14.121 Problem Set 2 Solutions

Question 1

a. Given Pareto weights λ_i, the Lagrangian for the social planner's problem is given by

$$\max_{c_i(s^t)} \sum_i \lambda_i \left(\sum_{t,s^t}^{\infty} \beta^t u_i(c_i(s^t)) Pr(s^t) \right) + \sum_{t=0}^{\infty} \theta(s^t) \left(y(s^t) - \sum_i c_i(s^t) \right)$$

The FOC for each $c_i(s^t)$ is

$$\begin{aligned}
\lambda_i \beta^t u_i'(c_i(s^t)) Pr(s^t) &= \theta(s^t) \\
\implies \lambda_i u_i'(c_i(s^t)) &= \lambda_1 u_1'(c_i(s^t)) \qquad (1) \\
\implies c_i(s^t) &= (u_i')^{-1} \left(u_1'(c_1(s^t)) \frac{\lambda_1}{\lambda_i} \right) \qquad (2)
\end{aligned}$$

Combine this with the budget constraint:

$$\begin{aligned}
y(s^t) &= \sum_i c_i(s^t) \\
&= \sum_i (u_i')^{-1} \left(u_1'(c_1(s^t)) \frac{\lambda_1}{\lambda_i} \right) \qquad (3)
\end{aligned}$$

This implicitly pins down $c_1(s^t)$ as a function of $Y(s^t)$ and the set of Pareto weights, and then we can obtain $c_i(s^t)$ (as a function of $Y(s^t)$ and the Pareto weights) from (2).

b. Using implicit and inverse differentiation on (3) obtains

$$\frac{\partial c_1(s^t)}{\partial y(s^t)} = \frac{1}{\sum_{i=1}^{I} \frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{\lambda_1}{\lambda_i}}$$

Now differentiate (2) to obtain

$$\begin{aligned}
\frac{\partial c_i(s^t)}{\partial y(s^t)} &= \frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{\lambda_1}{\lambda_i} \frac{\partial c_1(s^t)}{\partial y(s^t)} \\
&= \frac{\frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{\lambda_1}{\lambda_i}}{\sum_{j=1}^{I} \frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{\lambda_1}{\lambda_j}} \\
&\underset{(1)}{=} \frac{\frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{u_i'(c_i(s^t))}{u_1'(c_1(s^t))}}{\sum_{j=1}^{I} \frac{u_1''(c_1(s^t))}{u_i''(c_i(s^t))} \frac{u_j'(c_j(s^t))}{u_1'(c_1(s^t))}} \\
&= \frac{-\frac{u_i'(c_i(s^t))}{u_i''(c_1(s^t))}}{\sum_{j=1}^{I} -\frac{u_j'(c_j(s^t))}{u_i''(c_i(s^t))}} \\
&= \frac{\rho_i(s^t)}{\rho_0(s^t)}
\end{aligned}$$

c.

$$\begin{aligned}
\frac{d}{dy(s^t)}\left(\frac{c_i(s^t)}{y(s^t)}\right) \quad > \quad 0 &\iff \frac{dc_i(s^t)}{dy(s^t)}y(s^t) - c_i(s^t)\frac{dy(s^t)}{dy(s^t)} > 0 \\
&\iff \frac{dc_i(s^t)}{dy(s^t)} > \frac{c_i(s^t)}{y(s^t)} \\
&\underset{\text{part b}}{\iff} \frac{\rho_i(s^t)}{\rho_0(s^t)} > \frac{c_i(s^t)}{y(s^t)} \\
&\iff \frac{\rho_i(s^t)}{c_i(s^t)} > \sum_{j=1}^{I} \frac{\rho_j(s^t)}{c_j(s^t)}\frac{c_j(s^t)}{y(s^t)}
\end{aligned}$$

d. Substituting the utility function into (2) obtains

$$c_i(s^t) = c_1(s^t)\left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma}}$$

