Lecture 12. Portfolio choice

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Outline

Optimal Consumption/Portfolio Choice

Introduction

We now study an agent's optimal consumption/savings/portfolio choice.

- The securities market consists of N non-redundant securities/assets, with payoff matrix D and price vector P.
- An agent has endowment $e = [e_0; e_1]$ and expected utility function of the form: $u_0(c_0) + \mathbb{E}[u_1(c_1)]$, $u_t'(\cdot) > 0$, $u_t''(\cdot) < 0$ for $t = 0, 1$.
- Let $c = [c_0; c_1]$ be the agent's consumption plan and θ her portfolio holding.

An agent's optimization problem can be stated as:

$$\begin{aligned} \max_{\theta} \quad & u_0(c_0) + \mathbb{E}[u_1(c_1)] \\ \text{s.t.} \quad & c_0 = e_0 - P^\top \theta \\ & c_1 = e_1 + D\,\theta \end{aligned} \tag{1}$$

Here, $e_0 - c_0 = P^\top \theta$ is the agent's time-0 savings. We have omitted potential restrictions such as $c \geq 0$, which can be easily included (usually don't bind because standard utility functions satisfy the Inada condition $\lim_{c \downarrow 0} u_t'(c) = \infty$)

Existence of Optimal Portfolio

We first establish the existence of a solution to (1).

Theorem (Existence)

The optimization problem (1) has a solution iff there is no arbitrage in the financial market, defined by $\{D, P\}$.

- The intuition for necessity is intuitive. If there were arbitrage opportunities in the market, the agent can achieve unbounded consumption levels. With insatiability, there will be no optimum for any finite portfolio.
- Sufficiency is also intuitive. In the absence of arbitrage, any consumption (at any time and in any state) has a positive cost. With limited resources, given by her finite endowment, the agent's budget set is compact. The agent should be able to achieve optimum as allowed by the market, within the compact budget set.

Optimal Portfolio in a Complete Market

When the securities/asset market is complete, the optimal portfolio choice problem has a simple solution. This is an important benchmark case, as we will see later. Hence, we consider it here.

- For simplicity, assume the complete set of A-D securities are traded.
- Let ϕ denote the state price vector ($\phi \gg 0$).
- Let $e = [e_0; e_1]$ denote an agent's endowment.
- The agent's total wealth is $w = e_0 + \phi^\top e_1$, and her budget set is $B(e) = \{c : c_0 + \phi^\top c_1 = w\}$.
- The optimal consumption/portfolio problem then becomes:

$$\max_{c_0 + \phi^\top c_1 = w} u_0(c_0) + \sum_\omega p_\omega u_1(c_{1\omega}). \tag{2}$$

  Here, we have omitted the possible constraint $c \geq 0$ for simplicity. They can be easily added if needed.

Optimal Portfolio in a Complete Market

- Letting λ be the Lagrange multiplier from budget constraint, FOC is:

$$u_0'(c_0) = \lambda \qquad \leftarrow \lambda \text{ captures the marginal value of wealth},$$
$$p_\omega u_1'(c_{1\omega}) = \lambda \phi_\omega, \quad \forall\, \omega \in \Omega,$$

  - $p_\omega u_1'(c_{1\omega}) = \frac{\partial E[u_1(c_1)]}{\partial \theta_\omega}$ captures marginal benefit of increasing $c_{1\omega}$.
  - Marginal cost is ϕ_ω. Additional \$1 invested in asset ω increases $c_{1\omega}$ by $\frac{1}{\phi_\omega}$

- FOC equates marginal benefit of additional \$1 of θ_ω across assets $\Rightarrow$

$$\frac{p_\omega u_1'(c_{1\omega})}{u_0'(c_0)} = \phi_\omega, \quad \underbrace{\frac{u_1'(c_{1\omega})}{u_0'(c_0)}}_{\substack{\text{intertemporal}\\ \text{marginal rate}\\ \text{of substitution}}} = \underbrace{\frac{\phi_\omega}{p_\omega}}_{\substack{\text{state price}\\ \text{density}\\ \eta_\omega}}, \quad \frac{p_\omega u_1'(c_{1\omega})}{p_{\omega'} u_1'(c_{1\omega'})} = \frac{\phi_\omega}{\phi_{\omega'}} \quad \forall\, \omega, \omega' \in \Omega.$$

- Thus, at the optimum, relative marginal utilities for consumption in different dates/states are equal to their relative prices:

$$\frac{u_1'(c_{1\omega})}{u_1'(c_{1\omega'})} = \frac{\phi_\omega / p_\omega}{\phi_{\omega'} / p_{\omega'}} = \frac{\eta_\omega}{\eta_{\omega'}}$$

Optimal Portfolio in a Complete Market

- When $u_t(\cdot)$, $t = 0, 1$, is strictly concave, we have:
  - $u_t'(\cdot)$ is strictly monotonic, and
  - $u_t'^{-1}(\cdot)$ exists.

