**14.121 Problem Set 1 Solutions**

Question 1

a. The commodity space consists of a bundle for each state $\theta_1$, $\theta_2$. It has dimension $2L$.

b. The inequalities are

$$\begin{aligned} u(x(\theta_1),\theta_1) &\geq u(x(\theta_2),\theta_1) \\ u(x(\theta_2),\theta_2) &\geq u(x(\theta_1),\theta_2) \end{aligned}$$

c. Note that

$$\begin{aligned} u(\alpha x(\theta_2)' + (1-\alpha)x(\theta_2)'', \theta_1) &> \alpha u(x(\theta_2)',\theta_1) + (1-\alpha)u(x(\theta_2)'',\theta_1) \\ &= \alpha u(x(\theta_1),\theta_1) + (1-\alpha)u(x(\theta_1),\theta_1) \\ &= u(x(\theta_1),\theta_1) \end{aligned}$$

Thus the convex combination $(x(\theta_1), \alpha x(\theta_2)' + (1-\alpha)x(\theta_2)')$ is not an incentive compatible contract.

d. The inequalities are

$$\begin{aligned} \sum_{i=1}^{n} \pi(c_i,\theta_1)u(c_i,\theta_1) &\geq \sum_{i=1}^{n} \pi(c_i,\theta_2)u(c_i,\theta_1) \\ \sum_{i=1}^{n} \pi(c_i,\theta_2)u(c_i,\theta_2) &\geq \sum_{i=1}^{n} \pi(c_i,\theta_1)u(c_i,\theta_2) \end{aligned}$$

e. Suppose we have IC contracts $(\pi(c_1,\theta_1),...,\pi(c_n,\theta_2))$ and $(\pi'(c_1,\theta_1),...,\pi'(c_n,\theta_2))$. Then

$$\begin{aligned} \sum_{i=1}^{n}(\alpha\pi(c_i,\theta_j) + (1-\alpha)\pi'(c_i,\theta_j))u(c_i,\theta_j) &= \alpha\sum_{i=1}^{n}\pi(c_i,\theta_j)u(c_i,\theta_j) + (1-\alpha)\sum_{i=1}^{n}\pi'(c_i,\theta_j)u(c_i,\theta_j) \\ &\geq \alpha\sum_{i=1}^{n}\pi(c_i,\theta_{-j})u(c_i,\theta_j) + (1-\alpha)\sum_{i=1}^{n}\pi'(c_i,\theta_{-j})u(c_i,\theta_j) \\ &= \sum_{i=1}^{n}(\alpha\pi(c_i,\theta_{-j}) + (1-\alpha)\pi'(c_i,\theta_{-j}))u(c_i,\theta_j) \end{aligned}$$

So the convex combination is also IC. By similar steps, can check that a convex combination of distributions is also a distribution (nonnegative and sums to one).

2. If $py_j^* > 0$, then the firm could not be maximizing profits since for any $\alpha > 1$ we have $\alpha y_j^* \in Y_j$ and $p(\alpha y_j^*) = \alpha p y_j^* > p y_j^*$. If $py_j^* < 0$ then the firm could not be maximizing profits since for any $\alpha < 1$ we have $\alpha y_j^* \in Y_j$ and $p(\alpha y_j^*) = \alpha p y_j^* > p y_j^*$.

3. Suppose $u = (\bar{u}_1, .., \bar{u}_I) \in \mathcal{U}$. Then there exists a feasible allocation $(x, y)$ and $u_i(x_i) \geq \bar{u}_i$ for $i = 1, .., I$. Similarly, if $u' = (\bar{u}'_1, .., \bar{u}'_I) \in \mathcal{U}$ then there exists a feasible allocation $(x', y')$ and $u_i(x'_i) \geq \bar{u}'_i$

for $i = 1, .., I$. Then for $\alpha \in (0, 1)$ I will show that the allocation $(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y')$ supports $\alpha u + (1-\alpha)u'$. Note that this allocation is feasible: convexity of consumption sets implies that $\alpha x_i + (1-\alpha)x'_i$ is in agent $i$'s consumption set for each agent $i = 1, .., I$, convexity of production sets implies $\alpha y_j + (1-\alpha)y'_j$ is in firm $j$'s production set for each $j = 1, .., J$, and the resource constraints are satisfied

