Lecture 6. Arbitrage Pricing - Corporate Finance and Fixed Income Applications

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Neoclassical Corporate Finance

- In previous discussions, we only allow agents to invest in financial assets.
- Now, we also allow agents to invest in real productive opportunities.
- These productive opportunities are often created through corporations.
- Thus, our analysis here is about the behavior of corporations.
- We will assume a frictionless and complete securities market.
- Firms are only defined by the production technologies they possess.
- This simple framework provides an introduction to corporate finance.
- More comprehensive analysis of corporate finance beyond the neoclassical theory is provided in follow up courses.

Production Opportunities

- With a complete securities market, WLOG we assume a complete set of A-D securities are traded with price vector $\phi \gg 0$.
- In addition to endowment $e = [e_0; e_1]$, suppose that an agent also has an production opportunity defined by the following production function:

  $$y_{1\omega} = y_\omega(y_0), \quad \omega \in \Omega$$

  where
  - y_0 is the investment into the production opportunity at time 0,
  - $y_\omega(y_0)$ is the output from production at time 1 in state ω.
- The production function, assumed to be twice differentiable, satisfies:

  $$\begin{aligned} &(1) \quad y_\omega(0) = 0, \\ &(2) \quad y'_\omega(\cdot) \geq 0, \\ &(3) \quad y''_\omega(\cdot) < 0. \end{aligned}$$

  If $\lim_{x \downarrow 0} y'_\omega(x) = +\infty$ for all $\omega \in \Omega$ (Inada conditions), an interior solution is guaranteed.
- The agent is assume to maximize utility function $u(c)$

Optimal Production Policy

- Given investment level y_0, v denotes the time 0 market value of the agent's production technology:

$$v = \phi^\top y - y_0 = \sum_{\omega \in \Omega} \phi_\omega y_\omega(y_0) - y_0.$$

  We call v the net present value (NPV) of the investment.

- The agent's wealth at time 0 is:

$$w = e_0 - y_0 + \phi^\top (e_1 + y_1) = e_0 + \phi^\top e_1 + v \tag{1}$$

  where $y_1 \equiv [y_{11}; \ldots; y_{1M}]$ denotes the production vector.

- The agent's optimization problem now becomes:

$$\begin{aligned} \max_{y_0,\, c_0,\, c_1} \quad & u([c_0; c_1]) \\ \text{s.t.} \quad & c_0 + \phi^\top c_1 = w \end{aligned} \tag{2}$$

Optimal Production Policy

- The optimization problem can be solved in two steps:
  1. Choose y_0 to maximize current wealth, i.e., the NPV of production,
  2. Choose c to maximize utility.
- The FOC for maximizing market value of production is:

$$\substack{\text{marginal}\\\text{cost}} = 1 = \phi^\top y'(y_0) = E^{\mathbb{P}}[\eta_\omega \cdot y'_\omega(y_0)] = \frac{1}{1+r_F} E^{\mathbb{Q}}[y'_\omega(y_0)] = \substack{\text{expected}\\\text{discounted}\\\text{marginal}\\\text{benefit}}$$

- Given the concavity of $y(\cdot)$, FOC has a unique solution, which gives the optimal production decision.
- The optimal production decision is independent of the agent's consumption decisions. It only depends on the production function and the state prices.
- If a model implies the wrong state prices $\Rightarrow$ it generates counterfactual investment incentives

Corporate Investment Decisions

- Production opportunities are often owned by firms
- There $j = 1, \ldots, F$ firms. Firm j has production technology:

$$y_j(y_{j0}) = [y_{j1}(y_{j0}); \ldots; y_{jM}(y_{j0})].$$

- Each firm is owned by agents (i.e., this is part of the endowment). Let s_{kj} be the share of firm j owned by agent k:

$$\sum_k s_{kj} = 1, \quad j = 1, \ldots, F.$$

- For a firm investing y_{j0}, its net market value at time 0 is:

$$v_j = \phi^\top y_j(y_{j0}) - y_{j0}, \quad \forall\, k.$$

- Thus, the total wealth of agent k is

$$w_k = e_{k0} + \phi^\top e_{k1} + \sum_j s_{kj} v_j.$$

Corporate Investment Decisions

- We now consider the firms' investment decision.
- Suppose that firm j is solely owned by agent k. Her decision on the firm's investment is to maximize v_j:

$$\phi^\top y_j'(y_{j0}) = 1.$$

- This decision is independent of her endowment and preferences. The firm's optimal investment decision is independent of who owns it!

