Debreu's Theorem - Proof that the utility representation is continuous

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Introduction

This is the final part of Debreu's theorem that was covered in class.

$\succeq$ is a rational, continuous preference relation on $\mathbb{R}^n_+$. We have constructed a utility function $\alpha(x)$ defined by $\alpha(x)d \sim x$, where d is the n-vector of ones. For the purposes of this note, take it as already proven that $\alpha(x)$ represents $\succeq$. We aim to show here that $\alpha(x)$ is also continuous.

Proof

Take a sequence x_n such that $\lim_{n\to\infty} x_n = x$. We want to show that $\lim_{n\to\infty} \alpha(x_n) = \alpha(x)$.

Suppose towards contradiction that $\lim_{n\to\infty} \alpha(x_n) = \alpha' \neq \alpha(x)$. (How do we know that the limit exists at all? We don't, but for now let us assume it does, and I will deal with the possibility that it doesn't later on).

Then there are two cases: either $\alpha' < \alpha(x)$ or $\alpha' > \alpha(x)$. We will consider each in turn.

Case 1: $\alpha' > \alpha(x)$. Then there exists some $\tilde{\alpha}$ such that $\alpha' > \tilde{\alpha} > \alpha(x)$. Strict monotonicity then implies that $\alpha' d \succ \tilde{\alpha} d \succ \alpha(x)d$.

Now, $\alpha(x_n) \to \alpha'$, so $\exists N$ such that $\forall n > N,\ \alpha(x_n) > \tilde{\alpha}$. Hence, because $\alpha(x_n)d \sim x_n$, we have $x_n \succ \tilde{\alpha} d$ for all $n > N$.

On the other hand, $x \sim \alpha(x)d$. So $\tilde{\alpha}d \succ x$.

So, let us take the sequence $\{x_n\}_{n=N}^{\infty}$. Every element of this sequence is strictly preferred to $\tilde{\alpha}d$. But the limit of this sequence, x, is strictly worse than $\tilde{\alpha}d$. This violates continuity of preferences (why? The set of weakly preferred bundles does not contain all its limit points, so is not closed). Hence we have a contradiction.

Case 2: $\alpha' < \alpha(x)$. This goes very similarly to case 1. There exists some $\tilde{\alpha}$ such that $\alpha(x) > \tilde{\alpha} > \alpha'$. Strict monotonicity then implies that $\alpha(x)d \succ \tilde{\alpha}d \succ \alpha'd$.

Now, again because $\alpha(x_n) \to \alpha'$, $\exists N$ such that $\forall n > N, \;\; \alpha(x_n) < \tilde{\alpha}$. Hence, for all $n > N$, we have $\tilde{\alpha}d \succ x_n$.

On the other hand, $x \sim \alpha(x)d \succ \tilde{\alpha}d$.

So again let us take the sequence $\{x_n\}_{n=N}^{\infty}$. Every element of this sequence is strictly worse than $\tilde{\alpha}d$. But the limit of this sequence, x, is strictly better than $\tilde{\alpha}d$. Again this violates continuity of preferences.

Hence, in either case there is a contradiction. So we cannot have $\lim_{n\to\infty} \alpha(x_n) = \alpha' \neq \alpha(x)$.

Now, back to the complication I mentioned: how do we know $\{\alpha(x_n)\}_{n=1}^{\infty}$ has a limit at all? Well, take some $\epsilon > 0$. Because $x_n \to x$, we know that $\exists N$ such that $||x_n - x|| < \epsilon$ for all $n > N$. This means that in turn we can find some interval $[\alpha_0, \alpha_1]$ such that $\alpha(x_n) \in [\alpha_0, \alpha_1]$ for all $n > N$[1]. In other words, the sequence is bounded. But this means that it must have a convergent subsequence.

Then, we simply take *any* convergent subsequence, and apply the arguments above to show that its limit must be $\alpha(x)$. Finally, we note that if a sequence $\{\alpha(x_n)\}_{n=1}^{\infty}$ is bounded, and all its convergent subsequences converge to the same limit, that sequence must also have the same limit. So $\{\alpha(x_n)\}_{n=1}^{\infty}$ converges to $\alpha(x)$.

[1] If this is unclear to you, the diagram on p.48 of MWG is quite helpful.