Feasibility implies

$$\begin{aligned}
y(s^t) &= \sum_{i=1}^{I} c_i(s^t) \\
&= \sum_{i=1}^{I} c_1(s^t)\left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma}} \\
\implies c_1(s^t) &= \frac{1}{\sum_{i=1}^{I}\left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma}}}y(s^t) \\
\implies c_i(s^t) &= \frac{\left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma}}}{\sum_{j=1}^{I}\left(\frac{\lambda_1}{\lambda_j}\right)^{\frac{-1}{\gamma}}}y(s^t) \\
\implies c_i(s^t) &= \frac{\lambda_i^{\frac{1}{\gamma}}}{\sum_j \lambda_j^{\frac{1}{\gamma}}}y(s^t) \\
\implies \log c_i(s^t) &= \log\left(\lambda_i^{\frac{1}{\gamma}}\right) - \log\left(\sum_{j=1}^{I}\lambda_j^{\frac{1}{\gamma}}\right) + \log y(s^t) \\
\implies \log \tilde{c}_i(s^t) &= \underbrace{-\log\left(\sum_{j=1}^{I}\lambda_j^{\frac{1}{\gamma}}\right)^{-1}}_{=\zeta} + \underbrace{\log\left(\lambda_i^{\frac{1}{\gamma}}\right)}_{=\alpha_i} + \underbrace{\log y(s^t)}_{=\eta(s^t)} + \underbrace{0}_{=\theta}\log y_i(s^t) + \epsilon_i^m(s^t)
\end{aligned}$$

This can be viewed as a structural model or data generating process which describes how variation in $\tilde{c}_i(s^t)$ is generated from variation in 1_i, 1_{s^t}, $\log y_i(s^t)$, and $\epsilon_i^m(s^t)$. The error term, or all of the factors which affect consumption other than variables included in the regression, consists of $\epsilon_i^m(s^t)$. Since $\epsilon_i^m(s^t)$ is uncorrelated with the regressors, the coefficients, including $\theta = 0$ on $y(s^t)$, are identified via regression.

e. We now have

$$c_i(s^t) = c_1(s^t)^{\frac{\gamma_1}{\gamma_i}} \left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma_i}} \tag{4}$$

and $c_1(s^t)$ is now implicitly determined as a function of $y(s^t)$ and the Pareto weights by

$$y(s^t) = \sum_{i=1}^{I} c_1(s^t)^{\frac{\gamma_1}{\gamma_i}} \left(\frac{\lambda_1}{\lambda_i}\right)^{\frac{-1}{\gamma_i}}$$

This determines $c_1(s^t)$ implicitly as a (strictly increasing) function of $y(s^t)$. Then from (4) we have

$$\begin{aligned}
\log \tilde{c}_i(s^t) &= -\frac{1}{\gamma_i} \log\left(\frac{\lambda_1}{\lambda_i}\right) + \frac{\gamma_1}{\gamma_i} \log c_1(s^t) + \epsilon^m(s^t) \\
&= \underbrace{-\frac{1}{\gamma_i} \log\left(\frac{\lambda_1}{\lambda_i}\right)}_{\alpha_i} + \underbrace{0}_{\eta_t} + \underbrace{0}_{\theta} \log y_i(s^t) \underbrace{+\frac{\gamma_1}{\gamma_i} \log c_1(s^t) + \epsilon_i^m(s^t)}_{\equiv v_i(s^t)}
\end{aligned}$$

Now consumption is affected by a term which is not captured in the stated regression. In particular, the error in the data generating process after controlling for 1_i, 1_{s^t}, and $\log y_i(s^t)$ now corresponds to $v_i(s^t)$. Consider what happens when the aggregate endowment is relatively large. Then $c_1(s^t)$ is relatively large, so $\log c_1(s^t)\gamma_i^{-1}$ is relatively large for all the agents, but it is largest for agents with a higher risk-tolerance, so then the error $v_i(s^t)$ is greater on average for agents with higher risk-tolerance. But we have assumed that $y_i(s^t)$ is also on average higher for agents with higher risk-tolerance. So both $v_i(s^t)$ and $y_i(s^t)$ tends to be high when $y(s^t)$ is high, and vice versa, which implies that they are correlated. This means that the regression coefficient on $\log y_{it}$ would be biased upward. Schulholfer-Wohl (2011) shows that risk-tolerant workers typically hold jobs where earnings carry more aggregate risk, suggesting heterogeneity may be a problem with risk-sharing tests.

