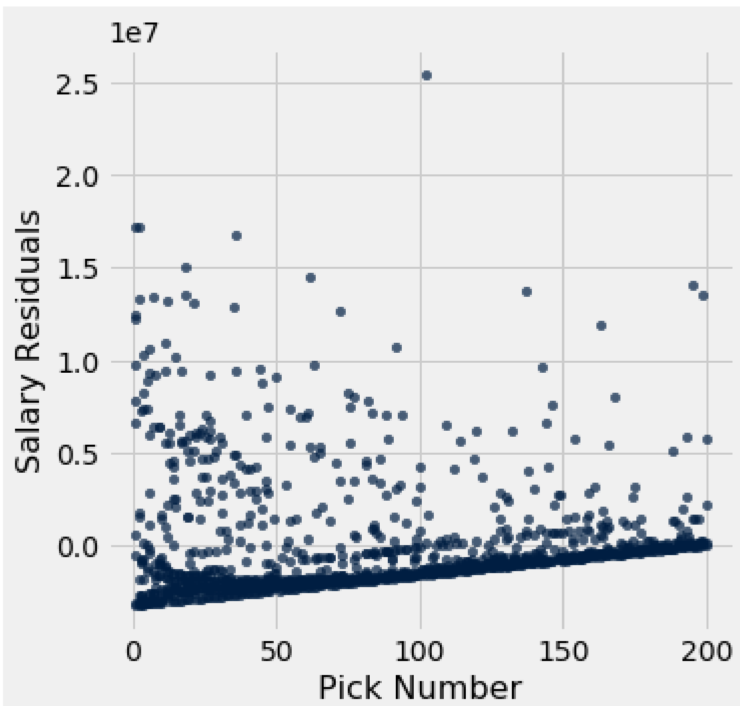
Hint: We want to get the predictions for every player in the dataset

Hint 2: This question is really involved, try to follow the skeleton code!

```
BEGIN QUESTION
name: q1_13
manual: true
```

In [28]:
```
predicted_career_lengths = career_length_slope * nfl.column("Pick Number") + career_length_intercept # SOLUTION
predicted_salaries = salary_slope * nfl.column("Pick Number") + salary_intercept # SOLUTION


career_length_residuals = nfl.column("Career Length") - predicted_career_lengths # SOLUTION
salary_residuals = nfl.column("Salary") - predicted_salaries # SOLUTION

nfl_with_residuals = nfl.with_columns("Career Length Residuals", career_length_residuals, "Salary Residuals", salary_residuals)

# Now generate two scatter plots!
nfl_with_residuals.scatter("Pick Number", "Career Length Residuals")
nfl_with_residuals.scatter("Pick Number", "Salary Residuals")
```





Here's a [link (https://www.inferentialthinking.com/chapters/15/6/Numerical_Diagnostics.html)](https://www.inferentialthinking.com/chapters/15/6/Numerical_Diagnostics.html) to properties of residuals in the textbook that could help out with some questions.

Question 14

Based on these plots of residuals, do you think linear regression is a good model for `Pick Number` vs `Career Length` and for `Pick Number` vs `Salary`? Explain for both.

```
BEGIN QUESTION
name: q1_14
manual: true
```

SOLUTION: There appear to be patterns in both the `Pick Number` vs `Career Length` and for `Pick Number` vs `Salary` residual graphs so linear regression is not a good model for either relationship. Note: it's incorrect to say that there appear to be *trends* in the graphs since residual plots should **never** have a trend.

Question 15

Assign `career_length_residual_corr` and `salary_residual_corr` to either 1, 2 or 3 corresponding to whether or not the correlation between `Pick Number` and `Career Length Residuals` is positive, zero, or negative, and to whether or not the correlation between `Pick Number` and `Salary Residuals` is positive, zero, or negative respectively.

1. Positive
2. Zero
3. Negative

```
BEGIN QUESTION
name: q1_15
manual: false
```

In [29]:
```
career_length_residual_corr = 2 # SOLUTION
salary_residual_corr = 2 # SOLUTION
```

In [30]:
```
# TEST
all([type(career_length_residual_corr) == int,
    type(salary_residual_corr) == int])
```

Out[30]: True

In [31]:
```
# HIDDEN TEST
career_length_residual_corr == 2 and salary_residual_corr == 2
```

Out[31]: True

It looks like the largest residuals are positive residuals, so let's investigate those more closely.

Question 16

Let's investigate where our regression line is making errors. Using the `nfl_with_residuals` table, assign `greatest_career_length_residual` to the string that is the name of the player with the largest positive residual for `Pick Number vs Career Length`.

```
BEGIN QUESTION
name: q1_16
manual: false
```

In [32]:
```
greatest_career_length_residual = nfl_with_residuals.sort("Career Length Residuals", descending=True).column("Player").item(0) # SOLUTION
greatest_career_length_residual
```

Out[32]:
```
'Tom Brady'
```

In [33]:
```
# TEST
type(greatest_career_length_residual) == str
```

Out[33]:
```
True
```

In [34]:
```
# HIDDEN TEST
greatest_career_length_residual == 'Tom Brady'
```

Out[34]:
```
True
```

Now let's investigate the residuals for salary. Run the cell below to see the players with the largest residuals for `Pick Number vs Salary`.

