Lecture 4. Arbitrage Pricing

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This week

- We'll cover some of the key results that follow no arbitrage assumptions
- Define some useful terms that appear throughout the course:
  - state-price
  - risk-neutral measure
  - state-price density (AKA stochastic discount factor / pricing kernel)
- Learn what no arbitrage implies for sources of risk premia

- Wednesday/next week: applications of the basic no arbitrage framework
  - Basic option pricing
  - Risk-free bond pricing
  - Corporate finance basics

Outline

Where We Left Off: Asset Pricing Model/Operator

- An asset pricing model is a mapping from a security's payoff vector d to its price P:

  $$P = V(d).$$

- $V(\cdot)$ is called the pricing/valuation operator/functional.

The NA principle imposes general properties on the pricing operator $V(\cdot)$.

Theorem (Positivity)

A portfolio with a positive payoff must have a positive price:

$$V(d) > 0 \quad \textit{if} \quad d > 0.$$

(Also, $V(d) = 0$ if $d = 0$.)

Theorem (Law of One Price)

Two portfolios with same payoffs must have the same price:

$$V(d_1) = V(d_2) \quad \textit{if} \quad d_1 = d_2.$$

Asset Pricing Properties

Theorem (Monotonicity)

Portfolios with higher payoffs must have higher prices:

$$V(d_1) \geq V(d_2) \quad \textit{if} \;\; d_1 \geq d_2.$$

Thus, $V(\cdot)$ is an increasing operator.

Theorem (Linearity)

In a frictionless market, the pricing operator is linear:

$$V(a\, d_1 + b\, d_2) = a\, V(d_1) + b\, V(d_2) \;\; \forall\; a, b \in R \;\textit{and}\; d_1, d_2 \in C_1(D)$$

The Fundamental Theorem of Asset Pricing (FTAP)

Theorem (The Fundamental Theorem of Asset Pricing)

There is no arbitrage in the securities market if and only if there exists $\phi \gg 0$ such that for all traded securities:

$$P^\top = \phi^\top D. \tag{1}$$

We call ϕ the state price vector implied from D and P.

Proof. Sufficiency is obvious. Proof by contradiction. Suppose arbitrage exists and $P^\top = \phi^\top D$. If it is 1^{st} type of arbitrage then $P^T\theta < 0$ and $D\theta \geq 0$. Then, $\phi^\top \times (D\theta) < 0$ and $D\theta \geq 0$, but this contradicts our assumptions that $\phi \gg 0$ and $D\theta >= 0$. One can make an analogous argument 2^{nd} type of arbitrage by assuming that $P^\top\theta \leq 0$ and $D\theta > 0$.

Necessity follows from Stiemke's Lemma, which we'll prove shortly

Discounted Cash Flow/Present Value Formula

- FTAP: $\exists \phi \gg 0 : \qquad P^T = \phi^T D$
- Given ϕ, we have the pricing equation for all traded securities:

$$P_n = \phi^\top D_n = \sum_{\omega \in \Omega} \phi_\omega D_{\omega n}, \quad i = 1, \ldots, N, \tag{2}$$

  also called Discounted Cash Flow (DCF) or Present Value (PV) formula.
- DCF formula expresses current price as the sum of the present value, i.e., market value at $t = 0$, of all future payoffs, using appropriate state prices.
- State price ϕ_ω: price of a hypothetical Arrow-Debreu security whose payoff may or may not be achievable. $\phi \gg 0 \Rightarrow$ all A-D prices are arbitrage-free
- Knowing ϕ, we can price all traded securities, without relying on $\mathbb{P}$, the physical probability measure.

Understanding state prices

Getting intuition: An example

Flip a coin.

$$\text{State 1} = \{\text{Heads}\} \qquad \text{State 2} = \{\text{Tails}\}$$

Asset	State 1 (prob = ½)	State 2 (prob = ½)
1	+ $55	- $50
2	+ $2	+ $2
3	- $50	+ $55

Which asset is more valuable? **Context matters!**

- Expected payoff = $2.50 for assets 1 and 3 vs. $2.00 for asset 2
- Variance of assets 1 and 3 is higher.
- Do I have to pick only 1? Risk averse investors dislike variance
- What if I can choose how much to buy of each?
  - 50/50 mixture of 1 and 3 gives me $2.50 for sure.
  - **Variance doesn't have to be bad if it can be diversified away**

Understanding state prices

Getting intuition: An example, slightly different context

In previous example, asset 1 seemed similar to asset 3. Let's redefine states...

State 1 = {Lose my job, take pay cut}

State 2 = {Keep my job, get raise}

Asset	State 1 (prob = ½)	State 2 (prob = ½)
1	+ $55	- $50
2	+ $2	+ $2
3	- $50	+ $55

Assume I can choose how much to buy of each. Which is more valuable?

