Lecture 2. A Basic Framework

Lawrence Schmidt

MIT Sloan School of Management

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Outline

Brief overview: Realized returns in the US

- Where we were: what were returns historically for different financial assets?

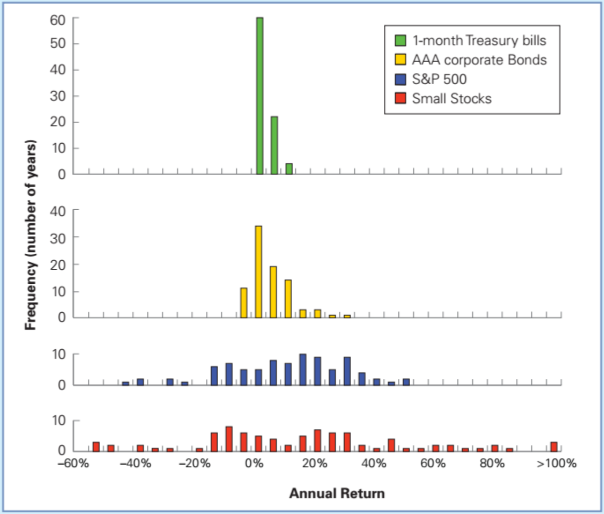
- Standard summary statistics to consider:
  - Average (arithmetic) returns $= \frac{1}{T}\sum_{t=1}^{T} R_t$, average return over one period
  - Average (geometric) returns / compound growth rates $= \left[\prod_{t=1}^{T} R_t\right]^{\frac{1}{T}} = \exp\left(\frac{1}{T}\sum_{t=1}^{T} \log R_t\right)$, per-period return associated with buying and holding the asset over the entire sample period
  - Sample variance: $\frac{1}{T-1}\sum_{t=1}^{T}\left[R_t - \frac{1}{T}\sum_{j=1}^{T} R_j\right]^2$
  - Standard deviation / **Volatility**: square root of sample variance
  - Higher moments: skewness, kurtosis
  - Nonparametric methods: e.g., summarize data with a histogram

One year return distributions for some major asset classes

FIGURE 10.5

The Empirical Distribution of Annual Returns for U.S. Large Stocks (S&P 500), Small Stocks, Corporate Bonds, and Treasury Bills, 1926–2011.

The height of each bar represents the number of years that the annual returns were in each 5% range. Note the greater variability of stock returns (especially small stocks) compared to the returns of corporate bonds or Treasury bills.



Source: Berk and Demarzo, 2017.

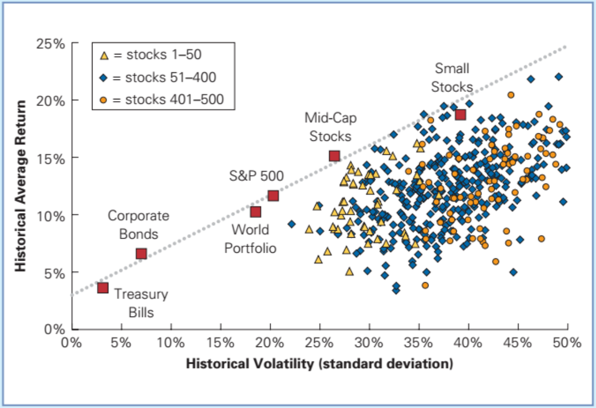
Means and standard deviations: portfolios and individual stocks

FIGURE 10.7

Historical Volatility and Return for 500 Individual Stocks, Ranked Annually by Size