Question 2

a. Note that preferences are concave and the consumption sets are convex, so the utility possibilities set is convex and Pareto optimal allocations are spanned by solving the planner's problem for different Pareto weights. The Lagrangian for the planner's problem can be written

$$\lambda_B(c_{B,1} + 2\sqrt{c_{B,2}}) + \lambda_L(c_{L,1} + 2\sqrt{c_{L,2}}) - \mu_1(c_{B,1} + c_{L,1} - 1) - \mu_2(c_{B,2} + c_{L,2} - 1) + \sum_{i=B,L} \sum_{t=1,2} \eta_{i,t} c_{i,t}$$

The FOCS are

$$
\begin{aligned}
c_{B,1} : \lambda_B &= \mu_1 - \eta_{B,1} \\
c_{B,2} : \frac{\lambda_B}{\sqrt{c^*_{B,2}}} &= \mu_2 - \eta_{B,2} \\
c_{L,1} : \lambda_L &= \mu_1 - \eta_{L,1} \\
c_{L,2} : \frac{\lambda_L}{\sqrt{c^*_{L,2}}} &= \mu_2 - \eta_{L,2}
\end{aligned}
$$

Note that the nonnegativity constraints for good 2 will not bind with equality since marginal utility explodes as the amount consumed approaches zero, so $\eta_{i,2} = 0$ for $i = B, L$. However, the nonnegativity constraints for good 1 can bind. First we consider Pareto optima where they don't bind. In that case we obtain

$$c^*_{B,2} = c^*_{L,2}$$

Then feasibility implies

$$
\begin{aligned}
c^*_{B,2} = c^*_{L,2} = \frac{1}{2} \\
c^*_{L,1} = 1 - c^*_{B,1}
\end{aligned}
$$

where $c^*_{B,1}$ ranges on $[0, 1]$. Now suppose the the nonnegativity constraint $c_{B,1} \geq 0$ binds. Then we must have $\lambda_B \leq \mu_1 = \lambda_L$. Applying the FOCs and feasibility obtains

$$
\begin{aligned}
c^*_{B,1} &= 0 \\
c^*_{L,1} &= 1 \\
c^*_{B,2} &= \frac{1}{1 + \left(\frac{\lambda_L}{\lambda_B}\right)^2} \\
c^*_{L,2} &= \frac{\left(\frac{\lambda_L}{\lambda_B}\right)^2}{1 + \left(\frac{\lambda_L}{\lambda_B}\right)^2}
\end{aligned}
$$

where λ_L/λ_B ranges on $[1, \infty]$. If the nonnegativity constraint $c_{L,1} \geq 0$ binds then similarly

$$
\begin{aligned}
c^*_{B,1} &= 1 \\
c^*_{L,1} &= 0 \\
c^*_{B,2} &= \frac{1}{1 + \left(\frac{\lambda_L}{\lambda_B}\right)^2} \\
c^*_{L,2} &= \frac{\left(\frac{\lambda_L}{\lambda_B}\right)^2}{1 + \left(\frac{\lambda_L}{\lambda_B}\right)^2}
\end{aligned}
$$

where λ_L/λ_B ranges on $[0,1]$. The set of Pareto allocations can be represented in an Edgeworth box by a vertical line from $(0,0)$ to $(0,1/2)$, a horizontal line from $(0,1/2)$ to $(1,1/2)$, and a vertical line from $(1,1/2)$ to $(1,1)$.