- When would you rather have a dollar?
  - Here: State 1 seems objectively worse than State 2
  - Asset 1 acts like *insurance*, pays off when I have less cash on hand
- Average of values of assets 1 and 3 will still be > value of asset 2
  $\Rightarrow$ value of asset 1 > value of asset 2

The Fundamental Theorem of Asset Pricing (FTAP)

Getting intuition: Constructing a ϕ in the payoff space

- Earlier, we showed that $\phi^T = P^T D^{-1}$ in complete market case
- In general, N.A. $\Rightarrow$ can find ϕ as a linear combination of traded assets.
- Some elements of ϕ could be ≤ 0, but can't trade A-D sec's to exploit them.

Theorem

If $rank(D) = N$, there exists a unique $\phi^* \in \mathbb{R}^M$ such that $P^T = \phi^{*T} D$, where $\phi^* = D\theta$ for some portfolio θ.

- Proof: by construction. Plug in candidate portfolio to desired result:

$$\Rightarrow P = D^\top D\theta$$

- Now solve for θ^*. Since $rank(D) = N$, $D^\top D$ is invertible:

$$\theta^* = (D^\top D)^{-1} P$$

$$\phi^* = D\theta^* = D(D^\top D)^{-1} P$$

- Note: $rank(D) < N$ & no arbitrage $\Rightarrow \theta^*$ may not be unique, but ϕ^* is

The Fundamental Theorem of Asset Pricing (FTAP)

Stiemke's Lemma

Theorem (Stiemke's Lemma)

Let D be an $m \times n$ matrix and P, θ, $\phi \in R^m$. No θ exists such that:

(1) $P^\top \theta \leq 0$,

(2) $D\,\theta \geq 0$, *and*

(3) *at least one of the inequalities is strict*

if and only if there exists $\phi \gg 0$ such that:

$$P^\top = \phi^\top D.$$

Proof. Result follows from Separating Hyperplane Thm. Sufficiency is trivial.

- Let $L \equiv R \otimes R^m$, $K \equiv R_+ \otimes R_+^m$, and $M \equiv \{m \in \mathbb{R}^{m+1} : m = [-P^\top \theta;\ (D\,\theta)^\top] \text{ for some } \theta \in \mathbb{R}^n\}$. K and M are both closed, convex subsets of L and M is a linear subspace of L.
- Let $A \equiv \{\theta : P^\top \theta < 0 \text{ and } D\,\theta \geq 0, \text{ or } P^\top \theta \leq 0 \text{ and } D\,\theta > 0\}$.

The Fundamental Theorem of Asset Pricing (FTAP)

Stiemke's Lemma

- $A = \emptyset$ implies $K \cap M = \{0\}$.
- From the Separating Hyperplane Theorem, there exists a linear functional, $F: \ L \mapsto R$ such that:

$$F(y) < F(x) \ \ \forall \ y \in M, \ x \in K \setminus \{0\}.$$

- Since F is a linear functional and M is a linear space, $F(ay) = aF(y)$ for any $a \in \mathbb{R}$ (since $y \in M \Rightarrow ay \in M$)
  - This implies $F(y) = 0 \ \forall \ y \in M$.
  - Otherwise, if $F(y) \neq 0$, we can always find a, s.t. $sign(a) = sign(F(y))$, to make $F(ay) > F(x)$ for some (any) $x \in K$
- Hence, $F(x) > 0 \ \forall \ x \in K \setminus \{0\}$.
- Since F is a linear functional, $F(x) = ax_1 + bx_{\setminus 1}$ where $a \in R$ and $b \in R^m$.
- Since x is any vector in $K \equiv \mathbb{R}_+ \times \mathbb{R}^m_+$, it must be $a > 0$ and $b \gg 0$. So,

$$\forall \ \theta \in R^n \ (y \in M): \quad F(y) = -aP^\top \theta + b^\top D\,\theta = 0 \quad \Rightarrow \quad P^\top = a^{-1}\,b^\top D.$$

Letting $\phi = a^{-1}\,b$, we obtain the lemma.

The Fundamental Theorem of Asset Pricing

Market Completeness and State Prices

In general, the state price vector is not unique, unless the market is complete.

Theorem (Market Completeness and Unique State Prices)

In a complete market with no arbitrage, there is a unique set of strictly positive state prices.

Proof. We only need to show uniqueness. Prove by contradiction. Suppose there are $\phi, \phi' \gg 0$ and $\phi \neq \phi'$. There exists $\omega \in \Omega$ such that $\phi_\omega \neq \phi'_\omega$. Consider the state-$\omega$ contingent claim, whose payoff is 1_ω. For its price at $t = 0$, we have:

$$\phi^\top 1_\omega \neq \phi'^\top 1_\omega,$$

which violates LOP.

