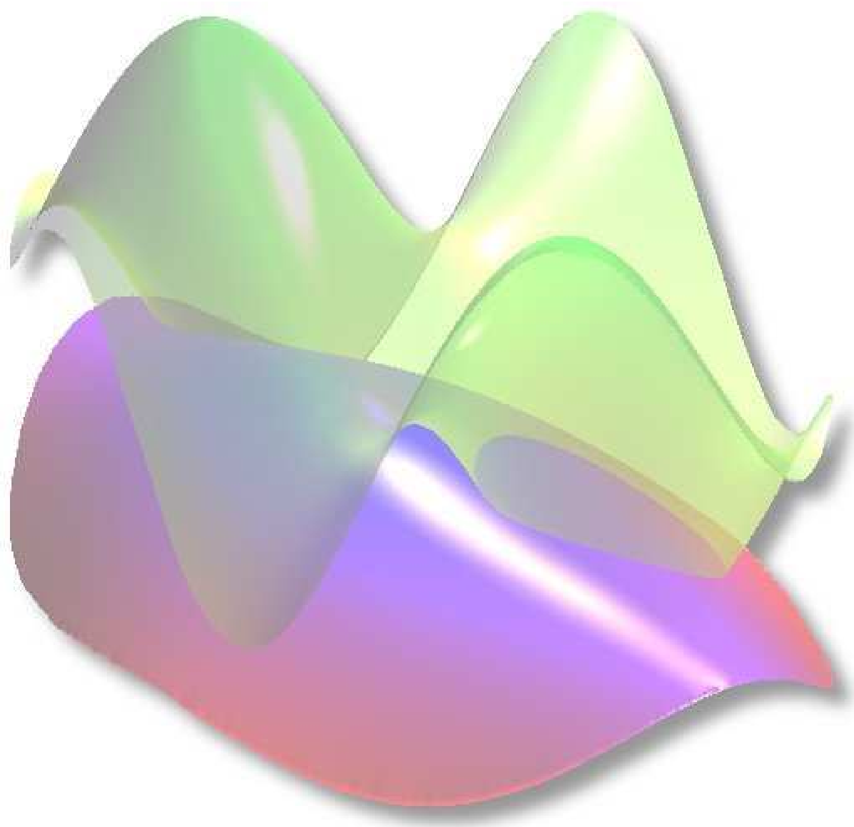
Elementary Theoretical Methods in PDE

Gregory T. von Nessi



Contents

Preface

The history of the writing of this text is, in many ways, a reflection of my own eduction in the subject of PDE. In December 2002, I started writing these notes as a personal guide in my graduate studies while at the University of Maryland. At this time, Dr. Georg Dolzmann was my teacher in PDE; this text was born out of the class notes I took from his wonderfully organized and insightful lectures. After my coursework in PDE was completed, I continued to expand upon those notes, adding proofs, exercises, rewriting parts, etc. I did this initially just to keep "my head in the subject"; but as time went on, I found myself referring more to those notes than the conventional textbooks in the subject. During the past two years the material forthcoming has waxed and waned with my interests and biases on what I think is important to convey in an introductory graduate course in PDE. Originally, I did not expect this text to be of use to anyone but myself; but as my fellow students learned of their existence, they soon became a reference for others. As a result, I now keep developing the following text with the hope that it might be, in some way, beneficial to others who have an interest in PDE.

With that sentiment conveyed, I acknowledge that a text such as this can in no way be regarded as a complete treatise on theoretical PDE; the subject is simply to vast. Instead, I hope this material to represent a well-rounded introduction to the subject of PDE concatenated with more modern concepts born out of, admittedly, my own interests in the field of PDE.

When writing this book, I tried to adhere to a few main objectives.

1. **Clarity:** My primary objective in writing the following is clarity as the subject of theoretical PDE contains many subtle arguments taken from an eclectic collection mathematical subjects. As a result, I have tried to write out proofs as clearly as possible utilizing more the reader's intuition rather than belaboring their patience with excessive technical

calculations. This is not to say that I've made excessive use of "the calculation is left as an exercises to the reader"; but rather, reference is frequently made to ideas conveyed elsewhere in the text. I've also tried to include imagery to convey geometric concepts when appropriate, i.e. without betraying the full generality or pathologies of what is being explained.

2. **Motivation:** In an eclectic subject such as PDE, it would be easy to bombard the reader with pages and pages of theory, only to present a handful of applications; but I've tried to resist this temptation in order to present as many examples as possible throughout all parts of the text. Also, the global organization of this book has be written to this end, with more concrete, classical theory presented first and modern, more abstract theory left until later in the book. After all, to present an introduction to PDE without significant emphasis on application, I believe would be betraying the fact that PDE originated out of studies in physics and engineering.

3. **Attainability:** I've tried to make this text as self contained as possible, in that what follows requires no previous background outside of some familiarity of measure theory. Even though, many real and functional analytic principles are conveyed throughout various points of the text, these can by no means substitute for a proper education in these subjects.

As to the local organization of this book, I've have arranged the material with primary emphasis on clarity and logical progression rather than grouping like topics/theorems together. Thus, one will notice theorems pertaining, say to, Maximum principles spread out over more than one chapter. This might seem like an unacceptable loss of continuity to the reader, but such is necessary to avoid presenting a theorem only to first use it 100 pages later in the text. Again, this is a trade-off one has to accept when writing on such an eclectic topic as PDE.

Finally, I would like to thank Dr. Georg Dolzmann whose lectures inspired the origins of this book and to Dave Bourne who very graciously allowed me to use his class notes to write the outline of what corresponds to the first few chapters of this material.

Greg von Nessi
Australian National University

Part I

Classical Approaches

Chapter 1

Physical Motivations

1.1 Introduction

As the chapter title indicates, the following material is meant to simply acquaint the reader with the concept of Partial Differential Equations (PDE). The chapter starts out with a few simple physical derivations that motivate some of the most fundamental and famous PDE. In some of these examples analytic techniques, such as Duhamel's Principle, are introduced to whet the reader's appetite for the material forthcoming. After this initial motivation, basic definitions and classifications of PDE are presented before moving on to the next chapter.

1.2 The Heat Equation

One of the most, if not THE most, famous PDE is the Heat Equation. Obviously, this equation is somehow connected with the physical concept of heat energy, as the name not-so subtly suggests. The natural questions now are: what is the heat equation and where does it come from? To answer these, we will first construct a physical model; but before we proceed with this, let us define some notation.

In our model, take Ω to be a heat conducting body, $\rho(x)$ as the mass density at $x \in \Omega$, $c(x)$ is the specific heat at $x \in \Omega$, and $u(x,t)$ is the temperature at point x and time t. Also take B to be an arbitrary ball in Ω.

In order to construct any physical model, one has to make some assumption about the reality they are trying to replicate. In our model we assume that heat energy is conserved in any subset of Ω; and in particular, is conserved in B.

With this in mind, we first consider the total heat energy in B, which is

$$\int_B \rho(x)c(x)u(x,t)\ dx;$$

and the rate of change of the heat energy in B is

$$\frac{d}{dt}\int_B \rho(x)c(x)u(x,t)\ dx.$$

If we know the heat flux rate, $\vec{F}(x,t)$ (a vector-valued function), across a surface element A centered at x with normal $\vec{\nu}(x)$, then we know the flux across a surface is given by:

$$|A|\cdot\vec{F}(x,t)\cdot\vec{\nu}(x).$$

Thus, using the assumption that heat energy is conserved in B, we calculate

$$\begin{aligned}\frac{d}{dt}\int_B \rho(x)c(x)u(x,t)\ dx &= -\int_{\partial B}\vec{F}(x,t)\cdot\vec{\nu}(x)\ dS\\ &= -\int_B \operatorname{div}\vec{F}(x,t)\ dx.\end{aligned}$$

Upon rewriting this equality, we see that

$$\int_B \underbrace{(\rho(x)c(x)u_t(x,t)+\operatorname{div}\vec{F}(x,t))}_{\text{turns out to be smooth}}\ dx = 0.$$

Since B is arbitrary, this last equation must be true for all balls $B\subset\Omega$. We can now say,

$$\rho(x)c(x)u_t(x,t)+\operatorname{div}\vec{F}(x,t)=0. \qquad \text{in } \Omega \tag{1.1}$$

We seek u, but we have only one equation that involves the unknown heat flux F; in order to proceed with our model, we must make another assumption based on what (we think) we know of reality. So we make the following *constitutive assumption* as to what the heat flux "looks like" mathematically:

$$\vec{F}(x,t) = -k(x)\cdot\vec{Du}(x,t).$$

(See Appendix A for notation). Plugging this constitutive assumption into (1.1), we get *Fourier's Law of Cooling*:

$$\rho(x)c(x)u_t(x,t) - \text{ div } (k(x) \cdot \vec{D_x} u(x,t)) = 0. \tag{1.2}$$

To further simplify the problem, we consider the special case that $\rho(x)$, $c(x)$ and $k(x)$ are all constant; take them to be ρ_1, c and k respectively. Now, we can further manipulate the second term of the LHS of (1.2):

$$\begin{aligned}
\text{div } Du &= \text{div } \left(\frac{\partial}{\partial x_1} u, \ldots, \frac{\partial}{\partial x_n} u \right) \\
&= \frac{\partial^2}{\partial x_1^2} u + \ldots + \frac{\partial^2}{\partial x_n^2} u \\
&= \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right) u \\
&= \Delta u \qquad (\equiv \nabla^2 u).
\end{aligned}$$

Thus we conclude that (1.2) can be written as

$$\begin{aligned}
\rho c \cdot u_t(x,t) &= k \cdot \Delta u \\
u_t(x,t) &= K \cdot \Delta u, \qquad (1.3)
\end{aligned}$$

where $K = \frac{k}{\rho c}$ is the diffusion constant. (1.3) is the legendary Heat Equation in all its glory.

We shall see later that the solutions of the Heat Equation will evolve to a *steady state* which does not change in time. These steady state solutions thus solve

$$\Delta u = 0. \tag{1.4}$$

This is the most ubiquitous equation in all of physics. This is *Laplace's Equation*.

1.2.1 A Note on Physical Assumptions

[The reader may skip this section if they feel comfortable with the concepts of conservation laws and constitutive assumptions.]

In our derivation of the Heat Equation, we used two physical observations to construct a model of heat flow. Specifically, we first assumed that heat energy was conserved; this is particular manifestation of what is known as a *conservation law* this was followed by a constitutive assumption based on the actual observed behavior of heat. Considering conservation laws, it is observed that many quantities are conserved nature; examples are conservation of mass/energy, momentum, angular momentum etc. It turns out that these conservation laws correspond to specific symmetries (via Noether's Theorem). Physics, as we know it, is constructed off of the assumed validity of these conservation laws; and it is through these laws that one finds that conclusions in physics manifest in the form of PDE. Some examples are Schrödinger's Equation, the Heat Equation (via the argument above), Einstein's Equations, and Maxwell's Equation's, to name a few.

In the above derivation, the use of the conservation of energy did not depend on the fact that we were trying to model *heat* energy. Indeed, are model lost it's generality (i.e. pertaining to the general flow of energy) via the constitutive assumption, which is based on the observations pertaining specifically to heat flow. In the above derivation, the constitutive assumption was simply a mathematical way of saying that heat energy is observed to move from a hot to a cold object, and that the rate of this flow will increase with the temperature differential present. It is worth trying to understand this physically, through the example of your own sense of "temperature". To do this, one first should realize that when one feels "temperature", they are really feeling the heat energy flux on the surface of their skin. This is why people with hypothermia cease to feel "cold". In this situation their body's internal temperature has dropped below the normal 98.6°F. As a result, the heat energy flux across the surface of the skin has decreased as the temperature differential between the person's body and outside air has decreased.

Now, let us move on to another model to try and solidify the ideas of conservation laws and constitutive assumptions.

1.3 Classical Model for Traffic Flow

We will now proceed to construct a model for (very) simple traffic flow in the same spirit as the heat equation was derived; i.e. our model will be born out of a conservation law and a constitutive assumption.

First, we define $\rho(x,t)$ to represent the density of cars on an ideal high (one that is straight, one lane with no ramps). Here, we take the highway to be represented by the x-axis. Since we do not expect cars to be created or destroyed (hopefully) on our highway, it is reasonable to assume that $\rho(x,t)$ represents a conserved quantity. The corresponding conservation law is written as

$$\rho_t(x,t) + \text{div } \vec{F}(x,t) = 0.$$

Again, $\vec{F}$ represents the flux of car density at any given point in space-time:

$$\vec{F}(x,t) = \rho(x,t) \cdot \vec{v}(x,t),$$

where $\vec{v}(x,t)$ is the velocity of cars at a given point in space-time. With that, we rewrite our conservation law as

$$\rho_t + (\rho \cdot v)_x = 0. \tag{1.5}$$

Now it is time to bring in a constitutive assumption based on traffic flow observations. The assumption we will use comes from the *Lighthill-Whitham-Richards model* of traffic flow, and this corresponds to $\vec{v}(x,t)$ assumed to have the form

$$\vec{v}(x,t) = v_{\max}\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right)\hat{x},$$

where $v_{\max}$ is the maximum car velocity (speed limit) and $\rho_{\max}$ is the maximum density of cars (parked bumper-to-bumper). Looking at this assumption, we see that this model rather näively indicates that car velocity is directly proportional to car density; cars approach the speed-limit as their density goes to zero and, conversely, slow to a stop as they get bumper-to-bumper. Using this constitutive assumption with our conservation law (1.5) yields

$$\begin{aligned} & \rho_t + v_{\max} \cdot \left(\rho - \frac{\rho^2}{\rho_{\max}}\right)_x = 0 \\ \Longrightarrow \quad & \rho_t + v_{\max} \cdot \left(1 - \frac{2\rho}{\rho_{\max}}\right)\rho_x = 0. \end{aligned} \tag{1.6}$$

It turns out that this PDE predicts that a continuous density profile can develop a discontinuity in finite time, i.e. a car pile-up is an inevitability on an ideal highway!

1.4 The Transport Equation

Here we will again present another PDE via a physical derivation. In this model we seek to understand the dispersion of an air-born pollutant in the presence of a constant and uni-directional wind. For this, we take $u(x,t)$ to be the pollutant density, and $\vec{b}$ to be the velocity of the wind, which we are assuming to be constant in both space and time.

<u>Notation</u>: From this point forward, the vector notation over the symbols will be dropped, as the context will indicate whether or not a quantity is a vector. With this in mind, we acknowledge that x may indeed represent a spatial variable in any number of dimenions (although for physical relevance, $x \in \mathbb{R}^3$; but this restriction is not necessary in our mathematical derivation).

If we assume that the pollutant doesn't react chemically with the surrounding air, then we its total mass is conserved:

$$u_t + \text{div } F(x,t) = 0,$$

where

$$F(x,t) \quad = \quad u(x,t)b.$$

Again, the form of the pollutant density flux is born out of a constitutive assuumption regarding the dispersion of pollutant. It is easy to see that this assumption corresponds to the pollutant simply being carried along by the wind. Combining the above conservation law and constitutive assumption yields

$$u_t + Du \cdot b = 0,$$

which is called the *Transport Equation*. Viewed in a space-time setting, this equation corresponds to the directional derivative of u (as a function of x and t) being zero in the direction $(b,1)$:

$$\underbrace{(Du, u_t) \cdot (b,1)}_{\text{Directional derivative}} = 0. \tag{1.7}$$

Before moving on to another PDE, let us see if we can make any progress in terms of solving (1.7).

1.4.1 Method of Characteristics: A First Look

Expect $u =$ constant on the integral curves. The curve through (x,t): $\gamma(s) = (x+sb, t+s)$

$$\begin{aligned} z(s) &= u(\gamma(s)) = u(x+sb,t+s) \\ \dot{z}(s) &= Du(x+sb,t+s)\cdot b + u_t(x+sb,t+s) \\ &= 0 \end{aligned}$$

If we know the concentration of the pollutant at $t=0$, i.e. we are solving the initial value problem

$$\begin{cases} u_t + Du\cdot b &= \quad 0 \\ u(x,0) &= g(x) \end{cases}$$

$$\begin{aligned} \text{then } u(x,t) &= u(\gamma(0)) \qquad \text{by definition of } \gamma \\ &= u(\gamma(s))\ \forall s. \text{ (because } u \text{ is constant on curve } \gamma(s)) \end{aligned}$$

Thus $u(x,t)$ is the value of g at the intersection of the line $\gamma(s)$ with the plane $\mathbb{R}^n \times \{t=0\}$ for $s=-t$.

$$u(x,t) = u(x-tb,0) = g(x-tb)$$

Remarks: If you want to check that $u(x,t) = g(x-t,b)$

1. is a solution of $u_t + Du\cdot b = 0$, then we need that g is differentiable. The formula also makes sense for g not differentiable, and $u(x,t) = g(x-tb)$ would be a so-called generalized or weak solution.

2. We were solving the PDE by solving the ODE

$$\begin{aligned} \dot{z}(s) &= 0 \\ z(-t) &= u(\gamma(-t)) = u(x-tb,0) \\ &= g(x-tb) \end{aligned}$$

This is a general concept which is called the method of characteristics.

1.4.2 Duhamel's Principle

If you can solve the homogeneous equation $u_t + Du\cdot b = 0$ then you can solve the inhomogeneous equation $u_t + Du\cdot b = f(x,t)$. $f(x,t)$ is the source of the pollutant (e.g. power plant).

Idea: Suppose the pollutant were released only once per hour, then you could trace the evolution of the pollutant by homogeneous equation, and you could add this evolution to the evolution of the initial data. Equation we want to solve

$$\begin{cases} u_t + Du \cdot b &= f \\ u(x,0) &= g(x) \end{cases}$$

Since the sum of two solutions is a solution (since the equation is linear) we can write $u = v + w$, where

$$\begin{cases} v_t + Dv \cdot b &= 0 \\ v(x,0) &= g(x) \end{cases}$$

$$\begin{cases} w_t + Dw \cdot b &= f \\ w(x,0) &= 0 \end{cases}$$

$$\begin{aligned} \text{Check: } u_t + Du \cdot b &= (v+w)_t + D(v+w) \cdot b \\ &= \underbrace{(v_t + Dv \cdot b)}_{=0} + \underbrace{(w_t + Dw \cdot b)}_{f} = f \end{aligned}$$

$$\begin{aligned} u(x,0) &= v(x,0) + w(x,0) \\ &= g(x) + 0 \\ &= g(x) \end{aligned}$$

$$\text{Suspect:} \quad w(x,t) = \int_0^t w(x,t;s)\ ds$$

where

$$\begin{cases} w_t(x,t;s) + Dw(x,t;s) \cdot b = 0 \quad \text{for } x \in \mathbb{R}^n, t > s \\ w(x,s;s) = f(x,s) \end{cases}$$

$$\begin{aligned} \text{Solutions:} \quad v(x,t) &= g(x - tb) \\ w(x,t;s) &= f(x + (s-t)b, s) \end{aligned}$$

$$\text{Expectation:} \quad u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\ ds$$

Theorem 1.1. *Let f and g be smooth. Then the solution of the initial value problem*

$$\begin{aligned} u_t + Du \cdot b &= f \\ u(x,0) &= g \end{aligned}$$

is given by

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\, ds$$

Proof: Differentiate explicitly. ■

1.5 Minimal Surfaces

$D =$ unit disk in the plane.
All surfaces that are graphs of functions on D with a given fixed boundary curve

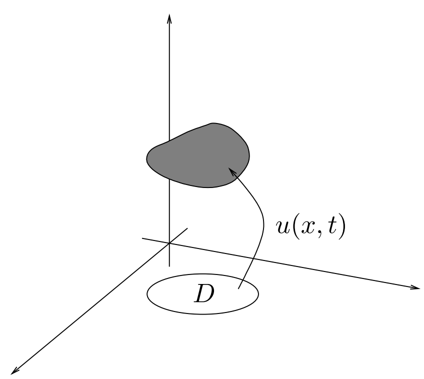


Figure 1.1:

Question: Find a surface with minimal surface are $A(u)$

$$A(u) = \iint_D \sqrt{1 + |Du|^2}$$

Idea: Find a PDE that is a necessary condition for $A(u)$ to be minimal. If you take any ϕ which is zero on ∂D, then $A(u + \phi) \geq A(u)$. Define

$$g(\epsilon) = A(u + \epsilon\phi) \geq A(u)$$

$\forall \epsilon \in \mathbb{R}$, and we have a minimum (as a function of ϵ!) if $\epsilon = 0$. Thus $g'(\epsilon) = 0$

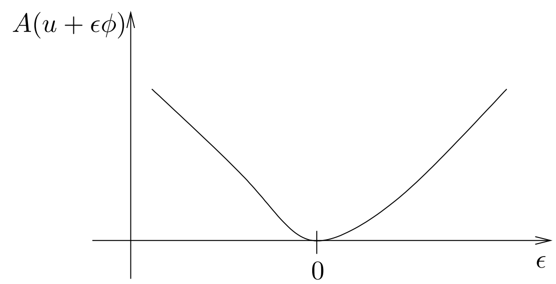


Figure 1.2:

$$
\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} g(\epsilon) = 0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \iint_D \sqrt{1+|D(u+\epsilon\phi)|^2} \\
&= \left.\frac{1}{2}\iint_D \frac{2Du\cdot D\phi + 2\epsilon|D\phi|^2}{\sqrt{1+|Du+\epsilon D\phi|^2}}\, dx\right|_{\epsilon=0} \\
&= \iint_D \frac{Du\cdot D\phi}{\sqrt{1+|Du|^2}}\, dx \\
&= -\iint_D \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\phi\, dx
\end{aligned}
$$

This holds $\forall\phi$

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{Minimal Surface Equation}$$

1.5.1 Divergence Theorem and Integration by Parts

Theorem 1.2 (Divergence Theorem). *$\vec{F}$ vector field domain Ω, smooth boundary $\partial\Omega = S$, outward unit normal $\vec{\nu}(x)$*

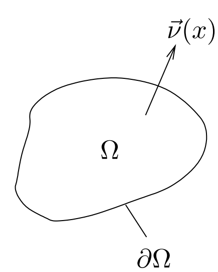


Figure 1.3:

$$\int_\Omega div\ \vec{F}\ dx = \int_{\partial\Omega} \vec{F}\cdot\vec{\nu}\ dS \qquad \textit{Divergence Theorem}$$

Example: Take $\vec{F}(x) = w(x)\vec{v}(x)$
$\vec{v}(x) =$ vector field
$w(x) =$ scalar

$$\begin{aligned}
\text{LHS} &= \int_\Omega \text{div } (w(x)\vec{v}(x))\ dx \\
&= \int_\Omega \sum_{i=1}^n \partial_i(w(x)v_i(x))\ dx \\
&= \int_\Omega \sum_{i=1}^n [(\partial_i w(x))v_i(x) + w(x)(\partial_i v_i(x))]\ dx \\
&= \int_\Omega [Dw\cdot\vec{v} + w(x)\cdot\text{div } v(x)]\ dx \\
\text{RHS} &= \int_{\partial\Omega} (w(x)\vec{v}(x))\cdot\vec{\nu}(x)\ dS \\
&= \int_{\partial\Omega} w(x)\vec{v}(x)\cdot\vec{\nu}(x)\ dS
\end{aligned}$$

Thus, we have

$$\int_\Omega Dw\cdot\vec{v}\ dx = \int_{\partial\Omega} w\vec{v}\cdot\vec{\nu}\ dS - \int_\Omega w\cdot\text{div } \vec{v}\ dx \tag{1.8}$$

Integration by parts

1 dimension: $$\int_a^b w'v\ dx = wv\Big|_a^b - \int_a^b wv'\ dx$$

Applications:

1. $\vec{v}(x) = Du(x)$
   $w(x) = \phi(x)$

$$\begin{aligned}
\int_\Omega D\phi\cdot Du\ dx &= \int_{\partial\Omega} \phi\ Du\cdot\vec{\nu}\ dS - \int_\Omega \phi \text{ div } Du\ dx \\
\text{i.e.} \quad \int_\Omega D\phi\cdot Du\ dx &= \int_{\partial\Omega} \phi\ \partial_\nu u\ dS - \int_\Omega \phi\ \Delta u\ dx
\end{aligned}$$

2. Soap bubbles = minimal surfaces

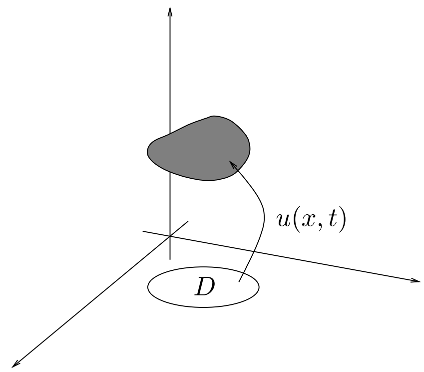


Figure 1.4:

$$\text{surface area} \qquad \int_D \sqrt{1+|Du|^2}\,dx.$$

Minimize.
Assume that u is a minimizer. i.e. that $A(u) \le A(v)$ for all functions v with the same boundary values. Family of functions $u + \epsilon\phi$ where ϕ has zero boundary values. Look at $g(\epsilon) = A(u+\epsilon\phi)$

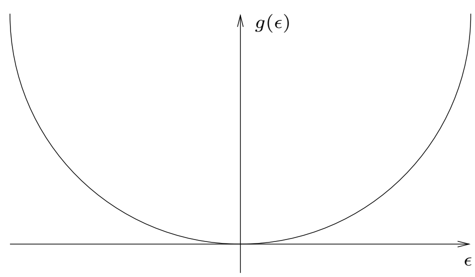


Figure 1.5:

$$\begin{aligned}
\frac{d}{d\epsilon} &= \int_\Omega \frac{d}{d\epsilon}\sqrt{1+|Du+\epsilon D\phi|^2}\,dx \\
&= \int_\Omega \frac{d}{d\epsilon}\sqrt{1+|Du|^2+2\epsilon Du\cdot D\phi+\epsilon^2|D\phi|^2}\,dx
\end{aligned}$$

$$= \frac{1}{2}\int_\Omega \frac{2Du\cdot D\phi + 2\epsilon|D\phi|^2}{\sqrt{1+|Du|^2+2\epsilon Du\cdot D\phi+\epsilon^2|D\phi|^2}}\,dx$$

Now, since u is a minimum,

$$\begin{aligned} \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int \ldots\, dx &= 0 \\ \implies \int_\Omega \frac{Du\cdot D\phi}{\sqrt{1+|Du|^2}}\,dx &= 0 \end{aligned}$$

If we identify

$$w=\phi \qquad \vec{v}(x) = \frac{Du}{\sqrt{1+|Du|^2}},$$

we can apply (1.8):

$$\begin{aligned} \int_\Omega D\phi\cdot\frac{Du}{\sqrt{1+|Du|^2}}\,dx &= \int_{\partial\Omega}\phi\frac{Du}{\sqrt{1+|Du|^2}}\cdot\vec{\nu}\,dS - \int_\Omega \phi\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\,dx \\ &= 0 \end{aligned}$$

The first term on the RHS is 0 since $\phi = 0$ on $\partial\Omega$. Thus,

$$\int_\Omega \phi\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\,dx = 0$$

for all ϕ that are zero on $\partial\Omega$. If u is smooth, then the integrand must be zero pointwise, i.e.

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{Minimal surface equation}$$

1.6 The Wave Equation

[?] Comments on ad hoc derivations (e.g. wave equation).
Outcome of this “incomprehensible” derivation is the wave equation

$$u_{tt} = c^2u_{xx}$$

Newton's law $f = ma$.

1.7 Classification of PDE

1. Diffusion equation (2nd order):

$$u_t = k\Delta u$$

2. Traffic flow (1st order):

$$\rho_t + v_{\max}\partial_x \left(\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)\right) = 0$$

3. Transport equation (1st order):

$$u_t + Du \cdot b = 0$$

4. Minimal surfaces (2nd order):

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

5. Wave equation (2nd order):

$$u_{tt} = c^2 u_{xx}$$

Definition 1.3. *A PDE is an expression of the form*

$$F(D^k u, D^{k-1}u, \ldots, Du, u, x) = 0,$$

$$\begin{array}{rrcl} \textit{where} & Du & = & \textit{gradient of } u \\ & D^2u & = & \textit{all 2nd order derivatives of } u \\ & & \vdots & \\ & D^k u & = & \textit{all kth order derivatives of } u \end{array}$$

k is called the order of the PDE.

Definition 1.4. *A PDE is linear if it's of the form*

$$\sum_{|\alpha|\le k} a_\alpha(x)\ D^\alpha u(x) = 0$$

where α is a multi-index.

Notation: $\alpha = (\alpha_1, \ldots, \alpha_n)$ with α_i non-negative integers. We define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

So,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Example:

- $n = 2$. $\alpha = (0,0)$ corresponds to $D^\alpha u = u$.
  $|\alpha| = 1$ is associated with $\alpha = (1,0)$ or $\alpha = (0,1)$.

$$\alpha = (1,0) : D^\alpha u = \frac{\partial u}{\partial x_1}$$
$$\alpha = (0,1) : D^\alpha u = \frac{\partial u}{\partial x_2}$$

  $|\alpha| = 2$ is associated with $\alpha = (2,0)$, $\alpha = (1,1)$ or $\alpha = (0,2)$.

$$\alpha = (2,0) : D^\alpha u = \frac{\partial^2 u}{\partial x_1^2}$$
$$\alpha = (1,1) : D^\alpha u = \frac{\partial^2 u}{\partial x_1 \partial x_2}$$
$$\alpha = (0,2) : D^\alpha u = \frac{\partial^2 u}{\partial x_2^2}$$

- $n = 3$. Just look at $\alpha = (1,1,2)$ with $u = u(x_1, x_2, x_3)$

$$D^\alpha u = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_3^2} u = \frac{\partial^4 u}{\partial x_1 \partial x_2 \partial x_3^2} u$$

Example: 2nd order linear equation in 2D]

$$\sum_{|\alpha| \le 2} a_\alpha(x)\ D^\alpha u(x)$$

$$\begin{aligned}
= \quad & a_{(2,0)}(x) u_{xx} + a_{(1,1)}(x) u_{x,y} + a_{(0,2)}(x) u_{yy} && |\alpha| = 2 \\
& + a_{(1,0)}(x) u_x + a_{(0,1)}(x) u_y && |\alpha| = 1 \\
& + a_{(0,0)}(x) u && |\alpha| = 0 \\
= \quad & f
\end{aligned}$$

Crucial term is the highest order derivative. Generalizations:

- a PDE is semilinear if it is linear in the highest order derivatives, i.e. it is of the form

$$\sum_{|\alpha|=k} a_\alpha(x)\ D^\alpha u(x) + a_0(D^{k-1}u, \ldots, Du, u, x)$$

- a PDE is quasilinear if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(D_{k-1}u, \ldots, Du, u, x)\ D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x)$$

Example:

- diffusion equation:

$$u_t = k\Delta u \qquad \text{linear}$$

- traffic flow:

$$\rho_t + v_{\max}\partial_x\left(\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)\right) = 0 \qquad \text{quasilinear}$$

- transport equation:

$$u_t + Du \cdot b = 0 \qquad \text{linear}$$

- minimal surface equation:

$$\begin{aligned} \text{dif}\ \frac{Du}{\sqrt{1+|Du|^2}} &= 0 \\ \Longrightarrow \sum \frac{\partial}{\partial x_i}\frac{\partial_{x_i}u}{\sqrt{1+|Du|^2}} &= 0 \quad \text{quasilinear} \end{aligned}$$

- wave equation:

$$u_{tt} = c^2 u_{xx} \qquad \text{linear}$$

1.8 Exercises

1.1: Classify each of the following PDEs as follows:

a) Is the PDE linear, semilinear, quasilinear or fully nonlinear?

b) What is the order of the PDE?

Here is the list of the PDEs:

i) Linear transport equation $u_t + \sum_{i=1}^{n} b_i D_i u = 0$.

ii) Schrodinger's equation $iu_t + \Delta u = 0$.

iii) Beam equation $u_t + u_{xxxx} = 0$.

iv) Eikonal equation $|Du| = 1$.

v) p-Laplacian $\text{div}(|Du|^{p-2}Du) = 0$.

vi) Monge-Ampere equation $\det(D^2 u) = f$.

1.2: Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain representing a heat conducting plate with density $\rho(x,t)$ and specific heat $c(x)$. Find an equation for the evolution of the temperature $u(x,t)$ at the point x at the time t under the assumption that there is a heat source $f(x)$ in Ω. Use Fourier's law of cooling as the constitutive assumption and assume that all functions are sufficiently smooth.

1.3:

a) Verify that the function $u(x,t) = e^{-kt}\sin x$ satisfies the heat equation $u_t = ku_{xx}$ on $-\infty < x < \infty$ and $t > 0$. Here $k > 0$ is the diffusion constant.

b) Verify that the function $v(x,t) = \sin(x - ct)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ on $-\infty < x < \infty$ and $t > 0$. Here $c > 0$ is the wave speed.

c) Sketch u and v. What is the most striking difference between the evolution of the temperature u and the wave v?

1.4: Suppose that u is a smooth function that minimizes the Dirichlet integral

$$I(u) = \frac{1}{2}\int_{\Omega} |Du|^2 \, dx$$

subject to given boundary conditions $u(x) = g(x)$ on $\partial\Omega$. Show that u satisfies Laplace's equation $\Delta u = 0$ in Ω.

1.5: [Evans 2.5 #1] Suppose g is smooth. Write down an explicit formula for the function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

1.6: Show that the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

can in two dimensions be written as

$$u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_x u_y u_{xy} = 0.$$

1.7: Suppose that $u = u(x,y) = v(r)$ is a radially symmetric solution of the minimal surface equation

$$u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_x u_y u_{xy} = 0.$$

Show that v satisfies

$$v_{rr} + \frac{v_r}{r}(1+v_r^2) = 0.$$

Use this result to show that $u(x,y) = \rho \log(r + \sqrt{r^2 - \rho^2})$ is a solution of the minimal surface equation for $r = \sqrt{x^2+y^2} \geq \rho > 0$.

Hint: You can verify this by differentiating the given solution or by solving the ordinary differential equation for v.

1.8: [Evans 2.5 #2] Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if R is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Rx), \quad x \in \mathbb{R}^n,$$

then $\Delta v = 0$.

Chapter 2

Laplace's Equation $\Delta u = 0$ and Poisson's Equation $-\Delta u = f$

2.1 Laplace's Equation in Polar Coordinates and for Radially Symmetric Functions

We say that u is harmonic if $\Delta u = 0$.

Example:
The real and imaginary part of holomorphic functions are harmonic

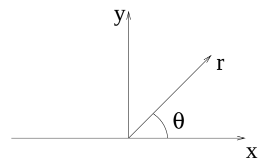
$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$\left.\begin{array}{rcl} u(x,y) & = & x^2 - y^2 \\ u(x,y) & = & 2xy \end{array}\right\} \text{are harmonic}$$

2.2 [

Laplace's Eqn. in Polar Coordinates]Laplace's Equation in Polar Coordinates and for Radially Symmetric Functions

$$\begin{array}{rcl} r & = & \sqrt{x^2+y^2} \\ \theta & = & \tan^{-1}\left(\dfrac{y}{x}\right) \\ u(x,y) & = & v(r,\theta) \end{array}$$



$$
\begin{aligned}
\partial_x f &= \partial_x \sqrt{x^2+y^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{r} \\
\partial_{xx} r &= \partial_x \left(\frac{x}{r}\right) = \frac{1}{r} + x\partial_x\left(\frac{1}{r}\right) = \frac{1}{r} - \frac{x^2}{r^3} \\
\partial_x \frac{1}{r} &= \partial_x (x^2+y^2)^{-1/2} = -\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}^3} = \frac{-x}{r^3} \\
\partial_{xy} r &= \partial_y \left(\frac{x}{r}\right) = \frac{-xy}{r^3}
\end{aligned}
$$

In Summary:

$$
\begin{aligned}
\partial_{xy} &= \frac{x}{r}, \qquad \partial_y r = \frac{y}{r} \\
\partial_{xx} r &= \frac{1}{r} - x^2 r^3 = \frac{y^2}{r^3}, \qquad \partial_{yy} r = \frac{x^2}{r^3} \\
\partial_{xy} r &= \frac{-xy}{r^3}
\end{aligned}
$$

Now, we look at derivatives with respect to θ.

$$
\begin{aligned}
\partial_x \theta &= \partial_x \tan^{-1}\frac{y}{x} = \frac{1}{1+\frac{y^2}{x^2}} - \frac{y}{x^2} = \frac{-y}{x^2+y^2} \\
&= \frac{-y}{r^2} \\
\partial_y \theta &= \frac{x}{r^2} \\
\partial_{xx}\theta &= \partial_x\left(-\frac{y}{r^2}\right) = \partial_x(-y(x^2+y^2)^{-1}) = \frac{2xy}{(x^2+y^2)^2} \\
&= \frac{2xy}{r^4}
\end{aligned}
$$

$$
\begin{aligned}
\partial_{xy}\theta &= \partial_y\left(-\frac{y}{r^2}\right) = -\frac{1}{r^2} - y\partial_y(x^2+y^2)^{-1} = -\frac{1}{r^2} + \frac{2y^2}{(x^2+y^2)^2} \\
&= -\frac{1}{r^2} + \frac{2y^2}{r^4}
\end{aligned}
$$

$$\begin{aligned}
\partial_{yy}\theta &= \partial_y\left(\frac{x}{r^2}\right) = x\partial_y(x^2+y^2)^{-1} \\
&= -\frac{2xy}{r^4}
\end{aligned}$$

In Summary:

$$\begin{aligned}
\partial_x\theta &= -\frac{y}{r^2}, \qquad \partial_y\theta = \frac{x}{r^2} \\
\partial_{xx}\theta &= \frac{2xy}{r^4}, \qquad \partial_{yy}\theta = -\frac{2xy}{r^4} \\
\partial_{xy}\theta &= \frac{y^2-x^2}{r^4}
\end{aligned}$$

Now, we consider

$$\begin{aligned}
u(x,y) &= v(r,\theta) \\
\partial_x u &= v_r r_x + v_\theta \theta_x \\
\partial_{xx} u &= v_{rr}(r_x)^2 + 2v_{r\theta}r_x\theta_x + v_{\theta\theta}(\theta_y)^2 + v_r r_{yy} + v_\theta\theta_{yy}
\end{aligned}$$

$$\begin{aligned}
\partial_y u &= v_r r_y + v_\theta \theta_y \\
\partial_{yy} u &= v_{rr}(r_y)^2 + 2v_{r\theta}r_y\theta_y + v_{\theta\theta}(\theta_y)^2 + v_r r_{yy} + v_\theta\theta_{yy}
\end{aligned}$$

$$\begin{aligned}
\Delta u &= u_{xx} + u_{yy} \\
&= v_{rr}(r_x^2 + r_y^2) + 2v_{r\theta}(r_x\theta_x + r_y\theta_y) + v_{\theta\theta}(\theta_x^2+\theta_y^2) \\
&\quad + v_r(r_{xx}+r_{yy}) + v\theta(\theta_{xx}+\theta_{yy}) \\
&= 0 \\
&= v_{rr}\left(\frac{x^2+y^2}{r^2}\right) + v_{\theta\theta}\left(\frac{x^2+y^2}{r^4}\right) \\
&\quad + v_r\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3}\right)
\end{aligned}$$

<u>Result:</u> If $u(x,y) = v(r,\theta)$ and u solves Laplace's equation, then v solves

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$$

Laplace's equation in polar coordinates

Now we consider Laplace's equation for radially symmetric functions:

$$u(x) = v(r), \qquad r = \sqrt{x_1^2 + \ldots + x_n^2}$$

$$
\begin{aligned}
\frac{\partial u}{\partial x_i} &= v_r(r)\frac{\partial r}{\partial x_i} = v_r(r)\frac{x_i}{r} \\
\frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i}\left(v_r(r)\frac{x_i}{r}\right) \\
&= v_{rr}\frac{x_i^2}{r^2} + v_r\left(\frac{1}{r} + x_i \partial_{x_i}\frac{1}{r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{x_i}\frac{1}{r} &= \partial_{x_i}(x_1^2 + \ldots + x_n^2)^{-1/2} = -\frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-3/2} 2x_i = -\frac{x_i}{r^3} \\
&= v_{rr}\frac{x_i^2}{r^2} + v_r\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta u &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u \\
&= \sum_{i=1}^{n} v_{rr}\frac{x_i^2}{r^2} + v_r\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \\
&= v_{rr} + \left(\frac{n}{r} - \frac{1}{r}\right) v_r
\end{aligned}
$$

Result: If $u(x) = v(r)$ is a radially symmetric solution of Laplace's equation then

$$
v_{rr} + \frac{n-1}{r} v_r = 0 \tag{2.1}
$$

2.3 The Fundamental Solution of the Laplacian

Special solution radially symmetric by solving (1) of the previous section. Assuming $v_r \neq 0$

$$
\begin{aligned}
\frac{v_{rr}}{v_r} &= \frac{1-n}{r} \qquad \text{integrate in } r \\
\Longrightarrow \ln v_r &= (1-n)\ln r + C_1 \\
\Longrightarrow v_r &= C_2 \cdot r^{1-n}
\end{aligned}
$$

Integrate again:

$$
\begin{aligned}
n = 2: &\quad v_r = \frac{C_2}{r} \Longrightarrow v = C_2 \cdot \ln r + C_3 \\
n \geq 3: &\quad v_r = \frac{C_2}{r^{n-1}} \Longrightarrow v = \frac{-1}{n-2}\frac{C_2}{r^{n-2}} + C_3
\end{aligned}
$$

Find special solutions

$$u(x) = v(r) = \begin{cases} C_2 \cdot \ln|x| + C_3 & n = 2 \\ -\frac{C_2}{n-2}\frac{1}{|x|^{n-2}} + C_3 & n \geq 3 \end{cases}$$

The fundamental solution is of this form with particular constants.

Definition 2.1 (Fundamental Solution).

