14.121 Problem Set 2

Due: 9/27 in class

1. Consider an economy populated by households $i = 1, .., I$ with preferences

$$U_i = E\left[\sum_{t=1}^{\infty} \beta^t u_i(c_i(s^t))\right]$$
$$= \sum_{t=1}^{\infty} \sum_{s^t} \beta^t u_i(c_i(s^t)) Pr(s^t)$$

Utility satisfies $u_i'(c) > 0$, $u_i''(c) < 0$ for all c, and $\lim_{c \to 0} u_i'(c) = \infty$. (You can assume $u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i}$ if you wish, although the results for parts a-c don't require specifying a functional form beyond the given assumptions.) Households receive endowment $y_i(s^t)$, and the aggregate endowment is denoted by $y(s^t) = \sum_{i=1}^{I} y_i(s^t)$.

a. Show that in a Pareto optimal allocation consumption follows a risk-sharing rule, $c_i(s^t) = g_i(y(s^t))$ with $\sum_{i=1}^{I} c_i(s^t) = y(s^t)$.

b. Define the individual (absolute) risk-tolerance $\rho_i(s^t)$ and aggregate (absolute) risk-tolerance $\rho_0(s^t)$ as

$$\rho_i(s^t) \equiv -\frac{u'(c_i(s^t))}{u''(c_i(s^t))}, \; \rho_0(s^t) = \sum_{i=1}^{I} \rho_i(s^t)$$

Show that in a Pareto optimal allocation increases in the aggregate endowment are shared among households according to their risk-tolerance:

$$\frac{\partial c_i(s^t)}{\partial y(s^t)} = \frac{\rho_i(s^t)}{\rho_0(s^t)}$$

c. Show that in a Pareto optimal allocation the share of the aggregate endowment consumed by agent i increases after positive shock if agent i's (relative) risk-tolerance is greater than aggregate (relative) risk-tolerance:

$$\frac{\rho_i(s^t)}{c_i(s^t)} > \sum_{j=1}^{I} \frac{\rho_j(s^t)}{c_j(s^t)} \frac{c_j(s^t)}{y(s^t)}$$

d. Suppose preferences are given by $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Suppose also that consumption is measured with error, so measured consumption $\tilde{c}_i(s^t)$ relates to actual consumption by $\tilde{c}_i(s^t) = \exp(\epsilon_i^m(s^t)) c_i(s^t)$, where ϵ_{it}^m is independent of the income process and the Pareto weights. Find $c_i(s^t)$ in a Pareto optimum as a closed-form expression of the Pareto weights and aggregate endowment. Conclude that in a Pareto optimum the coefficient θ is equal to zero in the regression

$$\log \tilde{c}_{it} = \alpha_i + \eta_t + \theta \log y_{it} + \epsilon_{it}$$

e. Suppose now that agents can have different levels of constant relative risk tolerance, or $u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i}$. Assume more risk-tolerant agents have endowments which respond more strongly to aggregate shocks. Show that in a Pareto optimum y_{it} is positively correlated with the error term in the regression

$$\log \tilde{c}_{it} = \alpha_i + \eta_t + \theta \log y_{it} + \epsilon_{it}$$

Clarification: From the model, you can write $\log \tilde{c}_i(s^t)$ as a sum

$$\log \tilde{c}_i(s^t) = \alpha_i + \eta(s^t) + \theta \log y_i(s^t) + \zeta_i(s^t) + \epsilon_i^m(s^t)$$

The error term corresponds to the part of this equation that is not captured by including individual income and individual and time fixed effects in the regression, or $\zeta_i(s^t) + \epsilon_i^m(s^t)$. Since $\epsilon_i^m(s^t)$ is independent of everything in the model, it suffices to find $\zeta_i(s^t)$ and argue that it is positively correlated with individual income $y_i(s^t)$ under the given assumptions. You don't need to solve for $\zeta_i(s^t)$ in terms of model primitives, but you can find a way to express it that should allow you to say how consumption of agents with different risk tolerance respond differentially to the aggregate endowment.

