Lecture 3. Arbitrage Pricing

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Outline

Replication

Following the basic framework, we start our analysis with the general properties of the securities market.

- Let D (payoff matrix) denote the market structure with N securities.
- In general, the payoffs of traded securities need not be linearly independent.
- A security is redundant if its payoff can be replicated by a portfolio of other traded securities.
  - Let D_n be the payoff vector of security n.
  - Let $D_{\setminus n}$ denote the payoff matrix excluding D_n:

$$D_{\setminus n} = [D_1, \ldots, D_{n-1}, D_{n+1}, \ldots, D_N].$$

  - Let $\theta_{\setminus n}$ denote a portfolio of securities excluding n.
  - If $\exists\ \theta_{\setminus n}$ such that $D_{\setminus n}\theta_{\setminus n} = D_n$, then security n is redundant.
- In a frictionless market, redundant securities do not influence equilibrium.

Payoff Space and Spanning

- Going forward, D includes only securities with linearly independent payoffs.
- Consequently, $rank(D) = N \leq M$.
- Given D, the set $C_1(D) = \{c_1 = D\,\theta : \theta \in R^n\}$ is called the payoff space.
- C_1 is a linear subspace of dimension N.
- The set of securities in D is said to span $C_1(D)$.
- Any payoff vector in $c_1 \in C_1(D)$ can be delivered by a portfolio θ.
- In this case, the payoff c_1 is said to be replicated or financed by θ.

Market Completeness

Definition

Definition (Market Completeness)

A securities market is complete if any payoff at $t = 1$ can be achieved from the payoff of a portfolio of traded securities.

- D defines the market structure, and θ denotes a portfolio.
- Let c_1 be an arbitrary payoff for $t = 1$.
- We have $c_1 \in R^M$ and $\theta \in R^N$.
- Market completeness: $\forall\ c_1 \in R^M$, $\exists\ \theta \in R^N$ such that $D\,\theta = c_1$.
- Clearly, if $N < M$, the market cannot be complete.

Market Completeness

Sufficient Condition for Market Completeness

Theorem (Sufficient Condition for Market Completeness)

A securities market, defined by D, is complete if and only if the rank of D equals the number of states at $t = 1$: $rank(D) = M$.

State-Contingent Claims

The security/portfolio that pays 1 in only one of the time-1 states has the simplest payoff and plays a fundamental role in our analysis.

Definition (State-Contingent Claims)

A state-ω contingent claim pays 1 in state ω and nothing otherwise.

The state-ω contingent clam has the following payoff tree:

$$
\begin{array}{l}
0, \quad m = 1 \\
\vdots \\
0, \quad m = \omega - 1 \\
1, \quad m = \omega \\
0, \quad m = \omega + 1 \\
\vdots \\
0, \quad m = M
\end{array}
$$

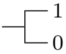
Arrow-Debreu Securities and Arrow-Debreu Market

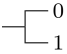
- A state-contingent claim is also called an Arrow-Debreu security.
- A securities market with a complete set of state-contingent/Arrow-Debreu securities is called the Arrow-Debreu market/economy.
- The market structure of an Arrow-Debreu market is given by

$$D^{A-D} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} = I$$

  where I denotes an identity matrix, and $N = M$.

Example. For the binomial tree case, we have two Arrow-Debreu securities:

State-1 A-D security 

State-2 A-D security 

Arrow-Debreu Market and Market Completeness

Theorem (Completeness of the Arrow-Debreu Market)

The Arrow-Debreu securities market is complete.

State Prices

- Let ϕ_ω denote the time-0 price of the state-ω contingent claim (when traded).
- The price ϕ_ω is also called the state price as it corresponds to the price at $t = 0$ for a payoff of 1 in state-ω at $t = 1$.
- Let

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_\omega \\ \vdots \\ \phi_M \end{bmatrix}$$

denote the vector of all state prices, the state-price vector.

Arbitrage

- Consider a securities market with market structure D and price vector P.
- A trading strategy is given by portfolio θ at $t = 0$.

Definition (Arbitrage)

An arbitrage is a trade in the securities market such that

1. $P^{\top}\theta \leq 0$,
2. $D\,\theta \geq 0$, and
3. at least one of the above inequalities is strict.

- An arbitrage relies only on prices and payoffs, but not probabilities.
- An arbitrage, if it exists, is available to everyone.
- An arbitrage is scalable in frictionless markets.

Arbitrage
Example

Example. Suppose two traded securities have the following prices/payoffs:

$$A:\quad 1 \; \begin{cases} 1 \\ 1 \end{cases} \qquad\qquad B:\quad 1 \; \begin{cases} 2 \\ 1 \end{cases}$$

Here, we have $P = [1;\ 1]$ and $D = [1,2;1,1]$.

