14.121, Fall 2018
Problem Set 5 Solutions

1) MWG 11.B.2

(a) The Pareto problem is:

$$\max_{h,T} \phi_1(h) + w_1 - T$$

$$\text{s.t. } \phi_2(h, w_2 + T) \geq \bar{u_2}$$

Let λ be the Lagrange multiplier on the constraint and assuming an interior solution, the Karush-Kuhn-Tucker first order conditions are:

$$\phi_1'(h) + \lambda \frac{\partial \phi_2(h, w_2 + T)}{\partial h} = 0$$

$$-1 + \lambda \frac{\partial \phi_2(h, w_2 + T)}{\partial w} = 0$$

Together, these conditions yield

$$\phi_1^{'}(h^0) = -\frac{\frac{\partial \phi_2(h^0, w_2+T^0)}{\partial h}}{\frac{\partial \phi_2(h^0, w_2+T^0)}{\partial w}}$$

.

Note that since the objective function is concave and the constraint set is convex, the necessary FOC is also sufficient.

(b) Consumer 2's problem is:

$$\max_{h} \phi_2(h, w_2 - p_h \cdot h)$$

Letting $h(p_h, w_2)$ denote his demand for h, the FOC satisfies:

$$\frac{\partial \phi_2(h(p_h, w_2), w_2 - p_h h(p_h, w_2))}{\partial h} - p_h \frac{\partial \phi_2(h(p_h, w_2), w_2 - p_h h(p_h, w_2))}{\partial w} = 0$$

Differentiating this with respect to w_2 we get:

$$\frac{\partial^2 \phi_2}{\partial h^2} \frac{\partial h(p_h, w_2)}{\partial w_2} + \frac{\partial^2 \phi_2}{\partial w_2 \partial h} \cdot (1 - p_h \frac{\partial h(p_h, w_2)}{\partial w_2}) - p_h \frac{\partial^2 \phi_2}{\partial h \partial w_2} \frac{\partial h(p_h, w_2)}{\partial w_2} - p_h \frac{\partial^2 \phi_2}{\partial w_2^2} \cdot (1 - p_h \frac{\partial h(p_h, w_2)}{\partial w_2}) = 0$$

We will assume that ϕ_2 is twice continuously differentiable, so that it's cross partials are equal. Thus, the above expression simplifes to:

$$\frac{\partial h(p_h, w_2)}{\partial w_2} = \frac{p_h \frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial^2 \phi_2}{\partial h \partial w_2}}{\frac{\partial^2 \phi_2}{\partial h^2} - 2p_h \frac{\partial^2 \phi_2}{\partial h \partial w_2} + p_h^2 \frac{\partial^2 \phi_2}{\partial w_2^2}}$$

From the FOC we have that $p_h = \frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w_2}}$ and substituting this into the above expression will express the wealth effect in the required terms.

(c) From part (a) above we that $\phi_1'(h^0) = -\frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w_2}}$. This first order condition can be equivalently expressed as

$$\frac{\partial \phi_1}{\partial h} + \frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w}} = 0$$

and using the implicit function theorem, we have:

$$\frac{\partial h^0}{\partial w_2} = \frac{\frac{\partial \phi_2}{\partial h}\frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial \phi_2}{\partial w_2}\frac{\partial^2 \phi_2}{\partial h \partial w_2}}{(\frac{\partial \phi_2}{\partial w_2})^2 \frac{\partial}{\partial h}(\frac{\partial \phi_1}{\partial h} + \frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w}})} = \frac{\frac{\partial \phi_2}{\partial h}\frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial \phi_2}{\partial w_2}\frac{\partial^2 \phi_2}{\partial h \partial w_2}}{(\frac{\partial \phi_2}{\partial w_2})^2 \phi_1''(h^0) + \frac{\partial \phi_2}{\partial w_2}\frac{\partial^2 \phi_2}{\partial h^2} - \frac{\partial \phi_2}{\partial h}\frac{\partial^2 \phi_2}{\partial h \partial w_2}} \tag{1}$$

We need to show that equation 1 is positive (for a positive externality). We first observe that since we are at a global maximum in part a, $\frac{\partial}{\partial h}(\frac{\partial \phi_1}{\partial h} + \frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w}}) < 0$. Thus, we need to show that the numerator in 1 is negative.

