Lecture 7. Arbitrage Pricing - Option Pricing Applications

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Today

- We'll discuss implications of the no arbitrage framework for derivative securities, especially options
- This isn't just about calculating asset prices. Many interesting economic decision problems involve option-like payoffs. For example: writing research papers / hiring job market candidates!
- I'll also emphasize some ways in which empirical researchers have used information from derivative markets + no arbitrage conditions to back out model-free measures of (risk-neutral) expectations from financial markets

What is a derivative security?

- A **derivative** security is a contract whose value derives from the price of another observable outcome (often a price of another security)
- Let X_T be the payoff of a given asset at time $T > t$. We call X_T the **underlying asset**. Example: $X_T = S_T$ the stock price
- A derivative security has a payoff equal to $f(X_T)$, where $f(\cdot)$ is a *known, potentially nonlinear* function specified as part of the contract
- Some derivatives have payoffs that depend on the whole path of $\{X_t\}_{t=t+1}^{T}$
- Not surprisingly, can price a derivative using the SDF:

$$\text{derivative price} = E_t\left[\eta_{t:T} \cdot f(X_T)\right]$$

- Derivatives are often redundant assets, so can often price them purely by no arbitrage via constructing replicating portfolios of other securities

Some examples of derivative securities

- A long forward contract corresponds with an obligation to buy the underlying asset at a pre-specified price K at time T

$$Payoff = \underbrace{S_T}_{\text{actual value of stock purchase}} - \underbrace{K}_{\text{pre-negotiated purchase price}}$$

  allows buyer to lock in a purchase price K for a stock at time T

- Credit default swaps insure debt-holder against losses from default
- Interest rate swaps allow investors to insure themselves against interest rate risk by exchanging a fixed set of cash payments for floating payments tied to interest rates
- An Arrow-Debreu security is an example of a derivative security: "underlying" is the state of nature.
  - For example, you can buy a security which pays off if Trump is re-elected in 2020 (in case you're curious: price today is \$0.36)
- Naturally, all derivative securities are just portfolios of A-D securities!
  - Discrete states: derivative price $= \sum_{\omega=1}^{M} \phi_\omega \cdot f(X_\omega)$
  - Continuous states: derivative price $= \int \phi(\omega) \cdot f(X(\omega))d\omega$

Outline

Introduction to Options

- Let S_1 denote the payoff of a security at $t = 1$ and S_0 it price at $t = 0$.
- We call this security "stock" or the underlying asset for convenience.

Definition (European Options)

A European call (put) option on the stock is a security/contract that gives the buyer the right (not the obligation) to buy (sell) the stock from (to) the seller of the option at a future date T and at a fixed price K.

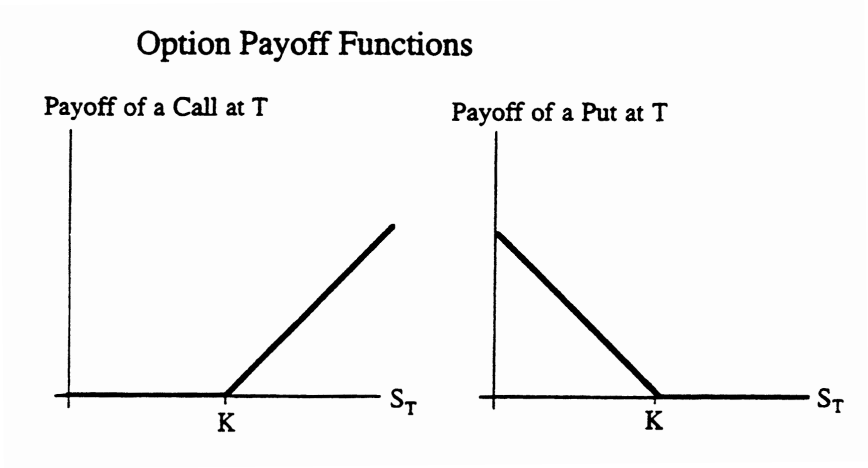
$T = 1$ is the maturity/exercise date and K the strike/exercise price.

