14.121: GE with Indivisibilities

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GE with Indivisible Goods: Some Simple Models

A **housing market** is a list $((a_k, h_k)_{k=\in\{1,\ldots,n\}}, P)$ such that

- $\{a_1, a_2, \ldots, a_n\}$ is a set of agents and $\{h_1, \ldots, h_n\}$ is a set of houses such that each agent a_k owns (or occupies) h_k.
- Each agent a has strict preferences over houses denoted by P_a. Let R_a be the weak-preference relation associated with P_a. Let $P = (P_a)_{a\in A}$.

An allocation in a housing market can be described as a **matching** $\mu : A \to H$, a one-to-one and onto function. House $\mu(a)$ is the assigned house of agent a under μ.

A matching μ is **Pareto-efficient** if there is no other matching ν such that $\nu(a) R_a \mu(a)$ for all $a \in A$ and $\nu(a) P_a \mu(a)$ for some agent $a \in A$.

Housing market: Basic Exchange Economy

Core: defined as before

The core is the set of matchings μ such that there exists no coalition $B \subset A$ and a matching ν such that

(a) for any $a \in B$, $\nu(a) = h_\ell$ for some $a_\ell \in B$, and
(b) for any $a \in B$, $\nu(a) R_a \mu(a)$ and for some $b \in B$, $\nu(b) P_b \mu(b)$. (we say that ν weakly dominates μ through B)

A matching is **individually rational** if every agent receives a house at least as good as her initial house.

- A core matching is individually rational (take B all $|A|$ singletons, one by one in the definition of core)
- A core matching is Pareto efficient (take $B = A$ in the definition of core).

Housing market

Is the core non-empty? If it exists, what does it look like?

Theorem (Shapley and Scarf 1974, Roth and Postlewaite 1977)

The core is nonempty for each housing market and it is unique.

Proof. (Alternative proof attributed to David Gale)

Using an algorithm known as **Gale's top trading cycles** (TTC) algorithm.

It is an iterative algorithm defined over a directed graph with houses and agents as nodes. We construct a matching as follows in several steps:

Step 1. Let each agent point to her first choice and each house point to its owner. In this graph, there is necessarily a cycle and no two cycles intersect (since preferences strict, it turns out that this restriction is quite important). Remove all cycles from the problem by assigning each agent the house that she is pointing to.

Step k. Let each remaining agent point to her first choice among the remaining houses and each remaining house point to its owner (note that houses leave with their owners and agents leave with their houses, so a house remaining in the problem implies that the agent is still in the problem and vice versa). There is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house that she is pointing to.

Let μ be the matching obtained as the result of Gale's TTC algorithm.

We prove that μ is in the core, first and then we prove that μ dominates any other matching through some coalition.

μ is in the core:

Let $C_1, C_2, ..., C_k$ be the agents in cycles (in the order they are removed) in Gale's TTC algorithm.

Note that no agent in C_1 can be in a blocking coalition, since they get their first choice under μ.

Given this, no agent in C_2 can be in a blocking coalition, since they get their first choice in $H \backslash \mu(C_1)$, iteratively we continue.

There is no other matching in the core:

Consider a matching $\nu \neq \mu$. We will show it is dominated by μ.

Let b be the first agent who satisfies $\nu(b) \neq \mu(b)$ (according to the order of the cycles $C_1, ..., C_k$, if there are multiple agents in a cycle like b, then choose one of them arbitrarily).

Let b be in cycle C_ℓ. Note that for every agent a assigned before the cycle C_ℓ, we have $\nu(a) = \mu(a)$. (by definition)

Given this, for every agent $a \in C_\ell$, $\mu(a) R_a \nu(a)$ for all $a \in C_\ell$, as the agents in C_ℓ will be pointing to their most preferred alternative among those available.

Then we have for $a \in C_\ell$, $\mu(a) P_a \nu(a)$, by strictness of preferences.

Moreover for each agent $a \in A$, $\mu(a) = h_m$ which is owned by some $a_m \in A$ by construction of μ and C_ℓ. (so μ is feasible)

Hence μ dominates ν through coalition C_ℓ, concluding the proof.

