

Advanced Algorithms Problems and Solutions

Mattia Setzu Giorgio Vinciguerra

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1 Range updates

Consider an array C of n integers, initially all equal to zero. We want to support the following operations:

- **update(i, j, c):** where $0 \leq i \leq j \leq n - 1$ and c is an integer: it changes C such that $C[k] = C[k] + c$ for every $i \leq k \leq j$.
- **query(i)** where $0 \leq i \leq n - 1$: it returns the value of $C[i]$.
- **sum(i,j)** where $0 \leq i \leq j \leq n - 1$: it returns $\sum_{k=i}^j C[k]$.

Design a data structure that uses $O(n)$ space and implements each operation above in $O(\log(n))$ time. Note that $query(i) = sum(i, i)$ but it helps to reason. [Hint to further save space: use an implicit tree such as the Fenwick tree (see wikipedia).]

1.1 Solution 1: Segment tree lazy a-b sums

Let T be a segmented binary tree over a continuous interval $I : [0, N - 1]$ s.t. its leafs are the points in I , and the parent of two nodes comprises of their interval:

$$n' \cup n'' = n, n' \cap n'' = \emptyset \text{ s.t. } n \text{ is the parent of } n', n''$$

T will keep track of the prefix sums for every interval. We define a function

$$s' : [0, n - 1] \rightarrow \mathbb{N} \quad (1)$$

that given a node in T returns the value associated with I , namely the cumulative sum of that interval.

In order to reduce the computational cost, we introduce a lazy algorithm that doesn't propagate sums over T as they are streamed in the input, which means $s'(i)$ might not be accurate at a given time t for any of the requested operation.

We'll instead either compute over T or update T as necessary. Let us define a function to do so:

$$l : \mathbb{N} \rightarrow (\mathbb{N} \cup \{\epsilon\}, \mathbb{N}) \quad (2)$$

to keep track of our lazy sums:

$$s(n) = \begin{cases} \epsilon, - & \text{if no lazy prefix sum is in that interval} \\ k, m & \text{if a lazy sum of } k \text{ is to be propagated to } m \end{cases} \quad (3)$$

The QUERY function is then trivial:

- 1: **function** QUERY(I, i, sum):
- 2: **if** $I.size = 1$ **then** ▷ Return found value
- 3: **return** $I.sum$
- 4: **if** lazy(I), $i \in I.left, i \notin I.right$ **then** ▷ Lazy on left child

```

5:      lazy(I) ← False
6:      QUERY(I.left, i, sum + I.sum)
7:      if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right child
8:          lazy(I) ← False
9:          QUERY(I.right, i, sum + I.sum)
10:     if lazy(I), i ∈ I.right, i ∈ I.left then          ▷ Lazy on both
11:         lazy(I) ← False
12:         QUERY(I.right, i, j, sum + I.sum) + QUERY(I.left, i, j, sum +
I.sum)
13:     if !lazy(I), i ∈ I.left then                      ▷ Not lazy on left child
14:         QUERY(I.left, i, sum)
15:     if !lazy(I), i ∈ I.right then                    ▷ Not lazy on right child
16:         QUERY(I.right, i, sum)
17:     if !lazy(I), i ∈ I.right, i ∈ I.left then        ▷ Not lazy both
18:         SUM(I.right, i, sum)

1: function SUM(I, i, j, sum):
2:     if I.size = 1 then                                ▷ Return
3:         return I.sum + sum
4:     if lazy(I), i ∈ I.left, i ∉ I.right then          ▷ Lazy on left
5:         lazy(I) ← False
6:         SUM(I.left, i, j, sum + I.sum)
7:     if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right
8:         lazy(I) ← False
9:         SUM(I.right, i, j, sum + I.sum)
10:    if lazy(I), i ∈ I.right, i ∈ I.left then           ▷ Lazy on both
11:        lazy(I) ← False
12:        SUM(I.right, i, j, sum + I.sum) + SUM(I.left, i, j, sum + I.sum)
13:    if !lazy(I), i ∈ I.left then                        ▷ Not lazy on both
14:        SUM(I.left, i, sum)
15:    if !lazy(I), i ∈ I.right then                      ▷ Not lazy on both
16:        SUM(I.right, i, sum)
17:    if !lazy(I), i ∈ I.right, i ∈ I.left then          ▷ Not lazy on both
18:        SUM(I.right, i, sum)

1: function UPDATE(I, i, j, k):
2:     if I.size = 1 then                                ▷ Return
3:         return I.val ← I.val + update
4:         return I.val+ = update
5:     if lazy(I), i ∈ I.left, i ∉ I.right then          ▷ Lazy on left
6:         lazy(I.left) ← True
7:         I.left.val ← k
8:     if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right
9:         lazy(I.right) ← True

```

```

10:      I.right.val  $\leftarrow k$ 
11:  if lazy(I), i  $\in$  I.right, i  $\in$  I.left then                                 $\triangleright$  Lazy on both
12:      lazy(I)  $\leftarrow True$ 
13:      I.val  $\leftarrow k$ 
14:  if  $\neg$ lazy(I), i  $\in$  I.left then                                             $\triangleright$  Not lazy
15:      UPDATE(I.left, i, update)
16:      update(I.left, i, update)
17:  if  $\neg$ lazy(I), i  $\in$  I.right then                                            $\triangleright$  Not lazy
18:      UPDATE(I.right, i, update)
19:      update(I.right, i, update)
20:  if  $\neg$ lazy(I), i  $\in$  I.right, i  $\in$  I.left then                             $\triangleright$  Not lazy
21:      UPDATE(I.right, i, update)
22:      update(I.right, i, update)