$$
\begin{aligned}
\sum_{i=1}^{I} (\alpha x_i + (1-\alpha)x'_i) &= \alpha \sum_{i=1}^{I} x_i + (1-\alpha) \sum_{i=1}^{I} x'_i \\
&= \alpha \left( \bar{\omega} + \sum_{j=1}^{J} y_j \right) + (1-\alpha) \left( \bar{\omega} + \sum_{j=1}^{J} y'_j \right) \\
&= \bar{\omega} + \sum_{j=1}^{J} \left( \alpha y_j + (1-\alpha) y'_j \right)
\end{aligned}
$$

Since utility functions are concave, we have

$$u_i(\alpha x_i + (1-\alpha)x'_i) \geq \alpha u_i(x_i) + (1-\alpha)u_i(x'_i) \geq \alpha \bar{u}_i + (1-\alpha)\bar{u}'_i$$

This shows that $\alpha u + (1-\alpha)u' \in \mathcal{U}$.

Question 4

a. Homogeneity implies $u(\alpha x_1, \alpha x_2) = \alpha u(x_1, x_2)$. Differentiate both sides by $x_l$ and apply the chain rule to get the result.

b. One way to calculate Pareto optimal allocations is to maximize a linear welfare function, which offers an opportunity to apply the result from question 3. To start, it's useful to compute some of the derivatives of utility function

$$
\begin{aligned}
u_{12}(x_1, x_2) &= \frac{d}{dx_2} u_1(x_1/x_2) = \frac{-x_1}{x_2^2} u'_1(x_1/x_2) \\
u_{21}(x_1, x_2) &= \frac{d}{dx_1} u_2(x_1/x_2) = \frac{1}{x_2} u'_2(x_1/x_2)
\end{aligned}
$$

Since $u_{12} = u_{21}$, this implies

$$u'_2(x_1/x_2) = \frac{-x_1}{x_2} u'_1(x_1/x_2) \tag{1}$$

Moreoever,

$$
\begin{aligned}
u_{11}(x_1, x_2) &= \frac{d}{dx_1} u_1(x_1/x_2) = \frac{1}{x_2} u'_1(x_1/x_2) \\
u_{22}(x_1, x_2) &= \frac{d}{dx_2} u_2(x_1/x_2) = \frac{-x_1}{x_2^2} u'_2(x_1/x_2) = \frac{x_1^2}{x_2^3} u'_1(x_1/x_2)
\end{aligned}
$$

Now we can compute the characteristic equation of the Hessian as

$$\begin{aligned}
(u_{11}-\lambda)(u_{22}-\lambda)-u_{12}u_{21} &= \left(\frac{1}{x_2}u_1'-\lambda\right)\left(\frac{x_1^2}{x_2^3}u_1'-\lambda\right)-\frac{x_1^2}{x_2^4}\left(u_1'\right)^2 \\
&= \lambda^2-\left(\frac{x_1^2}{x_2^3}+\frac{1}{x_2}\right)u_1'\lambda
\end{aligned}$$

This has solutions

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= \left(\frac{x_1^2}{x_2^3}+\frac{1}{x_2}\right)u_1'(x_1/x_2) \\
&= \left(\frac{x_1^2}{x_2^3}+\frac{1}{x_2}\right)x_2u_{11}(x_1,x_2)<0
\end{aligned}$$

The Hessian is a symmetric with nonpositive eigenvalues, so it is negative semi-definite, which implies $u$ is concave. Since consumption sets are convex, this implies that the utility possibilities set is convex and that all Pareto optima can be obtained by maximizing a linear welfare function. Since $\lim_{x_j\to 0}u_i(x_1,x_2)=\infty$, nonnegativity constraints will not bind in the planner's or the individual's optimal allocation. The Lagrangian for the planner's problem can therefore be written

$$\lambda_1u(x_1^1,x_2^1)+\lambda_2u(x_1^2,x_2^2)-\mu_1(x_1^1+x_1^2-\bar{\omega}_1)-\mu_2(x_2^1+x_2^2-\bar{\omega}_2)$$

