14.381: Statistics

Fall, 2012

Midterm Examination

Question 1(a)-1(e), Question 2 and Question 3 are required. Question 1(f) is optional - if you have time left. Good luck!

1. Suppose that the random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + e_i, i = 1, ..., n,$$

where $x_1, ..., x_n$ are fixed constants and $e_1, ..., e_n$ are i.i.d. normals with mean 0 and variance σ^2 (variance is unknown).

   (a) Find a two-dimensional sufficient statistic for (β, σ^2).

   (b) Find the MLE of β and show that it is unbiased.

   (c) Find the distribution of the MLE of β.

   (d) Is $\hat{\beta}_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}$ an unbiased estimator for β? Find its variance.

   (e) Is $\hat{\beta}_2 = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{x_i}$ an unbiased estimator for β? Find its variance.

   (f) Which of the three estimator $\hat{\beta}_{MLE}, \hat{\beta}_1$ and $\hat{\beta}_2$ has the smallest variance? *Hint: you may need the following inequalities. For any numbers $a_1, .., a_n$ we have*

$$(\sum_i a_i)^2 \leq n \sum_i a_i^2 \quad \textit{and} \quad \left(\frac{1}{n}\sum_i \frac{1}{a_i}\right)^{-1} \leq \frac{1}{n}\sum_i a_i$$

2. Suppose that the random variables $Y_1, ..., Y_n$ are independently drawn from a Poisson distribution and

$$Y_i \sim Poisson(\beta x_i),$$

where $x_1, ..., x_n$ are fixed constants.

A reminder if $Y \sim Poisson(\lambda)$, then it takes values $0, 1, 2, ...$ with probabilities

$$P\{Y = k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!},$$

and we know that $EY = \lambda$ and $Var(Y) = \lambda$.

   (a) Find the MLE estimator of β.

   (b) Is this estimator unbiased?

   (c) Find the Rao-Cramer bound for β.

   (d) Is the estimator you found in (a) efficient?

3. Suppose that we have two independent samples: $X_1, \ldots, X_n$ are iid exponential(θ) and $Y_1, \ldots, Y_m$ are iid exponential(μ). Both θ and μ are unknown. We want to test $H_0 : \mu = \theta$ vs $H_1 : \mu \neq \theta$. The goal of this problem is to come up with an LR test statistic.

   Reminder: if X is exponential with parameter β, then it has pdf

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 \leq \beta < \infty, \beta > 0,$$

   and $EX = \beta, Var(X) = \beta^2$.

   (a) Write down the likelihood function and find the unrestricted ML estimates of μ and θ.

   (b) Find the restricted ML estimate (via imposing the null).

   (c) Write down the LR test statistic and explain how would you perform the LR test.

   (d) Construct Wald confidence set for μ. Is it finite sample or asymptotic confidence set? Why?