```

2 Depth of a node in a random search tree

A random search tree for a set S can be defined as follows: if S is empty, then the null tree is a random search tree; otherwise, choose uniformly at random a key k as root, and the random search trees on $L = \{x \in S : x < k\}$ and $R = \{x \in S : x > k\}$ become, respectively, the left and right subtree of the root k . Consider the randomized QuickSort discussed in class and analyzed with indicator variables CLRS 7.3, and observe that the random selection of the pivots follows the above process, with indicator variables, prove that:

1. the expected depth of a node (i.e. the random variable representing the distance of the node from the root) is nearly $2 \ln n$;
2. the expected size of its subtree is nearly $2 \ln n$ too, observing that it is a simple variation of the previous analysis;
3. the probability that the expected depth of a node exceeds $2 \ln n$ is small for any given constant $c > 1$.

Chernoff's bounds.

2.1 Recursive balanced proof

Let n be the number of nodes in the input list l , $h = \log_2(n)$ the height of a balanced tree over l , $T(p)$ the tree built over the permutation p of pivots, $d(m)$ be the positional distance of a value m of a partition from the median value of the said partition. Then the following holds:

- $height(T) = h \iff |T.left| = |T.right| \pm 1$ Trivially, let r be the root of a 3-nodes partition: then, if the partition is unbalanced, the lesser one will comprise of 0 nodes, while the greater one of 2, which implies that $height(T.right) = 2$.
- $P = \text{pivot}, d(m) = \pm k \implies height(T.left) = height(T.right) \pm k$. Recursively from the previous statement, a partition unbalanced of one element generates subtrees whose levels differ on a factor of 1. By iterating recursively, their subtrees, if unbalanced by 1, will yield one more level difference. Over k unbalanced pivots on a single subtree, at most k levels will be added to h .
- By the previous statement, it follows that $\nexists T, T' : height(T) > height(T')$, T balanced, T' unbalanced. As stated, let T', T be the unbalanced/balanced tree respectively; let us cheat with T and switch the root pivot with the first element in its subtree. Now, let us prove by contradiction that T can't stay balanced and that its height will increase. By shifting the tree to the left we have deprived $T.right$ of either 0 levels (in case $T.right$ is able to switch every pivot in its tree with its right subtree root, ending with the rightmost leaf in its subtree) or 1, in case no rightmost leaf is present. Therefore $height(T) < height(T')$.

- The completely unbalanced tree is the tree with the most levels. By taking partitions of size 0 we constantly force, at each level, one subtree to disappear. Therefore, its level(s) has to be necessarily transferred to its brother. We then have exactly one node per level, therefore n levels.

Behaviour on random permutations Now let us analyse how the tree depth varies according to random pivot selection. We start by applying the 2.1k-distance to a tree T with $n = 3$ nodes. Trivially, $height(T)$ with balanced tree is equal to two. Now, let us pick either the lowest or the greatest pivot possible: the tree is unbalanced towards either the left or the right, but $height(T) = 2$ in both cases. As the reader can see from 2.1, the distance works in absolute value; it is then clear how, at every permutation for a pivot p , out of the n , there are 2 that generate a tree of the same height: $p = d(P) + k, p = d(P) - k$. Given that at every iteration a node x in a completely unbalanced tree T' has a probability of $\frac{1}{n-i}$, we can define the probability of x being a pivot at level l as:

$$P(x_k) = \frac{1}{n-l} \quad (4)$$

Now, in order for x not to be chosen as pivot in the previous $l - 1$ levels we have:

$$P(x_k) = \sum_{k=1}^{l-1} \left(\frac{1}{n-l+1} \right) \quad (5)$$

Given the height of T , the (harmonic) partial series converges to $\ln(n) + 1$. Let us now add a root r s.t. $T'.right = T, T'.left = T$. We now have to consider the mirror case $\ln(n') + \ln(n')$, given by the previous $n' = n/2$ in the logarithm, since the number of nodes doubled, the $+1$ removed for both, since now neither of $T'.left, T.right$ is the root, and a $+1$ added since a new level has been added.

2.2 Upper bound

By hypothesis,

$$E[d(x) > 2c \ln(n)] <<< 1 \quad (6)$$

By definition the ancestor of a node i are independent random variables, and we can apply the Chernoff bounds over the set $x : d(x) \geq 2 \ln(n)$ of random variables determining the expected distance of nodes.

$$P[X \geq cE[X]] < e^{-c \ln(\frac{c}{e})E[X]}$$

Let us consider $X = 1 \forall i == \ln(n)$, the expected depth of $\ln(n)$, then

$$P[X \geq c \ln(n)] < e^{-c \ln(\frac{c}{e}) \ln(n)}$$

2.3 Proof with indicator variable

Prove that the expected depth of a node is nearly $2 \ln n$.

Proof. Let z_m the m th smallest element in S and

$$X_{ij} = \begin{cases} 1 & \text{if } z_j \text{ is an ancestor of } z_i \text{ in the random search tree} \\ 0 & \text{otherwise} \end{cases}$$

The depth of the node i in the tree is given by the number of its ancestors:

$$X = \sum_{\substack{j=1 \\ j \neq i}}^n X_{ij} \quad (7)$$

Note that the depth of a node is also equal to the number of comparison it's involved in (in other words, the number of times it became the left or the right child of a randomly chosen pivot).