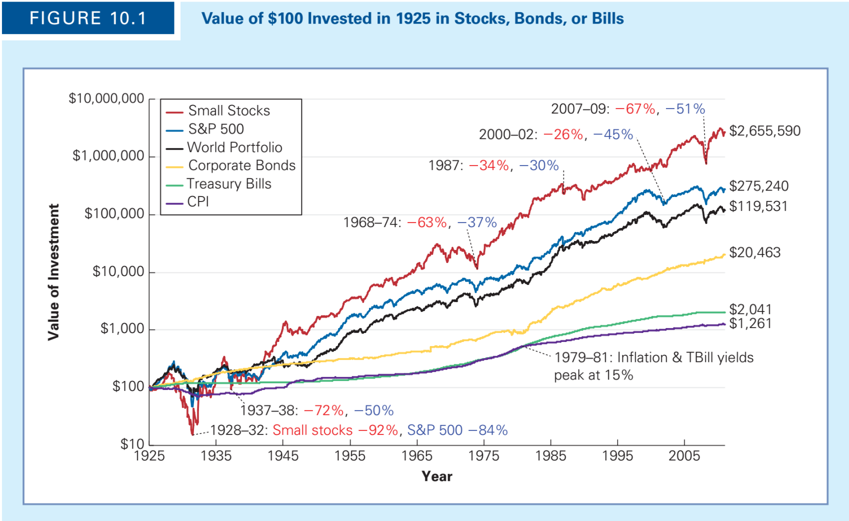
Unlike the case for large portfolios, there is no precise relationship between volatility and average return for individual stocks. Individual stocks have higher volatility and lower average returns than the relationship shown for large portfolios. (Annual data from 1926–2011.)

Source: CRSP



Source: Berk and Demarzo, 2017.

Cumulative returns across asset classes



Note: from these graphs, can get cumulative log return over any holding period by compare height of lines at any two points

Inflation makes future nominal payments risky

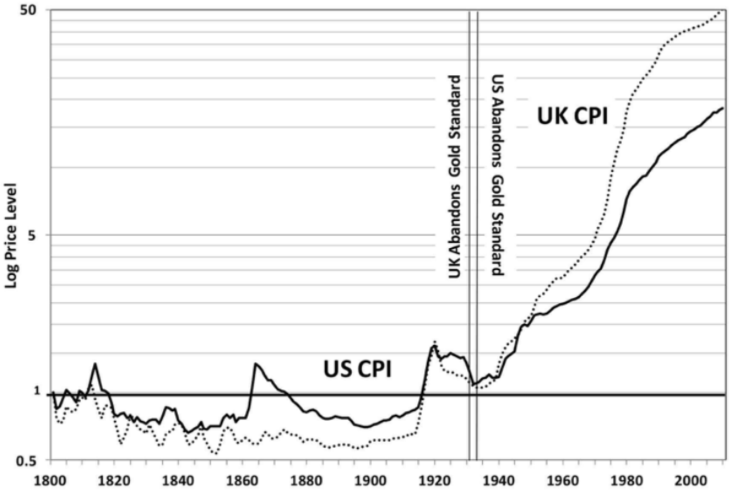


FIGURE 5-3
U.S. and U.K. Consumer Price Index 1800–2012

Source: Siegel, "Stocks for the Long Run"

Real returns across asset classes

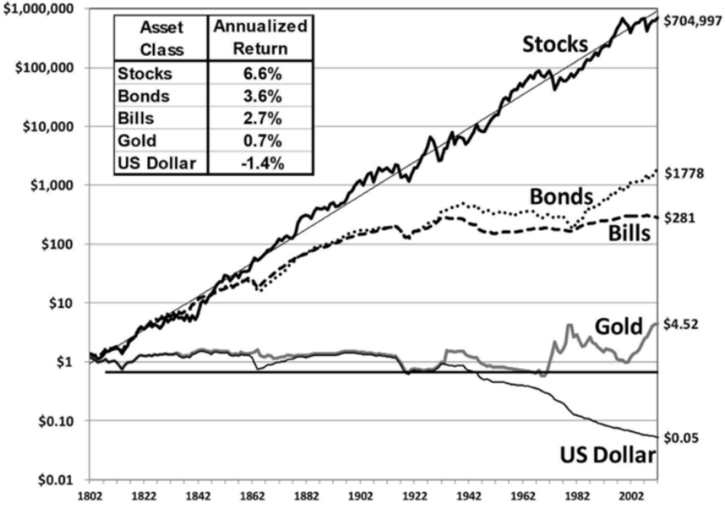
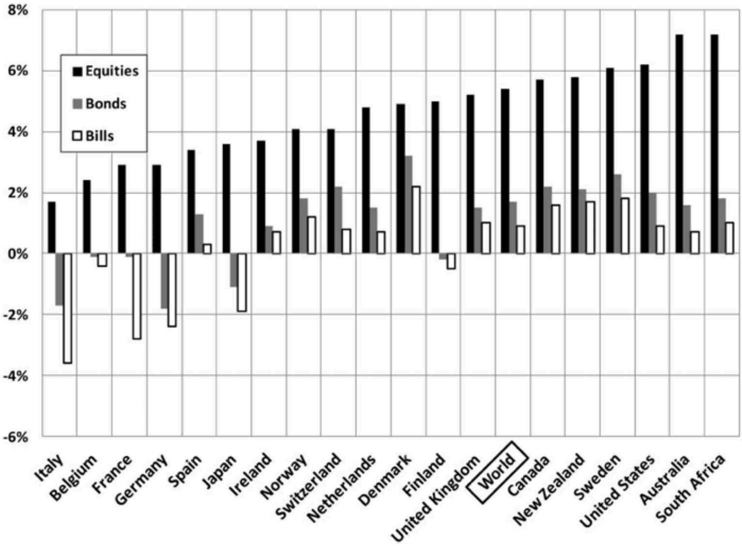