- Let $[x]_+ \equiv \max[0, x]$.
- The payoff of a European call and put, c_1 and p_1, are given by:

$$c_1 = [S_1 - K]_+, \quad p_1 = [K - S_1]_+.$$

- An American call (put) option allows the buyer to exercise at any time before and including the maturity date T.

Call/put option payoffs

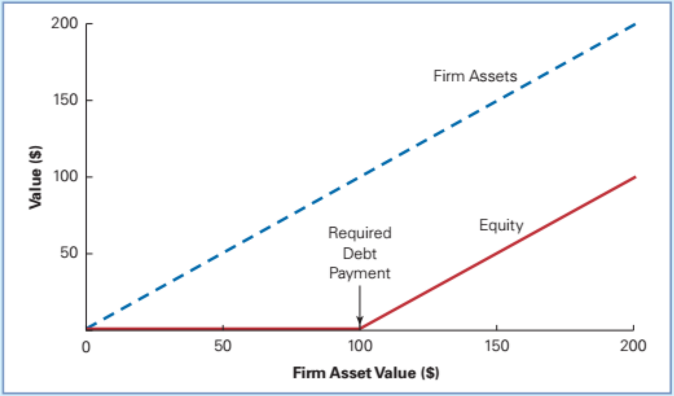


Equity is a *call option* on the assets of the firm

FIGURE 20.8

Equity as a Call Option

If the value of the firm's assets exceeds the required debt payment, the equity holders receive the value that remains after the debt is repaid; otherwise, the firm is bankrupt and its equity is worthless. Thus, the payoff to equity is equivalent to a call option on the firm's assets with a strike price equal to the required debt payment.

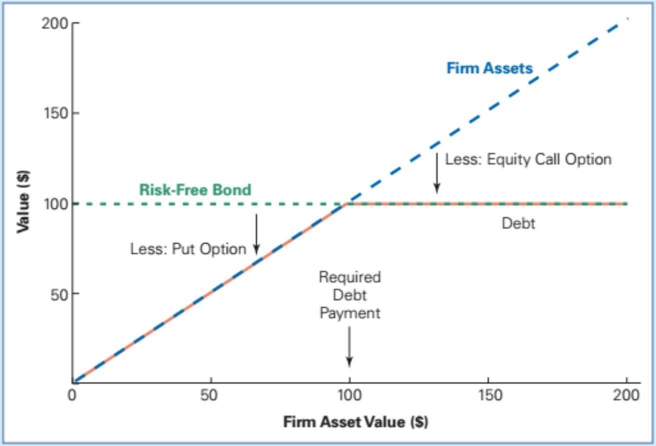


Debt = underlying minus a call option on assets of the firm

FIGURE 20.9

Debt as an Option Portfolio

If the value of the firm's assets exceeds the required debt payment, debt holders are fully repaid. Otherwise, the firm is bankrupt and the debt holders receive the value of the assets. Note that the payoff to debt (orange line) can be viewed either as (1) the firm's assets, less the equity call option, or (2) a risk-free bond, less a put option on the assets with a strike price equal to the required debt payment.



Equivalent way to construct payoff: risk free bond - put option

Introduction to Options

Intrinsic value

- For a call, $S - K$ is called its intrinsic value, i.e., the cashflow if exercised.
- The intrinsic value of a put is $K - S$.
- An option is said to be:
  - In the Money (ITM), if its intrinsic value is positive,
  - At the Money (ATM), if its intrinsic value is zero,
  - Out of the Money (OTM), if its intrinsic value is negative.

  At maturity, clearly it is suboptimal to exercise an option if its intrinsic value is negative.