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \cdot \ln|x| & n = 2 \\ -\frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

for $x \neq 0$, and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Note: $\Delta\Phi = 0$ for $x \neq 0$ but not defined if $x = 0$. $\Delta\Phi$ can be defined everywhere in the sense of distributions,

$$-\Delta\Phi = \delta_0 = \text{"Dirac Mass"}$$

placed at $x = 0$,

$$\begin{aligned} \delta_0(x) &= \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \\ -\Delta\Phi(y-x) &= \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases} \end{aligned}$$

Formally, this suggests a solution formula for Poisson's equation,

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n$$

namely,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy$$

Why?

$$\begin{aligned} \text{``} - \Delta u(x) &= \int_{\mathbb{R}^n} -\Delta_x\Phi(x-y)f(y)\, dy \\ &= \int_{\mathbb{R}^n} \delta_x(y)f(y)\, dy = f(x)\text{``} \end{aligned}$$

Lemma 2.2. *We have*

$$\begin{aligned} |D\Phi(x)| &\leq \frac{C}{|x|^{n-1}} \qquad x \neq 0 \\ |D^2\Phi(x)| &\leq \frac{C}{|x|^n} \qquad x \neq 0 \end{aligned}$$

$$\textit{and} \quad \frac{\partial \Phi}{\partial \nu}(x) = \frac{-1}{n\alpha(n)} \frac{1}{r^{n-1}} \qquad \forall x \in \partial B(0,r)$$

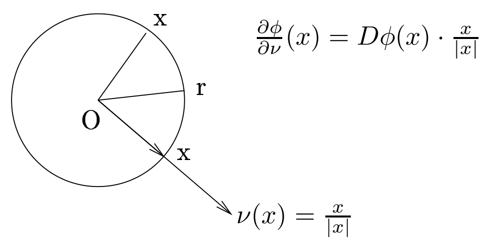


Figure 2.1:

Proof:

$$\begin{aligned} n = 2: \qquad \frac{\partial}{\partial x_i} \ln|x| &= \frac{1}{|x|}\frac{\partial}{\partial x_i}|x| = \frac{1}{|x|}\frac{x_i}{|x|} \\ D\Phi(x) &= \frac{-2}{2\pi}\frac{x}{|x|^2}, \qquad |D\Phi| \leq C\frac{1}{|x|} \quad \surd \end{aligned}$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \ln|x| = \frac{\partial}{\partial x_j}\frac{x_i}{|x|^2} = \frac{\delta_{ij}}{|x|^2} + x_i \frac{\partial}{\partial x_j}\frac{1}{|x|^2}$$

$$\begin{aligned} \frac{\partial}{\partial x_j}\frac{1}{|x|^2} &= \frac{\partial}{\partial x_j}\left(\sum x_k^2\right)^{-1} \\ &= -\left(\sum x_k^2\right)^{-2} 2x_j = -\frac{2x_j}{|x|^4} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} &= -\frac{1}{2\pi}\left(\frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4}\right) \\ \left|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\right| &\leq -\frac{1}{2\pi}\left(\frac{1}{|x|^2} - \frac{2|x_i||x_j|}{|x|^4}\right) \\ &\leq \frac{1}{2\pi}\frac{3}{|x|^2} \end{aligned}$$

$$\begin{aligned}
D\Phi(x) &= -\frac{1}{2\pi} D \ln|x| = -\frac{1}{2\pi}\frac{1}{|x|}\frac{x}{|x|} \\
\frac{\partial \Phi}{\partial \nu}(x) &= -\frac{1}{2\pi}\frac{x}{|x|^2}\frac{x}{|x|} \\
&= -\frac{1}{2\pi}\frac{|x|^2}{|x|^2} = -\frac{1}{2\pi}\frac{1}{|x|}
\end{aligned}$$

$$n \geq 3: \qquad \begin{aligned}
\frac{\partial}{\partial x_i}\frac{1}{|x|^{n-2}} &= \frac{\partial}{\partial x_i}\left(\sum x_k^2\right)^{-\frac{n-2}{2}} \\
&= -\frac{n-2}{2}\left(\sum x_k^2\right)^{-\left(\frac{n-2}{2}+1\right)} 2x_i \\
&= -(n-2)\frac{x_i}{|x|^n}
\end{aligned}$$

$$\begin{aligned}
D\Phi(x) &= \frac{-1}{n\alpha(n)}\frac{x}{|x|^n} \\
|D\Phi(x)| &\leq \frac{1}{n\alpha(n)}\frac{|x|}{|x|^n} = \frac{1}{n\alpha(n)}\frac{1}{|x|^{n-1}}
\end{aligned}$$

Second derivatives are similar

$$\begin{aligned}
\frac{\partial \Phi}{\partial \nu}(x) &= \frac{-1}{n\alpha(n)}\frac{x}{|x|^n}\frac{x}{|x|} \\
&= \frac{-1}{n\alpha(n)}\frac{|x|^2}{|x|^{n+1}} \qquad \blacksquare
\end{aligned}$$

Remark: $|D^2\Phi| \leq \frac{C}{|x|^n}$
i.e. $D^2\Phi$ are not integrable about zero.

Theorem 2.3. *Solution of Poisson's equation in $\mathbb{R}^n$. Suppose that $f \in C^2$ and that f has compact support, i.e. closure of $(\{x \in \mathbb{R}^n : f(x) \neq 0\}) \subseteq B(0,s)$ for some $s > 0$. Then*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$$

is twice differentiable and solves Poisson's equations $-\Delta u = f$ in $\mathbb{R}^n$.

Proof: Change variables: $z = x - y$

1. $y = x - z$ and $dy = dz$, i.e.,

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy &= \int_{\mathbb{R}^n} \Phi(z)f(x-z)\ dz \\ &= \int_{\mathbb{R}^n} \Phi(y)f(x-y)\ dy = u(x)\end{aligned}$$

2. u is differentiable

$$\lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h} = \lim_{h\to 0} \int_{\mathbb{R}^n} \Phi(y)\frac{f(x+he_i-y)-f(x-y)}{h}\ dy$$

Technical Step: If we can interchange $\lim_{h\to 0}$ and integration, then ($f \in C^2$):

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial f}{\partial x_i}(x-y)\ dy$$

and by the same arguments,

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j} &= \int_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\ dy \\ \text{and} \qquad \Delta u &= \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)\ dy \qquad \blacksquare\end{aligned}$$

So, what was the "black box" that enables use to switch the integration and limit?

Black Box: Lebesgue's Dominated Convergence Theorem: f_k be a sequence of integrable functions such that $|f_k| \le g$ almost everywhere, g integrable $f_k \to f$ pointwise almost everywhere. then

$$\lim_{k\to\infty} \int_\Omega f_k = \int_\Omega \lim_{k\to\infty} f_k = \int_\Omega f$$

Our application: Need

$$\left|\Phi(y)\frac{f(x+he_i-y)-f(x-y)}{h}\right| \le g$$

$$g(y) = |\Phi(y)| \underbrace{\max_{z\in\mathbb{R}^n}\left|\frac{\partial f}{\partial x_i}(z)\right|}_{\le M}$$

is integrable since f and hence $\frac{\partial f}{\partial x_i}$ have compact support.

$$\begin{aligned}
\Delta u &= \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)\ dy \\
&= \int_{\mathbb{R}^n} \Phi(y)\Delta_y f(x-y)\ dy \\
&= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\ dy + \int_{B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\ dy \\
&= I_1 + I_2
\end{aligned}$$

Now, we have

$$|I_2| \le \max_{\mathbb{R}^n} |D^2 f| \int_{B(0,\epsilon)} |\Phi(y)|\ dy$$

$\underline{n=2:}$

$$\begin{aligned}
\int_{B(0,\epsilon)} \ln|x|\ dx &= \int_0^{2\pi}\int_0^{\epsilon} \ln r \cdot r dr d\theta \\
&= 2\pi \int_0^{\epsilon} r\ln r\ dr \\
&= \int_0^{\epsilon} \left(\frac{1}{2}r'\right)' \ln r\ dr \\
&= \frac{1}{2}r^2\ln r\Big|_0^{\epsilon} - \int_0^{\epsilon} \frac{1}{2}r^2\frac{1}{r}\ dr \\
&= \frac{1}{2}\epsilon^2\ln\epsilon - \frac{1}{4}r^2\Big|_0^{\epsilon} \\
&= \frac{1}{2}\epsilon\ln\epsilon - \frac{1}{4}\epsilon^2
\end{aligned}$$

$$\therefore |I_2| \text{ is } O(\epsilon\ln\epsilon)$$

$\underline{n=3:}$

$$\begin{aligned}
\int_{B(0,\epsilon)} \frac{1}{|x|^{n-2}}\ dx &= \int_0^{\epsilon}\int_{\partial B(0,\epsilon)} \frac{1}{|r|^{n-2}}\ dSdr \\
&= \int_0^{\epsilon} \frac{1}{r^{n-2}} C r^{n-1}\ dr \\
&= C\int_0^{\epsilon} r\ dr \\
&= \frac{C}{2}\epsilon^2
\end{aligned}$$

$$
\begin{aligned}
\therefore |I_2| &\le \max_{\mathbb{R}^n} |D^2 f| \int_{B(0,\epsilon)} |\Phi(y)|\, dy \\
&= \begin{cases} O(\epsilon \ln \epsilon) & n = 2 \\ O(\epsilon^2) & n \ge 3 \end{cases}
\end{aligned}
$$

In particular: $I_2 \to 0$ as $\epsilon \to 0$

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_y f(x-y)\, dy \\
&= \int_{\partial B(0,\epsilon)} \Phi(y) Df(x-y) \cdot \nu\, dS - \int_{\mathbb{R}^n \setminus B(0,\epsilon)} D\Phi(y) Df(x-y)\, dy \\
&= J_1 + J_2
\end{aligned}
$$

Now,

$$
\begin{aligned}
|J_1| &= \left| \int_{\partial B(0,\epsilon)} \Phi(y) Df(x-y) \cdot \nu\, dS \right| \\
&\le \max |Df| \int_{\partial B(0,\epsilon)} \Phi(y)\, dS(y)
\end{aligned}
$$

$$
\begin{aligned}
&\underline{n = 2:} && \epsilon \ln \epsilon \to 0 \text{ as } \epsilon \to 0 \\
&\underline{n \ge 3:} && \frac{C}{\epsilon^{n-2}} \epsilon^{n-1} = \epsilon \to 0 \text{ as } \epsilon \to 0
\end{aligned}
$$

So we are left to look at

$$
\begin{aligned}
J_2 &= -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} D\Phi(y) Df(x-y)\, dy \\
&= -\int_{\partial B(0,\epsilon)} D\Phi(y) \cdot \nu_{\text{in}}(y) f(x-y)\, dS + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \underbrace{\Delta\Phi(y)}_{=0} f(x-y)\, dy \\
&= \int_{\partial B(0,\epsilon)} D\Phi(y) \cdot \nu_{\text{out}}(y) f(x-y)\, dS \\
&= \int_{\partial B(0,\epsilon)} \frac{-1}{|\partial B(0,\epsilon)|} f(x-y)\, dS \\
&= \fint_{\partial B(0,\epsilon)} -f(x-y)\, dS(y) \to -f(x) \qquad \text{as} \qquad \epsilon \to 0
\end{aligned}
$$

Notation: $\fint_u f = |u|^{-1} \int_u f$, where $|u|$ is the measure of u.

Finally,

$$\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y)\Delta f(x-y)\,dy \\
&= \text{terms tending to zero as } \epsilon \to 0 \\
&\quad -\fint_{B(0,\epsilon)} f(x-y)\,dS \to -f(x) \quad \text{as} \quad \epsilon \to 0 \\
\implies -\Delta u(s) &= f(x)
\end{aligned}$$

2.4 Properties of Harmonic Functions

2.4.1 WMP for Laplace's Equation

from AMSI summer notes, needs to be integrated properly into the material

To begin this section, we introduce the *Laplacian* differential operator:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \ u \in C^2(\Omega).$$

Note that Δu is sometimes written as $\text{div} \cdot \nabla u$ in classical notation. Coupled to this we have the following definition

Definition 2.4. *$u \in C^2(\Omega)$ is called* harmonic *in Ω if and only if $\Delta u = 0$ in Ω.*

Example 2.1. *$u(x_1, x_2) = x_1^2 - x_2^2$ is a nonlinear harmonic function in $\mathbb{R}^2$.*

In addition to the above, we have the next definition.

Definition 2.5. *$u \in C^2(\Omega)$ is called* subharmonic*(*superharmonic*) if and only if $\Delta u \geq 0 (\leq 0)$ in Ω.*

Remark: It is obvious, due to the linearity of the Laplacian, that if u is subharmonic, then $-u$ is superharmonic and vice-versa.

Example 2.2. *If $x \in \mathbb{R}^n$, then $\Delta|x|^2 = 2n$. Thus, $|x|^2$ is subharmonic in $\mathbb{R}^n$.*

Now, we come to our first theorem

Theorem 2.6 (Weak Maximum Principle for the Laplacian). *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with Ω bounded in $\mathbb{R}^n$. If $\Delta u \geq 0$ in Ω, then*

$$u \leq \max_{\partial\Omega} u \ \left(\iff \max_{\overline{\Omega}} u = \max_{\partial\Omega} u \right). \tag{2.2}$$

Conversely, if $\Delta u \leq 0$ in Ω, then

$$u \geq \min_{\partial\Omega} u \quad \left(\iff \min_{\overline{\Omega}} u = \min_{\partial\Omega} u \right). \tag{2.3}$$

Consequently, if $\Delta u = 0$ in Ω, then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u. \tag{2.4}$$

Proof: We will just prove the subharmonic result, as the superharmonic case is proved the same way. Given u subharmonic, take $\epsilon > 0$ and define $v = u + \epsilon|x|^2$, so that $\Delta v = \Delta u + 2\epsilon n > 0$ in Ω. If v takes attains a maximum in Ω, we have $\Delta v \leq 0$ at that point, a contradiction. Thus, by the compactness of Ω, we have

$$v \leq \max_{\partial\Omega} v.$$

Finally, we take $\epsilon \to 0$ in the above to get the result. ■

Corollary 2.7 (Uniqueness). *Take $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If $\Delta u = \Delta v = 0$ in Ω and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Before moving onto more general elliptic equations, we briefly remind ourselves of the *Dirichlet Problem*, which takes the following form.

$$\begin{aligned} &\Delta u = 0 \text{ in } \Omega \\ &u = \phi \text{ on } \partial\Omega \text{ for some } \phi \in C^0(\partial\Omega) \end{aligned}$$

2.4.2 Poisson Integral Formula

The best possible scenario one can hope for when dealing in theoretical PDE is to actually have an analytic solution to the equation. Obviously, it is easier to analyze a solution directly rather than having to do so through its properties. Fortunately, this turns out to be possible for the Dirichlet problem for Laplace's equation in a ball.

To start off, let us consider some other examples of harmonic functions:

- Since Laplace's equation is a constant coefficient equation, we know that if $u \in C^3(\Omega)$ is harmonic, then so is all its partial derivatives. Indeed,

$$D_i(\Delta u) = 0 \iff \Delta(D_i u) = 0.$$

- There are also radially symmetric solutions to Laplace's equation. To derive these, let $r = |x|$ and $u(x) = g(r)$. We calculate

$$\begin{aligned} D_i u &= g' \cdot \frac{x_i}{|x|} = \frac{g'}{r} \cdot x_i \\ \Delta u &= D_{ii}u = n \cdot \frac{g'}{r} + \frac{g''}{r^2} \cdot x_i^2 - \frac{g'}{r^3} x_i^2 \\ &= n \cdot \frac{g'}{r} + g'' - \frac{g'}{r} \\ &= (n-1) \cdot \frac{g'}{r} + g'' \\ &= \frac{(r^{n-1}g')'}{r^{n-1}} = 0. \end{aligned}$$

  This implies that $r^{n-1}g'(r) =$constant; integration then yields

$$g = \begin{cases} C_1 r^{2-n} + C_2 & n > 2 \\ C_1 \ln r + C_2 & n = 2 \end{cases}.$$

  These are all the radial solution of the homogeneous Laplace's equation outside of the origin. For simplicity we will take $C_1 = 1$ and $C_2 = 0$ in the above.

- It is easily verified that translating the above solutions so that y is taken to be the new origin, are also harmonic functions:

$$\begin{cases} |x-y|^{2-n} & n > 2 \\ C_1 \ln |x-y| & n = 2 \end{cases}.$$

  Since these functions are symmetric in x and y, it's obvious the above are harmonic with respect to either x or y provided $x \neq y$. Taking a derivative of the above indicates that

$$\frac{x_i - y_i}{|x_i - y_i|^n}$$

  is also harmonic. Next consider

$$\begin{aligned} V(x,y) &= \frac{|y|^2 - |x|^2}{|x-y|^n} = \frac{|y|^2 + |x|^2 - 2xy}{|x-y|^n} - \frac{2x(x-y)}{|x-y|^n} \\ &= \frac{1}{|x-y|^{n-2}} - \frac{2x_i(x_i - y_i)}{|x-y|^n}. \end{aligned}$$

  The last term is harmonic since it is the derivative with respect to y_i of $|x-y|^{2-n}$. Thus, $v(x,y)$ is harmonic in both x and y variables.

It would be difficult to verify the above aforementioned functions solved Laplace's equation, but they are readily constructed from simpler solutions.

Now, let us consider $|y| = R$ and define

$$\tilde{v}(x) = \int_{|y|=R} V(x,y)\, dS(y).$$

First, it is clear that $\tilde{v}(x)$ is harmonic for $x \in B_R(0)$. Indeed for any fixed x in the $B_R(0)$, $V(x,y)$ is bounded. So we may interchange the Laplacian with the integral to get that $\tilde{v}(x)$ is harmonic in $B_R(0)$. Next, we claim that $\tilde{v}(x)$ is radial, i.e. only depends on $|x|$. To show this we consider an arbitrary rotation transformation on $\tilde{v}(x)$. Sometimes this is called an orthonormal transformation as it corresponds to the transformation of $x = Px'$, where P is an orthonormal matrix (rows/columns are orthogonal basis in $\mathbb{R}^n$, $P^T = P^{-1}$ is another property). So, given the transformation, we have

$$\begin{aligned} \tilde{v}(Px) &= \int_{|y|=R} \frac{|y|^2 - |Px|^2}{|Px - y|^n}\, dS(y) \\ &= \int_{|y|=R} \frac{R^2 - |Px|^2}{|Px - y|^n}\, dS(y). \end{aligned}$$

Now, we change variables with $Pz = y$:

$$\begin{aligned} \tilde{v}(Px) &= \int_{|Pz|=R} \frac{R^2 - |Px|^2}{|Px - Pz|^n}\, dS(Pz) \\ &= \int_{|z|=R} \frac{R^2 - |x|^2}{|P(x - z)|^n} \det(P)\, dS(z) \\ &= \int_{|z|=R} \frac{R^2 - |x|^2}{|x - z|^n}\, dS(z). \end{aligned}$$

In the above, we have used the fact that rotations obviously do not change vector magnitude. Also, we know $\det(P) = 1$ as $1 = \det(I) = \det(PP^{-1}) = \det(P)\det(P^{-1}) = \det(P)\det(P^T) = \det(P)^2$. So, $\tilde{v}(x)$ is rotationally invariant. Thus, we calculate

$$\tilde{v}(0) = \int_{|y|=R} R^{2-n}\, dS(y) = R^{2-n} A(S_R(0)) = n\omega_n R.$$

It turns out the above result is valid for any $x \in B_R(0)$, this can be verified by carrying out the integration; but this is rather complicated.

Now, we can make the following definition

Definition 2.8. *The* Poisson Kernel*:*

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R} \frac{1}{|x-y|^n},$$

where $|y| = R$.

From the above, we already know that $K(x,y)$ is harmonic in $B_R(0)$ with respect to x. From the calculation for $\tilde{v}(x)$, we also have

$$\int_{|y|=R} K(x,y)\, dS(y) = 1$$

by construction. Next, we have

Definition 2.9. *The* Poisson Integral *for* $x \in B_R(0)$ *is*

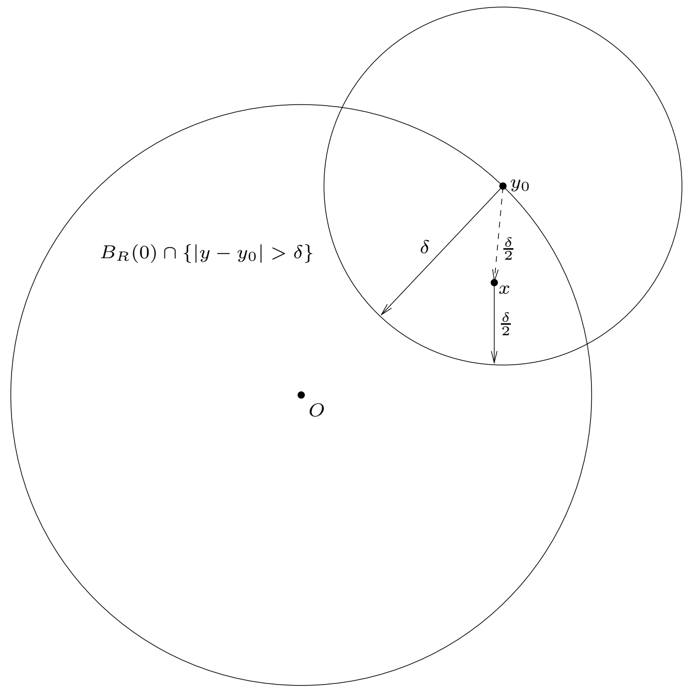
$$w(x) = \int_{|y|=R} K(x,y)\phi(y)\, dS(y).$$

Theorem 2.10. *(Properties of the Poisson Integral) Given the above definition,* $w \in C^\infty(B_R(0)) \cap C^0(\overline{B_R(0)})$ *with* $\Delta w = 0$ *in* $B_R(0)$*. Also* $w(x) \to \phi(y)$ *as* $x \to y \in \partial B_R(0)$.

Remark: Basically one has that w solves the Dirichlet problem for Laplace's equation:

$$\begin{aligned} \Delta w = 0 &\quad \text{in} \quad B_R(0) \\ w = \phi &\quad \text{on} \quad \partial B_R(0) \end{aligned}$$

Proof: $w \in C^\infty(B_R(0))$ is obvious since for fixed $x \in B_R(0)$, $K(x,y)$ is bounded and harmonic. Hence on may take derivatives of any order inside the integral (the derivatives will also be well behaved as $x \neq y$ for our fixed x). So, all we need to show is that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. Take an arbitrary $\epsilon > 0$ and choose δ such that $|\phi(y) - \phi(y_0)| < \epsilon$ implies



$|y - y_0| < \delta$. Keeping in mind the figure, we calculate:

$$
\begin{aligned}
|w(x) - \phi(y_0)| &= \left| \int_{\partial B_R(0)} K(x,y)(\phi(y) - \phi(y_0))\, dS(y) \right| \\
&\le \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\qquad + \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\le \epsilon \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)|\, dS(y) \\
&\qquad + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} \frac{R^2 - |x|^2}{|x-y|^n}\, dS(y).
\end{aligned}
$$

Now, if one chooses x such that $|x - y_0| \le \frac{\delta}{2}$, this implies that $|x - y| \ge \frac{\delta}{2}$ for $y \in \partial B_R(0) \cap \{|y - y_0| > \delta\}$. Thus, we have

$$
\begin{aligned}
|w(x) - \phi(y_0)| &\le \epsilon + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \frac{R^2 - |x|^2}{\frac{\delta}{2}^n} \\
&\le 2\epsilon,
\end{aligned}
$$

where we have chosen x so that $\cdot \max_{\partial B_R(0)} \phi \cdot \frac{R^2-|x|^2}{\frac{\delta}{2}^n} < \frac{\epsilon}{2}$. The last inequality comes from simply picking x close enough to y_0 (as this happens $R^2 - |x|^2$ clearly shrinks). So, we have now shown that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. This also implies the continuity of $w(x)$ on the closure of Ω. ■

A very important consequence to the last theorem is that if $u \in C^2(\Omega)$ is harmonic, then for any ball $B_R(z)$, one has

$$u(x) = \frac{R^2 - |x-z|^2}{n\omega_n R} \int_{\partial B_R(z)} \frac{u(y)}{|x-y|^n}\, dS(y).$$

In other words, the value of a harmonic function at any point is completely determined by the values it takes on any given spherical shell surrounding that point! Taking, $x = z$ in the above, we get the following:

Theorem 2.11. *(Mean Value Property) If $u \in C^2(\Omega)$ and is harmonic on Ω, then one has*

$$\begin{aligned} u(z) &= \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(z)} u(y)\, dS(y) \\ &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y), \end{aligned} \tag{2.5}$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

In addition, one has

Corollary 2.12. *If $u \in C^2(\Omega)$ and is subharmonic(superharmonic) on Ω, then one has*

$$u(z) \le (\ge) \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y), \tag{2.6}$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

Proof: This is a simple matter of solving the following Dirichlet Problem:

$$\begin{cases} \Delta v = 0 & \text{in} \quad B_R(z) \\ v = u & \text{on} \quad \partial B_R(z) \end{cases}$$

First, we know that $\Delta(u - v) \ge (\le)\, 0$. Thus, the weak maximum principle states that $u \le (\ge)\, v$ in $\partial B_R(z)$. So, putting everything together, one has

$$\begin{aligned} u \le (\ge)\, v &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} v(y)\, dS(y) \\ &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y). \qquad ■ \end{aligned}$$

2.4.3 Strong Maximum Principle

Theorem 2.13 (Strong Maximum Principle). *If $u \in C^2(\Omega)$ with Ω connected and $\Delta u \geq (\leq)\, 0$, then u can not take on interior maximum(minimum) in Ω, unless u is constant.*

Proof: Suppose $u(y) = M := \max_\Omega u$. Clearly, one has

$$\Delta(M - u) \leq (\geq)\, 0.$$

Thus, by the corollary to the mean value property,

$$(M-u)(y) \geq (\leq)\, \frac{1}{n\omega_n R} \int_{\partial B_R(y)} M - u(x)\, dS(x)$$

for any $B_R(y) \subset \Omega$. This implies that $u = M$ on $B_R(y)$, which in turn implies $u = M$ on Ω as Ω is connected. ■

Problem: Take $\Omega \subset \mathbb{R}^n$ bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If

$$\begin{aligned} \Delta u = u^3 - u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

Prove $-1 \leq u \leq 1$. Can u take either value ± 1?

2.5 Estimates for Derivatives of Harmonic Functions

Now, we will go through estimating the derivatives of harmonic functions. First, note that the mean value property immediately implies that

$$u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy$$

for any $B_R(x) \subset \Omega$ and u harmonic. This can simply be ascertained from preforming a radial integration on the statement of the mean value property. As we have shown derivatives to be harmonic, we have that

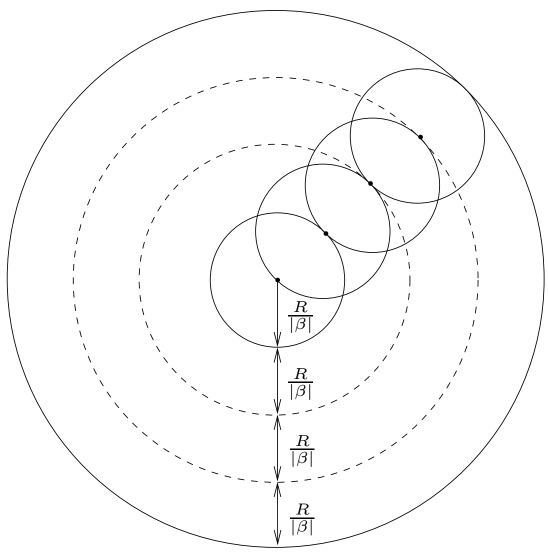
$$D_i u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} \operatorname{div} u^{,i}\, dy,$$

where $u^{,i}$ is a vector whose ith place is u and whose other components are zero. So applying the divergence theorem to the above, we have

$$\begin{aligned} D_i u(x) &= \frac{1}{\omega_n R^n} \int_{\partial B_R(x)} u^{,i} \cdot \nu \, dS(y) \\ &\leq \frac{1}{\omega_n R^n} \sup_{B_R(x)} |u| \cdot \int_{\partial B_R(x)} dS(y) \\ &= \frac{n}{R} \sup_{\partial B_R(x)} |u|. \end{aligned}$$

From this we see that any derivative is bounded by the final term in the above, thus

$$|Du(x)| \leq \frac{n}{R} \sup_{B_R(x)} |u|.$$



Applying this result iteratively over concentric balls whose radii increase by $R/|\beta|$ for each iteration, yields the following (refer to the figure):

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{R} \right)^{|\beta|} \sup_{B_R(x)} |u|.$$

From this equation, the following is clear.

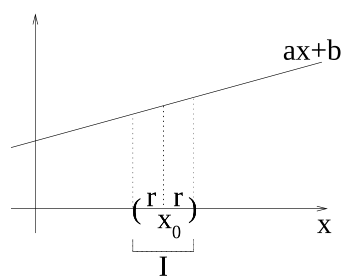
Theorem 2.14. *Consider Ω bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is harmonic in Ω, then for any $\Omega' \Subset \Omega$ the following estimate holds*

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{d_{\Omega'}}\right)^{|\beta|} \sup_{\overline{\Omega}} |u| \quad \forall x \in \Omega', \tag{2.7}$$

where $d_{\Omega'} = \mathrm{dist}(\Omega', \partial\Omega)$.

2.5.1 Mean-Value and Maximum Principles

$n = 1$: $u'' = 0$, $u(x) = ax + b$. u is affine.



$$\begin{aligned} u(x_0) &= \frac{1}{2}(u(x_0 + r) + u(x_0 - r)) \\ &= ⨍_{\partial I} u(y)\ dS(y) \\ u(x_0) &= ⨍_I u(y)\ dy \end{aligned}$$

Theorem 2.15 (Mean value property for harmonic functions). *Suppose that u is harmonic on U, and that $B(x,r) \subset U$. Then*

$$\begin{aligned} u(x) &= ⨍_{\partial B(x,r)} u(y)\ dS(y) \\ &= ⨍_{B(x,r)} u(y)\ dy \end{aligned}$$

<u>Proof:</u>

Idea: $\Phi(r) = \fint_{\partial B(x,r)} u(y)\ dS(y)$; Prove that $\Phi'(r) = 0$.

$$\begin{aligned}
\frac{d}{dr}\Phi(r) &= \frac{d}{dr}\fint_{\partial B(x,r)} u(y)\ dS(y) \\
&= \frac{d}{dr}\fint_{\partial B(0,1)} u(x+rz)\ dS(z) \\
&= \fint_{\partial B(0,1)} Du(x+rz)\cdot z\ dS(z) \\
&= \fint_{\partial B(x,r)} Du(y)\cdot\frac{y-x}{r}\ dS(y) \\
&= \fint_{\partial B(x,r)} Du(y)\cdot\nu(y)\ dS(y) \quad \text{(See below figure)} \\
&= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \underbrace{\text{div } Du(y)}_{\Delta u = 0}\ dy \\
\therefore \Phi'(r) &= 0
\end{aligned}$$

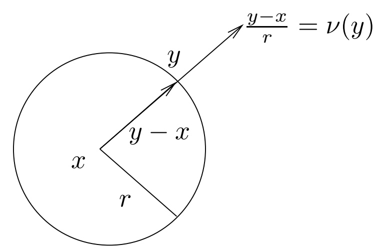


Figure 2.2:

Theorem 2.16 (Maximum principle for harmonic functions). *u harmonic ($u \in C^2(\Omega)\cap C(\overline{\Omega}), \Delta u = 0$) in open bounded Ω. Then*

$$\max_{u\in\overline{\Omega}} u = \max_{\partial\Omega} u$$

Moreover, if Ω is connected and if the maximum of u is obtained in Ω, then u is constant.

Remark:

- The second statement is also referred to as the strong maximum principle.
- The assertion holds also for the minimum of u, replace u by $-u$.

Proof: We prove the strong maximum principle.
Suppose that $M = \max_{\overline{\Omega}} u$ is attained at $x_0 \in \Omega$, then exists a ball $B(x_0, r) \subset \Omega$, $r > 0$. Mean-value property:

$$\begin{aligned} M = u(x_0) &= \fint_{B(x_0,r)} u(y)\, dy \\ &\leq \fint_{B(x_0,r)} M\, dy = M \end{aligned}$$

$\Longrightarrow$ we have equality.
$\Longrightarrow u \equiv M$ on $B(x_0, r)$
If $u(z) < M$ for a point $z \in B(x_0, r)$, then we get a strict inequality.
$\mathcal{M} = \{x \in \Omega, u(x) = M\}$
$\mathcal{M}$ is a closed set in Ω since u is constant; $\mathcal{M}$ is also open in Ω since it can be viewed as the union of open balls.
$\Longrightarrow \mathcal{M} = \Omega$, u is constant in Ω. ■

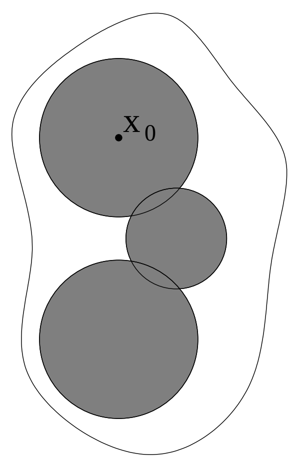


Figure 2.3:

Application of maximum principles: uniqueness of solutions of PDEs.

Theorem 2.17. *There is at most one smooth solution of the boundary-value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad\; u = g & \text{on } \partial\Omega \end{cases}$$

Proof: Suppose u_1 and u_2 are solutions, take $u_1 = u_2$

$$\begin{aligned} \max_{\overline{\Omega}}(u_1 - u_2) &= \max_{\partial\Omega}(u_1 - u_2) = 0 \\ \implies u_1 - u_2 &= \text{ in } \Omega \end{aligned}$$

Same argument for $-(u_1 - u_2)$ gives

$$\begin{aligned} & -(u_1 - u_2) \leq \text{ in } \Omega \\ \implies & |u_1 - u_2| \leq 0\text{in } \Omega \\ \implies & u_1 = u_2 \text{ in } \Omega \\ \implies & \text{there exists at most one solution!} \quad \blacksquare \end{aligned}$$

2.5.2 Regularity of Solutions

Goal: Harmonic functions are analytic (they can be expressed as convergent power series).

- Need that u is infinitely many times differentiable.
- Need estimates for the derivatives of u.

Next goal: Harmonic functions are smooth.

A way to smoothen functions is mollification.
Idea: (weighted) averages of functions are smoother.

Definition 2.18 (Standard Mollifier).

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

Fact: η is infinitely many times differentiable, $\eta \equiv 0$ outside $B(0,1)$.

We fix C such that $\int_{\mathbb{R}^n} \eta(y)\, dy = 1$.

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$$

- $\eta_\epsilon = 0$ outside $B(0,\epsilon)$
- $\int_{\mathbb{R}^n} \eta_\epsilon(y)\, dy = 1$

If u is integrable, then

$$\begin{aligned} u_\epsilon(x) &:= \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)\ dy \\ &= \int_{\mathbb{R}^n} \eta_\epsilon(y)u(x-y)\ dy \\ &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\ dy \end{aligned}$$

is called the mollification of u.

Theorem 2.19 (Properties of u_ϵ).
- *u_ϵ is C^∞ (infinitely many times differentiable)*
- *$u_\epsilon \to u$ (pointwise) a.e. as $\epsilon \to 0$*
- *If u is continuous then $u_\epsilon \to u$ uniformly on compact subsets*

Now since $\Phi'(r) = 0$,

$$\begin{aligned} \Longrightarrow \Phi(r) &= \lim_{\rho\to 0} \Phi(\rho) \\ &= \lim_{\rho\to 0} ⨍_{\partial B(x,\rho)} u(y)\ dS(y) \\ &= u(x) \end{aligned}$$

$\therefore \Phi(r) = u(x)\ \forall r$ such that $B(x,r) \subset U$.

$$\begin{aligned} ⨍_{B(x,r)} u(y)\ dy &= \frac{1}{|B(x,r)|}\int_0^r \int_{\partial B(x,\rho)} u(y)\ dS(y) d\rho \\ &= \frac{1}{|B(x,r)|}\int_0^r |\partial B(x,\rho)| \underbrace{⨍_{\partial B(x,\rho)} u(y)\ dS(y)}_{u(x)}\ d\rho \\ &= \frac{u(x)}{|B(x,r)|}\int_0^r n\alpha(n)\rho^{n-1}\ d\rho \\ &= \frac{u(x)}{r^n\alpha(n)} n\alpha(n)\frac{1}{n}\rho^n \Big|_0^r \\ &= u(x). \quad \blacksquare \end{aligned}$$

Note: $n\alpha(n)$ is the surface are of $\partial B(0,1)$ in $\mathbb{R}^n$.

Complex differentiable $\Longrightarrow$ analytic

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{a_n}{n!}(z-z_0)^n \\ a_n &= \frac{n!}{2\pi i} \oint_\gamma \frac{f(\omega)}{(\omega - z_0)^{n+1}}\, d\omega = f^{(n)}(z) \end{aligned}$$

Theorem 2.20. *If $u \in C(\Omega)$, Ω bounded and open, satisfies the mean-value property, then $u \in C^\infty(\Omega)$*

Proof: Take $\epsilon > 0$

$$\Omega_\epsilon = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \epsilon\}$$

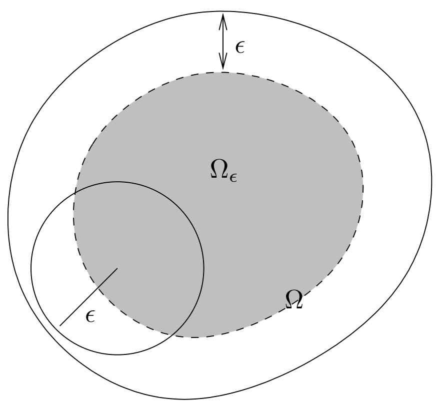


Figure 2.4:

In Ω_ϵ : $\qquad u_\epsilon(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)\ dy$

$$\eta(y) = \begin{cases} Ce^{\frac{1}{|y|^2-1}} & |y| \le 1 \\ 0 & |y| > 1 \end{cases}$$

$$\begin{aligned}
u_\epsilon(x) &= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\ dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \int_{\partial B(x,\rho)} \eta\left(\frac{\rho}{\epsilon}\right) u(y)\ dS(y) d\rho \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{\rho}{\epsilon}\right) |\partial B(x,\rho)| \underbrace{⨍_{\partial B(x,\rho)} u(y)\ dS}_{u(x)}\ d\rho \\
&= \frac{u(x)}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{\rho}{\epsilon}\right) n\alpha(n)\rho^{n-1}\ d\rho \\
&= \frac{u(x)}{\epsilon^n}\int_0^\epsilon \int_{\partial B(x,\rho)} \eta\left(\frac{\rho}{\epsilon}\right)\ dS d\rho \\
&= u(x)\int_{\mathbb{R}^n} \eta_\epsilon(y)\ dy \\
&= u(x)
\end{aligned}$$

In Ω_ϵ, $u_\epsilon(x) = u(x)$
$\Longrightarrow u \in C^\infty(\Omega_\epsilon)$
$\epsilon > 0$ arbitrary $\Longrightarrow u \in C^\infty(\Omega)$. ■

Corollary 2.21. *If $u \in C(\Omega)$ satisfies the mean-value property, then u is harmonic.*

Proof: Mean-value property $\Longrightarrow u \in C^\infty(\Omega)$. Our proof of harmonic $\Longrightarrow$ mean-value property gave

$$\Phi'(r) = \frac{r}{n}⨍_{B(x,r)} \Delta u(y)\ dy,$$

$$\text{where} \qquad \Phi(r) = ⨍_{\partial B(x,r)} u(y)\ dy$$

Now $\Phi'(r) = 0$ and thus

$$⨍_{B(x,r)} \Delta u(y)\ dy = 0$$

Since Δu is smooth, this is only possible if $\Delta u = 0$ i.e. u is harmonic. ■

2.5.3 Estimates on Harmonic Functions

Theorem 2.22 (Local estimates for harmonic functions). *Suppose $u \in C^2(\Omega)$, u harmonic. Then*

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \int_{B(x_0,r)} |u(y)|\, dy, \qquad |\alpha| = k$$

where

$$C_k = \begin{cases} \frac{1}{\alpha(n)} & k = 0 \\ \frac{(2^{n+1}nk)^k}{\alpha(n)} & k = 1, 2, \ldots \end{cases}$$

and $B(x_0, r) \subset \Omega$.

Remarks:

$$|D^\alpha(x_0)| \le \frac{C}{r^k} ⨍_{B(x_0,r)} |u(y)|\, dy \qquad |\alpha| = k,$$

Proof: (by induction)
$k = 0$:

$$\begin{aligned} u(x_0) &= ⨍_{B(x_0,r)} u(y)\, dy \\ \therefore |u(x_0)| &\le \frac{1}{\alpha(n)} \frac{1}{r^n} \int_{B(x_0,r)} |u(y)|\, dy \end{aligned}$$

$k = 1$: u harmonic $\implies u \in C^\infty(\Omega)$

$$\Delta u_{x_i} = \frac{\partial}{\partial x_i} \Delta u = 0,$$

i.e., all derivatives of u are harmonic. Use mean value property for $\frac{\partial}{\partial x_i} u = u_{x_i}$ on $B\left(x_0, \frac{r}{2}\right)$

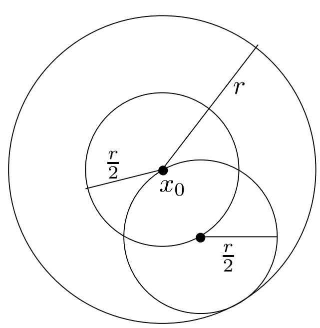


Figure 2.5:

$$u_{x_i} = \fint_{B\left(x_o, \frac{r}{2}\right)} \underbrace{u_{x_i}}_{\text{div } u} \; dy$$

Note: $u_{x_i} = \text{div }(0, \ldots, 0, u, 0, \ldots, 0)$ where u is in the ith component.
Use divergence theorem:

$$\begin{aligned} u_{x_i} &= \frac{1}{\left|B\left(x_0, \frac{r}{2}\right)\right|} \int_{\partial B\left(x_0, \frac{r}{2}\right)} u(y)\nu_i \; dS(y) \\ &\leq \frac{2^n}{\alpha(n)r^n} \underbrace{\left|\partial B\left(x_0, \frac{r}{2}\right)\right|}_{n\alpha(n)\frac{r^{n-1}}{2^{n-1}}} \max_{y \in \partial B\left(x_0, \frac{r}{2}\right)} |u(y)| \\ &\leq \frac{2n}{r} \frac{1}{\alpha(n)} \frac{1}{\left(\frac{r}{2}\right)^n} \max_{y \in \partial B\left(x_0, \frac{r}{2}\right)} \int_{B\left(y, \frac{r}{2}\right)} |u(z)| \; dz \\ &\leq \frac{2^{n+1}n}{\alpha(n)r^{n+1}} \int_{B(x_0, r)} |u(y)| \; dy \end{aligned}$$

<u>$k \geq 2$:</u>
$|\alpha| = k$, then

$$D^\alpha u(x) = D^\beta \frac{\partial}{\partial x_i} u(x),$$

$|\beta| = k - 1$ for some $1 \leq i \leq n$.