Consider the following portfolio: $\theta = [-1;\ 1]$. It yields payoff:

$$-P^\top\theta = -(-1+1) = 0 \; \begin{cases} (D\theta)_1 = -1+2 = 1 \\ (D\theta)_2 = -1+1 = 0 \end{cases}$$

This is an arbitrage: The return on A dominates that of B.

Arbitrage
Example

Example. Suppose three traded securities have the following prices/payoffs:

$$\text{A:}\quad 1 \left\langle \begin{array}{l} 1 \\ 1 \\ 1 \end{array} \right. \qquad \text{B:}\quad 1 \left\langle \begin{array}{l} 0 \\ 2 \\ 2 \end{array} \right. \qquad \text{C:}\quad 2 \left\langle \begin{array}{l} 2 \\ 0 \\ 0 \end{array} \right.$$

Thus,

$$P = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

Consider the following trade: $\theta_1 = [2; -1; -1]$. Its cashflow is:

$$-(2-1-2) = 1 \left\langle \begin{array}{l} 2-0-2=0 \\ 2-2-0=0 \\ 2-2-0=0 \end{array} \right.$$

This is an arbitrage: positive payoff at $t = 0$ and no future liabilities.

Arbitrage

Example

Example. (Continued) Still,

$$P = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

Consider the following trade: $\theta_2 = [2; 0; -1]$. Its cashflow is:

$$-(2-0-2) = 0 \quad \begin{cases} 2-0-2=0 \\ 2-0-0=2 \\ 2-0-0=2 \end{cases}$$

This is an arbitrage: zero cost at $t = 0$ and positive payoff at $t = 1$ (strict in state 2 and 3).

Principle of No Arbitrage (NA)

Principle (No Arbitrage Principle)

There is no arbitrage in a (frictionless) securities market.

Definition (Monotonic Preferences)

Agent k's preferences are strictly monotonic if for $c > c'$, $u_k(c) > u_k(c')$.

Theorem (No Arbitrage in Equilibrium)

In a market equilibrium, there are no arbitrage opportunities if there is at least one agent whose preference is strictly monotonic.

A simple example of an arbitrage strategy

- Suppose that you will be receiving a large paycheck in Euros next period and using it to buy a large asset in the US next period, but don't want to expose yourself to fluctuations in exchange rates

- What exchange rate can you lock in today? 2 options in financial markets
  1. Buy a forward contract in financial markets: agree to convert Euros to Dollars at a pre-specified price F
  2. Follow a cash-and-carry strategy
     - ★ Borrow Euros today at interest rate $r_{€}$
     - ★ Convert those euros to dollars at the current exchange rate S
     - ★ Invest those dollars at the US interest rate $r_{\$}$

  Under no arbitrage, these strategies should have the same price, which leads to the **covered interest parity** formula

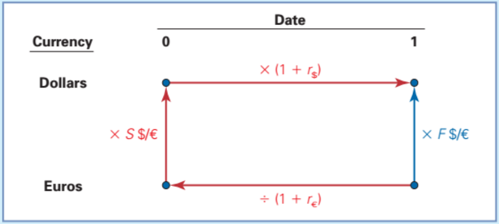
$$\underbrace{F}_{\frac{\$\text{ in one year}}{€\text{ in one year}}} = \underbrace{S}_{\frac{\$\text{ today}}{€\text{ today}}} \times \underbrace{\frac{1+r_{\$}}{1+r_{€}}}_{\frac{\$\text{ in one year}/\$\text{ today}}{€\text{ in one year}/€\text{ today}}}$$

The CIP strategy graphically

FIGURE 30.5

Currency Timeline Showing Forward Contract and Cash-and-Carry Strategy

The cash-and-carry strategy (three transactions in red) replicates the forward contract (in blue) by borrowing in one currency, converting to the other currency at the spot exchange rate, and investing in the new currency.



Source: Berk and Demarzo

CIP in the data: pre and post financial crisis

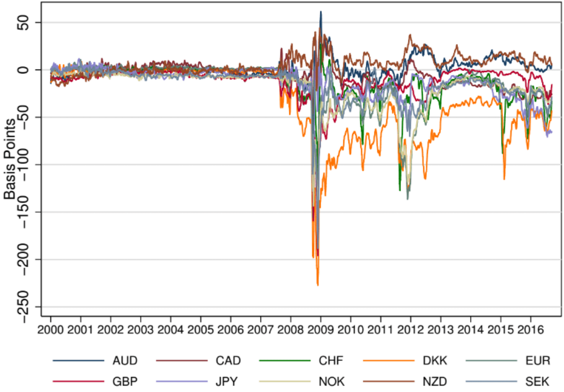


Figure 2. Short-term Libor-based deviations from covered interest rate parity. This figure plots the 10-day moving averages of the three-month Libor cross-currency basis, measured in bps for G10 currencies. Covered interest rate parity implies that the basis should be zero. The Libor basis is equal to $y^{\$,Libor}_{t,t+n} - (y^{Libor}_{t,t+n} - \rho_{t,t+n})$, where n = three months, $y^{\$,Libor}_{t,t+n}$ and $y^{Libor}_{t,t+n}$ denote the U.S. and foreign three-month Libor rates, and $\rho_{t,t+n} \equiv \frac{1}{n}(f_{t,t+n} - s_t)$ denotes the forward premium obtained from the forward $f_{t,t+n}$ and spot s_t exchange rates.

