14.121, Fall 2014
Problem Set 3 Solutions

1) MWG Exercise 3.I.3

Solution: The CV of the price change is

$$CV(p^0, p^1, w) = e(p^0, u^0) - e(p^1, u^0)$$

and the EV is

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^1, u^1)$$

where $u^1 := v(p^1, w)$ and $u^0 = v(p^0, w)$. Note that because $p^1 \leq p^0$ and $p^1 \neq p^0$, we must have $u^1 > u^0$.

Only the price of good l changed, so by Shephard's lemma, we can write

$$CV(p^0, p^1, w) = \int_{p_l^1}^{p_l^0} h_l(p_l, p_{-l}, u^0) dp_l$$

and

$$EV(p^0, p^1, w) = \int_{p_l^1}^{p_l^0} h_l(p_l, p_{-l}, u^1) dp_l$$

Next, note that $h_l(p, u) = x_l(p, e(p, u))$. Good l is inferior which means that x_l is strictly decreasing in its second argument. Furthermore, $e(p, u)$ is increasing in u (assuming local non-satiation). Hence $h_l(p, u)$ is strictly decreasing in u.

Therefore, $h_l(p, u^1) < h_l(p, u^0)$ for any p. Hence $EV(p^0, p^1, w) < CV(p^0, p^1, w)$.

MWG Exercise 3.I.4

Solution: Obviously, there are many possible answers here. We shall present two examples:

Example 1: Consider Leontief preferences that generate "L-shaped" indifference curves such that the vectors $(1, 1), (4, 2)$, and $(5, 3)$ are kinks of indifference curves. Let $u(1, 1) = 1$. Note that if one of the two prices is equal to zero, then the demand is not a singleton. We thus need to consider a demand correspondence $x(p, w)$. But this does not essentially change our argument because we are working on expenditure functions, which are single-valued by definition.

Let $p^0 = (1, 1), p^1 = (\frac{1}{2}, 0), p^2 = (0, \frac{2}{3})$, and $w = 2$. Then $(1, 1) \in x(p^0, w), (4, 2) \in x(p^1, w), (5, 3) \in x(p^2, w)$, and $v(p^2, w) > v(p^1, w)$. But $e(p^1, 1) = \frac{1}{2}$ and $e(p^2, 1) = \frac{2}{3}$. Thus

$$CV(p^0, p^1, w) = 2 - \frac{1}{2} = \frac{3}{2}, \quad CV(p^0, p^2, w) = 2 - \frac{2}{3} = \frac{4}{3}$$

Hence $CV(p^0, p^1, w) > CV(p^0, p^2, w)$.

Example 2: Consider quasi-linear preference $u(x) = x_1 + f(x_2)$. Let $p^0 = (1, 1), p^1 = (p_1^1, 1), p^2 = (1, p_2^2)$, where $p_1^1, p_2^2 < 1$, and such that $v(p^1, u) = v(p^2, u) = u^1 \geq u^0$.

In this case, we have

$$EV(p^0, p^1, w) := e(p^0, u^1) - w = EV(p^0, p^2, w)$$

By $p_1^1 < 1$, $CV(p^0, p^1, w) < EV(p^0, p^1, w)$. By quasi-linearity, $CV(p^0, p^2, w) = EV(p^0, p^2, w)$.
Hence $CV(p^0, p^1, w) < CV(p^0, p^2, w)$, but $v(p^1, u) = v(p^2, u)$.

MWG Exercise 3.I.5

If preferences are quasilinear then we can write

$$u(x) = x_1 + v(x_{-1})$$

Normalize $p_1 = 1$. Then

$$\begin{aligned}
e(p,u) &= \min_x x_1 + p \cdot x_{-1} \text{ s.t. } u(x) \geq u \\
&= \min_x x_1 + p \cdot x_{-1} \text{ s.t. } u(x) = u \\
&= \min_x x_1 + p \cdot x_{-1} \text{ s.t. } x_1 + v(x_{-1}) = u \\
&= \min_{x_{-1}} u - v(x_{-1}) + p \cdot x_{-1} \\
&= u + \min_{x_{-1}} p \cdot x_{-1} - v(x_{-1}) \\
&:= u + \tilde{e}(p)
\end{aligned}$$

Hence,

$$CV(p^0, p^1, w) = e(p^0, u^0) - e(p^1, u^0) = \tilde{e}(p^0) - \tilde{e}(p^1)$$

and

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^1, u^1) = \tilde{e}(p^0) - \tilde{e}(p^1)$$

2) Take a consumer with preferences over $X = \mathbb{R}^L_+$ given by a continuously differentiable, strictly increasing and concave utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$. Let $v(p,w)$ be the indirect utility function.

   (a) Show that $u(x) \leq v(p, p \cdot x)$ for all x

   Solution: Clearly, we have that $x \in B_{p,p\cdot x}$. Therefore

   $$v\left(p, px\right) = \max_{z \in B_{p,p\cdot x}} u\left(z\right) \geq u\left(x\right)$$

   as we wanted to show.

   (b) Show that for all $x \gg 0$, we have $u(x) = \min_{p \gg 0} v(p, p \cdot x)$

   Solution: See that, given x, Part (a) tells us that the function $\psi\left(p\right) \equiv v\left(p, px\right)$ is bounded from below by $u = u\left(x\right)$. Therefore, if there exist some $\widehat{p}$ such that $\psi\left(\widehat{p}\right) = u\left(x\right)$ then we must have that $u\left(x\right) = \min_{p \gg 0} v\left(p, px\right)$, which is what we want to show. See that $u\left(x\right) = v\left(\widehat{p}, \widehat{p}x\right)$ implies that if we want to solve the program

   $$\begin{aligned}
   &\max_{z \in \mathbb{R}^L_+} u\left(z\right) \\
   s.t \quad &: \quad \widehat{p}z \leq \widehat{p}x
   \end{aligned}$$

   then $z = x$ is an optimum for this program.