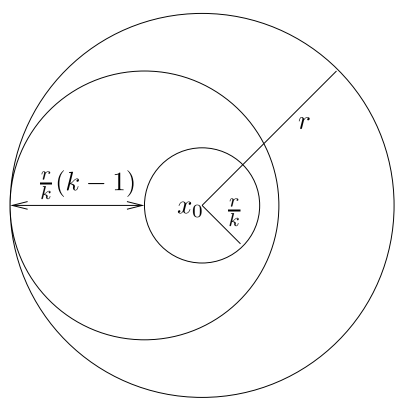


Figure 2.6:

$D^\alpha u$ is harmonic. MVF for $D^\alpha u$ on $B\left(x_0, \frac{r}{k}\right)$

$$\begin{aligned}
|D^\alpha u(x)| &= \left| \fint_{B\left(x_0,\frac{r}{k}\right)} \underbrace{D^\alpha u(y)}_{\frac{\partial}{\partial x_i} D^\beta u} \, dy \right| \\
&= \frac{1}{\left|B\left(x_0,\frac{r}{k}\right)\right|} \left| \int_{\partial B\left(x_0,\frac{r}{k}\right)} D^\beta u \cdot \nu_i \, dS \right| \\
&\le \frac{\left|\partial B\left(x_0,\frac{r}{k}\right)\right|}{\left|B\left(x_0,\frac{r}{k}\right)\right|} \max_{y\in\partial B\left(x_0,\frac{r}{k}\right)} |D^\beta u(u)| \\
&= \frac{n\alpha(n)\left(\frac{r}{k}\right)^{n-1}}{\alpha(n)\left(\frac{r}{k}\right)^n} \max_{y\in\partial B\left(x_0,\frac{r}{k}\right)} |D^\beta u(u)| \\
&= \frac{nk}{r} \max_{y\in\partial B\left(x_0,\frac{r}{k}\right)} |D^\beta u(u)|
\end{aligned}$$

For $y \in \partial B\left(x_0, \frac{r}{k}\right)$, we have $B\left(y, \frac{(k-1)r}{k}\right) \subset B(x_0, r)$ and we use the induction hypothesis on this ball for

$$\begin{aligned}
|D^\beta u(y)| &\le \frac{1}{\left(\frac{(k-1)r}{k}\right)^{n+k-1}} \frac{(2^{n+1}n(k-1))^{k-1}}{\alpha(n)} \int_{B(x_0,r)} |u(t)| \, dt \\
&= \frac{1}{r^{n+k-1}} \left(\frac{k}{k-1}\right)^{n+k-1} \frac{(2^{n+1}n(k-1))^{k-1}}{\alpha(n)} \int_{B(x_0,r)} |u(t)| \, dt \\
&= \frac{1}{r^{n+k-1}} \left(\frac{k}{k-1}\right)^{n} \frac{(2^{n+1}nk)^{k-1}}{\alpha(n)} \int_{B(x_0,r)} |u(t)| \, dt \\
&\le \frac{1}{r^{n+k-1}} 2^n \frac{(2^{n+1}nk)^{k-1}}{\alpha(n)} \int_{B(x_0,r)} |u(t)| \, dt
\end{aligned}$$

$$\begin{aligned}
\implies |D^\alpha u(x_0)| &\le \frac{nk}{r} \frac{1}{r^{n+k-1}} 2^n \frac{(2^{n+1}nk)^{k-1}}{\alpha(n)} \int_{B(x_0,r)} |u(t)| \, dt \\
&\le \frac{1}{r^{n+k}} \underbrace{\frac{(2^{n+1}nk)^k}{\alpha(n)}}_{C_k} \int_{B(x_0,r)} |u(t)| \, dt \qquad \blacksquare
\end{aligned}$$

As a consequence of this, we get

Theorem 2.23 (Liouville's Theorem). *If u is harmonic on $\mathbb{R}^n$ and u is bounded, then u is constant.*

Proof:

$$\begin{aligned} |Du(x_0)| &\leq \frac{C}{r^{n+1}} \int_{B(x_0,r)} |u(y)|\, dy \\ &\leq \frac{C}{r^{n+1}} M|B(x_0,r)|, \text{where } M := \sup_{y\in\mathbb{R}^n} |u(y)| \\ &\leq \frac{C}{r} M \end{aligned}$$

where $B(x_0, r) \subseteq \mathbb{R}^n$.
As $r \to \infty$, $Du(x_0) = 0$.
$\Longrightarrow$ u is constant. ■

2.5.4 Analyticity of Harmonic Functions

Theorem 2.24 (Analyticity of harmonic functions). *Suppose that Ω is open (and bounded) and that $u \in C^2(\Omega)$, u harmonic in Ω. Then u is analytic i.e. u can be expanded locally into a convergent power series.*

Proof:

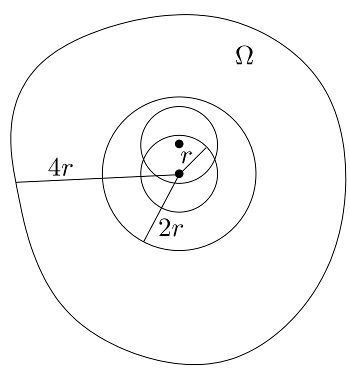


Figure 2.7:

Suppose that $x_0 \in \Omega$, show that u can be expanded into a power series at x_0. i.e.

$$u(x) = \sum_{\alpha} \frac{1}{|\alpha|} D^\alpha u(x_0)(x - x_0)^\alpha$$

Let

$$\begin{aligned} r &= \frac{1}{4}\text{dist}(x_0, \partial\Omega) \text{ and let} \\ M &= \frac{1}{\alpha(n)r^n}\int_{B(x_0,2r)} |u(y)|\, dy \end{aligned}$$

Local estimates: $\forall x \in B(x_0, r)$, we have $B(x,r) \subset B(x_0, 2r) \subset \Omega$, and thus

$$\begin{aligned} \max_{x\in B(x_0,r)} |D^\alpha u(x)| &\leq M\frac{1}{r^k}(2^{n+1}nk)^k \\ &= M\left(\frac{2^{n+1}nk}{r}\right)^k \\ &= M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|} \end{aligned}$$

Stirling's formula:

$$\begin{aligned} \lim_{k\to\infty}\frac{\sqrt{k}k^k}{k!e^k} &= \frac{1}{\sqrt{2\pi}} \\ \implies k^k &\leq \frac{C}{\sqrt{2\pi}}\frac{k!e^k}{\sqrt{k}} \\ &\leq C_1 k! e^k \end{aligned}$$

Multinomial theorem:

$$\begin{aligned} n^k &= \left(\sum_{i=1}^n 1\right)^k \\ &= \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!} \qquad \text{Multinomal Thm.} \\ &\geq \frac{|\alpha|!}{\alpha!} \end{aligned}$$

for every α with $|\alpha| = k$

$$\implies |\alpha|! \leq n^k \alpha!$$

$$\begin{aligned}
\max_{x\in B(x_0,r)} |D^\alpha u(x)| &\leq M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|} \\
&\leq CM\left(\frac{2^{n+1}n^2}{r}\right)^{|\alpha|} |\alpha|! e^{|\alpha|} \\
&= CM\left(\frac{2^{n+1}en^2}{r}\right)^{|\alpha|} |\alpha|!
\end{aligned}$$

Taylor Series:

$$\sum_{N=0}^{\infty}\sum_{|\alpha|=N} \frac{|D^\alpha u(x_0)|}{\alpha!}(x-x_0)^\alpha \leq \sum_{N=0}^{\infty}\sum_{|\alpha|=N} CM\left(\frac{2^{n+1}en^2}{r}\right)^N |x-x_0|^\alpha$$

——

there are no more that n^N multiindices α with $|\alpha| = N$, thus

——

$$\begin{aligned}
&\leq \sum_{N=0}^{\infty} CM\left(\frac{2^{n+1}en^2}{r}\right)^N n^N |x-x_0|^\alpha \\
&\leq \sum_{N=0}^{\infty} CM\left(\frac{2^{n+1}en^3}{r}\right)^N |x-x_0|^N \\
&\leq \sum_{N=0}^{\infty} CMq^N \qquad |q| < 1
\end{aligned}$$

The last inequality is true only if the series is converging, i.e. for

$$\begin{aligned}
\frac{2^{n+1}en^3}{r}|x-x_0| &< 1 \\
\Longrightarrow |x-x_0| < \frac{r}{2^{n+1}en^3}
\end{aligned}$$

Result: The Taylor series converges in a ball with radius $R = \frac{r}{2^{n+1}en^3}$ which only depends on $\text{dist}(x_0, \partial\Omega)$ and dimension. ■

2.5.5 Harnack's Inequality

Theorem 2.25 (Harnack's Inequality). *Suppose that Ω is open and that $\Omega' \subset\subset \Omega$ (Ω' is compactly contained in Ω), and Ω' is connected. Suppose that $u \in C^2(\Omega)$ and harmonic, and nonnegative, then*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

and this constant does not depend on u.

This means that non-negative harmonic functions can not have wild oscillations on Ω'.

Remark: $\inf_{\Omega'} u = 0$, then $\sup_{\Omega'} u = 0 \Longrightarrow u \equiv 0$ on Ω'.

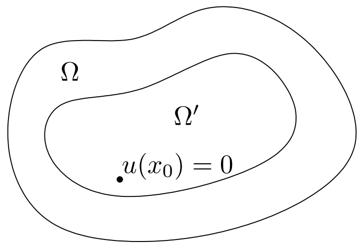


Figure 2.8:

This follows also from the minimum principle for harmonic functions.

Proof of Harnack:

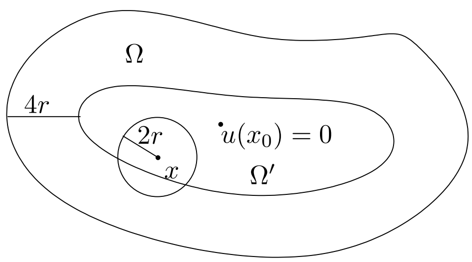


Figure 2.9:

Let $4r = \text{dist}(\Omega', \partial\Omega)$ and take $x, y \in \Omega'$, $|x-y| \le r \Longrightarrow B(x,2r) \subset \Omega$, $B(y,r) \subseteq B(x,2r)$.

$$\begin{aligned} u(x) &= \fint_{B(x,2r)} u(z)\,dz \\ &\ge \frac{1}{|B(x,2r)|}\int_{B(y,r)} u(z)\,dz \\ &= \frac{|B(y,r)|}{|B(x,2r)|}\fint_{B(y,r)} u(z)\,dz \\ &= \frac{1}{2^n}u(y) \end{aligned}$$

i.e.

$$u(x) \geq \frac{1}{2^n} u(y)$$

Same argument interchanging x and y yields

$$\begin{aligned} & u(y) \geq \frac{1}{2^n} u(x) \\ \implies\quad & 2^n u(x) \geq u(y) \geq \frac{1}{2^n} u(x) \quad \forall x, y \in \Omega',\ |x-y| \leq r \end{aligned}$$

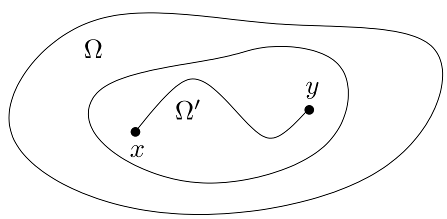


Figure 2.10:

$\overline{\Omega'}$ is compact, cover $\overline{\Omega'}$ by at most N balls $B(x_i, r)$. Take $x, y \in \Omega'$, take a curve γ joining x and y.

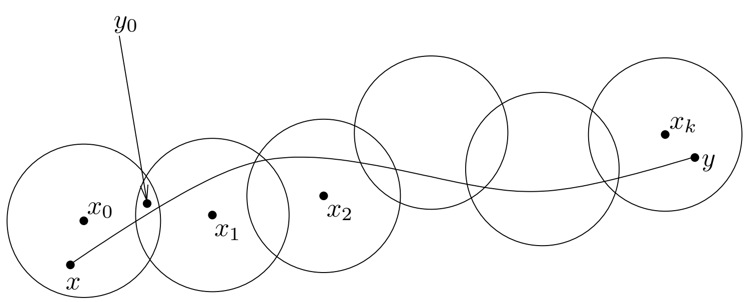


Figure 2.11:

Chain of balls covering γ, $B(x_0, r), \ldots, B(x_k, r)$, where $k \leq N$. Now, $|x -$

$x_0| \leq r$

$$\begin{aligned} u(x) &\geq \frac{1}{2^n} u(x_0) \\ &\geq \frac{1}{2^n}\left(\frac{1}{2^n} u(y_0)\right) \\ &= \frac{1}{2^{2n}} u(y_0) \\ &\geq \frac{1}{2^{3n}} u(x_1) \\ &\vdots \\ &\geq \frac{1}{2^{(2k+2)n}} u(y) \\ &\geq \frac{1}{2^{(2N+2)n}} u(y) \end{aligned}$$

$\forall x, y \in \Omega'$ we get

$$u(x) \geq \frac{1}{2^{(2N+2)n}} u(y)$$

$\Longrightarrow$ Take x_i such that $u(x_i) \to \inf_{x\in\Omega'} u(x)$
y_i such that $u(y_i) \to \sup_{x\in\Omega'} u(x)$
$\Longrightarrow$ assertion. ∎

Properties of harmonic functions

- mean-value property
- maximum principle
- regularity
- local estimates
- analyticity
- Harnack's inequality

2.6 Existence of Solutions

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \qquad \text{Laplace's equation}$$

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \qquad \text{Poisson's equation}$$

- Green's function $\rightarrow$ general representation formula

- Finding Green's function for special domains, half-space and a ball

- Perron's Method - get existence on fairly general domains from the existence on balls

2.6.1 Green's Functions

We have in $\mathbb{R}^n$

$$u(x) = \int_{\mathbb{R}^n} \Phi(y - x) f(y) \, dy$$

solves

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n$$

Issue: Construct fundamental solutions with boundary conditions.

General representation formula for smooth functions $u \in C^2$

Ω is a bounded, open domain. $x \in \Omega$, $\epsilon > 0$ such that $B(x, \epsilon) \subset \Omega$. Define $\Omega_\epsilon = \Omega \setminus B(x, \epsilon)$.

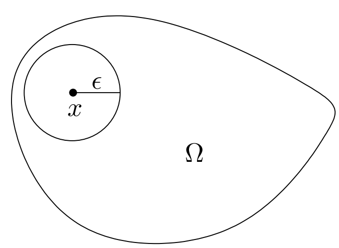


Figure 2.12:

$$
\begin{aligned}
0 &= \int_{\Omega_\epsilon} u(y)\Delta\Phi(y-x)\ dy \\
&= \int_{\partial\Omega_\epsilon} u(y)D\Phi(y-x)\cdot\nu\ dS(y) \\
&\quad -\int_{\Omega_\epsilon} Du(y)\cdot D\Phi(y-x)\ dy \\
&= \int_{\partial\Omega_\epsilon} [u(y)D\Phi(y-x)\cdot\nu - Du(y)\cdot\nu\Phi(y-x)]\ dS(y) \\
&\quad +\int_{\Omega_\epsilon} \Delta u(y)\Phi(y-x)\ dy
\end{aligned}
$$

Volume term for $\epsilon \to 0$ gives

$$
\int_{\Omega} \Delta u(y)\Phi(y-x)\ dy
$$

singularity of Φ integrable.
boundary terms:

$$
\begin{aligned}
= &\int_{\partial\Omega} [u(y)D\Phi(y-x)\cdot\nu - Du(y)\cdot\nu\Phi(y-x)]\ dS(y) \\
&+\int_{\partial B(x,\epsilon)} [\underbrace{u(y)D\Phi(y-x)\cdot\nu}_{I_1} - \underbrace{Du(y)\cdot\nu\Phi(y-x)}_{I_2}]\ dS(y)
\end{aligned}
$$

Remembering $Du(y) \le C\epsilon^{n-1}$ and

$$
\Phi \le \begin{cases} \frac{C}{\epsilon^{n-2}} & n \ge 3 \\ \ln\epsilon & n = 2 \end{cases},
$$

we see that

$$
\begin{aligned}
& I_2 \le \begin{cases} \epsilon & n \ge 3 \\ \epsilon\ln\epsilon & n = 2 \end{cases} \\
& \Longrightarrow I_2 \to 0 \text{ as } \epsilon \to 0.
\end{aligned}
$$

$$
I_1 = \int_{\partial B(x,\epsilon)} [u(y)D\Phi(y-x)\cdot\nu_{\text{in}}\ dS(y)
$$

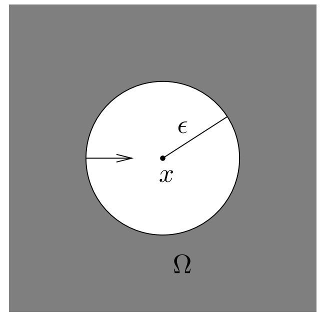


Figure 2.13:

$$\begin{aligned}
D\Phi(z)\cdot\nu_{\text{out}} &= \frac{-1}{n\alpha(n)\epsilon^{n-1}}, \qquad z\in\partial B(0,\epsilon)\\
D\Phi(y-x)\cdot\nu_{\text{in}} &= \frac{-1}{n\alpha(n)\epsilon^{n-1}}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{\partial B(x,\epsilon)} u(y)\frac{1}{n\alpha(n)\epsilon^{n-1}}\;dS(y)\\
&= ⨍_{\partial B(x,\epsilon)} u(y)\;dS(y) \to u(x) \text{ as } \epsilon\to 0
\end{aligned}$$

Putting things together:

$$\begin{aligned}
0 &= \int_{\partial\Omega}[u(y)D\Phi(y-x)\cdot\nu - Du(y)\cdot\nu\Phi(y-x)]\;dS(y)\\
&\quad +u(x) + \int_\Omega \Delta u(y)\Phi(y-x)\;dy\\
\Longrightarrow u(x) &= \int_{\partial\Omega}[Du(y)\cdot\nu\Phi(y-x) - u(y)D\Phi(y-x)\cdot\nu]\;dS(y)\\
&\quad -\int_\Omega \Delta u(y)\Phi(y-x)\;dy
\end{aligned}$$

<u>Recall:</u> Existence of harmonic functions.

Green's functions: $\Omega, u\in C^2$

$$0 = \int_{\Omega_\epsilon} u(y)\Delta\Phi(y-x)\;dy$$

Representation of $u(x)$

$$\begin{aligned} u(x) &= -\int_{\partial\Omega} \left[u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial u}{\partial \nu}(y)\Phi(y-x) \right] dS(y) \\ &\quad - \int_{\Omega} \Delta u(y)\Phi(y-x)\, dy \end{aligned}$$

Green's function = fundamental solution with zero boundary data.

Idea: Correct the boundary values of $\Phi(y-x)$ by a harmonic corrector to be equal to zero.

$$\text{Green's function} \qquad G(x,y) = \Phi(y-x) - \phi^x(y),$$

where

$$\begin{cases} \Delta_y \phi^x(y) &= 0 \quad \text{in } \Omega \\ \phi^x(y) &= \Phi(y-x) \quad \text{on } \partial\Omega \end{cases}$$

Same calculations as before:

$$\begin{aligned} u(x) &= -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) \\ &\quad - \int_{\Omega} G(x,y)\Delta u(y)\, dy \end{aligned}$$

Expectation: The solution of

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{cases}$$

is given by

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y)$$

Theorem 2.26 (Symmetry of Green's Functions). $\forall x, y \in \Omega$, $x \neq y$, $G(x,y) = G(y,x)$.

Proof: Define

$$\begin{aligned} v(z) &= G(x,z) \\ w(z) &= G(y,z) \end{aligned}$$

v harmonic for $z \neq x$, w harmonic for $z \neq y$.
$v, w = 0$ on $\partial\Omega$.

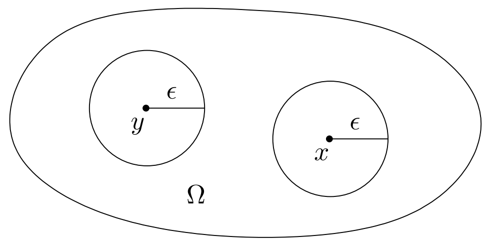


Figure 2.14:

ϵ small enough such that $B(x,\epsilon) \subset \Omega$ and $B(y,\epsilon) \subset \Omega$.
$\Omega_\epsilon = \Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \Delta v \cdot w \; dz \\ &= \int_{\partial\Omega_\epsilon} \frac{\partial v}{\partial \nu} w \; dS - \int_{\Omega_\epsilon} Dv \cdot Dw \; dz \\ &= \int_{\partial\Omega_\epsilon} \left(\frac{\partial v}{\partial \nu} w - v \frac{\partial w}{\partial \nu} \right) \; dS \\ &\quad + \int_{\Omega_\epsilon} v \cdot \Delta w \; dz \end{aligned}$$

Now, the last term on the RHS is 0. Thus

$$\int_{\partial B(x,\epsilon)} \left(\frac{\partial v}{\partial \nu} w - v \frac{\partial w}{\partial \nu} \right) \; dS = \int_{\partial B(y,\epsilon)} \left(v \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} w \right) \; dS$$

Now, since w is nice near x, we have

$$\left| \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v \; dS \right| \leq C\epsilon^{n-1} \sup_{\partial B(x,\epsilon)} |v| = o(1) \qquad \text{as } \epsilon \to 0$$

On the other hand, $v(z) = \Phi(z - x) - \phi^x(z)$, where ϕ^x is smooth in U. Thus

$$\lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu} w \; dS = \lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(x,z) w(z) \; dS = w(x)$$

by previous calculations. Thus, the LHS of (2.8) converges to $w(x)$ as $\epsilon \to 0$. Likewise the RHS converges to $v(y)$. Consequently

$$G(y,z) = w(x) = v(y) = G(x,y) \qquad \blacksquare$$

2.6.2 Green's Functions by Reflection Methods

Green's Function for the Halfspace

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial v}{\partial \nu}(y-x)\, dy$$

In this case: $\partial\Omega = \partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\}$

Idea: reflect x into $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$ and to take $\phi^x(y) = \Phi(y - \tilde{x})$

$$\begin{aligned} G(x,y) &= \Phi(y-x) - \phi^x(y) \\ &= \Phi(y-x) - \Phi(y-\tilde{x}) \end{aligned}$$

This works since $x \in \mathbb{R}^n_+$, $\tilde{x} \notin \mathbb{R}^n_+$
For $y \in \partial\mathbb{R}^n_+$, $y_n = 0$, $|y-x| = |y-\tilde{x}|$. Thus, $\Phi(y-x) = \Phi(y-\tilde{x})$ for $y \in \partial\mathbb{R}^n_+$. Guessing from what we showed on bounded domains:

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\, dy,$$

where

$$\frac{\partial G}{\partial \nu} = -\frac{\partial G}{\partial y_n}$$

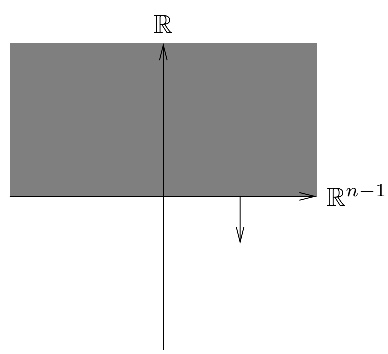


Figure 2.15:

$n \geq 3$:

$$\begin{aligned} \Phi(z) &= \frac{1}{n(n-2)\alpha(n)}\frac{1}{|z|^{n-2}} \\ \Longrightarrow D\Phi &= \frac{-1}{n\alpha(n)}\frac{z}{|z|^n} \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial G}{\partial y_n}(y-x) &= \frac{1}{n\alpha(n)}\frac{y_n - x_n}{|y-x|^n} \\
-\frac{\partial G}{\partial y_n}(y-\tilde{x}) &= \frac{1}{n\alpha(n)}\frac{y_n + x_n}{|y-x|^n}
\end{aligned}
$$

So if $y \in \partial\mathbb{R}^n_+$,

$$
\frac{\partial G}{\partial \nu}(x,y) = -\frac{\partial G}{\partial y_n}(x,y) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}
$$

So our expectation is

$$
\begin{aligned}
u(x) &= -\int_{\partial\mathbb{R}^n_+} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dy \\
&= \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}\,dy. \qquad (2.8)
\end{aligned}
$$

This is Poisson's formula on $\mathbb{R}^n_+$.

$$
K(x,y) = \frac{2x_n}{n\alpha(n)|x-y|^n}
$$

is called Poisson's kernel on $\mathbb{R}^n_+$.

Theorem 2.27 (Poisson's formula on the half space). *$g \in C^0(\partial\mathbb{R}^n_+)$, g is bounded and $u(x)$ is given by (2.8), then*

i.) $u \in C^\infty(\mathbb{R}^n_+)$

ii.) $\Delta u = 0$ *in* $\mathbb{R}^n_+$

iii.) $\lim_{x\to x_0} u(x) = g(x_0)$ *for* $x_0 \in \partial\mathbb{R}^n_+$

Proof: $y \mapsto G(x,y)$ is harmonic $x \neq y$.

i.) symmetric $G(x,y) = G(y,x)$
$\Longrightarrow$ G is harmonic in its first argument $x \neq y$
$\Longrightarrow$ derivatives of G are harmonic for $x \neq y$
$\Longrightarrow$ our representation formula involves

$$
\partial_\nu G(x,y) \quad y \in \partial\mathbb{R}^n_+,\ x \in \mathbb{R}^n_+ \Longrightarrow x \neq y
$$

$\Longrightarrow$ $\partial_\nu G$ is harmonic for $x \in \mathbb{R}^n_+$.

ii.) u is differentiable

- integrate on $n-1$ dimensional surface
- decay at ∞ is of order $\frac{1}{|x|^n} \Longrightarrow$ the function decays fast enough to be integrable, and the same is true for its derivatives (since g is bounded). $\Longrightarrow$ differentiate under the integral, $\Longrightarrow \Delta u(x) = 0$.

iii.) Boundary value for u. Fix $x_0 \in \partial\mathbb{R}^n_+$, choose $\epsilon > 0$, choose $\delta > 0$ small enough such that $|g(x) - g(x_0)| < \epsilon$ if $|x - x_0| < \delta$, $x \in \partial\mathbb{R}^n_+$, by continuity of g. For $x \in \mathbb{R}^n_+$, $|x - x_0| < \frac{\delta}{2}$

$$|u(x) - g(x_0)| = \left| \frac{2x_n}{m\alpha(n)} \int_{\mathbb{R}^n_+} \frac{g(y) - g(x_0)}{|x-y|^n}\, dy \right| \tag{2.9}$$

since

$$\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1 \qquad \text{(HW\# 4)} \tag{2.10}$$

$$\begin{aligned}(2.9) \quad \leq \quad & \underbrace{\int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y)\, \underbrace{|g(y) - g(x_0)|}_{\leq \epsilon}\, dy}_{\leq \epsilon \text{ by } (2.10)} \\ & + \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y) |g(y) - g(x_0)|\, d\end{aligned}$$

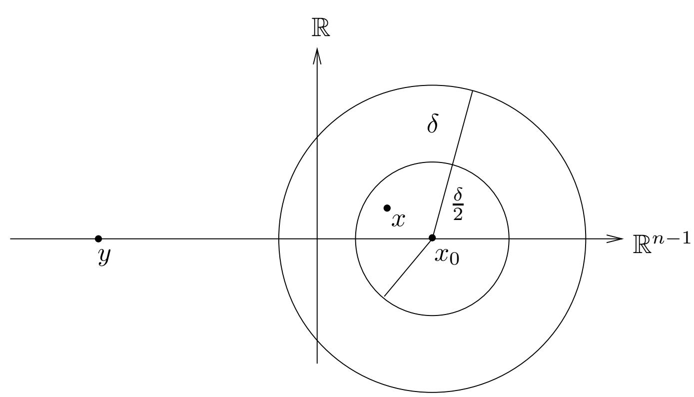


Figure 2.16:

$$
\begin{aligned}
|x - x_0| &< \frac{\delta}{2} \\
|x_0 - y| &> \delta
\end{aligned}
$$

$$
\begin{aligned}
|y - x_0| &\leq |y - x| + |x - x_0| \\
&\leq |y - x| + \frac{\delta}{2} \\
&\leq |y - x| + \frac{1}{2}|x_0 - y| \\
\Longrightarrow \frac{1}{2}|y - x_0| &\leq |y - x|
\end{aligned}
$$

2nd integral

$$
\begin{aligned}
&= \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} \frac{2x_n}{n\alpha(n)} \frac{|g(y) - g(x_0)|}{|x - y|^n}\, dy \\
&\leq \frac{4x_n^0 \max|g|}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} \left(\frac{2}{|x_0 - y|}\right)^n\, dy \\
&\leq \frac{4x_n^0 \max|g|}{n\alpha(n)} C(\delta) \leq \epsilon
\end{aligned}
$$

If x_n^0 is sufficiently small. Two estimates together: $|u(x) - g(x_0)| \leq 2\epsilon$ if $f|x - x_0|$ is small enough. $\Longrightarrow$ continuity at x_0 $u(x_0) = g(x_0)$ as asserted. ■

Green's Function on a Ball

1. unit ball

2. scale the result to arbitrary balls

3. inversion in the unit sphere

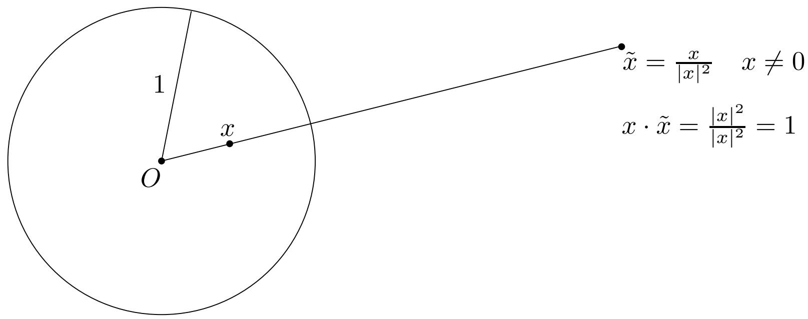


Figure 2.17:

$$\Phi(y-x) = \begin{cases} -\frac{1}{2\pi}\ln|y-x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x-y|^{n-2}} & n \geq 3 \end{cases}$$

For $x = 0$ this simplifies to

$$\begin{cases} -\frac{1}{2\pi}\ln|y| \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-2}} \end{cases} \underbrace{=}_{|y|=1} \begin{cases} 0 \\ \frac{1}{n(n-2)\alpha(n)} \end{cases}$$

We choose $\Phi(y)$ to be this constant, and we assume $x \neq 0$ from now on. For $x \in B(0,1)$, $\tilde{x} \notin B(0,1)$, and Φ centered at $\tilde{x}$ is a good function.

$$\begin{array}{lcll} \underline{n \geq 3} & y & \mapsto & \Phi(y-\tilde{x}) \text{ is harmonic} \\ & y & \mapsto & |x|^{2-n}\Phi(y-\tilde{x}) \text{ is harmonic} \\ & y & \mapsto & \Phi(|x|(y-\tilde{x})) \text{ is harmonic} \end{array}$$

Corrector $\phi^x(y) = \Phi(|x|(y-\tilde{x}))$ harmonic in $B(0,1)$

$$\begin{aligned} |y| = 1 \Longleftarrow y \in \partial B(0,1) \quad \Longrightarrow \quad & |x|^2|y-\tilde{x}|^2 \\ = \quad & |x|^2\left|y - \frac{x}{|x|^2}\right|^2 \\ = \quad & |x|^2\left||y|^2 - \frac{2(y,x)}{|x|^2} + \frac{|x|^2}{|x|^4}\right| \\ = \quad & |x|^2 - 2(y,x) + |y|^2 \\ = \quad & |x-y|^2 \\ \Longrightarrow \Phi(|x|(y-\tilde{x})) \quad = \quad & \Phi(y-x) \quad \text{if } y \in \partial B(0,1) \end{aligned}$$

Definition 2.28. $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$ *is the Green's function on* $B(0,1)$.

Expectation: The solution of

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad B(0,1) \\ u = g \quad &\text{on} \quad \partial B(0,1) \end{aligned}$$

is given by

$$\begin{aligned} u(x) &= -\int_{\partial B(0,1)} \partial_\nu G(x,y) g(y)\, dy \\ &= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dy \end{aligned}$$

Poisson's formula on $B(0,1)$

$$K(x,y) = \frac{1-|x|^2}{n\alpha(n)} \frac{1}{|x-y|^n}$$

is called Poisson's kernel on $B(0,1)$

So now, we want to scale this result.

$$\begin{aligned} &D\Phi(z) = \frac{-1}{n\alpha(n)} \frac{z}{|z|^n} \\ &\frac{\partial \Phi}{\partial y}(y-x) = \frac{-1}{n\alpha(n)} \frac{y-x}{|y-x|^n} \\ &\frac{\partial \Phi}{\partial y}(|x|(y-\tilde{x})) = \frac{-1}{n\alpha(n)} \frac{|x|(y-\tilde{x})}{||x|(y-\tilde{x})|^n}|x| \end{aligned}$$

on $\partial B(0,1)$. $|y-x|^2 = ||x|(y-\tilde{x})|^2$

$$\begin{aligned} \implies \frac{\partial \Phi}{\partial y}(|x|(y-\tilde{x})) &= \frac{-1}{n\alpha(n)} \frac{|x|^2\left(y - \frac{x}{|x|^2}\right)}{|y-x|^n} \\ \frac{\partial G}{\partial y}(y,x) &= \frac{-1}{n\alpha(n)} \left[\frac{y-x}{|y-x|^n} - \frac{y|x|^2 - x}{|y-x|^n}\right] \\ &= \frac{-1}{n\alpha(n)} \left[\frac{y(1-|x|^2)}{|y-x|^n}\right] \\ \frac{\partial G}{\partial \nu} &= y\frac{\partial G}{\partial y}(x,y) = \frac{-1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n} \end{aligned}$$

This gives for a solution of $\Delta u = 0$ in $B(0,1)$, $u(x) = g(x)$ on $\partial B(0,1)$

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, DS(y)$$

Transform this to balls with radius r

$$\begin{array}{llcll} \text{Solve} & \Delta u & = & 0 & \text{in } B(0,r) \\ & u & = & g & \text{on } \partial B(0,r) \end{array}$$

<u>Define</u> $v(x) = u(rx)$ on $B(0,1)$

$$\begin{aligned} \Longrightarrow \Delta v(x) &= r^2 \Delta u(rx) = 0 \\ v(x) &= u(rx) = g(rx) \text{ for } x \in \partial B(0,1) \end{aligned}$$

Poisson's formula on $B(0,1)$

$$\begin{aligned} v(x) &= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(ry)}{|x-y|^n}\, dS(y) \qquad \text{for } x \in B(0,1) \\ u(x) &= v\left(\frac{x}{r}\right) \\ &= \frac{1-\left|\frac{x}{r}\right|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{g(y)}{\left|\frac{x}{r}-\frac{y}{r}\right|^n} \frac{dS(y)}{r^{n-1}} \\ &= \frac{r^2-|x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} g(y)\, dS(y) \end{aligned}$$

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \tag{2.11}$$

Poisson's formula on $B(0,r)$

Theorem 2.29. *Suppose that $g \in C(\partial B(0,x))$. then $u(x)$ given by (2.11) has the following properties:*

i.) $u \in C^\infty(B(0,r))$

ii.) $\Delta u = 0$

iii.) $\forall x_0 \in \partial B(0,r)$

$$\lim_{x \to x_0} u(x) = g(x_0), \quad x \in B(0,r)$$

<u>Proof:</u> Exercise, similar to the proof in the half-space.

next subsection overlaps with following AMSI notes inclusion,fix this

2.6.3 Energy Methods for Poisson's Equation

Theorem 2.30. *There exists at most one solution of*

$$\begin{cases} -\Delta u = f & in \quad \Omega \\ u = g & on \quad \partial\Omega \end{cases}$$

if $\partial\Omega$ is smooth, f, g continuous.

Proof: Two solutions u_1 and u_2. Then define $w = u_1 - u_2$, thus

$$\begin{cases} -\Delta w = 0 & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\begin{aligned} 0 &= \int_\Omega -\Delta w \cdot w \, dx \\ &= -\int_{\partial\Omega} Dw \cdot \nu \underbrace{w}_{=0} \, dS(y) + \int_\Omega |Dw|^2 \, dx \\ &= \int_\Omega |Dw|^2 \, dx \end{aligned}$$

$\Longrightarrow Dw = 0$ in Ω
$\Longrightarrow w =$ constant in Ω
$\Longrightarrow w = 0$ on $\partial\Omega$
$\Longrightarrow w = 0$ in Ω
$\Longrightarrow u_1 = u_2$ in Ω. ■

2.6.4 Perron's Method of Subharmonic Functions

- Poisson's formula on balls
- compactness argument
- construct a solution of $\Delta u = 0$ in Ω
- Check boundary data, barriers.

Assumption: Ω open, bounded and connected.

Definition 2.31. *A continuous function u is said to be subharmonic on Ω, if for all balls $B(x, r) \Subset \Omega$, and all harmonic functions h with $u \leq h$ on $\partial B(x, r)$ we have $u \leq h$ on $B(x, r)$.*

Definition 2.32. *Superharmonic is analogous.*

Proposition 2.33. *Suppose u is subharmonic in Ω. Then u satisfies a strong maximum principle. Moreover, if v is superharmonic on Ω with $u \le v$ on $\partial\Omega$, then either $u < v$ on Ω, or $u = v$ on Ω.*

Proof: u satisfies a mean-value property:
$B(x,r) \Subset \Omega$. Let h be the solution of

$$\begin{aligned} \Delta h &= 0 \quad \text{in } B(x,r) \\ h &= u \quad \text{on } \partial B(x,r) \end{aligned}$$

Then

$$u(x) \le h(x) = \fint_{\partial B(x,r)} h(y)\, dy = \fint_{\partial B(x,r)} u(y)\, dy \qquad \blacksquare$$

Idea: Local modification of subharmonic functions.

Definition 2.34. *u subharmonic on Ω, $B \Subset \Omega$. Let $\overline{u}$ be the solution of*

$$\begin{aligned} \Delta \overline{u} &= 0 \quad \text{in } B \\ \overline{u} &= u \quad \text{on } \partial B \end{aligned}$$

Define

$$U(x) = \begin{cases} \overline{u}(x) & \text{in } B \\ u(x) & \text{on } \Omega \setminus B \end{cases}$$

U is called the harmonic lifting of u on B. U subharmonic.

In 1D:

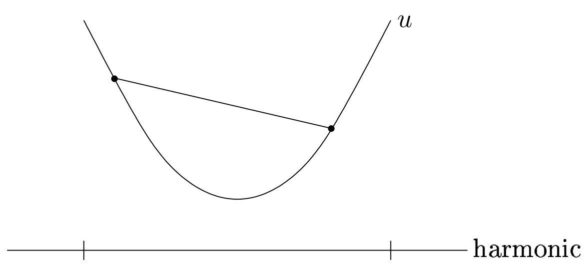


Figure 2.18:

Proposition 2.35. *If $u_1, \ldots, u_n$ are subharmonic, then $u = \max\{u_1, \ldots, u_n\}$ is subharmonic.*

Idea: Φ is bounded on $\partial\Omega$. A continuous subharmonic function u is called a subfunction relative to Φ if $u \leq \Phi$ on $\partial\Omega$.

Define superfunction analogously

$$\begin{array}{ll} \inf\limits_{\partial\Omega} \Phi & \text{subfunction} \\ \sup\limits_{\partial\Omega} \Phi & \text{superfunction} \end{array}$$

Theorem 2.36. *Φ bounded function on $\partial\Omega$.*

$$S_\Phi = \{v : \ v \textit{ subfunction relative to } \Phi\}$$

and define $u = \sup_{v \in S_\Phi} v$. Then u is harmonic in Ω.

Remark: u is well-defined since $w(x) = \sup_{\partial\Omega} \Phi$ is a superfunction.

All subfunction $\leq$ all superfunctions on $\partial\Omega$, and thus by Proposition 2.20, all subfunctions $\leq$ all superfunctions on Ω. Thus,

$$\text{all subfunctions } \leq \sup_{\partial\Omega} \Phi < \infty.$$

Proof: u is well-defined. Fix $y \in \Omega$, there exists $v_k \in S_\Phi$ such that $v_k(y) \to u(y)$. Replace v_k by $\max\{v_k, \inf \Phi\}$ if needed to ensure that v_k is bounded from below. Choose

$$B(y, R) \Subset \Omega, \qquad R > 0.$$

$v_k =$ harmonic lifting of v_k on $B(y, R)$
v_k uniformly bounded
$v_k(y) \to u(y)$
v_k is harmonic in $B(y, R)$.
HW1 (compactness result): $\exists$ subsequence v_{k_i} that converges uniformly on all compact subsets on $B(y, R)$ to a harmonic function v. By construction, $v(y) = u(y)$.

Assertion: $u = v$ on $B(y, R)$. $\implies u$ harmonic on $B(y, R) \implies u$ harmonic in Ω

Clearly: $v \leq u$ in $B(y, R)$

Suppose otherwise:

$$\begin{aligned} & \exists z \in B(y,R),\ v(z) < u(z) \\ \Longrightarrow\quad & \exists \overline{u} \in S_\Phi \text{ such that } v(z) < \overline{u}(z) \le u(z) \end{aligned}$$

Take $w_L = \max\{\overline{u}, v_{k_i}\}$

$$W_L = \text{ harmonic lifting of } w_L \text{ on } B.$$

As before, $\exists$ subsequence of $\{W_L\}$ that converges to a harmonic function w on B, by construction:

$$v \le w \le u$$

$v(y) = w(y) = u(y)$

Maximum principle $\Longrightarrow v = w$.

However, $w(z) > v(z)$, contradiction.
$\Longrightarrow u = v$. ∎

Boundary behavior

Definition 2.37. *Let $\xi \in \partial\Omega$. A function $\omega \in C(\overline{\Omega})$ is called a <u>barrier</u> at ξ relative to Ω if*

i.) w *is superharmonic in* Ω

ii.) $w > 0$ *in* $\overline{\Omega} \setminus \{\xi\}$

Theorem 2.38. *Φ bounded on $\partial\Omega$. Let u be Perron's solution of $\Delta u = 0$ as constructed before. Assume that ξ is a regular boundary point, i.e., there exists a barrier at ξ, and that Φ is continuous at ξ. Then u is continuous at ξ and $u(x) \to \Phi(\xi)$ as $x \to \xi$.*

<u>Proof:</u> $M = \sup_{\partial\Omega} |\Phi|$. $w =$ barrier at ξ. Φ continuous at ξ. Fix $\epsilon > 0$, $\exists \delta > 0$ such that $|\Phi(x) - \Phi(\xi)| < \epsilon$ if $|x - \xi| < \delta$, $x \in \partial\Omega$. Choose $k > 0$ such that $kw(x) > 2M$ for $x \in \overline{\Omega}$, and $|x - \xi| > \delta$.

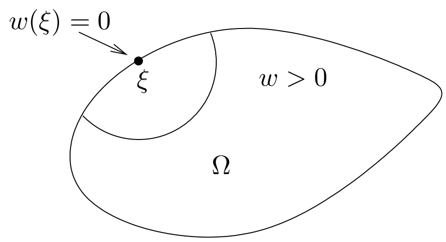


Figure 2.19:

$x \mapsto \Phi(\xi) - \epsilon - kw(x)$ is a subfunction.
$x \mapsto \Phi(\xi) + \epsilon + kw(x)$ is a superfunction.

w superharmonic $\Longrightarrow -w$ subharmonic.

By construction

$$\begin{aligned} \Phi(\xi) - \epsilon - kw(x) &\leq u(x) \\ &\leq \Phi(\xi) + \epsilon + kw(x) \\ \Longleftrightarrow |u(x) - \Phi(\xi)| &\leq \epsilon + kw(x) \end{aligned}$$

and $w(x) \to 0$ as $x \to \xi$

$$|u(x) - \Phi(\xi)| < \epsilon \quad \text{if} \quad |x - \xi| \text{ is small enough.} \quad \blacksquare$$

Theorem 2.39. *Ω bounded, open, connected and all $\xi \in \partial\Omega$ regular. g continuous on $\partial\Omega \Longrightarrow$ there exists a solution of*

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ h = g & \text{on} \quad \partial\Omega \end{cases}$$

Remark: A simple criterion for $\xi \in \partial\Omega$ to be regular is that Φ satisfies an exterior sphere condition: there exists $B(y, R)$, $R > 0$ such that $B(y, R) \cap \overline{\Omega} = \{\xi\}$

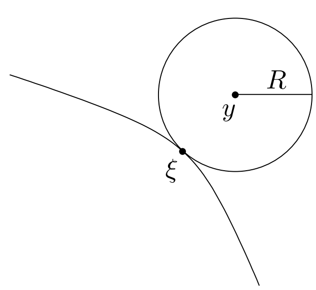


Figure 2.20:

Barrier:

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & n \geq 3 \\ \ln \frac{|x-y|}{R} & n = 2 \end{cases}$$

2.7 Solution of the Dirichlet Problem by the Perron Process

2.7.1 Convergence Theorems

Important to the proof of existence of solutions to Laplace's equation is the concept of convergence of harmonic functions.

