

Advanced Algorithms Problems and Solutions

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Contents

1	Range updates	3
1.1	Solution 1: Segment tree lazy a-b sums	3
2	Depth of a node in a random search tree	6
2.1	Recursive balanced proof	6
2.2	Upper bound	7
2.3	Proof with indicator variable	7
3	Karp-Rabin fingerprinting on strings	10
3.1	Solution 1: Cumulative shift	10
3.1.1	Construction	10
3.1.2	<code>equals(i, j, l)</code>	10
3.1.3	<code>lce(i, j)</code>	11
4	Hashing sets	12
4.1	$k \in S$	12
4.2	Expected number of 1 in B_s	13
5	Family of uniform hash functions	14
5.1	First proof	14
5.2	Second proof	14
6	Deterministic data streaming	16
7	Special case of most frequent item in the stream	17
7.1	Solution 1: Up & Down counter	17
8	Count-min sketch: extension to negative counters	19
8.1	First solution	19
8.2	Second solution	20
9	Count-min sketch: range queries	22
10	Count-min sketch: extension to negative counters	24
10.1	Wide bitvector	24
10.1.1	First level pointers	24
10.1.2	Hash functions optimization	25

1 Range updates

Consider an array C of n integers, initially all equal to zero. We want to support the following operations:

- **update(i, j, c):** where $0 \leq i \leq j \leq n - 1$ and c is an integer: it changes C such that $C[k] = C[k] + c$ for every $i \leq k \leq j$.
- **query(i)** where $0 \leq i \leq n - 1$: it returns the value of $C[i]$.
- **sum(i,j)** where $0 \leq i \leq j \leq n - 1$: it returns $\sum_{k=i}^j C[k]$.

Design a data structure that uses $O(n)$ space and implements each operation above in $O(\log(n))$ time. Note that $query(i) = sum(i, i)$ but it helps to reason. [Hint to further save space: use an implicit tree such as the Fenwick tree (see wikipedia).]

1.1 Solution 1: Segment tree lazy a-b sums

Let T be a segmented binary tree over a continuous interval $I : [0, N - 1]$ s.t. its leafs are the points in I , and the parent of two nodes comprises of their interval:

$$n' \cup n'' = n, n' \cap n'' = \emptyset \text{ s.t. } n \text{ is the parent of } n', n''$$

T will keep track of the prefix sums for every interval. We define a function

$$s' : [0, n - 1] \rightarrow \mathbb{N} \quad (1)$$

that given a node in T returns the value associated with I , namely the cumulative sum of that interval.

In order to reduce the computational cost, we introduce a lazy algorithm that doesn't propagate sums over T as they are streamed in the input, which means $s'(i)$ might not be accurate at a given time t for any of the requested operation.

We'll instead either compute over T or update T as necessary. Let us define a function to do so:

$$l : \mathbb{N} \rightarrow (\mathbb{N} \cup \{\epsilon\}, \mathbb{N}) \quad (2)$$

to keep track of our lazy sums:

$$s(n) = \begin{cases} \epsilon, - & \text{if no lazy prefix sum is in that interval} \\ k, m & \text{if a lazy sum of } k \text{ is to be propagated to } m \end{cases} \quad (3)$$

The QUERY function is then trivial:

- 1: **function** QUERY(I, i, sum):
- 2: **if** $I.size = 1$ **then** ▷ Return found value
- 3: **return** $I.sum$
- 4: **if** lazy(I), $i \in I.left, i \notin I.right$ **then** ▷ Lazy on left child

```

5:      lazy(I) ← False
6:      QUERY(I.left, i, sum + I.sum)
7:      if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right child
8:          lazy(I) ← False
9:          QUERY(I.right, i, sum + I.sum)
10:     if lazy(I), i ∈ I.right, i ∈ I.left then          ▷ Lazy on both
11:         lazy(I) ← False
12:         QUERY(I.right, i, j, sum + I.sum) + QUERY(I.left, i, j, sum +
I.sum)
13:     if !lazy(I), i ∈ I.left then                      ▷ Not lazy on left child
14:         QUERY(I.left, i, sum)
15:     if !lazy(I), i ∈ I.right then                    ▷ Not lazy on right child
16:         QUERY(I.right, i, sum)
17:     if !lazy(I), i ∈ I.right, i ∈ I.left then        ▷ Not lazy both
18:         SUM(I.right, i, sum)

1: function SUM(I, i, j, sum):
2:     if I.size = 1 then                                ▷ Return
3:         return I.sum + sum
4:     if lazy(I), i ∈ I.left, i ∉ I.right then          ▷ Lazy on left
5:         lazy(I) ← False
6:         SUM(I.left, i, j, sum + I.sum)
7:     if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right
8:         lazy(I) ← False
9:         SUM(I.right, i, j, sum + I.sum)
10:    if lazy(I), i ∈ I.right, i ∈ I.left then           ▷ Lazy on both
11:        lazy(I) ← False
12:        SUM(I.right, i, j, sum + I.sum) + SUM(I.left, i, j, sum + I.sum)
13:    if !lazy(I), i ∈ I.left then                        ▷ Not lazy on both
14:        SUM(I.left, i, sum)
15:    if !lazy(I), i ∈ I.right then                      ▷ Not lazy on both
16:        SUM(I.right, i, sum)
17:    if !lazy(I), i ∈ I.right, i ∈ I.left then          ▷ Not lazy on both
18:        SUM(I.right, i, sum)

1: function UPDATE(I, i, j, k):
2:     if I.size = 1 then                                ▷ Return
3:         return I.val ← I.val + update
4:         return I.val+ = update
5:     if lazy(I), i ∈ I.left, i ∉ I.right then          ▷ Lazy on left
6:         lazy(I.left) ← True
7:         I.left.val ← k
8:     if lazy(I), i ∈ I.right, i ∉ I.left then          ▷ Lazy on right
9:         lazy(I.right) ← True

```

```

10:      I.right.val  $\leftarrow$  k
11:  if lazy(I), i  $\in$  I.right, i  $\in$  I.left then                                 $\triangleright$  Lazy on both
12:      lazy(I)  $\leftarrow$  True
13:      I.val  $\leftarrow$  k
14:  if  $\neg$ lazy(I), i  $\in$  I.left then                                             $\triangleright$  Not lazy
15:      UPDATE(I.left, i, update)
16:      update(I.left, i, update)
17:  if  $\neg$ lazy(I), i  $\in$  I.right then                                           $\triangleright$  Not lazy
18:      UPDATE(I.right, i, update)
19:      update(I.right, i, update)
20:  if  $\neg$ lazy(I), i  $\in$  I.right, i  $\in$  I.left then                             $\triangleright$  Not lazy
21:      UPDATE(I.right, i, update)
22:      update(I.right, i, update)

