Proof Debreu's Theorem

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1 Statement of Theorem

We actually prove a special case of the theorem, the version stated as Proposition 3.C.1 in MWG:

Any rational, continuous and monotone preference relation $\succeq$ on a set $X = R_+^L$ can be represented by a continuous utility function $u : X \mapsto \mathbb{R}_+$.

It helps to write out the assumptions and their definitions, so we can refer back to them:

- **Rational:** complete and transitive.
- **Continuous preference relation:** For any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that $\lim x_n = x$, $\lim y_n = y$, and $x_n \succeq y_n \ \forall\ N$, we have $x \succeq y$.
- **Monotone preference relation:** If $y >> x$, then $y \succ x$.

For this proof it is also helpful to recall that a continuous function $f(x)$ is one where $\lim_{n\to\infty} f(x_n) = f(x)$ for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} = x$, .

2 Proof

The proof is constructive: we create a utility function $u : X \to \mathbb{R}_+$ and show that it represents preferences $\succeq$ and is continuous. The proof proceeds in four steps.

Step 1: We show that $\forall x$, there exists a scalar $u(x) \in \mathbb{R}_+$ s.t. $u(x)e \sim x$, where e is the $L \times 1$ vector of ones.

Proof: To show this, we need only to establish that for any x the sets

$$U^+ = \{u \in \mathbb{R}_+ : ue \succeq x\}$$

$$U^- = \{u \in \mathbb{R}_+ : x \succeq ue\}$$

overlap, i.e. $U^+ \cap U^- \neq \emptyset$.

Firstly, note that by monotonicity, there exist $\underline{u}, \bar{u}$ such that $\bar{u}e \succ x \succ \underline{u}e$[1]. Hence, U^+ and U^- are both nonempty. Second, continuity of preferences implies that U^+ and U^- contain all their limit points, and are therefore closed. Finally, preferences are complete, so $U^+ \cup U^- = \mathbb{R}_+$. But then, it is a fact that any two nonempty closed sets which cover $\mathbb{R}_+$ must have a nonempty intersection[2].

Step 2: $u(x)$ is unique.

Proof: Take any $\tilde{u} \neq u(x)$, then either $\tilde{u} > u(x)$ or $\tilde{u} < u(x)$. In the former case, then by monotonicity we have $\tilde{u}e \succ u(x)e \sim x$ and hence by transitivity $\tilde{u}e \succ x$. In the latter case we similarly have $x \sim u(x)e \succ \tilde{u}e$ and hence $x \succ \tilde{u}e$.

Step 3: We show that $u(x)$ represents x.

Proof: We need to show that for any $x, y \in X$, $u(x) \geq u(y) \Leftrightarrow x \succeq y$.

$\Rightarrow$: If $u(x) \geq u(y)$, we have two cases:

- $u(x) > u(y)$, in which case by monotonicity $u(x)e \succ u(y)e$, and then by transitivity $x \succ y$
- $u(x) = u(y)$, in which case by reflexivity $u(x)e \sim u(y)e$, and then by transitivity $x \sim y$

so overall, $x \succeq y$.

$\Leftarrow$: If $x \succeq y$, then by transitivity $u(x)e \succeq u(y)e$. But monotonicity means this can only hold when $u(x) \geq u(y)$ (because if $u(x) < u(y)$ we would have to have $u(x)e \prec u(y)e$).

Step 4: We show that $u(x)$ is continuous.

Proof: This part is technical and less insightful.

[1] To get this, we just take a $\bar{u}$ greater than the maximum element of x and a $\underline{u}$ lower than its minimum value.

[2] Formally, this is the connectedness of the real line.

Take a sequence x_n such that $\lim_{n\to\infty} x_n = x$. We want to show that $\lim_{n\to\infty} u(x_n) = u(x)$.

Firstly, we establish that $u(x_n)$ is bounded for large n. Take a given $\epsilon > 0$, then by definition there exists N such that for all $n > N$, $||x_n - x|| < \epsilon$. Hence by monotonicity there exists $\underline{u}, \bar{u}$ such that $\underline{u} < u(x_n) < \bar{u}$ for $n > N$.

As $\{u(x_n)\}_{n=1}^{\infty}$ is bounded, it must have a convergent subsequence. Take any convergent subsequence, and call it $\{u(x_{n_k})\}_{k=1}^{\infty}$.

Suppose towards contradiction that $\lim_{k\to\infty} u(x_{n_k}) = u' \neq u(x)$. Then there are two cases: either $u' < u(x)$ or $u' > u(x)$.

If $u' > u(x)$, there exists some $\tilde{u}$ such that $u' > \tilde{u} > u(x)$. Furthermore, because $u(x_{n_k}) \to u'$, $\exists N$ such that $\forall k > N$, $u(x_{n_k}) > \tilde{u}$. Then, monotonicity implies that $u(x_{n_k})e \succ \tilde{u}e \succ u(x)e$ for $k > N$. Hence, by transitivity, we have $x_{n_k} \succ \tilde{u}e \succ x$. But this contradicts continuity of preferences.

If $u' < u(x)$, we can make a similar argument: there exists some $\tilde{u}$ such that $u' < \tilde{u} < u(x)$. Then we can conclude that there exists N such that $u(x_{n_k})e \prec \tilde{u}e \prec u(x)e$ for $n > N$, and hence $x_{n_k} \prec \tilde{u}e \prec x$, contradicting continuity of preferences.

Therefore, every convergent subsequence has a limit equal to $u(x)$, which means in turn that $\lim_{n\to\infty} u(x_n) = u(x)$.