14.121: Introduction to General Equilibrium

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General Equilibrium: main issues

1) Welfare theorems
   - First welfare theorem: Every Walrasian Equilibrium is Pareto Optimal (Adam Smith)
   - Second welfare theorem: Every Pareto optimum is a Walrasian Equilibrium with Lump-Sum Transfers
2) Existence: Arrow-Debreu (1954), McKenzie (1954)
3) Properties of equilibrium
   - Uniqueness under strong assumptions, local comparative statics (Debreu)
   - Falsifiability of GE (Sonnenschein, Mantel, Debreu, 1970s)
4) Justifying price-taking and the Walrasian mechanism
   - Informational efficiency (Hayek 1945, Hurwicz 1973, Mount-Reiter 1974)
   - Core convergence (Edgeworth 1881, Shubik 1959, Debreu-Scarf 1964, Aumann 1964)
5) Indivisibilities (Gale-Shapley 1961, Shapley-Scarf 1974)
6) Uncertainty (Arrow, Debreu, Radner, Ross)

Exchange economy: no production

- Finite number of agents $i \in \mathcal{I} = \{1, ..., I\}$
- Finite commodities $\ell \in \mathcal{L} = \{1, ..., L\}$
- Commodities $x \in \mathbb{R}_+^L$
- Each agent has an endowment $\omega^i \in \mathbb{R}_+^L$ and utility $u^i : \mathbb{R}_+ \to \mathbb{R}$

Write as

$$\mathcal{E} = ((u^i, \omega^i)_{i \in \mathcal{I}})$$

Each agent chooses consumption given budget constraint:

$$\max_{x \in \mathbb{R}_+^L} u^i(x) \qquad \text{s.t.} \qquad p \cdot x \leq p \cdot \omega^i$$

Only difference with UMP is that consumer's "wealth" is $p \cdot \omega^i$, the amount she could obtain if she sold her entire endowment

$$B^i(p) = \{x : p \cdot x \leq p \cdot \omega^i\}$$

Walrasian equilibrium for economy $\mathcal{E}$ is a pair $(p, (x^i)_{i \in \mathcal{I}})$ such that:

- Agents are maximizing utilities: for all $i \in \mathcal{I}$

$$x^i \in \arg\max_{x \in B^i(p)} u^i(x)$$

- Markets clear: for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} \omega^i_\ell$$

An allocation $(x^i)_{i \in \mathcal{I}} \in \mathbb{R}^{I \cdot L}_+$ is **feasible** if for all $\ell \in \mathcal{L}$

$$\sum_{i \in \mathcal{I}} x^i_\ell \leq \sum_{i \in \mathcal{I}} \omega^i_\ell$$

Notes:

- Equilibrium involves relative, not absolute prices
- If one market clears, the other does as well

Welfare properties

Given economy $\mathcal{E}$, a feasible allocation x is **Pareto optimal** (or Pareto efficient) if there is no other feasible allocation $\hat{x}$ such that

- $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$
- $u^i(\hat{x}^i) > u^i(x^i)$ for some $i \in \mathcal{I}$

Discussion

- Pareto efficiency is a property of allocations and has nothing to do with which mechanism was used for implementation (e.g., price equilibrium, planning, etc.)
- Pareto efficiency says nothing about distributional justice or equity. e.g., one person can have everything

Assumptions on preferences:

A1) For all agents $i \in \mathcal{I}$, u^i is continuous

A2) For all agents $i \in \mathcal{I}$, u^i is increasing, i.e. $u^i(x') > u^i(x)$ whenever $x' \gg x$

A3) For all agents $i \in \mathcal{I}$, u^i is concave

A4) For all agents $i \in \mathcal{I}$, $\omega^i \gg 0$

First three assumptions are ones we have worked with before (and can be weakened somewhat – increasing to local nonsatiation, concavity to quasi-concavity)

Last assumption however is strong and important

We'll call these the *four assumptions*.

At this stage, we will not be interested in the minimal conditions needed for our results

2×2 exchange economy

Setup:

- $i = 1, 2$ consumers; $\ell = 1, 2$ commodities, consumption vector $x^i = (x_1^i, x_2^i)$
- Each agent is initially endowed with endowment $\omega^i = (\omega_1^i, \omega_2^i) \in \mathbb{R}_+^2$; ω_ℓ^i is i endowment of commodity ℓ.
- Aggregate endowment $\omega = \omega^1 + \omega^2$; Aggregate endowment of good ℓ: $\omega_\ell = \omega_\ell^1 + \omega_\ell^2$

exchange economy $\mathcal{E} = (u_i, \omega^i)_{i=1,2}$.