```

2 Depth of a node in a random search tree

A random search tree for a set S can be defined as follows: if S is empty, then the null tree is a random search tree; otherwise, choose uniformly at random a key k as root, and the random search trees on $L = \{x \in S : x < k\}$ and $R = \{x \in S : x > k\}$ become, respectively, the left and right subtree of the root k . Consider the randomized QuickSort discussed in class and analyzed with indicator variables CLRS 7.3, and observe that the random selection of the pivots follows the above process, with indicator variables, prove that:

1. the expected depth of a node (i.e. the random variable representing the distance of the node from the root) is nearly $2 \ln n$;
2. the expected size of its subtree is nearly $2 \ln n$ too, observing that it is a simple variation of the previous analysis;
3. the probability that the expected depth of a node exceeds $2 \ln n$ is small for any given constant $c > 1$.

Chernoff's bounds.

2.1 Recursive balanced proof

Let n be the number of nodes in the input list l , $h = \log_2(n)$ the height of a balanced tree over l , $T(p)$ the tree built over the permutation p of pivots, $d(m)$ be the positional distance of a value m of a partition from the median value of the said partition. Then the following holds:

- $height(T) = h \iff |T.left| = |T.right| \pm 1$ Trivially, let r be the root of a 3-nodes partition: then, if the partition is unbalanced, the lesser one will comprise of 0 nodes, while the greater one of 2, which implies that $height(T.right) = 2$.
- $P = \text{pivot}, d(m) = \pm k \implies height(T.left) = height(T.right) \pm k$. Recursively from the previous statement, a partition unbalanced of one element generates subtrees whose levels differ on a factor of 1. By iterating recursively, their subtrees, if unbalanced by 1, will yield one more level difference. Over k unbalanced pivots on a single subtree, at most k levels will be added to h .
- By the previous statement, it follows that $\nexists T, T' : height(T) > height(T')$, T balanced, T' unbalanced. As stated, let T', T be the unbalanced/balanced tree respectively; let us cheat with T and switch the root pivot with the first element in its subtree. Now, let us prove by contradiction that T can't stay balanced and that its height will increase. By shifting the tree to the left we have deprived $T.right$ of either 0 levels (in case $T.right$ is able to switch every pivot in its tree with its right subtree root, ending with the rightmost leaf in its subtree) or 1, in case no rightmost leaf is present. Therefore $height(T) < height(T')$.

- The completely unbalanced tree is the tree with the most levels. By taking partitions of size 0 we constantly force, at each level, one subtree to disappear. Therefore, its level(s) has to be necessarily transferred to its brother. We then have exactly one node per level, therefore n levels.

Behaviour on random permutations Now let us analyse how the tree depth varies according to random pivot selection. We start by applying the 2.1k-distance to a tree T with $n = 3$ nodes. Trivially, $height(T)$ with balanced tree is equal to two. Now, let us pick either the lowest or the greatest pivot possible: the tree is unbalanced towards either the left or the right, but $height(T) = 2$ in both cases. As the reader can see from 2.1, the distance works in absolute value; it is then clear how, at every permutation for a pivot p , out of the n , there are 2 that generate a tree of the same height: $p = d(P) + k, p = d(P) - k$. Given that at every iteration a node x in a completely unbalanced tree T' has a probability of $\frac{1}{n-i}$, we can define the probability of x being a pivot at level l as:

$$P(x_k) = \frac{1}{n-l} \quad (4)$$

Now, in order for x not to be chosen as pivot in the previous $l-1$ levels we have:

$$P(x_k) = \prod_{k=1}^{l-1} \left(\frac{1}{n-l+1} \right) \quad (5)$$

Given the height of T , the (harmonic) partial series converges to $\ln(n) + 1$. Let us now add a root r s.t. $T'.right = T, T'.left = T$. We now have to consider the mirror case $\ln(n') + \ln(n')$, given by the previous $n' = n/2$ in the logarithm, since the number of nodes doubled, the $+1$ removed for both, since now neither of $T'.left, T'.right$ is the root, and a $+1$ added since a new level has been added.

2.2 Upper bound

By hypothesis,

$$\mathbb{E}[d(x) > 2c \ln(n)] \ll 1 \quad (6)$$

By definition the ancestor of a node i are independent random variables, and we can apply the Chernoff bounds over the set $x : d(x) \geq 2 \ln(n)$ of random variables determining the expected distance of nodes.

$$\mathbb{P}[X \geq c\mathbb{E}[X]] < e^{-c \ln(\frac{c}{e})\mathbb{E}[X]}$$

Let us consider $X = 1 \forall i == \ln(n)$, the expected depth of $\ln(n)$, then

$$\mathbb{P}[X \geq c \ln(n)] < e^{-c \ln(\frac{c}{e}) \ln(n)}$$

2.3 Proof with indicator variable

Prove that the expected depth of a node is nearly $2 \ln n$.

Proof. Let z_m the m th smallest element in S and

$$X_{ij} = \begin{cases} 1 & \text{if } z_j \text{ is an ancestor of } z_i \text{ in the random search tree} \\ 0 & \text{otherwise} \end{cases}$$

The depth of the node i in the tree is given by the number of its ancestors:

$$X = \sum_{\substack{j=1 \\ j \neq i}}^n X_{ij} \quad (7)$$

Note that the depth of a node is also equal to the number of comparison it's involved in (in other words, the number of times it became the left or the right child of a randomly chosen pivot).

Once a pivot k is chosen from S , S is partitioned in two subsets L and R . The elements in the set L will not be compared with the elements in R at any subsequent time. The event $E_1 = z_j$ is an ancestor of z_i in the random search tree occurs if z_j and z_i belongs to the same partition *and* z_j was chosen as pivot before z_i . The probability that E_1 occurs, since it is the intersection of two events, can be upper bounded by:

$$\Pr\{z_j \text{ was chosen as pivot before } z_i\} = \frac{1}{\text{size of the partition}} \leq \frac{1}{|j - i| + 1}$$

because pivots are chosen randomly and independently, and because the partition that contains both z_j and z_i must contain *at least* the $|j - i| + 1$ numbers between z_j and z_i .

