14.121, Fall 2018
Problem Set 2 Solutions

1) MWG Exercise 3.G.5
Solution:

a) We consider the following two maximization problems:

$$\max_{(x,z)} \tilde{u}(x,z) \quad \text{s.t. } p \cdot x + \alpha z \leq w \tag{1}$$

$$\max_{(x,y)} u(x,y) \quad \text{s.t. } p \cdot x + (\alpha q_0) \cdot y \leq w \tag{2}$$

<u>Claim 1</u>: If (x^*, z^*) is a solution to (1), then there exists y^* such that $q_0 \cdot y^* = z^*$ and (x^*, y^*) is a solution to (2).

Proof of Claim 1: We want to show that (x^*, y^*) solves (2), that is, take any (x, y) that satisfies $p \cdot x + (\alpha q_0) \cdot y \leq w$, $u(x,y) \leq u(x^*, y^*)$.

First, by definition of $\tilde{u}$, we have $u(x,y) \leq \tilde{u}(x, q_0 \cdot y)$. Next, note that $p \cdot x + \alpha(q_0 \cdot y) \leq w$ (by the assumption that (x, y) is in the budget set), we have $\tilde{u}(x, q_0 \cdot y) \leq \tilde{u}(x^*, z^*)$ since (x^*, z^*) is the optimal solution to (1). Putting the two inequalities together, we have

$$u(x,y) \leq \tilde{u}(x, q_0 \cdot y) \leq \tilde{u}(x^*, z^*) = u(x^*, y^*)$$

<u>Claim 2</u>: If (x^*, y^*) is a solution to (2), then $(x^*, q_0 \cdot y^*)$ is a solution to (1).

Proof of Claim 2: For every y such that $q_0 \cdot y \leq q_0 \cdot y^*$, we have $p \cdot x^* + (\alpha q_0) \cdot y \leq p \cdot x^* + (\alpha q_0) \cdot y^* \leq w$. Hence $u(x^*, y) \leq u(x^*, y^*)$. Therefore, $u(x^*, y^*) = \tilde{u}(x^*, q_0 \cdot y^*)$.

Now let (x, z) satisfy $p \cdot x + \alpha z \leq w$. Then there exists y such that $q_0 \cdot y \leq z$ and $u(x, y) = \tilde{u}(x, z)$. Since $p \cdot x + (\alpha q_0) \cdot y \leq p \cdot x + \alpha z \leq w$, we have $u(x,y) \leq u(x^*, y^*)$ by definition of (x^*, y^*). Putting these inequalities together, we have

$$\tilde{u}(x,z) = u(x,y) \leq u(x^*, y^*) = \tilde{u}(x^*, q_0 \cdot y^*)$$

b,c) It suffices to check that $\tilde{u}$ satisfies all the following assumptions: 1. $\tilde{u}$ is continuous; 2. $\tilde{u}$ is locally non-satiated; 3. $\tilde{u}$ is strictly quasi-concave. We assume that u satisfies all these assumptions.

Continuity: Since u is continuous, and the budget set correspondence is compact and continuous (as we've shown in recitation), then by the theorem of maximum, $\tilde{u}$ is continuous.

Local non-satiation: Since u is locally non-satiated, the value function $\tilde{u}$ is strictly increasing in z. Thus $\tilde{u}$ also satisfies local non-satiation.

Strict quasi-concavity: Suppose $\tilde{u}(x,z) \geq u_0$ and $\tilde{u}(x', z') \geq u_0$, we want to show that $\tilde{u}(\alpha x + (1-\alpha)x', \alpha z + (1-\alpha)z') > u_0$. Let $u(x,y) = \tilde{u}(x,z)$ and $u(x', y') = \tilde{u}(x', z')$. We have

$$\tilde{u}(\alpha x + (1-\alpha)x', \alpha z + (1-\alpha)z') \geq u(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y')$$

because $(\alpha y + (1-\alpha)y') \cdot q_0 = \alpha(y \cdot q_0) + (1-\alpha)(y \cdot q_0) \leq \alpha z + (1-\alpha)z'$ (i.e. it is feasible), and $\tilde{u}(\alpha x + (1-\alpha)x', \alpha z + (1-\alpha)z')$ attains the maximum value by definition.

Since u is strictly quasi-concave, we have

$$u(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') > \min\{u(x,y), u(x', y')\} \geq u_0$$

Putting the two inequalities together, we have that $\tilde{u}(\alpha x + (1-\alpha)x', \alpha z + (1-\alpha)z') > u_0$.

Finally, since $\tilde{u}$ is continuous, satisfies local non-satiation, and is strictly quasi-concave, all the propositions follow immediately by taking $\tilde{u}$ to be the direct utility function.