At $\overline{p}_h$ and $\overline{w}_2$, we are assuming that consumer 2's demand for the externality derived in part b is normal. Note that the denominator of the expression derived in part b for $\frac{\partial h}{\partial w_2}$ is just the second order condition for the consumer's optimization problem, which is therefore negative. These two facts imply that the numerator of $\frac{\partial h}{\partial w_2}$ is negative as well, i.e.,

$$\begin{aligned} 0 &> \overline{p}_h \frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial^2 \phi_2}{\partial h \partial w_2} \\ &= \frac{\frac{\partial \phi_2}{\partial h}}{\frac{\partial \phi_2}{\partial w_2}} \frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial^2 \phi_2}{\partial h \partial w_2} \\ &\Rightarrow \quad 0 > \frac{\partial \phi_2}{\partial h}\frac{\partial^2 \phi_2}{\partial w_2^2} - \frac{\partial \phi_2}{\partial w_2}\frac{\partial^2 \phi_2}{\partial h \partial w_2} \end{aligned}$$

where the implication above is true due to the assumption that $\frac{\partial \phi_2}{\partial w_2} > 0$. But this is exactly what we needed to show. Hence, $\frac{\partial h^0}{\partial w_2} > 0$.

2) MWG 11.B.5

(a) The firm solves

$$\max_{q,h} p \cdot q - c(q,h)$$

and the FOCs are:

$$p \leq \frac{\partial c(q^*, h^*)}{\partial q}, \text{ with equality if } q^* > 0$$

$$0 \leq \frac{\partial c(q^*, h^*)}{\partial h}, \text{ with equality if } h^* > 0$$

(b) Since the consumer's utility function is quasilinear with respect to money, the utility possibility frontier is a linear with slope -1 and we can therefore find the Pareto optimal level of h and q by maximizing the sum of the profit function and the utility function without wealth, i.e.,

$$\max_{q,h} p \cdot q - c(q,h) + \phi(h)$$

which yields the FOCs,

$$p \leq \frac{\partial c(q^0, h^0)}{\partial q}, \text{ with equality if } q^0 > 0$$

$$\phi'(h^0) \leq \frac{\partial c(q^0, h^0)}{\partial h}, \text{ with equality if } h^0 > 0$$

(c) Let t denote the tax rate on output. Then the firm solves:

$$\max_{q,h} p \cdot q - c(q,h) - t \cdot q$$

which yields the FOCs,

$$p \leq \frac{\partial c(q,h)}{\partial q} + t, \text{ with equality if } q > 0$$
$$0 \leq \frac{\partial c(q,h)}{\partial h}, \text{ with equality if } h > 0$$

As we will show in part (d) it is possible to restore efficiency by taxing output in a special case. But, it is clear by the above first order conditions that in general the output tax will result in a level of $h \neq h^0$.

If, however, a tax is imposed on h, say τ, the firm solves:

$$\max_{q,h} p \cdot q - c(q,h) - \tau \cdot h$$

which yields the FOCs,

$$p \leq \frac{\partial c(q,h)}{\partial q}, \text{ with equality if } q > 0$$
$$-\tau \leq \frac{\partial c(q,h)}{\partial h}, \text{ with equality if } h > 0$$

and if we set $\tau = -\phi'(h^0)$ then this gives the level $h = h^0$ as in part (b) above and efficiency is restored.

(d) Let t denote the tax rate on output. The firm solves:

$$\max_{q} p \cdot q - c(q, \alpha q) - t \cdot q$$

which yields the FOC:

$$p \leq \frac{\partial c(q,\alpha q)}{\partial q} + \alpha \frac{\partial c(q,\alpha q)}{\partial h} + t \text{ with equality if } q > 0$$

and if we set $t = -\alpha\phi'(h^0)$ then the pair (q^0, h^0) will solve this FOC as in part (b) above and efficiency is restored.

3) MWG 11.D.5

(a) This is a model of free entry so fishermen will send out boats as long as there are positive profits from doing so. Therefore, the equilibrium number of boats, b^*, will be reached when $p\frac{f(b^*)}{b^*} = r$. This condition is that the average revenue equals the average cost. (We ignore integer problems, but if we are to given the integer equilibrium number then it is b^* such that $p\frac{f(b^*)}{b^*} - r \geq 0$ and $p\frac{f(b^*+1)}{b^*+1} - r < 0$.)

(b) To characterize the optimal number of boats we must solve for the maximum total surplus, i.e.,

$$\max_{b} pf(b) - rb$$

which yields the FOC:

$$pf'(b) - r \leq 0$$

which is necessary and sufficient since the SOC, $pf''(b) < 0$ is satisfied. Therefore, the condition for the optimal number of boats is $f'(b^0) = \frac{r}{p}$ (assuming interior solution), i.e., that the marginal revenue equals the marginal cost. Assuming that $f(0) = 0$ ensures that $b^0 \leq b^*$.