Taking expectations of both sides of (7), and then using linearity of expectation, we have:

$$\begin{aligned} \mathbb{E}[X] &= \sum_{\substack{j=1 \\ j \neq i}}^n \mathbb{E}[X_{ij}] \\ &= \sum_{\substack{j=1 \\ j \neq i}}^n \Pr\{z_j \text{ is an ancestor of } z_i \text{ in the random search tree}\} \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{|j - i|} \\ &= \sum_{j=1}^{i-1} \frac{1}{i - j} + \sum_{j=i+1}^n \frac{1}{j - i} \end{aligned}$$

With the change of variables $l = i - j$ and $m = j - i$:

$$= \sum_{l=1}^{i-1} \frac{1}{l} + \sum_{m=1}^n \frac{1}{m} \approx 2 \ln n$$

□

Prove that the expected size of its subtree is nearly $2\ln n$ too, observing that it is a simple variation of the previous analysis.

Proof. The size of the subtree of a randomly chosen pivot of $z_j \in S$ is given by the number of its descendants. Since (7) is the number of ancestors of a node z_i , we can find the number of descendants of z_j by changing the summation from $j = 1, \dots, n$ to $i = 1, \dots, n$. \square

3 Karp-Rabin fingerprinting on strings

Given a string $S : |S| = n$, and two positions $0 \leq i < j \leq (n - 1)$, the longest common extension $lceS(i, j)$ is the length of the maximal run of matching characters from those positions, namely: if $S[i] = S[j]$ then $lceS(i, j) = 0$; otherwise, $lceS(i, j) = \max l \geq 1 : S[i \dots i + l - 1] = S[j \dots j + i - 1]$. For example, if $S = \text{abracadabra}$, then $lceS(1, 2) = 0$, $lceS(0, 3) = 1$, and $lceS(0, 7) = 4$. Given S in advance for preprocessing, build a data structure for S based on the Karp-Rabin fingerprinting, in $O(n \ln(n))$ time, so that it supports subsequent

- $lceS(i, j)$: it computes the longest common extension at positions i
- $equals(i, j, c)$: it checks if $S[i \dots i + c - 1] = S[j \dots j + c - 1]$ in constant time.

Analyze the cost and the error probability. The space occupied by the data structure can be $O(n \log(n))$ but it is possible to use $O(n)$ space. [Note: in this exercise, a onetime preprocessing is performed, and then many online queries are to be answered on the fly.]

3.1 Solution 1: Cumulative shift

3.1.1 Construction

In order to save computational cycles on checks over ranges we use a similar structure to the one in the range updates: we compute the hashing on the first character in $O(1)$ time, then roll the hash through the $n - 1$ remaining characters through $nO(1)$ operations. We call H this array; we also denote h_k as the function ca^i computing the Rabin-Karp hash of a string s . The reader shall now see that $\exists h^{-1}(s)$: that is, h is invertible in $O(1)$. The entries $h[i] = \sum_{i \in [0, n-1]} (h(i))$ have cumulative hash and the following properties hold:

- $h[s[i]] = (h[i] - h[i - 1]) / a^{-1}, a^{-1} = a^1$
- $h[i \dots j] - h[k \dots l] = (h[l] - h[k - 1]) / a^{-1} - (h[j] - h[i - 1]) / a^{-1}, a^{-1} =$
modular inverse

3.1.2 equals(i, j, l)

EQUALS works on cumulative hashes, subtracting them and scaling them accordingly, as our *rabin* function multiplies by an a^i constant.

```

1: function EQUALS( $i, j, length$ ):
2:    $h_i = h[i + length] - h[i - 1]$ 
3:    $h_j = h[j + length] - h[j - 1]$ 
4:    $h^i = h_i / inv(a, i, l)$ 
5:    $h^j = h_j / inv(a, j, l)$ 
   return  $h^i - h^j == 0$ 
6: function INV( $h, k, l$ ): return  $h^{k-l}$ 
```

3.1.3 lce(i, j)

LCE works on cumulative hashes, checks the equality on the middle element of the strings and runs recursively on the half with different hashing. We define LCE as an auxiliary function

```

1: function LCE( $i, j, l$ ):
2:   eq = EQUALS( $i, j$ )
3:   if eq then return  $l$ 
4:   else if  $\neg$  EQUALS( $(j-i)/2, (n-j)/2, l$ ) then return EQUALS( $(j-i)/2,$ 
   ( $n-j)/2$ )
5:   else  $(j-i)/2$  return EQUALS( $(j-i)/2, (n-j)/2$ )
6:    $h_i = h[i + length] - h[i - 1]$ 
7:    $h_j = h[j + length] - h[j - 1]$ 
8:    $h^i = h_i / inv(a, i, l)$ 
9:    $h^j = h_j / inv(a, j, l)$ 
   return  $h^i - h^j == 0$ 
10: function INV( $h, k, l$ ): return  $h^{k-l}$ 

```

4 Hashing sets

Your company has a database $S \subseteq U$ of keys. For this database, it uses a randomly chosen hash function h from a universal family H (as seen in class); it also keeps a bit vector B_S of m entries, initialized to zeroes, which are then set $B_S[h(k)] = 1 \forall k \in S$ (note that collisions may happen). Unfortunately, the database has been lost, thus only B_S and h are known, and the rest is no more accessible. Now, given $k \in U$, how can you establish if k was in S or not? What is the probability of error? (Optional: can you estimate the size $|S|$ of S looking at h and B_S and what is the probability of error?) Later, another database R has been found to be lost: it was using the same hash function h , and the bit vector B_R defined analogously as above. Using h, B_S, B_R , how can you establish if k was in $S \cap R$ (union), $S \cup R$ (intersection), or $S \setminus R$ (difference)? What is the probability of error?

4.1 $k \in S$

Trivially for $B_S[h(k)] = 0$ we can answer FALSE with $\mathbb{P}[\text{error}] = 0$. Let us analyse the opposite case, $B_S[h(k)] = 1$. Let $i \in [0, m]$ be some index s.t. $B_S[i] = 1$, and let $cl_S(i)$ be the list of $k \in S : h(k) = i$ for some set $S \in \mathcal{P}(S)$. We can then denote the sets $cl_U := \{k_U : k_U \in U, h(k) = i\}$, $cl_S := \{k_S : k_S \in S, h(k) = i\}$; it is trivial to show that

- $cl_S \subseteq cl_U$ as $S \subseteq U$.
- $|cl_S(k)| \leq |cl_U(k)| \forall k \in U$ as $S \subseteq U$.

