Lecture 5. Arbitrage Pricing

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Outline

Riskless Security

- Suppose that security 1 is riskless (a bond), with a sure payoff of 1 at $t = 1$.
- The riskless security can be traded or implied (by a model of ϕ).
- The riskless payoff is denoted by $D_1 = [1; \ldots ; 1] = 1_M$, where 1_M is the vector of ones. Let P_1 denote its price at $t = 0$.
  - Note on notation: I'm dropping the time subscript, so this refers to a column (wlog, the first one) of the payoff matrix D.
- With state price vector ϕ, we have:

$$P_1 = \phi^\top 1_M = \sum_{\omega=1}^{M} \phi_\omega .$$

- The riskless interest rate, denoted by r_F, is defined as the net payoff from 1 unit of investment in, or the rate of return from, the riskless bond:

$$P_1 (1 + r_F) = 1 \quad \text{or} \quad r_F = \frac{1}{P_1} - 1.$$

Risk-Neutral Measure

- Given ϕ, we have the pricing equation:

$$P_n = \phi^\top D_n = \sum_{\omega \in \Omega} \phi_\omega D_{\omega n}, \quad n = 1, \ldots, N.$$

- Define:

$$q_\omega \equiv \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}}.$$

- Since $q_\omega > 0$ and $\sum q_\omega = 1$,

$$\mathbb{Q} \equiv \{q_\omega,\ \omega \in \Omega\}$$

  can be interpreted as a measure over Ω.

- Since $\mathbb{Q}$ and $\mathbb{P}$ agree on zero measure sets (i.e. have the same support), they are equivalent.

Risk-Neutral Measure

Risk-Neutral Measure and Risk-Neutral Pricing

- We can then rewrite the pricing equation as:

$$P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1+r_F}, \quad n = 1, \ldots, N, \tag{1}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation under $\mathbb{Q}$ and D_n here represents the payoff of security n as a random variable.

- (1) is called the risk-neutral pricing formula.
- $\mathbb{Q}$ is called the risk-neutral measure.
- "Risk-neutral pricing" reflects the analogy that if agents in the market were risk-neutral, they would price all securities by only considering their expected payoffs and the riskless interest rate.
- Usually, the market is not risk-neutral and $\mathbb{Q} \neq \mathbb{P}$. Relative to objective measure, $\mathbb{Q}$ is tilted towards states with higher state prices

Risk-Neutral Measure

Example

Example. (Continued)

Security 1: $1 \begin{cases} 1 \\ 1 \end{cases}$ Security 2: $1/2 \begin{cases} 2 \\ 0 \end{cases}$

- From the prices/payoffs of the two securities, we have

$$\phi_1 + \phi_2 = 1, \quad 2\phi_1 = 1/2.$$

- The state prices are $\phi_1 = 1/4$, $\phi_2 = 3/4$.
- The risk-neutral measure is $\mathbb{Q} = \{1/4, 3/4\}$.
- Using the risk-neutral pricing formula, we have:

$$\frac{\mathbb{E}^{\mathbb{Q}}[D_1]}{1+0} = \frac{(1/4)(1) + (3/4)(1)}{1} = 1,$$
$$\frac{\mathbb{E}^{\mathbb{Q}}[D_2]}{1+0} = \frac{(1/4)(2) + (3/4)(2)}{1} = 1/2,$$

which give the right prices.

State Price Density

- Risk-neutral measure $\mathbb{Q}$ represents "normalized" state prices.
- $\mathbb{P}$ is tied to data. Thus, we would like to relate $\mathbb{Q}$ and $\mathbb{P}$.

Given ϕ, we have

$$P_n = \sum_{\omega} \phi_\omega D_{\omega n} = \sum_{\omega} p_\omega \frac{\phi_\omega}{p_\omega} D_{\omega n}, \;\; n = 1, \ldots, N.$$

Definition (State Price Density)

The state-price density (SPD) for state ω, denoted by η_ω, is defined as:

$$\eta_\omega \equiv \phi_\omega / p_\omega, \quad \forall\, \omega \in \Omega.$$

SPD, strictly positive, gives the "state price per unit of probability." Then,

$$P_n = \sum_{\omega} p_\omega \eta_\omega D_{\omega n} = \mathbb{E}^{\mathbb{P}}\left[\eta D_n\right], \;\; n = 1, \ldots, N. \tag{2}$$

