14.121: Existence and Other Properties

Parag Pathak
MIT

October 2019

Existence of Walrasian Equilibrium

Theorem (Brouwer)

Suppose $A \subset \mathbb{R}^L$ is nonempty, convex, compact, and $f : A \to A$ is continuous. Then f has a fixed point:

$$\exists x^* \in A \text{ such that } f(x^*) = x^*.$$

Note: only useful for functions

von Neumann (1937): used to prove existence of "minimax" solution to two-person games

Scarf (1973): Developed an efficient algorithm to find approximate fixed points

$$\forall \epsilon > 0, \exists x^*_\epsilon \text{ such that } |f(x^*_\epsilon) - x^*_\epsilon| < \epsilon$$

See Scarf's Handbook of Mathematical Economics (1982) article

Theorem (Kakutani's Fixed Point theorem)

Suppose $A \subset \mathbb{R}^L$ *is nonempty, compact, convex, and* $f : A \to A$ *is a correspondence such that*

- f *is non-empty*
- f *is convex valued*
- f *is upper hemi-continuous,*

then f *has a fixed point, i.e.*

$$\exists x^* \in A \text{ such that } x^* \in f(x^*).$$

Nash (1950): proof of existence of Nash equilibrium

Arrow, Debreu, McKenzie (all mid-1950s): proofs of existence of Walrasian equilibrium

Rough idea of Kakutani:

If we could find a continuous selection g from f:

$$g : A \to A \text{ continuous }, \forall a, \quad g(a) \in f(a)$$

We can apply Brouwer: there exists $a^* \in A$ such that $g(a^*) = a^*$ so $a^* \in f(a^*)$, and we would be done. Unfortunately, cannot find a selection in general.

For each $n \in \mathbb{N}$, find a continuous function g_n whose graph is within $\frac{1}{n}$ of the graph of f.

By Brouwer's Theorem, we can find a fixed point a_n^* of g_n so (a_n^*, a_n^*) is in the graph of g_n. Therefore, there exists (x_n, y_n) in the graph of f such that

$$|a_n^* - x_n| < \frac{1}{n} \quad \text{and} \quad |a_n^* - y_n| < \frac{1}{n}$$

Since A is compact, $\{a_n^*\}$ has a convergent subsequence:

$$a_{n_k}^* \to a^*$$

for some $a^* \in A$.

$$\lim_{k\to\infty} x_{n_k} = a^*, \quad \lim_{k\to\infty} y_{n_k} = a^*$$

Since f has closed graph, and

$$(x_{n_k}, y_{n_k}) \to (a^*, a^*)$$

(a^*, a^*) is in the graph of f, so $a^* \in f(a^*)$.

What matters in Walrasian model are relative prices, so we can normalize to simplex. Define

$$\Delta = \{p \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1\} \qquad \Delta^0 = \{p \in \mathbb{R}^L_{++} : \sum_{\ell=1}^{L} p_\ell = 1\}$$

Focus on a pure exchange economy with our four assumptions, but strengthen to strong monotonicity

- **excess demand of consumer** i:

$$z^i(p) = x^i(p, p \cdot \omega^i) - \omega^i$$

- **aggregate excess demand**

$$z(p) = \sum_i z^i(p)$$

Price vector $p \in \mathbb{R}^L_+$ satisfies $z(p) = 0$ if and only if $(p, (x^i)_{i \in \mathcal{I}})$ is a Walrasian equilibrium

Hence, need to show that a solution to $z(p) = 0$ exists

Aside: Why add back **strong monotone**?