The FOC writes

$$\begin{aligned}
\lambda_iu_1(x_1^i,x_2^i) &= \mu_1 \\
\lambda_iu_2(x_1^i,x_2^i) &= \mu_2
\end{aligned}$$

for $i=1,2$. Dividing gives

$$\frac{u_2(x_1^1/x_2^1)}{u_1(x_1^1/x_2^1)}=\frac{\mu_2}{\mu_1}=\frac{u_2(x_1^2/x_2^2)}{u_1(x_1^2/x_2^2)} \tag{2}$$

Denote $MRS(y)=\frac{u_2(y)}{u_1(y)}$. Note that $MRS$ is strictly increasing since $u_1'(x_1/x_2)<0$ and $u_2'=\frac{-x_1}{x_2}u_1'(x_1/x_2)>0$. Therefore we can apply $MRS^{-1}$ to both sides of (2) to obtain

$$\frac{x_1^1}{x_2^1}=\frac{x_1^2}{x_2^2}$$

Substituting in the feasibility conditions obtains

$$\frac{x_1^1}{x_2^1}=\frac{\bar{\omega}_1-x_1^1}{\bar{\omega}_2-x_2^1}$$

and so

$$x_2^i=\frac{\bar{\omega}_2}{\bar{\omega}_1}x_1^i$$

for $i = 1, 2$. This shows that the Pareto optimal allocations are of the form

$$((x_1^1, x_2^1), (x_1^2, x_2^2)) = (\alpha\,(\bar{\omega}_1, \bar{\omega}_2)\,, (1-\alpha)\,(\bar{\omega}_1, \bar{\omega}_2))$$

for $\alpha \in [0, 1]$. On an Edgeworth box corresponds to the diagonal from the lower left corner to the upper right corner.

c. Individual rationality additionally requires

$$\begin{aligned} u(\omega_1^1, \omega_2^1) &\le u(\alpha\bar{\omega}_1, \alpha\bar{\omega}_2) = \alpha u(\bar{\omega}_1, \bar{\omega}_2) \\ u(\omega_1^2, \omega_2^2) &\le u((1-\alpha)\bar{\omega}_1, (1-\alpha)\bar{\omega}_2) = (1-\alpha) u(\bar{\omega}_1, \bar{\omega}_2) \end{aligned}$$

which can otherwise be summarized as

$$\alpha \in \left[\frac{u\,(\omega_1^1, \omega_2^1)}{u\,(\bar{\omega}_1, \bar{\omega}_2)}, 1 - \frac{u(\omega_1^2, \omega_2^2)}{u\,(\bar{\omega}_1, \bar{\omega}_2)}\right]$$

d. The Lagrangian for the utility maximization problem is

$$u(x_1^i, x_2^i) - \lambda(x_1 + px_2 - (\omega_1^i + p\omega_2^i))$$

The first-order condition obtains

$$\begin{aligned} u_1(x_1^i/x_2^i) &= \lambda \\ u_2(x_1^i/x_2^i) &= p\lambda \\ \implies \frac{u_2(x_1^1/x_2^1)}{u_1(x_1^1/x_2^1)} &= p = \frac{u_2(x_1^2/x_2^2)}{u_1(x_1^2/x_2^2)} \end{aligned}$$

By the same reasoning as in part b, this implies

$$\frac{x_1^1}{x_2^1} = \frac{x_1^2}{x_2^2}$$

and hence by feasibility

$$x_2^i = \frac{\bar{\omega}_2}{\bar{\omega}_1} x_1^i$$

for $i = 1, 2$. The price is then

$$p = \frac{u_2(\bar{\omega}_1/\bar{\omega}_2)}{u_1(\bar{\omega}_1/\bar{\omega}_2)}$$

Notice that with identical homothetic preferences the price only depends on aggregate quantities and does not depend on distributional characteristics. Substituting this price into the budget constraint determines

$$\begin{aligned} x_1^i &= \bar{\omega}_1 \frac{u_1(\bar{\omega}_1/\bar{\omega}_2)\omega_1^i + u_2(\bar{\omega}_1/\bar{\omega}_2)\omega_2^i}{u_1(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_1 + u_2(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_2} \\ x_2^i &= \bar{\omega}_2 \frac{u_1(\bar{\omega}_1/\bar{\omega}_2)\omega_1^i + u_2(\bar{\omega}_1/\bar{\omega}_2)\omega_2^i}{u_1(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_1 + u_2(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_2} \end{aligned}$$