Other names: stochastic discount factor, pricing kernel

Representation Theorem

Theorem (Representation Theorem)

The following are equivalent:

1. *Existence of a positive pricing operator (V);*
2. *Existence of a risk-neutral measure ($\mathbb{Q}$) and a riskless interest rate (r_F);*
3. *The existence of a strictly positive state price density (η).*

With a discrete number of states, the proof is immediate

Discounted Cash Flow/Present Value Formula Again

- The DCF/PV formula is usually expressed in the following form:

$$P_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{1+\bar{r}_n} = \frac{\sum_\omega p_\omega D_{\omega n}}{1+\bar{r}_n}, \quad i = 1, \ldots, N, \tag{3}$$

where $\bar{r}_n$ is called the discount rate for payoff D_n.

- Putting FTAP together with defn of $\mathbb{Q}$, we have:

$$\bar{r}_n = \frac{\sum_\omega p_\omega D_{\omega n}}{\sum_\omega \phi_\omega D_{\omega n}} - 1 = (1+r_f)\frac{E^{\mathbb{P}}[D_n]}{E^{\mathbb{Q}}[D_n]} - 1, \quad n = 1, \ldots, N.$$

- We can also rewrite the above as:

$$\bar{r}_n = \frac{\sum_\omega p_\omega D_{\omega n}}{P_n} - 1, \quad n = 1, \ldots, N.$$

Thus, $\bar{r}_n$ is also the expected rate of return on security n.

- Remember: all of this extends naturally to a continuum of states!

Time Value of Money

- The DCF for the riskless bond gives:

$$P_1 = \sum_{\omega} \phi_{\omega} = \frac{\sum_{\omega} q_{\omega} \times 1}{1 + r_F} = \frac{\sum_{\omega} p_{\omega} \times 1}{1 + r_F} = \frac{1}{1 + r_F}$$

$$\text{or} \quad \bar{r}_1 = \frac{\mathbb{E}^{\mathbb{P}}[D_1]}{P_1} - 1 = \frac{1}{P_1} - 1 = r_F.$$

  The return on the riskless security is also called the time value of money.

- The return on the riskless bond is certain:

$$r_{1\omega} = \frac{1}{P_1} - 1 = r_F, \ \forall\, \omega \in \Omega.$$

- Note the tight link between r_F and the SPD

$$\frac{1}{1 + r_F} = E^{\mathbb{P}}[\eta] \qquad \Longleftrightarrow \qquad 1 + r_F = \frac{1}{E^{\mathbb{P}}[\eta]}$$

Risk Premium

- The return on a risky security n $(n > 1)$ is in general uncertain:

$$r_{\omega n} = \frac{D_{\omega n}}{P_n} - 1, \quad \omega \in \Omega.$$

- Its expected value in excess of the riskless interest rate:

$$\pi_n \equiv \bar{r}_n - r_F$$

  is called the risk premium.

- From our earlier result:

$$\bar{r}_n = \frac{\sum_\omega p_\omega D_{\omega n}}{\sum_\omega \phi_\omega D_{\omega n}} - 1 = (1 + r_f)\frac{E^{\mathbb{P}}[D_n]}{E^{\mathbb{Q}}[D_n]} - 1, \quad n = 1, \ldots, N.$$

  the risk premium will be positive when $E^{\mathbb{Q}}[D_n] < E^{\mathbb{P}}[D_n]$

  - What does this mean in words? Give some intuition

Time Value of Money and Risk Premium

Example

Example. The binomial model with equal probability for the two states.

- There are with two traded securities:

Security 1: $1 \begin{cases} 1 \\ 1 \end{cases}$ Security 2: $1/2 \begin{cases} 2 \\ 0 \end{cases}$

- Security 1 is a riskless bond. Its price and payoff imply that $r_F = 0$.
- Security 2 has expected payoff of $(1/2)(2) + (1/2)(0) = 1$, same as security 1. Yet, it is price is 1/2, only half of the price of security. The lower price is to compensate for risk.
- The expected rate return on security 2 is:

$$\bar{r} = \frac{\mathbb{E}^{\mathbb{P}}[D]}{P} - 1 = 2 - 1 = 1 = 100\%.$$

- Since $r_F = 0$, security 2 carries a risk premium of 100%.

