14.121 Problem Set 4 Solutions

Question 1

a. Let $\tilde{x}_l = x_l - \underline{x}_l$. Then the consumer's problem is equivalent to

$$\max_{\tilde{x}_l} \left[\sum_{l=1}^{L} \alpha_l \tilde{x}_l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad \text{s.t.} \quad \sum_l p_l \tilde{x}_l = w - \sum_l p_l \underline{x}_l$$

The Lagrangian is

$$\left[\sum_{l=1}^{L} \alpha_l \tilde{x}_l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \lambda \left(w - \sum_l p_l \underline{x}_l - \sum_l p_l \tilde{x}_l \right)$$

FOCs

$$\alpha_l \tilde{x}_l^{\frac{-1}{\sigma}} \left[\sum_{l=1}^{L} \tilde{x}_l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} = \lambda p_l$$

Hence

$$\begin{aligned} \left(\frac{\tilde{x}_l}{\tilde{x}_l} \right)^{\frac{-1}{\sigma}} &= \frac{\alpha_1 p_l}{\alpha_l p_1} \\ \implies \tilde{x}_l &= \tilde{x}_1 \left(\frac{p_l}{p_1} \right)^{-\sigma} \left(\frac{\alpha_l}{\alpha_1} \right)^{\sigma} \end{aligned} \tag{1}$$

Now substitute (1) into the budget constraint to obtain

$$\begin{aligned} \tilde{x}_l &= \alpha_l^{\sigma} p_l^{-\sigma} \frac{w - \sum_k p_k \underline{x}_k}{\sum_k \alpha_k^{\sigma} p_k^{1-\sigma}} \\ &= \alpha_l^{\sigma} p_l^{-\sigma} \left(w - \sum_k p_k \underline{x}_k \right) P^{\sigma-1} \end{aligned}$$

where

$$P = \left[\sum_l \alpha_l^{\sigma} p_l^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

is commonly called the CES price index in applications. Indirect utility is then

$$\begin{aligned} v(p, w) &= \frac{w - \sum_l p_l \underline{x}_l}{\left[\sum_l \alpha_l^{\sigma} p_l^{1-\sigma} \right]^{\frac{1}{1-\sigma}}} \\ &= -\frac{\sum_l p_l \underline{x}_l}{\left[\sum_l \alpha_l^{\sigma} p_l^{1-\sigma} \right]^{\frac{1}{1-\sigma}}} + \frac{1}{\left[\sum_l \alpha_l^{\sigma} p_l^{1-\sigma} \right]^{\frac{1}{1-\sigma}}} w \end{aligned}$$

which could also be written

$$v(p,w) = \left(w - \sum_l p_l \underline{x}_l\right) P^{-1}$$

b. The limit as $\sigma \to \infty$ is straightforward. Note that

$$\log U = \frac{\log\left(\sum_l \alpha_l \tilde{x}_l^{\frac{\sigma-1}{\sigma}}\right)}{\left(\frac{\sigma-1}{\sigma}\right)}$$

By L'Hopital's rule,

$$\begin{aligned}
\lim_{\sigma\to 1} \log U &= \lim_{\sigma \to 1} \frac{\frac{1}{\sigma^2}\sum_l \alpha_l \log(\tilde{x}_l)\tilde{x}_l^{\frac{\sigma-1}{\sigma}}}{\frac{1}{\sigma^2}\left(\sum_l \alpha_l \tilde{x}_l^{\frac{\sigma-1}{\sigma}}\right)} \\
&= \sum_l \alpha_l \log(\tilde{x}_l)
\end{aligned}$$

Since log is continuous, this implies

$$\lim_{\sigma\to 1} U = \Pi_l \tilde{x}_l^{\alpha_l}$$

Suppose WLOG $\tilde{x}_1 \in \min_l\{\tilde{x}_l\}$. Then

$$\begin{aligned}
\lim_{\sigma\to 0} U &= \lim_{\sigma\to 0} \tilde{x}_1 \left[\sum_l \alpha_l \left(\frac{\tilde{x}_l}{\tilde{x}_1}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= \tilde{x}_1
\end{aligned}$$

c. The demand and indirect utility functions can be computed using the limit utility functions or by taking the demand and indirect utility functions to the limit. I do the latter. Note that we can write the demand as

$$\tilde{x}_l(p,w) = \frac{w - \sum_k p_k \underline{x}_k}{\sum_k \left(\frac{\alpha_k/p_k}{\alpha_l/p_l}\right)^{\sigma} p_k}$$