Once a pivot k is chosen from S , S is partitioned in two subsets L and R . The elements in the set L will not be compared with the elements in R at any subsequent time. The event $E_1 = z_j$ is an ancestor of z_i in the random search tree occurs if z_j and z_i belongs to the same partition *and* z_j was chosen as pivot before z_i . The probability that E_1 occurs, since it is the intersection of two events, can be upper bounded by:

$$\Pr\{z_j \text{ was chosen as pivot before } z_i\} = \frac{1}{\text{size of the partition}} \leq \frac{1}{|j - i| + 1}$$

because pivots are chosen randomly and independently, and because the partition that contains both z_j and z_i must contain *at least* the $|j - i| + 1$ numbers between z_j and z_i .

Taking expectations of both sides of (7), and then using linearity of expectation, we have:

$$\begin{aligned} E[X] &= \sum_{\substack{j=1 \\ j \neq i}}^n E[X_{ij}] \\ &= \sum_{\substack{j=1 \\ j \neq i}}^n \Pr\{z_j \text{ is an ancestor of } z_i \text{ in the random search tree}\} \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{|j - i|} \\ &= \sum_{j=1}^{i-1} \frac{1}{i - j} + \sum_{j=i+1}^n \frac{1}{j - i} \end{aligned}$$

With the change of variables $l = i - j$ and $m = j - i$:

$$= \sum_{l=1}^{i-1} \frac{1}{l} + \sum_{m=1}^n \frac{1}{m} \approx 2 \ln n$$

□

Prove that the expected size of its subtree is nearly $2\ln n$ too, observing that it is a simple variation of the previous analysis.

Proof. The size of the subtree of a randomly chosen pivot of $z_j \in S$ is given by the number of its descendants. Since (7) is the number of ancestors of a node z_i , we can find the number of descendants of z_j by changing the summation from $j = 1, \dots, n$ to $i = 1, \dots, n$. \square

3 Karp-Rabin fingerprinting on strings

Given a string $S : |S| = n$, and two positions $0 \leq i < j \leq (n - 1)$, the longest common extension $lceS(i, j)$ is the length of the maximal run of matching characters from those positions, namely: if $S[i] = S[j]$ then $lceS(i, j) = 0$; otherwise, $lceS(i, j) = \max l \geq 1 : S[i \dots i + l - 1] = S[j \dots j + i - 1]$. For example, if $S = \text{abracadabra}$, then $lceS(1, 2) = 0$, $lceS(0, 3) = 1$, and $lceS(0, 7) = 4$. Given S in advance for preprocessing, build a data structure for S based on the Karp-Rabin fingerprinting, in $O(n \ln(n))$ time, so that it supports subsequent

- $lceS(i, j)$: it computes the longest common extension at positions i
- $equals(i, j, c)$: it checks if $S[i \dots i + c - 1] = S[j \dots j + c - 1]$ in constant time.

Analyze the cost and the error probability. The space occupied by the data structure can be $O(n \log(n))$ but it is possible to use $O(n)$ space. [Note: in this exercise, a onetime preprocessing is performed, and then many online queries are to be answered on the fly.]

3.1 Solution 1: Cumulative shift

3.1.1 Construction

In order to save computational cycles on checks over ranges we use a similar structure to the one in the range updates: we compute the hashing on the first character in $O(1)$ time, then roll the hash through the $n - 1$ remaining characters through $nO(1)$ operations. We call H this array; we also denote h_k as the function ca^i computing the Rabin-Karp hash of a string s . The reader shall now see that $\exists h^{-1}(s)$: that is, h is invertible in $O(1)$. The entries $h[i] = \sum_{i \in [0, n-1]} (h(i))$ have cumulative hash and the following properties hold:

- $h[s[i]] = (h[i] - h[i - 1]) / a^{-1}, a^{-1} = a^1$
- $h[i..j] - h[k..l] = (h[l] - h[k - 1]) / a^{-1} - (h[j] - h[i - 1]) / a^{-1}, a^{-1} = \text{modular inverse}$

3.1.2 equals(i, j, l)

EQUALS works on cumulative hashes, subtracting them and scaling them accordingly, as our *rabin* function multiplies by an a^i constant.

```

1: function EQUALS( $i, j, length$ ):
2:    $h_i = h[i + length] - h[i - 1]$ 
3:    $h_j = h[j + length] - h[j - 1]$ 
4:    $h^i = h_i / inv(a, i, l)$ 
5:    $h^j = h_j / inv(a, j, l)$ 
   return  $h^i - h^j == 0$ 
6: function INV( $h, k, l$ ): return  $h^{k-l}$ 
```

3.1.3 lce(i, j)

LCE works on cumulative hashes, checks the equality on the middle element of the strings and runs recursively on the half with different hashing. We define LCE as an auxiliary function

```
1: function LCE( $i, j, l$ ):
2:   eq = EQUALS( $i, j$ )
3:   if eq then return  $l$ 
4:   else if  $\neg$  EQUALS( $(j-i)/2, (n-j)/2, l$ ) then return EQUALS( $(j-i)/2,$ 
   ( $n-j)/2$ )
5:   else  $(j-i)/2$  return EQUALS( $(j-i)/2, (n-j)/2$ )
6:    $h_i = h[i + length] - h[i - 1]$ 
7:    $h_j = h[j + length] - h[j - 1]$ 
8:    $h^i = h_i / inv(a, i, l)$ 
9:    $h^j = h_j / inv(a, j, l)$ 
   return  $h^i - h^j == 0$ 
10: function INV( $h, k, l$ ): return  $h^{k-l}$ 
```