In [35]:
```
# Just run this cell!
nfl_with_residuals.sort("Salary Residuals", descending=True).take(np.ara
nge(10)).drop(2,6)
```

Out[35]:

Player	Salary	Pick Number	Position	Career Length	Salary Residuals
Kirk Cousins	27500000	102	QB	8	2.54226e+07
Marcus Mariota	20922000	2	QB	5	1.72216e+07
Jameis Winston	20922000	1	QB	5	1.72054e+07
Derek Carr	19900000	36	QB	6	1.67514e+07
Joe Flacco	18500000	18	QB	12	1.50593e+07
Jimmy Garoppolo	17200000	62	QB	6	1.44734e+07
Antonio Brown	14625000	195	WR	10	1.4057e+07
Grady Jarrett	15209000	137	DT	5	1.36996e+07
Melvin Ingram	17000000	18	DE	8	1.35593e+07
Tom Brady	14000000	199	QB	20	1.34969e+07

Question 17

What patterns do you notice with these large residuals for salary? How could this affect our analysis?

```
BEGIN QUESTION
name: q1_17
manual: true
```

SOLUTION: 7 of the top 10 players with the largest pick number vs salary residuals all have the position QB (quarterback). This could affect our analysis because this position group might have a different distribution than the other groups and may be introducing a lot of error into our analysis. We could try analyzing the groups separately!

Prediction Intervals

Now, Evan wants to predict his salary based on his specific pick number, which is 169. Instead of using the best fit line generated from the sample, Evan wants to generate an interval for his predicted career length.

Question 18

Define the function `one_resample_prediction` that generates a bootstrapped sample from the `tbl` argument, calculates the line of best fit for `ycol` vs `xcol` for that resample, and predicts a value based on `xvalue`.

Hint: Remember you defined the `parameters` function earlier

```
BEGIN QUESTION
name: q1_18
manual: false
```

In [36]:
```
def one_resample_prediction(tbl, colx, coly, xvalue):
    # BEGIN SOLUTION
    resample = tbl.sample()
    resample_parameters = parameters(resample, colx, coly)

    slope = resample_parameters.item(0)
    intercept = resample_parameters.item(1)
    return slope * xvalue + intercept
    # END SOLUTION

evans_career_length_pred = one_resample_prediction(nfl, "Pick Number",
"Career Length", 169)
evans_career_length_pred
```

Out[36]: 4.06911211363613

In [37]:
```
# TEST
type(evans_career_length_pred) in set([float, np.float32, np.float64])
```

Out[37]: True

In [38]:
```
# HIDDEN TEST
np.random.seed(19)
one_resample_prediction(nfl, "Pick Number", "Career Length", 169)
```

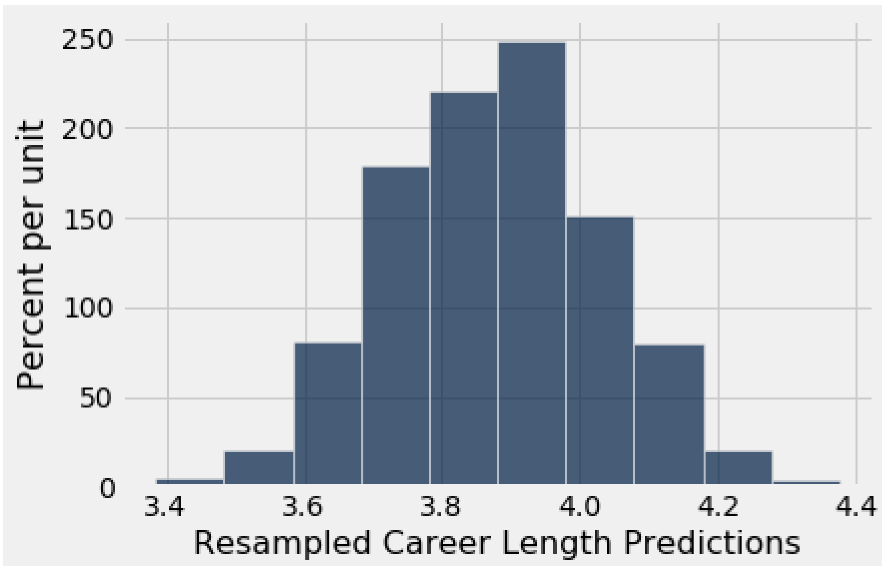
Out[38]: 4.00532790957706

Question 19

Assign `resampled_predictions` to be an array that will contain 1000 resampled predictions for Evan's career length based on his pick number, and then generate a histogram of it.

```
BEGIN QUESTION
name: q1_19
manual: true
```

In [39]:
```
resampled_predictions = make_array() # SOLUTION

# BEGIN SOLUTION
for i in np.arange(1000):
    resample_pred = one_resample_prediction(nfl, "Pick Number", "Career
 Length", 169)
    resampled_predictions = np.append(resampled_predictions, resample_pr
ed)
# END SOLUTION

# Don't change/delete the code below in this cell
Table().with_column("Resampled Career Length Predictions", resampled_pre
dictions).hist()
```



Question 20

Using `resampled_predictions` from Question 19, generate a 99% confidence interval for Evan's predicted career lengths.

```
BEGIN QUESTION
name: q1_20
manual: false
```

In [40]:
```
lower_bound_evan = percentile(0.5, resampled_predictions) # SOLUTION
upper_bound_evan = percentile(99.5, resampled_predictions) # SOLUTION

# Don't delete/modify the code below in this cell
print(f"99% CI: [{lower_bound_evan}, {upper_bound_evan}]")
```

```
99% CI: [3.4621698577747893, 4.273129963153647]
```

In [41]:
```
# TEST
all([type(lower_bound_evan) in set([float, np.float32, np.float64]),
    type(upper_bound_evan) in set([float, np.float32, np.float64])])
```