(c) To restore efficiency we need the equilibrium condition satisfied at b^0, i.e., we need the tax level to satisfy $\frac{f(b^0)}{b^0} = \frac{r+t}{p}$ or $t = p\frac{f(b^0)}{b^0} - r$.

(d) Clearly, if owned by a single individual, the problem to be solved is exactly that solved in part (b) above, which results in b^0.

4) MWG 11.E.1

(a) The optimal quantity h is determined by solving

$$\max_h E_\eta[\phi(h,\eta)] + E_\theta[\pi(h,\theta)]$$

which yields the FOC

$$E_\eta[\frac{\partial\phi(\hat{h},\eta)}{\partial h}] + E_\theta[\frac{\partial\pi(\hat{h},\theta)}{\partial h}] \leq 0$$

and substituting the functional form we have $\gamma - c\hat{h} + E[\eta] + \beta - b\hat{h} + E[\theta] \leq 0$, from which we solve: $\hat{h} \geq \frac{\gamma+\beta}{c+b}$, with equality for $\hat{h} > 0$. Thus, $\hat{h} = \frac{\gamma+\beta}{c+b}$ if $\gamma + \beta > 0$, otherwise, $\hat{h} = 0$.

(b) Given a tax t, the firm will maximize profits and will choose h according to the FOC $\frac{\partial\pi(h,\theta)}{\partial h} - t = 0$, which yields the firm's reaction function to a tax t, given θ, as $h(t,\theta) = \frac{\theta+\beta-t}{b}$. The optimal tax t^* will be given by the solution to:

$$\max_t E[\phi(h(t,\theta),\eta)] + E[\pi(h(t,\theta),\theta)]$$

which yields the FOC

$$E[\frac{\partial\phi(h(t,\theta),\eta)}{\partial h}\frac{\partial h(t,\theta)}{\partial t}] + E[\frac{\partial\pi(h(t,\theta),\theta)}{\partial h}\frac{\partial h(t,\theta)}{\partial t}] = 0$$

Since $\frac{\partial h(t,\theta)}{\partial t} = -\frac{1}{b}$ is a constant, we can cancel out the $\frac{\partial h(t,\theta)}{\partial t}$ from the FOC and substituting the functional form of our functions the FOC becomes:

$$\gamma - c\frac{E[\theta]+\beta-t}{b} + E[\eta] + \beta - b\frac{E[\theta]+\beta-t}{b} + E[\theta] = 0$$

from which we solve $t^* = \frac{\beta c-\gamma b}{c+b}$.

(c) The choices of $\hat{h}$ and t^* are depicted in figure 1. The intersection of the expected marginal profits and marginal utility curves solves for $\hat{h}$ and t^* as shown. Consider a realization of θ and η that gives rise to curves intersecting at the point x as shown. The optimal level of the externality is therefore h^*. If we use quantity regulation $\hat{h}$, the deadweight loss is the triangle xuv, while if we use tax regulation t^*, the firm will choose $h(t^*,\theta)$ and the loss is the triangle xyz. We need to compare the expected difference in losses to determine which policy is better.

Before we proceed, we will introduce a nonstandard way of calculating the area of a triangle. Consider the triangle in figure 2. The area is normally calculated using the formula $A = \frac{1}{2}ed$. Note that we can write $d = \frac{e_1}{b}$, where b is the slope of the top edge of the triangle. We can also write $e_1 = \frac{b}{b+c}e$, which we can then substitute into the above to get: $A = \frac{1}{2}\frac{e^2}{b+c}$.

Now, to calculate the area of the triange xuv, it is the edge uv squared, divided by twice the sum of the slopes of both marginal curves. The height of the edge uv is

$$\frac{\partial\pi(\hat{h},\theta)}{\partial h} - (-\frac{\partial\phi(\hat{h},\eta)}{\partial h}) = (\beta - b\frac{\gamma+\beta}{c+b} + \theta) + (\gamma - c\frac{\gamma+\beta}{c+b} + \eta) = \theta + \eta$$