4 Hashing sets

Your company has a database $S \subseteq U$ of keys. For this database, it uses a randomly chosen hash function h from a universal family H (as seen in class); it also keeps a bit vector B_S of m entries, initialized to zeroes, which are then set $B_S[h(k)] = 1 \forall k \in S$ (note that collisions may happen). Unfortunately, the database has been lost, thus only B_S and h are known, and the rest is no more accessible. Now, given $k \in U$, how can you establish if k was in S or not? What is the probability of error? (Optional: can you estimate the size $|S|$ of S looking at h and B_S and what is the probability of error?) Later, another database R has been found to be lost: it was using the same hash function h , and the bit vector B_R defined analogously as above. Using h, B_S, B_R , how can you establish if k was in $S \cap R$ (union), $S \cup R$ (intersection), or $S \setminus R$ (difference)? What is the probability of error?

4.1 $k \in S$

Trivially for $B_S[h(k)] = 0$ we can answer FALSE with $P(\text{error}) = 0$. Let us analyse the opposite case, $B_S[h(k)] = 1$. Let $i \in [0, m]$ be some index s.t. $B_S[i] = 1$, and let $cl_S(i)$ be the list of $k \in S : h(k) = i$ for some set $S \in \mathcal{P}(S)$. We can then denote the sets $cl_U := \{k_U : k_U \in U, h(k) = i\}$, $cl_S := \{k_S : k_S \in S, h(k) = i\}$; it is trivial to show that

- $cl_S \subseteq cl_U$ as $S \subseteq U$.
- $|cl_S(k)| \leq |cl_U(k)| \forall k \in U$ as $S \subseteq U$.

Let us not try and estimate $|cl_S(k)|$: given h is universal, we have an expected value of collisions of $E[|X_k|] \approx \frac{1}{m} \forall k \in S$, that is

$$\begin{aligned} P(h(k^0) = c) &= \frac{1}{m} \\ P(h(k^1) = c) &= \frac{1}{m^2} \\ P(h(k^{i-1}) = c, h(k^i) = c) &= \frac{1}{m^i} \\ P(h(k) = c, \forall k \in S) &= \frac{1}{m^{|S|}} \end{aligned}$$

We can similarly compute the probability of not collision by simply replacing $\frac{1}{m}$ with $(1 - \frac{1}{m})$: $P(h(k) \neq c, \forall k \in S) = (1 - \frac{1}{m})^{|S|}$.

Given our estimate of the collision list, we can now compute an estimate of the error probability: by 4.1, we give an erroneous answer whenever $k \in cl_U(k), k \notin cl_S(k)$, that is we have a margin of error of $|cl_U(k) \setminus cl_S(k)|$ whose size is $|cl_U(k)| - |cl_S(k)|$. Given the set of k for which $h(k) = i$ the *bad answers* are then

$$1 - P(\text{good answer}) = 1 - (1 - \frac{1}{m})^{|S|} \quad (8)$$

We now provide a lower and upper bound for the said value. The lower bound is given by the *perfect hash* with no collisions: given e = number of 1 in B_s , e is the lowerbound. Provided that $m \geq c|S|$ for some $c > 1$, $|S| \leq \frac{m}{c}$:

$$P(\text{collision over } i) = \alpha_S = \frac{\frac{m}{c}}{m} = \frac{1}{c} \quad (9)$$

Therefore

$$e \leq 1 - \frac{1}{m|S|} \leq \frac{1}{c} \quad (10)$$

4.2 Expected number of 1 in B_s

We define a random variable X_k :

$$X_k = \begin{cases} 1 & \text{if } B_s[k] = 1 \\ 0 & \text{otherwise} \end{cases}$$

and one over the ones in B_s , X :

$$X = \sum_{k=0}^{m-1} X_k \quad (11)$$

. We compute the relative expected value $E[X]$:

$$\begin{aligned} E[X] &= E\left[\sum_{k=0}^{m-1} X_k\right] \\ &= \sum_{k=0}^{m-1} P(B_s[k] = 1) \\ &= \sum_{k=0}^{m-1} \frac{\alpha}{m} \leq \frac{m}{c} \end{aligned}$$

where $P(B_s[k] = 1) = \frac{\alpha}{m}$ as α are the favorable cases (i.e. the collisions for a generic bucket $B_s[i]$), and m is the size of B_s .

5 Family of uniform hash functions

The notion of pairwise independence says that, for any $x_1 \neq x_2, c_1, c_2 \in \mathbb{Z}_p$, we have that

$$\mathbb{P}(h(x_1) = c_1, h(x_2) = c_2) = \mathbb{P}(h(x_1) = c_1) \times \mathbb{P}(h(x_2) = c_2) \quad (12)$$

In other words, the joint probability is the product of the two individual probabilities. Show that the family of hash functions $\mathcal{H} = \{h_{a,b}(x) = ((ax + b) \bmod p) \bmod m : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ is *pairwise independent* where p is a sufficiently large prime number ($m + 1 \leq p \leq 2m$).

