14.121 Problem Set 3 Solutions

Question 1

Local-nonsatiation fails. Imagine a picture like the one on slide 20 of lecture 6, except that the consumer has a "fat" indifference curve that extends into the purple shaded feasibility set. An allocation in the interior of the feasibility set and within this indifference curve is Pareto optimal, but it cannot be implemented as a Walrasian equilibrium since the firm would not be maximizing profits.

Question 2

Assumption 1: if $x, x \in l_\infty^+$, then they are bounded sequences with bounds M and M', respectively, so

$$|\alpha x_t + (1-\alpha)x_t'| \leq \alpha|x_t| + (1-\alpha)|x_t'| < \max\{M, M'\}$$

Assumption 2: if $U(x) < U(x')$ then

$$\begin{aligned} U(\alpha x + (1-\alpha)x') &= \sum_{t=0}^{\infty} \beta^t u(\alpha x_t + (1-\alpha)x_t') \\ &> \alpha \sum_{t=0}^{\infty} \beta^t u(x_t) + (1-\alpha) \sum_{t=0}^{\infty} \beta^t u(x_t') \\ &= \alpha U(x) + (1-\alpha)U(x') > U(x) \end{aligned}$$

Assumption 3: the first step is to show that U is continuous. Consider the measure space consisting of the natural numbers with the counting measure μ. Functions $\mathbb{N} \to \mathbb{R}$ can be denoted by sequences. Consider a sequence $x_n \to x$. Note that $f_n = (\beta^t u(x_{n,t}))_{t=0}^{\infty}$ converges pointwise to $f_\infty = (\beta^t u(x_t))_{t=0}^{\infty}$ by continuity of u, and the sequence f_n is dominated by an integrable function $g = (\beta^t \bar{u})_{t=0}^{\infty}$, where $\bar{u} > \sup_{c \geq 0} u(c)$ since u is bounded. The dominated convergence theorem implies

$$\lim_{n\to\infty} U(x_n) = \lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t u(x_{n,t}) = \lim_{n\to\infty} \int_{\mathbb{N}} f_n d\mu = \int_{\mathbb{N}} f_\infty d\mu = \sum_{t=0}^{\infty} \beta^t u(x_t) = U(x)$$

Now to show the main result, let $f(\alpha) = (1-\alpha)x' + \alpha x''$. The two sets in Assumption 3 can be written $(U \circ f)^{-1}[U(x), \infty)$ and $(U \circ f)^{-1}(-\infty, U(x)]$. Since $U \circ f$ is continuous, the preimages of these closed subsets must be closed.

Assumption 4: if $y, y' \in Y$, then there exist sequences k, k' with $k_0 = k_0' = \hat{k}$ and

$$\begin{aligned} k_{t+1} + y_t &\leq f(k_t) \\ k_{t+1}' + y_t' &\leq f(k_t') \end{aligned}$$

Note that $\alpha k_{t+1} + (1-\alpha)k_{t+1}'$ supports $\alpha y_t + (1-\alpha)y_t'$, as concavity of f implies

$$(\alpha k_{t+1} + (1-\alpha)k_{t+1}') + (\alpha y_t + (1-\alpha)y_t') \leq \alpha f(k_t) + (1-\alpha)f(k_t') \leq f(\alpha k_{t+1} + (1-\alpha)k_{t+1}')$$

This shows that $\alpha y_t + (1-\alpha) y_t \in Y$.

Assumption 5: Take $k_1 \in (0, \bar{k})$ and $y_0 = 0.5\left(f(\hat{k}) - k_1\right) \geq 0.5\left(f(k_1) - k_1\right) > 0$. For $t \geq 1$ take $k_{t+1} = k_1$ and $y_t = 0.5(f(k_1) - k_1)$. Then y_t satisfies $k_{t+1} + y_t \leq f(k_t)$ for all t, so y is in the production set. One can similarly check that $y_t' = y_t + r_t$ is also in the production set for $|r_t| < 0.25 \min\{f(\hat{k}) - k_1, f(k_1) - k_1\}$. The key property used here is free disposal, where $k_{t+1} + y_t \leq f(k_t)$ may hold with strict inequality.

The result in Debreu (1954) proves that versions of the welfare theorems hold for this infinite-dimensional economy. The SWT implies that the Pareto optimal allocation in this economy can be decentralized in the manner described in the theorem on slide 29. With some additional conditions, such as the Remark on slide 29, the Pareto optimal allocation can actually be decentralized as a valuation equilibrium. The FWT implies that any valuation equilibrium must yield this Pareto optimal allocation. Putting these together, and supposing the Remark on slide 29 holds, the results from Debreu (1954) imply that there exists a unique valuation equilibrium allocation for this economy.

