14.121, Fall 2014
Problem Set 1 Solutions

1) Suppose $X = \mathbb{R}^k_+$, for some $k \geq 2$, and define $x = (x_1, ..., x_k) \succeq y = (y_1, ..., y_k)$ if $x \geq y$; i.e., if for each $i = 1, ..., k$, $x_i \geq y_i$. (This is known as the *Pareto ordering* on $\mathbb{R}^k_+$.)

   a) Show that $\succeq$ is transitive but not complete.

   Solution: To see that $\succeq$ is transitive, consider $x, y, z \in X$ such that $x \succeq y$ and $y \succeq z$. This implies that $x_i \geq y_i \geq z_i$ for all i. It is immediate from the transitivity of the real line that $x \succeq z$. A counterexample to completeness can be found when $k = 2$, $x = (1, 0)$, and $y = (0, 1)$. In this case we do not have $x \succeq y$ or $y \succeq x$.

   b) Characterize $\succ$ defined from $\succeq$ in the usual fashion; i.e. $x \succ y$ if $x \succeq y$ and not $y \succeq x$. Is $\succ$ reflexive? transitive? symmetric? Prove your assertions.

   Solution: *Not Reflexive:* $x \succ x$ would require that $x \succeq x$ and $\neg(x \succeq x)$, a contradiction.

   Transitive: Let $x \succ y$ and $y \succ z$. Then $x_i \geq y_i$ for all i and $x \neq y$. Similarly, $y_i \geq z_i$ for all i and $y \neq z$. Thus, $x_i \geq z_i$ for all i by the transitivity of the reals and $x \neq z$ since otherwise $x = y = z$, a contradiction. Hence, $x \succ z$ and we have transitivity.

   Not Symmetric: $(1, 1) \succ (0, 0)$, but clearly $\neg((0, 0) \succ (1, 1))$.

   c) Characterize $\sim$ from $\succeq$ in the usual fashion; i.e. $x \sim y$ if $x \succeq y$ and $y \succeq x$. Is $\sim$ reflexive? transitive? symmetric? Prove your assertions.

   Solution: Note that $x \sim y$ requires that $x_i \geq y_i$ and $y_i \geq x_i$ for all i. Thus, $x \sim y$ iff $x = y$.

   Reflexive: It is clear that $x = x$, so $x \sim x$.

   Transitive: If $x \sim y$ and $y \sim z$, then $x = y$ and $y = z$, so $x = z$, and we have $x \sim z$.

   Symmetric: If $x \sim y$, then $x = y$, so $y = x$ and hence $y \sim x$.

2) MWG 1.D.5. [See textbook for body of question].

   a) Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ can be rationalized by preferences.

   Solution: There are six possible strict preference orderings ($x \succ y \succ z$, $x \succ z \succ y$, $z \succ x \succ y$, $z \succ y \succ x$, $y \succ x \succ z$, $y \succ z \succ x$). x is preferred to y in half of them; similarly for y and z, and x and z. So if we apply probability $\frac{1}{6}$ to each preference ordering, then this generates the choice function in question.

   b) This choice function implies that $Pr(x \succ y) = Pr(y \succ z) = Pr(z \succ x) = \frac{1}{4}$. So, $Pr(x \succ y \text{ or } y \succ z \text{ or } z \succ x) \leq \frac{3}{4}$. But in fact, one of those three relations always holds: if the first two don't, then $y \succ x$ and $z \succ y$, so by transitivity, $z \succ x$.

   c) Extending the argument from part *b*), we cannot have $\alpha < \frac{1}{3}$. By symmetry, then, we also cannot have $\alpha > \frac{2}{3}$ (just imagine relabeling the goods). So a necessary condition for rationalizability is $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

   This condition is also sufficient for rationalizability. Consider the following probability distribution:

$$Pr(x \succ y \succ z) = Pr(y \succ z \succ x) = Pr(z \succ x \succ y) = \alpha - \frac{1}{3}$$

$$Pr(x \succ z \succ y) = Pr(y \succ x \succ z) = Pr(z \succ y \succ x) = \frac{2}{3} - \alpha.$$

   This generates the choice function in the question. For instance, $Pr(x \succ y) = 2\left(\alpha - \frac{1}{3}\right) + \frac{2}{3} - \alpha = 2\alpha - \alpha = \alpha$.