b. The given allocation satisfies the characterization of Pareto optimal allocations in part a. It also satisfies individual rationality since

$$
\begin{aligned}
u(c_{B,1},c_{B,2}) &= 2-\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}} \geq 0+2\sqrt{1} \\
u(c_{L,1},c_{L,2}) &= \frac{2}{\sqrt{2}}-1+\frac{2}{\sqrt{2}} \geq 1+2\sqrt{0}
\end{aligned}
$$

In the 2-replica economy, consider the coalition consisting of 1 lender and 2 borrowers having coalition type allocation $((c'_{B,1},c'_{B,2}),(c'_{L,1},c'_{L,2})) = \left(\left(\frac{2}{3}\left(2-\frac{2}{\sqrt{2}}\right),\frac{2}{3}\right),\left(\frac{8}{3\sqrt{2}}-\frac{5}{3},\frac{2}{3}\right)\right)$. This is feasible since

$$
2*\frac{2}{3}\left(2-\frac{2}{\sqrt{2}}\right)+\frac{8}{3\sqrt{2}}-\frac{5}{3}=1
$$

for good 1 and

$$
2*\frac{2}{3}+\frac{2}{3}=2
$$

for good 2. Moreoever, it blocks the given allocation since

$$
\begin{aligned}
u(c'_{B,1},c'_{B,2}) &= \frac{2}{3}\left(2-\frac{2}{\sqrt{2}}\right)+2\sqrt{\frac{2}{3}} > 2 \\
u(c'_{L,1},c'_{L,2}) &= \frac{8}{3\sqrt{2}}-\frac{5}{3}+2\sqrt{\frac{2}{3}} > \frac{4}{\sqrt{2}}-1
\end{aligned}
$$

Note that in general a systematic way to search for a blocking coalition is to follow the constructive proof in the core equivalence theorem. In the theorem we identify a "favored" type, then consider the coalition consisting of everyone except one copy of the "favored' type,' where the allocation in the coalition can be obtained by adding a fraction of the "favored" type's net trade to everyone in the coalition. The coalition and allocation described above was obtained in such a manner.

c. When unconstrained, agent i's Lagrangian can then be written

$$
c_{i,1}+2\sqrt{c_{i,2}}-\lambda_i(c_{i,1}+pc_{i,2}-(y_{i,1}+py_{i,2}))+\eta_i c_{i,1}+\eta_{i,2}c_{i,2}
$$

Note that the nonnegativity constraint for good 2 will not bind since marginal utility goes to infinity as $c_{i,2}$ goes to zero. The FOCs and budget constraint imply

$$
\begin{aligned}
c_{i,2} &= \frac{1}{((1+\eta_i)p)^2} \\
c_{i,1} &= y_{i,1}+py_{i,2}-\frac{1}{(1+\eta_i)^2 p}
\end{aligned}
$$

Given the endowments, consumption in the first period is given by

$$c_{B,1} = p - \frac{1}{(1+\eta_B)^2 p}$$
$$c_{L,1} = 1 - \frac{1}{(1+\eta_L)^2 p}$$

Since we are considering $p > 1$, it's clear that $c_{i,1} > 0$ for $i = B, L$, which implies $\eta_i = 0$ and then

$$c_{B,1} = p - \frac{1}{p}$$
$$c_{L,1} = 1 - \frac{1}{p}$$

The collateral constraint implies that $c_{B,1} \leq \alpha p$, which binds when $p > (1-\alpha)^{\frac{-1}{2}}$. The demand with the collateral constraint is then

$$c_{B,1} = \begin{cases} p - \frac{1}{p} & p \leq (1-\alpha)^{\frac{-1}{2}} \\ \alpha p & p > (1-\alpha)^{\frac{-1}{2}} \end{cases}$$

$$c_{B,2} = 1 - \frac{1}{p} c_{B,1} = \begin{cases} \frac{1}{p^2} & p \leq (1-\alpha)^{\frac{-1}{2}} \\ 1-\alpha & p > (1-\alpha)^{\frac{-1}{2}} \end{cases}$$

