14.121 Lecture 2: Demand Theory

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Classical demand theory

Let $X \in \mathbb{R}^{L+}$ be the set of possible consumption vectors in a world with L goods.

Let $p \in \mathbb{R}^{L++}$ be the price vector (could allow 0 but won't).

We want $B_{p,w}$ for the **budget set** of a consumer with wealth w.

$$B_{p,w} = \left\{ x \in \mathbb{R}^{L+} \middle| \sum_{i=1}^{n} p_i x_i \leq w \right\}$$

The utility maximization problem (UMP) can be written as

$$\max_{x \in B_{p,w}} u(x)$$

or $\max_{x \in \mathbb{R}^{L+}} u(x)$ such that $p \cdot x \leq w$.

Comments:

1. (smallness) consumption decisions do not affect prices
2. linear prices typically justified (loosely) by possibility of resale
3. no savings

Definition

Walrasian demand correspondence $x(p, w) : \mathbb{R}^{L+} \times \mathbb{R} \rightrightarrows \mathbb{R}^{L+}$ is defined by:

$$x(p, w) = \{z \in B_{p,w} | u(z) = \max_{x \in B_{p,w}} u(x)\}.$$

Note: often a function, but people can be indifferent between several bundles, e.g. when goods are identical.

$x(p, w)$ is also often called the **Marshallian demand** (MWG for some reason don't like this name).

Proposition

Suppose u is continuous and satisfies local nonsatiation. Then

a) *The UMP problem has at least one solution*
b) *$x(p, w)$ is homogenous of degree 0, i.e. $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$,*
c) *$x(p, w)$ satisfies Walras's law, i.e. $p \cdot z = w$ for all $z \in x(p, w)$*
d) *If u is strictly quasi-concave, then $x(p, w)$ is a function, i.e. each $x(p, w)$ contains a single bundle.*

Kuhn-Tucker necessary conditions

Suppose u is continuously differentiable. If $x^* \in x(p, w)$ is a solution to the UMP, then $\exists$ a constant $\lambda \geq 0$, such that

$$\frac{du}{dx_i} = \lambda p_i \quad \text{for all } i \in \{1, 2, ..., L\} \text{ with } x_i^* > 0$$

and

$$\frac{du}{dx_i} \leq \lambda p_i \quad \text{for all } i \in \{1, 2, ..., L\} \text{ with } x_i^* = 0.$$

The constant λ is the "Lagrange multiplier."

Suppose x^* is an interior solution, and consider feasible perturbation x' where agent spends Δ less on j and Δ more on i

Taylor's theorem implies:

$$u(x') \approx u(x^*) + \frac{\partial u}{\partial x_j}(x^*)\left(-\frac{\Delta}{p_j}\right) + \frac{\partial u}{\partial x_i}(x^*)\left(\frac{\Delta}{p_i}\right)$$

Since x^* maximizes utility, perturbation x' cannot increase utility; hence, for small Δ,

$$\frac{\partial u}{\partial x_j}(x^*)\big/p_j \geq \frac{\partial u}{\partial x_i}(x^*)\big/p_i$$

Repeat for x'' where spend Δ more on j, and Δ less on i. Hence

$$\frac{\partial u}{\partial x_j}(x^*)\big/p_j = \frac{\partial u}{\partial x_i}(x^*)\big/p_i$$

Equalize "bang for buck"

Definition

The **elasticity of demand with income** is

$$\eta_i = \frac{\partial x_i}{\partial w} \cdot \frac{w}{x_i} = \frac{\partial \log x_i}{\partial \log w}$$

Elasticity of demand with price is

$$\epsilon_{ij} = \frac{\partial x_i}{\partial p_j} \cdot \frac{p_j}{x_i} = \frac{\partial \log x_i}{\partial \log p_j}$$

An appeal of elasticities is that they are unit-less (though it is important to remember they are evaluated at particular points)

