## 14.121 Problem Set 5 Solutions

Question 1

a. $u(x) \leq v(p, p \cdot x)$ follows from the fact that $x \in B(p, p \cdot x)$ and the definition of indirect utility $v(p, px) = \max_{x \in B(p, p \cdot x)} u(x)$.

b. By part a we know that $u(x) \leq v(p, p \cdot x)$, so it remains to show that there is some $p \gg 0$ such that $u(x) = v(p, p \cdot x)$. Consider $p = \nabla u(x)$. It suffices to show that $x$ is the Marshallian demand at prices $p$ and wealth $p \cdot x$. First note that $x \in B(p, p \cdot x)$. Then note that the Lagrangian for the consumer's problem can be written

$$\max_{z \in B(p, px)} u(z) + \lambda(px - pz)$$

The first order condition which must hold at a maximum is

$$\nabla u(z) = \lambda p = \lambda \nabla u(x)$$

for some $\lambda \geq 0$. This is clearly satisfied at $z = x$ (and $\lambda = 1$). Since $u$ is concave, $x$ must be a maximal point by sufficiency of KT conditions.

c. Normalize $w = 1$. By part b, we have

$$u(x_1, x_2) = \min_{p \gg 0} \left(p_1^{1-\sigma} + p_2^{1-\sigma}\right)^{-\frac{1}{1-\sigma}} \text{ s.t. } p_1 x_1 + p_2 x_2 = 1$$

The FOC and constraint determine

$$p_i = \frac{x_i^{-\frac{1}{\sigma}}}{x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}}$$

Then substitute in these prices to obtain CES utility

$$u(x_1, x_2) = \left[\frac{x_1^{\frac{-(1-\sigma)}{\sigma}} + x_2^{\frac{-(1-\sigma)}{\sigma}}}{\left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}\right)^{1-\sigma}}\right]^{-\frac{1}{1-\sigma}} = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

More generally, one can similarly show the result mentioned in slide 24 of lecture 10: if one has a primitively specified indirect utility function of the form

$$v^*(p, \bar{w}) = -\frac{\sum_{l=1}^{L} p_l \underline{x}_l^*}{\left(\sum_{l=1}^{L} p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}} + \left[\frac{1}{\left(\sum_{l=1}^{L} p_l^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}\right] \bar{w}$$

then this method can be used to derive a corresponding utility function

$$U^*(x) = \left[\sum_{l=1}^{L} (x_l - \underline{x}_l^*)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Question 2

Apply duality:

$$
\begin{aligned}
u_i &= v(p, e(p, u_i)) = a_i(p) + b(p)e(p, u_i) \\
\implies e(p, u_i) &= b(p)^{-1}u_i - b(p)^{-1}a_i(p)
\end{aligned}
$$

Question 3

a. Consider two budget sets $(p, w)$ and $(p, w')$. Since preferences are strictly convex and the consumption set is convex, demand at each budget set is unique. Suppose the demand at $(p, w)$ is given by $x(p, w) = (x_1(p, w), x_2(p, w), ..., x_L(p, w))$. I propose that

$$
x(p, w') = \underbrace{(x_1(p, w) + w' - w, x_2(p, w), .., x_L(p, w))}_{\equiv \tilde{x}}
$$

Note that $x(p, w) \in B(p, w)$ implies

$$
\begin{aligned}
w &\geq x_1(p, w) + \sum_{l=2}^{L} p_l x_l(p, w) \\
\iff w' &\geq (x_1(p, w) + w' - w) + \sum_{l=2}^{L} p_l x_l(p, w)
\end{aligned}
$$

and hence $\tilde{x} \in B(p, w')$. Since $x(p, w)$ is maximal at $(p, w)$, we have for $y = \left(w - \sum_{l=2}^{L} p_l y_l, y_2, .., y_L\right) \in B(p, w)$ that

$$
\begin{aligned}
x_1(p, w) + f(x_{-1}(p, w)) &\geq w - \sum_{l=2}^{L} p_l y_l + f(y_{-1}) \\
\iff (x_1(p, w) + w' - w) + f(x_{-1}(p, w)) &\geq w' - \sum_{l=2}^{L} p_l y_l + f(y_{-1})
\end{aligned}
$$

and thus $\tilde{x}$ maximizes utility within $B(p, w')$. Note that the demands for coordinates other than $l = 1$ are the same at $(p, w)$ and $(p, w')$, so we can denote them by $x_l(p)$ for $l = 2, .., L$, and we can write $x_1(p, w) = w - \sum_{l=2}^{L} p_l x_l(p)$. Finally, indirect utility can be written

$$
\begin{aligned}
v(p, w) &= x_1(p, w) + f(x_{-1}(p)) \\
&= w + \underbrace{\left(-\sum_{l=2}^{L} p_l x_l(p) + f(x_{-1}(p))\right)}_{\equiv \phi(p)}
\end{aligned}
$$

