14.121, Fall 2018
Problem Set 4 Solutions

1) *Note: there was a mistake in the question. The elasticity of substitution should be defined as*

$$\sigma = -\frac{\partial \log(K/L)}{\partial \log(F_K/F_L)}.$$

I have corrected this mistake in the wording of the question below.

The elasticity of substitution tells us the percentage change in the input ratio per percentage change in the marginal rate of substitution. A cost minimizing firm always sets

$$w/r = F_L/F_K = MRS,$$

where L is labor and K is capital, and F_L and F_K are the respective partial derivatives. We can therefore define the elasticity of substitution as:

$$\sigma = -\frac{\partial \log(K/L)}{\partial \log(F_K/F_L)}.$$

(a) Show that this expression is equivalent to

$$\sigma = \frac{F_K F_L}{F F_{LK}}.$$

Solution: Though the question did not make this clear, we have to assume that F has constant returns to scale, i.e. is homogeneous of degree 1.

We have

$$\begin{aligned}\sigma &= -\frac{\partial \log(K/L)}{\partial \log(F_K/F_L)} \\ &= -\frac{L}{K}\frac{F_K}{F_L}\frac{\partial(K/L)}{\partial(F_K/F_L)} \\ &= -\frac{L}{K}\frac{F_K}{F_L}\left(\frac{\partial(F_K/F_L)}{\partial(K/L)}\right)^{-1}\end{aligned}$$

Define $k := \frac{K}{L}$. F is homogeneous of degree one so its first derivatives are homogeneous of degree zero. Therefore, dividing through by L and K respectively, we can write

$$F_K(K,L) = F_K(1,k^{-1})$$

$$F_L(K,L) = F_L(k,1)$$

Therefore

$$\begin{aligned}\frac{\partial(F_K/F_L)}{\partial(K/L)} &= \frac{d}{dk}\frac{F_K(1,k^{-1})}{F_L(k,1)} \\ &= -\frac{F_{KL}(1,k^{-1})}{F_L}\frac{1}{k^2} - \frac{F_K}{F_L^2}F_{KL}(k,1)\end{aligned}$$

If F's first derivates are homogeneous degree zero then its second derivates are homogeneous degree -1. Hence

$$F_{KL}(1, k^{-1}) = KF_{KL}(K, L)$$
$$F_{KL}(k, 1) = LF_{KL}(K, L)$$

So

$$\begin{aligned}\frac{\partial(F_K/F_L)}{\partial(K/L)} &= -\frac{F_{KL}K}{F_L}\frac{L^2}{K^2} - \frac{F_{KL}LF_K}{F_L^2} \\ &= -\frac{LF_{KL}}{F_L}\left(\frac{L}{K} + \frac{F_K}{F_L}\right)\end{aligned}$$

Substitute this back into our expression for σ:

$$\begin{aligned}\sigma &= \frac{L}{K}\frac{F_K}{F_L}\frac{F_L}{LF_{KL}}\left(\frac{L}{K} + \frac{F_K}{F_L}\right)^{-1} \\ &= \frac{F_K}{F_{KL}}\frac{1}{K}\left(\frac{F_L L + F_K K}{KF_L}\right)^{-1} \\ &= \frac{F_K F_L}{FF_{KL}}\end{aligned}$$

where the last equality uses Euler's theorem (F h.o.d. $1 \Rightarrow F_L L + F_K K = F$).

(b) Let C be the cost function for the firm. Show that

$$\sigma = \frac{C_{12}C}{C_1 C_2}.$$

Solution: Using the firm's first order conditions, we can write

$$\begin{aligned}\sigma &= -\frac{\partial \log(K/L)}{\partial \log(F_K/F_L)} \\ &= -\frac{\partial \log(K^c/L^c)}{\partial \log(r/w)}\end{aligned}$$

where K^c and L^c are the firm's factor demands. By Shepard's lemma these are the first derivatives of the cost function. So let's write

$$\begin{aligned}\sigma &= -\frac{L}{K}\frac{r}{w}\frac{\partial(K^c/L^c)}{\partial(r/w)} \\ &= -\frac{L}{K}\frac{r}{w}\frac{\partial(C_1/C_2)}{\partial(r/w)}\end{aligned}$$