Theorem 2.40. *A bounded sequence of harmonic functions on a domain Ω contains a subsequence which converges uniformly, together with its derivatives, to a harmonic function.*

Proof: Take $\Omega' \subset \Omega$ compact. Now, we calculate via the mean value theorem of calculus:

$$\begin{aligned} |u_m(x) - u_n(y)| &\leq \max_{\substack{tx+(1-t)y \\ 0 \leq t \leq 1}} |Du_m| \cdot |x-y| \\ &\leq \max_{\Omega'} |Du_m| \cdot |x-y| \\ &\leq C \sup_{\Omega'} |u_m| \cdot |x-y| \\ &\leq C'(K,n,L)|x-y|, \end{aligned}$$

where L is a constant which bounds the sequence elements u_m. From this, we see that the sequence is equicontinuous. Next we recall that the Azalea-Ascoli theorem asserts that such a sequence will have a subsequence converging uniformly to a limit. Now, one may apply the same calculation above iteratively to any order of derivative (iteratively refining the subsequence for each new derivative). From this we conclude that there is a subsequence whose derivatives up to and including order $|\beta|$ all converge uniformly to a limit for any $|\beta|$. It now follows that the limit is indeed a solution to Laplace's equation. ■

2.7.2 Generalized Definitions & Harmonic Lifting

Before proving the existence of solution to the Dirichlet problem for Laplace's equation, one needs to extend the notions of subharmonic(superharmonic) functions to non-C^2 functions.

Definition 2.41. *$u \in C^0(\Omega)$ is* subharmonic(superharmonic) *in Ω if for any ball $B \Subset \Omega$ and harmonic function $h \in C^2(B) \cap C^0(\overline{B})$ with $h \geq (\leq)\, u$ on ∂B, one has $h \geq (\leq)\, u$ on B.*

Note: If u is subharmonic and superharmonic, then it is harmonic.

From this new definition, some properties are immediate.
Properties:

i.) The Strong Maximum Principle still applies to this extended definition. Since u is subharmonic, $u \leq h$ for h harmonic in B with the same boundary values, the Strong Maximum Principle follows by applying the Poisson Integral formula to this situation.

ii.) $w(x) = \max_m(\min_m)\{u_1(x), \ldots, u_m(x)\}$ is subharmonic(superharmonic) if $u_1, \ldots, u_m$ are subharmonic(superharmonic).

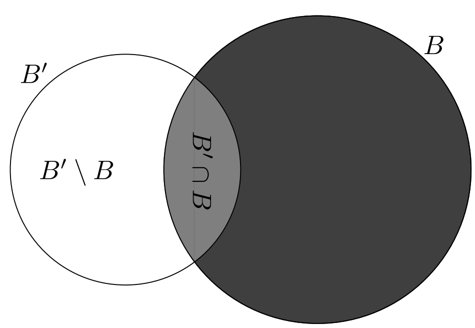
iii.) The *Harmonic Lifting* on a ball B for a subharmonic function u on Ω is defined as

$$U(x) = \begin{cases} u(x) & \text{for} \quad x \notin B \\ h(x) & \text{for} \quad x \in B \end{cases},$$

where $h \in C^2(B) \cap C^0(\overline{B})$ is harmonic in B with $h = u$ on ∂B.

Note: The key thing to realize about the definition of harmonic lifting is that the "lifted" part is harmonic in our classical C^2 sense; it is simply the Poisson integral solution on the ball with u taken as boundary data.

Proposition 2.42. *U is subharmonic on Ω, when h is harmonic in B.*



Proof: Refer to the figure on the next page as you read this proof. Take another ball $B' \Subset \Omega$ along with another harmonic function $h' \in C^2(B') \cap$

$C^0(\overline{B'})$ which is $\geq U$ on $\partial B'$. By definition of $u(x)$, one has that $h' \geq u$ on $B' \setminus B$. This, in turn, leads to the assertion that $h' \geq U$ on $\partial B \cap B'$, i.e. $h' \geq U$ on $\partial(B' \cap B)$. As, $h' - U$ is harmonic (in the classical sense), we can apply the Weak Maximum Principle to ascertain that $h' \geq U$ on $B \cap B'$. Since B' is arbitrary, the proposition follows. ■

2.7.3 Perron's Method

Now, we can start pursuit of the main objective of this section, namely to prove the existence of the Dirichlet Problem:

$$\begin{aligned} \Delta u = 0 &\quad \text{in } \Omega, \Omega \text{ bounded domain in } \mathbb{R}^n \\ u = \phi &\quad \text{on } \partial\Omega, \phi \in C^0(\partial\Omega) \end{aligned}$$

The first step to doing this requires the following definition

Definition 2.43. *A* subfunction(superfunction) *in Ω is subharmonic(superharmonic) in Ω and is in $C^0(\overline{\Omega})$ in addition to being $\leq \phi$ on $\partial\Omega$.*

Now, we define

$$S_\phi = \{\text{all subfunctions}\},$$

along with

$$\tilde{u}(x) = \sup_{v \in S_\phi} v(x),$$

where the supremum is understood to be taken in the pointwise sense. $\tilde{u}(x)$ is bounded as $v \in S_\phi$ implies that $v \leq \max_{\partial\Omega} \phi$ via the Weak Maximum Principle.

Remark: Essentially, the above definitions allow one to construct a weak formulation (or weak solution) to the Dirichlet Problem via Maximum Principles. Indeed, at this point, it has not been ascertained whether or not $\tilde{u}$ is continuous.

Theorem 2.44. *$\tilde{u}$ is harmonic in Ω.*

Proof: First, fix $y \in \Omega$. Since $u(y) = \sup_{S_\phi} v$, there is a sequence $\{v_m\} \subset S_\phi$ such that $v_m(y) \to u(y)$. Of course, we may assume that v_m is bounded from above by $\max_{\partial\Omega} \phi$. Now, consider a ball $B \Subset \Omega$ and replace all v_m by their harmonic liftings on B. Denoting the resulting sequence as $\{V_m\}$, it is clear that $\{V_m\} \subset S_\phi$ and that $V_m(y) \to u(y)$. Since $\{V_m\}$ are all harmonic (in the C^2 sense) in B, we apply our harmonic convergence theorem to extract a subsequence (again denoted $\{V_m\}$ after relabeling) which converges uniformly in compact subset of B to a harmonic function $\tilde{v}$ in B. Obviously, we have $\tilde{v}(y) = u(y)$, but we now make the following

Claim: $\tilde{v} = u$ in B.

Proof of Claim: Consider $z \in B$ such that $\tilde{v}(z) < u(z)$. This implies that there exists a function $\overline{v} \in S_\phi$ such that $\overline{v}(z) > \tilde{v}(z)$. Replace V_m by $\overline{v_m} = \max\{V_m, \overline{v}\}$. Next, take the harmonic liftings in B of these to get $\{\overline{V_m}\}$. Again, utilizing the harmonic convergence theorem, we extract a subsequence (still denoted $\overline{V_m}$ after relabeling) such that $\{V_m\}$ converges locally uniformly to a harmonic function $w > \tilde{v}$ in B, but $w(y) = v(y) = u(y)$. Thus, by the Strong Maximum Principle $w = \tilde{v}$, a contradiction.

Thus, $\tilde{v} = u$ in B which implies that u is indeed harmonic. ■

Now, to complete our existence proof, we need to see if the Perron solution of the Dirichlet Problem attains the proper boundary data. To do this we make the next definition.

Definition 2.45. *A* Barrier *at $y \in \partial\Omega$ is a function $w \in C^0(\overline{\Omega})$ that has the following properties:*

i.) w is superharmonic,

ii.) $w(y) = 0$,

iii.) $w > 0$ on $\overline{\Omega} \setminus \{y\}$.

Theorem 2.46. *Let u be a Perron solution. Then $u \in C^0(\overline{\Omega})$ with $u = \phi$ on $\partial\Omega$ if and only if there exists a barrier at each point $y \in \partial\Omega$.*

Proof:

(Only if) Solve with the BC $\phi = |x - y|$.

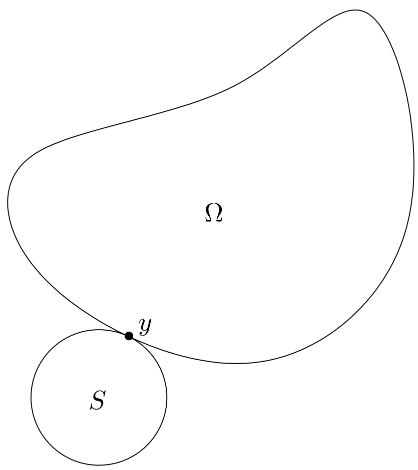
(If) Take $\epsilon > 0$ which implies there exists a $\delta > 0$ such that $|\phi(x) - \phi(y)| < \epsilon$ implies $|x - y| < \delta$ by continuity of ϕ. Now, choose a constant $K(\epsilon)$ large enough so that $Kw > 2 \cdot \max_{\partial\Omega} |\phi|$ if $|x - y| \geq \delta$. Next, define $w^\pm := \phi(y) \pm (\epsilon + Kw)$. It is clear that w^- is a subfunction and that

w^+ is a superfunction with $w^- \le u \le w^+$ on $\partial\Omega$. Application of the Weak Maximum Principle now yields

$$|u(x) - \phi(y)| \le \epsilon + Kw.$$

Taking $\epsilon \to 0$, one sees that $u(x) \to \phi(y)$ as $x \to y$ (since $w(x) \to 0$ as $x \to y$ by the definition). ■

Now, that it has been shown the existence of Perron solutions is equivalent to the existence of barriers on $\partial\Omega$, we wish to ascertain precisely what criterion on $\partial\Omega$ implies the existence of barriers. With that, we make the following definition.



Definition 2.47. *A domain* Ω *satisfies an* exterior sphere condition *if at every point* $y \in \partial\Omega$*, there exists a sphere* S *such that* $S \cap \overline{\Omega} = \{y\}$ *(refer to the above figure).*

Correspondingly, we have the following theorem.

Theorem 2.48. *If* Ω *satisfies an exterior sphere condition, then there exists a barrier at each point* $y \in \partial\Omega$.

Proof: Simply take

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & \text{for} \quad n > 2 \\ \ln \frac{R}{|x-y|} & \text{for} \quad n = 2 \end{cases},$$

as the barrier at $y \in \partial\Omega$. It is trivial to verify this function is indeed a barrier at $y \in \partial\Omega$. ■

Remarks:

i.) It is easy to see that if the boundary $\partial\Omega$ is smooth, in a sense that locally it is the graph of a C^2 function, then it satisfies an exterior sphere condition. Also, all convex domains satisfy an exterior sphere condition.

ii.) There are more general criterion for the existence of barriers, like the exterior cone condition; but the proofs are harder.

2.8 Exercises

2.1:

a) Find all solutions of the equation $\Delta u + cu = 0$ with radial symmetry in $\mathbb{R}^3$. Here $c \geq 0$ is a constant.

b) Prove that

$$\Phi(x) = -\frac{\cos(\sqrt{c}|x|)}{4\pi|x|}$$

is a fundamental solution for the equation, i.e., show that for all $f \in C^2(\mathbb{R}^3)$ with compact support a solution of the equation $\Delta u + cu = f$ is given by

$$u(x) = \int_{\mathbb{R}^3} \Phi(x-y)f(y)\ dy.$$

Hint: You can solve the ODE you get in a) by defining $w(r) = rv(r)$. Show that w solves $w'' + cw = 0$.

2.2: [Evans 2.5 #4] We say that $v \in C^2(\overline{\Omega})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

a) Suppose that v is subharmonic. Show that

$$v(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)\ dy \quad \text{for all } B(x,r) \subset \Omega.$$

b) Prove that subharmonic functions satisfy the maximum principle,

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v.$$

c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and let $v = \phi(u)$. Prove that v is subharmonic.

d) Let u be harmonic. Show that $v = |Du|^2$ is subharmonic.

2.3: Let $\mathbb{R}^n_+$ be the upper half space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0$.

a) Suppose that $\tilde{u}$ is a smooth solution of $\Delta u = 0$ in $\mathbb{R}^n_+$ with $\tilde{u} = 0$ on $\partial\mathbb{R}^n_+$.

b) Use the result in a) to show that bounded solutions of the Dirichlet problem in the half space $\mathbb{R}^n_+$,

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = g & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

are unique. Show by a counter example that the corresponding statement does not hold for unbounded solutions.

2.4: [Qualifying exam 08/98] Suppose that $u \in C^2(\mathbb{R}^2_+) \cap C^0(\mathbb{R}^2_+)$ is harmonic in $\mathbb{R}^2_+$ and bounded in $\mathbb{R}^2_+ \cup \partial\mathbb{R}^2_+$. Prove the maximum principle

$$\sup_{\mathbb{R}^2_+} u = \sup_{\partial\mathbb{R}^2_+} u$$

Hint: Consider $\epsilon > 0$ the harmonic function

$$v_\epsilon(x, y) = u(x, y) - \epsilon \ln(x^2 + (y + 1)^2).$$

2.5: Fundamental solutions and Green's functions in one dimension.

a) Find a fundamental solution Φ of Laplace's equation in one dimension,

$$-u'' = 0 \quad \text{in } (-\infty, \infty).$$

Show that

$$u(x) = \int_{-\infty}^{\infty} \Phi(x - y) f(y)\, dy$$

is a solution of the equation $-u'' = f$ for all $f \in C^2(\mathbb{R})$ with compact support.

b) Find the Green's function for Laplace's equation on the interval $(-1, 1)$ subject to the Dirichlet boundary conditions $u(-1) = u(1) = 0$.

c) Find the Green's function for Laplace's equation on the interval $(-1, 1)$ subject to the mixed boundary conditions $u'(-1) = 0$ and $u(1) = 0$, i.e., find a fundamental solution Φ with $\Phi'(-1) = 0$ and $\Phi(1) = 0$.

d) Can you find the Green's function for the Neumann problem on the interval $(-1, 1)$, i.e., can you find a fundamental solution Φ with $\Phi'(-1) = \Phi'(1) = 0$?

2.6: Use the method of reflection to find the Green's function for the first quadrant $R^2_{++} = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ in $\mathbb{R}^2$. Find a representation of a solution u of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^2_{++}, \\ u = g & \text{on} \quad \partial\mathbb{R}^2_{++}, \end{cases}$$

where g is a continuous and bounded function on $\partial\mathbb{R}^2_{++}$.

2.7: Let Ω_1 and Ω_2 be open and bounded sets in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary that are connected. Suppose that Ω_1 is compactly contained in Ω_2, i.e., $\overline{\Omega}_1 \subset \Omega_2$. Let $G_i(x,y)$ be the Green's functions for the Laplace operator on Ω_i, $i = 1, 2$. Show that

$$G_1(x,y) < G_2(x,y) \quad \text{for } x, y \in \Omega_1,\ x \neq y.$$

2.8: [Evans 2.5 #6] Use Poisson's formula on the ball to prove that

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}}u(0),$$

whenever u is positive and harmonic in $B(0,r)$. This is an explicit form of Harnack's inequality.

2.9: [Qualifying exam 08/97] Consider the elliptic problem in divergence form,

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u)$$

over a bounded and open domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. The coefficients satisfy $a_{ij} \in C^1(\overline{\Omega})$ and the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda_0|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

with a constant $\lambda_0 > 0$. A barrier at $x_0 \in \partial\Omega$ is a C^2 function w satisfying

$$\mathcal{L}w \geq 1 \text{ in } \Omega, \quad w(x_0) = 0, \text{ and } \quad w(x) \geq 0 \text{ on } \partial\Omega.$$

a) Prove the comparison principle: If $\mathcal{L}u \geq \mathcal{L}v$ in Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

b) Let $f \in C(\overline{\Omega})$ and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Show that there exists a constant C that does not depend on w such that

$$\left| \frac{\partial u}{\partial \nu}(x_0) \right| \leq C \left| \frac{\partial w}{\partial \nu}(x_0) \right|$$

where ν is the outward normal to $\partial\Omega$ at x_0.

c) Let Ω be convex. Show that there exists a barrier w for all $x_0 \in \partial\Omega$.

Chapter 3

The Heat Equation

$$u_t - \triangle u = 0$$

3.1 Fundamental Solution

Evan's approach: find a solution that is invariant under dilation scaling $u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$. Want $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$ for all $\lambda > 0$ and α, β constant that need to be determined.

<u>Idea:</u> $\lambda = \frac{1}{t}$, i.e. find

$$u(x,t) = \frac{1}{t^\alpha} u\left(\frac{x}{t^\alpha}, 1\right) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

and let $y = \frac{x}{t^\beta}$. Find an equation for v.

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \frac{-\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha} \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}\left(\frac{x}{t^\beta}\right) \cdot \frac{-\beta x_j}{t^{\beta+1}} \\
&= \frac{-\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) - \frac{\beta}{t^{\alpha+1}} D_x v\left(\frac{x}{t^\beta}\right) \cdot \frac{x}{t^\beta} \\
\frac{\partial}{\partial x_i} u(x,t) &= \frac{1}{t^\alpha} \frac{\partial v}{\partial x_i} \frac{1}{t^\beta} \\
\frac{\partial^2}{\partial x_i^2} u(x,t) &= \frac{1}{t^\alpha} \frac{\partial^2 v}{\partial x_i} \frac{1}{t^{2\beta}} \\
u_t - \triangle u &= 0
\end{aligned}$$

$$\Longrightarrow \frac{\alpha}{t^{\alpha+1}} v(y) + \frac{\beta}{t^{\alpha+1}} Dv(y) \cdot y + \frac{1}{t^{\alpha+2\beta}} \triangle v = 0$$

$\lambda = \frac{1}{t}$, $\beta = \frac{1}{2}$

$$\alpha v(y) + \frac{1}{2} Dv \cdot y + \triangle u = 0$$

<u>Assume:</u> v is radial, $v(y) = w(r) = w(|y|)$

$$\begin{aligned} D_i w(|y|) &= w'(|y|)\frac{y_i}{|y|} \\ Dv(y) \cdot y &= w'(|y|)\frac{|y|^2}{|y|} = w'(r)r \end{aligned}$$

Equation for w:

$$\alpha w + \frac{1}{2} w'(r) r + w'' + \frac{n-1}{r} w' = 0$$

$\alpha = \frac{n}{2}$:

$$\frac{n}{2} w + \frac{1}{2} w' r + w'' + \frac{n-1}{r} w' = 0$$

xr^{n-1}:

$$\underbrace{\frac{n}{2} r^{n-1} w + \frac{1}{2} r^n w'}_{\frac{1}{2}(r^n w)'} + \underbrace{r^{n-1} w'' + (n-1) r^{n-2} w'}_{(r^{n+1} w')'} = 0$$

Integrate:

$$\frac{1}{2} r^n w + r^{n-1} w' = a = \text{ constant}$$

<u>Assume:</u> $r^n w \to 0$, $r^{n-1} w' \to 0$ as $r \to \infty$, then $a = 0$ and

$$\begin{aligned} r^{n-1} w' &= -\frac{1}{2} r^n w \\ w' &= -\frac{1}{2} r w \Longrightarrow w(r) = b e^{-\frac{1}{4} r^2} \\ v(y) &= w(|y|) = b e^{-\frac{|y|^2}{4}} \qquad y = \frac{x}{\sqrt{t}} \\ u(x,t) &= \frac{1}{t^\alpha} v(y) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \end{aligned}$$

Definition 3.1. *The function*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t < 0 \end{cases}$$

is called the *<u>fundamental solution</u>* *of the heat equation.*

Motivation for choice of b:

Lemma 3.2.

$$\int_{\mathbb{R}^n} \Phi(y,t)\ dy = 1, \qquad t > 0$$

<u>Proof:</u>

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx \right)^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\ dydx \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\ drd\theta \\ &= 2\pi \left(-\frac{1}{2} e^{-r^2} \right) \Bigg|_0^{\infty} \\ &= \pi \end{aligned}$$

$$\int_{\mathbb{R}^n} \Phi(y,t)\ dy = \frac{1}{(4\pi t)^{n+2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}}\ dy$$

Change coordinates: $z = \frac{y}{\sqrt{4t}}$, $dz = \frac{1}{(4t)^{\frac{n}{2}}}\ dy$

$$\int_{\mathbb{R}^n} \Phi(y,t)\ dy = \frac{1}{\pi^{n+2}} \int_{\mathbb{R}^n} e^{-|z|^2}\ dz = 1. \qquad \blacksquare$$

Initial value problem for the heat equation in $\mathbb{R}^n$: want to solve

$$\begin{aligned} u_t - \triangle u &= 0 \text{ in } \mathbb{R}^n \times 0, \infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

Theorem 3.3. *Suppose g is continuous and* *<u>bounded</u>* *on $\mathbb{R}^n$, define*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} g(y)\ dy \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$. Then *i.)* $u \in C^\infty(\mathbb{R}^n \setminus (0,\infty))$

ii.) $u_t - \triangle u = 0$ *in* $\mathbb{R}^n \times (0,\infty)$

iii.) $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$ *if* $x_0 \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $t > 0$

Remark

i.) "smoothening": the solution is C^∞ for all positive times, even if g is only continuous.

ii.) Suppose that $g > 0$, $g \equiv 0$ outside $B(0,1) \in \mathbb{R}^n$, then $u(x,t) > 0$ in $(\mathbb{R}^n \times (0,\infty))$, "infinite speed of propagation".

Proof: $e^{\frac{-|x-y|^2}{4t}}$ is infinitely many times differentiable

i.) in x, t, the integral exists, and we can differentiate under the integral sign.

ii.) $u_t - \triangle u = 0$ since $\Phi(y,t)$ is a solution

iii.) Boundary data:
$\epsilon > 0$, choose $\delta > 0$ such that $|g(x_0) - g(x)| < \epsilon$ for $|x - x_0| < \delta$, Then for $|x - x_0| < \frac{\delta}{2}$

$$\begin{aligned}
|u(x,t) - g(x_0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y)[g(y) - g(x_0)]\, dy \right] \\
&= \int_{B(x_0,\delta)} \Phi(y+x,t)|g(y) - g(x_0)|\, dy \\
&\quad + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\underbrace{|g(y) - g(x_0)|}_{\leq C}\, dy
\end{aligned}$$

$|x - x_0| < \frac{\delta}{2}$, $|y - x_0| > \delta$

$\Longrightarrow$ (see Green's function on half space)

$\frac{1}{2}|x_0 - y| \leq |x - y|$ Then

$$\begin{aligned}
\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{\frac{-|x-y|^2}{4t}}\, dy &\leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{\frac{-|x_0-y|^2}{16t}}\, dy \\
&\leq \frac{C}{(4\pi t)^{\frac{n}{2}}} \int_0^\infty e^{\frac{-r^2}{16t}} r^{n-1}\, dr \to 0
\end{aligned}$$

as $t \to 0$

$|u(x,t) - g(x_0)| < 2\epsilon$ if $|x - x_0| < \frac{\delta}{2}$ and t small enough, if $|(x,t) - (x_0,t)|$ is small enough. ■

Formally: $\Phi_t - \triangle\Phi = 0$ in $\mathbb{R}^n \times (0,\infty)$
Φ as $t \to 0$

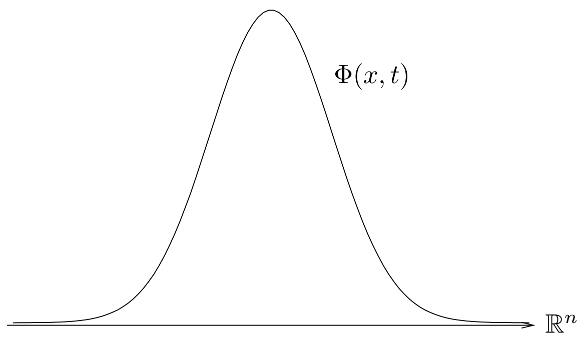


Figure 3.1:

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1 \qquad \text{for } t > 0.$$

Formally: $\Phi(x,0) = \delta_x$, i.e., the total mass is concentrated at $x = 0$.

3.2 The Non-Homogeneous Heat Equation

Non-homogeneous equation:

$$\left.\begin{array}{rcl} u_t - \Delta u = f & \text{in} & \mathbb{R}^n \times \{t > 0\} \\ u = 0 & \text{on} & \mathbb{R}^n \times \{t = 0\} \end{array}\right\} \tag{3.1}$$

3.2.1 Duhamel's Principle

$\Phi(x-y, t-s)$ is a solution of the heat equation. For fixed $x > 0$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\ dy$$

Solves

$$\left.\begin{aligned} u_t(\cdot;s) - \Delta u(\cdot;s) = 0 \quad &\text{in} \quad \mathbb{R}^n \times \{s, \infty\} \\ u(\cdot;s) = f(\cdot, s) \quad &\text{on} \quad \mathbb{R}^n \times \{t = s\} \end{aligned}\right\}$$

Duhamel's principle: the solution of (3.1) is given by

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t;s)\ ds \qquad \Longleftrightarrow \\ u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\ dyds \end{aligned} \tag{3.2}$$

Compare to Duhamel's principle applied to the transport equation.

Remark: The solution of

$$\left.\begin{aligned} u_t - \Delta u = f \quad &\text{in} \quad \mathbb{R}^n \times \{t > 0\} \\ u = g \quad &\text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{aligned}\right\} \tag{3.3}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\ dyds$$

Proof: See proof of following theorem.

Theorem 3.4. *Suppose that $f \in C_1^2(\mathbb{R}^n \times (0,\infty))$ (i.e. f is twice continuously differentiable in x, f is continuously differentiable in t), f has compact support, then the solution of the non-homogeneous equation is given by (3.2).*

Proof: Change coordinates

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\ dyds$$

We can differentiate under the integral sign:

$$\frac{\partial u}{\partial t}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial f}{\partial t}(x-y, t-s)\ dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0)\ dy$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y, t-s)\ dyds$$

and all these derivatives are continuous.

$$\begin{aligned}
u_t - \Delta u &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\ dyds \\
&\quad + \underbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\ dy}_{K} \\
&= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)(-\partial_s - \Delta_y) f(x-y,t-s)\ dyds \\
&\quad + \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)(-\partial_s - \Delta_y) f(x-y,t-s)\ dyds + K \\
&= I_\epsilon + J_\epsilon + K
\end{aligned}$$

$$\begin{aligned}
|J_\epsilon| &\leq (\sup|\partial_t f| + \sup|D^2 f|) \cdot \int_0^\epsilon \underbrace{\int_{\mathbb{R}^n} \Phi(y,s)\ dy}_{1} ds \\
&= \epsilon \to 0 \text{ as } \epsilon \to 0
\end{aligned}$$

$$\begin{aligned}
I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^n} \underbrace{(\partial_s - \Delta_y)\Phi(y,s)}_{=0} f(x-y,t-s)\ dyds \\
&\quad \underbrace{- \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\ dy}_{-K} \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\ dy
\end{aligned}$$

Collecting the terms:

$$\begin{aligned}
(u_t - \Delta u)(x,t) &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\ dy \\
&= f(x,t)
\end{aligned}$$

Calculation similar to last class. ■

3.3 Mean-Value Formula for the Heat Equation

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\ dyds$$

Analogy: Laplace $\partial B(x,r)$, $B(x,r)$
$\partial B(x,r) =$ level set of the fundamental solution.

Expectation: level sets of $\Phi(x,t)$ relevant.

Definition 3.5. *$x \in \mathbb{R}^n$. $t \in \mathbb{R}, r > 0$. Define the heat ball*

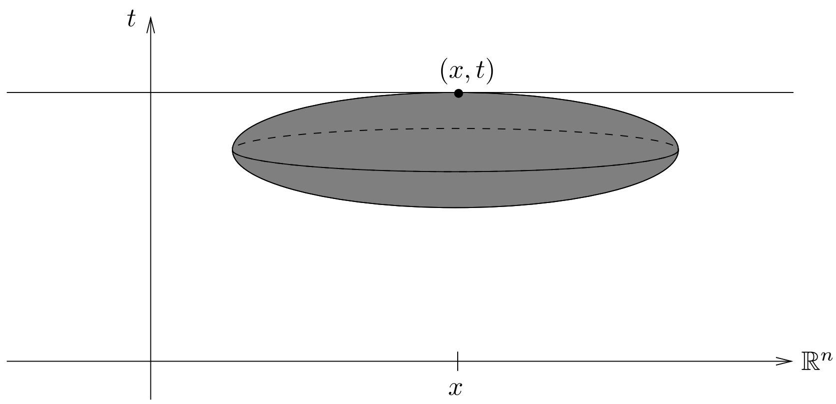


Figure 3.2:

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} \middle| \; s \leq t, \; \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$

x, t is in the boundary of $E(x,t;r)$

Definition 3.6. *1. Parabolic cylinder: Ω open, bounded set, then*
$\Omega_T = \Omega \times (0,T]$
Ω_T includes the top surface

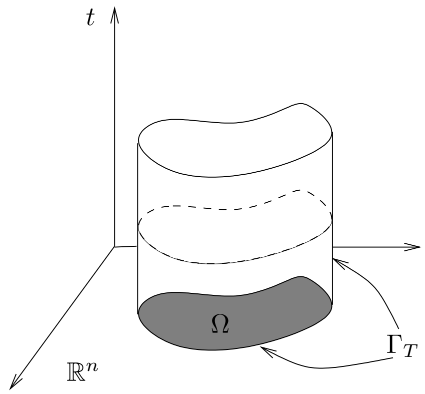


Figure 3.3:

2. *Parabolic boundary:* $\Gamma_T := \overline{\Omega}_T \setminus \Omega_T =$ *bottom surface + lateral boundary*

(This body is the relevant boundary for the maximum principle).

Theorem 3.7 (Mean value formula). Ω *bounded, open*
$u \in C_1^2(\Omega_T)$ *solution of the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds$$

$\forall E(x,t,r) \subset \Omega_T$

Proof: Translate in space and time to assume $x = 0, t = 0$.
$E(r) = E(0,0,r)$

$$\phi(r) = \frac{1}{4r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{|s|^2} \, dyds$$

Idea: compute $\phi'(r)$.

$E(r)$:

$$\begin{aligned}
& \phi(-y-s) > 1 \\
\iff \quad & \frac{1}{(4\pi(-s))^{\frac{n}{2}}} e^{\frac{-|y|^2}{-4s}} \geq 1 \\
\iff \quad & \frac{1}{\left(4\pi\frac{-s}{r^2}\right)^{\frac{n}{2}}} e^{-(y/r)^2/(-4(s/r^2))} \geq 1
\end{aligned}$$

$h(y,s) = (ry, r^2 s)$ maps $E(1)$ to $E(r)$.

Change coordinates:

$$\begin{aligned}
\frac{1}{4r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}\,dyds &= \frac{1}{4r^n}\iint_{h(E(1))} u(y,s)\frac{|y|^2}{s^2}\,dyds \\
\left[\int_{h(\Omega)} f(x)\,dx \right.&= \left.\int_{\Omega}(f\circ h)(x)\cdot|\det(Dh)|\,dx\right] \\
&= \frac{1}{4r^n}\iint_{E(1)} u(ry,r^2 s)\frac{|ry|^2}{(r^2 s)^2} r^{n+2}\,dyds \\
&= \frac{1}{4}\iint_{E(1)} u(ry,r^2 s)\frac{|y|^2}{s^2}\,dyds
\end{aligned}$$

$$\begin{aligned}
\phi'(r) &= \frac{1}{4}\iint_{E(1)}\left\{\sum_i \frac{\partial u}{\partial y_i}(ry_i,r^2 s)\cdot y_i\frac{|y|^2}{s^2} + \frac{\partial u}{\partial s}(ry,r^2 s)\cdot 2rs\frac{|y|^2}{s^2}\right\}\,dyds \\
&= \frac{1}{4}\iint_{E(r)}\left\{Du(y,s)\cdot\frac{y}{r}\frac{(y/r)^2}{(s/r^2)^2} + \frac{\partial u}{\partial s}(y,s)\cdot 2r\frac{|y/r|^2}{s/r^2}\right\}\frac{1}{r^{n+2}}\,dyds \\
&= \frac{1}{4r^{n+1}}\iint_{E(r)}\left\{Du(y,s)\cdot y\frac{|y|^2}{s^2} + 2\frac{\partial u}{\partial s}(y,s)\frac{|y|^2}{s}\right\}\,dyds \\
&=: A + B
\end{aligned}$$

Heat ball defined by $\phi(-y,-s) = \frac{1}{r^n}$

$$\iff \frac{1}{(r\pi(-s))^{\frac{n}{2}}}e^{-|y|^2/(-4s)} = \frac{1}{r^n}$$

Now, we define

$$\psi(y,s) = \frac{n}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + n\ln r$$

It's observed that $\psi = 0$ on $\partial E(0,0,r)$ which implies $\psi(y,s) = 0$ on $\partial E(s)$

$$D\psi(y) = \frac{y}{2s}$$

Rewrite B:

$$
\begin{aligned}
&\frac{1}{4r^{n+1}}\iint_{E(r)} 4\frac{\partial u}{\partial s}(y,s)\cdot y\cdot D\psi\ dyds \\
&\qquad = 0 - \frac{1}{4r^{n+1}}\iint_{E(r)}\sum_{i=1}^{n}\left(4\frac{\partial^2 u}{\partial s\partial y_i}y_i\psi + 4\frac{\partial u}{\partial s}\psi\right)\ dyds \\
&\qquad = -\frac{1}{4r^{n+1}}\iint_{E(r)}\left\{4\sum_{i=1}^{n}\frac{\partial^2 u}{\partial s\partial y_i}y_i\psi + 4n\frac{\partial u}{\partial s}\psi\right\}\ dyds \\
&\qquad = \frac{1}{4r^{n+1}}\iint_{E(r)}\left\{4\sum_{i=1}^{n}\frac{\partial u}{\partial y_i}y_i\frac{\partial \psi}{\partial s} - 4n\frac{\partial u}{\partial s}\psi\right\}\ dyds \\
&\qquad = \frac{1}{4r^{n+1}}\iint_{E(r)} 4\sum_{i=1}^{n}\left[\frac{\partial u}{\partial y_i}y_i\left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) - n\frac{\partial u}{\partial s}\psi\right]\ dyds \\
&\qquad = \frac{1}{4r^{n+1}}\iint_{E(r)}\left\{-4n\frac{\partial u}{\partial s}\cdot\psi - \sum_{i=1}^{n}\frac{2n}{s}\frac{\partial u}{\partial y_i}\cdot y_i\right\}\ dyds
\end{aligned}
$$

Collecting our terms

$$
\begin{aligned}
\phi'(r) &= A + B \\
&= \frac{1}{4r^{n+1}}\iint_{E(r)}\left\{-4n\Delta u\cdot\psi - \frac{2n}{s}\sum_{i=1}^{n}\frac{\partial u}{\partial y_i}\cdot y_i\right\}\ dyds \\
&= \frac{1}{4r^{n+1}}\iint_{E(r)}\left\{4nDu\cdot D\psi - \frac{2n}{s}\sum_{i=1}^{n}\frac{\partial u}{\partial y_i}\cdot y_i\right\}\ dyds \\
&= 0
\end{aligned}
$$

$\Longrightarrow \phi$ is constant

$$
\begin{aligned}
\Longrightarrow \phi(r) &= \lim_{r\to 0}\phi(r) \\
&= u(0,0)\lim_{r\to 0}\underbrace{\frac{1}{4r^n}\iint_{E(r)}\frac{|y|^2}{s^2}\ dyds}_{1} \\
&= u(0,0) \qquad \blacksquare
\end{aligned}
$$

Recall:

$$\text{MVP}: \quad \frac{1}{4r^n}\iint_{E(x,t,r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dyds = u(x,t)$$
$$\frac{1}{4r^n}\iint_{E(x,t,r)} \frac{|x-y|^2}{(t-s)^2}\,dyds = 1$$

Theorem 3.8 (Maximum principle for the heat equation on bounded and open domains). *$u \in C_1^2(\Omega_T)\cap C(\overline{\Omega_T})$ solves the heat equation in Ω_T. Then*

i.)

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u \qquad \textit{max. principle}$$

ii.) Ω connected, $\exists(x_0,t_0)\in\Omega_T$ such that $u(x_0,t_0)=\max_{\overline{\Omega_T}} u$ then u is constant on $\overline{\Omega_T}$ (strong max. principle).

Remarks:

1. In ii.) we can only conclude for $t \le t_0$.
2. Since $-u$ is also a solution, we have the same assertion with max replaced by min.
3. Heat conducting bar:

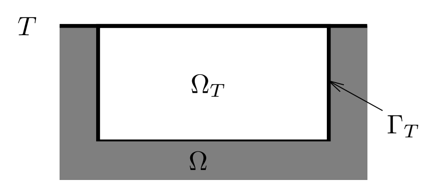


Figure 3.4:

No heat concentration in the interior of the bar for $t > 0$.

Proof: Suppose $u(x_0,t_0) = M = \max_{\overline{\Omega_T}} u$, with $x_0, t_0 \notin \Gamma_T$. For $r>0$ small enough, $E(x_0,t_0,r)\subset\Omega_T$. By the MVT:

$$\begin{aligned} M = u(x_0,t_0) &= \frac{1}{4r^n}\iint_{E(x_0,t_0,r)} u(y,r)\frac{|x-y|^2}{(t-s)^2}\,dydS \\ &\le M \end{aligned}$$

Suppose $(y_0, s_0) \in \Omega_T$, $s_0 < t_0$, line segment L connecting (y_0, s_0) and (x_0, t_0) in Ω_T.

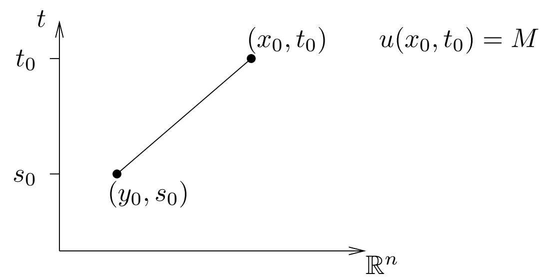


Figure 3.5:

Define r_0 as

$$\min\left\{ s \geq s_0 \;\middle|\; u(x,t) = M,\ \forall (x,t) \in L,\ s \leq t \leq t_0 \right\} = r_0$$

Either $r_0 = s_0 \Longrightarrow u(y_0, s_0) = u(x_0, t_0)$ or $r_0 > s_0$. Then for r small enough, $E(u_0, s_0, r) \subseteq \Omega_T$. Argument above $\Longrightarrow$ $u \equiv$ constant on $E(u_0, s_0, r)$. $E(y_0, s_0, t)$ contains a part of $L \Longrightarrow$ contradiction with definition of r_0.
$\Longrightarrow u \equiv M$ for all points (y_0, s_0) connected to (x_0, t_0) by line segments.
General Case: $(x,t) \in \Omega_T$, $t < t_0$, Ω connected.

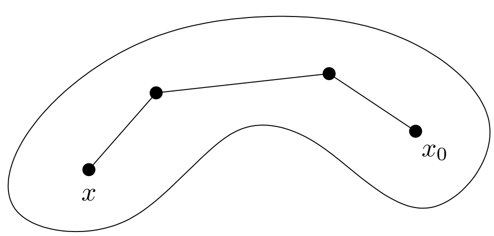


Figure 3.6:

Choose a polygonal path $x_0, x_1, \ldots, x_n = x$ in Ω.
$t_0 > t_1 > \cdots > t_n = t$

$\Longrightarrow$ segments $(x_i, t_i) + (x_{i+1}, t_{i+1})$ in Ω_T
$\Longrightarrow u(x,t) = u(x_0, t_0)$. ■

Theorem 3.9 (Uniqueness of solutions on bounded domains). *If $g \in C(\Gamma_T)$, $f \in C(\Omega_T)$ then there exists at most one solution to*

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Proof: If u_1 and u_2 are solutions then $v = \pm(u_1 - u_2)$ solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \Omega_T \\ u = 0 & \text{on } \Gamma_T \end{cases}$$

Now we use the maximum principle. ■

Theorem 3.10 (Maximum principle for solutions of the Cauchy problem).

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ is a solution of the heat equation with $u(x,t) \leq Ae^{a|x|^2}$ for $x \in \mathbb{R}^n$, $t \in [0,T]$, and $A, a > 0$ constants. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} u = \sup_{\mathbb{R}^n} g$$

Remark: Theorem not true without the exponential bound, [John].

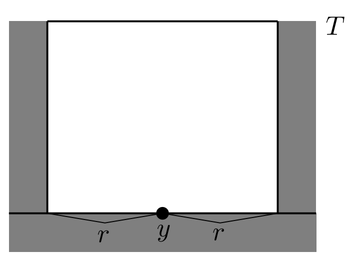
Proof: Suppose that $4aT < 1$, $4a(T+\epsilon) < 1$, $\epsilon > 0$ small enough, $a < \frac{1}{4(T+\epsilon)}$, there exists $\gamma > 0$ such that

$$a + \gamma = \frac{1}{4(T+\epsilon)}$$

Fix $y \in \mathbb{R}^n$, $\mu > 0$ and define

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \qquad x \in \mathbb{R}^n, t > 0$$

Check: v is a solution of the heat equation since since $\Phi(x,t)$ is a solution.