b. Using the result from part a, duality implies

$$
h(p, u) = x(p, e(p, u)) = x(p)
$$

and

$$\begin{aligned} u &= v(p, e(p,u)) = \phi(p) + e(p,u) \\ \implies e(p,u) &= u - \phi(p) \end{aligned}$$

c. Note that

$$\begin{aligned} EV(p, p', w) &= e(p, u') - e(p, u) \\ &= (u' + \psi(p)) - (u + \psi(p)) \\ &= u' - u \end{aligned}$$

and

$$\begin{aligned} CV(p, p', w) &= e(p', u') - e(p', u) \\ &= (u' + \psi(p)) - (u + \psi(p)) \\ &= u' - u \end{aligned}$$

So $EV = CV$.

Question 4

a. Note that

$$h_l(p_1, p_2, u_0) = \frac{\partial e(p_1, p_2, u_0)}{\partial p_l} = \left(\frac{p_{-l}}{p_1 + p_2}\right)^2 u_0$$

This implies $\lim_{p_l \to \infty} h_l(p_1, p_2, u_0) = 0$ and $\lim_{p_l \to \infty} h_{-l}(p_1, p_2, u_0) = u_0$. Since $u$ is continuous, no excess utility at the limit as $p_2 \to \infty$ implies

$$u_0 = u\left(\lim_{p_2 \to \infty} h_1(p_1, p_2, u_0), \lim_{p_2 \to \infty} h_2(p_1, p_2, u_0)\right) = u\left(u_0, 0\right)$$

I.e. this could be rewritten as $x_1 = u(x_1, 0)$, and a similar argument shows $x_2 = u(0, x_2)$.

b. If $x_1 > u_0$ then since $u$ is strictly increasing we have $u(x_1, x_2) > u(x_1, 0) = x_1 > u_0$, so no such $x_2$ exists. If $x_1 \leq u_0$, then $u(x_1, 0) = x_1 \leq u_0$ and $u(x_1, u_0) > u(0, u_0) = u_0$ by the fact that $u$ is strictly increasing, so by intermediate value theorem there exists $x_2$ such that $u(x_1, x_2) = u_0$.

c. Let $p = (1, p_2)$. Since utility is continuous, no excess utility holds, so $u_0 = u(h_1(p, u_0), h_2(p, u_0))$ for any $p$. The approach is to find $p_2(x_1, u_0)$ such that $x_1 = h_1(p, u_0)$, from which we can compute $\tilde{x}_2(x_1, u_0) = h_2(1, p_2(x_1, u_0), u_0)$. Using the Hicksian demands calculated in part a, we have

$$\begin{aligned} x_1 &= h_1(p, u_0) \\ &= \left(\frac{p_2}{1 + p_2}\right)^2 u_0 \\ \implies p_2 &= \frac{\sqrt{x_1}}{\sqrt{u_0} - \sqrt{x_1}} \end{aligned}$$

Thus

$$\begin{aligned}
\tilde{x}_2(x_1, u_0) &= h_2(1, p_2(x_1, u_0), u_0) \\
&= (1 + p_2(x_1, u_0))^{-2} u_0 \\
&= \left(\sqrt{u_0} - \sqrt{x_1}\right)^2
\end{aligned}$$

d. We can rearrange $\tilde{x}_2(x_1, u_0)$ to obtain $u_0$ as a function of $x_1$ and $x_2$, i.e.

$$\begin{aligned}
x_2 &= \left(\sqrt{u_0} - \sqrt{x_1}\right)^2 \\
\implies u_0 &= \left(\sqrt{x_1} + \sqrt{x_2}\right)^2
\end{aligned}$$

This is CES utility with $\sigma = 2$.

Question 5

a. By Walras' Law

$$\begin{aligned}
x_1(p_1, p_2, w) &= \frac{w}{p_1} - \frac{p_2}{p_1} x_2(p_1, p_2, w) \\
&= \frac{w}{p_1} - \frac{p_2}{p_1} a - \left(\frac{p_2}{p_1}\right)^2 b
\end{aligned}$$

It's straightforward to see that homogeneity of degree zero is satisfed. Note that the Slutsky matrix is

$$S = b \begin{pmatrix} \frac{p_2^2}{p_1^3} & -\frac{p_2}{p_1^2} \\ -\frac{p_2}{p_1^2} & \frac{1}{p_1} \end{pmatrix}$$

This is symmetric. The characteristic equation is

$$0 = \lambda \left(\lambda - b\left(\frac{1}{p_1} + \frac{p_2^2}{p_1^3}\right)\right)$$

which has all nonpositive eigenvalues, and hence is negative semi-definite, iff $b \leq 0$.