3) Let $\succeq$ be some complete, transitive preference on a non-empty convex set $X \subseteq \mathbb{R}^L$. We say that preferences are strictly convex when $\succeq$ is convex and for all x, y, z such that $y \succeq x$ and $z \succeq x$ and $y \neq z$, we have that for $\alpha \in (0,1)$:

$$\alpha y + (1-\alpha) z \succ x$$

a) Let $X^* \subseteq X$ be the set of maximal bundles of X:

$$X^* = \{x \in X : x \succeq y \text{ for all } y \in X\}.$$

Show if that $\succeq$ is complete, transitive, and convex, then X^* is convex.

Solution: Take $x, x' \in X^*$ and $z = \alpha x + (1-\alpha) x'$. We know $z \in X$ by convexity of the choice set X. We want to show that $z \succsim y$ for all $y \in X$. Now, given $y \in X$ we have that $x \succsim y$ and $x' \succsim y$ using that $x, x' \in X^*$. By convexity of $\succsim$, we must have that for all $\alpha \in [0,1]$

$$z \equiv \alpha x + (1-\alpha) x' \succsim y \text{ for all } y \in X$$

Thus $z \in X^*$. Therefore, X^* is a convex set.

b) Suppose that preferences are also strictly convex. Show that X^* has at most one element.

Solution: Suppose that there exist $x, x' \in X^*$ such that $x \neq x'$. Then, we know that $x \succsim x'$ and $x' \succsim x'$ by definition. But then, for $\alpha = \frac{1}{2}$ we must have that

$$z^* = \frac{1}{2}x + \frac{1}{2}x' \succ x'$$

so $x' \notin X^*$, a contradiction. Therefore, X^* cannot have more than one element.

c) Suppose that $\succeq$ is strictly monotone and that $X \subseteq \mathbb{R}^L$ is an open set. Show that $\succeq$ is also locally non-satiated.

Solution: Take any $x \in X$ and any $\varepsilon_1 > 0$. Since X is an open set, we know that there exists $\varepsilon_2 > 0$ such that if $\|y - x\| < \varepsilon_2 \implies y \in X$. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Define

$$\widetilde{y} = x + \frac{\varepsilon}{2\sqrt{L}}$$

See that

$$\|\widetilde{y} - x\| = \sqrt{\sum_{l=1}^{L}\left(x_l - x_l - \frac{\varepsilon}{2\sqrt{L}}\right)^2} = \sqrt{\sum_{l=1}^{L}\frac{\varepsilon^2}{4L}} = \sqrt{\left(\frac{\varepsilon}{2}\right)^2} = \frac{\varepsilon}{2} < \varepsilon_2$$

so $\widetilde{y} \in X$. Moreover, since $\widetilde{y} \gg x$, we have that $\widetilde{y} \succ x$ using monotonicity. Therefore, preferences are locally non satiated, as we wanted to show.

d) Suppose that X is a non-empty compact set and that $\succeq$ is complete, transitive, and continuous (but not necessarily convex). Prove that preferences cannot be locally non-satiated (Hint: show $X^* \neq \emptyset$). What does this imply about the relationship between monotonicity and local non-satiation?

Solution: Because preferences are complete, transitive and continuous, we can apply Debreu's Theorem and ensure the existence of a continuous utility function $u : X \to \mathbb{R}$ that represents the preferences. So, we can write the set X^* as

$$X^* = \arg\max_{x \in X} u(x)$$

Since $X \neq \varnothing$, X is compact and $u : X \to \mathbb{R}$ is a continuous function, we can apply Weierstrass' Theorem to ensure the existence of a maximum, so $X^* \neq \varnothing$, i.e. there exists $x^* \in X^*$ such that for all $y \in X$:

$$u(x^*) \geq u(y) \iff x^* \succsim y \text{ for all } y \in X$$

and see that these preferences do not satisfy locally non-satiation at $x = x^*$.