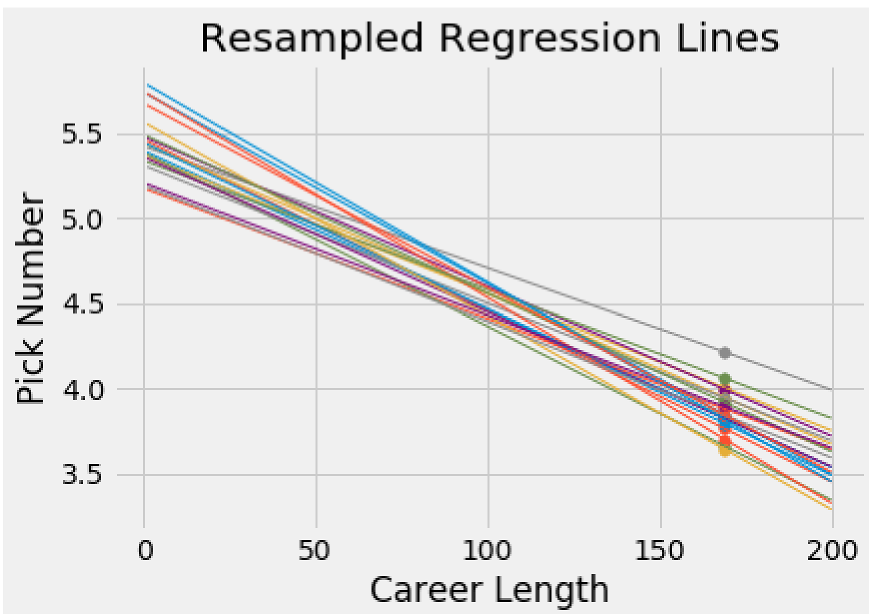
Out[41]: True

In [42]:
```
# HIDDEN TEST
all([lower_bound_evan == percentile(0.5, resampled_predictions),
     upper_bound_evan == percentile(99.5, resampled_predictions)])
```

Out[42]: True

Run the following cell to see a few bootstrapped regression lines, and the predictions they make for a career length of 169.

In [43]:
```
# Just run this cell!
# You don't need to understand all of what it is doing but you should re
cognize a lot of the code!
lines = Table(['slope','intercept'])
x=169
for i in np.arange(20):
    resamp = nfl.sample(with_replacement=True)
    resample_pars = parameters(resamp, "Pick Number", "Career Length")
    slope = resample_pars.item(0)
    intercept = resample_pars.item(1)
    lines.append([slope, intercept])

lines['prediction at x='+str(x)] = lines.column('slope')*x + lines.colum
n('intercept')
xlims = [min(nfl.column("Pick Number")), max(nfl.column("Pick Number"))]
left = xlims[0]*lines[0] + lines[1]
right = xlims[1]*lines[0] + lines[1]
fit_x = x*lines['slope'] + lines['intercept']
for i in range(20):
    plt.plot(xlims, np.array([left[i], right[i]]), lw=1)
    plt.scatter(x, fit_x[i], s=30)
plt.ylabel("Pick Number");
plt.xlabel("Career Length");
plt.title("Resampled Regression Lines");
```



Question 21

Does the Central Limit Theorem guarantee that the bootstrapped slopes or bootstrapped correlations will be normally distributed for any dataset that uses a large random sample? If you think yes, assign `True` to `clt_applies`, otherwise assign `False` to `clt_applies` if you think no. Are residuals normally distributed? If you think they are, assign `True` to `residuals_normal`, otherwise assign `False` to `residuals_normal`.

```
BEGIN QUESTION
name: q1_21
manual: false
```

In [44]:
```
clt_applies = False # SOLUTION
residuals_normal = False # SOLUTION
```

In [45]:
```
# TEST
type(clt_applies) == bool and type(residuals_normal) == bool
```

Out[45]: True

In [46]:
```
# HIDDEN TEST
clt_applies == False and residuals_normal == False
```

Out[46]: True

Question 22

What are some biases in this dataset that may have affected our analysis? Some questions you can ask yourself are: "is our sample a simple random sample?" or "what kind of data are we using/what variables are we dealing with: are they categorical, numerical, or both (both is something like ordinal data)?".

Hint: you might want to revisit the beginning of this assignment to reread how this data/ `nfl` table was generated.

```
BEGIN QUESTION
name: q1_22
manual: true
```

SOLUTION: Our dataset is not a simple random sample - it only contains players who played in the 2019 season, which may bias our results! Pick number is not a numerical value, it is a kind of mix between numerical and categorical data called "ordinal data", so our lines of best fit do not make sense for negative values or values larger than the max possible pick number.

(OPTIONAL, Out of Scope) Extending Linear Regression

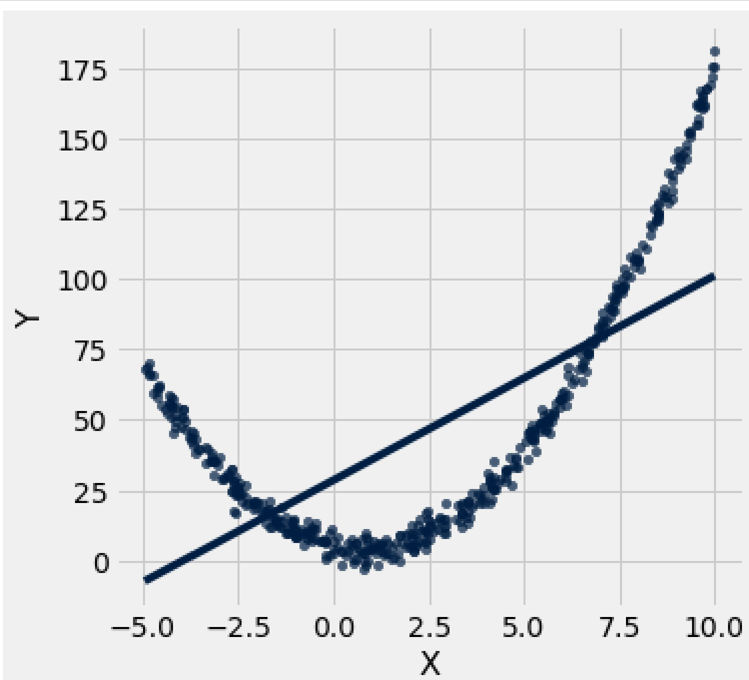
This following section is completely **optional**, meaning there's no code to be graded/filled in. Just run the cells/explore if you're interested.