FIGURE 5-4

Total Real Returns on U.S. Stocks, Bonds, Bills, Gold, and the Dollar, 1802–2012

Source: Siegel, "Stocks for the Long Run"

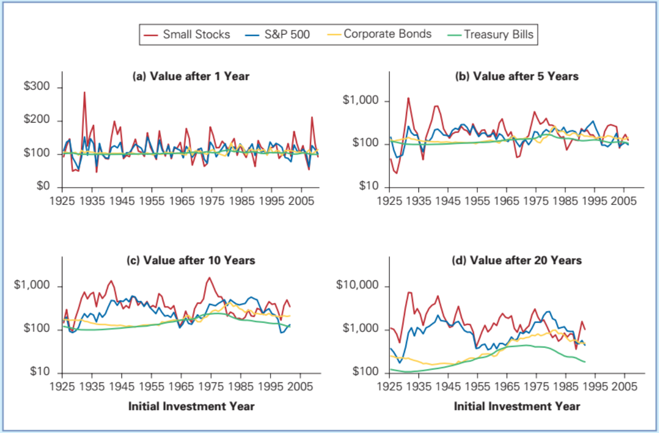
Rankings are fairly stable across countries



Source: Siegel, "Stocks for the Long Run"

Relative riskiness depends on the horizon

FIGURE 10.2 **Value of $100 Invested in Alternative Assets for Differing Horizons**



Each panel shows the result of investing $100 at the end of the initial investment year, in each investment opportunity, for horizons of 1, 5, 10, or 20 years. That is, each point on the plot is the result of an investment over the specified horizon, plotted as a function of the initial investment date. Dividends and interest are reinvested and transaction costs are excluded. Note that small stocks show the greatest variation in performance at the one-year horizon, followed by large stocks and then corporate bonds. For longer horizons, the relative performance of stocks improved, but they remained riskier.

Source Data: Chicago Center for Research in Security Prices, Standard and Poor's, MSCI, and Global Financial Data.

Relative riskiness depends on the horizon

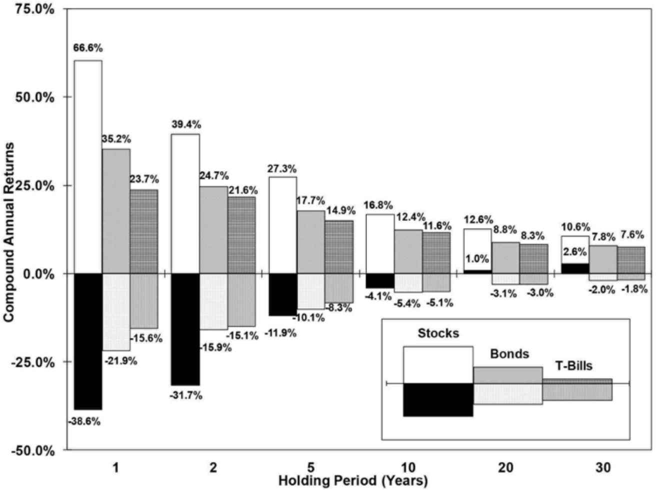


FIGURE 6-1
Highest and Lowest Real Returns on Stocks, Bonds, and Bills over 1-, 2-, 5-, 10-, 20-, and 30-Year Holding Periods 1802–2012

Source: Siegel, "Stocks for the Long Run"

What's next?