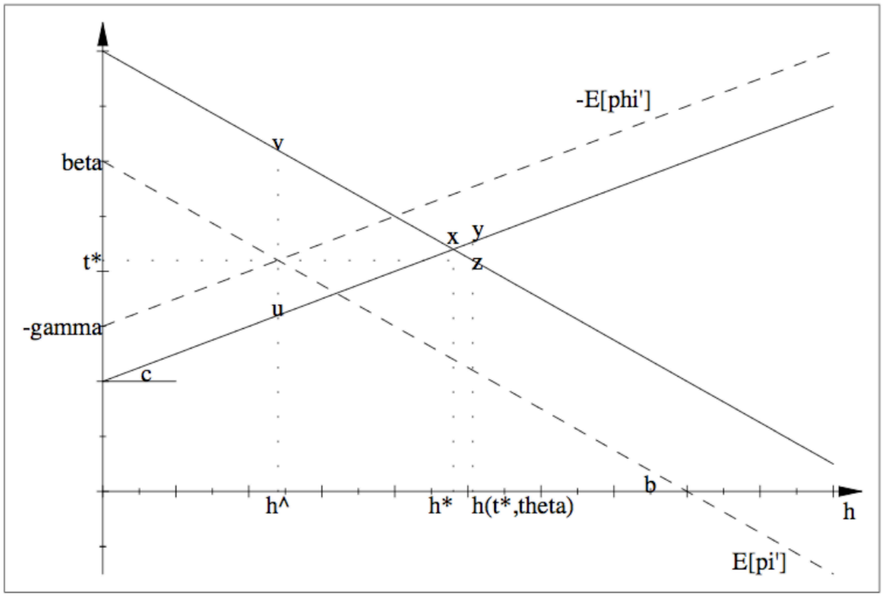

Figure 1

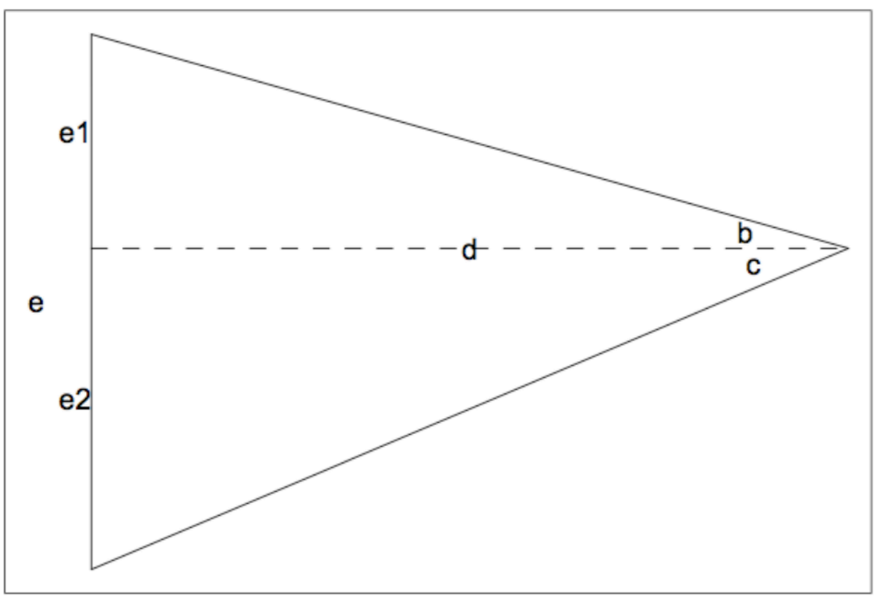

Figure 2

so that the loss from the quantity regulation is

$$L_h = \frac{(\theta+\eta)^2}{2(b+c)}.$$

To calculate the are of the loss from taxation, we need the height of the edge yz, which is calculated by

$$\begin{aligned}-\frac{\partial\phi(h(t^*,\theta),\eta)}{\partial h} - \frac{\partial\pi(h(t^*,\theta),\theta)}{\partial h} &= -(\gamma - c\frac{\theta+\beta-t^*}{b} + \eta) - (\beta - b\frac{\theta+\beta-t^*}{b} + \theta) \\ &= -\gamma + \frac{c}{b}(\theta+\beta-t^*) - \eta - t^* \\ &= c(h(t^*,\theta) - \hat{h}) - \eta\end{aligned}$$

where the last equality follows because we can rewrite $t^* = c\hat{h} - \gamma$ (follows by simple algebra). We need to find $h(t^*,\theta) - \hat{h}$, and again using simple algebra we get that $h(t^*,\theta) - \hat{h} = \frac{\theta}{b}$, which gives us the loss from taxation,

$$L_t = \frac{(\frac{\theta c}{b} - \eta)^2}{2(b+c)}.$$

We can now calculate the expected difference in losses,

$$E[L_h - L_t] = \frac{1}{2(b+c)}E[(\theta+\eta)^2 - (\frac{\theta c}{b} - \eta)^2] = \frac{\sigma_\theta^2(b-c)}{2b^2}$$

which implies that the optimal choice between quantity and tax regulation depends on the sign of $(b-c)$. When this term is positive, the economy is more sensitive to changes in the firm, and therefore taxation is better since it changes the level of the externality depending on the firm's realized marginal profits. The reverse is true when the term is negative.