One can check that

$$\begin{aligned}
&\lim_{\sigma\to 1} \tilde{x}_l(p,w) = \alpha_l \frac{w - \sum_k p_k \underline{x}_k}{p_l} \\
&\lim_{\sigma\to 1} v(p,w) = (w - \sum_k p_k \underline{x}_k)\Pi_l \left(\frac{\alpha_l}{p_l}\right)^{\alpha_l} \\
&\lim_{\sigma\to 0} \tilde{x}_l(p,w) = \frac{w - \sum p_k \underline{x}_k}{\sum_k p_k} \\
&\lim_{\sigma\to 0} v(p,w) = \frac{w - \sum p_k \underline{x}_k}{\sum_k p_k}
\end{aligned}$$

Taking the limit as $\sigma \to \infty$ obtains

$$\tilde{x}_l(p,w) = \begin{cases} 0 & \frac{\alpha_l}{p_l} \neq \max_k\{\alpha_k/p_k\} \\ \frac{w - \sum p_k \underline{x}_k}{\sum_{\alpha_r/p_r \in \text{argmax}\{\alpha_k/p_k\}_k} p_r} & \frac{\alpha_l}{p_l} \in \max_k\{\alpha_k/p_k\} \end{cases}$$

In this case, taking the limit only yields one element of the demand correspondence. In general, the demand correspondence consists of vectors $\tilde{x}$ such that

$$\sum_{\alpha_r/p_r \in \text{argmax}\{\alpha_k/p_k\}_k} p_r \tilde{x}_r = w - \sum p_k \underline{x}_k$$

Then

$$v(p,w) = \max_l \{\alpha_l/p_l\}_l \left(w - \sum p_k \underline{x}_k\right)$$

Question 2

a. The Lagrangian for the planner's problem is

$$\sum_i \lambda_i \left(\sum_l \alpha_l^i \log x_l^i\right) - \sum_l \mu_l \left(x_l^1 + x_l^2 - \bar{\omega}_l\right)$$

The FOC implies

$$\frac{\lambda_i \alpha_l^i}{x_l^i} = \mu_l$$

Market clearing implies

$$\mu_l = \frac{\sum_i \lambda_i \alpha_l^i}{\bar{\omega}_l}$$

So

$$x_l^i = \frac{\lambda_i \alpha_l^i}{\sum_i \lambda_i \alpha_l^i} \bar{\omega}_l$$

In a Walrasian equilibrium with transfers the price is given by

$$p_l(\lambda) = \mu_l = \frac{\sum_i \lambda_i \alpha_l^i}{\bar{\omega}_l}$$

and the wealth levels are

$$\begin{aligned} w^i(\lambda) &= \sum_l p_l(\lambda) x_l^i(\lambda) \\ &= \sum_l \lambda_i \alpha_l^i \\ &= \lambda_i \end{aligned}$$

In a Walrasian equilibrium (without transfers), the wealth levels are restricted to satisfy

$$\begin{aligned} w^i(\lambda) &= \sum_l p_l(\lambda)\omega_l^i \\ &= \sum_l \sum_j \lambda_j \alpha_l^j \frac{\omega_l^i}{\bar{\omega}_l} \\ &= \sum_j \lambda_j \sum_l \alpha_l^j \frac{\omega_l^i}{\bar{\omega}_l} \end{aligned}$$

Thus we have a system of equations

$$\lambda_i = \sum_j \lambda_j \sum_l \alpha_l^j \frac{\omega_l^i}{\bar{\omega}_l} \tag{2}$$

or

$$\lambda = A\lambda$$

where

$$A = \left(\sum_l \alpha_l^j \frac{\omega_l^i}{\bar{\omega}_l} \right)_{i,j}$$

b. The system becomes

$$\begin{aligned} \lambda_i &= \sum_j \lambda_j \sum_l \alpha_l^j \frac{1}{I} \\ &= \frac{1}{I} \sum_j \lambda_j \end{aligned}$$

which has solution $\lambda_i = \frac{1}{I}$.