We start with a parsimonious framework, aimed at capturing the basic economic function of financial transactions/activities – resource allocation.

- Start simple and gradually build up.
- Key components:
  - Financial environment
  - Agents
  - Financial/securities market
  - Market equilibrium
  - Efficiency
- While we will emphasize theory throughout most of this course, financial markets provide a wealth of data which constantly challenge our models qualitatively and (especially) quantitatively

Environment

- Two key elements of finance/resource allocation: time and risk.
- Consider two dates, $t = 0, 1$.
- There are M possible states at $t = 1$, denoted by: $\omega_1, \dots, \omega_M$. The set $\Omega = \{\omega_1, \dots, \omega_M\}$ is called the state space.
- There is a probability measure $\mathbb{P}$ over the state space:
  - The probability for state ω_m is $p_m > 0$, $m = 1, \dots, M$, and

$$\sum_{m=1}^{M} p_m = 1.$$

- The state-space model can also be described by the following "event-tree":

$$\{\Omega\} \longrightarrow \begin{cases} \omega_1, \text{ with probability } p_1 \\ \vdots \\ \omega_M, \text{ with probability } p_M \end{cases}$$

Environment (Alternative)

- Most of what we discuss will hold (even w/ the same notation) in alternative environments with multiple periods
- More generally, m can be an index for different combinations of future states and time periods
- Useful example: no uncertainty about the future, m indexes different time periods in the future
- Only substantive change will be that, in this case, probabilities may not sum to 1 (you may be surprised how infrequently probabilities appear)
- Interpretation: agents come to the market at time 0, trade to choose state- and time-contingent consumption bundles

Agents

We assume that the economy is populated by K agents, $k = 1, \ldots, K$.

Each agent can be defined by the following economic characteristics:

- Resources:
  - Information
  - Endowment
  - Production technology
- Choices: consumption (+ resource allocation)
- Preferences

We describe them sequentially.

Agents
Information

- In addition to $\mathbb{P}$, agents may receive information at $t = 0$ about the states at $t = 1$.
- Information can be public and private.
- For now, we assume that agents receive no information other than $\mathbb{P}$.
- All agents use $\mathbb{P}$ as their prior assessment of the likelihood of time-1 states.
    - They have rational expectations and homogenous beliefs
    - Note: this is actually stronger than what is needed
- We will later consider the situations where agents may:
    - Use priors different from $\mathbb{P}$, and/or
    - Receive public and/or private information about Ω.

Agents
Endowment

- There is only one perishable good/type of capital in the economy.
- Each agent k is endowed with the following endowment:
  - e_0^k at $t = 0$
  - $e_{1\omega}^k$ at $t = 1$ in state ω, $\omega \in \Omega$.
- Notation:
  - $[\cdot, \ldots, \cdot]$ denotes a row vector, and
  - $[\cdot; \ldots; \cdot]$ denotes a column vector.
- The endowment of agent k can then be by the following vector:

$$e^k \equiv [e_0^k; e_1^k] \equiv [e_0^k; [e_{11}^k; \ldots; e_{1M}^k]], \quad k = 1, \ldots, K.$$

- Useful special case: standard portfolio problem $\equiv e_0^k > 0$, $e_1^k = 0$.

Agents

Endowment

- Let $\mathbb{R}^{1+M}$ denote an $(1+M)$-dimensional real space and $\mathbb{R}_+^{1+M}$ its non-negative quadrant.
- Non-negative endowment means $e \in \mathbb{R}_+^{1+M}$.
- For a vector $a = [a_1; \ldots; a_n]$, define the following notation:

$$\begin{aligned} a \geq 0 &\ \text{ if } a_i \geq 0 \text{ for all } i, \\ a > 0 &\ \text{ if } a_i \geq 0 \text{ for all } i \text{ and } a_i > 0 \text{ for at least one } i, \\ a \gg 0 &\ \text{ if } a_i > 0 \text{ for all } i. \end{aligned}$$

- Then, $e \geq 0$ represents $\mathbb{R}_+^{1+M}$ and $e \gg 0$ represents the interior of $\mathbb{R}_+^{1+M}$.