An interesting difference in the infinite-dimensional case is that some continuous linear functionals on l_∞ do not take the form of a sequence of prices such that the value of a consumption stream can be written as the sum of the value of consumption in each period separately, as in $v(x) = \sum_{t=0}^{\infty} v_t x_t$.

Question 3

Note that preferences are concave and the consumption sets are convex, so the utility possibilities set is convex and Pareto optimal allocations are spanned by solving the planner's problem for different Pareto weights. The Lagrangian for the planner's problem can be written

$$\lambda_B(c_{B,1} + 2\sqrt{c_{B,2}}) + \lambda_L(c_{L,1} + 2\sqrt{c_{L,2}}) - \mu_1(c_{B,1} + c_{L,1} - 1) - \mu_2(c_{B,2} + c_{L,2} - 1) + \sum_{i,t} \eta_{i,t} c_{i,t}$$

The FOCS are

$$\begin{aligned}
c_{B,1} : \lambda_B &= \mu_1 - \eta_{B,1} \\
c_{B,2} : \frac{\lambda_B}{\sqrt{c^*_{B,2}}} &= \mu_2 - \eta_{B,2} \\
c_{L,1} : \lambda_L &= \mu_1 - \eta_{L,1} \\
c_{L,2} : \frac{\lambda_L}{\sqrt{c^*_{L,2}}} &= \mu_2 - \eta_{L,2}
\end{aligned}$$

Note that the nonnegativity constraints for good 2 will not bind with equality since marginal utility explodes as the amount consumed approaches zero, so $\eta_{i,2} = 0$ for $i = B, L$. However, the nonnegativity constraints for good 1 can bind. In particular, $\lambda_L/\lambda_B > 1$ implies the nonnegativity constraint $c_{B,1} \geq 0$ binds with equality (and hence $c_{L,1} = 1 > 0$ doesn't bind, so $\eta_{L,1} = 0$). Using the correspondence between Lagrange multipliers on resource constraints and prices in the equilibrium with transfers, we obtain from this system

$$p = \frac{\mu_2}{\mu_1} = \frac{1}{\sqrt{c^*_{L,2}}}$$

The wealth levels are pinned down by the budget constraints

$$w_L = \underbrace{c^*_{L,1}}_{=1} + pc^*_{L,2} = 1 + \sqrt{c^*_{L,2}}$$
$$w_B = \underbrace{c^*_{B,1}}_{=0} + pc^*_{B,2} = \frac{1 - c^*_{L,2}}{\sqrt{c^*_{L,2}}}$$

Note that the FOCs imply $c^*_{L,2} = c^*_{B,2}\lambda^2$, and the budget constraint implies $1 = c^*_{L,2} + c^*_{B,2}$, so putting these together obtains

$$c^*_{L,2} = \frac{\lambda^2}{1+\lambda^2}$$

Substituting this into the above gives

$$p = \sqrt{\frac{1+\lambda^2}{\lambda^2}}$$
$$w_L = 1 + \sqrt{\frac{\lambda^2}{1+\lambda^2}}$$
$$w_B = \left(1 - \frac{\lambda^2}{1+\lambda^2}\right)\sqrt{\frac{1+\lambda^2}{\lambda^2}}$$

Question 4

a. A type j consumer's endowment corresponds to a degenerate distribution 1_{c_j} which has all of its mass on c_j. A consumer of type j solves

$$\max_{\{\pi_j(c_i)\}_{i=1}^n} \sum_i \pi_j(c_i) u_j(c_i)$$

subject to

$$\sum_i p(c_i)\pi_j(c_i) \leq p(c_j)$$
$$\sum_i \pi_j(c_i) = 1$$
$$\pi_j(c_i) \geq 0 \ \forall i$$

Market clearing in this economy requires

$$\alpha_j = \sum_i \alpha_i \pi_i(c_j)$$

for $j = 1, .., n$.

b. Case 1: suppose $p(c_1) \neq p(c_2)$. WLOG suppose $p(c_1) < p(c_2)$. Then the *only* affordable bundle for type 1 which satisfies all of the distributional requirements is the endowment 1_{c_1}. Market clearing then implies that the other type must set $\pi_2(c_1) = 0$, and hence the other type must buy 1_{c_2}. Thus both types must consume their endowments. Note that in order for this case to hold we would need $u_2(c_1)/u_2(c_2) \leq p(c_1)/p(c_2) < 1$. Note also that this argument relies on the fact that agents must buy complete distributions over bundles in X, and they cannot elect to receive nothing with positive probability. It is therefore possible to have

$$u_1(c_1)/u_1(c_2) < u_2(c_1)/u_2(c_2) < p(c_1)/p(c_2)$$

and yet agents of type 1 still consume 1_{c_1}.