In the past few weeks you have learned one of the most powerful tools in a data scientist's arsenal: regression. At this point you may be wondering: what do we do when our data is not linear? You have learned that you shouldn't try and force models when they are bad fits: for example, if we detect heteroscedasticity in our residuals plot, we know that linear regression is a bad fit.

How can we fit data that is not linear then?

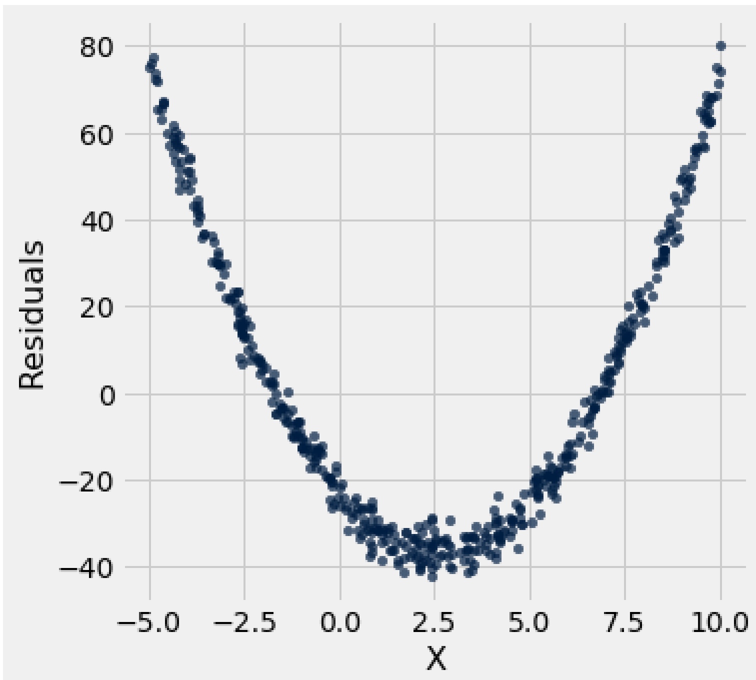
Let's increase our data's complexity a little: instead of linear data, let's look at data that you would naturally model with a parabola instead:

In [47]:
```
def parabola(x, a=1, b=0, c=0):
    random_noise = np.random.normal(size=len(x)) * 3
    return  a*(x**2) + b*(x) + c + random_noise

size = 500
x_values = np.random.uniform(-5, 10, size=size)
y_values = parabola(x_values, a=2, b=-3, c=5)

Table().with_columns("X", x_values, "Y", y_values).scatter("X","Y", fit_
line=True)
```



You can see that our line of best fit is a poor match for this data. Let's look at the residual plot:

In [48]:
```
def mse(slope, intercept):
    predicted_y = slope * x_values + intercept
    errors = y_values - predicted_y
    return np.mean(errors**2)


slope_and_intercept = minimize(mse, smooth=True)
predicted_y = slope_and_intercept.item(0) * x_values + slope_and_interce
pt.item(1)
residuals = y_values - predicted_y

Table().with_columns("X", x_values, "Residuals",residuals).scatter("X",
"Residuals")
```



Our residuals clearly have a pattern, confirming that linear regression is a bad fit for this data! In fact, our residuals actually look like our original data.

Linear regression generates a line that minimizes mean squared error. Using the `minimize` function on the `mse` function does all the work of finding values for us! Can we use `minimize` for more complicated models? Yes! In future data science classes, you will learn how to find these values yourself using the mathematical fields of Linear Algebra (note that it involves lines!) and calculus!

Let's take a look at the equation for a line:

$$y = ax + b$$

There are two parameters here that we can change: a, which is the slope, and b, which is the intercept.

How about the equation for a parabola?

$$y = ax^2 + bx + c$$

Now there are three parameters, a, b, c.

Let's change our mse function to incorporate these three parameters!

In [49]:
```
def mse_parabola(a, b, c):
    predicted_y = a * (x_values**2) + b * (x_values) + c
    errors = y_values - predicted_y
    return np.mean(errors**2)
```

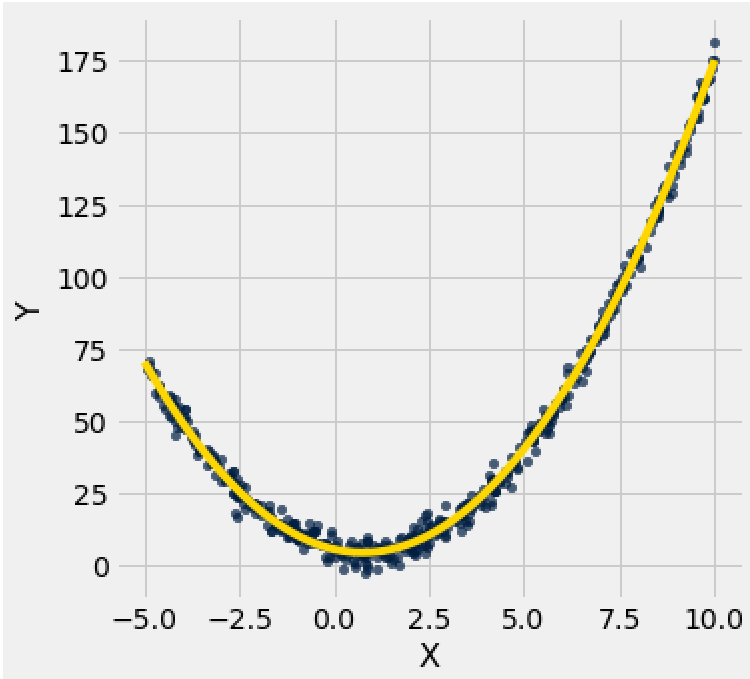
The function still returns the mean squared error of our predicted curve, just our curve is now a parabola with the parameters `a` , `b` , and `c` . Let's try and minimize this function!

