Lecture 8. Arbitrage Pricing Theory (APT), part 1

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Outline

Arbitrage Pricing

- Arbitrage pricing is a theory on relative pricing:
  - Start with the payoffs and prices of a set of base securities,
  - Determine the prices of other securities by NA.
- In the case of options, we can price them with the prices of the underlying security and the riskless bond.
- In general, we can price redundant securities using the prices of a set of base securities.
- We try to expand this idea to more general situations, again with the objective of doing so under the weakest assumptions possible

Recall: Irrelevance of Idiosyncratic risk

- Consider the price of an asset whose risky payoff D_n (and hence also its return r_n) is uncorrelated with the discount factor η.
  - Formula above $\Rightarrow$ this asset earns same return as the risk-free asset
  - Cash flows are simply discounted at the risk-free rate: $P_n = \frac{E[D_n]}{1+r_f}$
- This suggests a simple decomposition:

$$D_n = \underbrace{\text{proj}(D_n|\eta)}_{\substack{\text{"priced"} \\ \text{cash flow} \\ \text{component}}} + \underbrace{\epsilon_n}_{\substack{\text{"unpriced"} \\ \text{cash flow} \\ \text{component}}}$$

where $E[\epsilon_n] = 0$ and $\epsilon_n \perp \eta$. So, $\text{proj}(D_j, \eta) = \text{proj}(D_k, \eta) \iff P_j = P_k$

- There is no compensation for holding mean 0 risks orthogonal to the SPD

- Different asset pricing models are different specifications of η. **This week: we put conditions on η by restricting comovement of cash flows.**

A useful decomposition for expected returns

- Start with our fundamental asset pricing equation

$$\mathbb{E}^{\mathbb{P}}[r_n - r_F] = -(1+r_f)Cov^{\mathbb{P}}[\eta, r_n - r_F]$$

- Suppose I find a portfolio which "mimics" the SDF as follows How?

$$r_{-\eta} = E^{\mathbb{P}}[r_{-\eta}] - (\eta - E^{\mathbb{P}}[\eta])$$

- What is its risk premium?

$$\begin{aligned} E[r_{-\eta}] - r_F &\equiv \lambda = -(1+r_F)\cdot Cov^{\mathbb{P}}[\ \ \eta, -\eta] \\ \Rightarrow (1+r_F) &= \frac{\lambda}{Var[\eta]} \end{aligned}$$

Note: $r_{-\eta}$ attains maximum possible Sharpe ratio (achieves H-J bound)

- Remember that we can actually construct $r_{-\eta}$ as a traded portfolio. Rearranging definition of $r_{-\eta} \Rightarrow$

$$\eta = E^{\mathbb{P}}[r_{-\eta}] + E^{\mathbb{P}}[\eta] - r_{-\eta} = \lambda + \frac{1}{1+r_F} - (r_{-\eta} - r_F)$$

A useful decomposition for expected returns

- Now, substitute above expression into the formula for the risk premium:

$$\mathbb{E}^{\mathbb{P}}[r_n - r_F] = \underbrace{\frac{Cov^{\mathbb{P}}[\ r_n - r_F, r_{-\eta} - r_F]}{Var^{\mathbb{P}}[\eta]}}_{\text{SDF } \textbf{risk exposure} \ / \ \textbf{loading}} \times \underbrace{\lambda}_{r_{-\eta} \text{ risk premium}} \equiv \underbrace{\beta_n}_{\text{quantity of risk}} \times \underbrace{\lambda}_{\text{market price of risk}}$$

- Each asset's expected return depends solely on its loading β_n on $r_{-\eta} - r_F$
  - Suppose we observe many IID realizations of returns indexed by t
  - We could estimate the following statistical model:

$$r_{n,t} - r_F = \alpha_n + \beta_n(r_{-\eta,t} - r_F) + \varepsilon_{n,t} \qquad E[\varepsilon_n | r_{-\eta} - r_F] = 0,$$