Let us not try and estimate $|cl_S(k)|$: given h is universal, we have an expected value of collisions of $\mathbb{E}[X_k] \approx \frac{1}{m} \forall k \in S$, that is

$$\begin{aligned}\mathbb{P}[h(k^0) = c] &= \frac{1}{m} \\ \mathbb{P}[h(k^1) = c] &= \frac{1}{m^2} \\ \mathbb{P}\left[h(k^{i-1}) = c \mid h(k^i) = c\right] &= \frac{1}{m^i} \\ \mathbb{P}[h(k) = c, \forall k \in S] &= \frac{1}{m^{|S|}}\end{aligned}$$

We can similarly compute the probability of not collision by simply replacing $\frac{1}{m}$ with $(1 - \frac{1}{m})$: $\mathbb{P}[h(k) \neq c, \forall k \in S] = (1 - \frac{1}{m})^{|S|}$.

Given our estimate of the collision list, we can now compute an estimate of the error probability: by 4.1, we give an erroneous answer whenever $k \in cl_U(k)$, $k \notin cl_S(k)$, that is we have a margin of error of $cl_U(k) \setminus cl_S(k)$ whose size is $|cl_U(k)| - |cl_S(k)|$. Given the set of k for which $h(k) = i$ the *bad answers* are then

$$1 - \mathbb{P}[\text{good answer}] = 1 - \left(1 - \frac{1}{m}\right)^{|S|} \quad (8)$$

We now provide a lower and upper bound for the said value. The lower bound is given by the *perfect hash* with no collisions: given e = number of 1 in B_s , e is the lowerbound. Provided that $m \geq c|S|$ for some $c > 1$, $|S| \leq \frac{m}{c}$:

$$\mathbb{P}[\text{collision over } i] = \alpha_S = \frac{\frac{m}{c}}{m} = \frac{1}{c} \quad (9)$$

Therefore

$$e \leq 1 - \frac{1}{m|S|} \leq \frac{1}{c} \quad (10)$$

4.2 Expected number of 1 in B_s

We define a random variable X_k :

$$X_k = \begin{cases} 1 & \text{if } B_s[k] = 1 \\ 0 & \text{otherwise} \end{cases}$$

and one over the ones in B_s , X :

$$X = \sum_{k=0}^{m-1} X_k \quad (11)$$

. We compute the relative expected value $\mathbb{E}[X]$:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{k=0}^{m-1} X_k\right] \\ &= \sum_{k=0}^{m-1} \mathbb{P}[B_s[k] = 1] \\ &= \sum_{k=0}^{m-1} \frac{\alpha}{m} \leq \frac{m}{c} \end{aligned}$$

where $\mathbb{P}[B_s[k] = 1] = \frac{\alpha}{m}$ as α are the favorable cases (i.e. the collisions for a generic bucket $B_s[i]$), and m is the size of B_s .

5 Family of uniform hash functions

The notion of pairwise independence says that, for any $x_1 \neq x_2, c_1, c_2 \in \mathbb{Z}_p$, we have that

$$\mathbb{P}[h(x_1) = c_1, h(x_2) = c_2] = \mathbb{P}[h(x_1) = c_1] \times \mathbb{P}[h(x_2) = c_2] \quad (12)$$

In other words, the joint probability is the product of the two individual probabilities. Show that the family of hash functions $\mathcal{H} = \{h_{a,b}(x) = ((ax + b) \bmod p) \bmod m : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ is *pairwise independent* where p is a sufficiently large prime number ($m + 1 \leq p \leq 2m$).

5.1 First proof

By linear algebra, $a + (kp) \bmod p = a \forall k \in \mathbb{N}$. If we were to cap $a + pk \bmod p = a \forall k \in K, K_N = k_i : k_i < N$

$$\mathbb{P}[h(x_i) = c_i] = \frac{1}{m} = \mathbb{P}[h(x_1) = c_1, h(x_2) = c_2]$$

Given x_i, d , we define as $m_i = (ax_i + b)$; since $(ax_i + b)$ is a *linear transformation* m_i is unique. It follows trivially that there are $\frac{p}{m}$ values for which $m_i \bmod p = d$ since $p > m$ and by 5.1. The same goes for x_1, x_2 :

$$\begin{aligned} (ax_1 + b) \bmod p &= d \\ (ax_2 + b) \bmod p &= e \end{aligned}$$

By the Chinese remainder theorem the above system has only one solution for the variable (a, b) over the $(p)(p - 1)$ possible pairs, therefore $(ax_1 + b) \bmod p = d$ and $(ax_2 + b) \bmod p = e$ for $\approx \frac{p}{m}$ cases each. Since d, e are independent

$$((ax_2 + b) \bmod p = d) \wedge ((ax_2 + b) \bmod p = e) \text{ for } \frac{p}{m}, \frac{p}{m} = \frac{p^2}{m^2}$$

Over all the possible choices of (a, b) :

$$\mathbb{P}[(ax_2 + b) \bmod p = d \wedge (ax_2 + b) \bmod p = e] = \frac{\frac{p^2}{m^2}}{p(p - 1)}$$

We now prove $\mathbb{P}[h(x_1) = c_1] = \frac{1}{m}$. Of all the possible m buckets, one and only one is the one we look for: $\mathbb{P}[l \bmod m = c_1] = \frac{1}{m}$. As we've seen by 5.1 at most $\frac{p}{m}$ such l exist for $(ax + b) \bmod p$, therefore $\mathbb{P}[h(x_i) = c_i] = \frac{\frac{p}{m}}{m} \approx \frac{1}{m}$. Which computes $\approx \frac{1}{m^2}$ and proves the assumption under the said approximation.

5.2 Second proof

Proof. We will show that both sides of (12) are approximately equal to $\frac{1}{m^2}$.