2. Consider a pure exchange economy with 2 agents: B and L. They live for 2 periods, and consume a single consumption good in each period. Agents have quasilinear preferences given by

$$u_B(c_1, c_2) = u_L(c_1, c_2) = c_1 + 2\sqrt{c_2}$$

Each agent has an endowment of $y_{i,t} \geq 0$ for $i = B, L$ and $t = 1, 2$. We assume:

$$y_{B,1} + y_{L,1} = y_{B,2} + y_{L,2} = 1$$

and $y_{B,1} = y_{L,2} = 0$ and $y_{L,1} = y_{B,2} = 1$.

a. Characterize the set of Pareto optimal allocations in this economy.

b. Show that the allocation $((c_{B,1}, c_{B,2}), (c_{L,1}, c_{L,2})) = \left(\left(2 - \frac{2}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{2}{\sqrt{2}} - 1, \frac{1}{2}\right)\right)$ is in the core of this economy, but it is not in the core of the associated 2-replica economy.

c. Consider a competitive (collateral) equilibrium where agents cannot commit to repay any loan they borrowed. If an agent refuses to pay the loan $b = c_1 - y_1 > 0$, then the lender (who lent b to the borrower) can seize a fraction α of the remaining wealth of the borrower. Therefore, for the borrower to be willing to repay her debt, we must have that

$$b \leq \alpha p y_2 \iff c_1 \leq y_1 + a p y_2$$

where p is the relative price of consumption in period 2. Find the individual demands of the two agents as a function of $p > 1$.

d. Find the competitive (collateral) equilibrium prices $(1, p)$ and allocation of this modified economy. Show that if $\alpha \geq 1/2$ the equilibrium allocation is efficient. Find the relative Pareto weight that characterizes it as the maximum of a linear welfare function. Show that if $\alpha < 1/2$ then $i = B$ is always constrained and the equilibrium allocation is NOT Pareto Optimal. Why does the First Welfare Theorem fail?

e. Show that the borrowing constraint can be written as $c_2 \geq (1-\alpha)y_2$. Write down and solve the (constrained) Pareto problem where a feasible allocation must satisfy the borrowing constraint in addition to the resource constraint. Show that the competitive equilibrium allocation is *constrained* efficient, meaning it is also a solution to the constrained Pareto problem.

3. This question is based on Kilenthong and Townsend (2016). Consider a two-period economy, $t = 0, 1$, with two consumption goods, $j = 1, 2$. Good 1 cannot be stored, but good 2 is storable from $t = 0$ to $t = 1$. The return on the storable technology is given by R. The economy is populated by H households, indexed by $h = 1, ..., H$, with endowments $(e_t^h)_{t=0,1}$, where $e_t^h = (e_{1t}^h, e_{2t}^h)$. Preferences are homothetic and time separable:

$$U^h(c^h) = u_0(c_{10}^h, c_{20}^h) + \beta u_1(c_{11}^h, c_{21}^h)$$

Households can trade good 1 and good 2 in spot markets at both periods. The price of good 2 in period t (in terms of good 1) is denoted by p_t. In addition to the storage technology, households can save in a riskless bond. Let $Q = \frac{1}{1+r}$ denote the price of a bond paying one unit of good 1 in period 1. However, borrowing is limited by a collateral constraint:

$$p_1 R k^h + \theta^h \geq 0$$

where k^h denotes the amount put into storage by household h and θ^h is the net number of bonds bought by household h in period 0.

a. Consider a competitive (collateral) equilibrium in this economy. Write down the household's problem and derive the first-order conditions. Write down the market clearing conditions.

b. The price of good 2 at $t = 1$ can be written as $p_1 = p(z_1)$, where

$$z_1 = \frac{\sum_h e_{11}^h}{\sum_h e_{21}^h + R \sum_h k^h}$$

This follows from a similar calculation as in problem set 1, problem 4d. (Optional) Verify this.

c. Write down the (constrained) Pareto problem. The planner can choose the amount households put in storage, but it must take into account the fact that households can trade in spot markets in period 1 and that the price is given by $p(z_1)$.

d. Derive the first-order conditions. Does the competitive equilibrium allocation always satisfy the necessary conditions for the (constrained) Pareto problem? If not, discuss the source of inefficiency.