Theorem (Maximizing Current Market Value)

With a frictionless and complete securities market, there is unanimity among a firm's shareholders on its investment decisions, which is to maximize its current market value (NPV).

Corporate Investment Decisions

- The intuition for unanimity is simple:
  - In a frictionless and complete securities market, an agent can achieve any desirable consumption plan subject only to her budget constraint.
  - Maximizing current market value of the firm increases shareholders' wealth, hence their budget sets, which is desirable for all shareholders.
  - Firm owners can meet their own specific needs via financial transactions in the market.
- The unanimity also allows separation between ownership and management.

Financing Decisions

- So far, we assume that a firm is owned solely by its original shareholders, who provide the investment y_0 and share the output, according to their stock shares.
- In general, a firm can also raise outside funds by issuing securities such as bonds or new stocks. How a firm raises the funds for its investment is called its financing decision.
- The mix of securities a firm issues to finance its operations is called its capital structure.
- Different corporate securities represent different claims on the firm's assets/cashflows (liabilities):
  - timing and amount,
  - seniority,
  - control rights.

Financing Decisions

- Suppose a firm is financed by two types of securities: debt and equity.
- Let d_0 denote the amount of debt issued at time 0 (in market value).
- The remaining investment, $e_0 = y_0 - d_0$, is financed by equity holders.
- Then,

$$y_0 = d_0 + e_0.$$

- Let

$$d_1 = [d_{11}; \ldots ; d_{1M}], \quad e_1 = [e_{11}; \ldots ; e_{1M}]$$

be the payoff vectors for debt and equity at time 1, respectively.

- We have:

$$y_1 = d_1 + e_1.$$

- In order for the debt to be held by other agents, it must be that:

$$d_0 = \phi^\top d_1.$$

Financing Decisions

- Let D and E denote the current market value of debt and equity. Then,

$$D = \phi^\top d_1$$
$$E = \phi^\top e_1.$$

- The value of the firm's assets (payoffs) is:

$$V = D + E = \phi^\top y_1.$$

- The net value of the equity for shareholders is now:

$$\underbrace{\phi^\top (y_1 - d_1)}_{\text{PV of future dividends}} - \underbrace{e_0}_{\text{initial cash outlay from shareholders}} = \phi^\top y_1 - (d_0 + e_0) = \phi^\top y_1 - y_0 = NPV$$

- Maximizing the net value of equity is the same as before, independent of how the firm is financed, i.e., the choice of d_0 and d_1 (subject to $d_0 = \phi^\top d_1$).

Theorem (Modigliani-Miller)

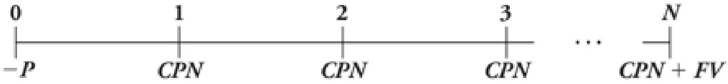
With a frictionless and complete securities market, a firm's value is determined solely by its investment decisions and is independent of how it is financed.

Fixed income securities

- Recall: **debt securities** are promises to pay back a fixed amount at a pre-specified time in the future. Investors receive full amount unless the entity goes bankrupt
- Big players in the bond market
  - Governments: US Treasury (bills < 1 year; notes 2 to 10 years; bonds > 10 years), states, municipalities, foreign countries
  - Corporations: commercial paper (short maturity) and corporate bonds
  - Agencies: issued by government-sponsored entities, e.g., Fannie Mae and Freddie Mac (mortgages), and student loans
- Today, we will focus on default-free securities (e.g., US Treasuries)

Payoffs

- A bond promises to pay...
  - Fixed coupon payments at pre-specified dates and
  - Fixed principal amount, also called the face value or par value, plus the coupon amount at the maturity date
- When there are no coupons, the security is called a pure discount bond or a zero coupon bond
- Otherwise, it is a **coupon bond**