If only increasing, suppose

$$u^1(x_1, x_2) = \min\{x_1, x_2\} \qquad \omega^1 = (2, 1)$$

$$u^2(x_1, x_2) = \min\{x_1, x_2\} \qquad \omega^2 = (1, 1)$$

For $p \gg 0$,

$$x^1(p) = (1, 1) \qquad x^2(p) = (1, 1) = \omega^2,$$

so

$$\sum_i x^i(p) \leq \bar{\omega} \qquad \text{or} \qquad z(p) \leq 0$$

With strongly monotone, consumers strictly value any increase in any single commodity so market clearing will be exact

Proposition

In an exchange economy, if preferences satisfy our four assumptions and are strongly monotone, then $z(p)$ defined for $p \in \Delta^0$ satisfies:

i) *z is a continuous function*

ii) *Homogeneity of degree 0*

iii) *Walras's law: $p \cdot z(p) = 0$ for all p*

iv) *Bounded below: $z(p) \geq -\bar{\omega}$*

v) *Boundary condition: $p^n \to p$ when $p \neq 0$ but $p_\ell = 0$ for some ℓ, then*

$$\max\{z_1(p^n), ..., z_L(p^n)\} \to \infty$$

Property v): as some prices go to zero, a consumer whose wealth goes to a positive limit (must exist since $p \cdot \sum_i \omega^i > 0$) will demand more and more because of strongly monotone preferences

Note: not necessarily the case that if $p_\ell \to 0$ and other prices bounded away, demand for good ℓ goes to infinity

Intuition: for two commodities, $L = 2$

$z(p)$ **homogenous of degree 0**: normalize $p_2 = 1$ and write

$$z(p) = z(p_1, 1)$$

Walras's Law: allows us to check market clearing in $L - 1$ markets

Therefore, we look at solutions to single equation: $z_1(p_1, 1) = 0$

For given $p_2 \neq 0$ when p_1 **very small**,

$$\underbrace{p_1 z_1(p) + p_2 z_2(p) = 0}_{\textbf{Walras}} \quad \text{together with} \quad \underbrace{\max\{z_1(p), z_2(p)\} \to \infty}_{\textbf{Boundary condition}}$$

imply that it is not the case that

$$z_2(p) \to \infty,$$

for then we violate Walras because z_1 is **bounded below**. Hence,

$$z_1(p_1, 1) > 0 \text{ for small enough } p_1.$$

When p_1 **very large**, homogeneity implies

$$z_1(p_1, 1) = z_1(1, 1/p_1),$$

so it is as if p_2 very small.

Hence, when p_1 very large, symmetric argument implies that

$$z_2(p_1, 1) > 0 \text{ for large enough } p_1$$

The only way this is consistent with Walras is if

$$z_1(p_1, 1) < 0 \text{ for large enough } p_1.$$

$z(p)$ **continuous**, must cross 0 $\Rightarrow$ existence

This is *almost* an existence result, but captures the main ideas

Proposition

If $z : \Delta \to \mathbb{R}^L$ *is a continuous function that satisfies Walras' law, then there exists some* p^* *in* Δ *such that* $z(p^*) \leq 0$.

- Note: $z_\ell(p^*) < 0$ is an equilibrium if $p_\ell = 0$
- Statement directly assumes that z is continuous on Δ
  - To ensure demand is continuous, need assumption like $\omega^i \gg 0$. If $\omega^i_\ell = 0$, budget correspondence will not be continuous in price, so we cannot apply Berge's theorem.
  - Not consistent with strongly monotone preferences: At boundary aggregate demand may not be defined when some price goes to zero
- Proper argument provides a way to "extend" the domain of the excess demand so that it is defined at zero

Additional Facts about GE

Properties of the set of equilibrium prices: $z(p) = 0$

1) relative prices are determined
2) only need to check market clearing in $L - 1$ markets
3) z is homogeneous of degree 0

Implication of homogeneity:

- $z_i(\alpha \cdot p)$ is independent of $\alpha > 0$ so $\frac{\partial}{\partial \alpha} z_i(\alpha p) = 0$
- When $\alpha = 1$,

$$\sum_{j=1}^{L} \frac{\partial z_i}{\partial p_j} \cdot p_j = Dz(p) \cdot p = 0$$

$$\text{where} \qquad Dz(p) = \begin{bmatrix} \frac{\partial z_1}{\partial p_1} & \cdots & \frac{\partial z_1}{\partial p_L} \\ \vdots & \cdots & \vdots \\ \frac{\partial z_L}{\partial p_1} & \cdots & \frac{\partial z_L}{\partial p_L} \end{bmatrix}$$