- Solving the FOC, we have:

  $$c_0 = u_0'^{-1}(\lambda), \quad c_{1\omega} = u_1'^{-1}(\lambda\phi_\omega/p_\omega), \quad \forall\, \omega \in \Omega$$

  where λ is determined by the budget constraint:

  $$c_0 + \phi^\top c_1 = e_0 + \phi^\top e_1 = w.$$

  The solution gives the optimal consumption/portfolio choice.

Theorem

In a complete market where agents have insatiable and strictly concave expected utility, $\forall\ k$ and $\omega, \omega' \in \Omega$, $c_{1\omega} < c_{1\omega'}$ iff $\phi_\omega/p_\omega > \phi_{\omega'}/p_{\omega'}$.

Thus, at the optimum, levels of consumption in different states are ranked inversely by the state price density. Intuition: high $\eta_\omega \Rightarrow$ LOW $c_{1\omega}$

Our first GE model: the Lucas (1978) thought experiment

- Consider a thought experiment proposed by Robert Lucas in his seminal paper "Asset prices in an exchange economy", Econometrica, 1978
- Suppose all agents have identical preferences and endowments
- Markets are complete: households can write contracts with one another to shift resources freely over time/states
- Market clearing condition: aggregate consumption = aggregate endowment
- **Question**: what are equilibrium state prices? What is the risk-free rate?
- **Answer**: Take first order condition + impose market clearing condition:

$$\frac{u_1'(c_{1\omega})}{u_0'(c_0)} = \frac{u_1'(e_{1\omega})}{u_0'(e_0)} = \frac{\phi_\omega}{p_\omega} = \eta_\omega$$

Denominator is known, so all random variation in η_ω depends on $u'(c_{1\omega})$ which is decreasing in $c_{1\omega}$. "Pain" (and marginal utility of consumption / state price) is high when $c_{1\omega} = e_{1\omega}$ is low

Our first GE model: the Lucas (1978) thought experiment

- Suppose endowment is constant $e_{1\omega} = e_1$ and $u_1(c) = \delta u_0(c) = \delta u(c)$
- Then, SPD is constant $\eta_\omega = \dfrac{\delta u'(c_{1\omega})}{u'(c_0)} = \frac{1}{1+r_F}$
- Remember FOC: high future marginal utility $\Rightarrow$ low future consumption
- Constant aggregate consumption $\Rightarrow 1 + r_F = \frac{1}{\delta}$
- $1 + r_F > \frac{1}{\delta}$ if $c_1 > c_0$ because people would prefer to smooth consumption (move resources from to the present). Prices adjust until markets clear
- Without uncertainty, real interest rates are low (relative to $\frac{1}{\delta}$) when resources are more abundant today and high when growth is high

Our first GE model: the Lucas (1978) thought experiment

- Suppose that investors have CRRA preferences: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$
  - $u'(c) = c^{-\gamma}$
  - $\frac{d}{dc} u'(c) = u''(c) = -\gamma c^{-\gamma-1}$
  - $\frac{d^2}{dc^2} u'(c) = u'''(c) = \gamma(1+\gamma)c^{-\gamma-2} > 0 \Rightarrow$ marginal utility $u'(c)$ is convex
- Add uncertainty: Suppose instead that $e_{1\omega} = \bar{c}_1 + \epsilon_\omega$, where $E[\epsilon] = 0$.
- Convex marginal utility + Jensen's inequality $\Rightarrow$

$$E[\eta_\omega] = \frac{1}{1+r_F} = E\left[\frac{\delta u'(c_{1\omega})}{u'(c_0)}\right] \geq \frac{\delta u'(E[c_{1\omega}])}{u'(c_0)} = \frac{\delta u'(\bar{c}_1)}{u'(c_0)}$$

- The possibility of high marginal utility states in the future makes the agent want to save more. This pushes the risk-free bond price upwards and the risk-free rate downwards
- This is called the precautionary savings effect

Characterization of Optimal Portfolio

- Substituting the budget constraint into the utility function, (1) becomes:

$$\max_{\theta} \; u_0(e_0 - P^\top \theta) + \mathbb{E}[u_1(e_1 + D\,\theta)].$$

  Here, e_1 denotes the endowment at $t = 1$ and $D\,\theta$ the portfolio payoff, both as random variables.