$|x - y| = r \qquad |x| \leq |y| + r$

Figure 3.7:

$\Omega = B(y,r)\ r > 0$
$\Omega_T = B(y,r) \times (0,T]$

Use the maximum principle for v on the bounded set Ω_T.

$$\max_{\overline{\Omega}_T} v \leq \max_{\Gamma_T} v$$

on the bottom:

$$\begin{aligned}
v(x,0) &= u(x,0) - \underbrace{\frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} \qquad x \in \mathbb{R}^n}_{\geq 0} \\
&\leq u(x,0) \\
&\leq \sup_{\mathbb{R}^n} g
\end{aligned}$$

$$\begin{aligned}
v(x,t) &= u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \qquad x \in \mathbb{R}^n \\
&\leq Ae^{a|x|^2} - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\epsilon)}} \qquad x \in \mathbb{R}^n \\
&\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\epsilon)}} \qquad x \in \mathbb{R}^n \\
\Bigg[a+\gamma &= \frac{1}{4(T+\epsilon)} \qquad \frac{1}{T+\epsilon} = 4(a+\gamma) \Bigg] \\
&= Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{\frac{n}{2}} e^{(a+\gamma)r^2} \qquad x \in \mathbb{R}^n \\
&\leq \sup_{\mathbb{R}^n} g \qquad \text{if we choose } r \text{ big enough.}
\end{aligned}$$

Leading order term $e^{(a+\gamma)r^2}$ dominates e^{ar^2} for r large enough. If r big enough, then

$$\max_{\overline{\Omega}_T} v \leq \sup_{\mathbb{R}^n} g \Longrightarrow \max_{\mathbb{R}^n \times [0,T]} v \leq \sup_{\mathbb{R}^n} g$$

Now $\mu \to 0$ and get

$$\max_{\mathbb{R}^n \times [0,T]} u \leq \sup_{\mathbb{R}^n} g$$

If $4aT > 1$, choose $t_0 > 0$ such that $4at_0 < 1$, apply above argument on $[0,t_0], [t_0, 2t_0], \ldots$ ■

Theorem 3.11 (Uniqueness of solutions for the Cauchy problem). *There exists at most one solution* $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ *of*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

which satisfies that

$$|u(x,t)| \leq Ae^{a|x|^2} \qquad x \in \mathbb{R}^n,\ t \in [0,T]$$

Proof: Suppose u_1 and u_2 are solutions. Define $w = \pm(u_1 - u_2)$ solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ w = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$|w(x,t)| \leq 2Ae^{a|x|^2}$
$\Longrightarrow$ max principle applies
$\Longrightarrow w = 0$
$\Longrightarrow$ uniqueness. ∎

3.4 Regularity of Solutions

Theorem 3.12. Ω *open,* $u \in C_1^2(\Omega_T)$*, is a solution of the heat equation. Then* $u \in C^\infty(\Omega_T)$*.*

Proof: Regularity is a local issue.
Parabolic cylinders

$$C(x,t,r) = \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; |x-y| \leq r,\ t - r^2 \leq s \right\}$$

Fix $(x_0, t_0) \in \Omega_T$

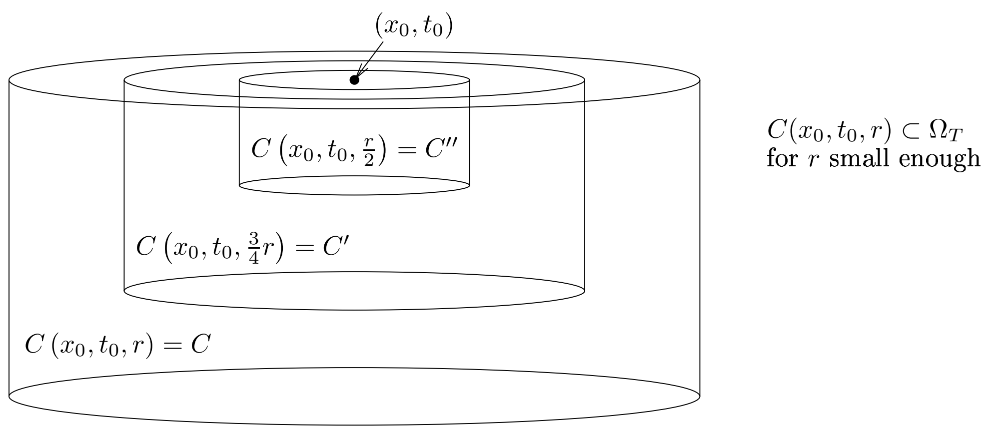


Figure 3.8:

$$v(x,t) = u(x,t) \cdot \xi(x,t)$$

ξ cut-off function.

$$\begin{aligned}
0 &\leq \xi && \text{on } \Omega_T \\
\xi &\equiv 1 && \text{on } C\left(x_0, t_0, \frac{3}{4}r\right) \\
\xi &\equiv 0 && \text{outside } C, \text{ i.e. on } \Omega_T \setminus C \\
\xi &\in C^\infty(\Omega_T)
\end{aligned}$$

<u>Goal:</u> Representation for u:

$$u(x,t) = \iint K(x,y,t,s)u(y,s)\ dyds$$

with a smooth kernel $\Longrightarrow$ u as good as K.

$$\begin{aligned}
v_t &= u_t\xi + u\xi_t \\
Dv &= Du \cdot \xi + uD\xi \\
\Delta v &= \Delta u \cdot \xi + sDu \cdot D\xi + u\Delta\xi \\
v_t - \Delta u &= u\xi_t - 2Du \cdot D\xi - u\Delta\xi = f
\end{aligned}$$

A solution $\tilde{v}$ of this equation is given by Duhamel's principle,

$$\tilde{v}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)f(y,s)\ dyds$$

Uniqueness for the Cauchy problem

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)f(y,s)\ dyds$$

Suppose for the moment being that u is smooth.
$(x,t) \in C''$, $\xi(x,t) = 1$

$\Longrightarrow v(x,t) = u(x,t)\xi(x,t) = u(x,t)$

$$\begin{aligned}
u(x,t) &= \underbrace{\int_0^t \int_{\mathbb{R}^n}}_{C\setminus C'} \Phi(x-y,t-s)[u\xi_s - 2Du\cdot D\xi - u\Delta\xi](y,s)\ dyds \\
&= \iint_{C\setminus C'} \Phi(x-y,t-s)[u\xi_s - u\Delta\xi + 2u\Delta\xi]\ dyds \\
&\quad + \iint_{C\setminus C'} D\Phi(x-y,t-s)2uD\xi\ dyds \\
&= \iint_{C\setminus C'} K(x,y,t,s)u(y,s)\ dyds
\end{aligned}$$

with

$$K(x,y,t,s) = [\xi_s + \Delta\xi](y,s)\Phi(x-y,t-s) + 2D\xi\cdot D\Phi(x-y,t-s)$$

We don't see the singularity of Φ, and K is C^∞
$\Longrightarrow$ representation of u,
$\Longrightarrow$ u is as good as K
$\Longrightarrow$ u is C^∞. ■

3.4.1 Application of the Representation to Local Estimates

Theorem 3.13 (local bounds on derivatives). *$u \in C_1^2(\Omega_T)$ solution of the heat equation:*

$$\max_{C\left(x,t,\frac{r}{2}\right)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \int_{C(x,t,r)} |u(y,s)|\ dyds$$

Proof: Scaling argument. Calculation on a cylinder with $r=1$.
$v(x,t) = u(x_0 + rx, t_0 + r^2 t)$

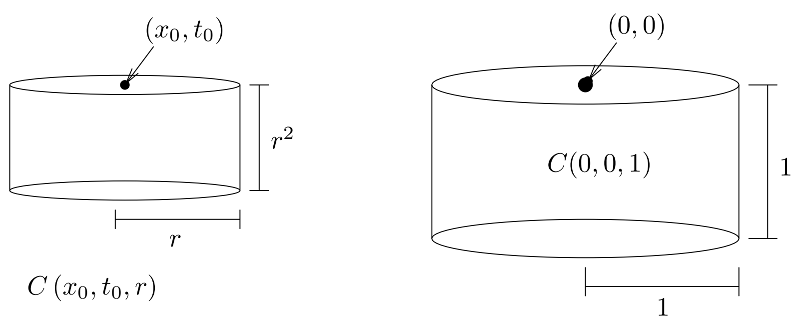


Figure 3.9:

v solves the hear equations since

$$\begin{aligned} v_t(x,t) &= r^2 u_t(x_0+rx, t_0+r^2t) \\ \Delta v(x,t) &= r^2 \Delta u(x_0+rx, t_0+r^2t) \end{aligned}$$

Representation of v:

$$v(x,t) = \iint_{C(0,0,1)} K(x,y,t,s)v(y,s)\ dyds$$

$x,t \in C\left(0,0,\frac{1}{2}\right)$

$$\begin{aligned} |D_x^k D_t^l u| &\le \iint_{C(0,0,1)} \underbrace{|D_x^k D_t^l K(x,y,t,s)|}_{C_{kl}} |v(y,s)|\ dyds \\ &= C_{kl} \iint_{C(0,0,1)} |v(y,s)|\ dyds \end{aligned}$$

$$\begin{aligned} &\Longrightarrow \max_{(x,t)\in C\left(0,0,\frac{1}{2}\right)} |D_x^k D_t^l v| \le C_{kl} \iint_{C(0,0,1)} |v(y,s)|\ dyds \\ &\Longrightarrow r^{k+2l} \max_{(x,t)\in C\left(x_0,t_0,\frac{r}{2}\right)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{n+2}} \iint_{C(x_0,t_0,r)} |u(y,s)|\ dyds \\ &\Longrightarrow \max_{(x,t)\in C\left(x_0,t_0,\frac{r}{2}\right)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \iint_{C(x_0,t_0,r)} |u(y,s)|\ dyds \qquad \blacksquare \end{aligned}$$

Remark: This reflects again the different scaling in x and t.

3.5 Energy Methods

Theorem 3.14 (Uniqueness of solutions). *Ω open and bounded, there exists at most one solution $u \in C_1^2(\Omega_T)$ of*

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Omega \times \{t=0\} \\ u = h & \text{on } \partial\Omega \times (0,T] \end{cases}$$

Proof: Suppose you have two solutions u_1, u_2. Let $w = u_1 - u_2$. w solves the homogeneous heat equations.

Let

$$
\begin{aligned}
e(t) &= \int_\Omega w^2(x,t)\ dx, \qquad 0 \le t \le T \\
\frac{d}{dt}e(t) &= \int_\Omega 2w(x,t)\cdot w_t(x,t)\ dx \\
&= 2\int_\Omega w(x,t)\cdot \Delta w(x,t)\ dx \\
&\overset{\text{I.P.}}{=} -2\int_\Omega Dw(x,t)\cdot Dw(x,t)\ dx + 0 \\
&= -2\int_\Omega |Dw(x,t)|^2\ dx \le 0
\end{aligned}
$$

$\Longrightarrow e(t)$ is decreasing in t.

$$e(0) = \int_\Omega w^2(x,0)\ dx = 0$$

$\Longrightarrow e(t) \equiv 0$ on $[0,T]$
$\Longrightarrow Dx(x,t) = 0\ \forall x,t$
$\Longrightarrow w(x,t) =$ constant
$\therefore w(x,t) = 0\ x \in \partial\Omega$

$\Longrightarrow w(x,t) = 0\ \forall x \in \Omega, \forall t \in [0,T]$
$\Longrightarrow u_1 = u_2$ in Ω_T. ■

3.6 Backward Heat Equation

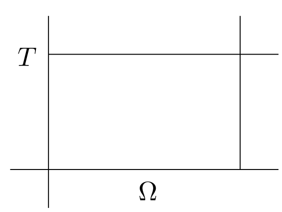


Figure 3.10:

$$
\begin{cases}
u_t - \Delta u = 0 & \text{in } \Omega_T \\
u(x,0) = g(x) & \text{in } \Omega
\end{cases}
$$

Backwards heat equation:

$$\begin{cases} u_t - \Delta = 0 & \text{in } \Omega_T \\ u(x,T) = g(x) & \text{in } \Omega \end{cases}$$

Reverses the smoothening, expect bad solutions in finite time even if $g(t)$ is smooth. Transform: $t \mapsto T - t$

$$\begin{aligned} v(x,t) &= u(x,T-t) \\ v_t(x,t) &= -u_t(x,T-t) \\ \Delta v(x,t) &= \Delta u(x,T-t) \end{aligned}$$

So, $u_t - \Delta u = 0$ transforms to

$$v_t + \Delta v = 0 \qquad \text{Backwards heat equation}$$

Theorem 3.15. *If u_1 and u_2 are solutions of the heat equation with $u_1(x,T) = u_2(x,T)$ with the same boundary values, then the solutions are identical.*

3.7 Exercises

3.1: Apply the similarity method to the linear transport equation

$$u_t + au_x = 0$$

to obtain the special solutions of the form $u(x,t) = c(at - x)^{-\alpha}$.

3.2: [Evans 2.5 #10] Suppose that u is smooth and that u solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$.

a) Show that $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

b) Use a) to show that $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well. The point of this problem is to use part a).

3.3: [Evans 2.5 #11] Assume that $n = 1$ and that u has the special form $u(x,t) = v(x^2/t)$.

a) Show that

$$u_t = u_{xx}$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0 \quad \text{for } z > 0.$$

b) Show that the general solution of the ODE in a) is given by

$$v(z) = c \int_0^z e^{-s^2/4} s^{-1/2}\, ds + d, \quad \text{with } c, d \in \mathbb{R}.$$

c) Differentiate the solution $v(x^2/t)$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for $n = 1$.

<u>3.4</u>: [Evans 2.5 #12] Write down an explicit formula for a solution of the initial value problem

$$\begin{cases} u_t - \Delta u + cu = f & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

Hint: Make the change of variables $u(x,t) = e^{at} v(x,t)$ with a suitable constant a.

<u>3.5</u>: [Evans 2.5 #13] Let $g : [0, \infty) \to \mathbb{R}$ be a smooth function with $g(0) = 0$. Show that a solution of the initial-boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\ u = 0 & \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = g & \text{on} \quad \{x = 0\} \times [0, \infty), \end{cases}$$

is given by

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-x^2/(t-s)} g(s)\, ds.$$

Hint: Let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.

Chapter 4

The Wave Equation

$$
\begin{aligned}
u_{tt} - \Delta u = 0 \text{ in } \Omega_T &\qquad \text{homogeneous} \\
u_{tt} - \Delta u = f \text{ in } \Omega_T &\qquad \text{non-homogeneous}
\end{aligned}
$$

$$
\begin{aligned}
u = g \text{ on } \Omega \times \{t = 0\} \\
u_t = h \text{ on } \Omega \times \{t = 0\}
\end{aligned}
$$

4.1 D'Alembert's Representation of Solution in 1D

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0 \\
\partial_t^2 u - \partial_x^2 u &= 0 \\
\left(\partial_t^2 - \partial_x^2\right) u &= 0 \\
(\partial_t + \partial_x) \underbrace{(\partial_t - \partial_x) u}_{v(x,t)} &= 0
\end{aligned}
$$

Solve $(\partial_t + \partial_x) v = v_t + v_x = 0$

Transport equation for v: $v(x,t) = a(x - t)$ an arbitrary function.
Thus $\partial_t u - \partial_x u = v(x,t) = a(x - t)$.

General solution formula for

$$
\begin{cases} u_t + b u_x = f(x,t) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}
$$

Solution (Duhamel's principle):

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\ ds$$

Solution u ($b = -1,\ f(x,t) = a(x-t)$)

$$\begin{aligned} u(x,t) &= g(x - t(-1)) + \int_0^t a(x + (s-t)(-1) - s)\ ds \\ &= g(x+t) + \int_0^t a\underbrace{(x + t - 2s)}_{y}\ ds \end{aligned}$$

$$\begin{aligned} a(x) = v(x,0) &= \partial_t u(x,0) - \partial_x u(x,0) \\ &= h(x) - g'(x) \end{aligned}$$

$$\begin{aligned} u(x,t) &= g(x+t) + -\frac{1}{2}\int_{x+t}^{x-t} a(y)\ dy \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\ dy \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))\ dy \end{aligned}$$

$$\Longrightarrow u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy$$

D'Alembert's formula.

Theorem 4.1 (D'Alembert's solution for the wave equation in 1D). $g \in C^2(\mathbb{R}),\ h \in C^1(\mathbb{R})$

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy$$

Then

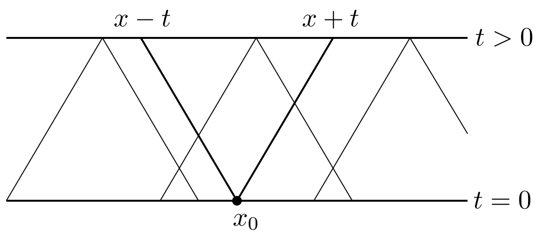
i.) $u \in C^2(\mathbb{R} \times (0,\infty))$

ii.)

$$\begin{cases} u_{tt} - u_{xx} &= 0 \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{cases}$$

Remark: We say $u(x,0) = g(x)$, if $\lim_{(x,t)\to(x_0,0),t>0} u(x,t) = g(x_0)\ \forall x_0 \in \mathbb{R}$. Remarks:

- finite speed of propagation.



information at x_0 travels with finite speed
c.f heat equation infinite speed of propagation

Figure 4.1:

- domain of influence

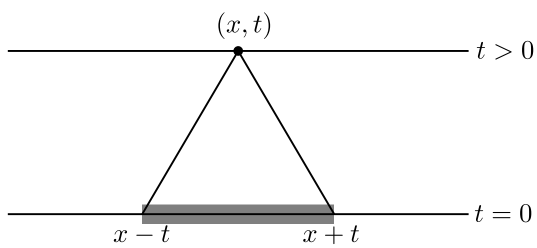


Figure 4.2:

The values of u depend only on the value of g and h on $[x-t, x+t]$

4.1.1 Reflection Argument

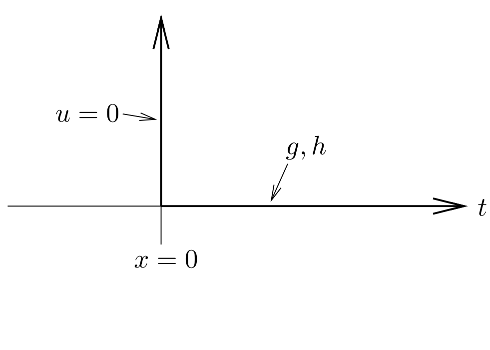


Figure 4.3:

Want to solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{0\} \times (0,\infty) \end{cases}$$

and suppose that $g(0) = 0$, $h(0) = 0$. Try to find a solution $\tilde{u}$ on $\mathbb{R}$ of a wave equation that is odd as a function of x, i.e.

$$\tilde{u}(-x,t) = -\tilde{u}(x,t)$$

$$\begin{aligned} \Longrightarrow \tilde{u}(0,t) &= -\tilde{u}(0,t) \\ \Longrightarrow \tilde{u}(0,t) &= 0 \end{aligned}$$

Extend g and h by odd reflection:

$$\tilde{g}(x) = \begin{cases} -g(-x) & x < 0 \\ g(x) & x \geq 0 \end{cases}$$

$$\tilde{h}(x) = \begin{cases} -h(-x) & x < 0 \\ h(x) & x \geq 0 \end{cases}$$

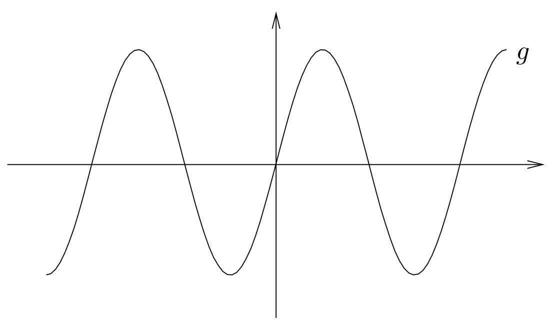


Figure 4.4:

$\tilde{u}$ solution of:

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g} & \text{on } \mathbb{R} \times \{t=0\} \\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

D'Alembert's formula:

$$\tilde{u}(x,t) = \frac{1}{2}(\tilde{g}(x-t) + \tilde{g}(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\, dy$$

Assertion: $\tilde{u}$ is odd, the restriction to $[0,\infty)$ is our solution u.

Proof:

$$\begin{aligned}
\tilde{u}(-x,t) &= \frac{1}{2}[\tilde{g}(-x-t)+\tilde{g}(-x+t)] + \frac{1}{2}\int_{-x-t}^{-x+t}\tilde{h}(y)\,dy \\
&= -\frac{1}{2}[\tilde{g}(x+t)+\tilde{g}(x-t)] - \frac{1}{2}\int_{-x-t}^{-x+t}\tilde{h}(-y)\,dy \\
(\text{sub. } z=-y) \quad &= -\frac{1}{2}[\tilde{g}(x+t)+\tilde{g}(x-t)] + \frac{1}{2}\int_{x+t}^{x-t}\tilde{h}(z)\,dz \\
&= -\frac{1}{2}[\tilde{g}(x+t)+\tilde{g}(x-t)] - \frac{1}{2}\int_{x-t}^{x+t}\tilde{h}(y)\,dy
\end{aligned}$$

$\Longrightarrow \tilde{u}(-x,t) = -\tilde{u}(x,t)$
$\Longrightarrow \tilde{u}$ is odd.

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x-t)+g(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \ge t \ge 0 \\ \frac{1}{2}[g(x+t)-g(t-x)] + \frac{1}{2}\int_{t-x}^{t+x} h(y)\,dy & t \ge x \ge 0 \end{cases}$$

$$\begin{aligned}
\int_{x-t}^{x+t}\tilde{h}(y)\,dy &= \int_{x-t}^{0}\tilde{h}(y)\,dy + \int_{0}^{x+t}\tilde{h}(y)\,dy \\
&= \int_{x-t}^{0} -h(-y)\,dy + \int_{x}^{x+t} h(y)\,dy \\
(\text{sub. } z=-y) \quad &= \int_{-x+t}^{0} h(z)\,dz + \int_{x}^{x+t} h(y)\,dy \\
&= \int_{t-x}^{x+t} h(y)\,dy
\end{aligned}$$

4.1.2 Geometric Interpretation ($h \equiv 0$)

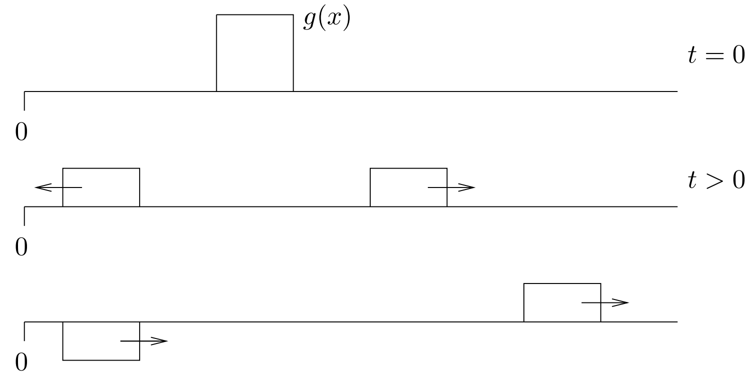


Figure 4.5:

Remark: D'Alembert:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy$$

The solution is generally of the form $u(x,t) = F(x-t) + G(x+t)$ with arbitrary functions F and G.

4.2 Representations of Solutions for $n = 2$ and $n = 3$

4.2.1 Spherical Averages and the Euler-Poisson-Darboux Equation

$$\begin{aligned} U(x;t,r) &= \fint_{\partial B(x,r)} u(y,t)\ dS(y) \\ G(x,r) &= \fint_{\partial B(x,r)} g(y)\ dS(y) \\ H(x,r) &= \fint_{\partial B(x,r)} h(y)\ dS(y) \end{aligned}$$

Recover $u(x,t) = \lim_{r\to 0} U(x;t,r)$

$x \in \mathbb{R}^n$ fixed from now on.

Lemma 4.2. *u a solution of the wave equation, then U satisfies*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r &= 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty) \\ U &= G \quad \text{on } \mathbb{R}_+ \times \{t = 0\} \\ U_t &= H \quad \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases} \tag{4.1}$$

Once this is established, we transform (4.1) into $\tilde{U}_{tt} - \tilde{U}_{rr} = 0$ and we use the representation on the half line to find $\tilde{U}$, U, u.

Calculation from MVP for harmonic functions:

$$\begin{aligned} U_r(x,t,r) &= \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\ dy \\ &= \frac{r}{n} \frac{1}{\alpha(n) r^n} \int_{B(x,r)} \Delta u(y,t)\ dy \\ &= \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta u(y,t)\ dy \end{aligned}$$

$$\begin{aligned} \therefore r^{n-1} U_r &= \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\ dy \\ \text{(since } u \text{ solves wave eq.)} &= \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y,t)\ dy \\ \text{(in polar)} &= \frac{1}{n\alpha(n)} \int_0^r \int_{\partial B(x,\rho)} u_{tt}\ dS d\rho \end{aligned}$$

$$\begin{aligned} \therefore [r^{n-1} U_r]_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\ dS \\ &= r^{n-1} ⨍_{\partial B(x,r)} u_{tt}\ dS \\ &= r^{n-1} U_{tt} \end{aligned}$$

$$\begin{aligned} \therefore \quad & U_{tt} - \frac{1}{r^{n-1}} [r^{n-1} U_r]_r = 0 \\ \iff \quad & U_{tt} - \frac{1}{r^{n-1}} [(n-1) r^{n-2} U_r + r^{n-1} U_{rr}] = 0 \\ \iff \quad & U_{tt} - \frac{n-1}{r} U_r - U_{rr} = 0 \end{aligned}$$

Euler-Poisson-Darboux Equation.

4.2.2 Kirchoff's Formula in 3D

Recall:

Representations for

$$\begin{cases} u_{tt} = \Delta u & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$\underline{n=3}$ Kirchhoff's formula

$$U(x,t,r) = ⨍_{\partial B(x,r)} u(y,t)\ dS(y)$$

$G(x,r),\ H(x,r)$

U satisfies Euler-Poisson-Darboux equation

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0$$

$U = G$, $U_t = H$

Transformation: $\tilde{U} = rU$

$$\begin{aligned} \tilde{U}_{tt} = rU_{tt} &= r[U_{rr} + \frac{2}{r}U_r] \\ &= rU_{rr} + 2U_r \\ &= [U + rU_r]_r \\ \tilde{U}_{rr} = [rU]_{rr} &= [U + rU_r]_r \end{aligned}$$

$$\Longrightarrow \begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G} & \text{at } t=0 \\ \tilde{U}_t = \tilde{H} & \text{at } t=0 \\ \tilde{U}(0,t) = 0 & t>0 \end{cases}$$

1D wave equation on the half line with Dirichlet condition at $x=0$. From D'Alembert's solution in 1D:

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \frac{\tilde{U}(x,t,r)}{r} \\ \text{(formula with } 0 \le r < t) &= \lim_{r\to 0} \frac{1}{2r}[\tilde{G}(t+r) + \tilde{G}(t-r)] + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\ dy \\ &= \tilde{G}'(t) + \tilde{H}(t) \end{aligned}$$

By definition

$$\begin{aligned}\tilde{G}(x;t) &= tG(x;t)\\ &= t\fint_{\partial B(x,t)} g(y)\ dS(y)\\ &= t\fint_{\partial B(0,1)} g(x+tz)\ dS(z)\end{aligned}$$

$$\begin{aligned}\tilde{G}'(x,t) &= \fint_{\partial B(0,1)} g(x+tz)\ dS(z) + t\fint_{\partial B(0,1)} Dg(x+tz)\cdot z\ dS(z)\\ &= \fint_{\partial B(x,t)} \left[g(y) + t\frac{\partial y}{\partial \nu}(y)\right]\ dS(y)\end{aligned}$$

$$\tilde{H}(t) = t\cdot H(t) = t\fint_{\partial B(x,t)} h(y)\ dy$$

$$u(x,t) = \fint_{\partial B(x,t)} \left[g(y) + t\frac{\partial g}{\partial \nu}(y) + t\cdot h(y)\right]\ dS(y)$$

This is Kirchhoff's Formula

Loss of regularity: need $g \in C^3$, $h \in C^2$, to get u in C^2.

Finite speed of propagation:

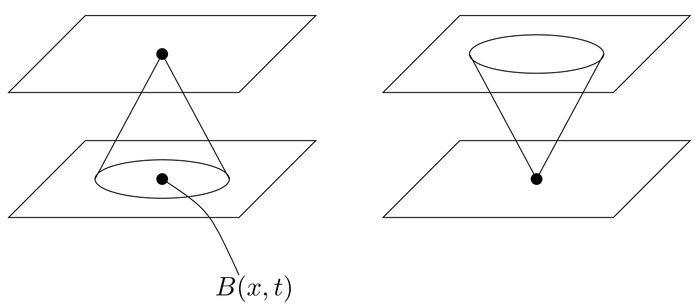


Figure 4.6:

4.2.3 Poisson's Formula in 2D

Method of descent: use the 3D solution to get the 2D solution by extending in 2D to 3D

u solves

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^2 \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^2 \times \{t=0\} \end{cases}$$

Define

$$\begin{aligned} \overline{u}(x_1,x_2,x_3,t) &= u(x_1,x_2,t) \\ \overline{g}(x_1,x_2,x_3) &= g(x_1,x_2) \\ \overline{h}(x_1,x_2,x_3) &= h(x_1,x_2) \end{aligned}$$

Then

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \overline{u} = \overline{g} & \text{on } \mathbb{R}^3 \times \{t=0\} \\ \overline{u}_t = \overline{h} & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

Kirchhoff's formula: $\overline{x} = (x_1,x_2,x_3)$

$$\overline{u}(\overline{x},t) = \fint_{\partial \overline{B}(\overline{x},t)} \left[\overline{g}(\overline{y}) + t\frac{\partial \overline{g}}{\partial \nu}(\overline{y}) + t \cdot \overline{h}(\overline{y}) \right] \, dS(\overline{y})$$

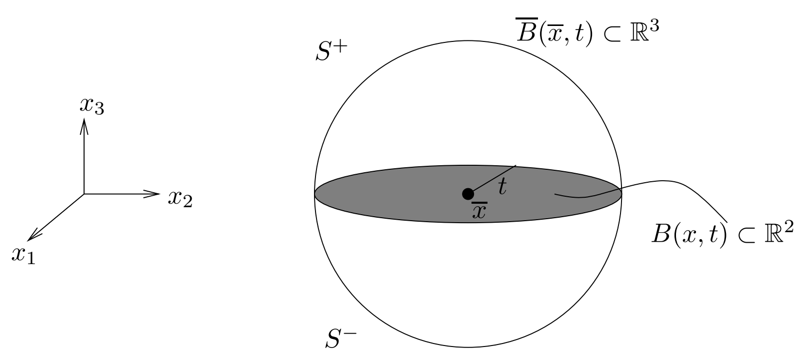


Figure 4.7:

$$u(x_1,x_2) = \overline{u}(x_1,x_2,0)$$

Assume $\overline{x} = (x_1, x_2, 0)$

Need to evaluate $\int_{\partial B(\overline{x},t)}$
$|\overline{y} - \overline{x}|^2 = t^2$ equation for the surface
$\iff (y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^2 = t^2$

Parametrization:

$$\begin{aligned} y_3^2 &= t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2 \\ &= t^2 - |y - x|^2 \end{aligned}$$

Parametrization: $f(y_1, y_2) = \sqrt{t^2 - |y - x|^2}$

$$dS = \sqrt{1 + |Df|^2}\ dy$$

$$\begin{aligned} \frac{\partial f}{\partial y_i} &= \frac{1}{2}\frac{1}{\sqrt{t^2 - |y-x|^2}}(-2)(y_i - x_i) \\ &= \frac{-(y_i - x_i)}{\sqrt{t^2 - |y-x|^2}} \end{aligned}$$

$$\begin{aligned} dS &= \sqrt{1 + \frac{|y-x|^2}{t^2 - |y-x|^2}}\ dy \\ &= \left(\frac{t^2}{t^2 - |y-x|^2}\right)^{\frac{1}{2}}\ dy \end{aligned}$$

$$dS = \frac{t}{\sqrt{t^2 - |y-x|^2}}\ dy$$

4.2.4 Representation Formulas in 2D/3D

- Kirchoff's formula

$$u(x,t) = \fint_{\partial B(x,t)} \left[g + t\frac{\partial y}{\partial \nu} + t \cdot h\right]\ dS(y)$$

- Poisson's formula $u \rightsquigarrow \overline{u}$
  $\overline{x} = (x_1, x_2, 0)$

$$\overline{u}(\overline{x}) = \fint_{\partial \overline{B}(\overline{x},t)} \left[\overline{g} + t\frac{\partial \overline{g}}{\partial \nu} + t \cdot \overline{h}\right]\ dS(y)$$

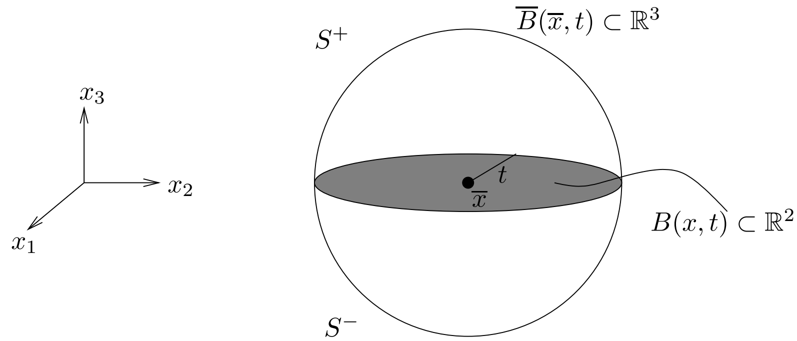


Figure 4.8:

Parameterize the surface integral
S^+, $f(y_1, y_2) = \sqrt{t^2 - |x-y|^2}$

$$\begin{aligned} dS &= \sqrt{1+|Df|^2}\ dy \\ &= \frac{t}{\sqrt{t^2 - |y-x|^2}} \end{aligned}$$

$$\begin{aligned} &\int_{S^+} \left[\overline{g} + t\frac{\partial \overline{g}}{\partial \nu} + t \cdot \overline{h}\right](y)\ dS(y) \\ &= \int_{B(x,t)} \left[\overline{g} + t\frac{\partial \overline{g}}{\partial \nu} + t \cdot \overline{h}\right](y_1, y_2, f(y_1, y_2)) \cdot \sqrt{t + |Df|^2}\ dy \\ &= \int_{B(x,t)} \left[g(y_1,y_2) + t\frac{Dg \cdot (y-x)}{t} + t \cdot h(y_1, y_2)\right] \frac{t\ dy}{\sqrt{t^2 - |y-x|^2}} \end{aligned}$$

———

$$\begin{aligned} \frac{\partial g}{\partial \nu} &= D_y \overline{g}(\overline{y}) \cdot \frac{\overline{y} - \overline{x}}{t} \\ &= D_y g \cdot \frac{y-x}{t} \end{aligned}$$

———

$$\overline{u}(\overline{x}) = ⨍_{\partial \overline{B}(\overline{x},t)} \left[\overline{g} + t\frac{\partial \overline{g}}{\partial \nu} + t \cdot \overline{h}\right]\ dS(y)$$

Since $\overline{g}$, $\overline{h}$ do not appear on

$$\begin{aligned}
u(x,t) &= \overline{u}(\overline{x},t) \\
&= \frac{2}{4\pi t^2}\cdot\int_{B(x,t)} [g(y_1,y_2) + Dg(y_1,y_2)\cdot(y-x) + t\cdot h(y_1,y_2)]\,\frac{t\ dy}{\sqrt{t^2-|x-y|^2}} \\
&= \frac{1}{2}⨍_{B(x,t)} \frac{t\cdot g(y) + t\cdot Dg(y)\cdot(y-x) + t^2 h(y)}{\sqrt{t^2-|x-y|^2}}\,dy \quad x\in\mathbb{R}^n, t>0
\end{aligned}$$

Poisson's formula in 2D.

$$\begin{aligned}
\overline{u}(x_1,x_2,x_3,t) &= u(x_1,x_2,t) \\
\overline{u}(x_1,x_2,0,t) &= u(x_1,x_2,t) \\
\overline{u}(\overline{x},t) &= u(x,t)
\end{aligned}$$

Remarks

- Used method of descent extend 2D to 3D. The same idea works in n-dimensions, extend from even to odd dimensions.
- Qualitative difference: integration is on the full ball in 2D.
- Loss of regularity:
  need $g\in C_1^3$, $h\in C^2$ to have $u\in C^2$.

4.3 Solutions of the Non-Homogeneous Wave Equation

$$\begin{cases} u_{tt}-\Delta u = f(x,t) & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n\times\{t=0\} \\ u_t = 0 & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$$

The fully non-homogeneous wave equation

$$\begin{cases} u_{tt}-\Delta u = f \\ u = g \\ u_t = h \end{cases}$$

can be solved by $u=v+w$, where

$$\begin{cases} v_{tt}-\Delta v = f \\ v = 0 \\ v_t = 0 \end{cases}$$

$$\begin{cases} v_{tt}-\Delta v = 0 \\ w = f \\ w_t = g \end{cases}$$

4.3.1 Duhamel's Principle

$u(x,t;s)$

$$\begin{array}{lll} u_{tt}(\cdot;s)-\Delta u(\cdot;s)=0 & \text{on } \mathbb{R}^n\times(x,\infty) & \\ u(\cdot;s)=0 & \text{on } \mathbb{R}^n\times\{t=s\} & (u(x,t,0)=0) \\ u_t(\cdot;s)=f(\cdot,s) & \text{on } \mathbb{R}^n\times\{t=s\} & (u_t(x,t,0)=f(\cdot,t)) \end{array}$$

Then

$$u(x,t)=\int_0^t u(x,t;s)\ ds, \qquad x\in\mathbb{R}^n, t>0$$

Proof: We differentiate

$$\begin{aligned} u_t(x,t) &= u(x,t;t)+\int_0^t u_t(x,t;s)\ ds \\ &= \int_0^t u_t(x,t;s)\ ds \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) &= u_t(x,t;t)+\int_0^t u_{tt}(x,t;s)\ ds \\ &= f(x,t)+\int_0^t u_{tt}(x,t;s)\ ds \end{aligned}$$

$$\begin{aligned} \Delta u(x,t) &= \int_0^t \Delta u(x,t;s)\ ds \\ &= \int_0^t u_{tt}(x,t;s)\ ds \end{aligned}$$

$\Longrightarrow u_{tt}(x,t)-\Delta u(x,t)=f(x,t).$ ■

Examples: 1D: D'Alembert's

Formula:

$$u(x,t)=\frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy$$

To get the solution for $u(x,t;s)$ translate the origin from 0 to s.

$$u(x,t;s)=\frac{1}{2}\left[g(x+t-s)+g(x-(t-s))\right]+\frac{1}{2}\int_{x-(t-s)}^{x+t-s} h(y)\ dy$$

Recall that for $u(x,t;s)$ we have $g \equiv 0$, $h(y) = f(y,s)$.

Duhamel's principle:

$$\begin{aligned} u(x,t) &= \int_0^t \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y,s)\ dyds \\ (\text{sub. } u = t-s) &= \frac{1}{2} \int_{x-u}^{u} f(y,t-u)\ dy(-du) \\ &= \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y,t-s)\ dyds \end{aligned}$$

Which values of f influence $u(x,t)$

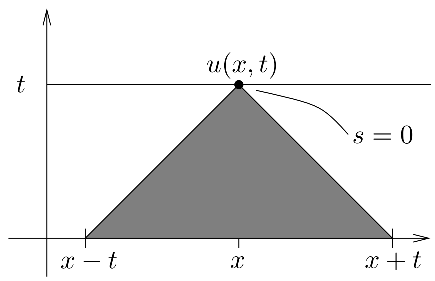


Figure 4.9:

Kirchhoff's formula (3D)

$$u(x,t) = \fint_{\partial B(x,t)} \left(g + t\frac{\partial g}{\partial \nu} + t \cdot h \right)\ dS(y)$$

Poisson's formula in 2D

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} \frac{t \cdot g(y) + t \cdot Dg \cdot (y-x) + t^2 h(y)}{\sqrt{t^2 - |y-x|^2}}\ dy$$

Non-homogeneous problems

$$\begin{cases} u_{tt} - \Delta u &= f \\ u &= 0 \\ u_t &= 0 \end{cases}$$

$$u(x,t) = \int_0^t u(x,t;s)\ ds$$

$u(\cdot; s)$ solution on $\mathbb{R}^n \times (s, \infty)$

$$\begin{cases} u(\cdot, s) = 0 \\ u_t(\cdot, s) = f(\cdot, s) \end{cases}$$

4.3.2 Solution in 3D

Initial data $g(y) = s$, $h(y) = f(y, s)$. Translate the origin from $t = 0$ to $t = s$

$$\begin{aligned} u(x, t; s) &= (t-s) \fint_{\partial B(x, t-s)} f(y, s)\, dS(y) \\ u(x, t) &= \int_0^t (t-s) \fint_{\partial B(x, t-s)} f(y, s)\, dS(y) ds \end{aligned}$$

$(r = t - s)$

$$\begin{aligned} &= \int_t^0 r \fint_{\partial B(x,r)} f(y, t-r)\, dS(y)(-dr) \\ &= \int_0^t r \fint_{\partial B(x,t)} f(y, t-r)\, dS(y) dr \\ &= \int_0^t \frac{r}{4\pi r^2} \int_{\partial B(x,r)} f(y, t-r)\, dS(y) dr \\ &= \int_{B(x,t)} \frac{1}{4\pi |x-y|} f(y, t - |x-y|)\, dy \\ &= u(x, t) \qquad \text{for } x \in \mathbb{R}^3,\ t > 0 \end{aligned}$$

4.4 Energy methods for the wave equation

$\Omega \subset \mathbb{R}^n$ bounded and open, $\Omega_T = \Omega \times (0, T)$

Theorem 4.3 (uniqueness). *There is at most one smooth solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{in } \Omega \times \{t = 0\} \\ u_t = h & \text{in } \Omega \times \{t = 0\} \\ u = k & \text{on } \partial\Omega \times (0, t) \end{cases}$$

Proof: Suppose u_1, and u_2 are solutions.
$w = u_1 - u_2$ satisfies

$$\begin{cases} w_{tt} - \Delta w &= 0 & \text{in } \Omega_T \\ w &= 0 & \text{in } \Omega \times \{t = 0\} \\ w_t &= 0 & \text{in } \Omega \times \{t = 0\} \\ w &= 0 & \text{on } \partial\Omega \times (0, t) \end{cases}$$

Multiply by w_t:

$$\begin{aligned} & w_{tt}w_t - (\Delta w)w_t = 0 \qquad \text{in}\Omega_T \\ & \int_\Omega (w_{tt}(x,t)w_t(x,t) + \Delta w(x,t) \cdot w_t(x,t)) \, dx = 0 \\ = & \int_\Omega (w_{tt}(x,t)w_t(x,t) + Dw(x,t) \cdot Dw_t(x,t)) \, dx \\ & \quad + \int_{\partial\Omega} (Dw \cdot \nu) w_t \, dS \\ = & \int_\Omega \left(\frac{1}{2}\partial_t |w_t|^2 + \frac{1}{2}\partial_t |Dw|^2 \right) \, dx \end{aligned}$$

Integrate from $t = 0$, to $t = T$

$$\begin{aligned} 0 &= \int_0^T \partial_t \int_\Omega \left(\frac{1}{2}|w_t|^2 + \frac{1}{2}|Dw|^2 \right) \, dxdt \\ &= \frac{1}{2} \int_\Omega \left(|w_t(x,T)|^2 + |Dw(x,T)|^2 \right) \, dx \\ & \quad - \frac{1}{2} \int_\Omega \left(|w_t(x,0)|^2 + |Dw(x,0)|^2 \right) \, dx = 0 \end{aligned}$$

$\Longrightarrow w_t,\ Dw = 0$ in Ω_T
$\Longrightarrow w = \text{constant} \Longrightarrow w = 0$
$\Longrightarrow u_1 = u_2$ in Ω_T. ■

Alternative: Define

$$e(t) = \frac{1}{2} \int_\Omega (|w_t(x,t)| + |Dw(x,t)|) \, dx$$

and show that $\dot{e}(t) = 0$, $t \geq 0$.

4.4.1 Finite Speed of Propagation via Energy Methods

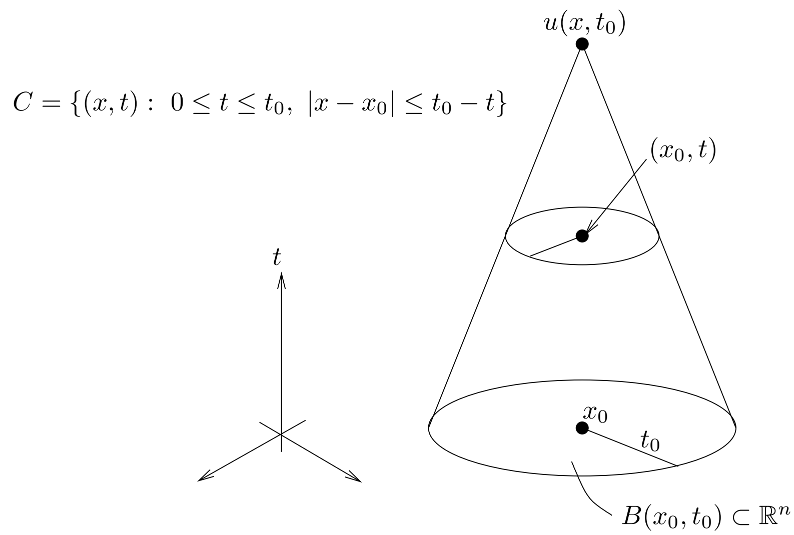


Figure 4.10:

If u is a solution of the wave equation with $u \equiv 0$, $u_t = 0$ in $B(x_0,t_0)$, then $u \equiv 0$ in C and in particular $u(x_0,t_0) = 0$.

$$\text{Define:} \qquad e(t) = \int_{B(x_0,t_0-t)} \left(u_t^2(x,t) + |Du(x,t)|^2\right)\, dx$$

$$\begin{aligned}
\text{Compute:} \qquad \dot{e}(t) &= \int_{B(x_0,t_0-t)} (2u_t u_{tt} + 2Du \cdot Du_t)\, dx \\
&\quad - \int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\, dS \\
&\overset{\text{I.P.}}{=} \int_{B(x_0,t_0-t)} (2u_t u_{tt} - 2\Delta u \cdot u_t)\, dx \\
&\quad + \int_{\partial B(x_0,t_0-t)} 2(Du \cdot \nu) u_t\, dS - \int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\, dS \\
&= 0 + \int_{\partial B(x_0,t_0-t)} (2(Du \cdot \nu)u_t - u_t^2 - |Du|^2)\, dS
\end{aligned}$$

$$|Du \cdot \nu| \overset{\text{C-S}}{\le} |Du| \cdot \underbrace{|\nu|}_{=1} = |Du|$$

$$\begin{aligned} |2(Du\cdot\nu)u_t| &\leq 2|Du|\cdot|u_t| \\ &\overset{\text{Young's}}{\leq} |Du|^2+|u_t|^2 \end{aligned}$$

Young's inequality: $2ab \leq a^2+b^2$, $a,b\in\mathbb{R}$. Comes from the fact $0 \leq a^2-2ab+b^2=(a-b)^2$.

$$\therefore \quad \dot{e}(t) \leq \int_{B(x_0,t_0-t)} (|Du|^2+|u_t|^2-u_t^2-|Du|^2)\ dx \leq 0$$

$\Longrightarrow e(t)$ is decreasing in time.
But,

$$e(0)=\int_{B(x_0,t_0)} (|u_t|^2+|Du|^2)\ dx = 0$$

$\Longrightarrow e(t)=0,\ t\geq 0$
$\Longrightarrow u_t(x,t),\ Bu(x,t)=0,\ (x,t)\in C$
$\Longrightarrow u\equiv 0$ in C
$\Longrightarrow u(x_0,t_0)=0.$ ■

4.5 Exercises

<u>4.1</u>: [Evans 2.6 #16] Suppose that $E=(E^1,E^2,E^3)$ and $B=(B^1,B^2,B^3)$ are a solution of Maxwell's equations

$$E_t = \operatorname{curl} B, \quad B_t = -\operatorname{curl} E, \quad \operatorname{div} E = 0, \quad \operatorname{div} B = 0.$$

Show that

$$u_{tt}-\Delta u = 0$$

where $u=E^i$ or $u=B^i$.