In [50]:
```
parameters = minimize(mse_parabola, smooth=True)
a = parameters.item(0)
b = parameters.item(1)
c = parameters.item(2)
a, b, c
```

Out[50]:
```
(2.0016803379360844, -3.0319901616253913, 5.140855142478964)
```

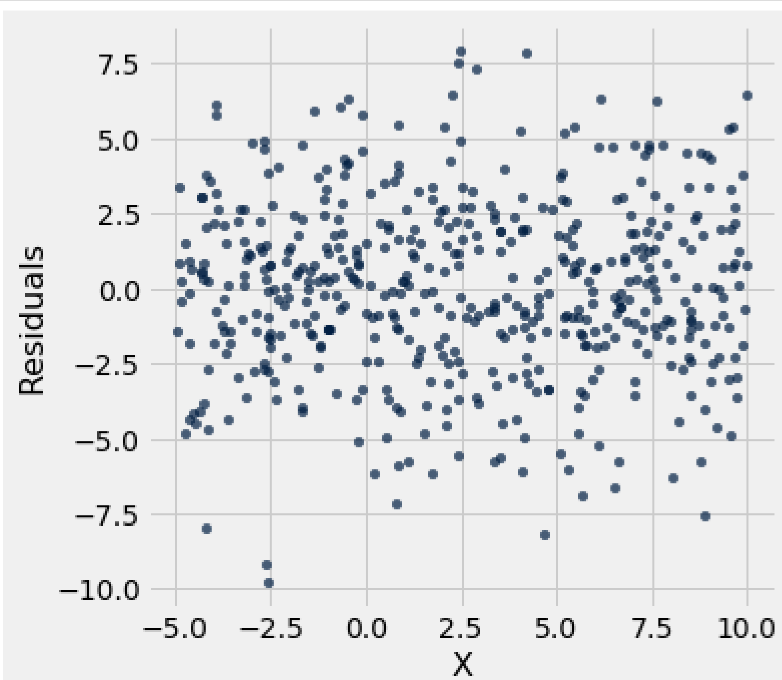
Let's plot our new curve with these values!

In [51]:
```
x_values_range = np.linspace(-5, 10, 1000)
predicted_y = a * (x_values_range**2) + b * (x_values_range) + c

Table().with_columns("X", x_values, "Y", y_values).scatter("X", "Y")
plt.plot(x_values_range, predicted_y, color='gold', markersize=1);
```



Our curve looks like a much better fit now! Let's double check the residuals plot to be sure.

In [52]:
```
residuals = y_values - (a * (x_values**2) + b * (x_values) + c)
Table().with_columns("X", x_values, "Residuals", residuals).scatter("X",
"Residuals")
```



A formless cloud, excellent!

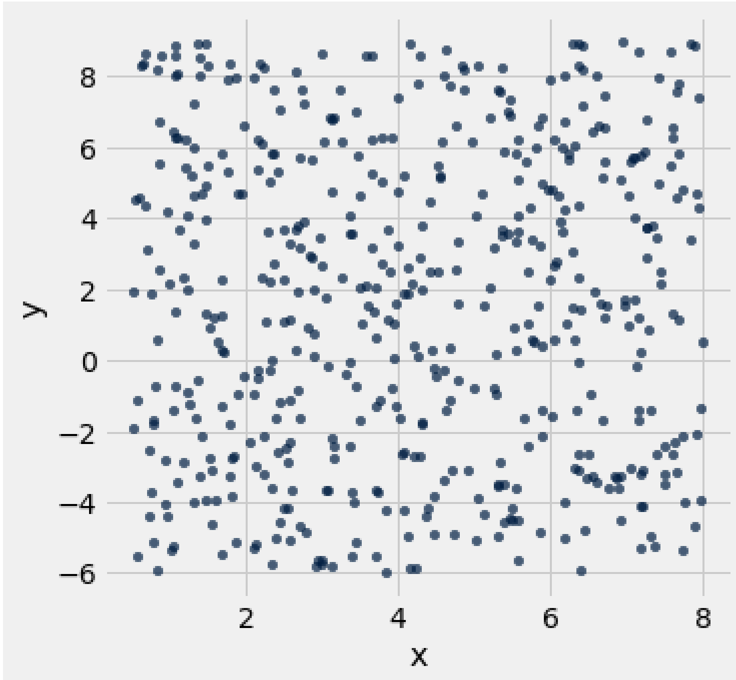
What else can the method of least squares do?

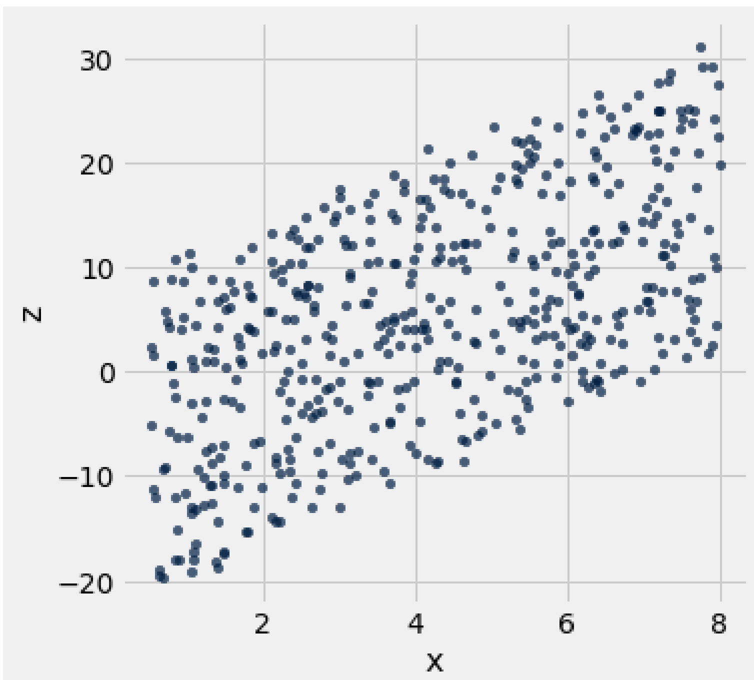
Can we predict a single variable based on the values of two other variables? Right now, we don't have a way of doing that.

