Lecture 11. Expected utility

Lawrence Schmidt

MIT Sloan School of Management

15.470/Fall 2019

Where we left off: Additional simplifications on preferences

- Last time, we considered further simplifications on expected utility
- State Independence: Utility over a consumption path depends only on the consumption levels along the path (state), not the path/state itself.
- That is:

$$u(c) = \sum_{\omega \in \Omega} p_\omega u(c_0, c_{1\omega}). \tag{1}$$

- Time Additivity: Utility over a consumption path is the sum of utility over consumption at each date:

$$u(c_0, c_{1\omega}) = u_0(c_0) + u_1(c_{1\omega}). \tag{2}$$

- Often, we even assume that:

$$u(c_0, c_{1\omega}) = u(c_0) + \rho\, u(c_{1\omega}), \quad \rho > 0.$$

ρ is called the time preference/discount coefficient.

Additional Simplifications

State-Independent and Time-Additive Expected Utility

A state-independent and time-additive, discounted expected utility function then takes the following form:

$$u(c_0) + \rho \sum_{\omega \in \Omega} p_\omega u(c_{1\omega}), \quad \rho > 0. \tag{3}$$

This will often be used as a canonical specification of an agent's utility function.

Going forward, we also assume that $u(\cdot)$ is twice differentiable.

- We will refer to $u'(c)$ as the marginal utility at consumption level c.
- Insatiability implies

  $$u'(\cdot) > 0,$$

  i.e., the agent has strictly increasing utility or positive marginal utility.

Critiques/Extensions: Behavioral Biases and Beyond

- Evidence inconsistent with expected utility
  - Allais paradox: evidence of direct violations of the independence axiom
  - Ellsberg paradox: people dislike ambiguous lotteries
  - Rabin critique: rejecting small mean zero bets implies ridiculous aversion to large bets.
  - ...
- More flexible utility functions – "behavioral economics/finance"
  - Habit formation
  - Catching up with the Jones
  - First order risk aversion (prospect theory/generalized disappointment aversion)
  - Uncertainty/ambiguity aversion
  - Difference in beliefs, general state-dependent preferences
  - ...
- Are these concerns really about preferences?

Definitions from last time

Axiom (Convexity of preferences)

$\forall\ a, b, c \in C$ and $\alpha \in (0,\ 1)$, if $a \succsim b$ and $c \succsim b$, then $\alpha a + (1-\alpha)c \succsim b$.

For continuous preferences, convexity implies that the sets of preferred bundles $\{a \in C : a \succsim c\}$ are convex. Strict convexity replaces $\succsim$ with $\succ$ above

Definition

A function $u(\cdot)$ is concave if $\forall\ x,\ x'$ and $\alpha \in [0, 1]$,

$$u(\alpha x + (1-\alpha)x') \geq \alpha u(x) + (1-\alpha)u(x').$$

Theorem (Concavity)

If a preference satisfies the Continuity, Independence and Convexity axioms, and can be represented by a discounted expected utility function of the form:

$$u(c_0) + \rho \sum_{\omega \in \Omega} p_\omega u(c_{1\omega}), \quad \rho > 0,$$

then $u(\cdot)$ is a concave function.

Definition of Risk Aversion

Definition

A random number x is a fair gamble if $\mathbb{E}[x] = 0$.

Definition

An agent with expected utility of $u(\cdot)$ is risk averse if

$$\mathbb{E}[u(w + x)] \;\leq\; \mathbb{E}[u(w)] \;\; \forall \; \mathbb{E}[x] = 0$$

and strictly risk averse if the inequality is strict.

A risk-averse agent always prefers a sure payoff over a risky payoff with the same mean.

Definition of Risk Aversion (RA)

Theorem

An agent is (strictly) risk averse if and only if $u(\cdot)$ *is (strictly) concave.*

Measures of Risk Aversion

Risk Premium

Definition

Let x be a fair gamble, $u(\cdot)$ be an agent's utility function with wealth w. The risk premium π required by the agent to take the gamble is given by:

$$\mathbb{E}[u(w+x)] = u(w-\pi). \tag{4}$$

- The risk premium defined above represents the amount of wealth an agent is willing to give up in order to get rid of the gamble/risk.
- The risk premium defined above (including the negative sign) is also called the certainty equivalent of the risky gamble.
- The definition of risk premium in (4) is not the only possible one. For example, we can also define the risk premium as $\hat{\pi}$, where

  $$\mathbb{E}[u(w+x+\hat{\pi})] = u(w).$$

  In this case, $\hat{\pi}$ represents the amount of wealth the agent is wiling to receive in order to take the gamble. In general, $\hat{\pi}$ and π are different.

Measures of Risk Aversion
Absolute Risk Aversion

- The risk premium π in general depends on $u(\cdot)$, w, and x.
- For a small gamble x, we expect π to be small and thus:

$$\mathbb{E}[u(w+x)] = u(w) + \tfrac{1}{2}u''(w)\mathbb{E}\left[x^2\right] + o\left(x^2\right) = u(w) - u'(w)\pi + o\left(\pi\right) \qquad (5)$$

where $o\left(\epsilon^n\right)$ denotes terms of higher order of ϵ^n.

- If we drop the $o(x^2)$ and $o(\pi)$ terms, we have:

$$\pi \approx \tfrac{1}{2}\left[-\frac{u''(w)}{u'(w)}\right]\mathbb{V}[x]. \qquad (6)$$

- See Campbell section 1.2.3 for a more formal argument which is due to Arrow and Pratt

Measures of Risk Aversion

Absolute Risk Aversion

- To the lowest order of x and π, we have:

$$\pi \approx \tfrac{1}{2}\left[-\frac{u''(w)}{u'(w)}\right]\mathbb{V}[x]. \tag{7}$$

- Thus, for small risks the risk premium π is proportional to the size of risk, as measured by its variance.
- The proportionality coefficient captures the agent's risk aversion:

$$A(w) \equiv -\frac{u''(w)}{u'(w)}. \tag{8}$$

  $A(w)$ is called the Arrow-Pratt measure of risk aversion. It depends not only on the utility function, but also on the wealth level.