This can be represented in the form $((x_1^1, x_2^1), (x_1^2, x_2^2)) = (\alpha(\bar{\omega}_1, \bar{\omega}_2), (1-\alpha)(\bar{\omega}_1, \bar{\omega}_2))$ where $\alpha = \frac{u_1(\bar{\omega}_1/\bar{\omega}_2)\omega_1^1 + u_2(\bar{\omega}_1/\bar{\omega}_2)\omega_2^1}{u_1(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_1 + u_2(\bar{\omega}_1/\bar{\omega}_2)\bar{\omega}_2}$. This satisfies the characterization of Pareto optimal allocations derived in part b.

e. Homogeneity implies $u(\alpha x_1, \alpha x_2) = \alpha u(x_1, x_2)$. Differentiate both sides by $\alpha$ at $\alpha = 1$ to obtain the result.

f. By part c, we need to show that

$$\alpha \in \left[\frac{u(\omega_1^1, \omega_2^1)}{u(\bar{\omega}_1, \bar{\omega}_2)}, 1 - \frac{u(\omega_1^2, \omega_2^2)}{u(\bar{\omega}_1, \bar{\omega}_2)}\right]$$

By the result in part e, can write

$$\alpha = \frac{u_1(\bar{\omega}_1/\bar{\omega}_2)\omega_1^1 + u_2(\bar{\omega}_1/\bar{\omega}_2)\omega_2^1}{u(\bar{\omega}_1, \bar{\omega}_2)}$$

Thus, it suffices to show that

$$\omega_1^1 u_1(\bar{\omega}_1/\bar{\omega}_2) + \omega_2^1 u_2(\bar{\omega}_1/\bar{\omega}_2) \geq u(\omega_1^1, \omega_2^1) \tag{3}$$

and

$$\omega_1^1 u_1(\bar{\omega}_1/\bar{\omega}_2) + \omega_2^1 u_2(\bar{\omega}_1/\bar{\omega}_2) \leq u(\bar{\omega}_1, \bar{\omega}_2) - u(\omega_1^2, \omega_2^2) \tag{4}$$

To show the first, let

$$f(x_1/x_2) = \omega_1^1 u_1(x_1/x_2) + \omega_2^1 u_2(x_1/x_2)$$

Since $u_2'(x_1/x_2) = -(x_1/x_2)u_1'(x_1/x_2)$ we have

$$\begin{aligned}
\frac{df}{d(x_1/x_2)} &= \omega_1^1 u_1'(x_1/x_2) + \omega_2^1(-x_1/x_2)u_1'(x_1/x_2) \\
&= u_1'(x_1/x_2)\omega_2^1\left(\frac{\omega_1^1}{\omega_2^1} - \frac{x_1}{x_2}\right)
\end{aligned}$$

This derivative is equal to zero iff $x_1/x_2 = \omega_1^1/\omega_2^1$. Moreoever, recalling $u_1'(x_1/x_2) < 0$, this derivative must be positive for $x_1/x_2 > \omega_1^1/\omega_2^1$ and negative for $x_1/x_2 < \omega_1^1/\omega_2^1$, so the minimum of $f$ must occur at $x_1/x_2 = \omega_1^1/\omega_2^1$.

Now to show the first inequality (3), we apply this result to obtain

$$\begin{aligned}
\omega_1^1 u_1(\bar{\omega}_1/\bar{\omega}_2) + \omega_2^1 u_2(\bar{\omega}_1/\bar{\omega}_2) &\geq \omega_1^1 u_1(\omega_1^1/\omega_2^1) + \omega_2^1 u_2(\omega_1^1/\omega_2^1) \\
&= \omega_1^1 u_1(\omega_1^1, \omega_2^1) + \omega_2^1 u_2(\omega_1^1, \omega_2^1) \\
&= u(\omega_1^1, \omega_2^1)
\end{aligned}$$

using the result from part e again. Note that (4) can be re-written (using the result from part e)

$$u(\omega_1^2, \omega_2^2) \leq \omega_1^2 u_1(\bar{\omega}_1/\bar{\omega}_2) + \omega_2^2 u_2(\bar{\omega}_1/\bar{\omega}_2)$$

which holds by a similar argument.