Introduction to Options

Option Payoffs

- The payoff of an option or a portfolio of options depends on the price of the underlying asset at maturity, S.
- A "straddle": long 1 call plus 1 put with the same strike K. Its payoff:

$$[S-K]_+ + [K-S]_+ = \begin{cases} K-S, & S \leq K \\ S-K, & S > K \end{cases}$$

- A "butterfly": long 1 call at strike $K-\delta$, short 2 calls at strike K, long 1 call at strike $K+\delta$. Its payoff

$$[S-K+\delta]_+ - 2[S-K]_+ + [S-K-\delta]_+ = \begin{cases} 0, & S \leq K-\delta \\ S-(K-\delta), & K-\delta < S \leq K \\ (K+\delta)-S, & K < S \leq K+\delta \\ 0, & S > K+\delta \end{cases}$$

Arbitrage Pricing Properties of Options

Let $c(S, K)$ and $p(S, K)$ denote the prices of a European call and put.

Theorem

Option prices are non-negative. In particular, $c(S, K) \geq 0$ *and* $p(S, K) \geq 0$.

Theorem

$c(S, K)$ *is non-increasing in* K *and* $p(S, K)$ *is non-decreasing in* K.

Theorem

$c(S, K)$ *and* $p(S, K)$ *are convex in* K.

S&P 500 (^SPX)
SNP - SNP Real Time Price. Currency in USD

Add to watchlist

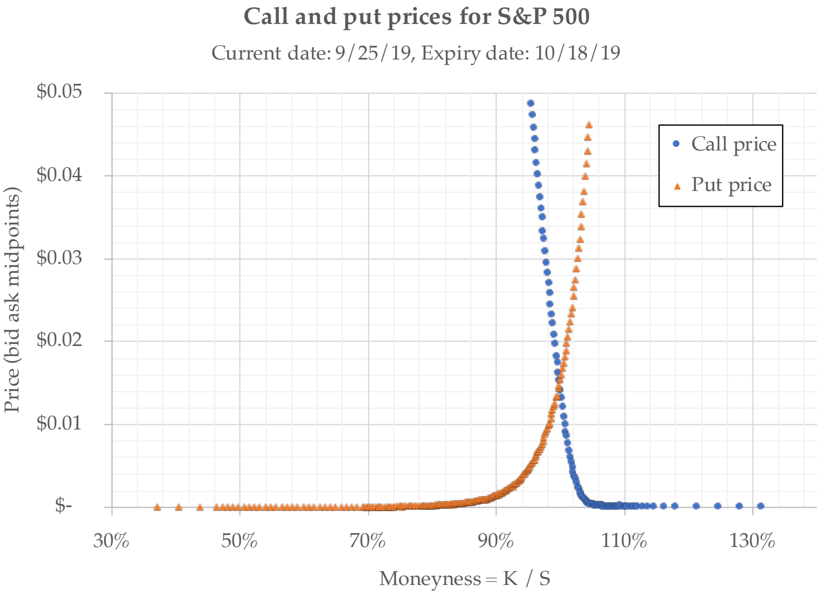
2,966.60 -25.18 (-0.84%)
At close: September 24 5:12PM EDT

Summary | Chart | Historical Data | **Options** | Components

October 18, 2019 | In The Money | Show: **List** | Straddle | Option Lookup

Calls for October 18, 2019

Contract Name	Last Trade Date	Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
SPXW191018C01100000	2019-09-20 3:41PM EDT	1,100.00	1,893.58	1,859.60	1,862.60	0.00	-	3	4	0.00%
SPXW191018C01200000	2019-08-21 3:34PM EDT	1,200.00	1,718.90	1,778.00	1,792.60	0.00	-	1	51	209.44%
SPX191018C01400000	2019-09-19 2:21PM EDT	1,400.00	1,612.85	1,560.10	1,563.10	0.00	-	4	76	0.00%
SPX191018C01500000	2019-09-13 4:13PM EDT	1,500.00	1,504.20	1,461.00	1,464.30	0.00	-	2	2,148	0.00%
SPX191018C01550000	2019-07-01 9:30AM EDT	1,550.00	1,420.00	0.00	0.00	0.00	-	-	1	0.00%
SPXW191018C01600000	2019-07-19 3:46PM EDT	1,600.00	1,381.20	1,307.30	1,314.30	0.00	-	2	2	0.00%