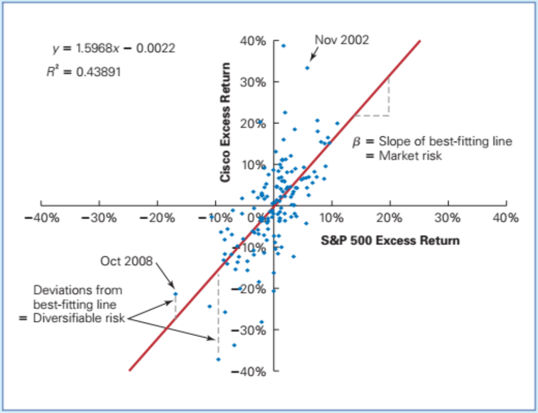
    which satisfies the conditions for classical OLS regression
  - OLS regression coefficient $\hat{\beta}_n = \dfrac{\widehat{Cov}(r_n - r_F, r_{-\eta} - r_F)}{\widehat{Var}(r_{-\eta} - r_F)}$
- λ is the market price of risk: captures the compensation that an investor receives per unit of exposure to the SDF

Geometric interpretation of beta

FIGURE 12.2

Scatterplot of Monthly Excess Returns for Cisco Versus the S&P 500, 2000–2012

Beta corresponds to the slope of the best-fitting line. Beta measures the expected change in Cisco's excess return per 1% change in the market's excess return. Deviations from the best-fitting line correspond to diversifiable, non-market-related risk. In this case, Cisco's estimated beta is approximately 1.60.



Suppose SDF return ($r_{-\eta}$) was the S&P 500 return. Estimating equation:

$$r_{n,t} - r_F = \underbrace{\alpha_n}_{\substack{\text{pricing error}\\ \text{abnormal return}}} + \underbrace{\beta_n(r_{-\eta,t} - r_F)}_{\substack{\text{comovement with}\\ \text{SDF excess return } r_{-\eta}}} + \underbrace{\varepsilon_{n,t}}_{\substack{\text{idiosyncratic}\\ \text{(unpriced) component}}}$$

β_n is part of a variance decomposition

$$r_{n,t} - r_F = \underbrace{\alpha_n}_{\substack{\text{pricing error}\\ \text{abnormal return}}} + \underbrace{\beta_n(r_{-\eta,t} - r_F)}_{\substack{\text{comovement with}\\ \text{SDF excess return } r_{-\eta}}} + \underbrace{\varepsilon_{n,t}}_{\substack{\text{idiosyncratic}\\ \text{(unpriced) component}}}$$

$$\begin{aligned} Var[r_n - r_F] &= \beta_n^2 Var[r_{-\eta} - r_F] + Var[\varepsilon_n] \\ &= \text{systematic, priced variance} + \text{unpriced residual variance} \end{aligned}$$

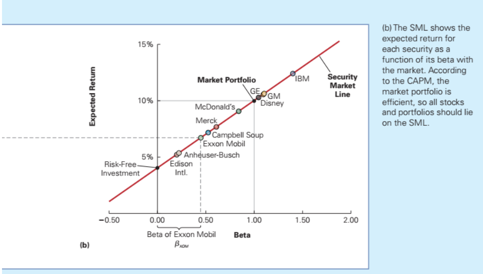
- Unpriced variance = asset-specific variance + variance of other common sources of variation which are uncorrelated with $r_{-\eta}$
- The joint hypothesis: No arbitrage + correct model of SDF $r_{-\eta,t} \Rightarrow$
  - Stocks w/ different $\varepsilon_{n,t}$ but same β_n earn same returns. Theory says $\alpha_n = 0$.
  - Finding $\alpha_n \neq 0 \Rightarrow$ we have the wrong model for the SDF or a stock is mispriced (no arbitrage fails). Can only ever reject **both** hypotheses.
  - Constant α_n captures mispricing/difference between the expected return of asset n and what we would predict if we had correct model for $r_{-\eta,t}$
- All asset pricing models imply an SDF $r_{-\eta,t}$ and involve a similar variance decomposition and a similar intuition!