Elasticities must satisfy "adding up constraints" implied by Walras' law

Proposition

Let s_i be the budget share of good i, $s_i = \frac{p_i x_i(p,w)}{w}$. Then

a) $\sum_{i=1}^{n} s_i \eta_i = 1$ *(Engel aggregation),*

b) $\sum_{i=1}^{n} s_i \epsilon_{ij} = -s_j$ *(Cournot aggregation)*

Engel: total expenditure must change by an amount equal to any wealth change

Cournot: total expenditure cannot change in response to change in price

Indirect Utility functions

Definition

The **indirect utility function** $v(p, w) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R}$ is defined by:

$$v(p, w) = \max_{x \in B_{p,w}} u(x)$$

Alternatively, this is $v(p, w) = u(x(p, w))$ if $x(p, w)$ is a function or $v(p, w) = u(x^*)$ for some $x^* \in x(p, w)$.

Proposition

Suppose u is continuous and satisfies local nonsatiation. Then the indirect utility function $v(p, w)$ is

a) *Homogenous of degree 0, i.e. $v(\alpha p, \alpha w) = v(p, w)$,*
b) *Strictly increasing in w and non-increasing in p_i for all i,*
c) *Quasi-convex, i.e. $\{(p, w) | v(p, w) \leq v\}$ is convex for any v*
d) *Continuous in p and w*

Math fact: **Theorem of the Maximum** (or Berge's theorem)

"continuity for correspondences"

Definition

Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f : A \rightrightarrows Y$ is **upper hemicontinuous** if it has a closed graph and images of compact sets are bounded.

- **closed graph**: for any two sequences $x^m \to x$ and $y^m \to y$ with $x^m \in A$ and $y^m \in f(x^m)$ for all m, then $y \in f(x)$
- images of compact sets are bounded: for every compact set $B \subset A$, the set $f(B) = \{y \in Y : y \in f(x) \text{ for some } x \in B\}$ is bounded

If f is a function, then it is upper hemicontinuous correspondence iff it is continuous. [see MWG for proof]

Definition

Given $A \subset \mathbb{R}^N$ and the compact set $Y \subset \mathbb{R}^K$, the correspondence $f : A \rightrightarrows Y$ is **lower hemicontinuous** if for every sequence $x^m \to x \in A$ with $x^m \in A$ for all m, and $y \in f(x)$, we can find a sequence $y^m \to y$ and an integer M such that $y^m \in f(x^m)$ for $m > M$.

Like asking reverse question of uhc.

Likewise can show that if f is a function, then it is lower hemicontinuous correspondence iff it is continuous.

If a correspondence is both upper and lower hemicontinuous, we say it is continuous.

$$\max f : \mathbb{R}^N \to R \text{ subject to } x \in C(q)$$

$C(q)$ is a nonempty constraint set and $q \in Q$ belongs to an admissible set $Q \in \mathbb{R}^S$.

Suppose f is continuous and $C(q)$ is compact for every $q \in Q$.

We already know that there exists a solution. When is the solution "continuous", when is the objective function "continuous"?

Let $x(q)$ be the set of solutions and $v(q)$ be the associated maximum value.

Theorem

Suppose that $f(\cdot)$ is a continuous function and the constraint correspondence $C : Q \rightrightarrows \mathbb{R}^N$ is continuous. Then the maximizer correspondence $x : Q \rightrightarrows \mathbb{R}^N$ is upper hemicontinuous and the value function $v : Q \to \mathbb{R}$ is continuous.

Expenditure Minimization Problem

The Expenditure Minimization Problem (EMP) is

$$\min_{x \in \mathbb{R}^{L+}} p \cdot x \quad \text{s.t.} u(x) \geq u_0$$

or

$$\min_{x \in C_{u_0}} p \cdot x \quad \text{where } C_{u_0} = \{z \in \mathbb{R}^+_L | u(z) \geq u_0\}$$

- minimal level of wealth required to achieve utility u_0

The solution to this is related to the solution to the UMP by a **basic duality result**:

Proposition

Suppose $u(x)$ is a continuous utility function satisfying local nonsatiation and $p \gg 0$. Then

a) *If x^* is a solution to the UMP with wealth w, then x^* is a solution to the EMP with utility $u(x^*)$.*
b) *If x^* is a solution to the EMP with utility u_0, then x^* is a solution to the UMP with wealth $p \cdot x^*$*

The EMP is used mostly to define two functions.