Call and put prices for S&P 500
Current date: 9/25/19, Expiry date: 10/18/19
$0.05
$0.04
$0.03
$0.02
$0.01
$-
Price (bid ask midpoints)
30%
50%
70%
90%
110%
130%
Moneyness = K / S
Call price
Put price

Arbitrage Pricing Properties of Options

Theorem

Let $\theta > 0$ be a portfolio of N securities with price vector $S = [S_1; \ldots; S_N] > 0$ and $K = [K_1; \ldots; K_N] > 0$. Then,

$$c\left(S^\top\theta, K^\top\theta\right) \le \sum_{i=1}^{N} \theta_i\, c(S_i, K_i), \quad \textit{and} \quad p\left(S^\top\theta, K^\top\theta\right) \le \sum_{i=1}^{N} \theta_i\, p(S_i, K_i).$$

Thus, an option on a portfolio is worth less than a portfolio of options on the assets in that portfolio.

Proof. The payoff of an option on the portfolio is:

$$\left[(S-K)^\top\theta\right]_+ = \left[\sum_i \theta_i(S_i - K_i)\right]_+ \le \sum_i \theta_i[S_i - K_i]_+.$$

The inequality follows from the convexity of the payoff function. Since the right-hand-side of the inequality is just the payoff of a portfolio of options on each assets. The theorem follows from NA.

Using portfolio - index spread to measure implicit guarantees

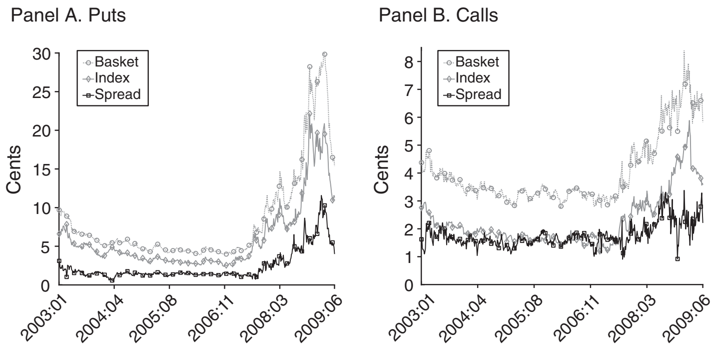


FIGURE 1. COST OF INSURING FINANCIAL SECTOR EQUITY

Notes: The cost of financial sector insurance based on options on the index (solid gray line) and options on the basket (dotted gray line), as well as the basket-index spread (black line). Units are cents per dollar insured. Delta is 25 and time to maturity is 365 days. Panel A uses put options and panel B uses call options.

Kelly, Lustig, and van Nieuwerburgh (2016), "Too-Systemic-to-Fail: What Option Markets Imply about Sector-Wide Government Guarantees", *American Economic Review*

Pricing Bounds

Theorem (Option Price Upper Bound)

$S \geq c(S, K)$.

Proof. $S_1 \geq [S_1 - K]_+$. By NA (positivity), $S \geq c(S, K)$.

Theorem (Option Price Lower Bound)

If there exists a riskless bond with interest rate r_F, then:

$$c(S, K) \geq \left[S - \frac{K}{1 + r_F}\right]_+ = \max\left\{0, S - \frac{K}{1 + r_F}\right\}.$$

Proof. Consider portfolio long 1 share of stock and short K units of RF bond. Payoff at $t = 1$ is $S_1 - K$ and its price at $t = 0$ is $S - K/(1+r_F)$.
Since $[S_1 - K]_+ \geq S_1 - K$, $c(S, K) \geq S - K/(1+r_F)$.
Also, $c(S, K) \geq 0$. The theorem follows.