Given a hash function $h_{a,b} \in \mathcal{H}$, and two distinct inputs $x_1, x_2 \in \mathbb{N}$, let:

$$\begin{aligned} r &= (ax_1 + b) \bmod p \\ s &= (ax_2 + b) \bmod p \end{aligned}$$

Notice that $r \neq s$ because $r - s \equiv a(x_1 - x_2) \pmod{p}$, both a and $x_1 - x_2$ are nonzero modulo a prime number, and so their product is nonzero.

Since there are a total of $p(p-1)$ pairs of (r, s) such that $r \neq s$ (p^2 choices subtracted the number of pairs where $r = s$), there is a one-to-one correspondence between pairs $(a, b) \in \mathbb{Z}_p^* \times \mathbb{Z}_p$ and pairs (r, s) . Therefore, for the left-hand side of (12):

$$\mathbb{P}_{h_{a,b} \in \mathcal{H}} [(h(x_1) = c_1, h(x_2) = c_2)] = \mathbb{P}_{r \neq s \in \mathbb{Z}_p^2} [r \bmod m = c_1, s \bmod m = c_2]$$

There are about p/m values of r that satisfy $r \bmod m = c_1$, and the same goes for c_2 and s . Hence, there are about $(p/m)^2$ choices for the pair $(r, s) \in \mathbb{Z}_p^2$ that satisfy $r \bmod m = c_1 \wedge s \bmod m = c_2$. Dividing the favorable cases with the all possible cases:

$$\frac{\lfloor p/m \rfloor^2}{p(p-1)} \leq \mathbb{P}[r \bmod m = c_1, s \bmod m = c_2] \leq \frac{(\lfloor p/m \rfloor + 1)^2}{p(p-1)}$$

Thus, $\mathbb{P}[h(x_1) = c_1, h(x_2) = c_2] \approx \frac{1}{m^2}$.

The right-hand side of (12) is equal to $\frac{1}{m^2}$, because both probabilities are $\frac{1}{m}$. Take for example $\mathbb{P}[h(x_2) = c_2]$: there are about p/m values of s that satisfy $s \bmod m = c_2$, out of p values for s . Hence:

$$\mathbb{P}[h(x_2) = c_2] = \mathbb{P}[s \bmod m = c_2] \approx \frac{\frac{p}{m}}{p} = \frac{1}{m}$$

□

6 Deterministic data streaming

Consider a stream of n items, where items can appear more than once. The problem is to find the most frequently appearing item in the stream (where ties are broken arbitrarily if more than one item satisfies the latter). For any fixed integer $k \geq 1$, suppose that only k items and counters can be stored, one item per memory cell, where each counter can use only $O(\text{polylog}(n))$ bits (i.e. $O(\log^c n)$ for any fixed constant $c > 0$): in other words, only $b = O(k \cdot \text{polylog}(n))$ bits of space are available. (Note that, even though we call them counters, they can actually contain any kind of information as long as it does not exceed that amount of bits.) Show that the problem cannot be solved deterministically under the following rules: the algorithm can only use b bits, and read the next item of the stream, one item at a time. You, the adversary, have access to all the stream, and the content of the b bits stored by the algorithm: you cannot change those b bits and the past, namely, the items already read by the algorithm, but you can change the future, namely, the next item to be read. Since the algorithm must be correct for any input, you can use any amount of streams to be fed to the algorithm and as many distinct items as you want. [Hint: it is an adversarial argument based on the fact that, for many streams, there can be a tie on the items.]

Solution. We, the adversaries, can decide the alphabet Σ and the content of the stream $s \in \Sigma^*$. The *idea* is to create a stream generator that forces any deterministic algorithm to reach a state q with at least two different streams. In fact, the number of states (i.e. configurations of memory) of this class of algorithms is bounded by b and, by the pigeonhole principle, there exist at least two streams that cause the algorithm to transition the same state.

Suppose that q is reached by the three streams:

1. $\alpha 0011$, where $\alpha \in \Sigma^*$ contains the same number of 0s and 1s
2. $\beta aabb$, where $\beta \in \Sigma^*$ contains the same number of a s and b s
3. $\gamma 00bb$, where $\gamma \in \Sigma^*$ contains the same number of 0s and b s

When we will emit 0 as next character of the stream and query the algorithm for the most frequent item, we will receive an answer \mathcal{A} . The algorithm is deterministic, therefore in q the answer will be the same for each stream. But \mathcal{A} will be wrong for at least one of the three streams. For example, if $\mathcal{A} = 0$, then the answer is wrong for the second stream.

7 Special case of most frequent item in the stream

Suppose to have a stream of n items, so that one of them occurs $> \frac{n}{2}$ times in the stream. Also, the main memory is limited to keeping just two items and their counters, plus the knowledge of the value of n beforehand. Show how to find deterministically the most frequent item in this scenario.

Hint: since the problem cannot be solved deterministically if the most frequent item occurs $\leq \frac{n}{2}$ times, the fact that the frequency is $> \frac{n}{2}$ should be exploited.

7.1 Solution 1: Up & Down counter

We'll use the notation of the previous section. Before illustrating our solution we prove that the following hold:

- **Given the j^1 element, $C(j^1) > C(j^i) \forall i \in S$.**
- Following the previous, $C(j^i) < C(j^1), i > 1$ element.
- **There are at most $\frac{n}{2} - 1$ elements in S :** given that j^1 appears at least $\frac{n}{2} + 1$ times, in the best case scenario every other $\frac{n}{2} - 1$ element is different, giving $\frac{n}{2} - 1$ different elements.
- **j^1 has sub-stream dominance:** $\exists S' \subseteq S : j^1$ is the most frequent item $\in S'$: trivially, since it is the most frequent element, S' is always a sub-stream dominant sub-stream.

Recursive sub-streams It is trivial to show that if we were to have $\frac{n}{2}$ counters, $\sum_{i=0, i \neq 1}^{\frac{n}{2}} (C(j^i) < C(j^1))$ by 7.1. Though trivial, by combining it with 7.1 we can assert that given $S' = S[0, \frac{n}{2}]$, $S'' = S[\frac{n}{2}, n]$, $k = \text{number of elements} \in S$

- j^1 is the sub-stream dominant element in either S', S'' : by contradiction, let us assume that is false. Then $\exists j_o \neq j^1 : C(j_o) \text{ in } S' > C(j^1, \exists j_o : C(j_o) \text{ in } S'' > C(j^1)$; given that $S'S'' = S$, that would make $C(j_o) = C(j^1)$: contradiction.
- More generally, given two sub-streams S', S'' , the sub-stream dominant element of $S'S''$ is one of the sub-stream dominants in S' or S'' and its frequencies are defined by

$$e - k - 1 \text{ in } S', \frac{n}{2} + 1 - (e - k - 1) \text{ in } S'' \quad (13)$$

where e is an arbitrary number of appearances of $j^2 : e < \frac{n}{2} + 1$ The reader will see as the $+1$ in the right side guarantees the sub-stream dominant to be maximum. The sides can be swapped without hurting generality.

