14.121 Final Exam

Alexander Wolitzky

MIT

The exam has 4 questions and lasts 90 minutes. Please try to answer each question so I can give partial credit. I also suggest that you don't spend too much time on any one question: if you get stuck, move on and come back to it later if time permits.

1. An individual must choose an alternative from a set $X \subseteq \mathbb{R}^n_+$.

a. What does it mean for a preference relation $\succsim$ on X to be convex? What does it mean for a utility function $u : X \to \mathbb{R}$ to be quasi-concave? What does it mean for a utility function $u : X \to \mathbb{R}$ to be concave?

b. Suppose utility function $u : X \to \mathbb{R}$ represents preference relation $\succsim$. Show that u is quasi-concave if and only if $\succsim$ is convex.

Now, suppose $n = 2$ and consider the utility function

$$u(x_1, x_2) = \begin{cases} x_1 x_2 \text{ if } x_1 x_2 \leq 1 \\ 1 \text{ if } x_1 x_2 \in (1, 2) \\ x_1 x_2 - 1 \text{ if } x_1 x_2 \geq 2 \end{cases}$$

c. Show that this utility function is quasi-concave.

d. Let $\succsim$ denote the preference relation represented by this utility function. Show that $\succsim$ cannot be represented by any concave utility function.

2. A consumer has a Cobb-Douglas utility function

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha_i},$$

where $\alpha_i \in (0,1)$ for all i and $\sum_{i=1}^{n} \alpha_i = 1$.

a. Derive the consumer's Marshallian demand $x(p,w)$.

b. Derive the consumer's Hicksian demand $h(p,u)$.

c. For any two goods $i \neq j$, what is $\partial x_i / \partial p_j$, the overall effect of a change in the price of good j on the Marshallian demand for good i? Use your answers to parts a and b and the Slutsky equation to decompose this overall effect into the substitution effect and the income effect.

d. In general, for any two goods $i \neq j$, what does it mean for them to be substitutes? What does it mean for them to be gross substitutes? With Cobb-Douglas utility, are any two goods $i \neq j$ substitutes? Are they gross substitutes?

3. Four friends—Ann, Bob, Chris, and Dana—go to a casino. The casino is an unusual one, in that each customer is only offered the opportunity to accept or reject a single gamble, and then must go home with her winnings (or losings). Thus, if an expected utility maximizer has initial wealth w, she gets expected utility $u(w)$ if she rejects the gamble she's offered, and gets expected utility $E[u(w+x)]$ if she accepts it (where x is the amount she wins or loses).

Assume that each of the friends is an expected utility maximizer with a strictly increasing and strictly concave von Neumann-Morgenstern utility function.

a. State as precisely as you can the assumptions about an individual's preferences over gambles that are needed for her behavior to be captured by expected utility maximization.

b. Show that if any of the friends is offered a gamble with a negative expected value $E[x]$, she rejects it.

c. Ann and Bob are offered the same gamble. Ann accepts the gamble, and Bob is less risk-averse than Ann. Show that Bob also accepts the gamble.

d. Chris and Dana have the same utility function. Chris rejects the gamble offered to him, and the gamble offered to Dana is a mean-preserving spread of the gamble offered to Chris. Show directly from first principles (that is, without referencing results from lecture) that Dana also rejects the gamble offered to her.

4. A firm produces output according to a production function $f(y,\theta)$ that depends on the firm's usage of a vector of inputs $y \in \mathbb{R}^n$ and an exogenous technological parameter $\theta \in \mathbb{R}$. The per-unit price of output is fixed at an exogenous level p by market forces outside the firm's control. The firm's cost of production is the sum of two terms, which we refer to as the variable cost and the fixed cost: the difference between the two terms is that the variable cost depends on total production $f(y,\theta)$ and the technological parameter θ, while the fixed cost depends only on the vector of utilized inputs y. Thus, the firm's profit function is

$$\pi(y,p,\theta) = pf(y,\theta) - c(f(y,\theta),\theta) - k(y),$$

where $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the variable cost function and $k : \mathbb{R}^n \to \mathbb{R}$ is the fixed cost function. The firm's problem is to choose the vector y from some set $Y \subseteq \mathbb{R}^n$ to maximize profit. Assume the following:

- f is increasing and supermodular in y with increasing differences in (y,θ).
- c has decreasing differences in its two arguments.
- c is concave in its first argument.
- $pz - c(z,\theta)$ is increasing in z for all θ.
- k is submodular in y.
- Y is a lattice.

a. Show that π is supermodular in y with increasing differences in $(y,(p,\theta))$. You may use any result from lecture or the problem sets, so long as it is cited correctly.

b. Let $Y^*(p,\theta)$ denote the set of solutions to the firm's problem. What can we conclude about how $Y^*(p,\theta)$ changes if either p increases or θ increases? What if both p and θ increase?

[Note: Even if you weren't able to answer part a, you should answer parts b, c, and d, taking the conclusion of part a as given.]

c. Assume that $n = 1$, so there is only a single input. What additional assumptions would imply that, for all $p < p'$, $y \in Y^*(p, \theta)$, and $y' \in Y^*(p', \theta)$, we have $y \leq y'$?

d. In addition to the assumptions from part c, what further assumptions would imply that $y < y'$?