The Risk Premium and the SPD

- Next, we'll derive a nice expression for the risk premium of any asset.
- Note on notation: let r_n denote the return realization, i.e., the return for the specific ω which realizes in the data. Useful when computing expectations.
- Divide both sides of the valuation formula with the state-price density by price (or recall that the price of any gross return is 1) to get:

$$1 = \mathbb{E}^{\mathbb{P}}[\eta(1 + r_n)]$$

- Rearranging slightly (see board), we get

$$\mathbb{E}^{\mathbb{P}}[r_n - r_f] = -(1 + r_f)Cov^{\mathbb{P}}[\eta, r_n - r_f]$$

Irrelevance of Idiosyncratic risk

- Recall our risk premium expression:

$$\mathbb{E}^{\mathbb{P}}[r_n - r_f] = -(1 + r_f)Cov^{\mathbb{P}}[\eta, r_n - r_f]$$

- Consider the price of an asset whose risky payoff D_n (and hence also its return r_n) is uncorrelated with the discount factor η.
  - Formula above $\Rightarrow$ this asset earns same return as the risk-free asset
  - Cash flows are simply discounted at the risk-free rate: $P_n = \frac{E[D_n]}{1+r_f}$
- This suggests a simple decomposition:

$$D_n = \underbrace{\text{proj}(D_n|\eta)}_{\substack{\text{"priced"} \\ \text{cash flow} \\ \text{component}}} + \underbrace{\epsilon_n}_{\substack{\text{"unpriced"} \\ \text{cash flow} \\ \text{component}}}$$

where $E[\epsilon_n] = 0$ and $\epsilon_n \perp \eta$. So, $\text{proj}(D_j, \eta) = \text{proj}(D_k, \eta) \iff P_j = P_k$

- There is no compensation for holding mean 0 risks orthogonal to the SPD

Redefining risk

$$\mathbb{E}^{\mathbb{P}}[r_n - r_f] = -(1+r_f)Cov^{\mathbb{P}}[\eta, r_n - r_f] = \frac{Cov^{\mathbb{P}}[-\eta, r_n - r_f]}{E^{\mathbb{P}}[\eta]}$$

- Prior classes / simple intuition (formalized later): people are risk averse, dislike risky gambles
- Might have expected an asset's variance to affect risk premium
  - Variance is necessary but not sufficient!

- Here: only **covariance** with the SDF matters
- Why the difference?
  - Investors can put an infinitesimal amount into a single asset
  - Can combine multiple assets into a single portfolio $\rightarrow$ asset-specific idiosyncratic risk can be diversified away

Intuition: Where does the risk premium come from?

- Central element of any macro-finance models is its specification of the state price density / stochastic discount factor η

$$\text{risk premium} = -\text{risk free rate} \times \text{comovement of return with SDF}$$

- Useful heuristic: "stochastic discount factor" $\approx$ "pain index"

  $\Rightarrow$The SDF is low in good macroeconomic states, high in bad states

- Intuition: assets that perform well in bad times earn lower returns
- Theory: preferences + fundamentals (e.g., cons/wealth) $\rightarrow$ SDF
- We'll learn more about specific models in the second half of the course. Each emphasizes a different aspect of "bad times"

Hansen-Jagannathan Bounds

- Hansen and Jagannathan had the brilliant insight that observed moments of asset prices place strong restrictions on moments of the SDF
- The Sharpe-ratio of any risky asset provides a lower bound on the coefficient of variation of the state price density.
- Specifically, they showed (see board):

$$\frac{\sqrt{Var^{\mathbb{P}}[\eta]}}{\mathbb{E}^{\mathbb{P}}[\eta]} \geq \frac{\mathbb{E}^{\mathbb{P}}[r_n - r_F]}{\sqrt{Var^{\mathbb{P}}[r_n - r_F]}} \equiv \left[\begin{array}{c} \text{Sharpe ratio} \\ \text{of asset n} \end{array} \right]$$