In order to better illustrate, we provide an algorithm that exploits the sub-stream dominance. Our idea is to *go up* with our counter when we are fed an object we've already met, and to *go down* when we are fed an object different

from us. Once we reach the bottom (0), we know by 7.1 that we either reached zero for an object $\neq j^1$, and we can ignore it, or we reached zero for j^1 , that being the most frequent element will have sub-stream dominance in the remaining sequence, which means it will *go upper* than any other element in that sub-stream. Given that any other element has lower frequency, the most frequent of them, j^2 will *go down* $e < \frac{n}{2} + 1$ times. Note that C holds only one element, for simplicity we'll denote its entries as a map in a Scala-like syntax.

```

1: function UP_AND_DOWN( $S, C$ ):
2:    $k \leftarrow S.next()$ 
3:   if  $\exists C(k)$  then
4:      $C \leftarrow \{k \Rightarrow C(k) + 1\}$        $\triangleright$  Increment current dominant element.
5:   else if  $C(k) == 0$  then
6:      $C \leftarrow \{k \Rightarrow 1\}$    $\triangleright$   $k$  is no more dominant, swap it with new element
    and initialize.
7:   else
8:      $C \leftarrow \{k \Rightarrow C(k) - 1\}$        $\triangleright$   $k$  is still dominant, but the stream.
9:   return  $C$ 

```

8 Count-min sketch: extension to negative counters

Check the analysis seen in class, and discuss how to allow $F[i]$ to change by arbitrary values read in the stream. Namely, the stream is a sequence of pairs of elements, where the first element indicates the item i whose counter is to be changed, and the second element is the amount v of that change (v can vary in each pair). In this way, the operation on the counter becomes $F[i] = F[i] + v$, where the increment and decrement can be now seen as $(i, 1)$ and $(i, -1)$.

8.1 First solution

Let F be the implicit vector of size n that receives updates (i, v) such that $F[i] = F[i] + v$ (initially, $F[i] = 0 \forall i \in [1, n]$). The Count-Min (CM) sketch data structure, with parameters ε and δ , consists of:

- a table T of size $r \times c$, where $r = \log \frac{1}{\delta}$ and $c = \frac{\varepsilon}{\delta}$, that we use to approximate queries on F .
- r hash functions h_1, \dots, h_r , chosen uniformly at random from a pairwise-independent family, that maps from $\{1, \dots, n\}$ to $\{1, \dots, c\}$.

The parameters specify that the error in answering a query is in within a factor of ε with probability $1 - \delta$. When an update (i, v) arrives, the table is updated as follows:

$$T[i, h_j(i)] = T[i, h_j(i)] + v \quad \forall j = 1, \dots, r$$

A point query returns an approximation $\tilde{F}[i]$ of $F[i]$. If we assume that $F[i] \geq 0$, then:

$$\tilde{F}[i] = \min_{j \in [1, r]} T[j, h_j(i)]$$

with the guarantees:

1. $\tilde{F}[i] \geq F[i]$
2. $\mathbb{P} \left[\tilde{F}[i] > F[i] + \varepsilon \|F\| \right] \leq \delta$.

The problem statement asks to remove the assumption $F[i] \geq 0$. In this case the point query becomes:

$$\tilde{F}[i] = \text{median}_{j \in [1, r]} T[j, h_j(i)]$$

with the guarantee that $\mathbb{P} \left[F[i] - 3\varepsilon \|F\| \leq \tilde{F}[i] \leq F[i] + 3\varepsilon \|F\| \right] < 1 - \delta^{1/4}$.

Proof. Let $I_{i,j,k}$ the indicator variable that is 1 if $i \neq k \wedge h_j(i) = h_j(k)$ and 0 otherwise. Observe that, by pairwise independence of the hash functions,

$$\mathbb{E} [I_{i,j,k}] = \mathbb{P} [h_j(i) = h_j(k)] \leq \frac{1}{c} = \frac{\varepsilon}{\delta}$$

and, given $X_{i,j} = \sum_{k=1}^n I_{i,j,k} F[k]$

$$\mathbb{E}[|X_{i,j}|] = \mathbb{E}\left[\sum_{k=1}^n I_{i,j,k} |F[k]| \right] \leq \sum_{k=1}^n |F[k]| \mathbb{E}[I_{i,j,k}] \leq \frac{\varepsilon}{e} \|F\| \quad (14)$$

The probability that any count is off by more than $3\varepsilon\|F\|$ is:

$$\begin{aligned} & \mathbb{P}[|T[i, h_j(i)] - F[i]| \geq 3\varepsilon\|F\|] \\ &= \mathbb{P}[|F[i] + X_{i,j} - F[i]| \geq 3\varepsilon\|F\|] && \text{by construction of } T \\ &= \mathbb{P}[|X_{i,j}| \geq 3\varepsilon\|F\|] \\ &\leq \frac{\mathbb{E}[|X_{i,j}|]}{3\varepsilon\|F\|} && \text{by the Markov inequality} \\ &\leq \frac{\varepsilon\|F\|}{e} \frac{1}{3\varepsilon\|F\|} && \text{by (14)} \\ &= \frac{1}{3e} < \frac{1}{8} \end{aligned}$$

Applying Chernoff bounds... □

8.2 Second solution

We trivially have some changes over X_{ji} and $\tilde{F}[i]$:

$$\begin{aligned} X_{ji} &= \Sigma^n(I_k F[k]) \\ \tilde{F}[i] &= F[i] + X_{ji} \end{aligned}$$

X_{ji} can vary as complementary increments $(+v_i, +v_j, +v_k, -v_k, -v_j, -v_i)$ can make it ≤ 0 without necessarily being updates on i . In order to have a minimum error estimate we'll pick the *median* value of X_{ji} .

