14.121 Final Exam: Solutions

Alexander Wolitzky

MIT

1. a. In lecture notes.

b. Quasi-concave utility implies convex preferences is in the lecture notes. For the converse, fix x and y with $u(x) \geq u(y)$. Then $x \succsim y$. Since $y \succsim y$, convexity implies that $\alpha x + (1-\alpha) y \succsim y$. Hence, $u(\alpha x + (1-\alpha) y) \geq u(y)$.

c. Fix $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $u(x) \geq u(y)$. Then either $x_1 x_2 \geq y_1 y_2$ or $x_1 x_2, y_1 y_2 \in (1, 2)$. Hence, either $(\alpha x_1 + (1-\alpha) y_1)(\alpha x_2 + (1-\alpha) y_2) \geq y_1 y_2$ or $(\alpha x_1 + (1-\alpha) y_1)(\alpha x_2 + (1-\alpha) y_2) \in (1, 2)$. In either case, $u(\alpha x + (1-\alpha) y) \geq u(y)$.

d. Suppose v represents $\succsim$. Then $v(\alpha, \alpha)$ must be constant on $\alpha \in [1, \sqrt{2}]$. For example, $v(1,1) = v\left(\frac{4}{3}, \frac{4}{3}\right) = v\left(\sqrt{2}, \sqrt{2}\right)$. On the other hand, it must also be the case that (for example) $v(2,2) > v\left(\sqrt{2}, \sqrt{2}\right)$. Hence, $\frac{2}{3} v(1,1) + \frac{1}{3} v(2,2) > v\left(\frac{4}{3}, \frac{4}{3}\right)$. So v is not concave.

2. a. The easiest way to do this is by setting up the Lagrangian and taking the FOC. The answer is $x_i = \frac{\alpha_i w}{p_i}$.

b. This can be solved the same way. The answer is $h_i = \frac{\alpha_i u}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}$.

c. $\partial x_i / \partial p_j = 0$. The substitution effect is $\partial h_i / \partial p_j = \frac{\alpha_j}{p_j} h_i = \frac{\alpha_i}{p_i} \frac{\alpha_j}{p_j} w$. The income effect is $-(\partial x_i / \partial w) x_j = -\frac{\alpha_i}{p_i} \frac{\alpha_j}{p_j} w$.

d. Substitutes means $\partial h_i / \partial p_j > 0$. Gross substitutes means $\partial x_i / \partial p_j > 0$. With Cobb-Douglas utility, distinct goods are substitutes, but not gross substitutes. (If you defined gross substitutes with a weak inequality and said goods are gross substitutes because $\partial x_i / \partial p_j = 0$, that's also fine.)

3. a. The required assumptions are rationality, continuity, and independence. Definitions

are in the lecture notes.

b. By Jensen's inequality, $E[u(w+x)] \leq u(w+E[x])$, so if $E[x] < 0$ then $E[u(w+x)] < u(w)$.

c. As Ann accepts, her certainty equivalent for the resulting lottery over final wealth levels is at least w. As Bob is less risk-averse than Ann, his certainty equivalent for any lottery is weakly greater than hers. Hence, he also accepts.

d. This amounts to reproving that if G is a mean-preserving spread of F, then F SOSD G. The proof is in the lecture notes.

4. a. For supermodularity, we saw in problem set 3 that if f is supermodular and increasing and h is increasing and convex, then $h \circ f$ is supermodular. Here, f is supermodular and increasing in y and $pz - c(z, \theta)$ is increasing and convex in z, so $pf(y, \theta) - c(f(y, \theta), \theta)$ is supermodular in y. Finally, $-k(y)$ is also supermodular in y, and (again by problem set 3) the sum of supermodular funcations is supermodular, so $\pi(y, z, \theta)$ is supermodular in y.

For increasing differences, $pf(y, \theta)$ has increasing differences because f has increasing differences in (y, θ) and pz has increasing differencs in (p, z); $-c(f(y, \theta), \theta)$ has increasing differences because f has increasing differences in (y, θ) and $-c(z, \theta)$ has increasing differences in (z, θ); and the sum of functions with increasing differences has increasing differences.

b. By Topkis' theorem, $Y^*(p, \theta)$ is increasing in (p, θ) in the strong set order. Hence, $Y^*(p, \theta)$ increases (in the strong set order) if either or both of p and θ increase.

c. By the monotone selection theorem, strictly increasing differences of f or strictly decreasing differences of c is enough.

d. By the Edlin-Shannon theorem, continuous differentiability of f, c, and k and interiority of the solution is enough (assuming strictly increasing differences).