Securities Market Line

Theory predicts that expected returns line up with β_n.

$$\mathbb{E}^{\mathbb{P}}[r_n] = r_F + \frac{Cov^{\mathbb{P}}[\; r_n - r_F, r_{-\eta} - r_F]}{Var^{\mathbb{P}}[\eta]} \times \lambda \equiv r_F + \underbrace{\beta_n}_{\substack{\text{quantity}\\ \text{of risk}}} \times \underbrace{\lambda}_{\substack{\text{market price}\\ \text{of risk}}}$$

- We can form empirical estimates of β_n and expected returns. Theory is (quite) testable!
- Plot linking them is called the securities market line.
- Slope of the line is market price of risk λ.
- More on this later!



Source: Berk & Demarzo (Figure 11.12b)

Factor Structure

- There are $n = 1, \ldots, N$ securities in the market, with payoff matrix D.
- Assume the first security is riskless. (This is not necessary but convenient.)
- Let $rank(D) = K$, which can be less than M, the number of states.
- Thus, some of the securities have redundant payoffs.
- Let F denote a basis for D: F is a set of K linearly independent vectors:

  $$F \equiv [F_1, \ldots, F_K],$$

  where each F_k is a $(M \times 1)$ vector and F is a $(M \times K)$ matrix.
- We refer to these vectors as factors and F as the factor structure.
- Note that the factors $(F_k,\ k = 1, \ldots, K)$ need not themselves be payoffs of securities in D. But they may be.

Factor Structure

- Let $C \equiv \{D\,\theta : \;\theta \in R^N\}$ denote the payoff space spanned by D.
- Then, for any $Z \in C$, there exists $\beta_Z \equiv [\beta_{Z1}; \ldots; \beta_{ZK}]$ such that:

$$Z = F\,\beta_Z. \tag{1}$$

The weights, $\beta_{Z1}, \ldots, \beta_{ZK}$, are called the betas of the payoff on the factors.

- The above simply states that any payoff in C can be spanned by the factors.
- The reverse is also true: Any factor can be replicated by securities in D:

$$\forall\; k = 1, \ldots, K, \;\; \exists\; \theta_k : \;\; F_k = D\,\theta_k.$$

This follows from the fact that $rank(D) = K$.

Factor Pricing

Theorem

There exists a set of coefficients $\lambda \equiv [\lambda_1; \ldots; \lambda_K]$ *such that* $\forall\ Z \in C$*:*

$$V(Z) = \lambda^\top \beta_Z = \sum_{k=1}^{K} \lambda_k \beta_{Zk}, \tag{2}$$

where $V(\cdot)$ *is the valuation operator.*

- The key point here is that λ_k, $k = 1, \ldots, K$, is independent of Z.
- Thus, using the factor structure, we have the following interpretation:
  - F_k represents a risk factor, and
  - λ_k represents the corresponding market price of risk.

Factor Pricing in Returns

Formulation in Returns

- For future convenience, we reformulate the above result in returns.
- Let R_n denote the gross return on security n:

$$R_n \equiv D_n/P_n,\ n = 1, \ldots, N.$$

- The market structure is then described by:

$$R \equiv [R_1, \ldots, R_N].$$

- Define:

$$\bar{R}_n \equiv \mathbb{E}[R_n], \quad \varepsilon_n \equiv R_n - \bar{R}_n, \quad \text{and} \quad \bar{R} \equiv [\bar{R}_1, \ldots, \bar{R}_N],\ \varepsilon \equiv [\varepsilon_1, \ldots, \varepsilon_N].$$

- We can then write the returns on the securities as:

$$R_n = \bar{R}_n + \varepsilon_n = \bar{R}_n + F\,\beta_n, \quad n = 1, \ldots, N, \quad \text{and} \quad \mathbb{E}[\varepsilon_n] = 0.$$