Definition

The **expenditure function** $e : \mathbb{R}^{L+} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$e(p, u) = \min_{x \in \mathbb{R}^{L+},\ s.t.\ u(x) \geq u} p \cdot x \quad \text{or} \quad \min_{x \in C_u} p \cdot x.$$

Definition

The **Hicksian demand correspondence** $h : \mathbb{R}^{L+} \times \mathbb{R} \rightrightarrows \mathbb{R}^{L+}$ is defined by

$$h(p, u) \equiv \{z \in \mathbb{R}^{L+} | p \cdot z = e(p, u) \text{ and } u(z) \geq u\}$$

or

$$h(p, u) \equiv \arg\min_{z \in C_u} p \cdot z$$

or

$$h(p, u) \equiv \{z \in \mathbb{R}^{L+} | z \text{ solves EMP at price } p \text{ utility } u\}.$$

Note that

$$e(p,u) = p \cdot z, \quad \forall z \in h(p,u)$$

$$h(p,u) = \{z \in \mathbb{R}^{L+} | p \cdot z = e(p,u) \quad \text{and } u(z) \geq u\}$$

The Hicksian demand is often called the **compensated** demand function. This is due to the fact that

$$h(p,u) = x(p, e(p,u)),$$

or the Hicksian demand can be used to answer:

"how would demand change if we change prices from p to p' **and** give whatever compensation is necessary so the utility level is unchanged."

Calculating Hicksian demands

FOCs for calculating Hicksian demands are

$$
\begin{aligned}
p_i &= \lambda \frac{du}{dx_i}(h(p,u)) \quad \text{if } h_i(p,u) > 0, \\
p_i &\geq \lambda \frac{du}{dx_i}(h(p,u)) \quad \text{if } h_i(p,u) = 0
\end{aligned}
$$

The following propositions give some basic properties of these functions.

Proposition

Suppose u is a continuous utility function satisfying local nonsatiation. Then, the expenditure function $e(p,u)$ is

a) *homogenous of degree 1 in p, i.e.* $e(\alpha p, u) = \alpha e(p, u)$
b) *strictly increasing in u and non-decreasing in each p_i*
c) *concave in p*
d) *continuous in p, u*

Properties of Hicksian Demand

Proposition

Suppose u is a continuous utility function satisfying local nonsatiation. Then the Hicksian demand correspondence $h(p, u)$ satisfies:

a) *h is homogenous of degree 0 in p, i.e.*
$h(\alpha p, u) = h(p, u), \quad \forall \alpha, p, u$

b) *no excess utility property: $u(x) = u$ for all $x \in h(p, u)$*

c) *If u is strictly quasi-concave, then each $h(p, u)$ contains a single element, i.e. $h(p, u)$ is a function.*

Notes:

a) is a little different from Walrasian demand. That was homogenous of degree 0 in p and x. Here if we just multiple p and keep u constant, budget is implicitly multiplied.

b) is like Walras's Law. If x gives extra utility you could have bought less.

Proofs of all of these are just like the properties of the Marshallian demand, so we'll skip this.

One new result we get with Hicksian demand is the **Compensated Law of Demand**:

Proposition (Compensated Law)

Suppose u is a continuous, strictly quasiconcave utility function satisfying local nonsatiation, then for all p' and p'', we have

$$(p'' - p') \cdot [h(p'', u_0) - h(p', u_0)] \leq 0$$

This says that in some weighted average, the demand goes down for products that have gone up in price.

Corollary

If p_i increases and all other goods' prices are unchanged, then the Hicksian demand for good i decreases, i.e. $h_i(p', u_0) \leq h_i(p, u_0)$ if $p'_i > p_i$ and $p'_j = p_j, \forall j \neq i$

We saw earlier that $e(p, u)$ is calculated easily from the Hicksian demand. There is also a nice theorem that lets one go in the other direction.