Housing markets: competitive equilibrium

What about an exchange economy and competitive equilibrium?

The price mechanism will also achieve the core matching:

Prices of houses in a vector $p = (p_1, ..., p_n)$.

A house h_m is affordable for agent a_ℓ at p if $p_m \leq p_\ell$ (budget set).

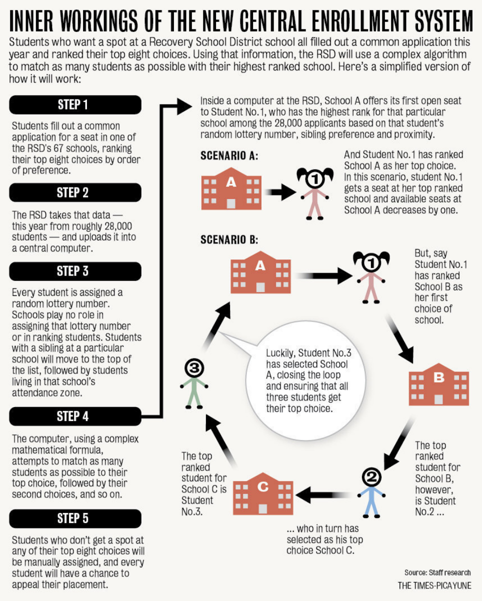
A matching μ and price vector p is a competitive equilibrium if for any agent a, $\mu(a)$ is the best house she can afford at prices p.

Theorem

There is a competitive equilibrium matching for a housing market which is given by the core matching.

Proof. Let P be a preference profile and let $C_1, C_2, ..., C_k$ be the cycles encountered in order in Gale's TTC algorithm for this market. Let price vector p be such that for any cycle C_m and for any agent $a_\ell \in A$ such that $p_\ell = q_m$ for some constant q_m (**each house in a cycle has the same price**) and let $q_m > q_{m+1}$ for any $m \in \{1, 2, ..., k-1\}$ (houses in earlier cycles have higher price). Let $\mu = \varphi[P]$.

Observe that (μ, p) is a competitive equilibrium. No agent a likes some house allocated in a later cycle more than $\mu(a)$. No agent can afford any house allocated in an earlier cycle. Hence every agent is allocated the best house she can afford. $\diamond$



Abdulkadiroğlu, Che, Pathak, Roth, Tercieux (2017)

House allocation problem: Collective Ownership Economy

Hylland and Zeckhauser (JPE 1979):

A house allocation problem is a triple $< A, H, P >$ such that

- $A = \{a_1, a_2, ..., a_n\}$ is a set of agents
- $H = \{h_1, h_2, ..., h_n\}$ is a set of houses (or any indivisible object)
- Each agent $a \in A$ has strict preferences over houses denoted by P_a. Weak preferences $R_a : hR_a g \Leftrightarrow hP_a g$ or $h = g$).
  $P = (P_a)_{a \in A}$ is the preference profile

House allocation problem

Real-life applications: organ allocation (deceased donor waiting list), dormitory room allocation at universities, parking space and office allocation at workplaces.

One of simplest resource allocation problems.

A (deterministic) mechanism is a procedure that assigns a matching for each house allocation problem.

A mechanism is **Pareto-efficient** if it always assigns a Pareto-efficient matching for each preference profile reported.

House allocation problem: serial dictatorship

A **serial dictatorship** is defined through a priority ordering of agents. A priority ordering is a function $f : \{1, 2, ..., n\} \to A$ that is one-to-one and onto.

$f(i)$ is the agent with the ith highest priority under f. Let $\mathcal{F}$ be the set of orderings.

Let φ^f be the serial dictatorship induced by ordering f.

The allocation $\varphi^f[P]$ is found as follows:

The first agent $f(1)$ gets her first choice, the second agent $f(2)$ gets her first choice excluding the house assigned to the first agent, ..., kth agent $f(k)$ gets her first choice excluding the house assigned to the house all agents before $f(k)$.