Agents
Production

- Each agent may possess a set of production technologies.
- A production technology turns an investment at $t = 0$ into output at $t = 1$.
- We can define a production technology by a production function:

$$y_1(I) = [f_1(I); \ldots ; f_M(I)]$$

or

$$-I \quad \begin{cases} f_1(I) \text{ for } \omega_1 \\ \vdots \\ f_M(I) \text{ for } \omega_M \end{cases}$$

- We will assume that:

$$f_\omega(0) = 0, \quad f'_\omega(I) \geq 0, \quad f''_\omega(I) \leq 0, \quad \forall\, \omega \in \Omega.$$

The condition on concavity implies decreasing return to scale.

Agents

Consumption

- Agents use their economic resources to best meet their economic needs.
- An agent's economic need is defined over her consumption: c_0 at $t = 0$ and $c_{1\omega}$ at $t = 1$ in state ω, $\omega \in \Omega$, or

$$c^k \equiv [c_0^k; c_1^k] \equiv [c_0^k; [c_{11}^k; \ldots; c_{1M}^k]], \quad k = 1, \ldots, K.$$

- A possible consumption choice, $c = [c_0; c_1]$, is called a consumption plan.
- A realization of a consumption plan, $(c_0,\ c_{1\omega})$ is called a consumption path.
- C, the consumption set, denotes the set of all possible consumption plans.
- C is a subset of $\mathbb{R}^{1+M}$. Standard choice of C is $\mathbb{R}_+^{1+M}$.

Agents

Contrast with "standard" application

- First application of consumer theory: different arguments of C are associated with different goods: e.g., good 1 is pizza and good 2 is beer
- Here, "good" is roughly the same $\rightarrow$ always talking about consumption
- Elements of c^k correspond with agent k's state-contingent consumption. Two reasons why states may differ:
  - Preferences may differ across states
  - Endowments may differ: different amounts of resources available to consume
  - Models will tend to emphasize the latter channel
- More general environment introduced above: consumption could be state and time-dependent

Agents

Consumption

Definition (Convex Set)

A set in R^n is convex if $\forall$ a and b in the set and $\alpha \in [0,1]$, $\alpha a + (1-\alpha)b$ is also in the set.

Definition (Closed Set)

A set in R^n is closed if a sequence of elements in the set, $a_i,\ i = 1, 2, \ldots$, has a as the limit, then a is also in the set.

Assumption (Consumption Set)

Consumption set C is a closed, convex subset of $\mathbb{R}^{1+M}$.

Agents

Preferences

An agent's economic need is described by her preferences, defined by her ranking of all possible consumption plans.

Definition (Preferences)

A preference is a set of binary relations $\succsim$ on C such that:

1. $\succsim$ is complete: $\forall\ a,\ b\ \in C$, $a \succsim b$ or $b \succsim a$ or both,
2. $\succsim$ is transitive: $a \succsim b$ and $b \succsim c$ implies $a \succsim c$.

Agents

Preferences

Assumption (Utility Function)

A preference relation $\succsim$ can be represented by a utility function:

$$u(\cdot): \; c \mapsto R,$$

such that:

$$\forall \; a, b \in C, \quad a \succsim b \quad \textit{iff} \quad u(a) \geq u(b).$$

- Under certain mild assumptions on a preference relation (continuity), we can show the existence of a corresponding utility function.
- In general, $u(\cdot)$ is an ordinal operator: any strictly increasing transformation of $u(\cdot)$ represents the same preferences
- We will return to this discussion of utility theory later in the course.

Agents

One last note about C

- In this context, consumption $\iff$ financial wealth. World ends after time 1, so agent will usually consume all wealth
- This can be a shorthand for a more complicated market structure with multiple goods. Agent k solves static consumer problem in background with financial resources $c^k_{1\omega}$
- In this case, we think of $c_{1\omega}$ as the indirect utility associated with choosing a consumption bundle a given amount of wealth, with prices given
- Another source of uncertainty: prices of goods could depend on ω

- Tight link between consumption and wealth will be broken in dynamic models later in the class