$\tilde{F}[i] = F[i] + X_{ji}$ by the above assertion the counter $\tilde{F}[i]$ could have no garbage, or negative garbage according to the other increments. Therefore by choosing the minimum $\tilde{F}[i]$ over the $\log(\frac{1}{\delta})$ we could actually pick the most perturbed result (think of a collision with the biggest negative increment over one row).

The last step is the probability proof: $\mathbb{P}[\tilde{F}[i] > F[i] + \epsilon\|F\|]$. Since by 8.2 X_{ji} can be either positive or negative we pick the *median* value *med* over the minimum m and maximum M . We are then able to split our error analysis in two of equivalent size $\frac{\epsilon}{2}$:

$$\begin{aligned} & \mathbb{P}[\tilde{F}[i] \in F[i] + \epsilon\|F\|] = \\ & \mathbb{P}[\tilde{F}[i] \geq F[i] - \epsilon\|F\| \wedge \tilde{F}[i] \leq F[i] + \epsilon\|F\|] = \\ & \mathbb{P}[\tilde{F}[i] \geq F[i] - \epsilon\|F\|] \mathbb{P}[\tilde{F}[i] \leq F[i] + \epsilon\|F\|] \end{aligned}$$

Let us define $p_0 = \mathbb{P} \left[\tilde{F}[i] \geq F[i] - |\epsilon| \|F\| \right]$ and $p_1 = \mathbb{P} \left[\tilde{F}[i] \leq F[i] + |\epsilon| \|F\| \right]$.
 We can report the analysis seen in class adjusted for the number of rows $\frac{r}{2}$

$$\begin{aligned} p_0 &= \prod^{\frac{r}{2}} \mathbb{P} [|Xji| \geq -|\epsilon| \|F\|] = \\ &= -\frac{1}{2^{\frac{r}{2}}} = -\delta^{\frac{1}{2}} \end{aligned}$$

Now for p_1 :

$$\begin{aligned} p_1 &= \mathbb{P} \left[\tilde{F}[i] \leq F[i] + |\epsilon| \|F\| \right] = \\ 1 - \mathbb{P} \left[\tilde{F}[i] \geq F[i] + |\epsilon| \|F\| \right] &= 1 - \frac{1}{2^{\frac{r}{2}}} = 1 - \delta^{\frac{1}{2}} \end{aligned}$$

Giving us a combined probability of $p = p_0 p_1 = (-\delta^{\frac{1}{2}})(1 - \delta^{\frac{1}{2}}) = -\sqrt{\delta} + \delta$.

9 Count-min sketch: range queries

Show and analyze the application of count-min sketch to range queries (i, j) for computing $\sum_{k=i}^j F[k]$. Hint: reduce the latter query to the estimate of just $t \leq 2 \log n$ counters c_1, c_2, \dots, c_t . Note that in order to obtain a probability at most δ of error (i.e. that $\sum_{l=1}^t c_l > \sum_{k=i}^j F[k] + 2\epsilon \log n \|F\|$), it does not suffice to say that it is at most δ the probability of error of each counter c_l : while each counter is still the actual wanted value plus the residual as before, it is better to consider the sum V of these t wanted values and the sum X of these residuals, and apply Markov's inequality to V and X rather than on the individual counters.

Solution. A range $[a, b]$ is a dyadic range if its length is a power of two ($l = 2^y$), and begins at a multiple of its own length: $[j2^y + 1, (j+1)2^y]$. For example, $[13, 16]$ can be written as $[3 \cdot 2^2 + 1, (3+1) \cdot 2^2]$. Any arbitrary range of size s can be partitioned into $O(\log s)$ dyadic ranges, for example [1]:

$$[18, 38] = [18, 18] \cup [19, 20] \cup [21, 24] \cup [25, 32] \cup [33, 36] \cup [37, 38]$$

Let n be the size of the implicit vector F whose entries we want to approximate. The idea is to maintain a collection C of $\log_2 n$ CM sketches, one for each set of dyadic ranges of length $2^y \forall y \in [0, \log_2 n - 1]$. The operations on C becomes:

- **Update** ($F[i] = F[i] + v$). Every sketch in C is updated, since each point $1 \leq i \leq n$ is member of $\log_2 n$ dyadic ranges.
- **Range queries** ($\sum_{i=l}^r F[i]$). The range $[l, r]$ is partitioned into at most $2 \log_2 n$ dyadic ranges. For each partition, a point query is made to the corresponding sketch in C ; the (estimated) result of the range query is the sum of the point queries. See Figure 1.

The time to compute the estimate or to make an update is $O(\log n \log \frac{1}{\delta})$. The space used is $O(\frac{1}{\epsilon} \log n \log \frac{1}{\delta})$, because each sketch requires $O(\frac{\epsilon}{\delta} \ln \frac{1}{\delta})$ space [2].

Let $F[l..r] = \sum_{i=l}^r F[i]$ be the answer to the range query and $\tilde{F}[l..r]$ the estimate. The guarantees are:

- $\tilde{F}[l..r] \geq F[l..r]$
- $\mathbb{P} \left[\tilde{F}[l..r] > F[l..r] + 2\epsilon \log n \|F\| \right] \leq \delta$

Proof. Let $I_{i,j,k}$ the indicator variable that is 1 if $i \neq k \wedge h_j(i) = h_j(k)$ and 0 otherwise. From the analysis of CM sketch, we know that $\mathbb{E}[I_{i,j,k}] \leq \frac{\epsilon}{e}$ and

$$\mathbb{E}[X_{i,j}] = \mathbb{E} \left[\sum_{k=1}^n I_{i,j,k} F[k] \right] \leq \frac{\epsilon}{e} \|F\|$$

A range query $\tilde{F}[l..r]$ is assembled by $t \leq 2 \log n$ point queries on dyadic intervals $[l_1, r_1], [l_2, r_2], \dots, [l_t, r_t]$. The expectation of the additive error X^j for any estimator (i.e. the j -th row of any of the t CM sketches queried) is

$$\mathbb{E}[X^j] = \mathbb{E}\left[\sum_{s=1}^t X_{i,j}\right] = \sum_{s=1}^t \mathbb{E}[X_{i,j}] \leq (2 \log n) \frac{\varepsilon}{e} \|F\| \quad (15)$$

Also observe that the j -th estimation of $F[l..r]$ is

$$\tilde{F}^j[l..r] = \sum_{s=1}^t \tilde{F}^j[l_s..r_s] = \sum_{s=1}^t (F[l_s..r_s] + X_{i,j}) = F[l..r] + X^j \quad (16)$$