c. For any λ, the right hand side of (2), $\sum_j \lambda_j \sum_l \alpha_l^j \frac{\omega_l^i}{\bar{\omega}_l}$, is strictly larger for $i = 1$ than for $i = 2$. Hence it must also be strictly larger at the solution λ^* which corresponds to the Walrasian equilibrium, which implies on the left hand side $\lambda_1^* > \lambda_2^*$.

Question 3

a. Denote the relative price of good 2 by p. Since agent 1 has Cobb-Douglas utility, his optimal bundle is determined based on the shares of his wealth $w_1 = \omega_1^1 + p\omega_2^1$, so

$$(x_1^1, x_2^1) \quad = \quad (\alpha(\omega_1^1 + p\omega_2^1), (1-\alpha)(p^{-1}\omega_1^1 + \omega_2^1))$$

Since agent 2 has linear utility, his optimal bundle is to invest everything in the good with the highest benefit/cost ratio, i.e.

$$(x_1^2, x_2^2) = \begin{cases} (0, p^{-1}\omega_1^2 + \omega_2^2) & \beta > p \\ (\omega_1^2 + p\omega_2^2, 0) & \beta < p \\ (x_1^2, x_2^2) | x_1^2 + \beta x_2^2 = \omega_1^2 + \beta\omega_2^2 & \beta = p \end{cases}$$

By Walras' law, if one of the goods market clears, then the other must clear as well, so we can determine the price using market clearing for either good.

If $p < \beta$, market clearing in good 1 implies

$$\begin{aligned} \omega_1^1 + \omega_1^2 &= x_1^1 + x_1^2 \\ &= \alpha(\omega_1^1 + p\omega_2^1) + 0 \\ \implies p &= \frac{(1-\alpha)\omega_1^1 + \omega_1^2}{\alpha\omega_2^1} \end{aligned}$$

Note that this requires the parametric restriction

$$\beta > \frac{(1-\alpha)\omega_1^1 + \omega_1^2}{\alpha\omega_2^1}$$

If $p > \beta$, market clearing in good 2 implies

$$\begin{aligned} \omega_2^1 + \omega_2^2 &= x_2^1 + x_2^2 \\ &= (1-\alpha)(p^{-1}\omega_1^1 + \omega_2^1) + 0 \\ \implies p &= \frac{(1-\alpha)\omega_1^1}{\omega_2^2 + \alpha\omega_2^1} \end{aligned}$$

Note that this requires the parametric restriction

$$\beta < \frac{(1-\alpha)\omega_1^1}{\omega_2^2 + \alpha\omega_2^1}$$

Finally, if $p = \beta$, the allocation is

$$\begin{aligned} x_1^1 &= \alpha(\omega_1^1 + \beta\omega_2^1) \\ x_2^1 &= (1-\alpha)(\beta^{-1}\omega_1^1 + \omega_2^1) \\ x_1^2 &= \omega_1^1 + \omega_2^1 - \alpha(\omega_1^1 + \beta\omega_2^1) \\ x_2^2 &= \omega_1^2 + \omega_2^2 - (1-\alpha)(\beta^{-1}\omega_1^1 + \omega_2^1) \end{aligned}$$

and the nonnegativity constraints require the parametric restrictions

$$\beta \le \frac{(1-\alpha)\omega_1^1 + \omega_1^2}{\alpha\omega_2^1}$$
$$\beta \ge \frac{(1-\alpha)\omega_1^1}{\alpha\omega_2^1 + \omega_2^2}$$

b. Take Harberger's convention and set $p = 1$. Since consumer 2 consumes positive quantities of both commodities, we must be in the $p = \beta$ case. Thus $\beta = p = 1$. To find α, substitute the endowments and $\beta = 1$ into the equation for x_1^2 to get

$$\begin{aligned} 5 &= 5(1-\alpha) + 10 - 25\alpha \\ \implies \alpha &= \frac{1}{3} \end{aligned}$$