- The first order condition (FOC) is:

$$u_0'(c_0)P_n = \mathbb{E}[u_1'(c_1)D_n], \quad n = 1, \ldots, N. \tag{3}$$

- (3) is also called the Euler equation.
  - The LHS of the FOC is the marginal utility from current consumption loss by paying P_n to invest in one unit of asset n.
  - The RHS is the marginal utility from future consumption gain by receiving payoff D_n from the investment in asset n.
  - The FOC simply states that at the optimum, the two must equal.

Characterization of Optimal Portfolio

- Another way to write the Euler equation is:

$$1 = \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)}\frac{D_n}{P_n}\right] = \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)}R_n\right], \quad n = 1, \ldots, N. \tag{4}$$

  Thus, the marginal utility from investing in the traded assets relative to the marginal utility from consuming today are all equal to 1.

- The FOC does not guarantee optimality. Optimality is obtained if the following second order condition is also met:

$$u_0''(c_0)P_n^2 + \mathbb{E}\big[u_1''(c_1)D_n^2\big] \leq 0, \quad n = 1, \ldots, N. \tag{5}$$

- (5) always holds when $u_0(\cdot)$ and $u_1(\cdot)$ are concave.

- Thus, going forward we only focus on FOC, which is both necessary and sufficient for optimality for a concave expected utility function.

Back to Lucas (1978)

- In the Lucas experiment, agents have complete markets and symmetric CRRA preferences. Therefore, each agent's consumption stream satisfies

$$1 = \mathbb{E}\left[\delta\left(\frac{c_1^k}{c_0^k}\right)^{-\gamma}\frac{D_n}{P_n}\right] = \mathbb{E}\left[\delta\left(\frac{c_1^k}{c_0^k}\right)^{-\gamma}R_n\right], \quad n = 1,\ldots,N, k = 1,\ldots,K.$$

- What is implied by the model?
  - Allow for heterogeneous endowments $\Rightarrow$ heterogenous levels of wealth
  - Complete markets $\Rightarrow$ all choose consumption rate per dollar of wealth
  - Everyone's consumption will grow at same rate as aggregate endowment, which equals growth rate of aggregate consumption
  - SDF $\eta = \delta\left(\frac{c_1^k}{c_0^k}\right)^{-\gamma} = \delta\left(\frac{\sum_k c_1^k}{\sum_k c_0^k}\right)^{-\gamma} = \delta\left(\frac{e_1}{e_0}\right)^{-\gamma} = \delta\left(\frac{c_1}{c_0}\right)^{-\gamma}$
  - Tests often work with a first order approximation of the SDF $\approx a + -\gamma\frac{c_1}{c_0}$ $\Rightarrow$ Risk premium $\pi_n \approx -\gamma \times Cov[R_n, \frac{c_1}{c_0}]$

Back to Lucas (1978)

- Euler equation is:

$$1 = \mathbb{E}\left[\delta\left(\frac{c_1^k}{c_0^k}\right)^{-\gamma}\frac{D_n}{P_n}\right] = \mathbb{E}\left[\delta\left(\frac{c_1^k}{c_0^k}\right)^{-\gamma} R_n\right], \ \ n = 1,\ldots,N, k = 1,\ldots,K.$$

- We can test the Lucas model w/ data on aggregate consumption. Call this the Consumption-CAPM
  - What happens? Can't match equity premium without astronomically high risk aversion: equity premium puzzle
  - Further note: we did not need to make assumptions about where endowments come from. Same FOC if e^k comes from a production model!
- Even if markets incomplete, Euler equation should hold at the micro level for traded assets like stocks and bonds.
  - Are people's consumption choices consistent with predictions of the model?
  - Euler equation above does not hold with equality when agents face borrowing constraints. More empirical success when focusing on unconstrained agents

Empirical evidence on the Consumption-CAPM

- First round of tests on Consumption-CAPM had little success, tended to use aggregate consumption
- More recently, some researchers have found more success:
  - Accounting for limited participation helps high net worth households' consumption more volatile and more correlated with stock returns, works better as a pricing factor: Vissing-Jorgensen (2002); Brav, Constantinides, and Geczy (2002)
  - + Measuring long-run changes in consumption works better:Parker and Julliard (2005); Hansen, Heaton, and Li (2008); Malloy, Moskowitz, and Vissing-Jorgensen (2009)
- More examples, taking seriously idea that consumption is hard to measure:
  - Use labor/entrepreneurial income: Jagannathan and Wang (1996); Heaton and Lucas (2000); Campbell, Delikouras, Jian, and Korniotis (2016)
  - Luxury goods (Ait-Sahalia, Parker, and Yogo, 2004): e.g., expensive wine, cars, art, etc, which are proxies for the spending of the very wealthy
  - Garbage (Savov, 2011) used the weight of garbage going into landfills as a proxy for consumption.
- Common themes: not everyone participates, measuring consumption is hard, and wealthy people play a disproportionately important role