Summarizing the above results, we have:

$$\left[S - \frac{K}{1 + r_F}\right]_+ \leq c(S, K) \leq S.$$

Put-Call Parity

Theorem (Put-Call Parity)

If there exists a riskless bond with interest rate r_F, then:

$$c(S,K) + \frac{K}{1+r_F} = p(S,K) + S. \tag{1}$$

American Options

- American options can be exercised early.
- For simplicity, suppose that we are 1 period from maturity. I either exercise today or wait (so option is equivalent to a European option)
- Let $C(S,K)$ and $P(S,K)$ denote the prices of an American call and put.
- The price of an American option can never be lower than the price of its European counterpart:

  $$C(S,K) \geq c(S,K), \quad P(S,K) \geq p(S,K).$$

  This is because an American option dominates its European counterpart in payoff.
- Strict inequality holds when early exercise has positive probability.
- Payoff prior from the stock prior to maturity is referred to as dividend.
- Dividends can influence early exercise and the value of American options.

Early Exercise without Dividends

American Call

- Without dividends, American calls should not be exercised early if $r_F > 0$
  - The value of early exercise is $S - K$.
  - But:

$$S - K \leq S - \frac{K}{1 + r_F} \leq \left[S - \frac{K}{1 + r_F} \right]_+ \leq c(S, K).$$

  - Thus, the value of early exercise never exceeds the value from selling the option as a European call for $c(S, K)$.
  - It is never optimal to exercise early.
- There are two costs for early exercising an American call:
  - Time value of money: By exercising early, we pay the strike price now instead of later. With $r_F > 0$, we have the first inequality.
  - Value of option: By exercising early, we give up the option not to exercise at maturity. This gives the last inequality.

Early Exercise without Dividends

American Put

- Without dividends, it can be optimal to exercise an American put early.
- The gain from early exercise is the time value of the exercise price.
- The cost of early exercise is the loss of option value.
- Note:

$$P(S, K) = \max[K - S,\ p(S, K)] = \max\left[K - S, \frac{K}{1 + r_F} - S + c(S, K)\right].$$

- Early exercise CAN be optimal if

$$K - S > \frac{K}{1 + r_F} - S + c(S, K) \quad \text{or} \quad \frac{r_F}{1 + r_F} K > c(S, K).$$

This is true when $K \gg S$, then $c(S, K)$ is very small and the put is deep in the money.

Early Exercise with Dividends

- Let D be the dividend at $t = 0$ and S the ex-dividend price.
- At $t = 0$, holder of an American call have two choices:
  1. Exercise and receive dividend D and S by selling the stock ex-dividend,
  2. Hold the option to maturity ($t = 1$).
- Under optimal exercise policy, we have:

$$C(S, D, K) = \max[S + D - K,\ c(S, K)],$$

  where $C(S, D, K)$ denotes the call price before the dividend.
- Similarly, for the put, we have:

$$P(S, D, K) = \max[K - S - D,\ p(S, K)].$$

- With dividends, the put-call parity formula becomes:

$$c(S, K) + \frac{K}{1 + r_F} + D = S + p(S, K).$$

- Dividends induce early exercise for calls and delay early exercise for puts.

Option Pricing in A Complete Market

- In a complete securities market, there exists a unique state price vector ϕ.
- Let $S_{1\omega}$ denote the time-1 stock price in state ω, $\omega \in \Omega$.
- The price of a European call option is given by:

$$c(S, K) = \sum_{\omega \in \Omega} \phi_\omega [S_{1\omega} - K]_+ = \frac{\mathbb{E}^{\mathbb{Q}}\left[[S_1 - K]_+\right]}{1 + r_F},$$

  where $\mathbb{Q}$ is the risk-neutral measure.
- In order to obtain more concrete results, we need to make additional assumptions on the stock price process. ("There is no free lunch.")

Option Pricing in A Binomial Model

Binomial Model

- The price of the stock is assumed to follow a binomial process:

$$S \;\begin{cases} uS \text{ with probability } p \\ dS \text{ with probability } 1-p \end{cases}$$

  Here, u and d are the gross return on the stock in the up and down state.