Proposition

Suppose u is a continuous, strictly quasiconcave utility function satisfying local nonsatiation. Then for $i = 1, 2, ..., L$, we have

$$h_i(p, u) = \frac{d}{dp_i} e(p, u)$$

There are a few ways to prove this result. We will use the Envelope Theorem.

Corollary

Suppose u is continuous, quasi-concave and satisfies local nonsatiation and h is continuously differentiable. Then $\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$

Envelope Theorem

Start with simple version

Proposition

Let $\phi(\alpha) = \max_{x \in \mathbb{R}^L} f(x;\alpha)$*. Suppose* ϕ *is differentiable at* $\bar{\alpha}$ *and* $f(x;\alpha)$ *is uniquely maximized at* $x(\bar{\alpha})$*. Then*

$$\frac{d\phi}{d\alpha_i} = \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha});\bar{\alpha})$$

Proof.

Follows directly from the chain rule:

$$\frac{d\phi}{d\alpha_i} = \frac{\partial f(x(\bar{\alpha});\bar{\alpha})}{\partial \alpha_i} + \sum_{j=1}^{L} \frac{\partial f}{\partial x_j}(x(\bar{\alpha});\bar{\alpha})\frac{\partial x_j}{\partial \alpha_i}(\bar{\alpha}) = \frac{\partial f(x(\bar{\alpha});\bar{\alpha})}{\partial \alpha_i}$$

as the second term drops out since $x(\bar{\alpha})$ maximizes f. $\square$

A variant that you may not have seen is what happens when you have constrained maximization.

Proposition

Suppose $\phi(\alpha) = \max_{x \in \mathbb{R}^L} f(x; \alpha)$ *such that* $g_i(x; \alpha) = b_i$ *for* $i = 1..m$. *Let* $x(\alpha)$ *be the maximizer to this problem. Assume that* ϕ *is differentiable and* $\lambda_1, ..., \lambda_m$ *are the Lagrange multiplier associated with the constraints. Then*

$$\frac{d\phi}{d\alpha_i}(\bar{\alpha}) = \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha}); \bar{\alpha}) - \sum_{m=1}^{M} \lambda_m \frac{dg}{d\alpha_i}(x(\bar{\alpha}); \bar{\alpha})$$

As before, we start with the chain rule:

$$\frac{d\phi}{d\alpha_i}(\bar{\alpha}) = \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha});\bar{\alpha}) + \sum_{j=1}^{L} \frac{\partial f}{\partial x_j}(x(\bar{\alpha});\bar{\alpha})\frac{dx_j}{d\alpha_i}(\bar{\alpha})$$

Now, however, we can only substitute KT conditions:

$$\frac{\partial f}{\partial x_j}(x(\bar{\alpha});\bar{\alpha}) = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m}{\partial x_j}(x(\bar{\alpha});\bar{\alpha})$$

Hence

$$\begin{aligned}
\frac{d\phi}{d\alpha_i}(\bar{\alpha}) &= \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha});\bar{\alpha}) + \sum_{j=1}^{L} \left(\sum_{m=1}^{M} \lambda_m \frac{\partial g_m}{\partial x_j}(x(\bar{\alpha});\bar{\alpha}) \right) \frac{\partial x_j}{\partial \alpha_i}(\bar{\alpha}) \\
&= \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha});\bar{\alpha}) + \sum_{m=1}^{M} \lambda_m \sum_{j=1}^{L} \frac{\partial g_m}{\partial x_j}(x(\bar{\alpha});\bar{\alpha})\frac{dx_j}{d\alpha_i}(\bar{\alpha})
\end{aligned}$$

Because $g_m(x(\bar{\alpha}); \bar{\alpha}) = b_m$ for all $\bar{\alpha}$ we have

$$0 = \frac{\partial g_m}{\partial \alpha_i} + \sum_{j=1}^{L} \frac{\partial g_m}{\partial x_j} \frac{dx_j}{d\alpha_i}$$

or

$$\sum_{j=1}^{L} \frac{\partial g_m}{\partial x_j}(x(\bar{\alpha}); \bar{\alpha}) \frac{dx_j}{d\alpha_i}(\bar{\alpha}) = -\frac{dg_m}{d\alpha_i}(x(\bar{\alpha}); \bar{\alpha})$$