Note that the collateral constraint never binds for L, so the demands are

$$c_{L,1} = 1 - \frac{1}{p}$$
$$c_{L,2} = \frac{1}{p^2}$$

d. If B is unconstrained, then market clearing requires that $p = \sqrt{2}$, which satisfies $p \leq (1-\alpha)^{\frac{-1}{2}}$ iff $\alpha \geq \frac{1}{2}$. (Note that $p > 1$ holds, so the price is consistent with the demands calculated in part c.) Consumption is given by

$$c_{B,1} = \frac{\sqrt{2}}{2}$$
$$c_{L,1} = 1 - \frac{\sqrt{2}}{2}$$
$$c_{B,2} = c_{L,2} = \frac{1}{2}$$

This belongs to the set of Pareto optimal allocations characterized above. Since the equilibrium is an interior solution, the FOC's from the planner's problem imply $\lambda_B = \lambda_L = \frac{1}{2}$.

If B is constrained then market clearing implies $p = \frac{1}{\sqrt{\alpha}}$, which satisfies $p > (1-\alpha)^{\frac{-1}{2}}$ iff $\alpha < \frac{1}{2}$. (Note that $p > 1$ still holds, so the price is consistent with the demands that were calculated in part c.) The allocation is

$$\begin{aligned} c_{B,1} &= \sqrt{\alpha} \\ c_{B,2} &= 1-\alpha \\ c_{L,1} &= 1-\sqrt{\alpha} \\ c_{L,2} &= \alpha \end{aligned}$$

This does not belong to the set of Pareto optimal allocations described above.

e. Combining the borrowing constrained with the budget constrained obtains $c_2 \geq (1-\alpha)y_2$. The planner's problem can be written

$$\lambda_B(c_{B,1} + 2\sqrt{c_{B,2}}) + \lambda_L(c_{L,1} + 2\sqrt{c_{L,2}}) - \mu_1(c_{B,1} + c_{L,1} - 1) - \mu_2(c_{B,2} + c_{L,2} - 1) + \eta(c_{B,2} - (1-\alpha))$$

The FOCs at an interior solution are

$$\begin{aligned} c_{B,1} : \lambda_B &= \mu_1 \\ c_{B,2} : \frac{\lambda_B}{\sqrt{c^*_{B,2}}} + \eta &= \mu_2 \\ c_{L,1} : \lambda_L &= \mu_1 \\ c_{L,2} : \frac{\lambda_L}{\sqrt{c^*_{L,2}}} &= \mu_2 \end{aligned}$$

Suppose that B is constrained in the planner's problem, so $c^*_{B,2} = 1-\alpha$. Market clearing implies

$$\begin{aligned} c^*_{B,2} &= 1-\alpha \\ c^*_{L,2} &= \alpha \\ c^*_{L,1} &= 1 - c^*_{B,1} \end{aligned}$$

where $c^*_{B,1}$ ranges on $[0,1]$. The competitive allocation found above is of this form. Note that the relative Pareto weight at this allocation is $\lambda_L/\lambda_B = 1$.

Quesion 3

a. The household problem is

$$\max_{c^h_{10}, c^h_{20}, c^h_{11}, c^h_{21}, k^h, \theta^h} u_0(c^h_{10}, c^h_{20}) + \beta u_1(c^h_{11}, c^h_{21})$$

$$\text{s.t. } (\lambda^h_1): \ c^h_{10} + p_0 c^h_{20} + \frac{1}{1+r}\theta^h + p_0 k^h \leq e^h_{10} + p_0 e^h_{20}$$

$$(\lambda_2^h):\ c_{11}^h + p_1 c_{21}^h \le e_{11}^h + p_1 e_{21}^h + \theta^h + p_1 R k^h$$
$$(\lambda_3^h):\ 0 \le p_1 R k^h + \theta^h$$

and nonnegativity constraints. FOCs for interior solution:

$$\begin{aligned}
c_{10}^h : u_{10}^h &= \lambda_1^h \\
c_{20}^h : u_{20}^h &= \lambda_1^h p_0 \\
c_{11}^h : \beta u_{11}^h &= \lambda_2^h \\
c_{21}^h : \beta u_{21}^h &= \lambda_2^h p_1 \\
\theta^h : \lambda_1^h \frac{1}{1+r} &= \lambda_2^h + \lambda_3^h \\
k^h : \lambda_1^h p_0 &= \lambda_2^h p_1 R + \lambda_3^h p_1 R
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{u_{20}^1}{u_{10}^1} &= p_0 = \frac{u_{20}^h}{u_{10}^h} \\
\frac{u_{21}^1}{u_{11}^1} &= p_1 = \frac{u_{20}^h}{u_{10}^h}
\end{aligned}$$

for $h = 1, .., H$, as well as

$$p_0 = \frac{R}{1+r} p_1$$
$$u_{20}^h = \beta R u_{21}^h + \lambda_3^h p_1 R \tag{5}$$

The market clearing conditions are

$$\begin{aligned}
\sum_h c_{10}^h &= \sum_h e_{10}^h \\
\sum_h c_{20}^h + \sum_h k^h &= \sum_h e_{20}^h \\
\sum_h c_{11}^h &= \sum_h e_{11}^h \\
\sum_h c_{21}^h &= \sum_h e_{21}^h + R \sum_h k^h \\
\sum_h \theta^h &= 0
\end{aligned}$$

b. Since preferences are homothetic, they can be represented by a homogeneous of degree one function. Applying some of the results from the last problem set about homogeneous of degree one functions, we can write

$$p_1 = \frac{u_{21}^h (c_{11}^h / c_{21}^h)}{u_{11}^h (c_{11}^h / c_{21}^h)}$$

and obtain the result that $z_1 \equiv c_{11}^h / c_{21}^h$ must be the same for all households. Adding up over households and applying feasibility obtains

$$z_1 = \frac{\sum_h c_{11}^h}{\sum_h c_{21}^h} = \frac{\sum_h e_{11}^h}{\sum_h e_{21}^h + R \sum_h k^h}$$

c. The planner allows households to determine period 1 consumption c_1^h in the spot markets. It turns out that, as long as the planner incorporates a household's period 1 budget constraint, it will choose the same allocation that the household would have chosen, i.e. both the household and planner will choose c_1^h to satisfy $c_{11}^h / c_{21}^h = z_1$ and the period 1 budget constraint. Since $p(z_1)$ clears the market for c_{21}, the planner does not need to explicitly take into consideration the associated resource constraint. By Walras' law, the market for c_{11} must also clear, so the planner can ignore that feasibility condition as well. To see this directly, note that aggregating the budget constraints and applying the optimizing conditions obtains

$$\underbrace{\sum_h c_{11}^h}_{=z_1 \sum_h c_{21}^h} + p_1(z_1) \sum_h c_{21}^h = \underbrace{\sum_h e_{11}^h}_{=z_1 \sum_h (e_{21}^h + Rk^h)} + p(z_1) \sum_h (e_{21}^h + Rk^h)$$

$$\Longrightarrow \sum_h c_{21}^h = \sum_h (e_{21}^h + Rk^h)$$

for $h = 1, .., H$, which also implies (multiplying both sides by z_1)

$$\sum_h c_{11}^h = \sum_h e_{11}^h$$

The planner still has to explicitly account for market clearing in period 0. One way to write the planner's problem is then

$$\max_{\{c_{10}^h, c_{20}^h, c_{11}^h, c_{21}^h, k^h, \theta^h\}_h} u_0(c_{10}^1, c_{20}^1) + \beta u_1(c_{11}^h, c_{21}^h)$$