Then we can proceed similarly to part (a). The cost function is homogeneous degree one in prices, so its derivatives are homogeneous degree zero. Let $R := \frac{r}{w}$. Then

$$\begin{aligned}\frac{\partial(C_1/C_2)}{\partial(r/w)} &= \frac{\partial(C_1(1, R^{-1})/C_2(R, 1))}{\partial(r/w)} \\ &= -\frac{C_{12}(1, R^{-1})}{C_2(R, 1)R^2} - \frac{C_1 C_{12}(R, 1)}{C_2^2} \\ &= -\frac{C_{12}w}{C_2}\left(\frac{w}{r} + \frac{C_1}{C_2}\right)\end{aligned}$$

Substituting this back into the expression for σ, we can cancel terms and then simplify using Euler's theorem, just as in part (a). This yields

$$\sigma = \frac{CC_{12}}{C_1 C_2}$$

(c) Are these three definitions equivalent when there are more than two factors of production?

Solution: No. As a counterexample consider the production function

$$F(K, L, Z) = K^\alpha L^{1-\alpha} + Z$$

This is homogeneous of degree one. Yet we can calculate that

$$-\frac{\partial \log(K/L)}{\partial \log(F_K/F_L)} = 1$$

while

$$\frac{F_K F_L}{F F_{KL}} = \frac{K^\alpha L^{1-\alpha}}{K^\alpha L^{1-\alpha} + Z}$$

As for the cost function expression, note that if the relative price of Z is low enough, the firm will demand only Z and zero of K and L. So the expression

$$\frac{CC_{12}}{C_1 C_2}$$

will not even be well-defined.

2) Strict monotonicity can be important for some economic analyses, in particular in information economics and contract theory. Suppose that f satisfies strictly increasing differences. That is, suppose that for all $x'' > x'$, $f(x'', \theta) - f(x', \theta)$ is strictly increasing in θ. Let $X^*(\theta) = \arg\max_{x \in \mathbb{R}} f(x, \theta) + g(x)$ be nonempty for each θ.

(a) Show that for $\theta'' > \theta$ if $z \in X^*(\theta')$ and $y \in X^*(\theta'')$, then $y \geq z$.

Solution: Using the notation in the problem, we will proceed by contradiction. Suppose that $y < z$. Since y is a maximizer at θ'' and z a maximizer at θ' we have

$$f(y, \theta'') + g(y) \geq f(z, \theta'') + g(z)$$

$$f(z, \theta') + g(z) \geq f(y, \theta') + g(y)$$

Combining the above inequalities we see that $f(z, \theta'') - f(y, \theta'') \leq g(y) - g(z) \leq f(z, \theta') - f(y, \theta')$. This, however, contradicts our assumption of strictly increasing differences: $f(z, \theta'') - f(y, \theta'') > f(z, \theta') - f(y, \theta')$. Thus, $y \geq z$.

(b) Now suppose that f is differentiable in x, so that $\frac{\partial}{\partial x} f(x, \theta)$ is strictly increasing θ, and suppose g is differentiable too. Show that in part (a), $y > z$. Can you draw a picture of $f(x, \theta')$ and $f(x, \theta'')$ as functions of x that show how this result fails if f is not differentiable in x at z.

Solution: With the differentiability assumptions, we know that since y is a maximizer at θ'' and z a maximizer at θ' we have

$$\frac{\partial}{\partial x} f(y, \theta'') + g'(y) = 0$$

$$\frac{\partial}{\partial x} f(z, \theta') + g'(z) = 0$$

Moreover, we know that $\frac{\partial}{\partial x} f(z,\theta'') > \frac{\partial}{\partial x} f(z,\theta')$ by assumption. From part (a) we know that $y \geq z$. Suppose that $y = z$. Then combining the first order conditions gives $\frac{\partial}{\partial x} f(z,\theta'') = \frac{\partial}{\partial x} f(z,\theta')$, a contradiction. Hence, $y > z$.

To see why the differentiabilty assumption is needed, let

$$f(x,\theta') = \begin{cases} x & \text{if } x \leq 1 \\ 3-2x & \text{if } x > 1 \end{cases} \quad \text{and } f(x,\theta'') = \begin{cases} 2x-1 & \text{if } x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Note that f has strictly increasing differences, however f is not differentiable in x at $x = 1$. It is clear that $X^*(\theta') = X^*(\theta'') = \{1\}$, so we do not have the result that the maximizer is strictly increasing in θ when the differentiability assumption is not met.

3) This question applies the results of the last one. Suppose that a worker's cost of going to school is $c(x,\theta)$, where x is the amount of school and θ is the worker's ability/productivity.

