14.380: Statistics

Problem Set 2

due Tuesday, September 28, 2019

You should hand the solution for problem 4 to the TA. Problems 1-3 are for practice.

1. Let $X_1, X_2, \ldots, X_n$ be iid observations. Find minimal sufficient statistics

   (a) $f(x \mid \theta) = \frac{2x}{\theta^2},\ 0 < x < \theta,\ \theta > 0;$

   (b) $f(x \mid \theta) = e^{-(x-\theta)} \cdot \exp\left\{-e^{-(x-\theta)}\right\},\ -\infty < x < \infty,\ -\infty < \theta < \infty;$

   (c) $f(x \mid \theta) = \frac{2!}{x!(2-x)!}\theta^x(1-\theta)^{2-x},\ x \in \{0,1,2\},\ 0 \le \theta \le 1.$

2. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ

$$P\{X = j\} = \frac{e^{-\lambda}\lambda^j}{j!} \qquad j = 0, 1, \ldots$$

   (a) Find a minimal sufficient statistic.

   (b) Assume that we are interested in estimating probability of a count of zero $\theta = P\{X = 0\} = \exp\{-\lambda\}$. Find an unbiased estimator of θ. *Hint:* $\theta = P\{X = 0\} = E\mathbb{I}\{X = 0\}$.

   (c) Is the estimator in (b) a function of a minimal sufficient statistic? Modify the estimator to make sure it is a function of a minimal sufficient statistic, while remaining unbiased. You may try to do analytical derivation (it can be done here). However, if it is too hard, then explain a Monte-Carlo procedure that you may use instead.

   (d) (Computer experiment) Implement the Monte-Carlo procedure hinted in (c). In particular assume that you have a sample of size $n = 100$, you calculate the statistic $Y = \frac{1}{n}\sum_{i=1}^n X_i$, and then report a new estimator $\widehat{\theta}$

for θ, which is unbiased and depends on the data only through Y. That is, $\widehat{\theta} = g(Y)$. Draw the graph of the function $g(\cdot)$ over a reasonable range (say $Y \in [0.5, 1.5]$), where you calculate the function $g(\cdot)$ by a proper Monte-Carlo procedure (note that you may need a large number of draws to ensure you can compute the function accurately at each Y).

3. Assume $X_1, \ldots, X_n$ are iid with mean μ and variance σ^2 (both unknown). Let us estimate the mean by

$$\hat{\mu} = \sum_{i=1}^{n} \omega_i X_i$$

   (i) Under what condition is $\hat{\mu}$ unbiased?

   (ii) Among all unbiased $\hat{\mu}$ find the one with the smallest variance.

   (iii) What $\{\omega_i\}$ whould lead to the smallest MSE?

4. *(Required problem)* Suppose that the random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + e_i, i = 1, ..., n,$$

where $x_1, ..., x_n$ are fixed constants and $e_1, ..., e_n$ are i.i.d. normals with mean 0 and variance σ^2 (variance is unknown).

   (a) Find a two-dimensional sufficient statistic for (β, σ^2).

   (b) Find the MLE of β and show that it is unbiased.

   (c) Find the distribution of the MLE of β.

   (d) Is $\hat{\beta}_1 = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i}$ an unbiased estimator for β? Find its variance.

   (e) Is $\hat{\beta}_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{Y_i}{x_i}$ an unbiased estimator for β? Find its variance.

   (f) Which of the three estimator $\hat{\beta}_{MLE}, \hat{\beta}_1$ and $\hat{\beta}_2$ has the smallest variance? *Hint: you may need the following inequalities. For any numbers* $a_1, .., a_n$ *we have*

$$(\sum_i a_i)^2 \leq n \sum_i a_i^2 \quad \textit{and} \quad \left(\frac{1}{n}\sum_i \frac{1}{a_i^2}\right)^{-1} \leq \frac{1}{n}\sum_i a_i^2$$