Now, for the j -th estimation we can compute

$$\begin{aligned} & \mathbb{P}\left[\tilde{F}^j[l..r] > F[l..r] + 2\varepsilon \log n \|F\|\right] \\ &= \mathbb{P}\left[X^j > 2\varepsilon \log n \|F\|\right] && \text{because of (16)} \\ &\leq \frac{\mathbb{E}[X^j]}{2\varepsilon \log n \|F\|} && \text{by the Markov inequality} \\ &\leq \frac{2\varepsilon \log n \|F\|}{e} \frac{1}{2\varepsilon \log n \|F\|} && \text{because of (15)} \\ &= e^{-1} \end{aligned}$$

Since we have $\ln \frac{1}{\delta}$ independent hash functions $\prod_{j=1}^{\ln(1/\delta)} e^{-1} = e^{-\ln(1/\delta)} = \delta \quad \square$

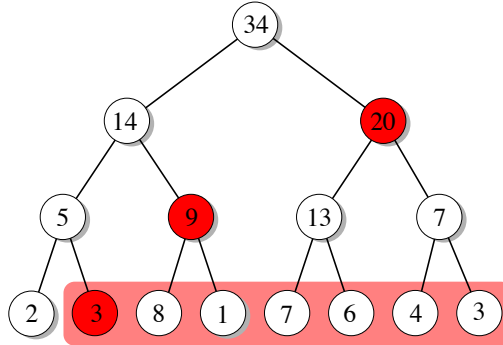


Figure 1: A hierarchy of dyadic ranges. The leaves are the ranges of length 2^0 , while the root corresponds to the single range of length 2^3 . Each level of the tree can be seen as a CM sketch table. To estimate $\sum_{k=2}^8 F[k]$, the range $[2, 8]$ is decomposed into dyadic ranges $[2, 2], [3, 4], [5, 8]$. Each node contains the sum of the values stored in its children. Red nodes are queried and their sum is returned. Adapted from [1].

10 Count-min sketch: extension to negative counters

Consider the two-level perfect hash tables presented in [CLRS] and discussed in class. As already discussed, for a given set of n keys from the universe U , a random universal hash function $h : U \rightarrow [m]$ is employed where $m = n$, thus creating n buckets of size $n_j \geq 0$, where $\sum_{j=0}^{n-1} (n_j) = n$. Each bucket j uses a random universal hash function $h_j : U \rightarrow [m]$ with $m = n_j^2$. Key x is thus stored in position $h_j(x)$ of the table for bucket j , where $j = h(x)$. This problem asks to replace each such table by a bitvector of length n_j^2 , initialized to all 0s, where key x is discarded and, in its place, a bit 1 is set in position $h_j(x)$ (a similar thing was proposed in Problem 4 and thus we can have a one-side error). Design a space-efficient implementation of this variation of perfect hash, using a couple of tips. First, it can be convenient to represent the value of the table size in unary (i.e., x zeroes followed by one for size x , so 000001 represents $x = 5$ and 1 represents $x = 0$). Second, it can be useful to employ a rank-select data structure that, given any bit vector B of b bits, uses additional $o(b)$ bits to support in $O(1)$ time the following operations on B :

- $rank_1(i)$: return the number of 1s appearing in the first i bits of B .
- $select_1(i)$: return the position i of the j^{th} 1, if any, appearing in B (i.e. $B[i] = 1$ and $rank_1(i) = j$).

Operations $rank_0(i)$ and $select_0(j)$ can be defined in the same way as above. Also, note that $o(b)$ stands for any asymptotic cost that is smaller than $O(b)$, $b \rightarrow \infty$.

10.1 Wide bitvector

The two-level hashing layers are comprised of:

1. An hash function and n pointers, each addressing one bitvector B_j .
2. n hash functions h_j hashing to their relative buckets B_j .

Given that by removing one hash function we would not be able to retrieve any element in its bucket, we try and optimize the pointers on the first level.

10.1.1 First level pointers

We merge the bitvectors B_j subsequently one after the other in a flat array B : note that the space is not reduced nor increased, as no further bits are used to separate two adjacent buckets.

$$B = 00101101 \dots 01$$

We then store their respective dimensions in unary form in an auxiliary bitvector L following the same principle.

$$L = 001|01|1|01|\dots 01|$$

It is trivial to see as given a substring S' in L the length it encodes ends at the very next 1. We can exploit this property to compute the size of any interval of buckets B_a and $B_b, b \geq a$ in $O(1)$:

- Pick the $b^{th}1 = \alpha$ with $select_1(b)$ in $O(1)$: this will give us the end of the precedent bucket.
- Pick the sum of the sizes of the preceding buckets by computing the number of 0s with $rank_0(\alpha)$: this will give us starting position of the desired bucket.

Given a bitvector B we employ a map structure to answer the $\mathcal{Q}(i)$ in $O(1)$:

- Hash on the first level in $O(1)$.
- Given $j = h(k)$, apply the above algorithm in $O(1)$ to select the corresponding bucket starting at index $B[k]$ and ending at $B[k + \delta]$, $\delta \geq 0$.
- Apply $h_j(k)$ to $B[k, k + \delta]$ and return $B[k + j]$.

10.1.2 Hash functions optimization

We now try and improve the space for the hash functions parameters a_j, b_j, p_j . First, we select the lowest p_j possible, that is the first $p_j > n_j^2$: by Bertrand postulate such a prime p_j exists for $n_j \leq p_j \leq 2n_j - 2$. We therefore need a space of at most $2n_j^2$ bits for each parameter in an unary encoding. Since we have $a \in \mathbb{Z}_{p_j}^*, b \in \mathbb{Z}_{p_j}$, we have a space of $3 \cdot 2n_j^2$ for a triple (a_j, b_j, p_j) of any given hash function. Let us now compute the expected space for the m hash functions:

$$\mathbb{E} \left[3 \sum_{j=0}^{m-1} 2n_j^2 \right] = 6 \cdot \mathbb{E} \left[\sum_{j=0}^{m-1} n_j^2 \right]$$

where $\mathbb{E} [n_j^2] \leq 2n$ as shown in the perfect hashing analysis. Therefore

$$6 \cdot \mathbb{E} \left[\sum_{j=0}^{m-1} n_j^2 \right] \leq 6 \cdot 2n = 12n$$

References

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