Case 2: suppose $p(c_1) = p(c_2)$. Then either type of agent can afford any distribution. Since $u_j(c_1) \neq u_j(c_2)$, either type of agent will either want to buy 1_{c_1} or 1_{c_2}, depending on whether $u_j(c_1) > u_j(c_1)$ or $u_j(c_1) < u_j(c_2)$. If type 1 agents want to buy 1_{c_2}, then the total ex-post demand for good c_2 would be either α_1 (if type 2 wants to buy 1_{c_1}) or 1 (if type 2 wants to buy 1_{c_2}), either of which is infeasible (since supply of good 2, α_2, is not equal to α_1 or 1). Hence the only feasible allocation is for type 1 agents to buy 1_{c_1}, i.e. to consume their endowment, and similarly type 2 agents must buy their endowment 1_{c_2}. Note that in order for $p(c_1) = p(c_2)$ to hold we would need $u_1(c_1)/u_1(c_2) \geq \underbrace{p(c_1)/p(c_2)}_{=1} \geq u_2(c_1)/u_2(c_2)$.

Finally, one can also see using the above price restrictions that if $u_i(c_{-i}) > u_i(c_i)$ for $i = 1, 2$ then there is no equilibrium in this economy.

Question 5

a. The symmetric PO allocation solves

$$\max_{c(\theta_1), c(\theta_2)} \lambda \underbrace{\sqrt{c(\theta_1)}}_{=U(c(\theta_1),\theta_1)} + (1-\lambda) \underbrace{c(\theta_2)}_{=U(c(\theta_2),\theta_2)}$$

$$\text{s.t } (\gamma) : \lambda c(\theta_1) + (1-\lambda) c(\theta_2) \leq 1$$

The FOCs and feasibility condition give

$$\begin{aligned}
c(\theta_1) : \frac{\lambda}{2\sqrt{c(\theta_1)}} &= \gamma\lambda \\
c(\theta_2) : (1-\lambda) &= \gamma(1-\lambda) \\
\implies c(\theta_1) &= \frac{1}{4} \\
c(\theta_2) &= \frac{1 - \frac{1}{4}\lambda}{1-\lambda}
\end{aligned}$$

b. Let $p(\theta_1)$ and $p(\theta_2)$ denote the prices on consumption conditional on type θ_1 or θ_2. Households sell their endowment $e = 1$ (for both states) in order to buy contingent consumption $c(\theta_i)$.

The household problem can be written

$$\max_{c(\theta_1),c(\theta_2)} \lambda\sqrt{c(\theta_1)} + (1-\lambda)c(\theta_2)$$

subject to

$$p(\theta_1)c(\theta_1) + p(\theta_2)c(\theta_2) \leq p(\theta_1) + p(\theta_2)$$

There is a broker dealer (like an insurance firm) who pools the endowments of the agents and redistributes ex-post based on state-contingent contracts that are bought by the households. The firm maximizes profits

$$\max_{(y(\theta_1),y(\theta_2))\in Y} p(\theta_1)y(\theta_1) + p(\theta_2)y(\theta_2)$$

where the production set is

$$Y = \left\{(y(\theta_1), y(\theta_2)) \in \mathbb{R}^2 : \lambda y(\theta_1) + (1-\lambda)y(\theta_2) \leq 0\right\}$$

In particular, since $y(\theta) < 0$ corresponds to the broker dealer taking in from those who announce θ and $y(\theta) > 0$ corresponds to the broker dealing giving out to those who announce θ, this constraint says that the broker dealer cannot give out more than it takes in. The firm's problem implies $p(\theta_1) = \lambda$ and $p(\theta_2) = 1-\lambda$ (up to a scalar multiple) and that the feasibility constraint binds with equality, $\lambda y(\theta_1) + (1-\lambda)y(\theta_2) = 0$. One can also see that with these prices the individual's problem is mathematically equivalent to the social planner's problem in part a and thus will have the same solution. Note that market clearing follows from checking feasibility in part a.

c. The allocation determined above is not incentive compatible: type 1 would like to pose as type 2 because $U(c_2, \theta_1) = \sqrt{\frac{1-\frac{1}{4}p}{1-p}} > \sqrt{\frac{1}{4}} = U(c_1, \theta_1)$.

d. Refer to the solution for the type i allocation from part a by c_i^* for short. Suppose consumption of type θ_2 is now given by a lottery x which gives some amount c_3 with probability $\alpha = \frac{c_2^*}{c_3}$ and 0 with probability $1-\alpha$. Note that under expected utility preferences $U(x, \theta_2) = c_3\alpha + 0*(1-\alpha) = c_2^*$ and $U(x, \theta_1) = \sqrt{c_3}\alpha + \sqrt{0}(1-\alpha) = \frac{c_2^*}{\sqrt{c_3}}$. Now choose c_3 large enough so that type 1 no longer wants to lie about his type, i.e.