Hansen-Jagannathan Bounds - Geometric interpretation

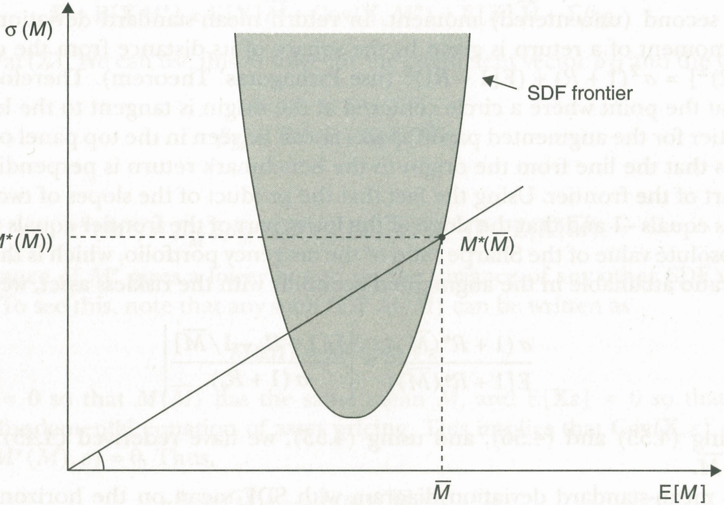

Figure 4.1. *Geometry of the Hansen and Jagannathan (1991) SDF Volatility Bound*

Entropy Bounds

Take a strictly positive random variable X. We define the entropy of X as:

$$L^{\mathbb{P}}(X) = \log \mathbb{E}^{\mathbb{P}}[X] - \mathbb{E}^{\mathbb{P}}[\log X] \geq 0$$

We can this definition to derive an alternative bound on the entropy of the state price density (see board) – useful with non-Gaussian shocks

$$L^{\mathbb{P}}(\eta(1+r_n)) = \log \underbrace{\mathbb{E}^{\mathbb{P}}[\eta(1+r_n)]}_{=1} - \mathbb{E}^{\mathbb{P}}[\log \eta(1+r_n)] \geq 0$$

$$\iff -\mathbb{E}^{\mathbb{P}}[\log \eta] \geq \mathbb{E}^{\mathbb{P}}[\log(1+r_n)]$$

Next, note that

$$L^{\mathbb{P}}(\eta) = \log \mathbb{E}^{\mathbb{P}}[\eta] - \mathbb{E}^{\mathbb{P}}[\log \eta] \quad \text{and} \quad \log \mathbb{E}^{\mathbb{P}}[\eta] = -\log(1+r_F)$$

Thus, it follows that

$$L^{\mathbb{P}}(\eta) \geq \mathbb{E}^{\mathbb{P}}[\log(1+r_n)] - \log(1+r_F)$$

So what?

- This may sound abstract, but it has powerful implications for our models!
- Why? We can find assets with annualized Sharpe ratios of 0.5, 0.8, or even higher $\Rightarrow$ volatility of SDF has to be VERY high
- Remember: any theory model will generate an SDF η
- Many models generate $\sigma(\eta)$ and $L^{\mathbb{P}}(\eta)$ substantially lower than the bound
- All models are wrong, but this is a VERY substantial failure...
  - State prices & welfare are tightly linked: see Alvarez & Jermann (2004-5)
  - Incorrect/unrealistic state prices potentially imply a nontrivial disconnect between the actual optimization problem being solved by agents in the real world vs those in the model.
  - Financial markets suggest that there are substantial tradeoffs across states, whereas many models might predict these tradeoffs are fairly trivial.
  - Incorrect objective functions can $\Rightarrow$ incorrect decision rules
- We can derive other model-free insights in dynamic settings

The law of iterated expectations

- Suppose that I want to calculate the expected value of a random variable X given two sets of conditioning information $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1 \subset \mathcal{F}_2$:
  - $\mathcal{F}_1$: Trump wins election
  - $\mathcal{F}_2$: Trump wins election and announces tariffs
- The law of iterated expectations says that

$$\mathbb{E}[X|\mathcal{F}_1] = \mathbb{E}[\ \ \mathbb{E}[X|\mathcal{F}_2]\ \ |\mathcal{F}_1]$$

- The best forecast one can make with limited information is the forecast of the forecast one would make with better information