<u>4.2</u>: [Evans 2.6 #17] (Equipartition of energy) Suppose that $u\in C^2(\mathbb{R}\times[0,\infty))$ is a solution of the initial value problem for the wave equation in one dimension,

$$\begin{cases} u_{tt}-u_{xx}=0 & \text{in} \quad \mathbb{R}\times(0,\infty), \\ u=g & \text{on} \quad \mathbb{R}\times\{t=0\}, \\ u_t=h & \text{on} \quad \mathbb{R}\times\{t=0\}. \end{cases}$$

Suppose that g and h have compact support. The kinetic energy is given by

$$k(t)=\int_{-\infty}^{\infty} u_t^2(x,t)\ dx$$

and the potential energy by

$$p(t) = \int_{-\infty}^{\infty} u_x^2(x,t)\ dx.$$

Prove that

a) the total energy, $e(t) = k(t) + p(t)$, is constant in t,

b) $k(t) = p(t)$ for all large enough times t.

4.3: [Evans 2.6 #18] Let u be a solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^3 \times \{t=0\}, \\ u_t = h & \text{on} \quad \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

where g and h are smooth and have compact support. Show that there exists a constant C such that

$$|u(x,t)| \le \frac{C}{t}, \quad x \in \mathbb{R}^3, t > 0$$

4.4: [Qualifying exam 01/01] Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and suppose that $g : \overline{\Omega} \to \mathbb{R}$ is smooth and positive. Given $f_1, f_2 : \Omega \to \mathbb{R}$, show that there can be at most one $C^2(\Omega)$ solution $u : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ to the initial value problem

$$\begin{cases} u_{tt} = g(x)\Delta u & \text{in} \quad \Omega \times (0,\infty), \\ u = f_1 & \text{on} \quad \Omega \times \{t=0\}, \\ u_t = f_2 & \text{on} \quad \Omega \times \{t=0\}, \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty). \end{cases}$$

Hint: Find an energy E for which $dE/dt \le CE$ and use Gronwall's inequality which asserts the following. If $y(t) \ge 0$ satisfies $dy/dt \le \phi y$, where $\phi \ge 0$ and $\phi \in L^1(0,T)$, then

$$y(t) \le y(0)\exp\left(\int_0^t \phi(s)\ dx\right).$$

4.5: [Qualifying exam 08/96] Let $h \in C(\mathbb{R}^3)$ with $h \ge 0$ and $h = 0$ for $|x| \ge a > 0$. Let $u(x,t)$ be the solution of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on} \quad \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = 0 & \text{on} \quad \mathbb{R}^3 \times \{t=0\}, \\ u_t(x,0) = 0 & \text{on} \quad \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

a) Show that $u(x,t) = 0$ in the set $\{(x,t) : |x| \le t - a, t \ge 0\}$.

b) Suppose that there exists a point x_0 such that $u(x_0,t) = 0$ for all $t \ge 0$. Show that $u(x,t) \equiv 0$.

<u>4.6</u>: [Qualifying exam 01/00] Suppose that $q : \mathbb{R}^3 \to \mathbb{R}^3$ is continuous with $q(x) = 0$ for $|x| > a > 0$. Let $u(x,t)$ be the solution of

$$\begin{cases} u_{tt} - \Delta u = e^{i\omega t} q(x) & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = 0 & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}, \\ u_t(x,0) = 0 & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

a) Show that $e^{-i\omega t} u(x,t) \to v(x)$ as $t \to \infty$ uniformly on compact sets, where

$$\Delta v + \omega^2 v = q,$$

Hint: You may take the result of Exercise 2.1 for granted.

b) Show that v satisfies the "outgoing" radiation condition

$$\left| \left(\frac{\partial}{\partial r} + i\omega \right) v(x) \right| \le \frac{C}{|x|^2}$$

as $r = |x| \to \infty$.

Chapter 5

Fully Non-Linear First Order PDEs

Solve $F(Du(x), u(x), x) = 0$ in Ω, Ω bounded and open in $\mathbb{R}^n$.

- Method of characteristics
- Often time variable (conservation laws)
- existence of classical solution only for short times
- global existence for conservation laws via weak solutions (broaden the class of solutions, solutions continuous and discontinuous).

Notation: Frequently, you set $p = Du$, $z = u$.

1st order PDE: $F(p, z, x) = 0$

$$F : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

always f is smooth.

Example: Transport equation

$$u_t(y,t) + a(y,t) \cdot Du(y,t) = 0 \tag{5.1}$$

Set $x = (y, t)$, $p = (Du, u_t)$. Then (5.1) is given by

$$
\begin{aligned}
F(p, z, x) &= 0 \quad \text{with} \\
F(p, z, x) &= \begin{pmatrix} a(y,t) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} Du \\ u_t \end{pmatrix} \\
&= \begin{pmatrix} a(x) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} Du(x) \\ u_t(x) \end{pmatrix} \\
F(p, z, x) &= \begin{pmatrix} a(x) \\ 1 \end{pmatrix} \cdot \rho
\end{aligned}
$$

5.1 Method of Characteristics

Idea: Reduce the PDE to a system of ODEs for $\underline{x}(s)$, $z(s) = u(\underline{x}(s))$, $\underline{p}(s) = Du(\underline{x}(s))$

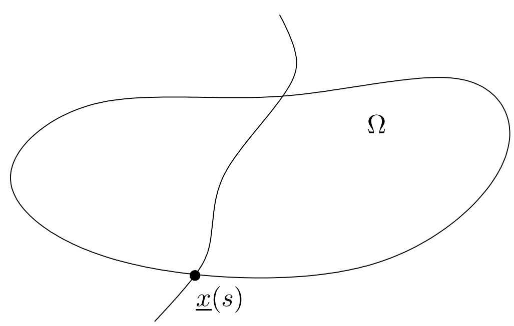


Figure 5.1:

Suppose u is a smooth solution of $F(Du, u, x) = 0$.

$$
\begin{aligned}
\dot{p}_i(s) &= \frac{\partial}{\partial s} u_{x_i}(\underline{x}(s)) \\
&= \sum_{j=1}^{n} u_{x_i x_j}(\underline{x}(s)) \cdot \dot{x}_j(s)
\end{aligned}
$$

1st order PDE, $u \in C^1$ natural, but we get $u_{x_i x_j}$.

Eliminate them!

$$\begin{aligned}
&F(Du,u,x)=0\\
&\frac{\partial}{\partial x_i}F(Du,u,x)=0\\
&=\sum_{j=1}^{n}\frac{\partial F}{\partial p_j}u_{x_ix_j}+\frac{\partial F}{\partial z}u_{x_i}+\frac{\partial F}{\partial x_i}=0
\end{aligned}$$

To eliminate the second derivatives, we impose that

$$\dot{x}_j=\frac{\partial F}{\partial p_j}(\underline{p}(s),z(s),\underline{x}(s))$$

Then

$$\begin{aligned}
\dot{p}_i(s) &= \sum_{j=1}^{n}u_{x_ix_j}(\underline{x}(s))\cdot\dot{x}_j(s)\\
&= \sum_{j=1}^{n}u_{x_ix_j}(\underline{x}(s))\cdot\frac{\partial F}{\partial p_j}(\underline{p}(s),z(s),\underline{x}(s))\\
&= -\frac{\partial F}{\partial z}u_{x_i}(\underline{x}(s))-\frac{\partial F}{\partial x_i}
\end{aligned}$$

Thus,

$$\dot{p}_i(s)=-\frac{\partial F}{\partial z}u_{x_i}(\underline{x}(s))-\frac{\partial F}{\partial x_i}$$

Now find

$$\begin{aligned}
\dot{z}(s) &= \frac{\partial}{\partial s}u(\underline{x}(s))\\
&= \sum_{j=1}^{n}\frac{\partial u}{\partial x_j}(\underline{x}(s))\cdot\dot{x}_j(s)\\
&= \sum_{j=1}^{n}p_j\cdot\dot{x}_j(s)
\end{aligned}$$

Thus,

$$\dot{z}(s)=\sum_{j=1}^{n}p_j\cdot\dot{x}_j(s)$$

Result: If u is a solution of $F(Du, u, x) = 0$, and if

$$\begin{aligned} \dot{x}_j &= \frac{\partial F}{\partial p_j}(\underline{p}(s), z(s), \underline{x}(s)), \quad \text{then} \\ \dot{p}(s) &= -\frac{\partial F}{\partial z} p - \frac{\partial F}{\partial x} \\ \dot{z}(s) &= \frac{\partial F}{\partial p} \cdot p, \end{aligned}$$

i.e., we get a system of $2n + 1$ ODEs

$$\begin{cases} \dot{x} &= F_p \\ \dot{p} &= -F_z p - F_x \\ \dot{z} &= F_p \cdot p \end{cases}$$

Characteristics equations for the 1st order PDE. The ODE $\dot{x} = F_p$ is called the projected characteristics equation. Issues:

- How do the boundary conditions relate to initial conditions for the ODEs?

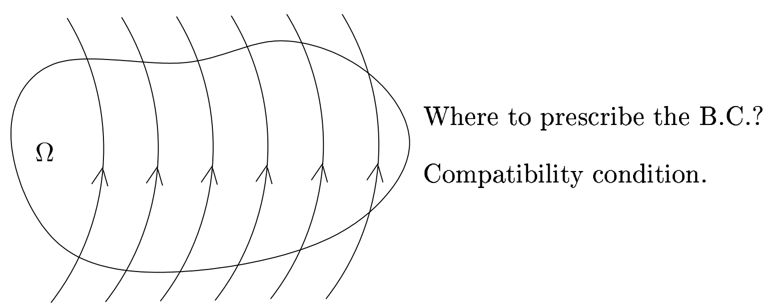


Figure 5.2:

- We can also have the following situation

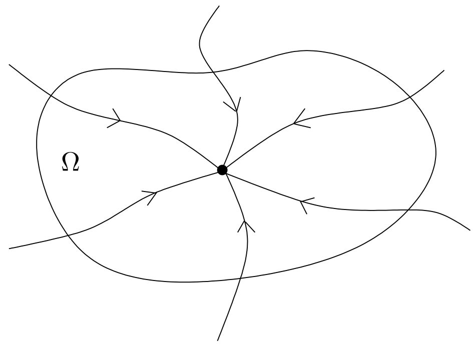


Figure 5.3:

Typically we can solve only locally, but not globally.

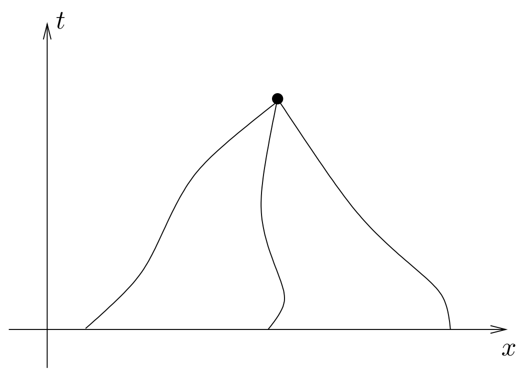


Figure 5.4:

- If we construct $u(\underline{x})$ from the values of u along the characteristic curves, do we get a solution of the PDE?

Examples:

1. Transport equation

$$\begin{aligned} u_t(y,t) + a(y,t) \cdot Du(y,t) &= 0 \\ F(p,z,x) = \begin{pmatrix} a(x) \\ 1 \end{pmatrix} \cdot p &= 0 \end{aligned}$$

Characteristic equations:

$$\begin{aligned} \dot{x} &= F_p = \begin{pmatrix} a(x) \\ 1 \end{pmatrix} \\ \dot{x}(s) &= \big(\dot{y}(s), \dot{t}(s)\big) = (a(y,t), 1) \end{aligned}$$

$\Longrightarrow \dot{t}(s) = 1 \Longrightarrow t = s$

$$\begin{aligned} \dot{y}(s) &= a(y(s), s) \\ \dot{z}(s) &= F_p \cdot p = \begin{pmatrix} a(s) \\ 1 \end{pmatrix} \cdot p = 0 \quad \text{(by the orig. PDE)} \end{aligned}$$

$\Longrightarrow z$ constant along characteristic curve
$\Longrightarrow u(y(s), s) =$ constant
$\Longrightarrow u$ determined from initial data.

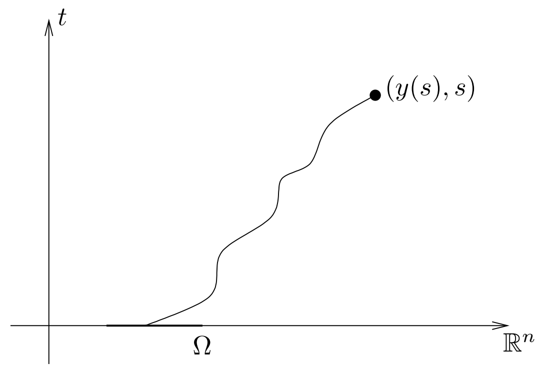


Figure 5.5:

2. Linear PDE: $F(p, z, x) = b \cdot p + cz$

$$\text{i.e.} \qquad b(x) \cdot Du(x) + c(x)u(x) = 0$$

Characteristic equations:

$$\begin{aligned} \dot{x}(s) &= F_p = b(x(s)) \\ \dot{z}(s) &= -c(x(s))u(x(s)) = -c(x(s))z(s) \end{aligned}$$

$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = -c(x(s))z(s) \end{cases}$$

can solve without using equation for $\dot{p}$.

Examples: $x_1 u_{x_2} - x_2 u_{x_1} = u = 0$ linear

$$\begin{aligned} F(p, z, x) &= 0 \\ F(p_1, p_2, z, x_1, x_2) &= x_1 p_2 - x_2 p_1 - z = 0 \\ &= x^{\perp} \cdot p - z, \end{aligned}$$

where

$$x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

- $\dot{x} = F_p$, $\dot{x} = x^\perp$

$$\left.\begin{array}{rcl} \dot{x}_1(s) & = & -x_2(s),\ x_1(0) = x_0 \\ \dot{x}_2(s) & = & x_1(s), \quad x_2(0) = 0 \end{array}\right\} \text{ I.C.}$$

  Solve subject to the initial data $u = g$ on $\Gamma = \{(x,0) \mid x_1 \geq 0\}$

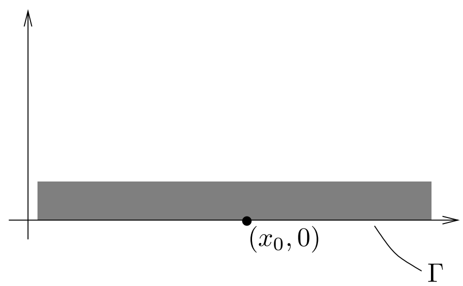


Figure 5.6:

- $\dot{z} = F_p \cdot p = x^\perp \cdot p = z$ by equation
  $z(0) = g(x_0)$

-

$$\dot{p} = -F_x - F_z p = \begin{pmatrix} p_2 \\ -p_1 \end{pmatrix} - (-1)p + \text{ suitable ICs, (see below)}$$

$$\begin{array}{rcl} \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} & = & \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) & = & \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(s) & = & e^{As} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \end{array}$$

$$\left\{\begin{array}{rcl} x_1(s) & = & x_0 \cos s \\ x_2(s) & = & x_0 \sin s \end{array}\right. \qquad \text{circular characteristics}$$

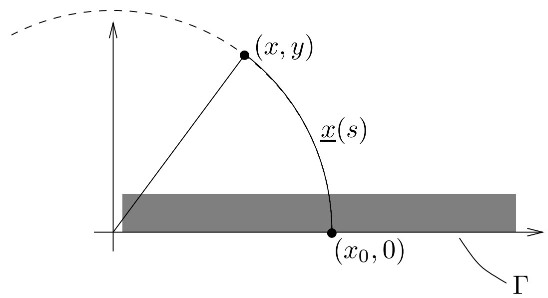


Figure 5.7:

$z(s) = Ce^s = g(x_0)e^s$

Reconstruct $u(x_1,x_2)$ from characteristics. Characteristic through (x_1,x_2) corresponds to $x_0 = \sqrt{x_1^2+x_2^2}$, $(x_1,x_2) = x_0e^{i\theta}$. $u(x_1,x_2) = g(x_0)e^{i\theta}$

$$g\left(\sqrt{x_1^2+x_2^2}\right)\exp\left(\arctan\frac{x_2}{x_1}\right) = u(x_1,x_2)$$

5.1.1 Strategy to Prove that the Method of Characteristics Gives a Solution of $F(Du,u,x)=0$

i.) Reduce initial data on Γ to initial data on a flat part of the boundary, $\mathbb{R}^{n-1}$

ii.) Check that the structure of the PDE does not change

iii.) Set up characteristic equations with the correct initial data

iv.) Solve the characteristic equations

v.) Reconstruct u from the solution

vi.) Check that u is a solution

Here we go!

i.) Flattening the boundary locally, in a neighborhood of $x_0 \in \Gamma$

Suppose that Γ is of class C_1^1 i.e. Γ is locally the graph of a C^1 function of $n-1$ variables

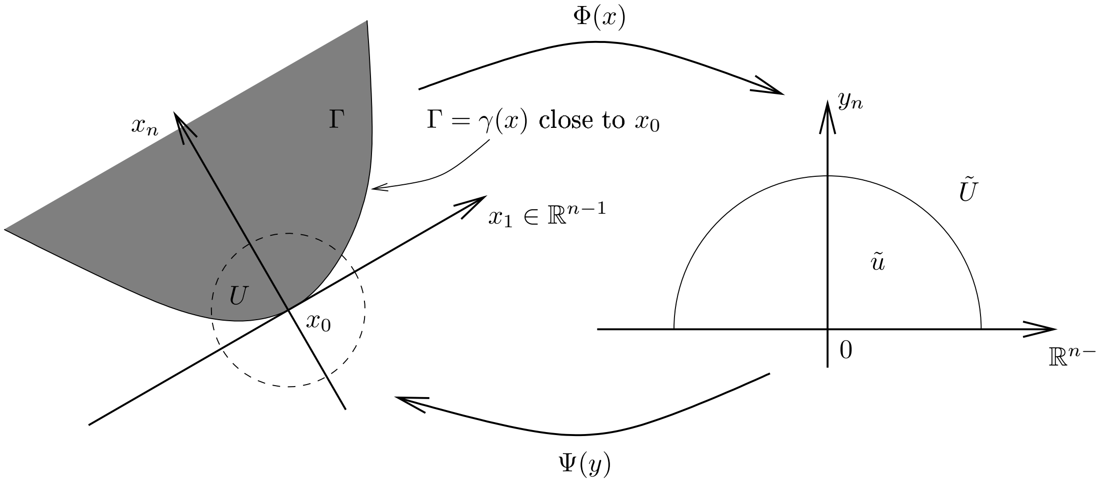


Figure 5.8:

Define transformations ($y = (y', y_n)$)

$$y = \Phi(x) : \begin{cases} y' = x' \\ y_n = x_n - \gamma(x') \end{cases}$$
$$x = \Psi(y) : \begin{cases} x' = y' \\ x_n = y_n + \gamma(x') \end{cases}$$

If $u : U \to \mathbb{R}$, then define
$\tilde{u} : \tilde{U} \to \mathbb{R}$

$$\tilde{u}(y) = u(x) = u(\Psi(y))$$

or vice versa $u(x) = \tilde{u}(y) = \tilde{u}(\Phi(x))$

If u satisfies $F(Du, u, x) = 0$ in U, what is the equation satisfied by $\tilde{u}$?

ii.)

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x) &= \frac{\partial}{\partial x_i}\tilde{u}(\Phi(x)) \\ &= \sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial y_i}\frac{\partial \Phi^j}{\partial x_i} \end{aligned}$$

$$\left[\Phi:\ \mathbb{R}^n \to \mathbb{R}^n, \quad D\Phi = \begin{pmatrix} \Phi^1_1 & \cdots & \Phi^1_n \\ \vdots & \ddots & \vdots \\ \Phi^n_1 & \cdots & \Phi^n_n \end{pmatrix}\right]$$

$$= \left((D\Phi)^T D\tilde{u}(\Phi(x))\right)_i$$

In matrix notation

$$\begin{aligned} Du(x) &= D\Phi^T(x) \cdot D\tilde{u}(\Phi(x)) \\ \Longrightarrow Du(x) &= D\Phi^T(\Psi(y) \cdot D\tilde{u}(y) \end{aligned}$$

$$F(Du(x), u(x), x) = 0$$

$$F(D\Phi^T(\Psi(y)) \cdot D\tilde{u}(y), \tilde{u}(y), \Psi(y)) = \tilde{F}(D\tilde{u}(y), \tilde{u}(y), y) = 0,$$

where $\tilde{F}(p, z, y)$ is $F(D\Phi^T(\Psi(y))p, z, \Phi(y)) = \tilde{F}$.

In particular, $\tilde{F}$ is a smooth function if the boundary of U is smooth.

From now on assume that $x_0 = 0$ and that $\Gamma \subseteq \mathbb{R}^{n-1}$

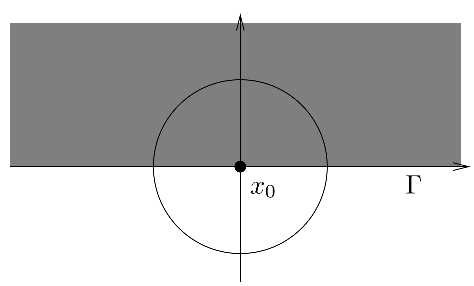


Figure 5.9:

<u>Goal:</u> construct u

Solution in a neighborhood of $x_0 = 0$.

iii.) Find initial conditions for the ODEs.

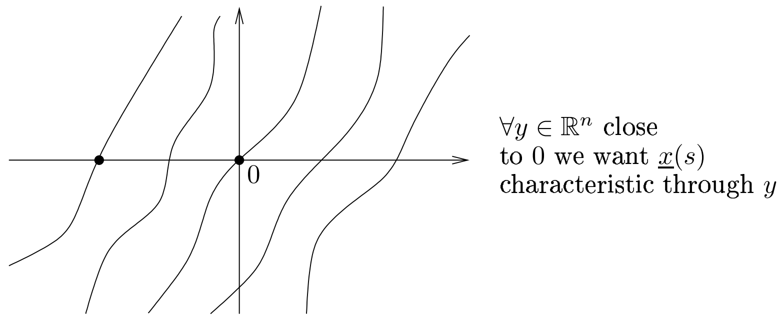


Figure 5.10:

$\implies \underline{x}(0) = y \in \mathbb{R}^{n-1}$
$\implies z(0) = u(\underline{x}(0)) = g(\underline{x}(0)) = g(y)$
$\implies p_i(0) = \frac{\partial g}{\partial y_i}\ i = 1, \dots, n-1$

Tangential derivatives, compatibility condition. One degree of freedom, $p_n(0)$. We must choose $p_n(0)$ to satisfy

$$F\left(\frac{\partial g}{\partial y_i}(y), \dots, \frac{\partial g}{\partial x_i}(y), p_n(0), g(y), y\right) = 0.$$

Definition 5.1. *A triple (p_0, z_0, x_0) of initial conditions is admissible at x_0 if $z_0 = g(x_0)$, $p_{0,i} = \frac{\partial g}{\partial y_i}(x_0)$, $i = 1, \dots, n-1$, $F(p_0, z_0, x_0) = 0$.*

iii.) Initial conditions

$$\left\{\begin{array}{rcl} \underline{x}(0) & = & \underline{y}\ (\in \mathbb{R}^{n-1}) \\ z(0) & = & g(y) \\ p_i(0) & = & \frac{\partial g}{\partial x_i}(y) \quad i \in \{1, \dots, n-1\} \\ p_n(0) & & \text{chosen such that } F(p_1(0), \dots, p_n(0), g(y), y) = 0 \end{array}\right. \tag{5.2}$$

(x_0, z_0, p_0) is called admissible if

$$\left\{\begin{array}{l} z_0 = g(x_0) \\ \left.\begin{array}{l} p_{0,i} = \frac{\partial g}{\partial x_i}(x_0) \quad i = 1, \dots, n-1 \\ F(p_0, z_0, x_0) = 0 \end{array}\right\} \begin{array}{l} n \text{ equations for} \\ \text{the } n \text{ ICs } p_1^0, \dots, p_n^0 \end{array} \end{array}\right.$$

Question: We want to solve close to x_0, can we find admissible initial data, y, $g(y)$, $q(y)$ for y close to x. $F(q(y), g(y), y) = 0$?

Lemma 5.2. *(p_0, z_0, x_0) admissible, and suppose that $F_p(p_0, z_0, x_0) \neq 0$. Then there exists a unique solution of (5.2) for y sufficiently close to x. The triple (p_0, z_0, x_0) is said to be non-characteristic.*

Remark: If Γ is a smooth boundary then (p_0, z_0, x_0) is non-characteristic if $F_p(p_0, z_0, x_0) \cdot \nu(x_0) \neq 0$, where $\nu(x_0)$ is the normal to Γ at x_0.

Proof: Use the implicit function theorem, $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $G(p, y) = (G^1(p, y), \ldots, G^n(p, y))$ with $G^i(p, y) = p_i - \frac{\partial g}{\partial x_i}(y)$. $G^n(p, y) = F(p, g(y), y)$.

The triple (p_0, z_0, x_0) admissible, $G(p_0, x_0) = 0$. Want to solve locally for $p = q(y)$, we can do this if $\det(G_p(p_0, y_0)) \neq 0$.

$$G_p(p, y) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ F_{p_1} & \cdots & \cdots & \cdots & \cdots & F_{p_n} \end{pmatrix}$$

$\det(G_p(p, y)) = F_{p_n}(p, g(y), y)$
$\Longrightarrow \det(G_p(p_0, x_0) \neq 0$
$\Longrightarrow$ there exists a unique solution $p = q(y)$ close to x_0

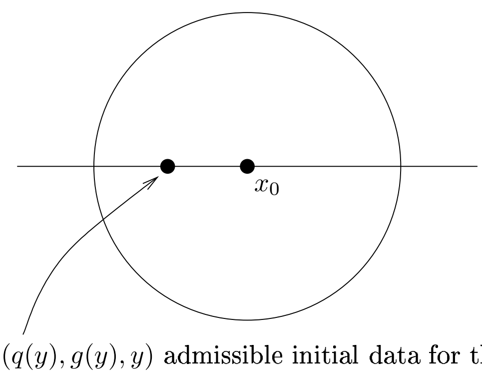


Figure 5.11:

iv.) Solution of the system of ODEs

$$\begin{aligned} \dot{x}(s) &= F_p(\underline{p}(s), z(s), \underline{x}(s)) \\ \dot{z}(s) &= F_p(\underline{p}(s), z(s), \underline{x}(s)) \cdot \underline{p}(s) \\ \dot{p}(s) &= -F_x(\underline{p}(s), z(s), \underline{x}(s)) - F_z(\underline{p}(s), z(s), \underline{x}(s)) p(s) \end{aligned}$$

Initial data

$$\begin{cases} \underline{x}(0) &= y \\ z(0) &= g(y) \qquad\qquad y \in \mathbb{R}^{n-1} \text{ parameter} \\ \underline{p}(0) &= q(y) \end{cases}$$

Let

$$\tilde{X}(y,s) = \begin{pmatrix} x(s) \\ z(s) \\ p(s) \end{pmatrix}$$

System:

$$\begin{cases} \dot{\tilde{X}}(y,s) &= G(\tilde{X}(y,s)) \\ \tilde{X}(y,0) &= (q(y), g(y), y) \end{cases} \tag{5.3}$$

Lemma 5.3 (Existence of solution for (5.3)). *If $G \in C^k$, $\tilde{X}(y,0)$ is C^k (as a function of y) then there exists a neighborhood N of $x_0 \in \mathbb{R}^{n-1}$ and an $s_0 > 0$ such that (5.3) has a unique solution $X : N \times (-s_0, s_0) \to \mathbb{R}^n \times \mathbb{R} \times \Omega$ and X is C^k.*

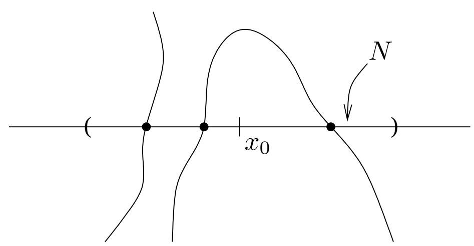


Figure 5.12:

v.) Reconstruction of u from the solutions of the characteristic equations

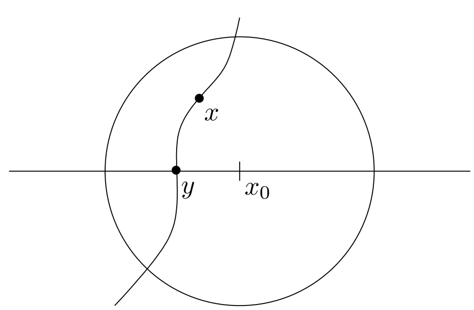


Figure 5.13:

given x: find characteristic through x, i.e. find $y = Y(x)$ and $s = S(x)$ such that the characteristic through y passes through x at time s.

Define: $u(x) = z(Y(x), S(x))$

Check: Invert $x \mapsto Y(x), S(x)$

Lemma 5.4. *Suppose that (p_0, z_0, x_0) are non-characteristic initial data. Then there exists a neighborhood V of x_0 such that $\forall x \in V$ there exists a unique Y and a unique S with*

$$\underline{x} = X(Y(x), S(x))$$

Proof: We need to invert X. Inverse functions. We can invert $X(y,s)$ if $DX(x_0) \neq 0$.

$$DX(x_0) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & F_{p_1} \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & F_{p_n} \end{pmatrix}$$

To find $\partial_s X$ recall the characteristic equation, $\dot{x} = F_p \Longrightarrow \det(DX) = F_{p_n} \neq 0$ since (p_0, z_0, x_0) is non-characteristic. ■

vi.) prove that u is a solution of the PDE

$$\begin{aligned} u(x) &= Z(Y(x), S(x)) \\ \underline{p}(x) &= P(Y(x), S(x)) \end{aligned}$$

Crucial step $Du(x) = p(x)$

Step 1: $f(y,s) := F(P(y,s), Z(y,s), X(y,s)) = 0$

Proof:

$$\begin{aligned} f(y,0) &= F(P(y,0), Z(y,0), X(y,0)) \\ &= F(q(y), g(y), y) \\ &= 0 \end{aligned}$$

since $(q(y), g(y), y)$ is admissible.

$$\begin{aligned}\dot{f}(y,s) &= F_p\dot{P} + F_z\dot{Z} + F_x\dot{X} \\ &= F_p(-F_x - F_z p) + F_z F_p p + F_x F_p \\ &= 0\end{aligned}$$

$\Longrightarrow F(p(y,s), u(x), x) = 0.$ ■

<u>Recall:</u> Use capital letters to denote the dependence on y,

$$P(y,s),\ Z(y,s),\ X(y,s)$$

solutions with initial data $(q(y), g(y), y)$.
Invert the characteristics:

$$x \rightsquigarrow Y(x),\ S(x) \text{ such that } X(Y(x), S(x)) = x$$

Convention today: all gradients are rows.

Define:

$$\begin{aligned}u(x) &:= Z(Y(x), S(x)) \\ p(x) &:= P(Y(x), S(x))\end{aligned}$$

1. $f(y,s) = F(P(y,s), Z(y,s), X(y,s)) \equiv 0$ as a function of s.

2. We assert the following:

   i.) $Z_s = P \cdot X_s^T = \langle P, X_s\rangle$
   ii.) $Z_y = P \cdot X_y = P \cdot D_y X$

$$\frac{\partial z}{\partial y_i} = \sum_{j=1}^{n-1} P_j \frac{\partial X^j}{\partial y_i} \qquad i = 1, \ldots, n-1$$

<u>Proof:</u>

i.) $Z_s = F_p \cdot p = X_s \cdot p$ by the characteristic equations.

ii.) Before we prove the second identity we find

$$\begin{aligned}D_y D_s Z &= D_y Z_s \\ &= P \cdot D_y D_s X + D_s X \cdot D_y P\end{aligned}$$

Define

$$r(s) := D_y Z - P \cdot D_y X$$

and show that $r(s) = 0$.

Idea: find ODE for r and show that $r = 0$ is the unique solution.

$$r_i(0) = \frac{\partial g}{\partial x_i}(y) - q(y) \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad \text{for } i = 1, \ldots, n-1$$

$$\Longrightarrow r_i(0) = \frac{\partial g}{\partial x_i}(y) - q_i(y) = 0$$

Since the triple $(q(y), g(y), y)$ is admissible.

$$\begin{array}{rcl} \dot{r}(s) & = & D_s D_y Z - D_s P \cdot D_y X - P \cdot D_s D_y X \\ & = & P \cdot D_y D_s X + D_s X \cdot D_y P \\ & & -D_s P \cdot D_y X - P \cdot D_s D_y X \\ & = & D_s X \cdot D_y P - D_s P \cdot D_y X \\ \text{char. eq.} & = & F_p \cdot D_y P - (-F_x - F_z P) D_y X - F_z \cdot D_y Z \\ & & -(F_x - F_z P) D_y X \end{array}$$

$(D_y(F(P, Z, X)) = 0)$

$$\begin{array}{rcl} & = & F_z P \cdot D_y X - F_z \cdot D_y Z \\ & = & -F_z (D_y Z - P \cdot D_y X) \\ & = & -F_z \cdot r(s) \end{array}$$

$$\therefore \quad \left\{ \begin{array}{rcl} \dot{r}(s) & = & -F_z \cdot r(s) \\ r(0) & = & 0 \end{array} \right.$$

Linear ODE for r with $r(0) = 0 \Longrightarrow r = 0$ for $s > 0$. ■

3. Prove that $Du(x) = p(s)$

$\Longrightarrow F(P, Z, X) = F(Du(x), u(x), x) = 0$

$\Longrightarrow u$ is a solution of the PDE

$$u(x) = Z(Y(x), S(x))$$

$$\begin{aligned}
\Longrightarrow D_x u &= D_y Z \cdot D_x Y + D_s Z \cdot D_x S \\
\text{(by 2.)} &= P \cdot D_y X \cdot D_x Y + P \cdot D_s X \cdot D_x S \\
&= P(D_y X \cdot D_x Y + D_s X \cdot D_x S) \\
&= P \cdot D_x X(Y, S) \\
&= P \cdot Id = p
\end{aligned}$$

$\Longrightarrow D_x u = P.$ ■

Examples:

1. Linear, homogeneous 1st order PDE:

$$b(x) \cdot Du(x) + c(x)u(x) = 0$$

Local existence of solutions:

Initial data non-characteristic:

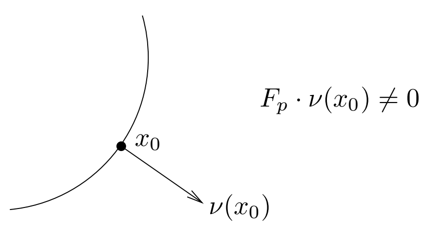


Figure 5.14:

$$\begin{aligned}
F(p, z, x) &= b(x) \cdot p + c(x)z \\
F_p &= b
\end{aligned}$$

Can solve if $F_p(p_0, z_0, x_0) \cdot \nu(x_0) = b(x_0) \cdot \nu(x_0) \neq 0$

Local existence of $b(x_0)$ is not tangential

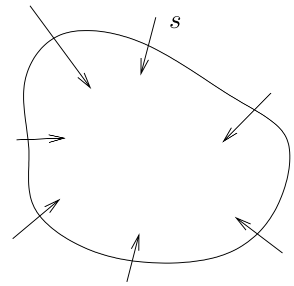


Figure 5.15:

Characteristic equation:

$$\dot{x} = F_p = b$$

($x(s)$ would be tangential to Γ if the initial conditions fail to be non-characteristic).

2. Quasilinear equation

$$b(x, u) \cdot Du(x) + c(x, u) = 0$$

Initial data non-characteristic $F_p = b$,

$$F_p(p_0, z_0, x_0) \cdot \nu(x_0) = b(x_0, z_0) \cdot \nu(x_0) \neq 0.$$

Characteristics:

$$\begin{aligned} \dot{x} &= F_p, \qquad \dot{x}(s) = b(x(s), z(s)) \\ \dot{z} &= F_p \cdot p \\ &= b(x(s), z(s)) \cdot p(s) \\ &= -c(x(s), z(s)) \end{aligned}$$

Closed system

$$\begin{aligned} \dot{x}(s) &= b(x(s), z(s)) \qquad x(0) = x_0 \\ \dot{z}(s) &= -c(x(s), z(s)) \qquad z(0) = g(x_0) \end{aligned}$$

for x and z, do not need the equation for p.

5.2 Exercises

5.1: [Evans 3.5 #2] Write down the characteristic equations for the linear equation

$$u_t + b \cdot Du = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

where $b \in \mathbb{R}^n$ and $f = f(x,t)$. Then solve the characteristic equations with initial conditions corresponding to the initial conditions

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

5.2: [Evans 3.5 #3] Solve the following first order equations using the method of characteristics. Derive the full system of the ordinary differential equations including the correct initial conditions before you solve the system!

a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u$ with $u(x_1, 1) = g(x_1)$.

b) $u u_{x_1} + u_{x_2} = 1$ with $u(x_1, x_2) = \frac{1}{2} x_1$.

c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ with $u(x_1, x_2, 0) = g(x_1, x_2)$.

5.3: [Qualifying exam 08/00] Suppose that $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a smooth solution of $u_t + u u_x = 0$ that is periodic in x with period L, i.e., $u(x + L, t) = u(x, t)$. Show that

$$\max_{x \in \mathbb{R}} u(x, 0) - \min_{x \in \mathbb{R}} u(x, 0) \le \frac{L}{T}.$$

5.4: Solve the nonlinear PDE of the first order

$$u_{x_1}^3 - u_{x_2} = 0, \quad u(x_1, 0) = g(x_1) = 2x_1^{3/2}.$$

5.5: [Qualifying exam 01/90] Find a classical solution of the Cauchy problem

$$\begin{cases} (u_x)^2 + u_t = 0, \\ u(x, 0) = x^2. \end{cases}$$

What is the domain of existence of the solution?

Chapter 6

Conservation Laws

6.1 Introduction and the Rankine-Hugoniot Condition

$$\begin{aligned}\text{Conservation law:} \quad & u_t + F(u)_x = 0 \\ \text{initial data:} \quad & u(x,0) = u_0(x) = g(x)\end{aligned}$$

Example: Burger's equation, $F(u) = \frac{1}{2}u^2$

$$u_t + F(u)_x = u_t + \left(\frac{u^2}{2}\right)_x = u_t + u \cdot u_x = 0$$

Solutions via Method of Characteristics.

$$u_t + F'(u)u_x = 0$$

Standard form: $G(p, z, y) = 0$, where $p = (u_x, u_t)$ and $y = (x, t)$.

$$G(p, z, y) = p_2 + F'(z)p_1 = 0$$

Characteristic equations: $\dot{y} = G_p$
$\dot{z} = G_p \cdot p$

$$\begin{aligned}\dot{y} &= (F'(z), 1) \\ \dot{z} &= (F'(z), 1) \cdot p = p_1 F'(z) + p_2 = 0 \quad \text{(equation)}\end{aligned}$$

Rewritten, we have

$$\begin{aligned}\dot{y_1}(s) &= F'(z(s)) \qquad y_1(0) = x_0 \qquad z(0) = g(x_0) \\ \dot{y_2}(s) &= 1 \qquad\qquad\quad y_2(0) = 0\end{aligned}$$

$\Longrightarrow y_2(s) = s$ ($\Longrightarrow$ identification $t = s$)
$\Longrightarrow \dot{z}(s) = 0 \Longrightarrow z(s) = \text{constant} = g(x_0)$

$$\dot{y_1}(s) = F'(z(s)) = F'(g(x_0))$$

$$\begin{aligned} y_1(s) &= F'(g(x_0))s + x_0 \\ &= F'(g(x_0))t + x_0 \qquad (s = t) \end{aligned}$$

$\Longrightarrow$ characteristic curves are straight lines.

Example: Burgers' equation with $g(x) = u(x, 0)$

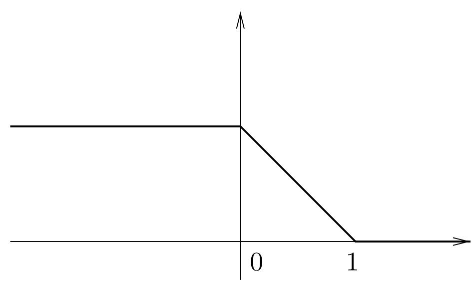


Figure 6.1:

$$g(x) = \begin{cases} 1 & x \le 0 \\ 1 - x & 0 \le x < 1 \\ 0 & x \ge 0 \end{cases}$$

Speed of the characteristic equations

$$F'(g(x_0)) = g(x_0)$$

Characteristics:

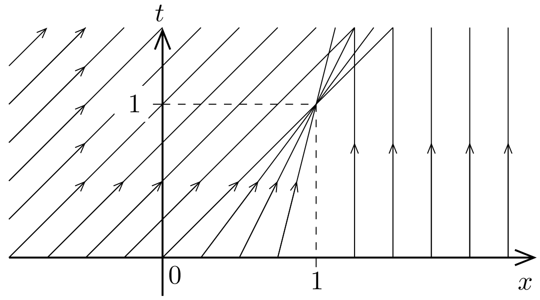


Figure 6.2:

Solution (local existence):

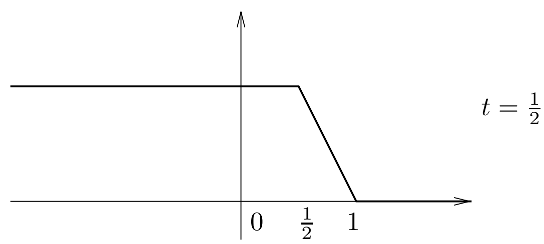


Figure 6.3:

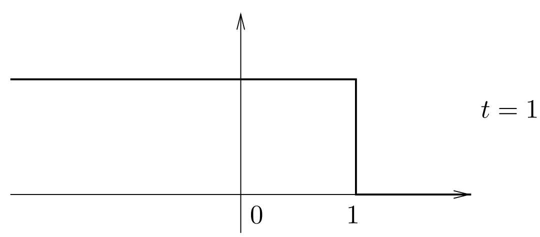


Figure 6.4:

- Discontinuous "solutions" in finite time.
- Break down of classical solution.
- Same effect if you take smooth (C^1, or more) initial data.
- Equation $u_t + F(u)_x = 0$ involving derivatives is not defined.
- Weak solutions (or integral solutions).
- Go back from the PDE to an integrated version.