Securities Market

- A security is a financial claim yielding a payoff at $t = 1$: $D_{1\omega}$, $\forall\, \omega \in \Omega$, or

$$D_1 = [D_{11}; D_{12}; \ldots; D_{1M}].$$

- We can also express the payoff by a tree (process):

D_{11} for ω_1

$\vdots$

D_{1M} for ω_M

- For convenience, we will refer to D as dividend or cash flow
- Define $\mathbb{R}^M$ as the payoff space. A security is then defined by a vector in the payoff space.
- Note: since we are starting with 2 period case here, I will sometimes omit the first (time) subscript

Securities Market

- Suppose there are N securities traded: $n = 1, \ldots, N$.
- Each security has payoff vector:

$$\underset{[M\times 1]}{D_{1n}} \equiv [D_{11n}; D_{12n}; \ldots ; D_{1Mn}].$$

- The payoff matrix:

$$D_1 \equiv [D_{11}, D_{12}, \ldots, D_{1N}] = \begin{bmatrix} D_{111} & \cdots & D_{11n} & \cdots & D_{11N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ D_{1\omega 1} & \cdots & D_{1\omega n} & \cdots & D_{1\omega N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ D_{1M1} & \cdots & D_{1Mn} & \cdots & D_{1MN} \end{bmatrix}.$$

defines the market structure.

Securities Market

- Let P_n denote the price of security n at $t = 0$ and $P = [P_1; \dots; P_N]$ the price vector.
- Let $\theta = [\theta_1; \dots; \theta_N]$ denote a portfolio, representing holdings of traded securities.
- The cost of portfolio (a cash outflow) θ at $t = 0$ is:

$$P^\top \theta$$

- Its payoff at $t = 1$ is:

$$D_1 \theta.$$

- Agents can trade in the securities market to allocate resources.
- $\theta_j < 0$ correspond with short sales: borrow θ_j units of asset j at time 0 and promise to pay back $\theta_j D_{1\omega j}$ at time 1.

Securities Market

Frictionless Market

A securities market is frictionless if it features

1. No access cost
2. No transactions costs
3. No position limits (constraints)
4. No market impact
5. No information asymmetry
6. No taxes.

The Economy

Summarizing the components, we have defined a model of an economy as follows:

- There are two dates, 0 and 1, with possible states $\omega \in \Omega$ at $t = 1$ and probability measure $\mathbb{P}$;
- There is only one perishable good;
- There are K agents in the economy, $k = 1, \ldots, K$. Each agent has:
  - Same prior as $\mathbb{P}$ about the states at 1,
  - Endowment $e^k \in \mathbb{R}_+^{1+M}$,
  - Preference relation $\succsim^k$ over consumption set $C = \mathbb{R}_+^{1+M}$, represented by a utility function $u_k(c)$;
- There is a frictionless securities market with market structure D_1.

Optimization

Budget Set

- In the economy defined above, agents can trade in the securities market to best meet her economic needs, given her resources.
- By purchasing portfolio θ at $t = 0$, she can achieve consumption:

$$c_0 = e_0 - P^\top \theta,$$
$$c_1 = e_1 + D_1 \theta.$$

- The set of consumption plans available to her is given by:

$$B(e, \{D_1, P\}) \equiv \{c \geq 0 : c_0 = e_0 - P^\top \theta,\ c_1 = e_1 + D_1 \theta,\ \theta \in R^N\}.$$

$B(e, \{D_1, P\})$ is called the agent's budget set.

Optimization

Each agent solves the following optimization problem:

$$\begin{aligned} \max_{\theta} \quad & u_k(c) \\ \text{s.t.} \quad & c_0^k = e_0^k - P^\top \theta \\ & c_1^k = e_1^k + D_1 \theta \\ & c^k \geq 0. \end{aligned}$$

Or equivalently:

$$\max_{c \in B(e^k, \{D_1, P\})} u_k(c).$$

We denote the solution by $\theta^k(P, e)$.