Source: Du, Tepper, and Verdelhan, "Deviations from Covered Interest Rate Parity", Journal of Finance, 2017.

CIP in the data: pre and post financial crisis

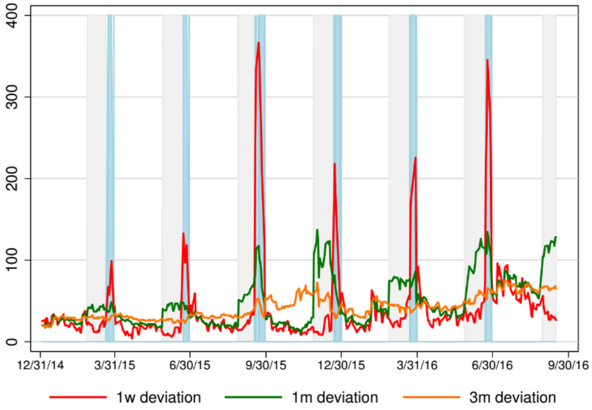


Figure 5. Illustration of quarter-end dynamics of CIP deviations. The blue shaded area denotes the dates for which the settlement and maturity of a one-week contract spans two quarters. The grey shaded area denotes the dates for which the settlement and maturity dates of a one-month contract spans two quarters but excludes the dates in the blue shaded area. The figure plots the one-week, one-month, and three-month Libor CIP deviations for the yen (in absolute values) in red, green, and orange, respectively.

Source: Du, Tepper, and Verdelhan, "Deviations from Covered Interest Rate Parity", Journal of Finance, 2017.

Reality check: Limits to arbitrage

- In its purest form, arbitrage requires no capital and is risk free.
  - By selling and purchasing identical securities at favorably different prices, arbitrageur captures an immediate payoff with no up-front capital.
  - No short sale constraints $\Rightarrow$ infinite profits!
- Pure arbitrage exists only in perfect capital markets.
  - Requires capital and posting collateral in practice.
  - People are worried about others keeping promises $\Rightarrow$ shorting is costly
- Why might we observe CIP violations in the data? In practice, arbitrage is associated with opportunity costs
  - Regulations imply that only financial institutions can close the CIP arbitrage
  - Making trades to close it increases regulatory capital requirements (these costs are really high at quarter ends)

Reality check: Limits to arbitrage

- **Imperfect information** and **market frictions** make arbitrage strategies both **capital intensive** and **risky**.
- Arbitrageurs often use leverage to invest more than their own money.
  - Face possibility that price gaps will become larger before they converge
  - Remember: even though most financial assets have limited liability (non-negative payoffs), losses from short-selling can be infinite
  - If strategy has lost money, may receive a **margin call**: post additional collateral or close the position at a loss
- Even with active arbitrageurs, opportunities may persist while arbitrageurs learn how to best exploit them. Length of time is unknown ex-ante
- Mitchell, Pulvino & Stafford (2002) consider returns from a particular set of arbitrage strategies with realistic leverage and costs
  - Out of 82 cases studied, 30% (!) never feature convergence of this spread.

Hypothetical trading strategy: Mitchell, Pulvino, & Stafford

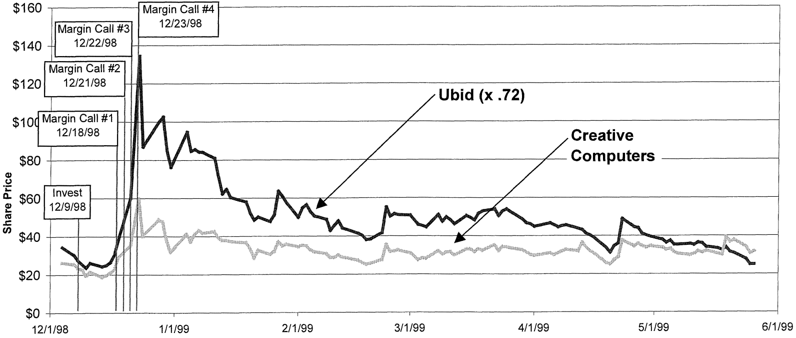


Figure 1. Paths of stock prices for Creative Computers and Ubid.