House allocation problem: serial dictatorship

Theorem. A serial dictatorship is Pareto-efficient.

Proof. Suppose that $\mu = \varphi^f[P]$ is the outcome of a serial dictatorship φ^f for preference profile P.

We prove the theorem by contradiction. Suppose that the serial dictatorship is not Pareto-efficient. Then there exists a matching ν that Pareto-dominates μ under reported preferences P.

In particular, there exists some agent $a \in A$ such that $\nu(a) P_a \mu(a)$. Let a be the highest priority agent in f with this property. Let $f(i) \equiv a$.

Since for any other agent $b \in A\backslash\{a\}$, $\nu(b)R_b\mu(b)$, for any agent $f(j)$ with $j < i$ (that is $f(j)$ has higher priority than $f(i) = a$), we have $\nu(f(j)) = \mu(f(j))$.

Therefore, in serial dictatorship φ^f, when it is $f(i)$'s turn to choose, houses $\nu(a)$ and $\mu(a)$ are still available.

However, she chooses $\mu(a) = \varphi^f[P](a)$ in the serial dictatorship, contradicting $\nu(a)P_a\mu(a)$. So there is no matching that can Pareto-dominate μ.

Hence $\mu = \varphi^f[P]$ is Pareto efficient.

Welfare theorems

Turns out we can also use the core mechanism for a house allocation problem, by simply starting with an arbitrary assigned endowment, and running the core mechanism. Known as **Core from assigned endowments**

How are these two mechanisms related?

Theorem (Abdulkadiroğlu and Sönmez 1998): For any ordering f, and any matching μ, simple serial dictatorship induced by f and core from assigned endowments μ both yield *Pareto efficient* matchings. Moreover, for any *Pareto efficient* matching η, there is a *simple serial dictatorship* and a *core from assigned endowments* that yields it.

Q: Why are these welfare theorems?

Proof

Let μ be a Pareto-efficient matching for a house allocation problem $\langle I, H, \succ \rangle$.

First, we will construct a priority order f so that $\varphi^f[\succ] = \mu$.

- Suppose that no agent receives his top choice in μ.
  - We can construct a cycle as follows with two or more agents.
  - Suppose each agent points to his top choice house and is pointed by the house he received in μ.
  - Then, by finiteness of the problem there exists a cycle.
  - Observe that there is no cycle with a single agent in it, as we assumed nobody receives his top choice in μ.
  - Then, we can improve every agent in the cycle by assigning him the house he is pointing to and otherwise leaving μ unchanged.
- This contradicts μ being Pareto efficient.

- We showed that there exists an agent who receives his top choice in μ. Let him be $f(1)$. Let's exclude house $\mu(f(1))$ from the problem.

- Show in the same manner that there exists an agent who receives his top choice excluding $\mu(f(1))$, let him be $f(2)$

- We construct the remaining of f in an iterative manner similarly.

- It is straightforward to observe that $\varphi^f[\succ] = \mu$.

Next, we will construct an endowment matching ω for the same problem.

Let $\omega = \mu$. Then core from endowment $\omega = \mu$.

To see this, observe that the agent $f(1)$, constructed above, will point to his endowment in Gale's TTC algorithm in round 1, and will receive it. Similarly $f(2)$ will receive his endowment in round 1 or 2, and so on.

This proves that $\phi^{\omega}[\succ] = \mu$.

Lottery Systems

A **lottery** is a probability distribution over matchings.

A **lottery mechanism** is a systematic procedure to select a lottery for each problem.

Examples:

- **Random serial dictatorship (RSD)**: Randomly select an ordering with uniform distribution and use the induced simple serial dictatorship.
- **Core from random endowments**: Randomly select a matching (to be interpreted as initial endowment) and select the core of the induced housing market.

How are these two mechanisms related?

Turns out that they produce exactly the same outcome!

Theorem (Pathak and Sethuraman (2011)): The *random serial dictatorship* is equivalent to the the *core from random endowments.* That is, for each problem they choose the same lottery.

- Surprising equivalence between a "market" mechanism and a dictatorship