Edgeworth Box (1881), also called the Edgeworth-Bowley Box

An **allocation** $x \in \mathbb{R}^4$ assigns a non-negative consumption vector to each consumer:

$$x = (x^1, x^2) = ((x_1^1, x_2^1), (x_1^2, x_2^2))$$

Allocation is **feasible** if

$$x_\ell^1 + x_\ell^2 \leq \omega_\ell^1 + \omega_\ell^2, \quad \forall \ell$$

Allocation is **non-wasteful** if the relation holds as an equality

$$x^1 + x^2 = \omega$$

Given price vector $p = (p_1, p_2)$

$$p \cdot \omega^i = p_1 \omega_1^i + p_2 \omega_2^i$$

Terminology

- **Budget set**: $B^i(p) = \{x \in \mathbb{R}^2_+ : p \cdot x \leq p \cdot \omega^i\}$.
  Observe that $B^i(p)$ may extend outside the Edgeworth Box.
- **Offer curve**: vary the price, so the budget constraint changes. At each point on offer curve, budget constraint is tangent to indifference curve.
- **Excess demand**: total agent demand for a commodity minus what is available in the economy
- **Pareto set**: allocations where indifference curves are tangent. If utilities are differentiable, Pareto allocations are characterized by:
$$\frac{\partial u^1}{\partial x_1} / \frac{\partial u^1}{\partial x_2}(x^1) = \frac{\partial u^2}{\partial x_1} / \frac{\partial u^2}{\partial x_2}(\omega - x^1)$$
- **Contract curve**: part of Pareto set where consumers do at least as well as initial endowment

Theorem (First Welfare Thm)

Let $(p,(x^i)_{i\in\mathcal{I}})$ be a Walrasian equilibrium for economy $\mathcal{E}$. If utility is increasing, then allocation $(x^i)_{i\in\mathcal{I}}$ is Pareto optimal

- Provides formal support for Adam Smith's claim that individuals acting in their own interest result in efficient allocation from a societal standpoint
- Even without explicit coordination, decentralized markets where agents simply maximize their utilities given prices are efficient
- Only needed utility is increasing (could get away with LNS)
- Should be emphasized that the model has a number of heroic assumptions; e.g., agents face the same prices, agents are price takers, markets exist for all goods, and we haven't said where prices come from

Theorem (Second Welfare Thm)

Let economy $\mathcal{E}$ satisfy our four assumptions. If $(\omega^i)_{i\in\mathcal{I}}$ is Pareto optimal, then there exists a price vector $p \in \mathbb{R}^L_+$ such that $(p, (\omega^i)_{i\in\mathcal{I}})$ is a Walrasian equilibrium for $\mathcal{E}$.

Implications:

- Does not say that starting from a given endowment, every Pareto optimal allocation is a Walrasian equilibrium.
- Rather, says if we start from a given endowment, then for any Pareto optimal allocation, there is a way to redistribute resources and prices that makes the allocation a Walrasian equilibrium outcome
- Unlike the FWT, convexity plays a role here
- Need strictly positive endowments
- SWT's economic significance more limited than FWT – I'd say that the SWT is an over-rated theorem

Discussion

- Might be tempting to say that while FWT shows "sufficiency" of price equilibrium, SWT shows "necessity" of price equilibrium
  - SWT actually says: once we find an efficient allocation, we could support it with a price equilibrium
  - But in that case, why not simply implement it directly without prices?
- Perhaps a more interesting question may be how do we find an efficient allocation
  - If we had all the preference information, we could feed into an algorithm to compute the efficient allocation without necessarily needing prices
  - Hayek (1945): if preference information is not available to the planner, then we somehow need to elicit enough information to find the efficient allocation; we'll return to discuss a formal sense in which price revelation is necessary for implementing an efficient allocation

- SWT has been used to say that any redistributional goals can be achieved from lump-sum reallocations of endowments and letting the market play out
  - Price equilibrium resulting from initial endowments may have undesirable distributional effects, even though it is Pareto optimal
  - SWT says we can redistribute some endowments to disadvantaged agents so that the price equilibrium with new endowments gets us to a more favorable allocation