Sample payoffs of a coupon bond

- Unlike stock (which tends to be unique), there can be many different bonds outstanding at a given time for a given issuer

Zero Coupon Bonds

- Usually normalize face value = \$1 (denominations are higher in reality)
- Let $P_{N,t}$ denote the price of an N-period zero coupon bond at time t.
  - This is the price of security, at time t, that pays \$1 at $t + N$
  - Notation follows Campbell: maturity is 1st subscript, calendar time is 2nd
  - If I hold the N period bond for 1 period, I enter next period with an $N - 1$ period bond with price $P_{N-1,t+1}$
- Almost always, $P_{N,t} < 1 = P_{0,t}$ (cash, numeraire)
- When the Law of One Price Holds (no arbitrage), any N period default-free coupon bond can be expressed as a portfolio of zero coupon bonds

$$\text{bond price} = \sum_{j=1}^{N} \underbrace{CF_{t+j}}_{\substack{\text{known}\\ \text{coupon}\\ \text{payment}}} \cdot P_{j,t},$$

so we can mostly restrict attention to zero coupon bonds

Link between zero coupon bond price and the SDF

- Let's place a bit of structure on the problem:
  - SDF $\eta(s)$at time t depends on $s_t \in \{1, \ldots, S\}$
  - s_t follows a 1st order Markov chain (virtually all models are Markovian):

    $$P[s_{t+1} = s | s_t, s_{t-1}, ..., s_0] = P[s_{t+1} = s | s_t] \equiv \pi(s_{t+1}, s_t)$$

    Probability that $s_{t+1} = s$ just depends on s_t, doesn't change over time
  - Under this structure, prices will depend on maturity and state variable. Notation $P_{N,t} \equiv P_N[s_t]$

- No arbitrage: price = expected discounted payoff

- If $k = 1$ and $s_t = j$ we know that

$$\begin{aligned} P_{1,t} = P_1[j] &= \sum_{s=1}^{S} Prob[s_{t+1} = s | s_t = j] \cdot \eta(s) \cdot \underbrace{\$1}_{\text{known payoff}} \equiv \sum_{s=1}^{S} \pi(s,j)\eta(s) \\ &\equiv \underbrace{E_t}_{\substack{\text{conditional expectation} \\ \text{with info up to t (i.e., } s_t)}} [\eta_{t+1} \cdot 1] = E_t\left[\eta_{t+1} \cdot P_0[j]\right] = E_t[\eta_{t+1}] \end{aligned}$$

Link between zero coupon bond price and the SDF

- How do I get $P_2[s_t]$? Iterate on the FTAP
- At $t+1$, investor receives $P_1[s_{t+1}]$: payoff that depends on s_{t+1}
- Then, plugging the payoff into the pricing formula, we get

$$\begin{aligned}
P_2[s_t] &= \sum_{s_{t+1}=1}^{S} \pi(s_{t+1}, s_t) \cdot \eta(s_{t+1}) \cdot \underbrace{P_1[s_{t+1}]}_{\text{future payoff}} = E_t[\eta_{t+1} \cdot E_{t+1}[\eta_{t+2}]] \\
&= \sum_{s_{t+1}=1}^{S} \pi(s_{t+1}, s_t) \cdot \eta(s_{t+1}) \cdot \left[\sum_{s_{t+2}=1}^{S} \pi(s_{t+2}, s_{t+1}) \eta(s_{t+2}) \cdot \$1 \right] \\
&= \sum_{s_{t+1}=1}^{S} \sum_{s_{t+2}=1}^{S} \pi(s_{t+2}, s_{t+1}) \pi(s_{t+1}, s_t) \cdot \eta(s_{t+1}) \cdot \eta(s_{t+2}) \\
&= E_t[\eta_{t+1} \cdot \eta_{t+2}]
\end{aligned}$$

where this is an example of the law of iterated expectations

- In general: $P_{N,t} = E_t[\eta_{t+1} P_{N-1,t+1}] = E_t[\eta_{t+1} \times \cdots \times \eta_{t+N}] \equiv E_t[\eta_{t:t+N}]$