5.1 First proof

By linear algebra, $a + (kp) \bmod p = a \forall k \in \mathbb{N}$. If we were to cap $a + pk \bmod p = a \forall k \in K, K_N = k_i : k_i < N$

$$\mathbb{P}(h(x_i) = c_i) = \frac{1}{m^2} = \mathbb{P}(h(x_1) = c_1, h(x_2) = c_2)$$

Given x_i, d , we define as $m_i = (ax_i + b)$; since $(ax_i + b)$ is a *linear transformation* m_i is unique. It follows trivially that there are $\frac{p}{m}$ values for which $m_i \bmod p = d$ since $p > m$ and by 5.1. The same goes for x_1, x_2 :

$$\begin{aligned} (ax_1 + b) \bmod p &= d \\ (ax_2 + b) \bmod p &= e \end{aligned}$$

By the Chinese remainder theorem the above system has only one solution for the variable (a, b) over the $(p)(p - 1)$ possible pairs, therefore $(ax_1 + b) \bmod p = d$ and $(ax_2 + b) \bmod p = e$ for $\approx \frac{p}{m}$ cases each. Since d, e are independent

$$((ax_2 + b) \bmod p = d) \wedge ((ax_2 + b) \bmod p = e) \text{ for } \frac{p}{m}, \frac{p}{m} = \frac{p^2}{m^2}$$

Over all the possible choices of (a, b) :

$$\mathbb{P}((ax_2 + b) \bmod p = d \wedge ((ax_2 + b) \bmod p = e) = \frac{\frac{p^2}{m^2}}{(p)(p - 1)}$$

We now prove $\mathbb{P}(h(x_1) = c_1) = \frac{1}{m}$. Of all the possible m buckets, one and only one is the one we look for: $\mathbb{P}(l \bmod m) = c_1 = \frac{1}{m}$. As we've seen by 5.1 at most $\frac{p}{m}$ such l exist for $(ax + b) \bmod p$, therefore $\mathbb{P}(h(x_i) = c_i) = \frac{\frac{p}{m}}{m} \approx \frac{1}{m}$. Which computes $\approx \frac{1}{m^2}$ and proves the assumption under the said approximation.

5.2 Second proof

Proof. We will show that both sides of (12) are approximately equal to $\frac{1}{m^2}$.

Given a hash function $h_{a,b} \in \mathcal{H}$, and two distinct inputs $x_1, x_2 \in \mathbb{N}$, let:

$$\begin{aligned} r &= (ax_1 + b) \bmod p \\ s &= (ax_2 + b) \bmod p \end{aligned}$$

Notice that $r \neq s$ because $r - s \equiv a(x_1 - x_2) \pmod{p}$, both a and $x_1 - x_2$ are nonzero modulo a prime number, and so their product is nonzero.

Since there are a total of $p(p-1)$ pairs of (r, s) such that $r \neq s$ (p^2 choices subtracted the number of pairs where $r = s$), there is a one-to-one correspondence between pairs $(a, b) \in \mathbb{Z}_p^* \times \mathbb{Z}_p$ and pairs (r, s) . Therefore, for the left-hand side of (12):

$$\mathbb{P}_{h_{a,b} \in \mathcal{H}}(h(x_1) = c_1, h(x_2) = c_2) = \mathbb{P}_{r \neq s \in \mathbb{Z}_p^2}(r \bmod m = c_1, s \bmod m = c_2)$$

There are about p/m values of r that satisfy $r \bmod m = c_1$, and the same goes for c_2 and s . Hence, there are about $(p/m)^2$ choices for the pair $(r, s) \in \mathbb{Z}_p^2$ that satisfy $r \bmod m = c_1 \wedge s \bmod m = c_2$. Dividing the favorable cases with the all possible cases:

$$\frac{\lfloor p/m \rfloor^2}{p(p-1)} \leq \mathbb{P}(r \bmod m = c_1, s \bmod m = c_2) \leq \frac{(\lfloor p/m \rfloor + 1)^2}{p(p-1)}$$

Thus, $\mathbb{P}(h(x_1) = c_1, h(x_2) = c_2) \approx \frac{1}{m^2}$.

The right-hand side of (12) is equal to $\frac{1}{m^2}$, because both probabilities are $\frac{1}{m}$. Take for example $\mathbb{P}(h(x_2) = c_2)$: there are about p/m values of s that satisfy $s \bmod m = c_2$, out of p values for s . Hence:

$$\mathbb{P}(h(x_2) = c_2) = \mathbb{P}(s \bmod m = c_2) \approx \frac{\frac{p}{m}}{p} = \frac{1}{m}$$

□

6 Deterministic data streaming

Consider a stream of n items, where items can appear more than once in the stream. The problem is to find the most frequently appearing item in the stream (where ties broken arbitrarily if more than one item satisfies the latter). Suppose that only k items can be stored, one item per memory cell, where the available storage is $k + O(1)$ memory cells. Show that the problem cannot be solved deterministically under the following rules: the algorithm can allocate only $O(\text{poly} \log(n))$ bits for each of the k items that it can store (i.e. $O(\log^c(n))$ for any fixed constant $c > 0$), and can read the next item of the stream; you, the adversary, have access to all the stream, and the content of the k items stored by the algorithm, and can decide what is the next item that the algorithm reads (please note that you cannot change past, namely, the items already read by the algorithm).

Hint: it is an adversarial argument based on the k items chosen by the hypothetical deterministic streaming algorithm, and the fact that there can be a tie on $> k$ items till the last minute.

7 Special case of most frequent item in the stream

Suppose to have a stream of n items, so that one of them occurs $> \frac{n}{2}$ times in the stream. Also, the main memory is limited to keeping just two items and their counters, plus the knowledge of the value of n beforehand. Show how to find deterministically the most frequent item in this scenario.