Remark: There is actually another simple approach by applying Weitzman (1974) directly. To map this question to the notation of Weitzman, we define $B(q,\eta) = \phi(h,\eta)$, and $C(q,\theta) = -\pi(h,\theta)$. We have

$$\begin{aligned}B'' &= \frac{\partial}{\partial h}(r - c\hat{h} + \eta) = -c \\ C'' &= -\frac{\partial}{\partial h}(\beta - b\hat{h} + \theta) = b\end{aligned}$$

Applying Weitzman's result, when $|C''| > |B''|$, that is $b > c$, tax is preferred.

(Note: Weitzman obtained this result using a second-order approximation of welfare loss around the optimal quantity. In our case, however, because all the first derivatives are linear, the approximation is exact).

5) MWG 11.E.2

The optimal quantities $(\hat{h_1}, \hat{h_2})$ are determined by solving

$$\max_{h_1,h_2} E_\eta[\phi(h_1+h_2,\eta)] + E_{\theta_1}[\pi(h_1,\theta_1)] + E_{\theta_2}[\pi(h_2,\theta_2)]$$

which yields the FOCs:

$$E_\eta(\frac{\partial\phi(h_1+h_2,\eta)}{\partial h}) + E_{\theta_1}(\frac{\partial\pi(h_1,\theta_1)}{\partial h_1}) \le 0$$

$$E_\eta(\frac{\partial\phi(h_1+h_2,\eta)}{\partial h}) + E_{\theta_2}(\frac{\partial\pi(h_2,\theta_2)}{\partial h_2}) \leq 0$$

Substituting the functional forms we get $\hat{h_1} = \hat{h_2} \geq \frac{\gamma+\beta}{2c+b}$ with equality for $\hat{h_i} > 0$.

Similarly, given a pair of taxes (t_1, t_2), each firm will maximize profits and will choose h_i according to the FOC $\frac{\partial\pi_i(h_i,\theta_i)}{\partial h_i} = t_i$, which yields each firm's reaction function to a tax t_i, given θ_i, as $h_i(t_i, \theta_i) = \frac{\theta_i+\beta-t_i}{b}$. The optimal taxes t_i^* will be given by

$$\max_{t_1,t_2} E[\phi(h_1(t_1,\theta_1) + h_2(t_2,\theta_2),\eta)] + E[\pi_1(h_1(t_1,\theta_1),\theta_1)] + E[\pi_2(h_2(t_2,\theta_2),\theta_2)]$$

This can be calculated in a similar way as in exercise 11.E.1. The first-order conditions are

$$\mathbb{E}\left[\frac{\partial\phi(h_1+h_2,\eta)}{\partial h}\frac{\partial h_1(t_1,\theta)}{\partial t_1}\right] + \mathbb{E}\left[\frac{\partial\pi_1(h_1,\theta)}{\partial h_1}\frac{\partial h_1(t_1,\theta)}{\partial t_1}\right] = 0 \tag{t_1}$$

$$\mathbb{E}\left[\frac{\partial\phi(h_1+h_2,\eta)}{\partial h}\frac{\partial h_2(t_2,\theta)}{\partial t_2}\right] + \mathbb{E}\left[\frac{\partial\pi_2(h_2,\theta)}{\partial h_2}\frac{\partial h_2(t_2,\theta)}{\partial t_2}\right] = 0 \tag{t_2}$$

which yields

$$\gamma - c\mathbb{E}\left[h_1(t_1,\theta_1) + h_2(t_2,\theta_2)\right] + t_1 = 0$$
$$\gamma - c\mathbb{E}\left[h_1(t_1,\theta_1) + h_2(t_2,\theta_2)\right] + t_2 = 0$$

which, after substituting in for h_1 and h_2 and simplifying, implies that

$$t_1 = t_2 = \frac{2c\beta - b\gamma}{2c+b}$$

When comparing between the two instruments, things become a little more cumbersome and for an analysis, see Section V in Weitzman (1974). It is worthwhile to convey the intuition for the effects of σ_{12}. If the marginal profits of each firm are perfectly correlated, then it is as if we have one producer and the analysis of the previous exercise follows through. As σ_{12} falls (they become less correlated) then it is more likely that taxes, which yield negatively correlated output decisions when the shocks are negatively correlated, will be more efficient than quantity quotas. Taxes allow for the individual levels of externalities generated by each firm to be responsive to the realized values of the $\theta_j's$, as opposed to the case with quotas.