Yield-to-Maturity

- Zero coupon bond price: $P_{Nt} = E_t[\eta_{t:t+N}]$ is a number a bit less than 1
- Often, we quote prices in units that are easier to think about
- The **yield to maturity** for a zero coupon bond is the geometric mean of the gross return that investor gets if holding bond to maturity

$$\left[\frac{\text{Payoff}}{\text{Price}}\right]^{\frac{1}{N}} = \left[\frac{1}{P_{Nt}}\right]^{\frac{1}{N}} \equiv Y_{Nt} \quad \Longleftrightarrow \quad P_{Nt} = \left[\frac{1}{Y_{Nt}}\right]^{N}$$

- **Prices and yields (returns) move in opposite directions!**
- This is just a definition: a nice way of describing the price. It is the per-period (gross) discount rate on the bond
- If investor buys and holds the bond to maturity, its *cumulative return is known* and has an geometric mean of Y_{Nt}

Yield-to-Maturity - zero coupon bond in logs

- Zero coupon bonds are easy to work with analytically when we take logs
- From previous slide: $\left[\frac{1}{P_{Nt}}\right]^{\frac{1}{N}} \equiv Y_{Nt} \quad \Longleftrightarrow \quad P_{Nt} = \left[\frac{1}{Y_{Nt}}\right]^{N}$
- Let $p_{Nt} \equiv \log P_{Nt}$ and $y_{Nt} \equiv \log Y_{Nt}$. Then

$$y_{Nt} = -\frac{1}{N} \times p_{Nt} \quad \Longleftrightarrow \quad p_{Nt} = -N \cdot y_{Nt}$$

Notation switch: p is a log bond price, not a probability!

- Therefore, the elasticity of the bond price with respect to the yield is

$$\frac{dp_{Nt}}{dy_{Nt}} = \frac{d\log P_{Nt}}{d\log Y_{Nt}} = -N = -1 \times \text{the investment horizon}$$

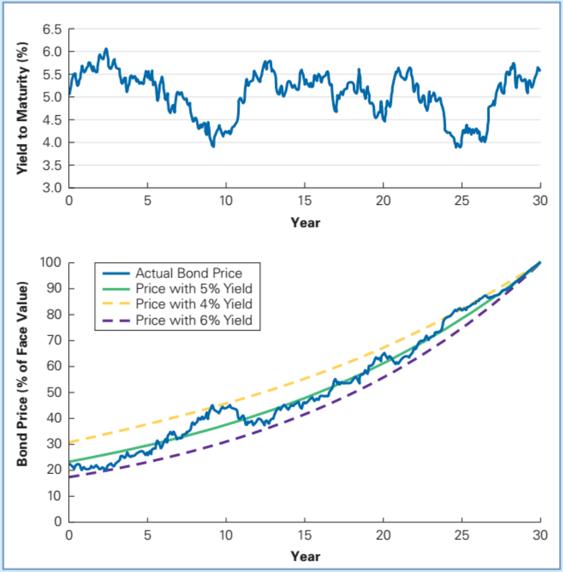
- Longer maturity bonds are **more responsive** to the same change in yield

Interest rate sensitivity over time: graphical example

FIGURE 6.2

Yield to Maturity and Bond Price Fluctuations over Time

The graphs illustrate changes in price and yield for a 30-year zero-coupon bond over its life. The top graph illustrates the changes in the bond's yield to maturity over its life. In the bottom graph, the actual bond price is shown in blue. Because the yield to maturity does not remain constant over the bond's life, the bond's price fluctuates as it converges to the face value over time. Also shown is the price if the yield to maturity remained fixed at 4%, 5%, or 6%.