- Problem is most characteristics we'd want to base redistribution on are choice variables or are private information
  - If the goal is to help the "poor" and poverty is related to agents' choices, then redistribution will not be lump sum because it will reduce agents' incentives to avoid poverty and hence undermine Pareto efficiency
  - If we wish to help people with low "ability," this characteristic may be private information, and agents may have an incentive to misreport their ability

Separating and Supporting Hyperplanes

Theorem (Separating Hyperplane Theorem)

If $B, C \subseteq \mathbb{R}^L$ convex, non-empty, and $B \cap C = \emptyset$, then there exists a price $p \in \mathbb{R}^L$, $p \neq 0$, such that

$$\sup p \cdot B \leq \inf p \cdot C$$

Note if $b \in B$ and $c \in C$, then $p \cdot b \leq p \cdot c$.

Theorem (Supporting Hyperplane Theorem)

Suppose that $B \subseteq \mathbb{R}^L$ is convex and that x is not an element of the interior of the set B. Then there is a $p \in \mathbb{R}^L$ with $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for all $y \in B$

Note: There are many versions of these results. Sometimes known as Minkowski's theorem or the support function theorem

FOC characterization of efficiency

Goal: use assumptions about utility functions to characterize Pareto optimal allocations

We'll work with our four assumptions and for simplicity, we also assume that for all x,

$$\nabla u^i(x^i) > 0$$

and normalize $u^i(0) = 0$

Define the **utility possibility set**:

$$\mathcal{U} = \{u^1(x^1), ..., u^I(x^I) : x^1, ..., x^I \in \mathbb{R}^L_+, \sum_i x^i \leq \omega^i\}$$

Pareto optimal allocations are those on the "northeast" frontier of the UPS

Consider the following program:

$$\max_{x \in \mathbb{R}_+^{LI}} u^1(x_1^1, ..., x_L^1)$$

$$s.t. \qquad u^i(x_1^i, ..., x_L^i) \geq \bar{u}^i, \qquad i = 2, ..., I$$

$$\sum_i x_\ell^i \leq \sum_i \omega_\ell^i \qquad \ell = 1, ..., L$$

Pareto optimal allocations $x = (x^1, ..., x^I)$ solve this program for different values of $(\bar{u}^2, ..., \bar{u}^I) \geq 0$

We maximize utility of the first consumer subject to feasibility and to other consumers getting at least some pre-specified level of utility

By varying the level of required utility for consumers $2, ..., I$ we can recover the full set of Pareto optimal allocations under monotonicity

Suppose we restrict attention to allocations where $x^i \neq 0$ for all i, hence $(\bar{u}^2, ..., \bar{u}^I) \gg 0$

- Under A1-A3, all of the constraints must be binding
- e.g., if the utility constraint for i were slack, we could reduce x^i a little and increase x^1 the same amount; if the resource constraint were slack, we could increase x^1

Can verify that Kuhn-Tucker theorem is applicable

- λ^i denote the multiplier on the agent i's constraint μ_ℓ be the constraint on commodity ℓ.
- Lagrangian is:

$$\sum_i \lambda^i u^i(x^i) + \mu \cdot \left(\sum_i \omega^i - \sum_i x^i \right)$$

where we adopt the convention that $\lambda^1 = 1$

Since all constraints bind, $\lambda^i > 0$ and $\mu_\ell > 0$

$$\lambda^i \frac{\partial u^i}{\partial x^i_\ell} - \mu_\ell \leq 0, \quad x^i_\ell \geq 0, \quad \left(\lambda^i \frac{\partial u^i}{\partial x^i_\ell} - \mu_\ell\right) x^i_\ell = 0$$

Plus

$$\sum_i x^i = \sum_i \omega^i$$

Can also interpret the Lagrangian as arising from maximizing a social welfare function

$$W(u) = \sum_i \lambda^i u^i(x^i)$$

subject to resource constraints

Suppose that each consumer consumes a positive amount of each good, $x_\ell^i > 0, \quad \forall i, \ell$

KT conditions simplify to:

$$\frac{\partial u^i(x^i)}{\partial x_\ell^i} = \frac{\mu_\ell}{\lambda^i}$$

Or,

$$\text{MRS}_{kl}^i = \frac{\partial u^i / \partial x_k^i}{\partial u^i / \partial x_\ell^i} = \frac{\partial u^j / \partial x_k^j}{\partial u^j / \partial x_\ell^j} = \text{MRS}_{k\ell}^j = \frac{\mu_k}{\mu_\ell}.$$