Two special cases

$$\max_{\theta} \quad u_k(c)$$
$$\text{s.t.} \quad c_0^k = e_0^k - P^\top \theta$$
$$c_1^k = e_1^k + D_1\theta$$
$$c^k \geq 0.$$

1. Basic intertemporal substitution problem: $M = 1, N = 1$.
   - Consumer trades off consumption today with consumption tomorrow
   - $D_1 = 1$, $P = \frac{1}{1+r}$: can borrow/lend at interest rate r
2. 2 Asset, 2 State portfolio problem: $M = 2, N = 2$
   - Simplest analytically when $e_0^k > 0$ and $e_1^k = 0$. Drop k subscripts. Suppose that agent has already fixed c_0 and is choosing how to allocate remaining wealth $w_0 = e_0 - c_0$ between c_{11} and c_{12}
   - Budget set: $c_{1\omega} = w_0 \left[\alpha \frac{D_{1\omega 1}}{P_1} + (1-\alpha)\frac{D_{1\omega 2}}{P_2}\right] \equiv w_0 \left[\alpha R_{1\omega 1} + (1-\alpha)R_{1\omega 2}\right]$
   - Vector notation: $c_1 = w_0 \underbrace{D_1 \text{diag}(P)^{-1}}_{R_1} \underbrace{\begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}}_{\equiv\tilde{\theta}} \equiv w_0 R_1 \tilde{\theta}$

     subject to $\tilde{\theta}' 1_N = 1$, where 1_N is an $N \times 1$ vector of ones.

Case 2 graphically

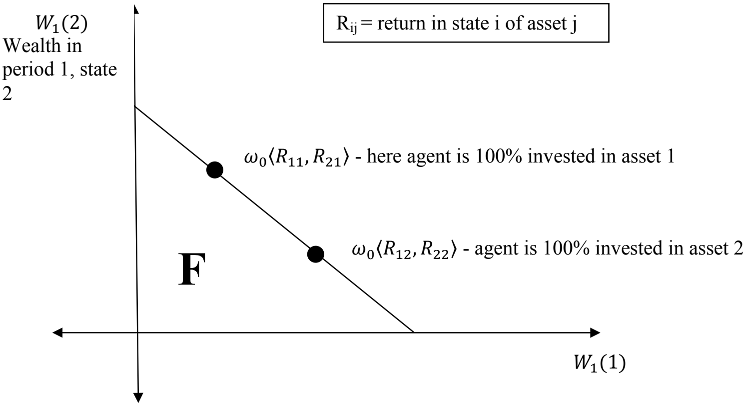


Figure 1

Note: if the agent is higher up the line than $\omega_0\langle R_{11}, R_{21}\rangle$ he has sold short asset 2, and invested the proceeds in asset 1. To see this consider that asset 1 does better in state 2 (vertical axis), so to have a greater payoff in state 2 than by investing 100% in asset 1 the investor must invest more than 100% in asset 1 by short selling asset 2. The cost of this is a reduced payoff in state 1, as asset 2 does relatively better in state 1.

Market Equilibrium

- Securities market reaches equilibrium when demand equals supply:

$$\sum_{k=1}^{K} \theta^k(P, e^k) = 0.$$

This is the market clearing condition.

- Clearing of the securities market leads to the clearing of the goods market:

$$\sum_{k=1}^{K} c_0^k = \sum_{k=1}^{K} e_0^k, \quad \sum_{k=1}^{K} c_{1\omega}^k = \sum_{k=1}^{K} e_{1\omega}^k, \quad \forall\, \omega \in \Omega$$

or

$$\sum_{k=1}^{K} c^k = \sum_{k=1}^{K} e^k.$$

- Solution to the securities market clearing condition gives their equilibrium prices:

$$P = P(D_1; \mathbb{P}, \{u^k, e^k : k = 1, \ldots, K\}),$$

which depend on the fundamentals or the primitives of the economy.

Optimality/Allocational Efficiency

Definition (Feasible Allocation)

An allocation $\{c^k \ \forall \ k\}$ is feasible given the total endowment of the economy $\{e^k \ \forall \ k\}$ if:

$$\sum_k c^k = \sum_k e^k.$$

Definition (Pareto Dominance)

Allocation $\{c_k \ \forall \ k\}$ Pareto dominates allocation $\{c^{k\prime} \ \forall \ k\}$ if:

$$\forall \ k: \quad u^k(c^k) \geq u^k(c^{k\prime})$$

and the inequality is strict for at least one agent.

Definition (Pareto Optimality)

An allocation of the economy is optimal if it is feasible and there is no other feasible allocation that Pareto dominates it.