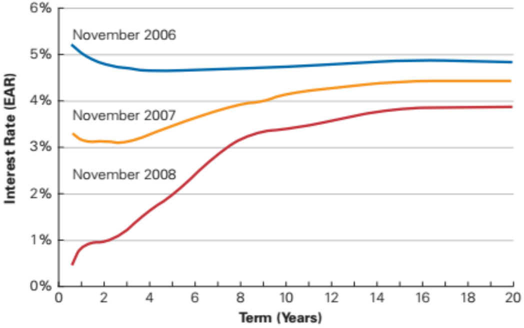
Should we compare yields across maturities?

- Yield captures an average rate of return over the life of the loan → yields across maturities are measured in different units
- Zero coupon bond prices are exchange rates: P_{Nt} is the rate at which I can exchange cash today for \$1 in $t+N$
- Maybe the value of having \$1 in $t+2$ just happens to be much lower for some reason, relative to $t+1$ and t
  - $\frac{P_{2t}}{P_{1t}}$ is relatively low, so it is cheap to transfer resources from $t+1$ to $t+2$
  - Real world example of this: maybe we know that we're currently in a recession, which will end by $t+2$
  - Buying and holding the 2 year bond at time t offers a higher return, but investor is forgoing the opportunity to tap into savings during the recession
- Arrow-Debreu security logic: \$1 need not have the same marginal value in all states of the world (here, states are time periods)

The yield curve

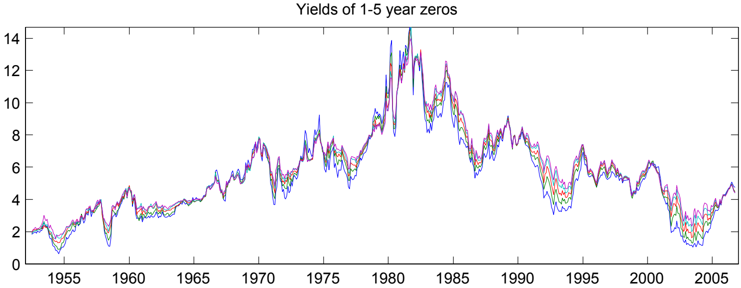
- At a given point in time, there are many US treasury securities outstanding $\Rightarrow$ can observe essentially any desired zero-coupon yield
- The **yield curve** plots the currently available sets of zero coupon bond yields on offer from the US government. It also changes a lot over time.

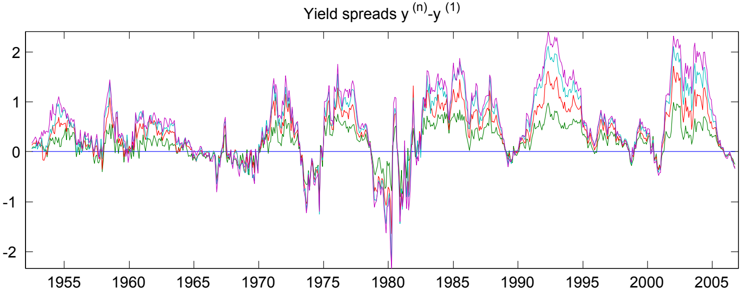
Term (years)	Date Nov-06	Nov-07	Nov-08
0.5	5.23%	3.32%	0.47%
1	4.99%	3.16%	0.91%
2	4.80%	3.16%	0.98%
3	4.72%	3.12%	1.26%
4	4.63%	3.34%	1.69%
5	4.64%	3.48%	2.01%
6	4.65%	3.63%	2.49%
7	4.66%	3.79%	2.90%
8	4.69%	3.96%	3.21%
9	4.70%	4.00%	3.38%
10	4.73%	4.18%	3.41%
15	4.89%	4.44%	3.86%
20	4.87%	4.45%	3.87%



The figure shows the interest rate available from investing in risk-free U.S. Treasury securities with different investment terms. In each case, the interest rates differ depending on the horizon. (Data from U.S. Treasury STRIPS.)

