14.121 Problem Set 1

Due: 9/18 in class

1. Consider a consumer facing an uncertain outcome $\theta \in \{\theta_1, \theta_2\}$. The consumer has a utility function $u(\cdot, \theta) : \mathbb{R}_+^L \to \mathbb{R}$ on bundles $x = (x_1, .., x_L)$ for each realization of θ. The consumer can contract ex ante to receive a bundle $x(\theta) = (x_1(\theta), .., x_L(\theta))$ for each potential realization of θ. The realization of θ is private information. In particular, ex post the agent can claim that their type is $\bar{\theta}$ and consequently receive $x(\bar{\theta})$, even if their true type is $\theta \neq \bar{\theta}$. We say that a contract is *incentive compatible* (IC) if ex post the agent never has an incentive to lie about their type.

a. Describe the commodity space. What is the number of commodities M?

b. Write down two inequalities that must be satisfied by an incentive compatible contract $(x(\theta_1), x(\theta_2))$.

c. The consumer's consumption set X is the subset of $\mathbb{R}_+^M$ consisting of incentive compatible contracts. Suppose that there exist incentive compatible contracts $(x(\theta_1), x(\theta_2)')$ and $(x(\theta_1), x(\theta_2)'')$ such that $u(x(\theta_1), \theta_1) = u(x(\theta_2)', \theta_1)$, $u(x(\theta_1), \theta_1) = u(x(\theta_2)'', \theta_1)$, and $x(\theta_2)' \neq x(\theta_2)''$. Show that if $u(\cdot, \theta_1)$ is strictly concave, then X is not convex.

Suppose now that the underlying commodity space is a finite set $C = \{c_1, .., c_n\} \subseteq \mathbb{R}_+^L$. Suppose the consumer can contract ex ante to receive c_i with probability $\pi(c_i, \theta)$ for each potential realization of θ. Consumers have expected utility preferences over lotteries, $U(\pi, \theta) = \sum_{i=1}^n \pi(c_i) u(c_i, \theta)$

d. The probability distributions must satisfy $\pi(c_i, \theta_j) \geq 0$ for each $i = 1, .., n$ and $j = 1, 2$, and $\sum_{i=1}^n \pi(c_i, \theta_j) = 1$ for $j = 1, 2$. Write down two additional inequalities that must be satisfied by an incentive compatible contract $(\pi(c_1, \theta_1), ..., \pi(c_n, \theta_2))$.

e. The consumer's consumption set $\tilde{X}$ is the subset of $\mathbb{R}_+^{2n}$ satisfying incentive compatibility and the probability distribution requirements (the probabilities in each state sum to one). Show that $\tilde{X}$ is convex.

2. Take a private ownership economy $\mathcal{E} = \left\{ \{X_i, \succsim_i, \omega_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\theta_{ij}\}_{i=1,j=1}^{I,J} \right\}$ and a firm j such that $Y_j \subseteq \mathbb{R}^L$ exhibits constant returns to scale. Show that, in any Walrasian equilibrium (x^*, y^*, p) firm j must have zero profits, i.e. $py_j^* = 0$.

3. MWG 16.E.2

4. Consider a pure exchange economy with 2 agents and 2 commodities x_1 and x_2. Each consumer has an endowment $\omega^i = (\omega_1^i, \omega_2^i)$ and identical homothetic preferences on $\mathbb{R}_+^2$ that can be represented by a continuously differentiable utility function $u : \mathbb{R}_+^2 \to \mathbb{R}$ that is homogeneous of degree 1 (or satisfying $u(\alpha x_1, \alpha x_2) = \alpha u(x_1, x_2)$ for $\alpha > 0$), strictly increasing, and exhibits diminishing marginal utility. In particular, $\frac{\partial u(x_1, x_2)}{\partial x_l} > 0$ and $\frac{\partial^2 u(x_1, x_2)}{\partial x_l^2} < 0$ for all (x_1, x_2) for $l = 1, 2$, and $\lim_{x_l \to 0} \frac{\partial u(x_1, x_2)}{\partial x_l} = \infty$ for $l = 1, 2$.

a. Let $u_l(x_1, x_2)$ denote the partial derivative of u with respect to the l coordinate. Show that a partial derivative of homogeneous of degree 1 function is homogeneous of degree 0, or satisfying

$u_l(\alpha x_1, \alpha x_2) = u_l(x_1, x_2)$ for all $\alpha > 0$. We can thus effectively write the derivative as a function of the ratio, or $u_l(x_1, x_2) = u_l(x_1/x_2, 1)$

b. Characterize the set of all Pareto optimal allocations in this economy. In an Edgeworth box, draw the set of Pareto optimal allocations. Hint: you may find the result from part a to be useful. Clarification: please describe all the allocations $((x_1^1, x_2^1), (x_1^2, x_2^2))$ which are in the Pareto set as functions of the endowments

c. Characterize the core of this economy. Clarification: please describe all the allocations $((x_1^1, x_2^1), (x_1^2, x_2^2))$ which are in the core as functions of the endowments

d. Find the Walrasian equilibrium prices $(1, p)$ and allocation $((x_1^1, x_2^1), (x_1^2, x_2^2))$ for this economy, where the price of good 1 is normalized to 1. Show that the allocation is Pareto optimal. Clarification: please express the price p and the allocation $((x_1^1, x_2^1), (x_1^2, x_2^2))$ as functions of the endowments. Note that these functions may involve the derivatives u_1 and u_2, but please be clear where they are evaluated.

e. (Euler's Theorem) Show that a homogeneous of degree 1 function satisfies $u(x_1, x_2) = x_1 u_1(x_1, x_2) + x_2 u_2(x_1, x_2)$.

f. Show directly that the Walrasian equilibrium belongs to the core. Hint: you may find the result from part e to be useful. Clarification: using part d, you can write an agent's utility at the Walrasian equilibrium as a function of the endowments. Show directly that this is greater than or equal to the utility evaluated at that agent's endowment.