$$\begin{aligned}
\text{s.t. } (\lambda_1^h) : u_0(c_{10}^h, c_{21}^h) + \beta u_1(c_{11}^h, c_{21}^h) &\geq \bar{u}^h \text{ for } h = 2, .., H \\
(\lambda_2^h) : p(z_1) Rk^h + \theta^h &\geq 0 \text{ for } h = 1, ..., H \\
(\lambda_3^h) : \; c_{11}^h + p(z_1) c_{21}^h &\leq e_{11}^h + p(z_1) e_{21}^h + \theta^h + p(z_1) Rk^h \text{ for } h = 1, .., H
\end{aligned}$$

$$\begin{aligned}
(\lambda_4) : \sum_h c_{10}^h &= \sum_h e_{10}^h \\
(\lambda_5) : \sum_h c_{20}^h + \sum_h k^h &= \sum_h e_{20}^h \\
(\lambda_6) : \sum_h \theta^h &= 0
\end{aligned}$$

The FOCs for an interior solution are

$$
\begin{aligned}
c_{10}^h : \lambda_1^h u_{10}^h &= \lambda_4 \\
c_{20}^h : \lambda_1^h u_{20}^h &= \lambda_5 \\
c_{11}^h : \lambda_1^h \beta u_{11}^h &= \lambda_3^h \\
c_{21}^h : \lambda_1^h \beta u_{21}^h &= \lambda_3^h p(z_1) \\
\theta^h : \lambda_2^h + \lambda_3^h &= \lambda_6 \\
k^h : \lambda_5 &= (\lambda_3^h + \lambda_2^h) R p\left(z_1\right) + \frac{\partial z_1}{\partial k^h} p'\left(z_1\right) \sum_{\tilde{h}} \left[\lambda_2^{\tilde{h}} R k^{\tilde{h}} + \lambda_3^{\tilde{h}} (R k^{\tilde{h}} + e_{21}^{\tilde{h}} - c_{21}^{\tilde{h}}) \right]
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{u_{20}^1}{u_{10}^1} &= \frac{u_{20}^h}{u_{10}^h} \; \forall h \\
p(z_1) &= \frac{u_{21}^h}{u_{11}^h} \; \forall h \qquad (6) \\
\lambda_1^h u_{20}^h &= \lambda_1^h R \beta u_{21}^h + R p(z_1) \lambda_2^h + \frac{\partial z_1}{\partial k^h} p'\left(z_1\right) \sum_{\tilde{h}} \left[\lambda_2^{\tilde{h}} R k^{\tilde{h}} + \lambda_3^{\tilde{h}} (R k^{\tilde{h}} + e_{21}^{\tilde{h}} - c_{21}^{\tilde{h}}) \right] \\
&= \lambda_1^h R \beta u_{21}^h + R p(z_1) \lambda_2^h + \frac{\partial z_1}{\partial k^h} p'\left(z_1\right) \sum_{\tilde{h}} \left[\lambda_2^{\tilde{h}} R k^{\tilde{h}} + (\lambda_6 - \lambda_2^{\tilde{h}}) (R k^{\tilde{h}} + e_{21}^{\tilde{h}} - c_{21}^{\tilde{h}}) \right] \\
&= \lambda_1^h R \beta u_{21}^h + R p(z_1) \lambda_2^h + \frac{\partial z_1}{\partial k^h} p'\left(z_1\right) \sum_{\tilde{h}} \lambda_2^{\tilde{h}} \left[c_{21}^{\tilde{h}} - e_{21}^{\tilde{h}} \right] \qquad (7)
\end{aligned}
$$

Comparing (5) and (7), one can see that the solution to the competitive equilibrium will not satisfy the conditions of a constrained Pareto optimum if $\lambda_2^h > 0$ for some household, that is if a collateral constraint is binding. There is a *pecuniary externality* in this economy as agents do not internalize how their actions affect prices and ultimately the collateral constraints of other agents in the economy. In particular, if an agent wants to increase storage this decreases the price of the collateral good and reduces the borrowing ability of other agents. The economy is inefficient insofar as those impacted agents (whose collateral constraints bind with equality) would be willing to offer compensation for refraining from over-storing, but there is no opportunity to do so in this economy. A benevolent social planner would mandate less storage.