- There exists a riskless bond, with a sure payoff of 1 at $t = 1$.
- Let its time-0 price be B and the corresponding interest rate be r_F:

$$B = \frac{1}{1 + r_F}.$$

- NA requires that:

$$d < 1 + r_F < u.$$

Option Pricing in A Binomial Model

Binomial Model

- With two states at $t = 1$ and two securities with independent payoffs, the market is complete.
- Let $\phi = [\phi_u; \phi_d]$ denote the unique state price vector.
- From the prices of the stock and bond, we have:

$$S = \phi_u uS + \phi_d dS,$$

$$B = \phi_u + \phi_d.$$

- It then follows that:

$$\phi_u = \frac{1}{1+r_F}\frac{1+r_F-d}{u-d} \quad \text{and} \quad \phi_d = \frac{1}{1+r_F}\frac{u-(1+r_F)}{u-d}. \tag{2}$$

- For the option price, we then have:

$$c = \phi_u[uS-K]_+ + \phi_d[dS-K]_+. \tag{3}$$

Option Pricing in A Binomial Model

Replication

- We can price an option because the market is complete: we can use the stock and bond to replicate the payoff of any security, including options.
- Note that the payoff of the option is given by:

$$c(S,K) \quad \begin{cases} c_u = [uS - K]_+, & S_1 = uS, \\ c_d = [dS - K]_+, & S_1 = dS. \end{cases}$$

- Consider a portfolio θ: θ_S shares of stock and θ_B units of bond. Its payoff:

$$x = \begin{cases} \theta_S\, uS + \theta_B, & S_1 = uS \\ \theta_S\, dS + \theta_B, & S_1 = dS \end{cases}$$

- Choose θ such that $x_u = c_u$ and $x_d = c_d$. The solution is:

$$\theta_S = \frac{c_u - c_d}{(u-d)S}, \qquad \theta_B = \frac{u\, c_d - d\, c_u}{(u-d)}.$$

- The value of the portfolio at $t = 0$ is:

$$c = \theta_S S + \theta_B \frac{1}{1+r_F} = \frac{1}{1+r_F}\left[\frac{(1+r_F)-d}{u-d} c_u + \frac{u-(1+r_F)}{u-d} c_d\right].$$

Option Pricing in A Binomial Model

Risk-Neutral Pricing

- Given the state prices, we can define the risk-neutral measure:

$$q = \frac{\phi_u}{\phi_u + \phi_d} = \frac{(1 + r_F) - d}{u - d}, \quad 1 - q = \frac{\phi_d}{\phi_u + \phi_d} = \frac{u - (1 + r_F)}{u - d}.$$

- We can rewrite the option pricing formula as:

$$c = \frac{1}{1 + r_F} [q\, c_u + (1 - q)\, c_d] = \frac{\mathbb{E}^{\mathbb{Q}}[c_1]}{1 + r_F},$$

which is the risk-neutral pricing formula.

Option and Market Structure

Completing Market with Options

- We now consider how derivative securities can help to complete markets.
- Suppose that there exists a security with state-separating payoff X:

$$X_\omega \neq X_{\omega'} \ \text{ if } \ \omega \neq \omega', \ \ \forall\, \omega, \omega' \in \Omega.$$

  This security could be a portfolio of securities.

- We call this security an state index security.
- Without loss of generality, assume that $X_\omega < X_{\omega'}$ if $\omega < \omega'$.
- Let us now introduce European options on the state index security.
- A European call on the state index security with strike X_ω yields payoff:

$$\begin{array}{cccccccc} [X - X_\omega]_+ & = & [0; \cdots; & 0; & X_{\omega+1} - X_\omega; & X_{\omega+2} - X_\omega; & \cdots; & X_M - X_\omega] \\ & & & \omega & \omega+1 & \omega+2 & \cdots & M \end{array}$$

Option and Market Structure

Completing Market with Options

- Now consider the following set of securities:

  (1) one share of the state index security,

  (2,...,M) calls on the state index security with strikes $X_1, \ldots, X_{M-1}$, respectively.