Substituting this in gives us the desired result:

$$\frac{d\phi}{d\alpha_i}(\bar{\alpha}) = \frac{\partial f}{\partial \alpha_i}(x(\bar{\alpha}); \bar{\alpha}) - \sum_{m=1}^{M} \lambda_m \frac{dg}{d\alpha_i}(x(\bar{\alpha}); \bar{\alpha})$$

Slutsky substitution

Slutsky thought about compensation a little differently. Instead of just giving enough to achieve the same utility, imagine giving $\Delta p \cdot x$ so people can afford same bundle. This usually makes them strictly better off.

Definition

The **Slutsky substitution matrix** $S(p, w)$ is the $L \times L$ matrix with

$$s_{ij} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j.$$

This means the change in x_i resulting from changing p_j **and** giving consumer wealth $x_j dp_j$ to make old bundle affordable.

What happens to the consumption of good j as the price of good i rises?

1) change in relative price of the k goods (substitution), and
2) reduction of real income of consumer (income effect)

Proposition

Suppose u is continuous, locally nonsatiated, and strictly quasi-concave. Let $w = e(p, u_0)$. Then

$$\frac{\partial h_i}{\partial p_j}(p, u_0) = \frac{\partial x_i}{\partial p_j}(p, w) + \frac{\partial x_i}{\partial w}(p, w) x_j(p, w) = s_{ij}$$

Rewriting:

$$\frac{\partial x_i}{\partial p_j}(p, w) = \underbrace{\frac{\partial h_i}{\partial p_j}(p, u_0)}_{\text{relative price effect}} - \underbrace{\frac{\partial x_i}{\partial w}(p, w) x_j(p, w)}_{\text{real income effect}}$$

Corollary

Under the conditions above, the Slutsky substitution matrix is symmetric and negative semi-definite.

Symmetry - can understand why same sign: if rise in price of good i causes demand for good j to rise, then j is in some sense substituting for i. If j substitutes for i, then i must substitute for j.

No easy intuition for why they are equal: if 1 cent rise in price of good i means a fall in demand for good j of 10 units, then a 1 cent rise in price of good j means a fall in demand for good i of 10 units.

We saw how $h(p, u)$ is the derivative with respect to p of the EMP's value function $e(p, u)$.

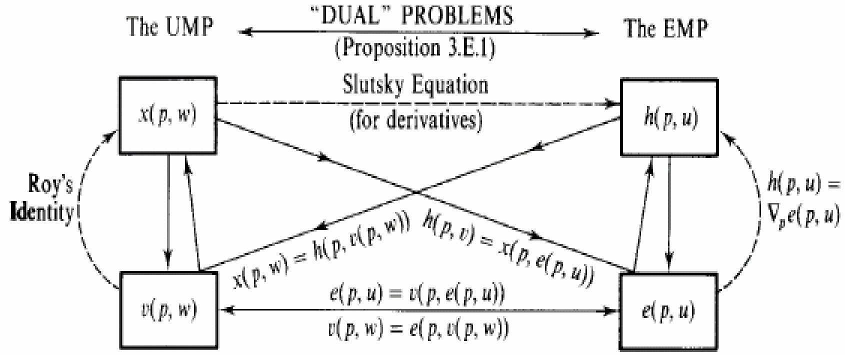
Do not have exact same statement for UMP because Walrasian demand is an ordinal concept.

Roy's Identity - provides analogous result when we normalize utility by marginal utility of wealth.

Proposition

Suppose that $u(\cdot)$ is continuous utility function and preferences are locally nonsatiated and strictly convex. Suppose that the indirect utility function is differentiable at $(p^*, w^*) \gg 0$. Then for every $i = 1, ..., L$,

$$x_i(p^*, w^*) = -\frac{\partial v(p^*, w^*)}{\partial p_i} \Bigg/ \frac{\partial v(p^*, w^*)}{\partial w}.$$



MWG Figure 3.G.3