  Here, $\mathbb{E}[\cdot] \equiv \mathbb{E}^{\mathbb{P}}[\cdot]$ denotes the expectation under probability measure $\mathbb{P}$.
- In matrix form, we have:

$$R = \bar{R} + \varepsilon, \quad \text{where } \bar{R} \equiv [\bar{R}_1, \ldots, \bar{R}_N],\ \varepsilon \equiv [\varepsilon_1, \ldots, \varepsilon_N]. \tag{3}$$

Factor Pricing in Returns

Formulation in Returns

- We can then write the returns on the securities as:

$$R_n = \bar{R}_n + \varepsilon_n = \bar{R}_n + F\,\beta_n, \quad n = 1, \ldots, N$$

or

$$R = \bar{R} + \varepsilon = \bar{R} + F\,\beta, \quad \beta \equiv [\beta_1, \ldots, \beta_N]. \tag{4}$$

- Note that R and $\bar{R}$ are $(M \times N)$, F is $(M \times K)$, and β is $(K \times N)$.
- The factors in (4) are similar as before:
  - The first factor is $\iota = [1; \ldots; 1]$;
  - The remaining factors can be the same as before, but demeaned;
  - These demeaned factors can be interpreted as risk factors.
  - The risk factors drive ε, the unanticipated return.
  - There are a total of $rank(D) - 1$ risk factors.
  - For simplicity, we still use K to denote the number of risk factors.

Factor Pricing in Returns

Formulation in Returns

Example. The binomial example: two securities with gross returns:

Security 1: $1 \begin{cases} 1 \\ 1 \end{cases}$ Security 2: $1 \begin{cases} 1/2 \\ 2 \end{cases}$

- Assume the two states are equally likely: $\mathbb{P} = \{1/2, 1/2\}$.
- The expected gross return on the two securities are:

$$\bar{R}_1 = \frac{1}{2}(1+1) = 1, \quad \bar{R}_2 = \frac{1}{2}(1/2+2) = 5/4.$$

- We then have:

$$R_n = \bar{R}_n + \varepsilon_n, \quad n = 1, 2, \quad \text{and} \quad \varepsilon_1 = [0; 0], \ \varepsilon_2 = [-3/4; 3/4].$$

- Introduce two factors: $F_0 = [1; 1]$ and $F_1 = [-1; 1]$. We then have:

$$R = [F_0, F_1][\beta_1, \beta_2] = \bar{R} + F_1\beta_2 = \bar{R} + \varepsilon, \quad \text{with } \beta_1 = [\bar{R}_1; 0], \ \beta_2 = [\bar{R}_2; 3/4].$$

- Factor F_1 thus represents the only risk factor, driving ε.

The Exact APT

Lemma

The value of any gross return is 1.

Lemma

The value of the sure gross return $R_F \equiv 1 + r_F$ *is 1.*

The Exact APT

Lemma

Under the risk-neutral measure, all securities yield the riskless gross return:

$$\mathbb{E}^{\mathbb{Q}}[R_n] = R_F \quad \textit{or} \quad \mathbb{E}^{\mathbb{Q}}[R_n] - R_F = 0, \quad n = 1, \ldots, N.$$

Theorem (Exact APT)

Let R_n be the gross return on security n, $\bar{R}_n$ its expected value, and

$$R_n = \bar{R}_n + F\,\beta_n, \quad n = 1, \ldots, N,$$

where $F = [F_1, \ldots, F_K]$ are the K risk factors and $\beta_n \equiv [\beta_{n1}; \ldots; \beta_{nK}]$ is security n's betas. Then, no arbitrage requires that:

$$\bar{R}_n - R_F = \bar{r}_n - r_F = \sum_{k=1}^{K} \lambda_k \beta_{nk} = \lambda^\top \beta_n, \quad n = 1, \ldots, N, \tag{5}$$

where $\lambda \equiv [\lambda_1; \ldots; \lambda_K]$ and

$$\lambda_k = -\mathbb{E}^{\mathbb{Q}}[F_k], \quad k = 1, \ldots, K.$$

The Exact APT

Proof.