For every agent and for every commodity pair, MRS must be equal to each other and the ratio of shadow prices μ_k and μ_ℓ

FOC characterization also provides a way to link Pareto optimal allocations to Walrasian equilibria

Utility maximization:

$$\max_{x^i} u^i(x^i) \quad \text{s.t. } p \cdot x^i \leq p \cdot \omega^i$$

Let γ^i be the multiplier on budget constraint. Then to solve each consumer's UMPs at given prices, we have for all i, ℓ:

$$\frac{\partial u^i(x^i)}{\partial x^i_\ell} - \gamma^i p_\ell \leq 0, \qquad x^i_\ell \geq 0, \qquad \left(\frac{\partial u^i(x^i)}{\partial x^i_\ell} - \gamma^i p_\ell\right) x^i_\ell = 0$$

x can be supported as a price equilibrium iff we can find a price vector p and some multiplier vector $\gamma \gg 0$ such that the KT conditions hold

Endowments can then be taken to be $\omega^i = x^i$

These UMP KT conditions are exactly the same as the KT conditions for Pareto optimality!

Last observation implies the Two Welfare theorems

- **First Welfare Theorem**

  If ω and p are given, and each agent maximizes utility, must be the case that at the solution, the consumption bundles $x^1, ..., x^I$, the UMP KT conditions hold and each consumer's budget constraint is satisfied

  Consider the Pareto problem with $\bar{u}^i = u^i(x^i)$ for agents $2, ..., I$.

  Define $\mu_\ell = p_\ell$, $\lambda^i = \frac{1}{\gamma^i}$.

  Then KT conditions for the Pareto problem are solved.

  Hence, any Walrasian equilibrium is Pareto optimal.

- **Second Welfare Theorem**

  If KT conditions for the Pareto program are satisfied, then define prices $p_\ell = \mu_\ell$ and Lagrange multipliers $\gamma^i = \frac{1}{\lambda^i}$

  Note we are now simply going in the other direction

  KT conditions for the UMP tell us that (p, x) is a Walrasian equilibrium with endowment $\omega^i = x^i$

  Also have the interpretation of equilibrium price p_ℓ of good ℓ as the shadow price μ_ℓ of the aggregate endowment of good ℓ in the linear social welfare function

Robinson Crusoe Model: Incorporating Production

Setup:

- 1 consumer, 1 firm, owned by consumer
- Both the consumer and firm act as price-takers (silly, but illustrative)
- 2 goods:
  - Leisure x_1, endowment $\bar{L}$ (24 hours a day)
  - consumption good x_2, endowment $= 0$
- p: price of output
- w: wage rate $=$ price of labor
- Production function $f(z)$, strictly concave
- Firm's profit:

$$pf(z) - wz$$

Firm maximizes profit, taking prices as given:

$$z(p, w) = \arg\max_{\tilde{z}} pf(\tilde{z}) - w\tilde{z}$$

FOC:

$$pf'(z) - w = 0$$

- Output $q(p, w) = f(z(p, w))$
- Profit $\pi(p, w) = pq(p, w) - wz(p, w)$

Consumer owns the firm, so she receives the profit. Budget constraint is:

$$px_2 \leq w(\bar{L} - x_1) + \Pi(p, w)$$

Walrasian equilibrium prices (p^*, w^*) clear the markets:

$$\begin{aligned} x_2(p^*, w^*) &= q(p^*, w^*) && \text{[product market]} \\ z(p^*, w^*) &= \bar{L} - x_1(p^*, w^*) && \text{[labor market]} \end{aligned}$$

Crusoe economy is a special case of a **private ownership economy** with I consumers each of whom shares s_{ij} of firm j's profits

2×2 Production Economy

Setup:

- 2 goods and 2 factors
- Factor 1 is labor and factor 2 is capital, with input prices w and r
- Total endowment L and K (note consumers are in background here)
- Factors produced by constant returns to scale technology

$$q^1 = f^1(\ell^1, k^1) \qquad q^2 = f^2(\ell^2, k^2)$$

- Recall our PMP when production technology is CRS

$$\max pf(k, \ell) - c(w, r, q)$$

Since $q = f(\lambda k, \lambda \ell) = \lambda f(k, \ell)$ and $c(w, r, q) = qc(w, r, 1)$, if p is large we'd have unbounded profits

- With CRS, the FOC of PMP does not provide us with a level of output:

$$p = \frac{\partial c(w, r, q)}{\partial q} = \frac{\partial [qc(w, r, 1)]}{\partial q} = c(w, r, 1)$$