Hint: since the problem cannot be solved deterministically if the most frequent item occurs $\leq \frac{n}{2}$ times, the fact that the frequency is $> \frac{n}{2}$ should be exploited.

7.1 Solution 1: Up & Down counter

We'll use the notation of the previous section. Before illustrating our solution we prove that the following hold:

- **Given the j^1 element, $C(j^1) > C(j^i) \forall i \in S$.**
- Following the previous, $C(j^i) < C(j^1), i > 1$ element.
- **There are at most $\frac{n}{2} - 1$ elements in S :** given that j^1 appears at least $\frac{n}{2} + 1$ times, in the best case scenario every other $\frac{n}{2} - 1$ element is different, giving $\frac{n}{2} - 1$ different elements.
- **j^1 has sub-stream dominance:** $\exists S' \subseteq S : j^1$ is the most frequent item $\in S'$: trivially, since it is the most frequent element, S' is always a sub-stream dominant sub-stream.

Recursive sub-streams It is trivial to show that if we were to have $\frac{n}{2}$ counters, $\sum_{i=0, i \neq 1}^{\frac{n}{2}} (C(j^i) < C(j^1))$ by 7.1. Though trivial, by combining it with 7.1 we can assert that given $S' = S[0, \frac{n}{2}]$, $S'' = S[\frac{n}{2}, n]$, $k = \text{number of elements} \in S$

- j^1 is the sub-stream dominant element in either S', S'' : by contradiction, let us assume that is false. Then $\exists j_o \neq j^1 : C(j_o) \text{ in } S' > C(j^1, \exists j_o : C(j_o) \text{ in } S'' > C(j^1)$; given that $S'S'' = S$, that would make $C(j_o) = C(j^1)$: contradiction.
- More generally, given two sub-streams S', S'' , the sub-stream dominant element of $S'S''$ is one of the sub-stream dominants in S' or S'' and its frequencies are defined by

$$e - k - 1 \text{ in } S', \frac{n}{2} + 1 - (e - k - 1) \text{ in } S'' \quad (13)$$

where e is an arbitrary number of appearances of $j^2 : e < \frac{n}{2} + 1$ The reader will see as the $+1$ in the right side guarantees the sub-stream dominant to be maximum. The sides can be swapped without hurting generality.

In order to better illustrate, we provide an algorithm that exploits the sub-stream dominance. Our idea is to *go up* with our counter when we are fed an object we've already met, and to *go down* when we are fed an object different

from us. Once we reach the bottom (0), we know by 7.1 that we either reached zero for an object $\neq j^1$, and we can ignore it, or we reached zero for j^1 , that being the most frequent element will have sub-stream dominance in the remaining sequence, which means it will *go upper* than any other element in that sub-stream. Given that any other element has lower frequency, the most frequent of them, j^2 will *go down* $e < \frac{n}{2} + 1$ times. Note that C holds only one element, for simplicity we'll denote its entries as a map in a Scala-like syntax.

```

1: function UP_AND_DOWN( $S, C$ ):
2:    $k \leftarrow S.next()$ 
3:   if  $\exists C(k)$  then
4:      $C \leftarrow \{k \Rightarrow C(k) + 1\}$        $\triangleright$  Increment current dominant element.
5:   else if  $C(k) == 0$  then
6:      $C \leftarrow \{k \Rightarrow 1\}$      $\triangleright$   $k$  is no more dominant, swap it with new element
    and initialize.
7:   else
8:      $C \leftarrow \{k \Rightarrow C(k) - 1\}$      $\triangleright$   $k$  is still dominant, but the stream.
9:   return  $C$ 

```

8 Count-min sketch: extension to negative counters

Check the analysis seen in class, and discuss how to allow $F[i]$ to change by arbitrary values read in the stream. Namely, the stream is a sequence of pairs of elements, where the first element indicates the item i whose counter is to be changed, and the second element is the amount v of that change (v can vary in each pair). In this way, the operation on the counter becomes $F[i] = F[i] + v$, where the increment and decrement can be now seen as $(i, 1)$ and $(i, -1)$.

8.1 Proof

We trivially have some changes over X_{ji} and $\tilde{F}[i]$:

$$\begin{aligned} X_{ji} &= \sum^n (I_k F[k]) \\ \tilde{F}[i] &= F[i] + X_{ji} \end{aligned}$$

X_{ji} can vary as complementary increments $(+v_i, +v_j, +v_k, -v_k, -v_j, -v_i)$ can make it ≤ 0 without necessarily being updates on i . In order to have a minimum error estimate we'll pick the *median* value of X_{ji} .

$\tilde{F}[i] = F[i] + X_{ji}$ by the above assertion the counter $\tilde{F}[i]$ could have no garbage, or negative garbage according to the other increments. Therefore by choosing the minimum $\tilde{F}[i]$ over the $\log(\frac{1}{\delta})$ we could actually pick the most perturbed result (think of a collision with the biggest negative increment over one row).