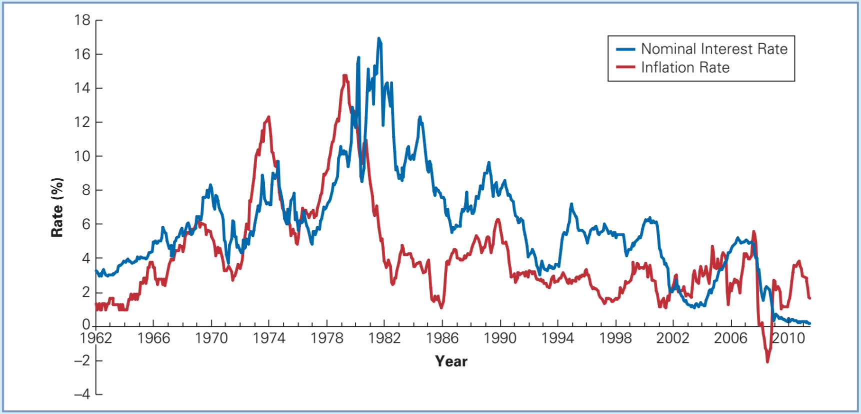
Yields fluctuate a lot over time





Some, but not all of this relates to inflation

FIGURE 5.1 U.S. Interest Rates and Inflation Rates, 1962–2012



Interest rates are one-year Treasury rates, and inflation rates are the increase in the U.S. Bureau of Labor Statistics' consumer price index over the coming year, with both series computed on a monthly basis The difference between them thus reflects the approximate real interest rate earned by holding Treasuries. Note that interest rates tend to be high when inflation is high.

How does inflation enter the picture?

- Most of the time, our models of the SDF η_t is denominated in units of real, not nominal consumption
- The nominal bond price is the real price $P_{n,t}$ times a price index CPI_t
  - Common notation $P^{\$}_{n,t} = P_{n,t} \times CPI_t$, $\Pi_t = CPI_t / CPI_{t-1}$
  - The real purchasing power of a nominal bond declines with inflation (like a negative dividend). Assumptions about terminal value $P_{0,t}$ capture this.
- With inflation, the asset pricing equation becomes

$$
\begin{aligned}
1 &= E_t\left[\eta_{t+1}\frac{P_{n-1,t+1}}{P_{n,t}}\right] = E_t\left[\eta_{t+1}\frac{P^{\$}_{n-1,t+1}/CPI_{t+1}}{P^{\$}_{n,t}/CPI_t}\right] \\
&= E_t\left[\frac{\eta_{t+1}}{\Pi_{t+1}}\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}\right] \equiv E_t\left[\underbrace{\eta^{\$}_{t+1}}_{\text{nominal SDF}} \quad \underbrace{\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}}_{\text{nominal return}}\right]
\end{aligned}
$$

- So, choose your own adventure, but keep track of units!
  - Use a real SDF + real prices, but account for inflation Π
  - Use a nominal SDF + nominal yields

A recipe for the yield curve

- Recall: $P_{Nt} = E_t[\eta_{t:t+N}]$. Yield curve $\iff$ moments of the SDF
- No-arbitrage models of the yield curve combine the following ingredients
  - Introduce one or many state variables x_t (data $\Rightarrow$ we need at least 3). SDF η_t is assumed to be a function of x_t.
  - Characterize law of motion for x_t under $\mathbb{P}$. This is similar to our earlier example $\pi(s_{t+1}, s_t)$. (Alternative: law of motion under $\mathbb{Q}$ + risk-free rate)
  - Solution: iterate on pricing equation $P_N[x_t] = E[\eta(z_{t+1})P_{N-1}[x_{t+1}] \mid x_t]$
  - Usually choose dynamics so that log bond prices are affine in x_t. Find coefficients using the method of undetermined coefficients
  - More examples, discussion, details in Campbell Chapter 8 (especially 8.3)
- Since zero coupon bonds are just moments of the SDF over different horizons, observing the entire term structure on an essentially tells us a lot of information about η and its dynamics.
- Remember: any dynamic asset pricing model will have implications for the yield curve. Easy to compare with the actual behavior of yields in the data!

Example: the Vasicek (1977) model

- Introduce a single state variable x_t which follows the law of motion

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$

where $\varepsilon_{t+1} \sim N(0,1)$ which are iid over time.