2) MWG Exercise 3.G.6 parts (a), (b), and (c).

Solution:

a) We can find the demand for x_3 by using Walras law: $x_3 = (w - p_1 x_1 - p_2 x_2)/p_3$

b) Yes. $\forall \lambda > 0$:

$$x_1 = 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{w}{p_3} = 100 - 5\frac{\lambda p_1}{\lambda p_3} + \beta\frac{\lambda p_2}{\lambda p_3} + \delta\frac{\lambda w}{\lambda p_3}$$

$$x_2 = \alpha + \beta\frac{p_1}{p_3} + \gamma\frac{p_2}{p_3} + \delta\frac{w}{p_3} = \alpha + \beta\frac{\lambda p_1}{\lambda p_3} + \gamma\frac{\lambda p_2}{\lambda p_3} + \delta\frac{\lambda w}{\lambda p_3}$$

thus the demand functions are homogeneous of degree zero in (p, w), that is, $x_i(p, w) = x_i(\lambda p, \lambda w)$.

c) We first normalize by letting $p_3 = 1$. Using the definition of the entries of the Slutsky substitution matrix, $\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p, w)$ for all l and k, and that the derivatives of Hicksian demand with respect to the own price (the diagonal entries of the Slutsky matrix) will be non-positive, we have (for all p_1, p_2, and w)

$$-5 + \delta(100 - 5p_1 + \beta p_2) + \delta^2 w \leq 0$$

But, if $\delta \neq 0$, $\delta^2 > 0$ and we can always find some (p_1, p_2, w) such that the above expression is positive. Thus, $\delta = 0$. (**Note**: the assertion in the heading of the problem is wrong.)

Since there is no income effect, we have that

$$h_1 = x_1 = 100 - 5p_1 + \beta p_2$$

and

$$h_2 = x_2 = \alpha + \beta p_1 + \gamma p_2$$

Then, $\frac{\partial h_2}{\partial p_2} \leq 0$ implies that $\gamma \leq 0$.

We can try to get more restrictions on α, β, and γ by using the fact that the Slutsky matrix is symmetric, but it is clear from the above expressions that this implies that $\beta = \beta$, a vacuous restriction.

Another way to look for restrictions on the parameters is to use the Compensated Law of Demand. In particular, let $p' = (p'_1, p'_2, p'_3)$ and $p'' = (p'_1 + \Delta_1, p'_2 + \Delta_2, p'_3)$. Then, we must have that

$$(p'' - p') \cdot (h(p'', u_0) - h(p', u_0)) \leq 0$$

Using the definitions of p', p'', and Hicksian demand, this implies that

$$(\Delta_1, \Delta_2, 0) \cdot (-5\Delta_1 + \beta\Delta_2, \beta\Delta_1 + \gamma\Delta_2, h_3(p'', u_0) - h_3(p', u_0)) \leq 0$$

$$\Rightarrow -5\Delta_1^2 + 2\beta\Delta_1\Delta_2 + \gamma\Delta_2^2 \leq 0 \text{ for all } \Delta_1, \Delta_2$$

If $\Delta_1 = 0$, then this implies that $\gamma \leq 0$. One can also find the Δ_1 (and symmetrically the Δ_2) that maximizes the left hand side of the above inequality, plug that value into the inequality and find that $\beta^2 \leq -5\gamma$. It may be possible to put further restrictions on the parameters by finding the Hicksian demand for good 3 and using the negative semidefiniteness of the Slutsky matrix, but this is messy and does not contain much in the way of economics.

3) MWG Exercise 3.G.17

Note: The minus sign at the beginning of the right-hand side of the indirect utility function should be deleted. That is, it should be $v(p,w) = (w/p_2 + (1/b)\,(ap_1/p_2 + a/b + c))\exp(-bp_1/p_2)$.

Also, in (b), the minus sign in front of the first term of the right-hand side of the expenditure function should be deleted. That is, it should be $e(p,u) = p_2 u \exp(bp_1/p_2) - (1/b)(ap_1 + ap_2/b + p_2 c)$.

Finally, in (c), the minus sign in front of the first term of the right-hand side of the Hicksian function should be deleted. That is, it should be $h(p,u) = ub\exp(bp_1/p_2) - a/b$.