The last step is the probability proof: $P(\tilde{F}[i] > F[i] + \epsilon \|F\|)$. Since by 8.1 X_{ji} can be either positive or negative we pick the *median* value med over the minimum m and maximum M . We are then able to split our error analysis in two of equivalent size $\frac{r}{2}$:

$$\begin{aligned} P(\tilde{F}[i] \in F[i] + \epsilon \|F\|) &= \\ P(\tilde{F}[i] \geq F[i] - |\epsilon| \|F\| \wedge \tilde{F}[i] \leq F[i] + |\epsilon| \|F\|) &= \\ P(\tilde{F}[i] \geq F[i] - |\epsilon| \|F\|) P(\tilde{F}[i] \leq F[i] + |\epsilon| \|F\|) \end{aligned}$$

Let us define $p_0 = P(\tilde{F}[i] \geq F[i] - |\epsilon| \|F\|)$ and $p_1 = P(\tilde{F}[i] \leq F[i] + |\epsilon| \|F\|)$. We can report the analysis seen in class adjusted for the number of rows $\frac{r}{2}$

$$\begin{aligned} p_0 &= \prod^{\frac{r}{2}} (P(|X_{ji}| \geq |\epsilon| \|F\|)) = \\ &= -\frac{1}{2^{\frac{r}{2}}} = -\delta^{\frac{1}{2}} \end{aligned}$$

Now for p_1 :

$$\begin{aligned} p_1 &= \mathbf{P}(\tilde{F}[i] \leq F[i] + |\epsilon|\|F\|) = \\ 1 - \mathbf{P}(\tilde{F}[i] \geq F[i] + |\epsilon|\|F\|) &= 1 - \frac{1}{2^{\frac{r}{2}}} = 1 - \delta^{\frac{1}{2}} \end{aligned}$$

Giving us a combined probability of $p = p_0 p_1 = (-\delta^{\frac{1}{2}})(1 - \delta^{\frac{1}{2}}) = -\sqrt{\delta} + \delta$.

9 Count-min sketch: range queries

Show and analyze the application of count-min sketch to range queries (i, j) for computing $\sum_{j=k=i}^k F[k]$.

Hint: reduce the latter query to the estimate of just $t \leq 2 \log(n)$ counters c_1, c_2, \dots, c_t . Note that the probability at most δ of error (i.e. that $tl = 1cl > jk = iF[k] + 2 \log(n) \|F\|$), it does not suffices to say that it is at most δ the probability of error of each counter c_l : while each counter is still the actual wanted value plus the residual as before, it is better to consider the sum V of these t wanted values and the sum X of these residuals, and apply Markov's inequality to V and X rather than on the individual counters.

A Hogwarts

The Hogwarts School¹ is modeled as a graph $G = (V, E)$ where V is the set of castle's rooms and $E \subseteq V \times V$ is the set of the stairs. Each stair is labelled with the time of appearance and disappearance, and can be walked in both directions, therefore the graph is undirected. The goal is to find, if possible, the minimum amount of time required to go from the first to the last room.

A.1 Solution 1: Preprocessing-then-Dijkstra

Dijkstra is able to find the shortest path in a graph with non-negative weights on its edges. Our main idea is to create a Dijkstra compatible graph through a NORMALIZE function, then apply Dijkstra to it in order to find the shortest path. The core of the preprocessing is the NORMALIZE function which computes traversal times between nodes at a given time *time*:

```
1: function NORMALIZE(from, to, time):
2:    $t \leftarrow \infty$ 
3:   if  $start[v'] \leq t < end[v']$  then                                ▷ No waiting time
4:      $t \leftarrow t + 1$ 
5:   else if  $t < start[v']$  then                                       ▷ Waiting time
6:      $t \leftarrow start[v'] + 1$ 
7:   else
8:      $t \leftarrow \infty$                                               ▷ Available time already expired
9:   return  $t$ 
```

The normalize function is then applied to a node traversal:

A.1.1 Pseudo-code

```
1: create vertex set  $Q$  of unvisited nodes
2: create vertexes set  $E'$  of edges weight
3:  $time \leftarrow 0$                                                     ▷ Initial time for traversal
4:  $edges \leftarrow STAIRS\_OF(0)$  ▷ Get incoming/outgoing edges of the source node
5: function PROCESS(node, time)
6:   if  $edge \in visited\_edges$  then
7:     return
8:    $traversal\_time \leftarrow \infty$ 
9:   for all  $neighbor \in neighbors\_of\_node$  do
10:     $traversal\_time \leftarrow TRAVERSAL\_TIME(node, neighbor, time)$ 
11:     $E'[0][node] \leftarrow traversal\_time$  ▷  $E'[i][j]$  holds the weight/traversal
12:                                         ▷ time for the stair between  $i$  and  $j$ 
13:    for all  $new\_neighbor \in neighbors\_of\_neighbor$  do
14:      NORMALIZE( $neighbor$ ,  $new\_neighbor$ ,  $traversal\_time$ )
15:   if  $DIJKSTRA(V, E') = \infty$  then
```

¹http://didawiki.cli.di.unipi.it/lib/exe/fetch.php/magistraleinformatica/alg2/algo2_16/hogwarts.pdf

```

16:     return -1
17: else
18:     return t

```

Computational cost: $\Theta(n^2)$ if the vertex set in DIJKSTRA is implemented as an array. $O(|E| + |V| \log |V|)$ with Fibonacci heap.