- "To avoid the costly margin calls, the arbitrageur would have had to post \$4.53 of excess cash for every \$1 of long position."
- Return with transaction costs/margin limits: 9.5%
- Return ignoring frictions: 45.9%

Asset Pricing Model/Operator

- An asset pricing model is a mapping from a security's payoff vector d to its price P:

$$P = V(d).$$

- $V(\cdot)$ is called the pricing/valuation operator/functional.

The NA principle imposes general properties on the pricing operator $V(\cdot)$.

Asset Pricing Properties Under No Arbitrage

Theorem (Positivity)

A portfolio with a positive payoff must have a positive price:

$$V(d) > 0 \quad \textit{if} \quad d > 0.$$

(Also, $V(d) = 0$ if $d = 0$.)

Theorem (Law of One Price)

Two portfolios with same payoffs must have the same price:

$$V(d_1) = V(d_2) \quad \textit{if} \quad d_1 = d_2.$$

Examples/Counterexamples

- Freeway lanes / driving routes
- "Happy meal theorem"

"Lawsuit Claims McDonald's Value Meals Are No Value", Time (12/20/2016)
He claims it would have been cheaper to order items separately
One man is questioning the value of McDonalds combo deals in court.

James Gertie of Des Plaines, Ill., has filed a consumer fraud class action against a Chicago-area McDonald's franchisee's pricing of value meals, the Cook County Register reported. The suit says that its pricing of the "Extra Value Meal"–which includes two cheeseburgers, fries and a drink–is 41 cents more expensive than if customers ordered the items separately.

The defendant, Karis Management Company, owns more than 10 McDonald's restaurants near Chicago. In the complaint, Gertie says he purchased an "Extra Value Meal" from at least five of Karis' restaurants between Oct. 14 and Nov. 13.

Each time, he was charged \$5.90 for the meal. However, the suit claims that if he purchased them individually, he would have paid \$2.50 for the two burgers, \$1.99 for the fries and \$1 for the drink–or about \$0.41 less than the value deal.

Examples/Counterexamples

- An old joke:

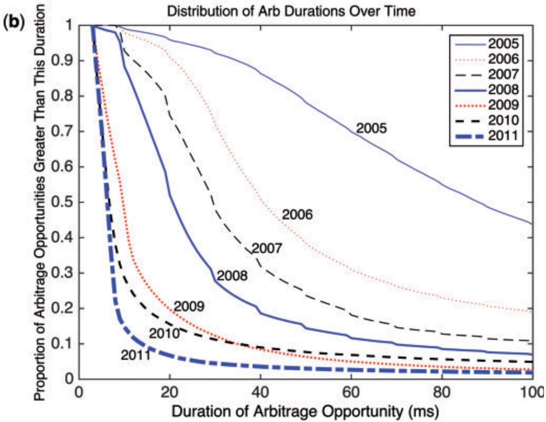
  A finance professor and a student are walking down a street. The student notices a $100 bill lying on the pavement and leans down to pick it up. The finance professor immediately intervenes and says, "Don't bother; there is no free lunch. If that were a real $100 bill lying there, somebody would already have picked it up!"

  - Have you ever actually seen a $100 bill lying on the sidewalk?
  - Free $100 bills, like arbitrage opportunities, are extremely rare:
    1. Since $100 is a large amount of money, people are careful not to lose it
    2. If someone *does* drop $100, prob you find it before someone else ≈ 0

- Real world example of this is ETF-futures arbitrage (Budish, Cramton, and Shim, 2017):
  - High-frequency traders buy an exchange-traded fund (ETF; a basket of securities traded on an exchange) or buy a futures contract (a promise to receive the index shortly in the future)
  - Strategy buys the cheaper of the two and sells the more expensive one

- Main point: agents will trade to quickly eliminate these opportunities

Some recent data on duration of arbitrage opportunities



"Competition has not affected the size or frequency of the arbitrage opportunities, it has only raised the bar for how fast one has to be to capture them"

Source: Budish, Cramton, and Shim, "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response", *Quarterly Journal of Economics*, 2017.

Asset Pricing Properties

Theorem (Monotonicity)

Portfolios with higher payoffs must have higher prices:

$$V(d_1) \geq V(d_2) \quad \textit{if} \;\; d_1 \geq d_2.$$

Thus, $V(\cdot)$ is an increasing operator.

Theorem (Linearity)

In a frictionless market, the pricing operator is linear:

$$V(a\, d_1 + b\, d_2) = a\, V(d_1) + b\, V(d_2) \quad \forall\; a, b \in R.$$

The Fundamental Theorem of Asset Pricing (FTAP)

Theorem (The Fundamental Theorem of Asset Pricing)

There is no arbitrage in the securities market if and only if there exists $\phi \gg 0$ such that for all traded securities:

$$P^\top = \phi^\top D. \tag{1}$$

We call ϕ the state price vector implied from D and P.

Proof. We'll discuss this next week!