- Since $A(w)$ is associated with the risk premium per unit of absolute risk, it is also called the absolute risk aversion (measures curvature of utility)
- The inverse of risk aversion is often referred to as risk tolerance $T(w)$:

$$T(w) \equiv \frac{1}{A(w)} = -\frac{u'(w)}{u''(w)}.$$

Measures of Risk Aversion

Relative Risk Aversion

- $A(w)$ is related to absolute risk and does not take into account the significance of the risk relative to the agent's total wealth.
- We now consider a risk x proportional to the wealth level and its risk premium π_R as fraction of total wealth:

$$\mathbb{E}[u(w(1+x))] = u(w(1-\pi_R)). \tag{9}$$

- For small risks (the support of wx is small), we have

$$\pi_R(w) = \tfrac{1}{2}\left[-\frac{wu''(w)}{u'(w)}\right]\mathbb{V}[x]. \tag{10}$$

- We define an agent's relative risk aversion, $R(w)$, by:

$$R(w) \equiv -\frac{wu''(w)}{u'(w)}. \tag{11}$$

- For a (small) risk as a fraction of an agent's wealth, the corresponding risk premium is proportional to the size of the risk, as a fraction of her wealth, and the proportionality coefficient is her relative risk aversion.

Measures of Risk Aversion

- Measures of risk aversion defined above are all for small risks.
- It is harder to define measures of risk aversion beyond small risks.
- The utility function is said to exhibit:
  - constant absolute risk aversion, CARA, if $A'(w) = 0$,
  - increasing or decreasing absolute risk aversion, IARA or DARA, if $A'(w) > 0$ or $A'(w) < 0$, respectively.
  - constant relative risk aversion, CRRA, if $R'(w) = 0$,
  - increasing or decreasing relative risk aversion, IRRA or DRRA, if $R'(w) > 0$ or $R'(w) < 0$, respectively.

Measures of Risk Aversion

Examples

We now consider a few examples of utility function and risk/aversion.

1. Linear utility function:

   $$u(w) = w.$$

   The risk aversion is:

   $$A(w) = R(w) = 0.$$

   The agent is risk neutral and demands no premium for taking risks.

2. Negative exponential utility function:

   $$u(w) = -e^{-aw}, \quad a > 0.$$

   The risk aversion is:

   $$A(w) = a, \quad R(w) = aw.$$

   Negative exponential utility function exhibits constant absolute risk aversion, or CARA. For CARA utility function, the relative risk aversion increases linearly with wealth.

Measures of Risk Aversion

Examples

3. Quadratic utility function:

$$u(w) = w - \tfrac{1}{2}aw^2, \quad a > 0, \quad w \in [0,\, 1/a].$$

The range for w guarantees $u'(w) > 0$. The risk aversion is:

$$A(w) = \frac{a}{1 - aw}, \quad R(w) = \frac{aw}{1 - aw}.$$

Here, $A(w)$ increases with w.

4. Logarithmic utility function:

$$u(w) = \log w.$$

The risk aversion is:

$$A(w) = \frac{1}{w}, \quad R(w) = 1.$$

The log utility function has some unique properties we explore later.

Measures of Risk Aversion

Examples

5. Power utility function:

$$u(w) = \frac{1}{1-\gamma} w^{1-\gamma}, \quad \gamma > 1.$$

The risk aversion is

$$A(w) = \frac{\gamma}{w}, \quad R(w) = \gamma.$$

Thus, power utility function has decreasing absolute risk aversion (DARA) but constant relative risk aversion, i.e., CRRA.

For $\gamma = 0$, we have the risk-neutral case. For $\gamma \to 1$, we have the log utility case.

Measures of Risk Aversion

Examples

6. Hyperbolic absolute risk aversion, HARA, utility function:

$$u(w) = a + b\left(d + \frac{w}{\gamma}\right)^{1-\gamma},$$

where a and b are free parameters.

$$A(w) = \frac{1}{d + w/\gamma} \quad \text{or} \quad T(w) = \frac{1}{A(w)} = d + \frac{w}{\gamma} \quad \text{or} \quad R(w) = \frac{w}{d + w/\gamma},$$

i.e., linear risk tolerance. This class of utility functions is directly defined by their risk aversion. It contains the cases consider earlier:

- Risk neutral: $d = \infty$.
- Quadratic: $\gamma = -1$.
- Negative exponential: $\gamma \to \infty$ and $d = 1/a$.
- Log: $d = 0$ and $\gamma = 1$.
- Power: $d = 0$ and $\gamma < 1$.

Comparing Risk Aversion

We now consider how measures of risk aversion can help us compare the attitude towards risk between two agents.

Let $u_1(\cdot)$ and $u_2(\cdot)$ denote the utility functions of two agents, 1 and 2, and $A_1(w)$ and $A_2(w)$ their absolute risk aversion, respectively.

Theorem (Pratt)

The following statements are equivalent:

1. $A_1(w) \geq A_2(w) \ \ \forall \ w$,
2. $u_1\left(u_2^{-1}(\cdot)\right)$ *is concave,*
3. $\exists \ f(\cdot)$ *with* $f'(\cdot) > 0$ *and* $f''(\cdot) \leq 0$ *such that* $u_1(w) = f(u_2(w))$,
4. $\pi_1 \geq \pi_2$ *for all* w *and fair gambles.*