We could also check that

$$
\begin{aligned}
x_1^1 &= \alpha(\omega_1^1 + \beta\omega_2^1) = \frac{1}{3}(5+25) = 10 \\
x_2^1 &= (1-\alpha)(\beta^{-1}\omega_1^1 + \omega_2^1) = (1-\frac{1}{3})(5+25) = 20 \\
x_1^2 &= \omega_1^1 + \omega_2^1 - \alpha(\omega_1^1 + \beta\omega_2^1) = (5+10) - 10 = 5 \\
x_2^2 &= \omega_1^2 + \omega_2^2 - (1-\alpha)(\beta^{-1}\omega_1^1 + \omega_2^1) = (25+10) - 20 = 15
\end{aligned}
$$

c. Take the calibration as above, i.e. $\beta = 1$ and $\alpha = \frac{1}{3}$. With a value-added tax, the budget constraint becomes

$$
\begin{aligned}
1.1x_1^i + px_2^i &\leq \omega_1^i + p\omega_2^i \\
\iff x_1^i + \frac{p}{1.1}x_2^i &\leq \frac{1}{1.1}(\omega_1^i + p\omega_2^i)
\end{aligned}
$$

Then the demands for agent 1 are given by

$$(x_1^1, x_2^1) = (\alpha\frac{1}{1.1}(\omega_1^1 + p\omega_2^1), (1-\alpha)(p^{-1}\omega_1^1 + \omega_2^1))$$

and the allocation for agent 2 depends on the relationship between $\frac{p}{1.1}$ and $\beta = 1$. Suppose for a contradiction $\frac{p}{1.1} < 1$. Then

$$(x_1^2, x_2^2) = (0, p^{-1}\omega_1^2 + \omega_2^2)$$

Market clearing for the first good implies

$$
\begin{aligned}
\omega_1^1 + \omega_1^2 &= \alpha\frac{1}{1.1}(\omega_1^1 + p\omega_2^1) + 0 \\
5 + 10 &= \frac{1}{3}\frac{1}{1.1}(5 + 25p) \\
\implies p &= 1.78 > 1.1
\end{aligned}
$$

Contradicting $p_2 < 1.1$. Suppose for a contradiction that $\frac{p_2}{1.1} > 1$. Then

$$(x_1^2, x_2^2) = (\frac{1}{1.1}(\omega_1^2 + p\omega_2^2), 0)$$

Market clearing for the second good implies

$$
\begin{aligned}
\omega_2^1 + \omega_2^2 &= (1-\alpha)(p^{-1}\omega_1^1 + \omega_2^1) + 0 \\
25 + 10 &= (1-\frac{1}{3})(5p^{-1} + 25) \\
\implies p &= \frac{2}{11} < 1.1
\end{aligned}
$$

Contradicting $p > 1.1$. So it must be that $\frac{p}{1.1} = 1$, or $p = 1.1$. The relative price (net of tax) remains the same. Note that the allocation is feasible:

$$
\begin{aligned}
x_1^1 &= \alpha(\frac{1}{1.1}\omega_1^1 + \omega_2^1) = \frac{1}{3}(\frac{1}{1.1} * 5 + 25) \approx 9.85 \\
x_2^1 &= (1-\alpha)(\frac{1}{1.1}\omega_1^1 + \omega_2^1) \approx (1 - \frac{1}{3})(\frac{1}{1.1} * 5 + 25) \approx 19.7 \\
x_1^2 &= \frac{1}{1.1}(\omega_1^1 + \omega_1^2) - \alpha(\frac{1}{1.1}\omega_1^1 + \omega_2^1) = \frac{1}{1.1}(5 + 10) - 9.85 \approx 3.79 \geq 0 \\
x_2^2 &= \omega_2^1 + \omega_2^2 - (1-\alpha)(\frac{1}{1.1}\omega_1^1 + \omega_2^1) = (25 + 10) - 19.7 \approx 15.3 \geq 0
\end{aligned}
$$