- The payoff matrix of these M securities is:

$$X = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ X_2 & X_2 - X_1 & 0 & \cdots & 0 \\ X_3 & X_3 - X_1 & X_3 - X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ X_M & X_M - X_1 & X_M - X_2 & \cdots & X_M - X_{M-1} \end{bmatrix}.$$

- X has full rank of M. Thus, the following securities market is complete:
  - state index security, plus
  - $M - 1$ call options on the state index security.

Option and Market Structure

Completing Market with Options

- Let the state-separating security represents aggregate payoff/consumption.
- For state $\omega = 1, 2, \ldots, M$, let the payoff of the state-separating security be δ, 2δ, …, $M\delta$.
- Consider calls on aggregate payoff with strikes 0, δ, …, $(M-1)\delta$, respectively.
- Note that the call with strike 0 is the underlying security.
- The payoff matrix of these M options is:

$$X = \begin{bmatrix} \delta & 0 & \cdots & 0 \\ 2\delta & \delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M\delta & (M-1)\delta & \cdots & \delta \end{bmatrix}.$$

Option and Market Structure

Completing Market with Options

- Consider the following butterfly position:
  - long 1 call with strike 0
  - short 2 calls with strike δ
  - long 1 call with strike 2δ.
- Its payoff is:

$$[\delta;\delta;\ldots;\delta] - [0;\delta;\ldots;\delta] = [\delta;0;\ldots;0].$$

  This is state-1 contingent claim as it pays δ in state-1 and 0 otherwise.
- Similarly, a butterfly with three strikes being $(\omega-1)\delta,\ \omega\delta,\ (\omega+1)\delta$ pays δ in state ω and nothing otherwise. This is a state-ω contingent claim.
- Thus, we can replicate all state contingent claims with the set of options.
- The option prices will then give the state prices and risk-neutral measure.

Option and Market Structure

Option Prices and State Prices

- Let ϕ_ω denote the state price of ω. Then:

$$\phi_\omega = \frac{1}{\delta}\left\{\left[c(K_{\omega+1}) - c(K_\omega)\right] - \left[c(K_\omega) - c(K_{\omega-1})\right]\right\}.$$

- For any security with payoff $\{D_\omega,\ \omega \in \Omega\}$, its price should be:

$$P = \sum_\omega \frac{\left[c(K_{\omega+1}) - c(K_\omega)\right] - \left[c(K_\omega) - c(K_{\omega-1})\right]}{\delta^2} D_\omega \delta \to \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} D(K) dK.$$

- For the stock (aggregate payoff) and riskless bond, we have:

$$S = \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} X(K) dK, \qquad B = \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} dK.$$

- The risk-neutral density function is:

$$q(K) = \frac{\partial^2 c(K)}{\partial K^2} \Bigg/ \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} dK.$$

Options can be used to extract the $\mathbb{Q}$ density of stock index returns

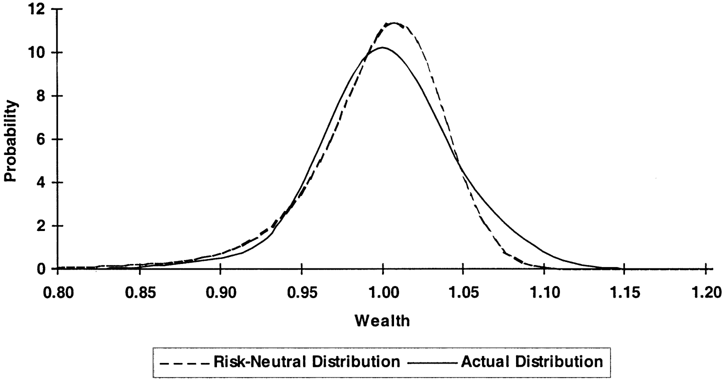


Figure 2
Postcrash risk-neutral and actual distributions.
The risk-neutral distribution is the 31-day option implied distribution on April 15, 1992. The actual distribution is a kernel density of the 31-day nonoverlapping returns over the 4 years prior to April 15, 1992.