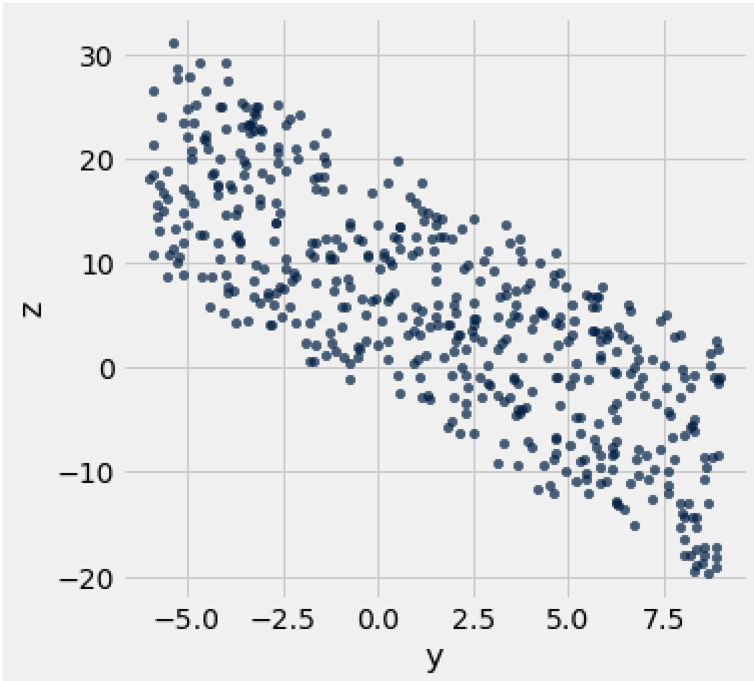
If you look at the previous example, you could say that the x^2 term is actually a second variable.

Let's generate a dataset to work with. We are going to try and predict `z` based on `x` and `y`.

In [53]:
```
x_values_range = np.linspace(-5, 10, 1000)

x = 0.5 * np.random.uniform(-5, 10, size=size) + 3
y = np.random.uniform(-5, 10, size=size) - 1
z = 3*x  + (-2*y) -4 + np.random.normal(size=size)

data = Table().with_columns("x", x, "y", y, "z", z)
data.scatter("x", "y")
data.scatter("x", "z")
data.scatter("y", "z")
```