4c and 4d show that satiation points of X must be on the upper boundary of X if preferences are monotone.

4) Let X be a convex subset of $\mathbb{R}^n_+$.

a) Suppose utility function $u : X \mapsto \mathbb{R}$ represents preference relation $\succsim$. Show that u is quasi-concave if and only if $\succsim$ is convex.

Solution: If:

Fix a utility level $\bar{u}$. We want to show that the set $B(\bar{u}) := \{x \in X : u(x) \geq \bar{u}\}$ is convex. So, pick any two points $x, y \in B(\bar{u})$. Without loss of generality, suppose $u(x) \geq u(y)$. Hence $x \succsim y$. Then, by convexity of $\succsim$, for any $\alpha \in [0, 1]$ we must have $\alpha x + (1 - \alpha)y \succsim y$. So $u(\alpha x + (1 - \alpha)y) \geq u(y) \geq \bar{u}$. Hence, $B(\bar{u})$ is convex, so u is quasiconcave.

Only if:

Fix a point z and pick any two points x and y such that $x \succsim z$ and $y \succsim z$. u represents $\succsim$ so we have $u(x) \geq u(z)$ and $u(y) \geq u(z)$. Therefore, by quasi-concavity of u, we have $u(\alpha x + (1 - \alpha)y) \geq u(z)$ for $\alpha \in [0, 1]$. So $\alpha x + (1 - \alpha)y \succsim z$. Hence, the set $\{x : x \succsim z\}$ is convex. So $\succsim$ is convex.

Now, suppose $n = 2$ and consider the utility function

$$u(x_1, x_2) = \begin{cases} x_1 x_2 \text{ if } x_1 x_2 \leq 1 \\ 1 \text{ if } x_1 x_2 \in (1, 2) \\ x_1 x_2 - 1 \text{ if } x_1 x_2 \geq 2 \end{cases}$$

b) Show that this utility function is quasi-concave.

Solution: Note that any monotone transformation of a quasi-concave function is quasi-concave. $u(x_1, x_2)$ is a monotone transformation of $\tilde{u}(x_1, x_2) := \ln(x_1, x_2) = \ln x_1 + \ln x_2$. In turn, $\tilde{u}(x_1, x_2)$ is concave: its Hessian matrix is

$$H_{\tilde{u}} = \begin{pmatrix} -x_1^{-2} & 0 \\ 0 & -x_2^{-2} \end{pmatrix}$$

which is negative semi-definite for all x_1, x_2. Therefore, $u(x_1, x_2)$ is quasi-concave.

c) Let $\succsim$ denote the preference relation represented by this utility function. Show that $\succsim$ cannot be represented by any concave utility function.

Solution: Consider any utility function w that represents $\succsim$. Take two points, x and y, such that $x_1 x_2 \in (1, 2)$ and $y_1 y_2 \geq 2$. Then we must have $y \succ x$. Hence, $w(y) > w(x)$.

Now, because $x_1 x_2 < 2$, there exists $\alpha \in (0, 1)$ such that $(\alpha x_1 + (1 - \alpha)y_1)(\alpha x_2 + (1 - \alpha)y_2) < 2$. (We can just pick α as $1 - \epsilon$ for small enough ϵ). For this α we must have $\alpha x + (1 - \alpha)y \sim x$. Hence, $w(\alpha x + (1 - \alpha)y) = w(x)$.

However, $\alpha w(x) + (1 - \alpha)w(y) > w(x)$. So we have found x, y and α such that

$$\alpha w(x) + (1 - \alpha)w(y) > w(\alpha x + (1 - \alpha)y)$$

Hence, w cannot be concave.