1) The elasticity of substitution tells us the percentage change in the input ratio per percentage change in the marginal rate of substitution. A cost minimizing firm always sets:

$$w/r = F_L/F_K = MRS,$$

where L is labor and K is capital, and F_L and F_K are the respective partial derivatives. We can therefore define the elasticity of substitution as:

$$\sigma = \frac{\partial \log(K/L)}{\partial \log(F_K/F_L)}.$$

(a) Show that this expression is equivalent to

$$\sigma = \frac{F_K F_L}{F F_{LK}}.$$

(b) Let C be the cost function for the firm. Show that

$$\sigma = \frac{C_{12} C}{C_1 C_2}.$$

(c) Are these three definitions equivalent when there are more than two factors of production?

2) Strict monotonicity can be important for some economic analyses, in particular in information economics and contract theory. Suppose that f satisfies strictly increasing differences. That is, suppose that for all $x'' > x'$, $f(x'', \theta) - f(x', \theta)$ is strictly increasing in θ. Let $X^*(\theta) = \arg\max_{x\in\mathbb{R}} f(x,\theta) + g(x)$ be nonempty for each θ.

(a) Show that for $\theta'' > \theta$ if $z \in X^*(\theta')$ and $y \in X^*(\theta'')$, then $y \geq z$.

(b) Now suppose that f is differentiable in x, so that $\frac{\partial}{\partial x} f(x,\theta)$ is strictly increasing θ, and suppose g is differentiable too. Show that in part (a), $y > z$. Can you draw a picture of $f(x,\theta')$ and $f(x,\theta'')$ as functions of x that show how this result fails if f is not differentiable in x at z.

3) This question applies the results of the last one. Suppose that a worker's cost of going to school is $c(x,\theta)$, where x is the amount of school and θ is the worker's ability/productivity.

(a) Suppose that c is differentiable and $\frac{\partial^2}{\partial x \partial \theta} c(x,\theta) \leq 0$. Interpret this condition in words in the context of the model.

(b) Now suppose that firms can observe education but not ability, and thus offer wages $w(x)$ which depend only on education. The worker's utility is given by $w(x) - c(x,\theta)$. Is there any wage function which will induce higher ability workers to choose lower levels of education?

(c) Now suppose that $\frac{\partial^2}{\partial x \partial \theta} c(x,\theta) < 0$. Based on your answers above, what is a sufficient condition on $w(x)$ such that two workers of different abilities choose different levels of education?

(d) Conclude from this that even if education is unproductive, firms may be willing to pay higher wages for higher levels of education.

4) Prove that $F : X \to \mathbb{R}$ on a product set $X_1 \times ... \times X_n \subset \mathbb{R}^n$ is supermodular if and only if it has increasing differences in (x_i, x_j) for all $i \neq j$ holding the other variables x_{-ij} fixed.