Motivates working with "per unit" cost and factor demands

- With CRS, total cost

$$C^j(w, r, q^j) = \underbrace{\min\{w\ell + rk | f(\ell, k) \geq 1\}}_{c^j(w,r)= \text{ "unit cost"}} \cdot q^j$$

- Denote $a^j_\ell(w, r)$ and $a^j_k(w, r)$ as the unit factor demands, the input combination solving the cost minimization problem

- Shepard's lemma:

$$\frac{\partial c^j(w, r)}{\partial w} = a^j_\ell(w, r) \qquad \frac{\partial c^j(w, r)}{\partial r} = a^j_k(w, r)$$

- Total amount of labor used for producing q^j is: $a^j_\ell q^j$.

$$a^1_\ell q^1 + a^2_\ell q^2 = L$$
$$a^1_k q^1 + a^2_k q^2 = K$$

  Known as "full employment conditions" for economy

- If only one good is produced in equilibrium, we say the economy is *specialized*
- Suppose both goods are produced (interior solutions), then given goods prices, factor prices (w^*, r^*) determined by FOC:

$$p^1 = c^1(w, r) \qquad p^2 = c^2(w, r)$$

- We say production of good 1 is **relatively more intensive** in labor than the production of good 2 if:

$$\frac{a^1_\ell(w, r)}{a^2_\ell(w, r)} > \frac{a^1_k(w, r)}{a^2_k(w, r)}, \qquad \forall (w, r)$$

Assume no reversal of factor intensities and interior solutions

- Factor prices are determined by goods prices (endowment does not matter)
  - **Samuelson (1949)**: Two countries with identical technologies facing same product prices will have same factor prices even if factor endowment differ (version of FPE)
- If output prices change, how do factor prices change?
  - **Stolper-Samuelson (1941)**: If p_j increases, then the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases
- If endowments change, how do outputs change?
  - **Rybczinski (1955)**: if the endowment of a factor increases, then the production of the good that uses this factor relatively more intensively increases and the production of the other good decreases
- Only some of these types of results still hold when there are more factors and goods

Public Goods

A **public good** is a good for which use of a unit by one consumer does not preclude its use by others.

Samuelson (1954, p.387): "each individual's consumption of such a good leads to no subtraction from any other individual's consumption"

This property is known as non-rivalness in consumption.

Classic examples are:

- lighthouse - one boat seeing the light does not prevent other boats from seeing it
- a radio broadcast - one consumer listening to it does not prevent another from doing so
- national defense, air quality

Public goods can either be excludable or non-excludable.

With a non-excludable public good, once it has been provided to one consumer, it is impossible (or prohibitively costly) to prevent others from consuming it. For example, a lighthouse is a non-excludable public good as is national defense.

Pay per TV broadcasts are an excludable public good.

Public goods are sometimes defined to be goods that are both non-rival in consumption and non-excludable. This is a matter of taste.

Many goods lie between the extremes of public good and a private good in the sense that they can be shared, but eventually additional consumers impose negative externalities on others. Like a swimming pool or park. This type of good is sometimes referred to as a public good with congestion or an impure public good.

Public goods are interesting to study because common sense suggests that they will be under-provided by the usual market mechanism.

The neo-classical theory of public goods developed by Samuelson in the 1950s is the starting point to consider this argument.

Neo-classical Theory of Public Goods

Consider a society consisting of n consumers indexed by $i = 1...n$

Suppose there are two goods - a numeraire private good x and a public good G.

Each consumer i has utility given by

$$u^i(x^i, G),$$

which is differentiable, increasing in both arguments, quasi-concave, and $u^{i'}(x^i, 0) > 0$ (this last condition guarantees an interior solution)

Suppose that each consumer i has some endowment of numeraire w^i, so total endowment of private good is $W = \sum_i w^i$. Public good endowment is zero.

Suppose public good may be produced from the private good according to a production function f. If z is the total units of private goods that are used as inputs, the level of public good is:

$$G = f(z).$$

An **allocation** for this community consists of a level of public good G, and allocation of private goods $x = (x^1, ..., x^n)$.