FTAP $\rightarrow$ Fundamental value of a stock

- Recall the one period stock payoffs:
  - Small **dividend** (cash flow) D_{t+1} paid
  - Also receive share price P_{t+1}
  - $X_{t+1} = D_{t+1} + P_{t+1}$ and $R_{t+1} = \frac{D_{t+1}+P_{t+1}}{P_t}$
- We will suppose that no arbitrage holds period-by-period $\Rightarrow$ SDF η_{t+k} exists for all periods
- Can derive the present value of future stock payoffs:

$$
\begin{aligned}
P_t &= \mathbb{E}_t[\eta_{t+1}(P_{t+1} + D_{t+1})] \\
\text{Also, } P_{t+1} &= \mathbb{E}_{t+1}[\eta_{t+2}(P_{t+2} + D_{t+2})] \\
\Rightarrow P_t &= \mathbb{E}_t\left[\sum_{k=1}^{T}(\eta_{t+1}\cdots\eta_{t+k})D_{t+k}\right] + \underbrace{\mathbb{E}_t\left[\eta_{t+1}\cdots\eta_{t+T}P_{t+T}\right]}_{\text{Usually assume}\rightarrow 0 \text{ as } T\rightarrow\infty} \\
&= \mathbb{E}_t\sum_{j=1}^{\infty}\left[\prod_{k=1}^{j}\eta_{t+k}\right]D_{t+j} \equiv \mathbb{E}_t\sum_{j=1}^{\infty}\eta_{t:t+j}D_{t+j}
\end{aligned}
$$

General DCF representation

- Next, we derive a general DCF representation
- What is the value of a single dividend k periods in the future?
  - Dividend strip price: $P_t^{(k)} = \mathbb{E}_t[\eta_{t:t+k} D_{t+k}]$
  - Expected payoff if I wait k periods: $\mathbb{E}_t[D_{t+k}]$
  - $\mathbb{E}[\text{k period return}] = \dfrac{\text{expected payoff}}{\text{price}} = \dfrac{\mathbb{E}_t[D_{t+k}]}{\mathbb{E}_t[\eta_{t:t+k} D_{t+k}]} \equiv \mathbb{E}_t[R_{t:t+k}^{(k)}]$
  - $\Rightarrow \mathbb{E}_t[\eta_{t:t+k} D_{t+k}] = \dfrac{\mathbb{E}_t[D_{t+k}]}{\mathbb{E}_t[R_{t:t+k}^{(k)}]} = \dfrac{\text{expected cash flow}}{\text{horizon/div-specific expected return}}$
- Same logic applies to multiple dividends $\Rightarrow$

$$P_t = \mathbb{E}_t \sum_{j=1}^{\infty} \eta_{t:t+j} D_{t+j} = \sum_{j=1}^{\infty} \frac{\mathbb{E}_t[D_{t+j}]}{\mathbb{E}_t[R_{t:t+j}^{(j)}]} = \text{sum of discounted cash flows!}$$

- Can define firm objective function / NPV accordingly

Appendix: The Risk Premium (lognormal case)

- Often, models are highly tractable if we assume that variables such as $\log \eta$ and $\log(1+r_n)$ are jointly multivariate normally-distributed
- In this case, we can derive a similar expression for the risk premium.
- Using the formula for the expectation of log-normal random variable:

$$\begin{aligned} 0 &= \mathbb{E}^{\mathbb{P}}[\log \eta] + \mathbb{E}^{\mathbb{P}}[\log(1+r_n)] \\ &+ \frac{1}{2} Var^{\mathbb{P}}[\log \eta] + \frac{1}{2} Var^{\mathbb{P}}[\log(1+r_n)] + Cov^{\mathbb{P}}[\log \eta, \log(1+r_n)] \end{aligned}$$

- Applying this formula to the risk-free asset $(1 + r_F = 1/E[\eta])$ and differencing yields the following expression for the risk premium in logs,

$$\mathbb{E}^{\mathbb{P}}[\log(1+r_n)] - \log(1+r_F) + \frac{1}{2} Var^{\mathbb{P}}[\log(1+r_n)] = -Cov^{\mathbb{P}}[\log \eta, \log(1+r_n)]$$

which is very similar to the expression in levels except that we have to apply a correction arising from Jensen's inequality.