Hence, the columns of $Dz(p)$ are linearly dependent:

$$\frac{\partial z}{\partial p_L} = -\sum_{j=1}^{L-1} \frac{p_j}{p_L} \frac{\partial z}{\partial p_j}$$

Consider again

$$Dz(p) = \begin{bmatrix} \frac{\partial z_1}{\partial p_1} & \cdots & \frac{\partial z_1}{\partial p_L} \\ \vdots & \ldots & \vdots \\ \frac{\partial z_L}{\partial p_1} & \cdots & \frac{\partial z_L}{\partial p_L} \end{bmatrix}$$

We can remove last column, and Walras' law allows us to remove last row. Let $D\hat{z}(p)$ be the truncated matrix.

An equilibrium price vector p^* is **regular** if the Jacobian matrix $Dz(p^*)$ has rank $L-1$. Whenever equilibrium is regular, the economy is **regular**.

Regularity is the key condition for comparative statics

Let $p^*_L = 1$ so q are the relative prices, let $\Omega = \{\omega^i\}$ and write

$$\hat{z}(q_1, ..., q_{L-1}, p^*_L; \Omega).$$

Question: How do prices change with changes in the endowment?

Totally differentiate equilibrium condition $\hat{z}(q, p; \Omega) = 0$:

$$\frac{\partial \hat{z}}{\partial q} \cdot dq + \frac{\partial \hat{z}}{\partial \Omega} \cdot d\Omega = 0$$

or

$$\frac{\partial q}{\partial \Omega} = -\left[D\hat{z}(p^*)\right]^{-1} \frac{\partial \hat{z}}{\partial \Omega},$$

so regularity allows us to invert the matrix to do comparative statistics.

Properties of Walrasian equilibrium

Equilibrium price vector $p \in \Delta^0$ is **locally unique** if $\exists \epsilon > 0$ such that $z(p') \neq 0$ for all $p' \in \Delta^0$, $||p - p'|| < \epsilon$.

Three main results:

1) Any regular equilibrium is locally unique.

2) A regular economy has a finite number of equilibria.

3) For almost every vector of initial endowments, the exchange economy is regular. [Debreu 1970]

Last result is important because it says regularity, or foundations for local comparative statics, are **generic**.

Can we anything more about aggregate demand?

Theorem (Sonnenschein-Mantel-Debreu)

Suppose z is a continuous, homogeneous of degree 0 *and satisfies Walras' Law. Then for any $\epsilon > 0$, there exist k consumers with continuous, strictly convex, and non-decreasing preferences such that z is the aggregate excess demand for those k consumers, for all p such that $p_i/||p|| > \epsilon$ for all i*

"Anything Goes": any potential excess demand function can be in fact a demand function

Interpretation

Recall with individual demand, rationality is equivalent to Slutsky symmetry and negative semi-definiteness.

Sonnenschein, Mantel, and Debreu tells us that z is not restricted by these rationality restrictions (due to wealth effect). Aggregation dissipates the restrictions that rationality imposes. That is, we (more or less) can do no better than continuity, homogeneity, and Walras' Law.

Ken Arrow (1991): "in the aggregate, the hypothesis of rational behavior has in general no implications."

Interpreted as proof that general equilibrium theory had no testable implications. That is, one could not test hypothesis that agents are or are not trading in a Walrasian way by observing price data, unless one makes assumptions about the preferences of agents who were trading

Two types of responses:

1) "Too much freedom"

   SMD allows us to choose any combination of utilities and endowments.

   Aggregation theorems place restrictions on the types of endowments and preferences we allow. SMD tells us that the utility hypothesis tells us nothing about market demand unless it is augmented by these additional requirements.

2) Why look only at excess demand functions?

   Mas-Collel (1977) showed that the SMD theorem can be extended to a statement about equilibrium prices, not just excess demand functions.

   But Brown and Matzkin (EMA 1996) argue that the negative conclusion of SMD doesn't apply if endowments are observable.