An allocation is **feasible** if there is some $z \geq 0$ such that

(i) $\sum_i x^i + z \leq W$

(ii) $G \leq f(z)$

An allocation $(\mathbf{x}, G)$ is Pareto optimal if there exists no other feasible allocation (x', G') such that

$$
\begin{aligned}
u^i(x^{i'}, G') \geq u^i(x^i, G), &\quad \text{for all } i \\
u^i(x^{i'}, G') > u^i(x^i, G), &\quad \text{for some } i
\end{aligned}
$$

Proposition

An allocation $(\mathbf{x}, G)$ is Pareto efficient if and only if

$$
\sum_{i=1}^{n} \frac{\partial u^i(x^i, G)/\partial G}{\partial u^i(x^i, G)/\partial x^i} = \frac{1}{f'(z)}
$$

Known as the **Samuelson condition** (or sometimes as the Lindahl-Samuelson or Bowen-Lindahl-Samuelson condition)

Sums of the marginal rates of substitutions between the agents is equal to the amount of private good required to produce an additional unit of public good (aka marginal rate of transformation).

Market Provision of Public Good

Suppose there is a competitive market for the private and public good.

p: price of public good (in terms of numeraire-private good)
g^i: quantity of public good purchased by agent i

Assume that there is a single price-taking profit maximizing firm that operates in the market.

How much of the public good would each consumer choose to buy? The key issue is that the amount demanded by an individual consumer depends on what he expects the other consumers to demand. So this means we have a strategic problem, as opposed to a decision-theoretic problem.

A **competitive equilibrium** consists of a p^* and $G^* = (g^{1*}, ..., g^{n*})$ such that

- Each agent's choice of g_i maximizes utility given p^* and $\mathbf{g}^*_{-i}$,
- The firm optimizes given p:

$$max_{z \geq 0} \quad pf(z) - z$$

or

$$p^* = \frac{1}{f'(f^{-1}(\sum_{i=1}^{n} g^{i*}))}.$$

Note this is a bit of an uncomfortable model because agents are price-takers, but they feel like their purchase of the public good can affect the aggregate level of public goods

Proposition

In the competitive equilibrium, there is under-provision of the public good relative to the level prescribed by the Samuelson condition.

Intuition: each agent when deciding how much to purchase, does not consider the benefit to other agents of the output he purchased. Since this is true for each agent, the group of agents purchase less than the amount desirable for Pareto optimality.

The problem with the market is one of **free-riding**: everyone free rides on every else's provision

Personalized Prices

There is a famous "market institution" that can be used to achieve efficiency. This idealized concept is called a **Lindahl equilibrium**

Idea: think of the amount purchased by each agent as a distinct commodity and have each agent face a **personalized price** p^i.

Set these prices so that all agents agree on the level of the public good.

Let $s^i \in [0,1]$ be agent i's share of the firm's profit: $\sum_{i=1}^{n} s^i = 1$

A **Lindahl equilibrium** is a vector $p^* = (p^{1*}, ..., p^{n*})$ and allocation $(x^{1*}, ..., x^{n*}, G^*)$ such that

- Firm maximizes profits $pf(z) - z$:

$$G^* = \arg\max_{G \geq 0} \left(\sum_i p^{i*} \right) G - f^{-1}(G)$$

- Each consumer maximizes utility:

$$(x^{i*}, G^*) = \arg\max_{x^i, G} u^i(x^i, G)$$

  such that

$$w^i + s^i \left(\sum_i p^{i*} G^* - f^{-1}(G^*) \right) - x^i - p^{i*} G \geq 0$$

- Market clears:

$$\sum_{i=1}^{n} x^{i*} + f^{-1}(G^*) \leq \sum_{i=1}^{n} w^i$$

This is a competitive equilibrium in a fictitious economy where the space of goods has been expanded to $(n + 1)$ goods, the private good and n personalized public goods.

The logic of Lindahl equilibrium is that each consumer faces personalized prices which induces him to choose the same level of public good.

Proposition

Any Lindahl equilibrium is Pareto optimal.

The Lindahl equilibrium is unrealistic in the sense that everybody faces a personalized price, which makes them demand the same level of public good - this is like perfect price discrimination.

This is more of a normative benchmark, than a positive description of the market.

By the definition of personalized price in the Lindahl equilibrium, an agent will not behave competitively. He will have an incentive to mis-report his desire for the public good.

Note: we also have the same problem with the case of private goods. However, the crucial distinction is that in the case of private goods, the incentive to reveal false demand functions decreases with the number of agents (see e.g, Roberts and Postlewaite EMA 1976), while an increase in the number of agents in the case of a public good only aggravates the problem.

Conclusion
Market mechanisms are unlikely to provide public goods efficiently; this provides a rationale for government involvement