The Exact APT

- For a portfolio θ, its beta on risk factor k is:

$$\beta_k = \sum_n \theta_n \beta_{nk}.$$

- A portfolio θ_k is called a factor portfolio or factor mimicking portfolio if

$$\beta_{kk} = 1 \quad \text{and} \quad \beta_{kj} = 0 \ \forall \ j \neq k.$$

Theorem (Factor Portfolio)

For each factor F_k, $k = 1, \ldots, K$, there exists a factor portfolio θ_k.

- Let the gross return on factor-k portfolio be R_k and its mean $\bar{R}_k$. Then,

$$R_k = \bar{R}_k + F_k \quad \text{or} \quad F_k = R_k - \bar{R}_k = r_k - \bar{r}_k.$$

- The expected excess return on the factor portfolio gives the factor premium:

$$\bar{R}_k - R_F = \bar{r}_k - r_F = \lambda_k.$$

- We then have the following exact APT:

$$\bar{r}_n - r_F = \sum_k \beta_{nk}(\bar{r}_k - r_F), \quad n = 1, \ldots, N. \tag{6}$$

For next time: Ross' Arbitrage Pricing Theory

- Deriving pricing formulas with an exact factor structure may require a large number factors.
- It works if the market is complete: $K = M$.
- In general, M is large. (Imagine the state space needed to describe the returns of 1,000 stocks.)
- The model loses its appeal if K is too large.
- Want to explore a model with a large number of states and assets but only a small number of factors
- To do this, next time, we will introduce a concept of "approximate/asymptotic arbitrage" opportunities and derive simple and meaningful pricing results that are implied by additionally ruling them out

Factor Model for Returns

We will work with a generalized model for security returns.

Definition (Factor Model for Returns)

In a factor model, security returns are given as follows:

$$r_n = \bar{r}_n + \sum_{k=1}^{K} \beta_{nk} F_k + \varepsilon_n, \quad n = 1, \ldots, N, \tag{7}$$

where

(1) $\mathbb{E}[F_k] = \mathbb{E}[\varepsilon_n] = \mathbb{E}[\varepsilon_n | F_k] = 0 \ \ \forall\, k, n$

(2) $\mathbb{E}[\varepsilon_n^2] = \sigma_n^2 < v < \infty, \quad \mathbb{E}[\varepsilon_n \varepsilon_{n'}] = 0 \ \ \forall\, n \neq n'$.

- $F_1, \ldots, F_K$ define the risk factors common to all securities.
- $\varepsilon_1, \ldots, \varepsilon_N$ capture the risks specific/idiosyncratic to individual securities.
- Setting $\varepsilon_n = 0 \ \forall\, n$, we return to the exact factor model.
- Here, we assume N, M very large (approaching infinity) while K small.

Appendix: Constructing an SDF-mimicking portfolio

Theorem

When D has full rank, there exists a unique $\phi^* \in \mathbb{R}^M$ such that $P^T = \phi^{*T} D$, where $\phi^* = D\theta$ for some portfolio θ.

- To see this, combine the above equalities $\Rightarrow P^T = \theta^T D^T D$
- Now solve for θ^*. As before, suppose $rank(D) = N$, so $D^T D$ is invertible:

$$\theta^* = (D^T D)^{-1} P$$

$$\phi^* = D\theta^* = D(D^T D)^{-1} P \qquad \Rightarrow \qquad \eta^* = \phi^* diag(p)^{-1}$$

- We wanted $r_{-\eta} = \text{constant} - \eta^*$. We do this in 2 steps (assume r_F exists):
    1. Construct a portfolio with payoff $-\eta^*$, which has price $-\sum_{\omega=1}^{M} \phi^*_\omega \eta^*_\omega$
    2. Add position in the risk free asset to make cost of the portfolio equal to 1 (so payoff = return): i.e., put $\$(1 + \sum_{\omega=1}^{M} \phi^*_\omega \eta^*_\omega)$ in the risk-free asset

Back