   Now, because u is differentiable and concave, we know that we can use the Kuhn Tucker conditions to characterize the optimum at prices p and income px. The Lagrangian is

   $$\mathcal{L}\left(z\right) \ = u\left(z\right) + \lambda\left[px - pz\right]$$

   so, $z = x$ is an optimum if and only if

   $$\frac{\partial \mathcal{L}}{\partial x_l} = 0 \Longleftrightarrow \frac{\partial u}{\partial x_l}\left(x\right) = \lambda \widehat{p}_l \tag{1}$$

$$\lambda\left(\widehat{p}x - \widehat{p}x\right) = 0, \lambda \geq 0, x \geq 0 \tag{2}$$

Now, complementary slackness is always satisfied, so we only need some $\lambda \geq 0$. But then, if we define

$$\widehat{p}_l = \frac{\partial u}{\partial x_l}(x) \text{ for all } l = 1, 2, ..., L$$

then x and $\lambda = 1$ satisfy (1) and (2). Therefore, x is optimal when prices are $\widehat{p}$, so

$$v\left(\widehat{p}, \widehat{p}x\right) = u\left(x\right)$$

as we wanted to show.

(c) Suppose that $v(p, w) = (p_1^\rho + p_2^\rho)^{-\frac{1}{\rho}} w$ with $\rho > 0$. Find a utility function u that has v as the indirect utility function. Check that u is concave, strictly increasing and differentiable.

Solution: W.l.o.g, let's normalize $w = 1$. We have $v(p, 1) = (p_1^\rho + p_2^\rho)^{-\frac{1}{\rho}}$. From part (b), we know

$$u(x_1, x_2) = \min_{p_1, p_2} (p_1^\rho + p_2^\rho)^{-\frac{1}{\rho}} \quad \text{s.t. } p_1 x_1 + p_2 x_2 = 1$$

Solving the first order conditions, we get

$$p_1 = \frac{x_1^{\frac{1}{\rho-1}}}{x_1^{\frac{\rho}{\rho-1}} + x_2^{\frac{\rho}{\rho-1}}}, \qquad p_2 = \frac{x_2^{\frac{1}{\rho-1}}}{x_1^{\frac{\rho}{\rho-1}} + x_2^{\frac{\rho}{\rho-1}}}$$

Substitute this back, we obtain

$$u(x_1, x_2) = \left[\frac{x_1^{\frac{\rho}{\rho-1}} + x_2^{\frac{\rho}{\rho-1}}}{\left(x_1^{\frac{\rho}{\rho-1}} + x_2^{\frac{\rho}{\rho-1}}\right)^\rho}\right]^{-\frac{1}{\rho}} = \left(x_1^{\frac{\rho}{\rho-1}} + x_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

We assume $\rho < 1$ here. Define $\gamma = \frac{\rho}{\rho-1} < 0$, we have

$$u(x_1, x_2) = (x_1^\gamma + x_2^\gamma)^{\frac{1}{\gamma}}$$

This is CES utility function with elasticity of substitution $\frac{1}{1-\gamma}$.

4) Consider a population of K consumers with Marshallian demand $x^k(p, w^k)$. Show that if preferences are homothetic, but not identical, and each consumer has a fixed share of total aggregate income (as prices and total income are varied), then there exists a single preference ordering that generates the aggregate demand. Is aggregate demand homothetic?

Solution: Let $u_1(x_1), \ldots, u_K(x_K)$ be the agents' utility functions. Since preferences are homothetic, w.l.o.g, we can assume the individual utility functions are homogeneous of degree 1. Let $a_1, \ldots, a_K$ be the wealth shares. Define a representative consumer's utility function as follows:

$$u(x) = \max_{\{x_i\}_{i=1}^K} \prod_{i=1}^K u_i(x_i)^{a_i} \quad \text{s.t. } \sum_{i=1}^K x_i \leq x$$

We need to show that for any (p, w), the maximizer of $u(x)$ subject to $p \cdot x \leq w$ is given by $\sum_{i=1}^K x_i(p, a_i w)$. Because of the way $u(x)$ is defined, we are actually solving the following:

$$\max_{\{x_i\}_{i=1}^K} \prod_{i=1}^K u_i(x_i)^{a_i} \quad \text{subject to} \sum_{i=1}^K p \cdot x_i \leq w$$

This program can be decomposed into two steps. First allocate income across individual consumers,

$$\max_{\{w_i\}_{i=1}^K} \prod_{i=1}^K v_i(p, w_i)^{a_i} \quad \text{s.t.} \sum_{i=1}^K w_i \le w$$

where v denotes the indirect utility function. Next maximize u_i separately,

$$\forall i = 1, \ldots, K \quad \max_{x_i} u_i(x_i) \quad \text{s.t. } p \cdot x_i \le w_i$$

Because each u_i is homogenous of degree 1, each v_i is homogenous of degree 1 in wealth. Thus the first-step maximization yields optimal wealth allocation $w_i = a_i w$. The second step then gives us $x_i = x_i(p, w_i) = x_i(p, a_i w)$. Hence the aggregate Marshallian demand x is indeed the sum of individual Marshallian demand. It's easy to see that the aggregate demand is also homothetic.

Remark: An alternative approach is to show that the aggregate demand function satisfies Walras' law, the Slutsky matrix is symmetric and negative semi-definite (note that the proof for the last property is non-trivial). Then we can apply Antonelli, Hurwicz-Uzawa's proposition. See Chipman (1974) for more details.