Empirical evidence on the Consumption-CAPM

Panel A: Mean returns versus consumption covariances

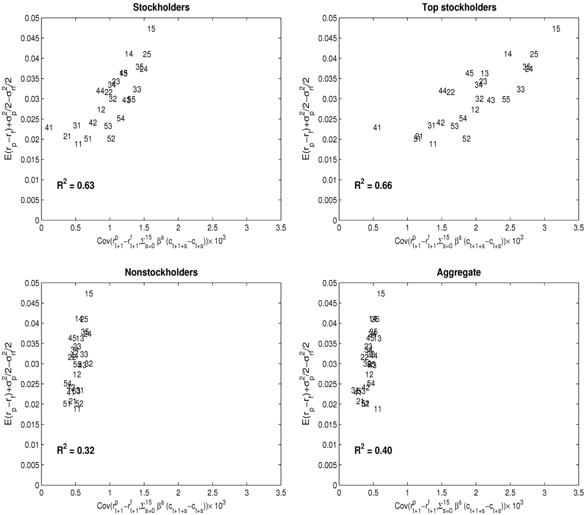

Figure 1 A from Malloy, Moskowitz, and Vissing-Jorgensen (2009), "Long-Run Stockholder Consumption Risk and Asset Returns", Journal of Finance

Decomposing the Portfolio Choice Problem

- We now provide a useful way of decomposing the portfolio problem in a general asset market
- Denote the value of portfolio an agent chooses by

$$w \equiv e_0 - c_0 = P^\top \theta.$$

  Clearly, w is the agent's savings from time 0 for investment.
- The optimal consumption/portfolio choice can be expressed in two parts:

$$\max_w \left\{ u_0(e_0 - w) + \max_{\{\theta : P^\top \theta = w\}} \mathbb{E}\left[u_1(e_1 + D\,\theta)\right] \right\}$$
$$= \max_w \; u_0(c_0 - w) + v_1(w)$$

  where

$$v_1(w) \equiv \max_{\{\theta : P^\top \theta = w\}} \mathbb{E}\left[u_1(e_1 + D\,\theta)\right]$$

  defines the portfolio choice problem given the total amount to invest w.

Decomposing the Portfolio Choice Problem

- To simplify, let $e_1 = 0$. Thus, the agent is endowed only with e_0 (cash).
- The portfolio choice problem then reduces to:

$$v(w) \equiv \max_{\{\theta : P^\top \theta = w\}} \mathbb{E}\left[u(D\,\theta)\right] \tag{6}$$

where the time index 1 is omitted for brevity.

- Assume that asset N is riskless, yielding a gross return of $R_N = 1 + r_F$.
- Let $a_n = \theta_n P_n$ denote the (dollar) amount invested in asset n.
- Then, $w = \sum_n a_n$ is the total initial investment.

- The resulting payoff from the portfolio is:

$$\tilde{w} \equiv D\,\theta = \sum_{n=1}^{N} a_n R_n = w(1 + r_F) + \sum_{n=1}^{N-1} a_n (r_n - r_F).$$

Here, $r_n - r_F$ gives the excess return of asset n.

General Problem

- Let $r \equiv [r_1, \ldots, r_{N-1}]$ denote the (row) vector of returns on the risky assets.
- Let $a = [a_1; \ldots; a_{N-1}]$ the (column) vector of investments in risky assets.
- The optimal portfolio choice problem now becomes

$$\max_{a} \mathbb{E}[u(\tilde{w})] = \mathbb{E}\left[u\left(w(1+r_F) + (r - r_F \iota^\top)a\right)\right]. \tag{7}$$

- The FOC for (7) with respect to a is

$$\mathbb{E}[u'(\tilde{w})(r_n - r_F)] = 0, \quad n = 1, \ldots, N-1. \tag{8}$$

Solution to these $N-1$ equations gives a ($N-1$ elements) as function of w.

Properties of Optimal Portfolio

One Risky Asset

Theorem

If the agent is strictly risk averse, then

- $a = 0$ *iff* $\bar{r} = r_F$
- $a > 0$ *iff* $\bar{r} > r_F$
- $a < 0$ *iff* $\bar{r} < r_F$.