- v test function.
  v smooth, with compact support in $\mathbb{R} \times [0, \infty)$.

- Multiply equation and integrate by parts.

$$
\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}} v(u_t + F(u)_x)\ dxdt \\
&= -\int_0^\infty \int_{\mathbb{R}} v_t \cdot u\ dxdt + \int_{\mathbb{R}} v \cdot u\ dx \Big|_{t=0}^{t=\infty} \\
&\quad -\int_0^\infty \int_{\mathbb{R}} v_x \cdot F(u)\ dxdt + \int_0^\infty F(u) \cdot v\ dt \Big|_{x=-\infty}^{x=\infty} \\
&= -\int_0^\infty \int_{\mathbb{R}} (v_t \cdot u + v_x \cdot F(u))\ dxdt - \int_{\mathbb{R}} v(x,0) \cdot u(x,0)\ dx
\end{aligned}
$$

So, we have

$$
\int_0^\infty \int_{-\infty}^\infty v_t \cdot u + v_x \cdot F(u)\ dxdt + \int_{-\infty}^\infty v(x,0) \cdot g(x)\ dx = 0 \tag{6.1}
$$

Definition 6.1. *u is a* <u>*weak solution*</u> *of the conservation law $u_t + F(u)_x = 0$ if (6.1) holds for all test functions v.*

<u>Question:</u> What information is hidden in the weak formulation?

If $u \in C^1$ is a weak solution, reverse the integration by parts and get

$$
\int_0^\infty \int_{\mathbb{R}} v(u_t + F(u)_x)\ dxdt = 0
$$

for all smooth test functions.

$$
\Longrightarrow u_t + F(u)_x = 0
$$

6.1.1 [

Information about Discontinuous Solutions, The Rankine-Hugonoit Condition]Information about Discontinuous Solutions of (6.1), The Rankine-Hugonoit Condition

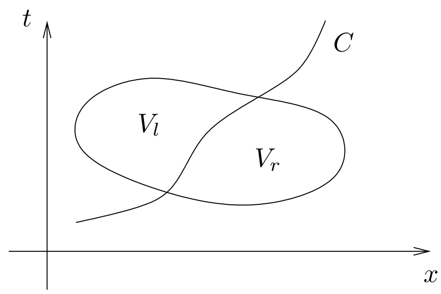


Figure 6.5:

u is a solution in $V = V_l \cup V_r$, u smooth in V_l and V_r and u can have a discontinuity across C (e.g. $u \equiv 1$ on V_l and $u \equiv 0$ on V_r).

Is this possible?

v test function. Support of v contained in V_l (V_r) $\Longrightarrow u_t + F(u)_x = 0$ in V_l (V_r).

Support for v contains a part of C, and that $v(x,0) = 0$.

$$
\begin{aligned}
(6.1) \qquad & \iint_V (v_t \cdot u + v_x \cdot F(u))\ dxdt = 0 \\
= \quad & \iint_{V_l} (v_t \cdot u + v_x \cdot F(u))\ dxdt + \iint_{V_r} (v_t \cdot u + v_x \cdot F(u))\ dxdt = 0
\end{aligned}
$$

Now, let us look at

$$
\begin{aligned}
\iint_{V_l} (v_t \cdot u + v_x \cdot F(u))\ dxdt \quad & \overset{\text{I.P.}}{=} \quad -\iint_{V_l} v\, \underbrace{(u_t + F(u)_x)}_{0}\ dxdt - \int_C \begin{pmatrix} v \cdot F(u) \\ v \cdot u \end{pmatrix} \cdot \nu\ dS \\
& = \quad \int_C (v \cdot F(u) \cdot \nu_1 + v \cdot u \cdot \nu_2)\ dS
\end{aligned}
$$

Note that u here is the trace of u from V_l on the curve C, call this u_l:

$$
\int_C v(\cdot F(u_l) \cdot \nu_1 + u_l \cdot \nu_2)\ dS
$$

Same calculation for V_r with ν replaced by $-\nu$

$$
\iint_{V_r} (v_t \cdot u + v_x \cdot F(u))\ dxdt = -\int_C v(F(u_r) \cdot \nu_1 + u_r \cdot \nu_2)\ dS
$$

Grand total:

$$\int_C v[F(u_l) - F(u_r)]\nu_1 + (u_l - u_r)\nu_2]\, dS = 0$$

$\implies [F(u_l) - F(u_r)]\nu_1 + (u_l - u_r)\nu_2 = 0$ along C.

Suppose now that C is given by $(x = s(t), t)$

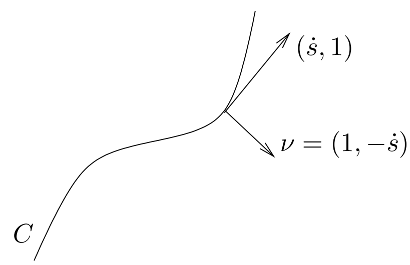


Figure 6.6:

Then $\nu = (1, -\dot{s})$ and

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r)$$

this is typically written as (adopting new notation)

$$[\![F(u)]\!] = \sigma \cdot [\![u]\!],$$

where $\sigma = \dot{s}$ and $[\![w]\!] = w_l - w_r =$ jump.

This is the Rankine-Hugoniot Condition

6.2 Shock Waves, Rarefaction Waves and Entropy Conditions

We return to our original example:

$$u_t + u \cdot u_x = 0 \qquad g(x) = \begin{cases} 1 & x \le 0 \\ 1 - x & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

A discontinuity develops at $t = 1$, specifically, there is a jump for $u_l = 1$ to $u_r = 0$.

In the general situation, one solves the Riemann problem for the conservation laws

$$u_t + F(u)_x = 0 \qquad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & 0 \leq x > 0 \end{cases}$$

Example: The Riemann problem for Burger's equation:

1. $u_l = 1$, $u_r = 0$. The discontinuity can propagate along a curve C with

$$\dot{s} = \frac{[\![F(u)]\!]}{[\![u]\!]} = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \frac{\frac{1}{2} - 0}{1} = \frac{1}{2}$$

   The characteristics are going into the curve C, a so-called shock wave.

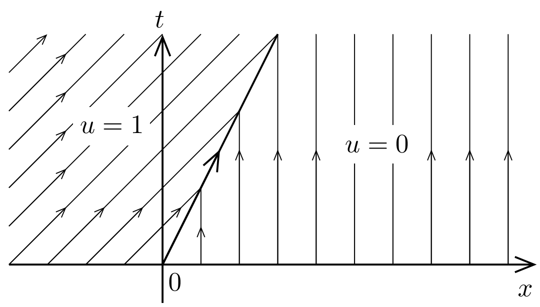


Figure 6.7:

2. $u_l = 0$, $u_r = 1$. Again, we have $\dot{s} = \frac{1}{2}$; but this time the characteristics are emerging from the shock.

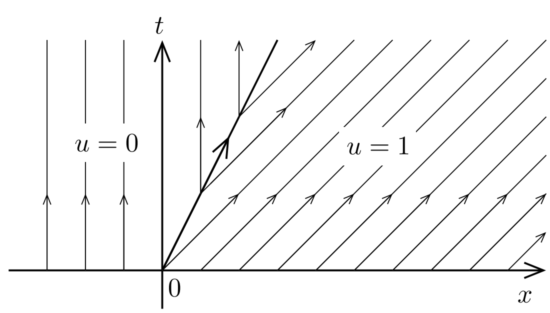


Figure 6.8:

Is this the only solution? We can find another solution of Burger's equation by the similarity method, i.e. by selecting solutions of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

This leads to $u(x,t) = \frac{x}{t}$. We know construct a different solution:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 \le x < t \\ 1 & x > t \end{cases}$$

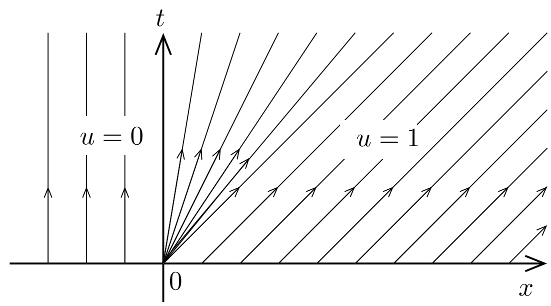


Figure 6.9:

This solution is continuous and called a rarefaction wave. It is a weak solution.

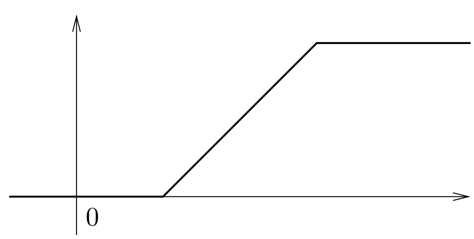


Figure 6.10:

<u>Question:</u> Which solution is the physically relevant solution?

<u>No Ideas:</u> The solution should be stable under perturbations, i.e. should look similar if we smooth the initial data . The solution should be the limit of a viscous approximation of the original equation (more later).

Lax entropy condition: $F'(u_l) > \dot{s} > F'(u_r)$
i.e., characteristics impinge into the shock.

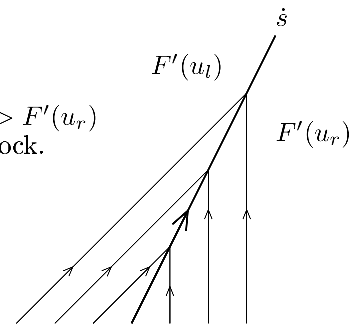


Figure 6.11:

Recall that $\dot{y} = G(F'(z), 1)$, i.e., $F'(u_l)$ is the speed of the characteristics with initial data u_l.

Recall: Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + u \cdot u_x = 0$$

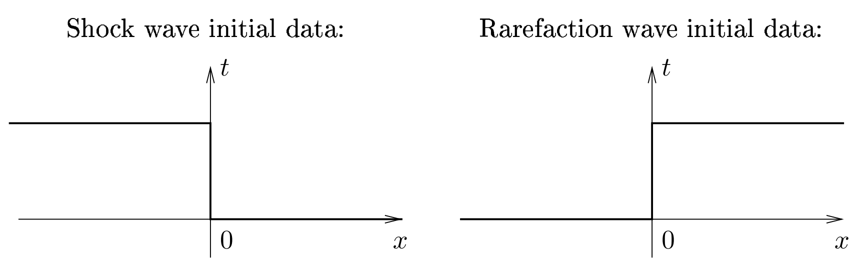


Figure 6.12:

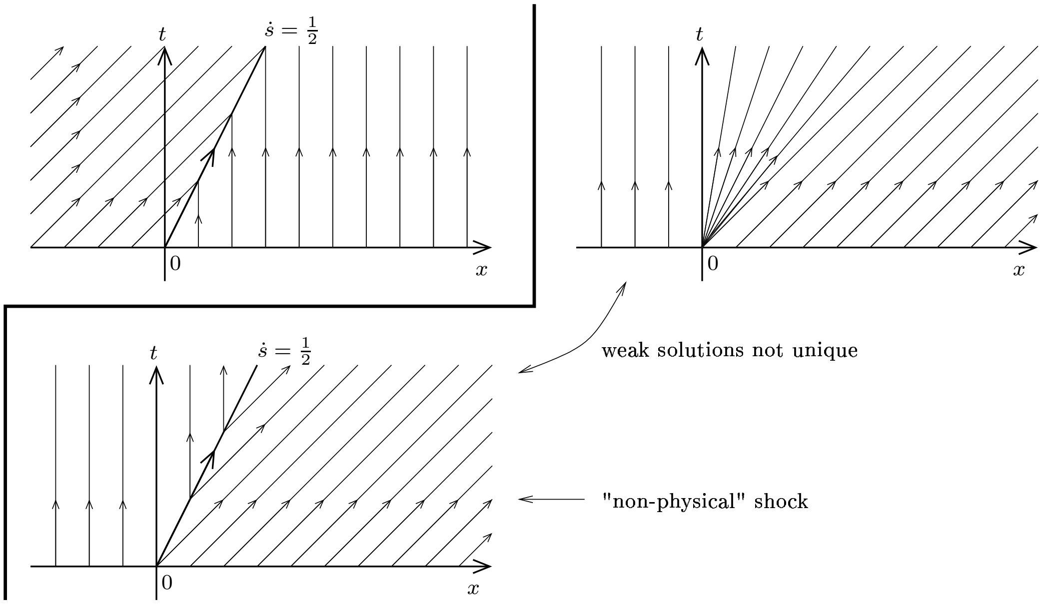


Figure 6.13:

Odd entropy conditions to select solutions:

Lax Characteristics going into the shock.

Viscous approximation: u as a limit of u_ϵ,

$$u_t^\epsilon + u^\epsilon \cdot u_x^\epsilon + \epsilon \cdot u_{xx}^\epsilon = 0$$

Hopf-Cole: Change of the dependent variables, $w = \Phi(u)$

$$\begin{cases} u_t - a\Delta u + b|Du|^2 & = & 0 \\ u & = & g \end{cases}$$

$w_t = a\Delta w - |Du|^2(a\Phi''(u) - b\Phi'(u))$

Choose Φ as a solution of $a\Phi'' - b\Phi' = 0 \implies w$ solves the heat equation. One solution is given by

$$\Phi(t) = \exp\left(-\frac{bt}{a}\right)$$

Choosing this Φ, we have

$$\begin{cases} w_t - a\Delta w & = & 0 \\ w(x,0) & = & \exp\left(-\frac{b \cdot g(x)}{a}\right) \end{cases}$$

6.2.1 Application to Burger's Equation

Define

$$v(x,t) := \int_{-\infty}^{x} u(y,t)\ dt$$

assuming that the integral exists and that $u, u_x \to 0$ as $x \to -\infty$.
The equation for v:

$$v_t - \epsilon \cdot v_{xx} + \frac{1}{2}v_x^2 = 0$$

Proof: Compute

$$\begin{aligned}
\int_{-\infty}^{x} u_t\ dy - \epsilon \cdot u_x + \frac{1}{2}u^2 &= \int_{-\infty}^{x} (\epsilon \cdot u_{xx} - \underbrace{u \cdot u_x}_{\left(\frac{1}{2}u^2\right)_x})\ dy - \epsilon \cdot u_x + \frac{1}{2}u^2 \\
&= \epsilon \cdot u_x - \frac{1}{2}u^2\Big|_{-\infty}^{x} - \epsilon \cdot u_x + \frac{1}{2}u^2 \\
&= \epsilon \cdot u_x - \frac{1}{2}u^2 - \epsilon \cdot u_x + \frac{1}{2}u^2 \\
&= 0
\end{aligned}$$

$\Longrightarrow v$ solves $v_t - \epsilon \cdot v_{xx} + \frac{1}{2}v_x^2 = 0$

$$v(x,0) = h(x) = \int_{-\infty}^{x} g(y)\ dy$$

This is of the form of the model equation with $a = \epsilon$, $b = \frac{1}{2}$.

$$w = \exp\left(-\frac{b \cdot v}{a}\right) \quad \text{solves} \quad \left\{ \begin{array}{rcl} w_t - \epsilon \cdot w_{xx} &=& 0 \\ w(x,0) &=& \exp\left(-\frac{b \cdot h(x)}{a - \epsilon}\right) \end{array} \right.$$

The unique bounded solution is

$$w(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right) \exp\left(-\frac{b \cdot h(y)}{\epsilon}\right)\ dy$$

Now, $v(x,t) = -\frac{\epsilon}{b}\ln w$

$$v(x,t) = -\frac{\epsilon}{b}\ln\left\{\frac{1}{\sqrt{4\pi\epsilon t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right) \exp\left(-\frac{b \cdot h(y)}{\epsilon}\right)\ dy\right\}$$

and $u(x,t) = v_x(x,t)$

$$u(x,t) = \frac{1}{b}\frac{\int_{\mathbb{R}} \frac{x-y}{4t} \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right) \exp\left(-\frac{b\cdot h(y)}{\epsilon}\right)\, dy}{\int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right) \exp\left(-\frac{b\cdot h(y)}{\epsilon}\right)\, dy}$$

Formula for u^ϵ, let $\epsilon \to 0$

Asymptotics of the formula for $\epsilon \to 0$: Suppose that k and l are continuous functions, l grows linearly, k grows at least quadratically. Suppose that k has a unique minimum point,

$$k(y_0) = \min_{y\in\mathbb{R}} k(y)$$

Then

$$\lim_{\epsilon\to 0} \frac{\int_{\mathbb{R}} l(y) \cdot \exp\left(-\frac{k(y)}{\epsilon}\right)\, dy}{\int_{\mathbb{R}} \exp\left(-\frac{k(y)}{\epsilon}\right)\, dy} = l(y_0)$$

Proof: See Evans

Evaluation for viscous approximation with

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\begin{aligned} l(y) &= \frac{x-y}{t} \\ k(y) &= \frac{(x-y)^2}{4t} + \frac{h(y)}{2} \\ h(y) &= \int_{-\infty}^{y} g(x)\, dx = \begin{cases} 0 & y < 0 \\ y & y \geq 0 \end{cases} \quad =: y^+ \end{aligned}$$

Find minimum of k:

Case 1: $y < 0$, $k(y) = \frac{(x-y)^2}{4t}$

$$\text{Minimum:} \quad \begin{array}{ll} y_0 = x & \text{if} \quad x \leq 0 \\ y_0 = 0 & \text{if} \quad x \geq 0 \end{array}$$

Case 2: $y \geq 0$, $k(y) = \frac{(x-y)^2}{4t} + \frac{y}{2}$

$$\begin{aligned} k'(y) &= \frac{x-y}{2t} + \frac{1}{2} \\ k''(y) &= \frac{1}{2t} > 0 \Longrightarrow \min \end{aligned}$$

$k'(y) = 0$, $y = x - t$

$$\begin{aligned} y_0 = x - t \quad &\text{if} \quad x \geq t \\ y_0 = 0 \quad &\text{if} \quad x \leq t \end{aligned}$$

Evaluation of the formula:

(1): $x > t$:
$y \leq 0$: $\Longrightarrow y_0 = 0$, $k(y_0) = \frac{x^2}{4t}$
$y \geq 0$: $\Longrightarrow u_0 = x - t$, $k(x-t) = \frac{t}{4} + \frac{x-t}{2} = \frac{-t}{4} + \frac{x}{2}$

$$\begin{aligned} &\frac{x^2}{4t} > \frac{x}{2} \iff x^2 > 2xt - t^2 \iff (x-t)^2 \geq 0 \qquad \checkmark \\ &y_0 = x - t, \; l(y) = \frac{x-y}{t}, \; l(x-t) = \frac{t}{t} = 1 \end{aligned}$$

Analogous arguments for:

(2): $0 \leq x \leq t$
$y_0 = 0$, $l(y) = \frac{x}{t}$

(3): $x < 0$, $y_0 = x$, $l(x) = 0$

$$u(x,t) = \lim_{\epsilon \to 0} u^\epsilon(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 \leq x \leq t \\ 1 & x > t \end{cases}$$

Rarefaction wave!

6.3 Viscous Approximation of Shocks; Traveling Wave Solutions

Burger's equation: Shocks (RH condition), rarefaction waves

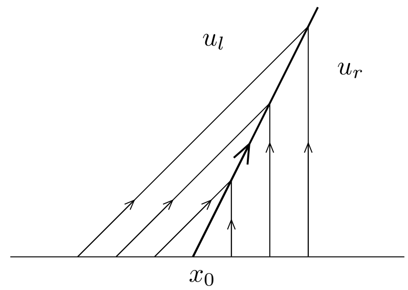


Figure 6.14:

$$u_t + F(u)_x = u_t + F'(u) \cdot u_x = 0$$

Lax criterion: $F'(u_l) > \dot{s} > F'(u_r)$

Since $F'(u_{l/r})$ is the speed of the characteristics on the left/right of x_0.

More generally, find shock solutions of the Riemann problem

$$\begin{aligned} u_t + F(u)_x &= 0 \\ u(x,0) &= \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \end{aligned}$$

Building block for numerical schemes that approximate initial data by piecewise constant functions.

Viscous approximation: $u_t + F(u)_x = \epsilon \cdot u_{xx}$

Special solutions: traveling waves

$$u(x,t) = v(x - st), \qquad s = \text{ speed of profile.}$$

Find $v = v(y)$ such that $v(y) \to u_l$ as $y \to -\infty$
$v(y) \to u_r$ as $y \to +\infty$

Equation for v: $u_t = -s \cdot v'$
$u_x = v'$
$u_{xx} = v''$

$$\begin{aligned} u_t + F'(u) \cdot u_x &= \epsilon \cdot u_{xx} \\ -s \cdot v' + F'(v) \cdot v' &= \epsilon \cdot v'' \end{aligned}$$

Define $w(y) = v\left(\frac{y}{\epsilon}\right)$, $v(y) = w(\epsilon \cdot y)$,

$$\Longrightarrow \begin{cases} -s \cdot w' + F'(w) \cdot w' = w' \\ w(y) \to u_l \quad \text{as} \quad y \to -\infty \\ w(y) \to u_r \quad \text{as} \quad y \to +\infty \end{cases}$$

Suppose for the moment being that $u_l > u_r$.

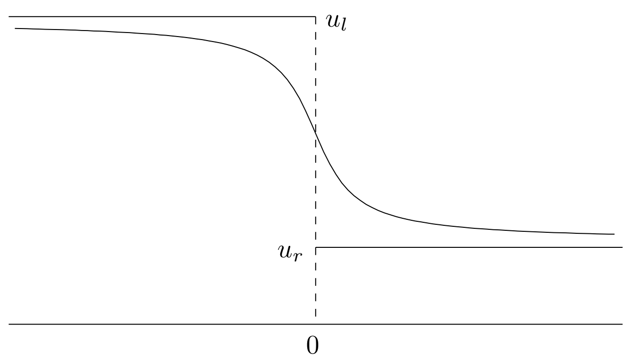


Figure 6.15:

Suppose we have a unique solution

$$-s \cdot w' + F'(w) \cdot w' = (-s \cdot w + F(w))'$$

Integrate $-s \cdot w + F(w) + C_0 = w'$

$$\begin{aligned} C_0 &= \text{constant of integration} \\ \text{Let } \Phi(w) &= -s \cdot w + F(w) + C_0 \end{aligned}$$

If $w(y) \to u_l$, $w'(y) \to 0$ as $y \to -\infty$, then

$$-s \cdot u_l + F(u_l) + C_0 = 0$$

$\Longleftrightarrow \; C_0 = s \cdot u_l - F(u_l)$. Then

$$\Phi(w) = -s(w - u_l) + F(w) - F(u_l)$$

If $w \to u_r$, $w' \to 0$ as $y \to +\infty$, $\Longrightarrow \Phi(u_r) = 0$

$$\begin{aligned} &= \quad -s(u_r - u_l) + F(u_r) - F(u_l) = 0 \\ &\Longleftrightarrow \quad s = \frac{F(u_r) - F(u_l)}{u_r - u_l} \qquad \text{R.H.} \end{aligned}$$

w defined by the ODE $w' = \Phi(w)$

Φ at least Lipschitz $\Longrightarrow$ local existence and <u>uniqueness</u> of solutions. Moreover, $\Phi(u_r) = 0$, $\Phi(u_l) = 0 \Longrightarrow w \equiv u_r$, $w \equiv u_l$ are solutions.
$\Longrightarrow u_l > w > u_r$ for all times.

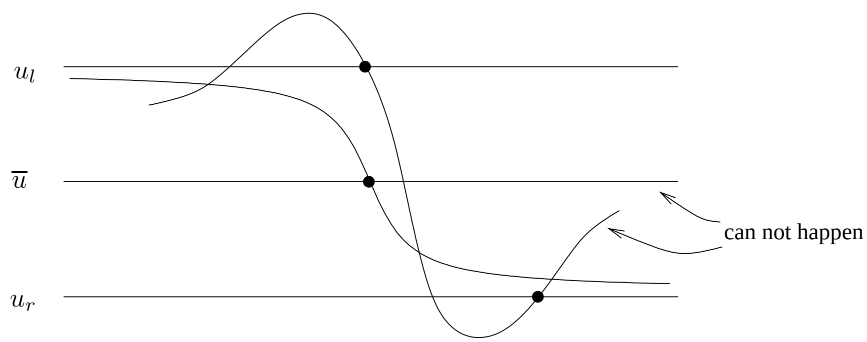


Figure 6.16:

Analogously, there is no $\overline{u}$ with $u_l > \overline{u} > u_r$ with $\Phi(\overline{u}) = 0$. Φ has a sign on $[u_r, u_l] \Longrightarrow w$ decreasing $\Longrightarrow \Phi(u) < 0$ on (u_r, u_l)

$$\Phi(u) = -s(u - u_l) + F(u) - F(u_l)$$

$\Phi(u) < 0$ for $u \in (u_r, u_l)$. So,

$$F(u) < s(u - u_l) + F(u_l)$$

$$F(u) < F(u_l) + \underbrace{\frac{F(u_r) - F(u_l)}{u_r - u_l}}_{\sigma}(u - u_l)$$

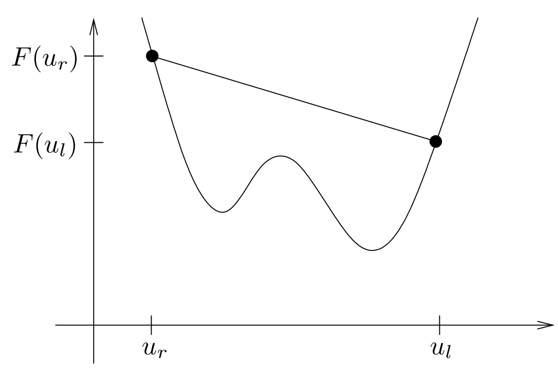


Figure 6.17:

$\Longrightarrow F$ lies under the line segment connecting $(u_r, F(u_r))$ and $(u_l, F(u_l))$.

Remarks:

1. This is always true (for all choices of u_l and u_r) if F is convex.

2. Take the limit $u \nearrow u_l$

$$\begin{aligned} s(u-u_l) &> F(u) - F(u_l) \\ \Longleftrightarrow\ s &< \frac{F(u)-F(u_l)}{u-u_l} \end{aligned}$$

   In the limit $\underline{s \leq F'(u_l)}$
   As $u \searrow u_r$,

$$\begin{aligned} F(u) - F(u_r) &< F(u_l) - F(u_r) + s(u_r - u_l + u - u_r) \\ &= s(u-u_r) \end{aligned}$$

$$\Longrightarrow s > \frac{F(u)-F(u_r)}{u-u_r}$$

   and in the limit $\underline{s \geq F'(u_l)}$

   Two conditions together yield: $\underline{F'(u_l) \geq s \geq F'(u_r)}$

   (Lax Shock admissibility criterion). So seeking traveling wave solution, we recover the R.H. and Lax conditions.

3. Combining the inequalities

$$\frac{F(u)-F(u_l)}{u-u_l} > \dot{s} > \frac{F(u)-F(u_r)}{u-u_r} \qquad \forall u \in (u_r, u_l)$$

   Oleinik's entropy criterion.

6.4 Geometric Solution of the Riemann Problem for General Fluxes

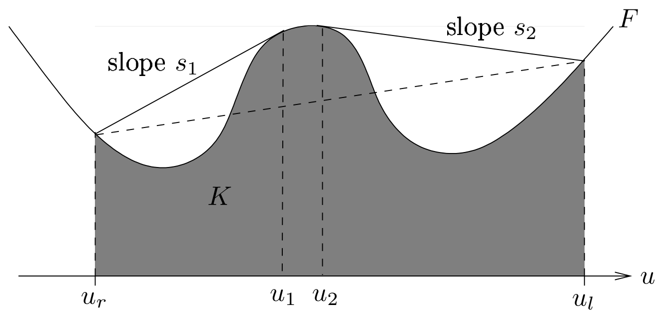


Figure 6.18:

Let $K = \{u, y: \ u_r \leq u \leq u_l, \ y \leq F(u)\}$
The convex hull is bounded by line segments tangent to the graph of f and arcs on the graph of f. The entropy solution consists of shocks (corresponding to the line segments) and rarefaction waves (corresponding to the arcs).

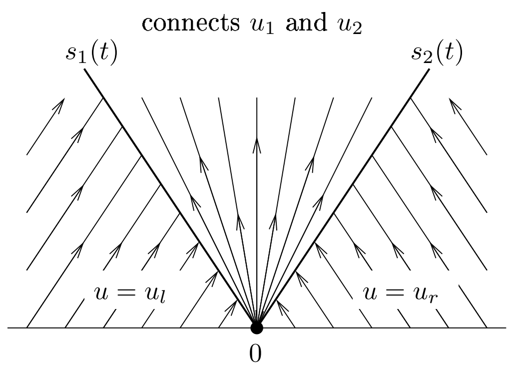


Figure 6.19:

6.5 A Uniqueness Theorem by Lax

Conservation Law $u_t + F(u)_x = 0$
A weak solution (say piecewise smooth with discontinuities) is an <u>entropy solution</u>

if Oleinik's entropy criterion holds.

$$\frac{F(u)-F(u_l)}{u-u_l} > \dot{s} > \frac{F(u)-F(u_r)}{u-u_r}$$

for all discontinuities of u.

Theorem 6.2 (Lax). *Suppose u, v are two entropy solutions. Then*

$$\mu(t) = \int_{\mathbb{R}} |u(x,t)-v(x,t)|\; dx$$

is decreasing in time.

Corollary 6.3 (Uniqueness). *There exists at most on entropy solution with given initial data, $u(x,0) = u_0(x)$, $x \in \mathbb{R}$*

Proof: $\mu(0) = 0$, μ decreasing $\Longrightarrow \mu(t) = 0\ \forall t \Longrightarrow u(x,t) = v(x,t)$. ■

Proof of Theorem (6.2): Take $u - v$

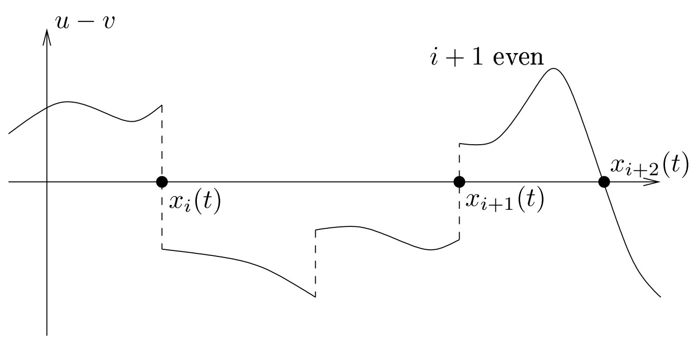


Figure 6.20:

Choose $x_i(t)$ in such a way that $u - v$ changes its sign at the points x_i and has a sign on $x_i, x_{i+1}]$, namely $(-1)^i$.
Now

$$\mu(t) = \sum_n (-1)^n \int_{x_n}^{x_{n+1}} (u(x,t)-v(x,t))\; dx$$

Find $\dot{\mu}(t)$, check that the jumps in the intervals (x_i, x_{i+1}) make no difficulties

$$\begin{aligned} \dot{\mu}(t) &= \sum_n (-1)^n \int_{x_n}^{x_{n+1}} (u_t - v_t)\; dx \\ &\quad + (u(x_{n+1}^-,t) - v(x_{n-1}^-,t))\cdot \dot{x}_{n+1} - (u(x_n^+,t) - v(x_n^+,t))\cdot \dot{x}_n \end{aligned} \tag{6.2}$$

$$
\begin{aligned}
\int_{x_n}^{x_{n+1}} (u-v)_t \, dx &= -\int_{x_n}^{x_{n+1}} (F(u)-F(v))_x \, dx \\
&= -F(u(x_{n+1}^-,t)) \pm F(v(x_{n+1}^-,t)) \pm F(u(x_n^+,t)) \pm F(v(x_n^+,t))
\end{aligned}
$$

Rewrite (6.2) formally:

$$
\begin{aligned}
\dot{\mu}(t) &= \sum_n (-1)^n \left\{ -(F(u)-F(v) \Big|_{x_n^+}^{x_{n+1}^-} + (u-v)\cdot \dot{x} \Big|_{x_n^+}^{x_{n+1}^-} \right\} \\
&= \sum_n (-1)^n \left\{ F(u)-F(v)-(u-v)\cdot\dot{x} \Big|_{x_n^-} + F(u)-F(v)-(u-v)\cdot\dot{x} \Big|_{x_n^+} \right\}
\end{aligned}
$$

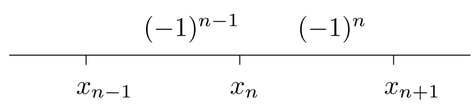


Figure 6.21:

Case 1: $u-v$ continuous at x_n. $u(x_n,t)=v(x_n,t) \Longrightarrow$ no contribution to the sum at x_n

Case 2: u jumps from u_l to u_r with $u_r < u_l$ and v continuous, $u-v$ changes sign at $x_n \Longrightarrow u_r < \overline{v} = v(x_n) < u_l$

$u-\overline{v} < 0$ on $[x_n, x_{n+1}] \Longrightarrow n$ is odd.

Term at x_n:

$$
-\{\underbrace{F(u_l)-F(\overline{v})-(u_l-\overline{v})}_{\frac{F(u_l)-F(\overline{v})}{u_l-\overline{v}}>\dot{s}\geq 0}\cdot\dot{x}_n + \underbrace{F(u_r)-F(\overline{v})-(u_r-\overline{v})}_{\dot{s}>\frac{F(u_r)-F(\overline{v})}{u_r-\overline{v}}\geq 0}\cdot\dot{x}_n\}
$$

$\Longrightarrow$ total contribution at x_n is non-positive
$\Longrightarrow \dot{\mu}(t) \leq 0$
$\Longrightarrow \mu$ is decreasing.

Analogous discussion for the remaining cases. $\blacksquare$

6.6 Entropy Function for Conservation Laws

Idea: Additional equations to be satisfied by smooth solutions, inequalities for non-smooth solutions.

u smooth solution of $u_t + F(u)_x = 0$. Suppose you find a pair of functions (η, ψ)

$$\begin{array}{l} \eta \text{ entropy}, \ \eta \text{ convex}, \ \eta'' > 0 \\ \psi \text{ entropy flux} \end{array}$$

such that

$$\begin{aligned} \eta(u)_t + \psi(u)_x &= 0 \\ \eta'(u) \cdot u_t + \psi'(u) \cdot u_x &= 0 \\ u_t + F'(u) \cdot u_x &= 0 \end{aligned}$$

$\Longrightarrow \eta'(u) \cdot u_t + F'(u) \cdot \eta'(u) \cdot u_x = 0$. This holds if

$$\psi' = F' \cdot \eta'$$

Suppose that u^ϵ is a solution of the viscous approximation

$$u_t^\epsilon + F(u^\epsilon)_x = \epsilon \cdot u_{xx}^\epsilon$$

suppose that (η, ψ) is an entropy-entropy flux pair with η convex.

$\Longrightarrow u_t^\epsilon \cdot \eta'(u^\epsilon) + F'(u^\epsilon) \cdot \eta'(u^\epsilon) \cdot u_x^\epsilon = \epsilon \cdot \eta'(u^\epsilon) \cdot u_{xx}^\epsilon$

$\Longleftrightarrow \ \eta(u^\epsilon)_t + \psi(u^\epsilon) \cdot u_x^\epsilon = \epsilon(\eta' \cdot \dot{u}_x)_x - \epsilon \cdot \eta''(u^\epsilon) \cdot \dot{u}_x^2$

Integrate this on $[x_1, x_2] \times [t_1, t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (\eta(u^\epsilon)_t + \psi(u^\epsilon)_x) \ dxdt &= \epsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} (\epsilon \cdot \eta'(u^\epsilon) \cdot u_x^\epsilon)_x \ dxdt \\ &\quad \underbrace{- \epsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \epsilon \cdot \eta''(u^\epsilon) u_x^{\epsilon\,2} \ dxdt}_{\geq 0} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \int_{t_1}^{t_2} \int_{x_1}^{x_2} (\eta(u^\epsilon)_t + \psi(u^\epsilon)_x) \ dxdt &\leq \epsilon^2 \int_{t_1}^{t_2} - \Big[\eta'(u^\epsilon(x_2, t)) \cdot u_x^\epsilon(x_2, t) \\ &\quad - \eta'(u^\epsilon(x_1, t)) \cdot u_x^\epsilon(x_1, t) \Big] \ dxdt \end{aligned}$$

Suppose that $u^\epsilon \to u$, and let $\epsilon \to 0$:

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} (\eta(u)_t + \psi(u)_x)\ dxdt \leq 0 \tag{6.3}$$

if u_x^ϵ is uniformly integrable, i.e.

$$\left| \int_{t_1}^{t_2} \int_{x_1}^{x_2} (\eta'(u^\epsilon) \cdot u^\epsilon)_x\ dxdt \right| \leq C.$$

This means that a solution u of $u_t + F(u_x) = 0$ has to satisfy

$$\eta(u)_t + \psi(u)_x = 0$$

in a suitable weak sense.

Example: Burger's equation.
$\eta(u) = u^2$, $F(u) = \frac{u^2}{2}$

$$\begin{aligned} \psi'(u) &= 2u \cdot u = 2u^2 \\ \psi(u) &= \frac{2}{3}u^2 \end{aligned}$$

Possible choice: $(\eta, \psi) = \left(u^2, \frac{2}{3}u^2\right)$

Suppose u has a jump across a smooth curve.

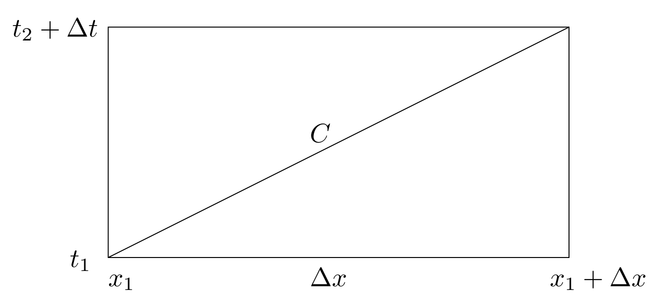


Figure 6.22:

$\dot{s}$ = speed of the curve

$\dot{s} = \frac{\Delta x}{\Delta t}$

$$\begin{aligned}
(6.3) \quad &= \int_{x_1}^{x_2} \eta(u)\Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \psi(u)\Big|_{x_1}^{x_2} dt \\
&\approx (x_2 - x_1)(\eta(u_l) - \eta(u_r)) + (t_2 - t_1)(\psi(u_r - u_l)) \\
&= \dot{s} \cdot \Delta t \cdot (u_l^2 - u_r^2) + \Delta t \frac{2}{3}(u_r^3 - u_l^3) \\
&= \ldots = -\frac{1}{6}\Delta t \underbrace{(u_l - u_r)^3}_{\geq 0} \leq 0
\end{aligned}$$

$\Longrightarrow$ jump admissible only if $u_l > u_r$.

6.7 Conservation Laws in nD and Kružkov's Stability Result

Conservation laws in n dimensions

$$u_t + \mathrm{div}_x \vec{F}(u) = 0,$$

where $\vec{F} : \mathbb{R} \to \mathbb{R}^n$ vector value flux.

$$\vec{F}(u) = \begin{pmatrix} F_1(u) \\ \vdots \\ F_n(u) \end{pmatrix}$$

Definition 6.4. *A pair (η, ψ) is called an entropy-entropy flux pair if $\vec{\psi}'(u) = \vec{F}'(u)\eta'(u)$, where*

$$\begin{aligned}
&\vec{\psi} : \mathbb{R} \to \mathbb{R}^n, \ \psi_i'(u) = F_i'(u)\eta'(u) \\
&\eta : \mathbb{R} \to \mathbb{R}
\end{aligned}$$

and if η is convex.

Take a viscous approximation:

$$u_t^\epsilon + \mathrm{div}_x \vec{F}(u^\epsilon) = \epsilon \cdot \Delta u^\epsilon$$

$$u_t^\epsilon \cdot \eta'(u^\epsilon) + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(u^\epsilon) \cdot \eta'(u^\epsilon) = \epsilon \cdot \Delta u^\epsilon \eta' \tag{6.4}$$

$$
\begin{aligned}
\left(\frac{\partial}{\partial x_i}F_i(u^\epsilon)\right)\eta'(u^\epsilon) &= F_i'(u^\epsilon)\cdot\eta'(u^\epsilon)\frac{\partial}{\partial x_i}u^\epsilon \\
&= \psi_i'(u^\epsilon)\frac{\partial}{\partial x_i}u^\epsilon \\
&= \frac{\partial}{\partial x_i}\vec{\psi}(u^\epsilon)
\end{aligned}
$$

(6.4) is equivalent to

$$
\begin{aligned}
\eta(u^\epsilon)_t + \operatorname{div}(\vec{\psi}(u^\epsilon)) &= \epsilon\cdot\operatorname{div}(Du^\epsilon\cdot\eta'(u^\epsilon)) - \epsilon\cdot Du^\epsilon\cdot D\eta'(u^\epsilon) \\
&= \epsilon\cdot\operatorname{div}(D(\eta(u^\epsilon))) - \epsilon\cdot Du^\epsilon\cdot Du^\epsilon\cdot\eta''(u^\epsilon)
\end{aligned}
$$

Multiply this by a test function $\phi \geq 0$ with compact support.

$$
\int_0^\infty\int_{\mathbb{R}^n}\phi[\eta(u^\epsilon)_t + \operatorname{div}\vec{\psi}(u^\epsilon)]\;dxdt = \int_0^\infty\int_{\mathbb{R}^n}\phi[\epsilon\cdot\Delta\eta(u^\epsilon) - \epsilon\cdot\eta''(u^\epsilon)|Du^\epsilon|^2]\;dxdt
$$

$$
\int_0^\infty\int_{\mathbb{R}^n}\phi[\eta(u^\epsilon)_t + \operatorname{div}\vec{\psi}(u^\epsilon)]\;dxdt \leq \int_0^\infty\int_{\mathbb{R}^n}\epsilon\cdot\phi\cdot\Delta\eta(u^\epsilon)\;dxdt
$$

If $u^\epsilon \to u$ as $\epsilon \to 0$, then

$$
\int_0^\infty\int_{\mathbb{R}^n}\phi[\eta(u)_t + \operatorname{div}\vec{\psi}(u)]\;dxdt \leq 0
$$

Definition 6.5. *A bounded function u is a weak entropy solution of*

$$
\begin{cases} u_t + div\,\vec{F}(u) = 0 \\ u(x,0) = u_0(x) \end{cases}
$$

if

$$
\int_0^\infty\int_{\mathbb{R}^n}(\eta(u)\cdot\phi_t + \vec{\psi}(u)\cdot D\phi)\;dxdt \leq \int_{\mathbb{R}^n}\phi(x,0)\eta(u_0(x))\;dx
$$

for all test functions $\phi \geq 0$ with compact support in $\mathbb{R}^n \times [0,\infty)$ and all entropy-entropy flux pairs $(\eta, \vec{\psi})$.