A.2 Solution 2: HogwartsDijkstra

```

1: function HOGWARTSDIJKSTRA( $G$ ):
2:   create vertex set  $Q$  of unvisited nodes
3:   for all vertex  $v \in V$  do                                      $\triangleright$  initialization
4:      $time[v] \leftarrow \infty$                                     $\triangleright$  unknown time from source to  $v$ 
5:     add  $v$  to  $Q$                                                 $\triangleright$  all nodes initially in  $Q$ 
6:    $time[0] \leftarrow 0$                                           $\triangleright$  time from source to source
7:   while  $Q \neq \emptyset$  do
8:      $u \leftarrow x \in Q$  with  $\min\{time[x]\}$ 
9:     remove  $u$  from  $Q$ 
10:    for all neighbor  $v$  of  $u$  do:
11:      if  $time[u] \leq appear[v]$  then
12:         $alt \leftarrow appear[v] + 1$                               $\triangleright$  wait the appearance of the stair
13:      else if  $time[u] < disappear[v]$  then
14:         $alt \leftarrow time[u] + 1$                                 $\triangleright$  use the stair
15:      else
16:         $alt \leftarrow \infty$                                       $\triangleright$  the stair has already disappeared
17:      if  $alt < time[v]$  then
18:         $time[v] \leftarrow alt$                                     $\triangleright$  a quicker path to  $v$  has been found
19:   return  $time[|N| - 1]$ 

```

Computational cost. See the previous section.

A.3 Solution 3: BFS-like traversal

```

1: function REACH( $N, M, A[], B[], appear[], disappear[]$ )
2:   for  $i = 0$  to  $M - 1$  do
3:      $edges\_A[i].push\_back(make\_pair(i, B[i]))$ 
4:      $edges\_B[i].push\_back(make\_pair(i, A[i]))$ 
5:   for  $i = 0$  to  $N - 1$  do
6:      $done\_ [i] \leftarrow false$ 
7:      $distance\_ [i] \leftarrow \infty$ 
8:    $reached\_ [0].push\_back(0)$ 
9:    $distance\_ [0] \leftarrow 0$ 

```

```

10:   for  $t = 0$  to  $MAX\_TIME$  do
11:     for all  $v \in reached\_ [t]$  do
12:       if not  $done\_ [v]$  then
13:         for all  $edge \in edges\_ [v]$  do
14:            $staircase \leftarrow edge.first$ 
15:            $neighbor \leftarrow edge.second$ 
16:            $time \leftarrow \max(distance\_ [v], appear[staircase]) + 1$ 
17:           if not  $done\_ [neighbor]$ 
18:             and  $distance\_ [v] < disappear[staircase]$ 
19:             and  $time < distance\_ [neighbor]$  then
20:                $distance\_ [neighbor] \leftarrow time$ 
21:                $reached\_ [time].push\_back(neighbor)$ 
22:              $done\_ [v] \leftarrow true$ 
23:   return  $(distance\_ [N - 1] = \infty) ? -1 : distance\_ [N - 1]$ 

```

<p>Computational cost: $O(m + MAX_TIME)$.</p>
--

B Paletta

Paletta ordering² is a peculiar ordering technique: given a 3-tuple of elements, paletta takes the central element as pivot and swaps the two elements right before and next to it. To make an example:

$$(3, 2, 1) \xrightarrow{\text{paletta}} (1, 2, 3)$$

We now want to develop an algorithm to order any array through paletta ordering with the minimal number of swaps. You should see as not every array can be ordered (e.g. $[1, 3, 2]$).

B.1 Solution 1: Split and count-inversions

We should note that the following properties hold:

1. Every element can be a pivot, but the first and the last one, as they have respectively no elements before and after them.
2. Every element can be swapped as many times as necessary, but only with elements of the same 2-remainder (numbers in even positions can only be swapped with numbers in even positions, the same holds for odd indexes). More formally, if n is the size of the array A we want to sort, $i, j \in [1, n-2]$, $A[i]$ can be swapped with $A[j]$ if and only if $i \equiv j \pmod{2}$.
3. The least number of swaps does not backtrack any element. Formally, let k be the minimal number of swaps applied to an array, backtracks included. By hypothesis, k is minimal, but at least m , $m > 0$ backtrack swaps have been operated, therefore we found a $k' = k - m : k' < k$, a new minimal number of swaps: contradiction.

Given item 2, we can split our array in two, even and odd numbers, and order them counting the swaps. In our example we'll use *mergesort*, as it runs in $O(n \log n)$, does backtrack elements, and is very well-known. Clearly, given an array, a swap happens when an element is pushed back, pulling the one between its new position and the old one ahead: we can map this behaviour in the merge routine of mergesort: the array merged is able to push back elements from its right pointer to the new array, moving them back of $(m - i) + (j - m)$ positions, where m is the dimension of the current two sub-arrays to merge. Provided that our edited version of mergesort ran successfully on both the even-index and odd-index, we now need to verify if by merging them we obtain an ordered array. Intuitively, the merged array will start with the first element of the even-index arrays, followed by the first of the odd-index array, followed by the second of the even-index array, and so on. To check for these elements is pretty trivial and can be done in linear time. Follows the pseudo-code for the edited version and SNAKE_CHECK function:

²http://didawiki.cli.di.unipi.it/lib/exe/fetch.php/magistraleinformatica/alg2/algo2_16/paletta.pdf

```

1: function MERGE_WITH_PALETTE(left, right, k):
2:   ...                                     ▷ merge instructions
3:   if right > left then
4:     palette_count ← palette_count + 1
5:     ...
1: function SNAKE_CHECK
2:   even, odd ← 0
3:   for ;even, odd < N; even = even + 1, odd = odd + 1 do
4:     if a[even] > a[odd] then
5:       return -1
6:   return palette_count

```

Computational cost: $\Theta(n \log n)$.