(a) Suppose that c is differentiable and $\frac{\partial^2}{\partial x \partial \theta} c(x,\theta) \leq 0$. Interpret this condition in words in the context of the model.

Solution: This condition states that the marginal cost of an extra unit of school is lower for higher ability workers

(b) Now suppose that firms can observe education but not ability, and thus offer wages $w(x)$ which depend only on education. The worker's utility is given by $w(x) - c(x,\theta)$. Is there any wage function which will induce higher ability workers to choose lower levels of education?

Solution: Let $x^*(\theta) = \sup \arg\max_x w(x) - c(x,\theta)$. Then since $-c(x,\theta)$ has increasing differences we know that $x^*(\theta)$ is weakly increasing in θ. Thus, for any wage function, the optimal choice of education is weakly increasing in ability, so we cannot induce higher ability workers to choose lower levels of education.

Note that if we had considered $X^*(\theta) = \arg\max_x w(x) - c(x,\theta)$, then it follows by Theorem 2.3 in Athey, Milgrom, and Roberts that $X^*(\theta)$ is nondecreasing in θ for all wage functions. The notion of set order used in this result implies that a higher ability individual could choose a lower level of education than a lower ability worker, but in that case the education level chosen by the lower ability worker would also be a maximizer for the higher ability worker as well. In summary, if we allow for multiple optimal education choices, we may observe the higher ability type choose a lower education level than the lower ability type. However, if this is the case, there exists another optimal education choice for the high type that is at least as large as that chosen by the low type.

(c) Now suppose that $\frac{\partial^2}{\partial x \partial \theta} c(x,\theta) < 0$. Based on your answers above, what is a sufficient condition on $w(x)$ such that two workers of different abilities choose different levels of education?

Solution: In this case we have that $-c(x,\theta)$ has strictly increasing differences and is differentiable, so if $w(x)$ is differentiable, then that is sufficient by 1b to guarantee that a higher ability worker will choose a strictly larger level of education.

(d) Conclude from this that even if education is unproductive, firms may be willing to pay higher wages for higher levels of education.

Solution: The above result suggests that education attainment can provide a signal of unobservable ability. Thus, even if education is unproductive, its ability to reveal underlying ability allows firms to identify higher ability workers and subsequently pay them more. In game theory you will see a more complete model of this idea, taking into account the strategic decision workers will make, knowing that their education level will lead to different wages.

4) Prove that $F : X \to \mathbb{R}$ on a product set $X_1 \times ... \times X_n \subset \mathbb{R}^n$ is supermodular if and only if it has increasing differences in (x_i, x_j) for all $i \neq j$ holding the other variables x_{-ij} fixed.

Solution:

Supermodularity $\to$ ID: Take any x_i,$y_i \in X_i$, x_j,$y_j \in X_j$ with $x_i > y_i$ and $y_j > x_j$ and $x_{-ij} = y_{-ij} \in X_{-ij} = \prod_{l \neq i,j} X_l$, and write the supermodular inequality for (x_i, x_j, x_{-ij}) and (y_i, y_j, y_{-ij}):

$$F(x_i, y_j, x_{-ij}) + F(y_i, x_j, y_{-ij}) \geq F(x_i, x_j, x_{-ij}) + F(y_i, y_j, y_{-ij})$$

ID $\to$ Supermodularity : Let $m = x \wedge y$, $M = x \vee y$. Then we can write

$$\begin{aligned}
F(M) - F(x) &= \sum_{i=1}^{n} \left[F(M_1, ..., M_i, x_{i+1}, ..., x_n) - F(M_1, ..., M_{i-1}, x_i, ..., x_n) \right] \\
&\geq \sum_{i=1}^{n} \left[F(y_1, ..., y_{i-1}, M_i, m_{i+1}, ..., m_n) - F(y_1, ..., y_{i-1}, x_i, m_{i+1}, ..., m_n) \right] \\
&= \sum_{i=1}^{n} \left[F(y_1, ..., y_{i-1}, y_i, m_{i+1}, ..., m_n) - F(y_1, ..., y_{i-1}, m_i, m_{i+1}, ..., m_n) \right] \\
&= F(y) - F(m).
\end{aligned}$$

The inequality is by repeated application of ID, since $M_j \geq y_j$ and $x_j \geq m_j$ for all j. For the second equality, note that for each i, either $M_i = y_i$ and $m_i = x_i$, in which case the equality trivially holds, or $M_i = x_i$ and $m_i = y_i$, in which case both differences equal zero.