$$\begin{aligned}
U(x, \theta_1) &\leq U(c_1, \theta_1) \\
\iff \frac{c_2^*}{\sqrt{c_3}} &\leq \sqrt{c_1^*} \\
\iff c_3 &\geq \frac{(c_2^*)^2}{c_1^*} = 4\left(\frac{1-\frac{1}{4}\lambda}{1-\lambda}\right)^2
\end{aligned}$$

Now we can assign a contract which gives c_1^* if the agent reports type 1 and which gives the lottery x (with $c_3 \geq 4\left(\frac{1-\frac{1}{4}\lambda}{1-\lambda}\right)^2$) if the agent reports type 2. As shown above, this contract is incentive compatible. Moreover, it is feasible since

$$E[pc_1^* + (1-p)x] = pc_1^* + (1-p)c_2^* = 1$$

Note that the utility of type 2 is still equal to $U(x,\theta_2) = U(c_2,\theta_2) = c_2^*$ and the utility of type 1 with this incentive compatible contract is equal to $U(c_1,\theta_1) = \sqrt{c_1^*}$. Since this allocation is feasible and replicates the utilities of the PO allocation in Part A, it must be PO.

e. Note that the consumption set of incentive compatible distribution contracts is a subset of $\mathbb{R}^{2n}$, where n is the number of elements of C. In the context of this problem, the household problem writes

$$\max_{x(c,\theta)} \lambda \sum_c x(c,\theta_1)\sqrt{c} + (1-\lambda)\sum_c x(c,\theta_2)c$$

subject to

$$\begin{aligned} x(c_i,\theta_j) &\geq 0 \ \forall i,j \\ \sum_c x(c,\theta_j) &= 1 \ \forall j \\ \sum_c x(c,\theta_1)\sqrt{c} &\geq \sum_c x(c,\theta_2)\sqrt{c} \\ \sum_c x(c,\theta_2)c &\geq \sum_c x(c,\theta_1)c \\ \sum_c p(c,\theta_1)x(c,\theta_1) + \sum_c p(c,\theta_2)x(c,\theta_2) &\leq p(1,\theta_1) + p(1,\theta_2) \end{aligned}$$

where the last line reflects the common endowment $e = 1$. The production set for firms is given by

$$Y = \left\{ y(c,\theta) \in \mathbb{R}^{2n} | \lambda \sum_c c y(c,\theta_1) + (1-\lambda)\sum_c c y(c,\theta_2) \leq 0 \right\}$$

The profits maximization problem for firms is

$$\max_{y\in Y} \sum_c p(c,\theta_1)y(c,\theta_1) + \sum_c p(c,\theta_2)y(c,\theta_2)$$

Since the firm's problem is linear, the equilibrium price system is

$$\begin{aligned} p(c,\theta_1) &= \lambda c \\ p(c,\theta_2) &= (1-\lambda)c \end{aligned}$$

Substituting these prices into the household's problem allows us to write it as

$$\max_{x(c,\theta)} \lambda \sum_c \left(\sqrt{c} - c\right) x(c,\theta_1)$$

Suppose $c = 1/4$ belongs to C. The household would like to give all weight to the value of c that maximizes $\sqrt{c} - c$, i.e. $c = 1/4$. From the budget constraint, we obtain

$$\sum_c x(c,\theta_2)c = \frac{1 - \frac{1}{4}\lambda}{1-\lambda} > \frac{1}{4} \tag{1}$$

Hence, type 2 will not want to misrepresent its type. Type 1 will not want to mispresent its type as long as $x(\cdot, \theta_2)$ satsifies

$$\sqrt{\frac{1}{4}} \geq \sum_c x(c, \theta_2)\sqrt{c} \tag{2}$$

Since type 2 is risk-neutral, the household is indifferent between all lotteries $x(\cdot, \theta_2)$ that satisfy (1) and (2). The example given above which assigns $x(c_3, \theta_2) = \frac{c_2^*}{c_3}$ and $x(0, \theta_2) = 1 - \frac{c_2^*}{c_3}$, where $c_3 \geq 4\left(\frac{1-\frac{1}{4}\lambda}{1-\lambda}\right)^2$, satisfies these conditions (assuming c_3 and 0 both belong to C). Note that market clearing follows from checking feasibility of the allocation in part d. Thus, the allocation in part d can be supported as a Walrasian equilibrium when we allow agents to contract lotteries.

f. No. As seen above, the full insurance allocation is not feasible with private information (even in the case with a risk neutral agent). In order for the full insurance allocation to be achieved, we must give a risky allocation to at least one type. If both agents are risk averse, then in order to reach the same utility we would have to include a compensation for the risk, or a higher expected value. However, considering that the full insurance allocation already exhausts the total resources of the economy, this would be infeasible.