The Fundamental Theorem of Asset Pricing (FTAP)

Example

Example. Continue with the example before with $P = [1; 1; 2]$:

$$\text{A:} \quad 1 \begin{cases} 1 \\ 1 \\ 1 \end{cases} \qquad \text{B:} \quad 1 \begin{cases} 0 \\ 2 \\ 2 \end{cases} \qquad \text{C:} \quad 2 \begin{cases} 2 \\ 0 \\ 0 \end{cases}$$

- Consider a market with only A. Any state-price vector $\phi \gg 0$ satisfying:

  $$1 = \phi_1 + \phi_2 + \phi_3$$

  prices security A. The solution is given by the set:

  $$\Phi_A \equiv \{\phi \gg 0 : \ \phi_1 + \phi_2 + \phi_3 = 1\}.$$

- Now suppose the market has only B. Any $\phi \gg 0$ satisfying

  $$1 = 2\phi_2 + 2\phi_3$$

  prices security B. The solution is given by the set:

  $$\Phi_B \equiv \{\phi \gg 0 : \ 2\phi_2 + 2\phi_3 = 1\}.$$

The Fundamental Theorem of Asset Pricing (FTAP)

Example

- Now consider a market with both securities A and B. Any state-price vector $\phi \gg 0$ satisfying both conditions are given by the intersection of A and B:

  $$\mathit{\Phi}_{A+B} = \mathit{\Phi}_A \cap \mathit{\Phi}_B = \{\phi \gg 0 : \ \phi_1 = 1/2, \ \phi_2 + \phi_3 = 1/2\}.$$

  $\mathit{\Phi}_{A+B}$ prices both A and B.

- Now suppose the market further includes security C. The state-price vectors implied by the price of C satisfy:

  $$2 = 2\phi_1,$$

  and the solution is given by the set:

  $$\mathit{\Phi}_C \equiv \{\phi \gg 0 : \ \phi_1 = 1\}.$$

- The state price vector that can price all three securities is given by the intersection of $\mathit{\Phi}_A$, $\mathit{\Phi}_B$ and $\mathit{\Phi}_C$, which is empty:

  $$\mathit{\Phi}_A \cap \mathit{\Phi}_B \cap \Phi_C = \emptyset.$$

- Given D and P, there does not exists a $\phi \gg 0$ that prices all securities. This is because the prices/payoffs from traded securities allow arbitrage.

Riskless Security

- Suppose that security 1 is riskless (a bond), with a sure payoff of 1 at $t = 1$.
- The riskless security can be traded or implied (by a model of ϕ).
- Let P_1 denote its price at $t = 0$.
- The riskless payoff is denoted by $D_1 = [1; \ldots; 1] = 1_M$, where 1_M is the vector of ones.
- With state price vector ϕ, we have:

$$P_1 = \phi^\top 1_M = \sum_{\omega=1}^{M} \phi_\omega.$$

- The riskless interest rate, denoted by r_F, is defined as the net payoff from 1 unit of investment in, or the rate of return from, the riskless bond:

$$P_1 (1 + r_F) = 1 \quad \text{or} \quad r_F = \frac{1}{P_1} - 1.$$

Risk-Neutral Measure

- Given ϕ, we have the pricing equation:

$$P_n = \phi^\top D_n = \sum_{\omega \in \Omega} \phi_\omega D_{\omega n}, \quad n = 1, \ldots, N.$$

- Define:

$$q_\omega \equiv \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}}.$$

- Since $q_\omega > 0$ and $\sum q_\omega = 1$,

$$\mathbb{Q} \equiv \{q_\omega, \ \omega \in \Omega\}$$

  can be interpreted as a measure over Ω.

- Since $\mathbb{Q}$ and $\mathbb{P}$ agree on zero measure sets, they are equivalent.
- In general, $\mathbb{Q} \neq \mathbb{P}$.

Risk-Neutral Measure

Risk-Neutral Measure and Risk-Neutral Pricing

- We can then rewrite the pricing equation as:

$$P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1 + r_F}, \quad n = 1, \ldots, N, \tag{3}$$

  where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation under $\mathbb{Q}$ and D_n here represents the payoff of security n as a random variable.

- (3) is called the risk-neutral pricing formula.
- $\mathbb{Q}$ is called the risk-neutral measure.
- "Risk-neutral pricing" reflects the analogy that if agents in the market were risk-neutral, they would price all securities by only considering their expected payoffs and the riskless interest rate.
- In general, the market is not risk-neutral and $\mathbb{Q} \neq \mathbb{P}$.

Risk-Neutral Measure
Example

Example. (Continued)

$$\text{Security 1:}\quad 1 \;\begin{array}{l} 1 \\ 1 \end{array} \qquad \text{Security 2:}\quad 1/2 \;\begin{array}{l} 2 \\ 0 \end{array}$$

- From the prices/payoffs of the two securities, we have

$$\phi_1 + \phi_2 = 1, \quad 2\phi_1 = 1/2.$$

- The state prices are $\phi_1 = 1/4$, $\phi_2 = 3/4$.
- The risk-neutral measure is $\mathbb{Q} = \{1/4, 3/4\}$.
- Using the risk-neutral pricing formula, we have:

$$\frac{\mathbb{E}^{\mathbb{Q}}[D_1]}{1+0} = \frac{(1/4)(1) + (3/4)(1)}{1} = 1,$$
$$\frac{\mathbb{E}^{\mathbb{Q}}[D_2]}{1+0} = \frac{(1/4)(2) + (3/4)(2)}{1} = 1/2,$$

which give the right prices.