Solution:

a) Use

$$\frac{\partial v(p,w)}{\partial p_1} = -p_2^{-1}(a\frac{p_1}{p_2} + b\frac{w}{p_2} + c)\exp(-b\frac{p_1}{p_2})$$
$$\frac{\partial v(p,w)}{\partial w} = p_2^{-1}\exp(-b\frac{p_1}{p_2})$$

and apply Roy's formula to obtain:

$$x_1(p,w) = a\frac{p_1}{p_2} + b\frac{w}{p_2} + c.$$

b) We can use identity $u = v(p, e(p,u))$

$$u = v(p,e(p,u)) = (e(p,u)/p_2 + (1/b)\,(ap_1/p_2 + a/b + c))\exp(-bp_1/p_2)$$

to solve for the expenditure function:

$$e(p,u) = p_2 u \exp(bp_1/p_2) - (1/b)(ap_1 + ap_2/b + p_2 c).$$

c) Use the fact that $h(p,u) = \nabla_p e(p,u)$ to obtain the Hicksian demand function for the first good:

$$h(p,u) = ub\exp(bp_1/p_2) - a/b.$$

4) A consumer has a continuous, strictly increasing, quasi-concave utility function. You know that the consumer's expenditure function is $e(p_1,p_2,u) = \frac{p_1 p_2}{p_1+p_2}u$.

a) Show that by considering what happens when one price goes to infinity it is easy to find $u(x_1,0)$ and $u(0,x_2)$.

Solution: Note that the expenditure function

$$e(p_1,p_2,u) = \frac{p_1 p_2}{p_1+p_2}u$$

yields the following Hicksian demand functions:

$$\begin{aligned} h_1(p_1,p_2,u) &= \left(\frac{p_2}{p_1+p_2}\right)^2 u \\ h_2(p_1,p_2,u) &= \left(\frac{p_1}{p_1+p_2}\right)^2 u \end{aligned}$$

$\lim_{p_1\to\infty} e(p_1,p_2,u_0) = p_2 u_0$ and $\lim_{p_1\to\infty} h_1(p_1,p_2,u_0) = 0$. Thus the expenditure is focused on x_2 and so we have $p_2 u_0 = p_2 x_2$, which implies that

$$u(0,x_2) = u_0 = x_2$$

We can analogously show that

$$u(x_1,0) = x_1$$

b) For what values of x_1 does there exist a non-negative x_2 such that $u(x_1, x_2) = u_0$?

Solution: For a given u_0, we know that $u(0, u_0) = u_0$, and $u(u_0, 0) = u_0$. Because u is continuous in x_1 and x_2, we know that we can also find an x_2 such that $u(x_1, x_2) = u_0$ for any $x_1 \in [0, u_0]$. Also, because u is strictly increasing in both variables, we know that $x_1 = u_0$ is the largest value for which such an x_2 can be found. Assuming $x_1 < 0$ is uninteresting (or impossible), the desired range is:

$$x_1 \in [0, u_0].$$

c) Let x_1 be in the range you identified in part (b). By fixing $p_1 = 1$ and considering what happens when the consumer faces prices $(1, p_2)$ and has wealth $e(1, p_2, u_0)$ you can find the set of values of x_2 for which $u(x_1, x_2) \leq u_0$. What do you know about the largest x_2 in this set? Use this approach to describe the indifference curve $u(x_1, x_2) = u_0$, i.e. find the function $x_2(x_1, u_0)$ such that $u(x_1, x_2(x_1, u_0)) = u_0$?

Solution: For any given $x_1 \in [0, u_0]$, u_0, there is some price p_2 such that x_1 will solve the EMP/UMP for that price, since we showed that there are a x_1, x_2 such that $u(x_1, x_2) = u_0$. We can identify this p_2 from the hicksian demand for good 1:

$$\begin{aligned} x_1 &= \left(\frac{p_2}{p_1 + p_2}\right)^2 u_0 = \left(\frac{p_2}{1 + p_2}\right)^2 u_0 \\ &\iff \\ p_2 &= \frac{\sqrt{x_1}}{\sqrt{u_0} - \sqrt{x_1}} \end{aligned}$$

Now this p_2 represents the inverse price ratio (for $p_1 = 1$) for which the x_1 chosen is optimal (in that it leads to a maximal utility or minimal expenditure), given a level of utility. Now, given that we have the price ratio we can plug it into the Hicksian demand function for good 2 to find x_2 such that $u(x_1, x_2) = u_0$. This will be the largest x_2 such that $u(x_1, x_2) \leq u_0$ since u is strictly increasing in x_2 and the point on the envelope of the budget constraints corresponding to different slopes:

$$\begin{aligned} x_2 &= \frac{1}{(1 + p_2)^2} u_0 = \frac{u_0}{\left(\frac{\sqrt{u_0}}{\sqrt{u_0} - \sqrt{x_1}}\right)^2} = \left(\sqrt{u_0} - \sqrt{x_1}\right)^2 \\ &\Longrightarrow \\ x_2(u_0, x_1) &= \left(\sqrt{u_0} - \sqrt{x_1}\right)^2 \end{aligned}$$

d) Can you see how to use this information to very quickly find $u(x_1, x_2)$?