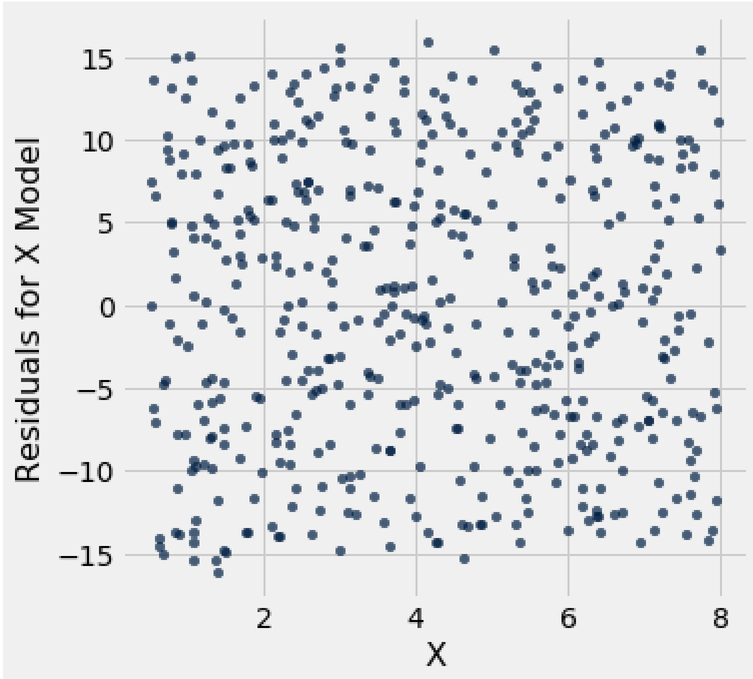

y
x
8
6
4
2
0
−2
−4
−6
2
4
6
8

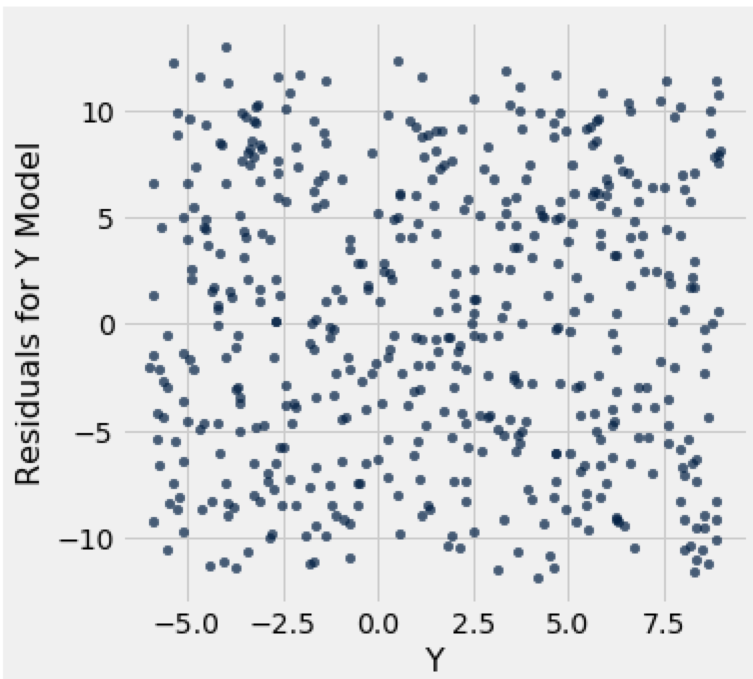

z
x
30
20
10
0
−10
−20
2
4
6
8



We can see that `x` and `y` would both be very helpful to predict `z` by themselves! However, if we combined them we could predict `z` even better. Since our goal is to minimize mean squared error, let's find the mean squared error of the models that only use `x` and `y` by themselves (using an intercept).

In [54]:
```
from scipy import stats
def su(x):
    return (x-np.mean(x)) / np.std(x)
def r(x, y):
    return np.mean(su(x) * su(y))

def mse_x(slope, intercept):
    predicted_z = slope * x + intercept
    errors = z - predicted_z
    return np.mean(errors**2)

def mse_y(slope, intercept):
    predicted_z = slope * y + intercept
    errors = z - predicted_z
    return np.mean(errors**2)


slope_and_intercept_x = minimize(mse_x, smooth=True)
predicted_z_x = slope_and_intercept_x.item(0) * x + slope_and_intercept_
x.item(1)
residuals_x = z - predicted_z_x

Table().with_columns("X", x, "Residuals for X Model", residuals_x).scatt
er("X", "Residuals for X Model")

slope_and_intercept_y = minimize(mse_y, smooth=True)
predicted_z_y = slope_and_intercept_y.item(0) * y + slope_and_intercept_
y.item(1)
residuals_y = z - predicted_z_y

Table().with_columns("Y", y, "Residuals for Y Model", residuals_y).scatt
er("Y", "Residuals for Y Model")
```





Both of the residual plots show no trend, so using these `x` or `y` by themselves would work, but how good are these models? Let's calculate their actual mse.

In [55]:
```
x_only_mse = mse_x(slope_and_intercept_x.item(0), slope_and_intercept_x.
item(1))
y_only_mse = mse_y(slope_and_intercept_y.item(0), slope_and_intercept_y.
item(1))

print(f"X only model MSE: {x_only_mse}, Y only model MSE: {y_only_mse}")
```

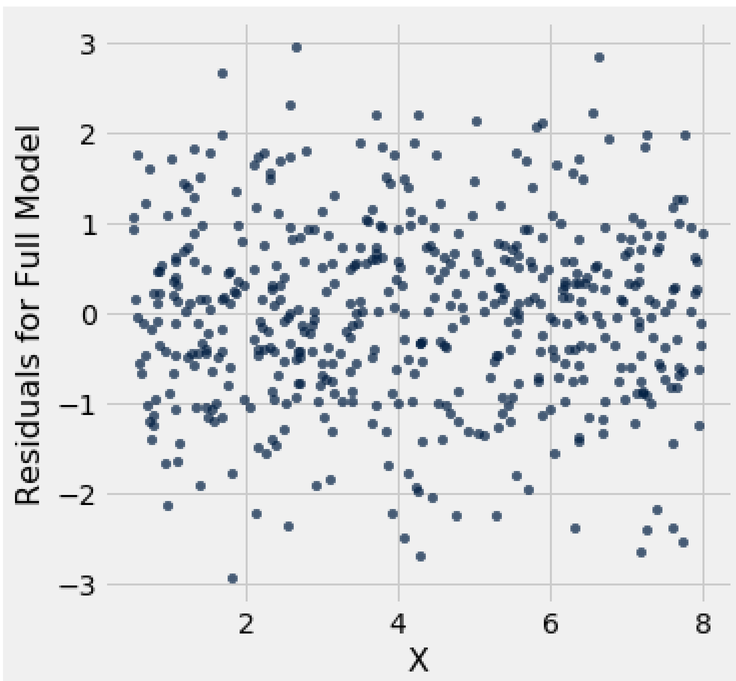
```
X only model MSE: 76.83961332517617, Y only model MSE: 43.2014780276691
1
```

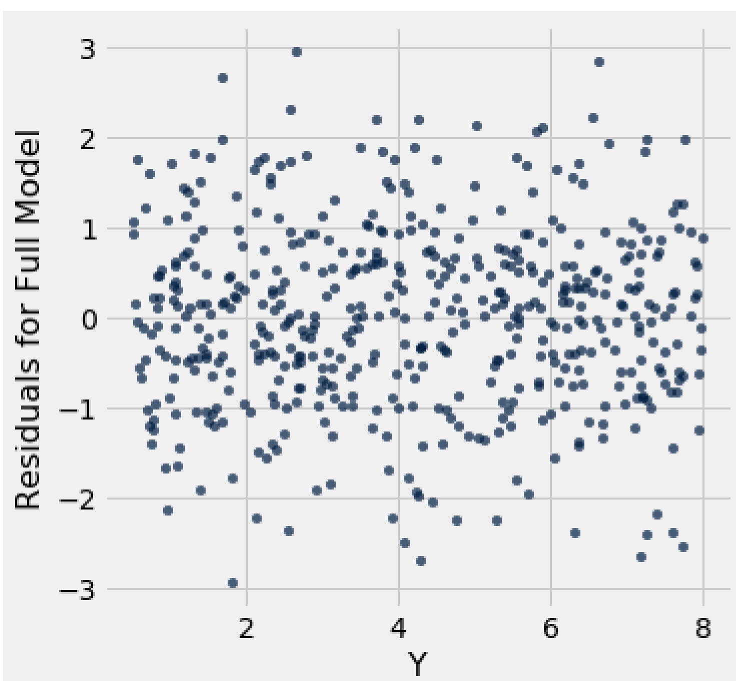
Looks like the y only model has lower MSE, so we should try and use that if we can only use `x` or `y`.

