14.381: Statistics
Fall, 2012

Suggested answers

1. (a) Likelihood:

$$L(\beta,\sigma^2|y_1,...,y_n)=\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\{-\frac{1}{2\sigma^2}\sum_i(y_i-\beta x_i)^2\}=$$

$$=\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\{-\frac{1}{2\sigma^2}\sum_i x_i^2\}\exp\left\{-\frac{1}{2\sigma^2}\left(\sum_i y_i^2-2\beta\sum_i y_ix_i\right)\right\}$$

Given that x_i are constants, we see from factorization theorem that $(\sum_i y_i^2, \sum_i y_ix_i)$ is two-dimensional sufficient statistics.

(b) Write the first order conditions for $L(\beta,\sigma^2)\to\max$ and find that $\hat\beta=\frac{\sum y_ix_i}{\sum x_i^2}$.

$$E\hat\beta=E\frac{\sum y_ix_i}{\sum x_i^2}=\frac{\sum E(y_i)x_i}{\sum x_i^2}=\frac{\sum \beta x_ix_i}{\sum x_i^2}=\beta.$$

We used that x_i are constants and that $Ey_i=\beta x_i$.

(c) $\hat\beta=\frac{\sum y_ix_i}{\sum x_i^2}$ is a linear combination of normal random variables y_i, thus it is normal. We need to find its variance:

$$Var(\hat\beta)=Var\left(\frac{\sum y_ix_i}{\sum x_i^2}\right)=\frac{Var(\sum y_ix_i)}{(\sum x_i^2)^2}=\frac{\sum Var(y_i)x_i^2}{(\sum x_i^2)^2}=\frac{\sum\sigma^2x_i^2}{(\sum x_i^2)^2}=\frac{\sigma^2}{\sum x_i^2}.$$

Thus, $\hat\beta\sim N(\beta,\frac{\sigma^2}{\sum x_i^2})$

(d) $E\hat\beta_1=\frac{\sum_{i=1}^n EY_i}{\sum_{i=1}^n x_i}=\frac{\sum\beta x_i}{\sum x_i}=\beta$. Yes, it is unbiased.

$$Var(\hat\beta_1)=\frac{\sum_{i=1}^n Var(Y_i)}{(\sum_{i=1}^n x_i)^2}=\frac{n\sigma^2}{(\sum x_i)^2}$$

(e) $E\hat\beta_2=\frac{1}{n}\sum_{i=1}^n\frac{EY_i}{x_i}=\beta$, yes, it is unbiased.

$$Var(\hat\beta_2)=\frac{1}{n^2}Var\left(\sum_{i=1}^n\frac{Y_i}{x_i}\right)=\frac{1}{n^2}\sum_{i=1}^n\frac{Var(Y_i)}{x_i^2}=\frac{\sigma^2}{n^2}\sum\frac{1}{x_i^2}.$$

(f) $\hat\beta_{MLE}$ has the smallest variance. Indeed, the first inequality gives us that $(\sum_i x_i)^2\le n\sum_i x_i^2$, and thus $\frac{\sigma^2}{\sum x_i^2}\le\frac{n\sigma^2}{(\sum x_i)^2}$. The second inequality implies that $\left(\frac{1}{n}\sum_i\frac{1}{x_i^2}\right)^{-1}\le\frac{1}{n}\sum_i x_i^2$, or $n^2\left(\sum_i\frac{1}{x_i^2}\right)^{-1}\le\sum_i x_i^2$, or $\frac{\sigma^2}{\sum x_i^2}\le\frac{\sigma^2}{n^2}\sum\frac{1}{x_i^2}$.

2. (a)

$$l(\beta)=-\beta\sum x_i+\sum log(\beta x_i)y_i-\sum log(y_i!)=$$

$$=-\beta\sum x_i+\sum log(x_i)y_i+\log(\beta)\sum y_i-\sum log(y_i!)\to\max$$

First order condition gives $\hat\beta=\frac{\sum y_i}{\sum x_i}$

(b) We know that $EY_i = \beta x_i$. As a result, yes, this estimator is unbiased (the proof is the same as 1(d)).

(c)

$$\frac{\partial l}{\partial \beta} = -\sum x_i + \frac{\sum y_i}{\beta}, \qquad \frac{\partial^2 l}{\partial \beta^2} = -\frac{\sum y_i}{\beta^2}$$

$$I_n(\beta) = E\frac{\sum y_i}{\beta^2} = \frac{\sum x_i}{\beta}$$

The Rao-Cramer bound: for any unbiased estimator $\tilde{\beta}$ we have $Var(\tilde{\beta}) \geq \frac{\beta}{\sum x_i}$

(d) We use that $Var(y_i) = \beta x_i$:

$$Var(\hat{\beta}) = \frac{\sum Var(y_i)}{(\sum x_i)^2} = \frac{\beta}{\sum x_i}.$$

Yes, $\hat{\beta}$ is an efficient estimator in a class of unbiased estimators as it hits the Rao-Cramer bound.

3. (a) $l(\theta, \mu) = -n \log \theta - \frac{1}{\theta} \sum X_i - m \log \mu - \frac{1}{\mu} \sum Y_i$.

First order conditions:

$$-\frac{n}{\theta} + \frac{1}{\theta^2} \sum X_i = 0, \qquad -\frac{n}{\mu} + \frac{1}{\mu^2} \sum Y_i = 0$$

So, the unrestricted MLE is $\hat{\theta} = \overline{X}, \hat{\mu} = \overline{Y}$.

(b) If $\theta = \mu$, then $\hat{\theta}_R = \hat{\mu}_R = \frac{1}{n+m}(\sum X_i + \sum Y_i)$

(c)

$$\begin{aligned} LR = 2 \Bigg(& -n \log \hat{\theta} - \frac{1}{\hat{\theta}} \sum X_i - m \log \hat{\mu} - \frac{1}{\hat{\mu}} \sum Y_i + \\ & +(n+m) \log \hat{\theta}_R + \frac{1}{\hat{\theta}_R} (\sum X_i + \sum Y_i) \Bigg) = \\ & = 2 \left(-n \log \hat{\theta} - m \log \hat{\mu} + (n+m) \log \hat{\theta}_R \right) \end{aligned}$$

Test compares LR statistics with the $1-\alpha$ quantile of χ^2_1, and rejects when LR statistics exceeds it.

(d) One may go and calculate information, but we look on $\hat{\mu} = \overline{Y}$ and notice that it satisfies the Central Limit Theorem: $\sqrt{m}(\hat{\mu} - \mu_0) \Rightarrow N(0, \mu^2)$ as $m \to \infty$.

Wald test accepts if $z_{\alpha/2} \leq \sqrt{m}\frac{\hat{\mu}-\mu_0}{\hat{\mu}} \leq z_{1-\alpha/2}$. As a result, the confidence set is $[\hat{\mu} - z_{1-\alpha/2}\frac{\hat{\mu}}{\sqrt{m}}, \hat{\mu} - z_{\alpha/2}\frac{\hat{\mu}}{\sqrt{m}}]$. It is an asymptotic confidence since it is based on an asymptotic statement. In finite samples the statistic $\sqrt{m}\frac{\hat{\mu}-\mu_0}{\hat{\mu}}$ does not have a normal distribution.