Jackwerth (2000), "Recovering Risk Aversion from Option Prices and Realized Returns Jens Carsten Jackwerth", *Review of Financial Studies*

The VIX index is the standard deviation of the $\mathbb{Q}$ distribution

Source: CBOE

Options can be used to extract the $\mathbb{Q}$ measure of "crash risk"

Authors compare measure of implied risk of large stock market declines under $\mathbb{P}$ and $\mathbb{Q}$. Argue this is a proxy for investors' fear

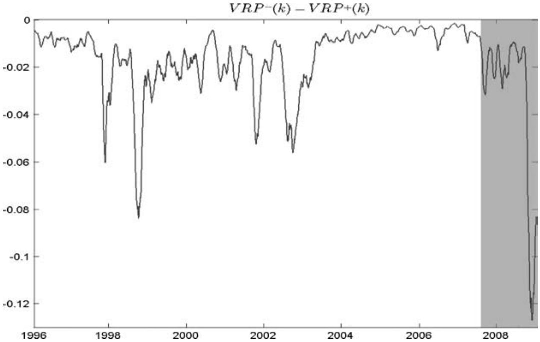


Figure 5. Investor fears. The estimates for the $FI(k)$ Investor Fears index are based on 5-minute S&P 500 futures prices and options. The values in the shaded area, corresponding to July 2007 through December 2008, are based on the parameter estimates for the $\mathbb{P}$ and $\mathbb{Q}$ measures obtained using data through June 2007. The log-moneyness of the options used for the left and right tails is fixed at $k = 0.9$ and $k = 1.1$, respectively.

Bollerslev and Todorov (2011), "Tails, Fears, and Risk Premia", *Journal of Finance*

Futures markets ⇒ measure monetary policy surprises

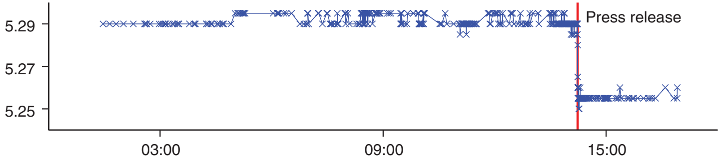


FIGURE 2. INTRADAY TRADING IN GLOBEX FEDERAL FUNDS FUTURES

Note: This figure plots the tick-by-tick trades in the Globex federal funds futures for the FOMC press release on August 8, 2006, with release time at 2:14 PM.

- These measures of changes in expectations are frequently used to estimate the effects of monetary policy

Gorodnichenko and Weber (2015), "Are Sticky Prices Costly? Evidence from the Stock Market", *American Economic Review*

Interest rate options ⇒ changes in expectations about distribution of future real interest rates

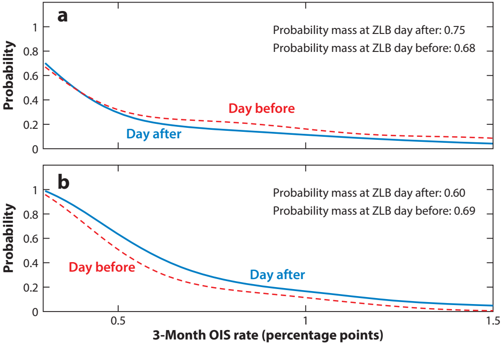


Figure 2

The (forward) risk-neutral PDFs for 3-month OIS rates at the 1-year-horizon as of the days before and after two Federal Open Market Committee meetings: (*a*) August 9, 2011; (*b*) June 19, 2013. These are constructed by local linear regressions using Eurodollar options, as described in Section 2.4. The densities are shifted by the amount of the LIBOR–OIS spread so as to be interpretable as PDFs for 3-month OIS rates. Panels *a* and *b* each show the continuous densities above 30 basis points along with the value of the cumulative distribution function at 30 basis points (probability mass at the zero lower bound). Abbreviations: OIS, overnight index swap; PDF, probability density function.

Wright (2017), "Forward-Looking Estimates of Interest-Rate Distributions", *Annual Review of Financial Economics*