Solution: We can find $u(x_1, x_2)$ by solving the above expression for u_0. We find that

$$u(x_1, x_2) = \left(\sqrt{x_1} + \sqrt{x_2}\right)^2$$

so we have CES utility with $\rho = \frac{1}{2}$.

5) There are two individuals in a household. You observe prices $p = p_1, ..., p_n$, total household income w, and total household demand $x_1^H, ..., x_n^H$ for each good. Only individual A benefits from good 1 and only individual B benefits from good 2, while goods 3 through n are private goods in the sense that each benefits only from the amount the individual consumes of the jointly observed amount of this good.

For instance, they order pizza and A benefits only from the number of slices that A consumed and similarly for B. You get to see how much pizza was ordered, but now how much each ate.

The preferences of A are constant in amount of goods $2, ..., n$ consumed by B, and the preferences of B are constant in the amount of goods $1, 3, ..., n$ consumed by A. We therefore can write their utilities as:

$$u^A(x_1, x_3^A, ..., x_n^A) \quad \text{and} \quad u^B(x_2, x_3^B, ..., x_n^B)$$

Assume that preferences are locally non-satiated and continuous, and you can assume that we are always at an interior solution.

1) Prove that the following two problems are *equivalent*, in the sense that for any function $\bar{u}^B(p, w)$ in part a), there is a function $f(p, w)$ in part b) that gives the same solution, and vice versa.

   a) The household maximizes the utility of person A subject to the constraint that person B obtains at least a certain utility level $\bar{u}^B(p, w)$ and subject to the household budget constraint, where the minimal utility level for B can depend on income and prices, but the function is homogenous degree zero.

   b) The household divides the household income among the two individuals according to some **sharing rule** that can depend on income and prices, but is homogenous degree 1, and individuals then each use their own income to maximize their utility subject to the induced budget constraint (That is, $w^A = f(p, w)$ where $0 \le f(p, w) \le w$ and $w^B = w - f(p, w)$).

   Solution: Consider the following two optimization problems:

$$\begin{aligned} \max_{\{x_i^A, x_i^B\}} \quad & u^A(x_1^A, x_3^A, \ldots, x_n^A) \\ \text{s.t.} \quad & u^B(x_2^B, x_3^B, \ldots, x_n^B) \ge \overline{u}^B(p, w) \\ & p \cdot x^A + p \cdot x^B \le w \end{aligned} \tag{1}$$

   And

$$\begin{aligned} \max_{x_i^A} \quad & u^A(x_1^A, x_3^A, \ldots, x_n^A) \quad \text{s.t. } p \cdot x^A \le f(p, w) \\ \max_{x_i^B} \quad & u^B(x_2^B, x_3^B, \ldots, x_n^B) \quad \text{s.t. } p \cdot x^B \le w - f(p, w) \end{aligned} \tag{2}$$

   First note that the choice variables in the maximization problem of (1) are all distinct, we can re-write (1) as the following:

$$\begin{aligned} \max_{x_i^A} \quad & u^A(x_1^A, x_3^A, \ldots, x_n^A) \quad \text{s.t. } p \cdot x^A \le w - p \cdot x^B \\ \min_{x_i^B} \quad & p \cdot x^B \quad \text{s.t. } u^B(x_2^B, x_3^B, \ldots, x_n^B) \ge \overline{u}^B(p, w) \end{aligned} \tag{1'}$$

   In another word, problem (1) is equivalent to first solving the EMP for person B with minimum utility $\overline{u}^B$, and then give the remaining wealth to person A, who then maximizes his or her utility.

   Next, we can go from (1') to (2) using the Duality result proved in lecture. In particular, the EMP with minimum utility $\overline{u}^B$ is equivalent to the UMP with wealth $e(p, \overline{u}^B)$.

   Furthermore, the sharing rule satisfies $w - f(p, w) = e(p, \overline{u}^B)$. Since the expenditure function is homogeneous of degree 1 in price, we have $f(\alpha p, \alpha w) = \alpha w - e(\alpha p, \overline{u}^B) = \alpha(w - e(p, \overline{u}^B)) = \alpha f(p, w)$. Thus we've verified that $f()$ is homogeneous of degree 1.

2) True or False:

a) The function specifying the minimal utility for person B's utility in part 1a) is increasing in w if and only if the income given to B in part 1b) is increasing in w.
Solution: True. The income given to person B is the expenditure function, which is strictly increasing in the minimal utility level required.

b) The function specifying the minimal utility for person B's utility in part 1a) is linear in w if and only if the income given to B in part 1b) is linear in w.
Solution: False. Utility is a ordinal concept and any monotonic transformation of the utility function will represents the same preference.