- x_t is called a first-order autoregressive (AR(1)) process

- Next, suppose that the log SDF $m_{t+1} \equiv \log \eta_{t+1}$ satisfies

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \left(\frac{\lambda}{\sigma}\right)\varepsilon_{t+1}$$

Note: many authors (e.g., Campbell & Cochrane) use M_{t+1} for SDF and $m_{t+1} \equiv \log M_{t+1}$. Switching notation to match 8.3.1 in Campbell book

- Goal: characterize analytically the set of log bond prices p_{nt}

Example: the Vasicek (1977) model

$$
\begin{aligned}
x_{t+1} &= \mu + \phi x_t + \sigma \varepsilon_{t+1} \\
m_{t+1} &= -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \left(\frac{\lambda}{\sigma}\right)\varepsilon_{t+1}
\end{aligned}
$$

- Note that $x_{t+1}|\mathcal{F}_t \sim N(\mu + \phi x_t, \sigma)$ and $m_{t+1}|\mathcal{F}_t \sim N(-x - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2, \left(\frac{\lambda}{\sigma}\right))$
- Property of normal: $X \sim N(\mu, \sigma) \Rightarrow E[\exp(t \cdot X)] = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$
- Therefore, we can easily compute the 1 period ahead bond price:

$$
p_{1t} = \log E_t[\exp(m_{t+1})] = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 = -x_t = 0 - 1 \cdot x_t
$$

- Note, therefore, that $x_t = y_{1t} = \log(1 + r_{Ft})$, which is also often referred to as the short rate. So, the model implies that the short rate follows a mean-reverting AR(1) process.

Example: the Vasicek (1977) model

$$
\begin{aligned}
x_{t+1} &= \mu + \phi x_t + \sigma \varepsilon_{t+1} \\
m_{t+1} &= -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \left(\frac{\lambda}{\sigma}\right)\varepsilon_{t+1}
\end{aligned}
$$

- What about other horizons? Our earlier argument implies that

$$
\begin{aligned}
p_{2t} &= \log E_t[\exp(m_{t+1} + p_{1,t+1})] = \log E_t[\exp(m_{t+1} - x_{t+1})] \\
&= \log E_t\left[\exp\left(-x_t - \tfrac{1}{2}\left(\tfrac{\lambda}{\sigma}\right)^2 - \left(\tfrac{\lambda}{\sigma}\right)\varepsilon_{t+1} - [\mu + \phi x_t + \sigma\varepsilon_{t+1}]\right)\right] \\
&= \log E_t\left[\exp\left(-(1+\phi)x_t - [\tfrac{1}{2}\left(\tfrac{\lambda}{\sigma}\right)^2 + \mu] - \left(\tfrac{\lambda}{\sigma} + \sigma\right)\varepsilon_{t+1}\right)\right] \\
&= -(1+\phi)x_t - [\tfrac{1}{2}\left(\tfrac{\lambda}{\sigma}\right)^2 + \mu] + \tfrac{1}{2}\left(\tfrac{\lambda}{\sigma} + \sigma\right)^2 \equiv A_2 + B_2 x_t
\end{aligned}
$$

- Note the pattern: $p_{1t} = 0 + (-1)\cdot x_t$ and $p_{2t} = A_2 + B_2 x_t$

Example: the Vasicek (1977) model

- Next, we will guess and verify that all bond prices are affine in x_t

$$p_{n,t} = A_n + B_n x_t$$

- To verify, we need to show that $p_{k,t} = A_k + B_k x_t \Rightarrow p_{k+1,t} = A_{k+1} + B_{k+1} x_t$
- Solution (left as an exercise) yields the following recursion for coefficients:

$$\begin{aligned} B_n &= -1 + \phi B_{n-1} = -\left(\frac{1-\phi^n}{1-\phi}\right) \\ A_n &= A_{n-1} + B_{n-1}(\mu - \lambda) + \tfrac{1}{2} B_{n-1}^2 \sigma^2 \end{aligned}$$

- This model doesn't describe the data very well, but richer models follow the same approach. See remainder of Campbell 8.3 for more details