Instead, let's try to build a model that is a combination of `x`, `y` and an intercept `c` to predict `z`!

$$z = ax + by + c$$

In [56]:
```
def mse_both(a, b, c):
    predicted_z = (a * x) + (b * y) + c
    errors = z - predicted_z
    return np.mean(errors**2)

slope_and_intercept_both = minimize(mse_both, smooth=True)
predicted_z = (slope_and_intercept_both.item(0) * x) + (slope_and_interc
ept_both.item(1) * y) + slope_and_intercept_both.item(2)
residuals = z - predicted_z

Table().with_columns("X", x, "Residuals for Full Model", residuals).scat
ter("X", "Residuals for Full Model")
Table().with_columns("Y", x, "Residuals for Full Model", residuals).scat
ter("Y", "Residuals for Full Model")
```





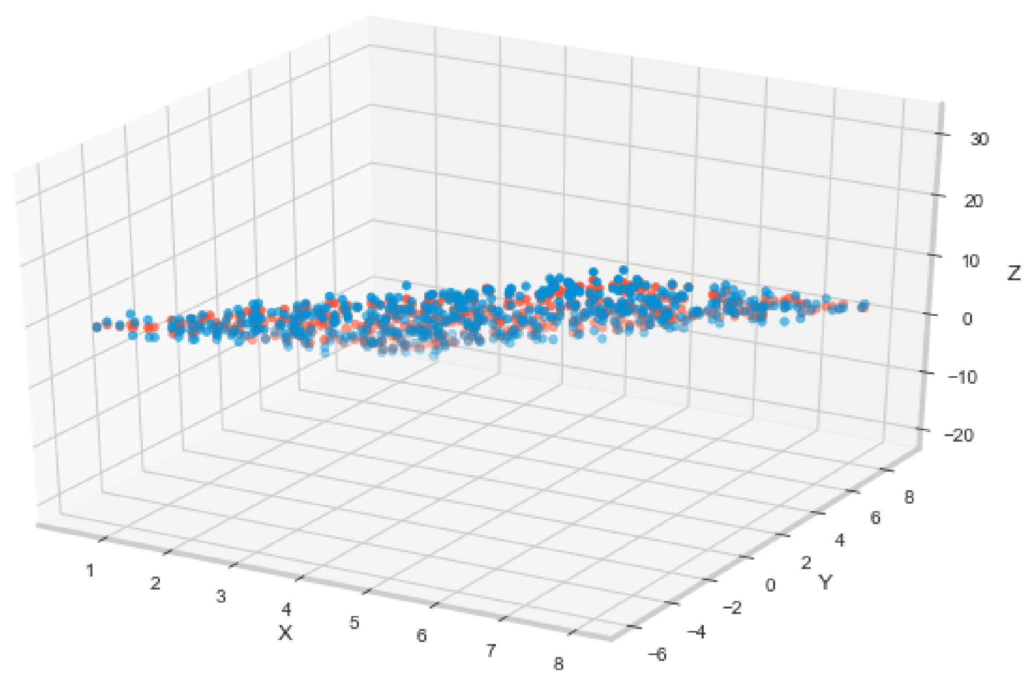
This model is also a good fit looking at the residuals with respect to both `x` and `y` ! What is this model's mse?

In [57]:
```
full_model_mse = mse_both(slope_and_intercept_both.item(0), slope_and_in
tercept_both.item(1), slope_and_intercept_both.item(2))

print(f"X only model MSE: {x_only_mse}, Y only model MSE: {y_only_mse},
 Both X and Y MSE: {full_model_mse}")
```

```
X only model MSE: 76.83961332517617, Y only model MSE: 43.2014780276691
1, Both X and Y MSE: 1.0218692933980662
```

That MSE is much lower! We should definitely use this model instead of either the x only or y only model independently! Let's try and visualize what this model looks like with a 3D graph!

In [58]:
```
import matplotlib
%matplotlib inline
import matplotlib.pyplot as plt

from mpl_toolkits.mplot3d import Axes3D
import seaborn as sns
sns.set_style("whitegrid", {'axes.grid' : False})

fig = plt.figure(figsize=(10,7));
ax = fig.add_subplot(111, projection='3d');
ax.scatter(x, y, z);
ax.set_xlabel('X');
ax.set_ylabel('Y');
ax.set_zlabel('Z');

ax.scatter(x,y,predicted_z);
```



Once we start working in more dimensions, visualization becomes increasingly difficult and useless. Instead of predicting a line, our prediction is actually a plane of values (the red values)!

2. Submission

Once you're finished, select "Save and Checkpoint" in the File menu and then execute the `submit` cell below. The result will contain a link that you can use to check that your assignment has been submitted successfully. If you submit more than once before the deadline, we will only grade your final submission. If you mistakenly submit the wrong one, you can head to okpy.org (https://okpy.org/) and flag the correct version. To do so, go to the website, click on this assignment, and find the version you would like to have graded. There should be an option to flag that submission for grading!

In [59]:
```
_ = ok.submit()
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-59-cc46ca874451> in <module>
----> 1 _ = ok.submit()

NameError: name 'ok' is not defined
```

In [60]:
```
# For your convenience, you can run this cell to run all the tests at once!
import os
print("Running all tests...")
_ = [ok.grade(q[:-3]) for q in os.listdir("tests") if q.startswith('q') and len(q) <= 10]
print("Finished running all tests.")
```

```
Running all tests...
Finished running all tests.
```