Additional conservation laws:

$$
\eta(u)_t + \operatorname{div}_x\vec{\psi}(u) \leq 0
$$

Analogue of the Rankine-Hugoniot jump condition

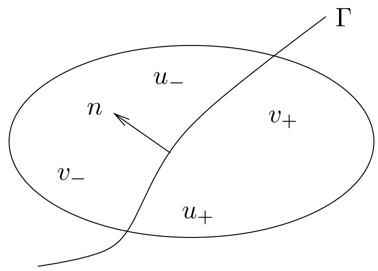


Figure 6.23:

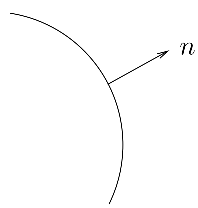
Similar arguments to the 1D case show

$$\nu_{n+1}[\![\eta]\!] + [\![\vec{\psi}]\!] \cdot \nu_0 \leq 0$$

where $\nu = (\nu_0, \nu_{n+1})$ is the normal in space-time. Define $\tilde{n} = \frac{\nu_0}{|\nu_0|}$, $V = -\frac{\nu_{n+1}}{|\nu_0|}$. Then

$$V[\![\eta]\!] \geq [\![\vec{\psi}]\!] \cdot \tilde{n}$$

<u>$n = 2$</u>



$V =$ normal velocity of the interface.

Figure 6.24:

By approximation, we can extend this setting to Lipschitz continuous η, and in particular to <u>Kružkov's entropy</u>, $\eta(u) = |u - k|$ k is constant.

$$\vec{\psi}(u) = \operatorname{sgn}(u - k)(\vec{F}(u) - \vec{F}(k))$$

<u>Check:</u> R.H. jump inequality $\dot{s}[\![\eta]\!] \geq [\![\vec{\psi}]\!]$ with Kružkov's entropy in 1 dimension.

$$\dot{s}(|u_l - k| - |u_r - k|) \geq \operatorname{sgn}(u_l - k)(\vec{F}(u_l) - \vec{F}(k)) - \operatorname{sgn}(u_r - k)(\vec{F}(u_r) - \vec{F}(k))$$

Suppose that $u_r < k < u_l$:

$$\begin{array}{rrcl} & \dot{s}(u_l - k + u_r - k) & \geq & \vec{F}(u_l) - \vec{F}(k) + \vec{F}(u_r) - \vec{F}(k) \\ \Longleftrightarrow & \dot{s}(u_l + u_r - 2k) & \geq & \vec{F}(u_l) + \vec{F}(u_r) - 2\vec{F}(k) \\ \Longleftrightarrow & \vec{F}(k) & \geq & \frac{\vec{F}(u_l) + \vec{F}(u_r)}{2} + \left(k - \left(\frac{u_l + u_r}{2}\right)\right)\dot{s} \end{array}$$

$$\dot{s} = \frac{\vec{F}(u_l) - \vec{F}(u_r)}{u_l - u_r}$$

Sign error. Should recover Oleinik's Entropy Criterion.

Correction: $u_+ = u_r$, $u_- = u_l \to$ Oleinik's entropy criterion.

Theorem 6.6 (Kružkov). *Suppose that u_1 and u_2 are two weak entropy solutions of $u_t + div\,\vec{F}(u) = 0$ which satisfies $u_1, u_2 \in C^0([0,\infty);\ L^1(\mathbb{R}^n)) \cap L^\infty((0,\infty) \times \mathbb{R}^n)$. Then*

$$||u_1(\cdot, t_2) - u_2(\cdot, t_2)||_{L^1(\mathbb{R}^n)} \leq ||u_1(\cdot, t_1) - u_2(\cdot, t_1)||_{L^1(\mathbb{R}^n)}$$

for all $0 \leq t_1 \leq t_2 < \infty$

Remark: This implies uniqueness if u_1 and u_2 have the same initial data.

Notation: $L^1(\mathbb{R}^n)$ = space of all integrable functions with

$$||f||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f|\ dx$$

L^∞ = space of all bounded measurable functions.

The first assumption means that the map

$$t \mapsto \Phi(t) = u(\cdot, t) \text{ is continuous}$$

i.e. for t_0 fixed, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||\Phi(t) - \Phi(t_0)||_{L^1(\mathbb{R}^n)} \leq \epsilon$ if $|t - t_0| < \delta$. i.e.

$$\int_{\mathbb{R}^n} |u(x, t_0) - u(x, t)|\ dx < \epsilon$$

Proof: (formal)
Kružkov entropy-entropy flux:

$$\eta(u) = |u - k|, \qquad \vec{\psi}(u) = \text{sgn}\,(u - k)(\vec{F}(u) - \vec{F}(k))$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^+} \underbrace{\eta(u)}_{|u_1 - k|}\ \phi_t + \text{sgn}\,(u_1 - k)(\vec{F}(u_1) - \vec{F}(k)) \cdot D\phi\ dxdt \geq 0$$

for all ϕ with compact support in $\mathbb{R}^n \times (0, \infty)$. Suppose you can choose $k = u_2(x,t)$ (it is not clear you can do this since k is constant)

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^+} |u_1(x,t) - u_2(x,t)|\phi_t + \operatorname{sgn}(u_1 - u_2)[F(u_1) - F(u_2)](x,t) \cdot D_x\phi \, dxdt \geq 0$$

Define $K \subset \mathbb{R}^n \times \mathbb{R}^+$

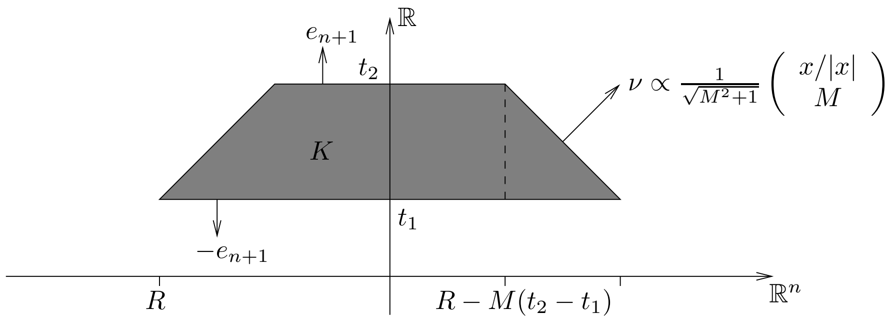


Figure 6.25:

Choose $\phi = \chi_K$, i.e.

$$\phi(x,t) = \begin{cases} 1 & (x,t) \in K \\ 0 & \text{otherwise} \end{cases}$$

(This will also have to be considered further as this function isn't smooth).

Expectation: $(D_x, D_t)\chi_K = -\nu\mathcal{H}^n \cdot L \cdot \partial K$ ($\mathcal{H}^n$ is the surface measure)

——

One way to make this precise is to use integration by parts:

$$\int (D_x\phi \cdot D_t\phi)\vec{\psi} = -\int \phi \cdot \operatorname{div}\vec{\psi}$$

——

Suppose all this is correct:

$$
\begin{aligned}
\text{Top:} \quad & -\int_{\{t=t_1,\ |x|\le R-M(t_2-t_1)\}} |u_1(x,t)-u_2(x,t)|\ dx \\
\text{Bottom:} \quad & +\int_{\{t=t_1,\ |x|\le R\}} |u_1(x,t)-u_2(x,t)|\ dx \\
& +\int_A \left\{ \frac{-M}{\sqrt{M^2+1}} |u_1(x,t)-u_2(x,t)| \right. \\
& \quad \left. - \frac{1}{\sqrt{M^2+1}} \operatorname{sgn}(u_1-u_2)(\vec{F}(u_1)-\vec{F}(u_2)) \cdot \frac{x}{|x|} \right\} d\mathcal{H}^n
\end{aligned}
$$

where $A := \{t_1 < t < t_2,\ |x| = R - M(t_2-t_1)\}$.

3rd integral:

$$
\begin{aligned}
&\int_A -M|u_1(x,t)-u_2(x,t)| - \operatorname{sgn}(u_1-u_2)(\vec{F}(u_1)-\vec{F}(u_2)) \cdot \frac{x}{|x|}\ d\mathcal{H}^n \\
&\quad \le \int_A -M|u_1(x,t)-u_2(x,t)| - |\vec{F}(u_1)-\vec{F}(u_2)|\ d\mathcal{H}^n \le 0
\end{aligned}
$$

By assumption, $|u_1|, |u_2| \le N$. Take $M = \sup_{|u|\le N} |F'(u)|$.

$$
\int_{|x|\le R-M(t_2-t_1)} |u_1(x,t_2)-u_2(x,t_2)|\ dx \le \int_{|x|\le R} |u_1(x,t)-u_2(x,t_2)|\ dx + \le 0
$$

<u>Remark:</u>
The proof shows more:

$$
\int_{|x|\le R} |u_1(x,t_2)-u_2(x,t_2)|\ dx \le \int_{|x|\le R+M(t_2-t_1)} |u_1(x,t_1)-u_2(x,t_1)|\ dx
$$

This shows again finite speed of propagation.

<u>Fixing the proof:</u>

- $\phi = \chi_K$. Fix by taking a regularization of χ_K, i.e. with standard mollifiers.
- Choice $k = u_2(x,t)$
  "Doubling of the variables". Take $u_2(y,s)$ and integrate in y and s for

the entropy inequality for u_1:

$$\begin{aligned} 0 \;\leq\; & \int_{\mathbb{R}^n\times\mathbb{R}^+}\int_{\mathbb{R}^n\times\mathbb{R}^+} \Big\{|u_1(x,t)-u_2(y,s)|\phi_t \\ & \quad + \operatorname{sgn}(u_1-u_2)(\vec{F}(u_1)-\vec{F}(u_2))\cdot D_x\phi\Big\}\, dxdtdyds \end{aligned}$$

Entropy inequality for u_2, written in y and s. Take $k = u_1(x,t)$, integrate in x and t:

$$\begin{aligned} 0 \;\leq\; & \int_{\mathbb{R}^n\times\mathbb{R}^+}\int_{\mathbb{R}^n\times\mathbb{R}^+} \Big\{|u_1(x,t)-u_2(y,s)|\phi_s \\ & \quad + \operatorname{sgn}(u_1-u_2)(\vec{F}(u_1)-\vec{F}(u_2))\cdot D_y\phi\Big\}\, dxdtdyds \end{aligned}$$

Add the two inequalities:

$$\begin{aligned} 0 \;\leq\; & \int_{\mathbb{R}^n\times\mathbb{R}^+}\int_{\mathbb{R}^n\times\mathbb{R}^+} |u_1-u_2|(\phi_s+\phi_t) \\ & \quad + \operatorname{sgn}(u_1-u_2)(\vec{F}(u_1)-\vec{F}(u_2))\cdot (D_x\phi + D_y\phi)\, dxdtdyds \end{aligned}$$

Trick: Chose ϕ_ϵ to enforce $y = x$ as $\epsilon \to 0$

$\rho \in C_0^\infty(-1,1)$

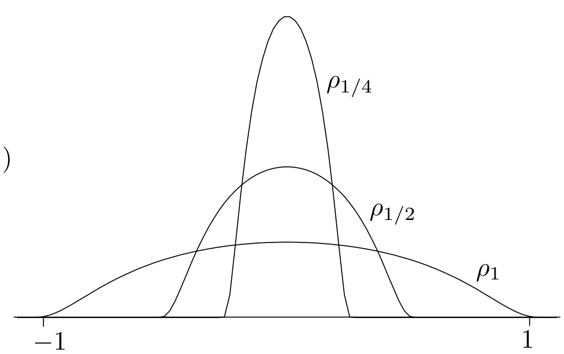


Figure 6.26:

$\int_{\mathbb{R}} \rho_\epsilon = 1$

$$\begin{aligned} \rho_\epsilon(t) &= \frac{1}{\epsilon}\rho\left(\frac{t}{\epsilon}\right) \\ \hat{\rho}_\epsilon(x) &= \frac{1}{\epsilon^n}\prod_{i=1}^{n}\rho\left(\frac{x_i}{\epsilon}\right) \end{aligned}$$

Choose

$$\phi(x,t,y,s) = \vec{\psi}\left(\frac{x+y}{2},\frac{s+t}{2}\right)\cdot\hat{\rho}_\epsilon\left(\frac{x-y}{2}\right)\cdot\rho_\epsilon\left(\frac{t-s}{2}\right)$$

$\vec{\psi}$ smooth, compact support in $\mathbb{R}^n \times (0,\infty)$.

Find

$$\begin{aligned}\phi_t + \phi_s &= \vec{\psi}_t(\hat{x},\hat{t})\cdot\hat{\rho}_\epsilon(\hat{y})\cdot\rho_\epsilon(\hat{s})\\ D_x\phi + D_y\phi &= D_x\vec{\psi}(\hat{x},\hat{t})\cdot\hat{\rho}_\epsilon(\hat{y})\cdot\rho_\epsilon(\hat{s})\end{aligned}$$

$$\hat{x} = \frac{x+y}{2},\quad \hat{y} = \frac{x-y}{2},\quad \hat{t} = \frac{t+s}{2},\quad \hat{s} = \frac{t-s}{2}$$

$\to$ weighted integral, as $\epsilon \to 0$ the integration is on $\hat{y} = 0$, $\hat{s} = 0$ i.e. $x = y$ and $t = s$. You get exactly the inequality we get formally with $k = u_2(x,t)$. ■

6.8 Exercises

6.1: Let $f(x)$ be a piecewise constant function

$$f(x) = \begin{cases} 1 & x < -1, \\ 1/2 & -1 < x < 1, \\ 3/2 & 1 < x < 2, \\ 1 & x > 2. \end{cases}$$

Construct the entropy solution of the IVP, $u_t + uu_x = 0$, $u(x,0) = f(x)$ for small values of $t > 0$ using shock waves and rarefaction waves. Sketch the characteristics of the solution. why does the structure of the solution change for large times?

6.2: Motivation of Lax's entropy condition via stability. Consider Burgers' equation

$$u_t + uu_x = 0$$

subject to the initial conditions

$$u_0(x) = \begin{cases} 0 & \text{for} \quad x > 0 \\ 1 & \text{for} \quad x < 0 \end{cases}, \quad u_1(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases}.$$

The physically relevant solution should be stable under small perturbations in the initial data. Which solution (shock wave or rarefaction wave) do you obtain in the limit as $n \to \infty$ if you approximate u_0 and u_1 by

$$u_{0,n}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ nx & \text{for} \quad 0 \le x \le \frac{1}{n} \\ 1 & \text{for} \quad x > \frac{1}{n} \end{cases}, \quad u_{1,n}(x) = \begin{cases} 1 & \text{for} \quad x < 0 \\ 1 - nx & \text{for} \quad 0 \le x \le \frac{1}{n} \\ 0 & \text{for} \quad x > \frac{1}{n} \end{cases} ?$$

6.3: Let the initial data for Burgers' equation $u_t + uu_x = 0$ be given by

$$f(x) = \begin{cases} u_l & \text{for} \quad x < 0, \\ u_l - \frac{u_l - u_r}{L} x & \text{for} \quad 0 \le x \le L, \\ u_r & \text{for} \quad x > L, \end{cases}$$

where $0 < u_r < u_l$. Find the time t_* when the profile of the solution becomes vertical. More generally, suppose that the initial profile is smooth and has a point x_0 where $f'(x_0) < 0$. Find the time t_* for which the solution of Burger's equation will first break down (develop a vertical tangent).

6.4: [Profile of rarefaction waves] Suppose that the conservation law

$$u_t + f(u)_x = 0$$

has a solution of the form $u(x,t) = v(x/t)$. Show that the profile $v(s)$ is given by

$$v(s) = (f')^{-1}(s).$$

6.5: Verify that the viscous approximation of Burgers' equation

$$u_t + uu_x = \epsilon u_{xx}$$

has a traveling wave solution of the form $u(x,t) = v(x - st)$ with

$$v(y) = u_R + \frac{1}{2}(u_L - u_R)\left(1 - \tanh\left(\frac{(u_L - u_R)y}{4\epsilon}\right)\right)$$

where $s = (u_L + u_R)/2$ is the shock speed given by the Rankine Hugoniot condition. Sketch this solution. What happens as $\epsilon \to 0$?

6.6: Consider the Cauchy problem for Burgers' equation,

$$u_t + uu_x = 0$$

with the initial data

$$u(x,0) = \begin{cases} 1 & \text{for} \quad |x| > 1, \\ |x| & \text{for} \quad |x| \le 1. \end{cases}$$

a) Sketch the characteristics in the (x,t) plane. Find a classical solution (continuous and piecewise C^1) and determine the time of breakdown (shock formation).

b) Find a weak solution for all $t > 0$ containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by $(x = s(t), t)$ in the (x,t)-plane.

6.7: [Traffic flow] A simple model for traffic flow on a one lane highway without ramps leads to the following first order equation which describes the conservation of mass,

$$\rho_t + v_{\max}\left(\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)\right)_x = 0, \quad x \in \mathbb{R}, t > 0.$$

In this model, the velocity of cars is related to the density by

$$v(x,t) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right),$$

where we assume that the maximal velocity $v_{\max}$ is equation to one. Solve the Riemann problem for this conservation law with

$$\rho(x,0) = \begin{cases} \rho_L & \text{if} \quad x < 0, \\ \rho_R & \text{if} \quad x > 0. \end{cases}$$

Under what conditions on ρ_L and ρ_R is the solution a shock wave and a rarefaction wave, respectively? Sketch the characteristics. Does this reflect your daily driving experience? Find the profile of the rarefaction wave.

6.8: [LeVeque] The Buckley-Leverett equations are a simple model for two-phase flow in a porous medium with nonconvex flux

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}.$$

In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore u represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem for the conservation law $u_t + f(u)_x = 0$ with initial states

$$u(x,0) = \begin{cases} 1 & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x > 0. \end{cases}$$

Hint: The line through the origin that is tangent to the graph of f on the interval $[0,1]$ has slope $1/(\sqrt{3}-1)$ and touches at $u = 1/\sqrt{3}$. If you are proficient with Mathematica or Matlab you could try to plot the profile of the rarefaction wave.

<u>6.9</u>: Let $u_0(x) = \operatorname{sgn}(x)$ be the sign function in $\mathbb{R}$ and let a be a constant. Let $u(x,t)$ be the solution of the transport equation

$$u_t + au_x = 0,$$

and let $u^\epsilon(x,t)$ be the solution of the so-called viscous approximation

$$u_t^\epsilon + au_x^\epsilon - \epsilon u_{xx}^\epsilon = 0,$$

both subject to the initial condition $u(\cdot,0) = u^\epsilon(\cdot,0) = u_0$.

a) Derive representation formulas for u and u^ϵ!

   Hint: For the viscous approximation, consider $v^\epsilon(x,t) = u^\epsilon(x+at,t)$.

b) Prove the error estimate

$$\int_{-\infty}^{\infty} |u(x,t) - u^\epsilon(x,t)|\, dx \le C\sqrt{\epsilon t}, \quad \text{for} \quad t > 0.$$

Chapter 7

Elliptic, Parabolic and Hyperbolic Equations

7.1 Classification of 2nd Order PDE in Two Variables

$$a(x,y)\cdot u_{xx} + 2b(x,y)\cdot u_{xy} + c(x,y)\cdot u_{yy} + d(x,y)\cdot u_x + e(x,y)\cdot u_y + g(x,y)\cdot u = f(x,y)$$

General mth order PDE

$$\mathcal{L}(x,D)u = \sum_{|\alpha|\leq m} a_\alpha(x)\cdot D^\alpha u$$

where $u : \mathbb{R}^n \to \mathbb{R}$
Principle part:

$$\mathcal{L}_p(x,D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha$$

The symbol of the PDE is defined to be

$$\mathcal{L}(x,i\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\cdot(i\xi)^\alpha$$

(you take $i\xi$ because of Fourier transform)

$\xi \in \mathbb{R}^n$, replace $\frac{\partial}{\partial x_j}$ by $i\xi_j$.

Examples: Laplace $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2} = \mathcal{L}$

$$\mathcal{L}(x, i\xi) = \sum_{j=1}^{n} (i\xi_j)^n = -|\xi|^2$$

Heat Equation: $t = n + 1$.

$$\mathcal{L}u = \frac{\partial u}{\partial x_{n+1}} - \Delta u =$$

Symbol:

$$\mathcal{L}(x, i\xi) = i\xi_{n+1} - \sum_{j=1}^{n} (i\xi_j)^2$$

Wave equation:

$$\mathcal{L} = \frac{\partial^2}{\partial x_{n+1}^2} - \Delta$$

$$\mathcal{L}(x, i\xi) = (i\xi_{n+1})^2 - \sum_{j=1}^{n} (i\xi_j)^2$$

———

Principle part of the symbol:

$$\mathcal{L}_p(x, i\xi) = \sum_{|\alpha|=m} a_\alpha(x) \cdot (i\xi)^\alpha$$

Examples:

$$\begin{array}{rcl} \Delta: \ \mathcal{L}_p & = & -|\xi|^2 \\ \text{heat:} \ \mathcal{L}_p & = & \xi_1^2 + \cdots + \xi_n^2 \\ \text{wave:} \ \mathcal{L}_p & = & -\xi_{n+1}^2 + \xi_1^2 + \cdot + \xi_n^2 \end{array}$$

2nd order equation in two variables

$$\begin{aligned} \mathcal{L}_p(x, D) &= a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2} \\ \mathcal{L}_p(x, i\xi) &= a(x,y)\xi_1^2 + 2b(x,y)\xi_1\xi_2 + c(x,y)\xi_2^2 \\ &= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^T \underbrace{\begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}}_{A} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \end{aligned}$$

Definition 7.1. *The PDE is*

$$\begin{array}{rcl} \underline{\textit{elliptic}} & \textit{if} & \det A > 0 \\ \underline{\textit{parabolic}} & \textit{if} & \det A = 0 \\ \underline{\textit{hyperbolic}} & \textit{if} & \det A < 0. \end{array}$$

Typical situation:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Equation: $\lambda_1 u_{xx} + \lambda_2 u_{yy}+$ lower order terms

$$\begin{array}{rll} \underline{\text{elliptic}}: & -A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & u_{xx} + u_{yy} = \Delta \\ \underline{\text{parabolic}}: & -A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & u_{xx} + \text{lower order terms} \\ \underline{\text{hyperbolic}}: & -A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & u_{xx} - u_{yy} = \Box = \text{ wave} \end{array}$$

Remarks: This definition even allows an ODE in the parabolic case (not good).
Better definition: parabolic is u_t- elliptic $= f$.

$$u_t - \Delta u = f$$

Two reasons why this determinant is important:

(A) Given Cauchy data along a curve, $C(x(s), y(s))$ parametrized with respect to arc length, that is

$$\begin{aligned} u(x(s), y(s)) &= u_0(s) \\ \frac{\partial u}{\partial n}(x(s), y(s)) &= v_0(s). \end{aligned}$$

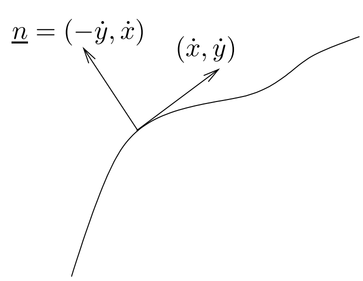


Figure 7.1:

Question: Conditions of C that determine the second derivatives of a smooth solution.

Tangential derivative

$$Du \cdot \vec{T} = u_x \dot{x} + u_y \dot{y} = w_0$$

Normal derivative $\quad Du \cdot \vec{n} = -u_x \dot{y} + u_y \dot{x} = v_0$

——

$$\begin{aligned} &u_{xx}\dot{x}^2 + 2u_{xy}\dot{x}\dot{y} + u_{yy}\dot{y}^2 + u_x\ddot{x} + u_y\ddot{y} = \dot{w}_0(s) \\ &-u_{xx}\dot{x}\dot{y} - u_{xy}\dot{y}^2 + u_{xy}\dot{y}^2 + u_{xy}\dot{x}^2 + u_{yy}\dot{x}\dot{y} - u_x\ddot{y} + u_y\ddot{x} = \dot{v}_0(s) \end{aligned}$$

——-

$$\begin{aligned} &u_{xx}\dot{x}^2 + 2u_{xy}\dot{x}\dot{y} + u_{yy}\dot{y}^2 = p(s) \\ &-u_{xx}\dot{x}\dot{y} + (\dot{x}^2 - \dot{y}^2)u_{xy} + u_{yy}\dot{x}\dot{y} = r(s) \\ &au_{xx} + 2bu_{xy} + cu_{yy} = q(s) \end{aligned}$$

We can solve for the 2nd derivatives along the curve if

$$\det \begin{vmatrix} \dot{x}^2 & 2\dot{x}\dot{y} & \dot{y}^2 \\ -\dot{x}\dot{y} & \dot{x}^2 - \dot{y}^2 & \dot{x}\dot{y} \\ a & 2b & c \end{vmatrix} \neq 0$$

$$\begin{aligned} \det \begin{vmatrix} \dot{x}^2 & 2\dot{x}\dot{y} & \dot{y}^2 \\ -\dot{x}\dot{y} & \dot{x}^2 - \dot{y}^2 & \dot{x}\dot{y} \\ a & 2b & c \end{vmatrix} &= a(2\dot{x}^2\dot{y}^2 - \dot{x}^2\dot{y}^2 - \dot{x}^2\dot{y}^2 + \dot{y}^4) \\ &\quad -2b(\dot{x}^3 y + \dot{x}y^3) + c(\dot{x}^4 - \dot{x}^2\dot{y}^2 + 2\dot{x}^2\dot{y}^2) \\ &= a(\dot{y}^4 + \dot{x}^2\dot{y}^2) - 2b(\dot{x}^3\dot{y} + \dot{x}\dot{y}^3) \\ &\quad +c(\dot{x}^4 + \dot{x}^2\dot{y}^2) \neq 0 \end{aligned}$$

Divide by $(\dot{x}\dot{y})^2$:

$$\begin{aligned} & a\left(\frac{\dot{y}^2}{\dot{x}^2} + 1\right) - 2b\left(\frac{\dot{y}}{\dot{x}} + \frac{\dot{x}}{\dot{y}}\right) + c\left(\frac{\dot{x}^2}{\dot{y}^2} + 1\right) \neq 0 \\ \iff\quad & a\left(\frac{dy^2}{dx^2} + 1\right) - 2b\left(\frac{dy}{dx} + \frac{dx}{dy}\right) + c\left(\frac{dx^2}{dy^2} + 1\right) \neq 0 \\ \iff\quad & \left(1 + \left(\frac{dx}{dy}\right)^2\right) \cdot \underbrace{\left(a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c\right)}_{\neq 0} \neq 0 \end{aligned}$$

$$at^2 - 2bt + c = 0$$

$$a\left(t^2 - \frac{2b}{a}t + \frac{c}{a}\right) = 0$$

$$\iff \quad \left(t - \frac{b}{a}\right)^2 - \frac{b^2}{a^2} + \frac{ac}{a^2} = 0$$

$$\iff \quad t = \frac{b}{a} \pm \frac{\sqrt{-ac + b^2}}{a}$$

Non-trivial solutions if

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 \leq 0$$

elliptic equation $\rightarrow$ 2nd derivative are determined.

Definition 7.2. *A curve C is called a* <u>*characteristic curve*</u> *for $\mathcal{L}$ if*

$$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0.$$

(b) <u>Curves of discontinuity</u>
$a, \ldots, f, g$ continuous.
u solution of $\mathcal{L}u = f$, u, u_x, u_y continuous, u_{xx}, u_{xy}, u_{yy} continuous except on a curve C, jump along C. What condition has C to satisfy.

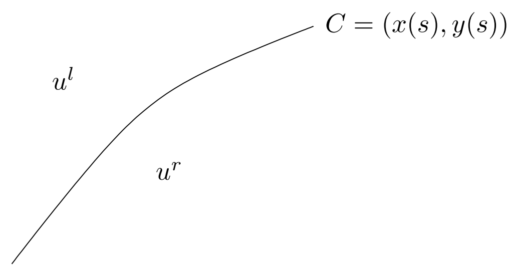


Figure 7.2:

$u^l_x(x(s), y(s)) = u^r_x(x(s), y(s))$.
Differentiate along the curve:

$$u^l_{xx}\dot{x} + u^l_{xy}\dot{y} = u^r_{xx}\dot{x} + u^r_{xy}\dot{y}$$

$$[\![u_{xx}]\!]\,\dot{x} + [\![u_{xy}]\!]\,\dot{y} = 0 \tag{7.1}$$

Analogously: $u_y^l(x(s), y(s)) = u_y^r(x(s), y(s))$

$$\Longrightarrow \qquad [\![u_{xy}]\!]\,\dot{x} + [\![u_{yy}]\!]\,\dot{y} = 0 \tag{7.2}$$

Equation holds outside the curve, all 1st order derivatives and coefficients are continuous $\Longrightarrow$ along the curve

$$a\,[\![u_{xx}]\!] + 2b\,[\![u_{xy}]\!] + c\,[\![u_{yy}]\!] = 0 \tag{7.3}$$

$\Longrightarrow$ linear system.

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & 2b & c \end{pmatrix} \begin{pmatrix} [\![u_{xx}]\!] \\ [\![u_{xy}]\!] \\ [\![u_{yy}]\!] \end{pmatrix} = 0$$

Non-trivial solution only if

$$\det \begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & 2b & c \end{pmatrix} = a\dot{y}^2 - 2b\dot{x}\dot{y} + c\dot{x}^2 = 0$$

$$\Longleftrightarrow \; a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0$$

Jump discontinuity only possible if C is characteristic,
Elliptic: no characteristic curve
Parabolic: one characteristic
Hyperbolic: two characteristic

$$\frac{dy}{dx} = \frac{b}{a} \pm \frac{\sqrt{b^2 - ac}}{a}$$

Laplace: $a = c = 1$, $b = 0$ $\frac{dy}{dx} = \pm\sqrt{-1}$. No solution

Parabolic: $a = 1$, $c = 0$, $b = 0$ $\frac{dy}{dx} = 0$ $y = \text{const.}$ characteristic.

Hyperbolic: $a = 1$, $c = -1$, $b = 0$ $\frac{dy}{dx} = \pm 1$, $y = \pm x$ characteristic.

7.2 Maximum Principles for Elliptic Equations of 2nd Order

Definition 7.3. *$\mathcal{L}$ is an elliptic operator in divergence form if*

$$\mathcal{L}(x, D)u = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u(x)$$

and if the matrix (a_{ij}) *is positive definite, i.e.,*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for all $x \in \Omega$*, all* $\xi \in \mathbb{R}^n$*,* $\lambda(x) > 0$

<u>Remarks:</u> Usually elliptic means uniformly elliptic, i.e.

(1) $\lambda(x) \geq \lambda_0 > 0 \; \forall x \in \Omega$.

(2) Divergence form because $\mathcal{L}(x,D)u = -\text{div}\,(ADu) + \vec{b}\cdot Du + cu$ where $A = (a_{ij})$

Definition 7.4. *$\mathcal{L}$ is an elliptic operator in <u>non-divergence</u> form if*

$$\mathcal{L}(x,D) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u(x)$$

$$\textit{and} \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j > \lambda(x)|\xi|^2$$

for all $x \in \Omega$*,* $\xi \in \mathbb{R}^n$

<u>Remark:</u> If $a_{ij} \in C^1$, then an operator in non-divergence form can be written as an operator in divergence form. We may also assume that $a_{ij} = a_{ji}$.

Definition 7.5. *A function* $u \in C^2$ *is called a <u>subsolution</u> (<u>supersolution</u>) if* $\mathcal{L}(x,D)u \leq 0$ $(\mathcal{L}(x,D)u \geq 0)$

<u>Remark:</u> $n = 1$, $\mathcal{L}u = -u_{xx}$
subsolution $-u_{xx} \leq 0 \iff u_{xx} \geq 0 \iff u$ convex $\implies u \leq v$ if $v_{xx} = 0$ on $v = u$ at the endpoint of an interval.

<u>Remark:</u> We can not expect maximum principles to hold for general elliptic equations. Obstruction: the existence of eigenvalues and eigenfunctions.

<u>**Example:**</u> $-u'' = \lambda u$
$\lambda = 1$, $u(x) = \sin(x)$ on $[0, 2\pi]$
$\implies \sin x$ solves $-u'' = u$

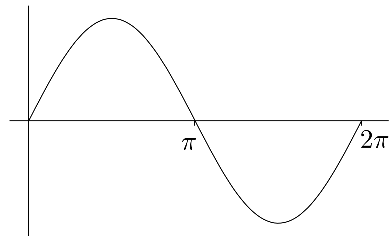


Figure 7.3:

There is no maximum principle for $-u_{xx} - u = \mathcal{L}$.
Big difference between $x \geq 0$ and $c < 0$.

7.2.1 Weak Maximum Principles

the next two subsections are from the AMSI summer notes and need to be incorporated properly

7.2.2 WMP for Purely 2nd Order Elliptic Operators

Again we take $\Omega \subset \mathbb{R}^n$. We now consider the following differential operator:

$$Lu := \sum_{i,j=1}^{n} a^{ij} D_{ij} u + \sum_{i=1}^{n} b^i D_i u + cu,$$

where $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. Now, we state the following.

Definition 7.6. *The operator L is said to be* elliptic(degenerate elliptic) *if $\mathcal{A} := [a^{ij}] > 0 (\geq 0)$ in Ω.*

Remarks:

i.) In the above definition $\mathcal{A} > 0$ means the minimum eigenvalue of the matrix $\mathcal{A}$ is > 0. More explicitly, this means that

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j > 0, \ \forall \xi \in \mathbb{R}^n.$$

ii.) The last definition indicates that parabolic PDE (such as the Heat Equation) are really degenerate elliptic equations.

For convenience, we will be using the Einstein summation convention for repeated indicies. This means that in any expression where letters of indicies appear more than once, a summation is applied to that index from 1 to n. With that in mind, we can write our general operator as

$$Lu := a^{ij} D_{ij} u + b^i D_i u + cu.$$

Now, we move onto the weak maximum principle for our special case operator:

$$Lu = a^{ij} D_{ij} u.$$

Theorem 7.7 (Weak Maximum Principle). *Consider* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *and* $f : \Omega \to \mathbb{R}$. *If* $Lu \geq f$, *then*

$$u \leq \max_{\partial\Omega} u + \frac{1}{2}\left(\sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr}\mathcal{A}}\right) \cdot (\operatorname{diam}\Omega)^2, \tag{7.4}$$

where $\operatorname{Tr}\mathcal{A}$ *is the trace of the coefficient matrix* $\mathcal{A}$ *(i.e.* $\operatorname{Tr}\mathcal{A} = \sum_{i=1}^n a^{ii}$*).*

Note: If $f \equiv 0$, the above reduces to the result of the first weak maximum principle we proved. Namely, $u \leq \max_{\partial\Omega} u$.

Proof: Without loss of generality we may translate Ω so that it contains the origin. Again, we consider an auxiliary function:

$$v = u + k|x|^2,$$

where k is an arbitrary constant to be determined later. Since the Hessian Matrix D^2u is symmetric, we may choose a coordinate system such that D^2u is diagonal. In this coordinate system, we calculate

$$Lv = Lu + 2ka^{ij}S_{ij} = Lu + 2k \cdot \operatorname{Tr}\mathcal{A},$$

where $S_{ii} = 1$ and $S_{ij} = 0$ when $i \neq j$. Now, suppose that v attains an interior maximum at $y \in \Omega$. From calculus, we then know that $D^2v(y) \leq 0$, which implies that $a^{ij}D_{ij} \leq 0$ as $\mathcal{A}$ is positive definite. Now, if we choose

$$k > \frac{1}{2} \sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr}\mathcal{A}},$$

our above calculation indicates that $Lv > 0$; a contradiction. Thus,

$$v \leq \max_{\partial\Omega} v$$

which implies the result as Ω contains the origin (which indicates that $|x|^2 \leq (\operatorname{diam}\Omega)^2$). ■

Example 7.1. *Now we will apply the weak maximum principle to the second most famous elliptic PDE, the* Minimal Surface Equation*:*

$$(1+|Du|^2)\Delta u - D_iuD_juD_{ij}u = 0.$$

This is a quasilinear PDE as only derivatives of the 1st order are multiplied against the 2nd order ones (a fully nonlinear PDE would have products of second order derivatives). We will now show that this is indeed an elliptic PDE. Recalling our special case $Lu = a^{ij}D_{ij}u$, we can rewrite the minimal surface equation in this form with

$$a^{ij} = (1+|Du|^2)S_{ij} + D_iuD_ju,$$

with S being as it was in the previous proof. Now, we calculate

$$\begin{aligned} a^{ij}\xi_i\xi_j &= (1+|Du|^2)S_{ij}\xi_i\xi_j - D_iuD_ju\xi_i\xi_j \\ &= (1+|Du|^2)|\xi|^2 - D_iuD_ju\xi_i\xi_j \\ &= (1+|Du|^2)|\xi|^2 - (D_iu\xi_i)^2 \\ &\geq (1+|Du|^2)|\xi|^2 - |Du|^2|\xi|^2 \\ &= |\xi|^2. \end{aligned}$$

To get the third equality in the above, we simply relabeled repeated indicies. Since one sums over a repeated index, it's representation is a dummy index, whose relabeling does not affect the calculation.

From this calculation, we conclude this equation is indeed elliptic. Now, upon taking $f = 0$ and replacing u by $-u$, we can now apply the weak maximum principle to this equation.

7.2.3 WMP for General Elliptic Equations

Now, we will go over the weak maximum principle for operators having all order of derivatives ≤ 2.

Theorem 7.8 (Weak Maximum Principle). *Consider $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$; $a^{ij}, b^i, c : \Omega \to \mathbb{R}$; $c \leq 0$; and Ω bounded. If u satisfies the elliptic equation (i.e. $[a^{ij}] = \mathcal{A} > 0$)*

$$Lu = a^{ij}D_{ij} + b^iD_iu + cu \geq f, \tag{7.5}$$

then

$$u \leq \max_{\partial\Omega} u^+ + C\left(n, \operatorname{diam}\Omega, \sup_{\overline{\Omega}} \frac{|b|}{\lambda}\right) \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}, \tag{7.6}$$

where

$$\lambda = \min_{|\xi|=1} a^{ij}\xi_i\xi_j \quad \textit{and} \quad u^+ = \max\{u, 0\}.$$

Note: If a^{ij} is not constant, then λ will almost always be non-constant on Ω as well.

Consequences:

- If $f = 0$, then $u \leq \max_{\partial\Omega} u^+$.
- If $f = c = 0$, then $u \leq \max_{\partial\Omega} u^+$.
- If $Lu \leq f$ one gains a lower bound

  $$u \geq \max_{\partial\Omega} u^- - C \sup_{\overline{\Omega}} \frac{|f|}{\lambda},$$

  where $u^- = \min\{u, 0\}$.
- If we have the PDE $Lu = f$, then one has both an upper and lower bound:

  $$|u| \leq \max_{\partial\Omega} |u| + C \sup_{\overline{\Omega}} \frac{|f|}{\lambda}.$$

- Uniqueness of the Dirichlet Problem. Recall, that the problem is as follows

  $$\begin{cases} \text{PDE:} \quad Lu = f & \text{in} \quad \Omega \\ \qquad\quad\;\; u = \phi & \text{on} \quad \partial\Omega \end{cases}.$$

  So if one has $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $Lu = Lv$ in Ω, and $u = v$ on $\partial\Omega$, then $u \equiv v$ in Ω. The proof is the simple application of the weak maximum principle to $w = u - v$, which is obviously a solution of PDE as it's linear.

Proof of Weak Max. Principle: As in previous proofs, we analyze a particular auxiliary function to prove the result.

Aside: $v = |x|^2$ will not work in this situation as $Lv = 2 \cdot \text{Tr}\,\mathcal{A} + 2b^i x_i + c|x|^2$; the $b^i x_i$ may be a large negative number for large values of $|x|$.

Let us try

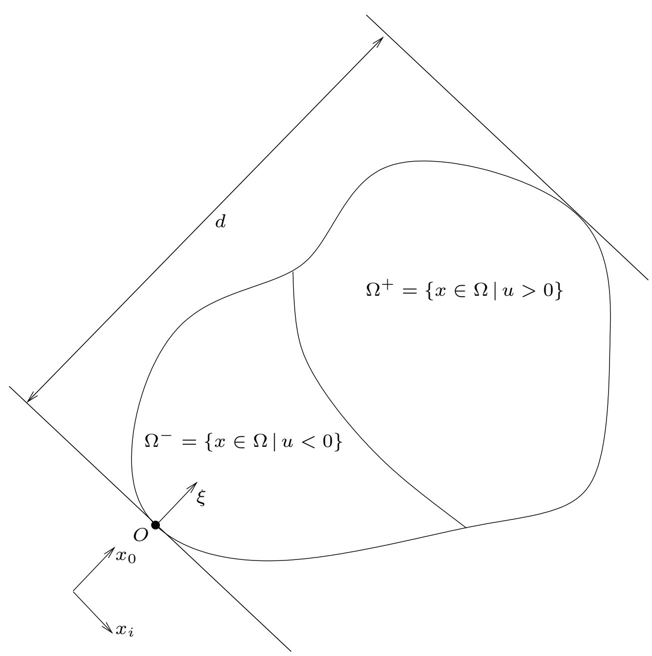
$$v = e^{\alpha(x,\xi)},$$

where ξ is some arbitrary vector. Also, we will consider $\Omega^+ = \{x \in \Omega \,|\, u > 0\}$ instead of Ω, for the rest of the proof. Clearly, this does not affect the result as we are seeking to bound u by $\max_{\partial\Omega} u^+$. With that, we have on Ω^+

$$\begin{aligned} L_0 u := a^{ij} D_{ij} u + b^i D_i u &\geq -cu + f \\ &\geq f. \end{aligned}$$

Now, we wish to specify which vector ξ is. For the following, please refer to the above figure. We first choose our coordinates so that the origin corresponds to a boundary point of Ω such that there exists a hyperplane through that point with Ω lying on one side of it. ξ is simply taken to be the unit vector normal to this hyperplane. Next, we calculate

$$\begin{aligned} L_0 v &= (\alpha^2 a^{ij} \xi_i \xi_j + \alpha b^i \xi_i) e^{\alpha(x,\xi)} \\ &= (\alpha^2 a^{ij} \xi_i \xi_j + \alpha b^i \xi_i) e^{\alpha(x,\xi)} \\ &\geq \alpha \lambda e^{\alpha(x,\xi)}, \end{aligned}$$



provided we choose $\alpha \geq \left(\sup_{\overline{\Omega}} \frac{|b|}{\lambda} + 1\right)$.

Aside: The calculation for the determination of α is as follows:

$$\alpha^2 a^{ij}\xi_i\xi_j + \alpha b^i\xi_i \quad \geq \quad \alpha^2\lambda + \alpha b^i\xi_i.$$

So, $\alpha^2\lambda + \alpha b^i\xi_i \geq \alpha\lambda$ implies that

$$\begin{aligned} \alpha\lambda &\geq \lambda - b^i\xi_i \\ \Longrightarrow \alpha &\geq \sup_{\overline{\Omega}}\left(1 - \frac{b^i\xi_i}{\lambda}\right) \end{aligned}$$

So, clearly we have the desired inequality if we choose $\alpha \geq \tilde{b} + 1$, where $\tilde{b} := \sup_{\overline{\Omega}} \frac{|b|}{\lambda}$.

Given our choice of ξ, it is clear that $(x, \xi) \geq 0$ for all $x \in \Omega$. Thus, one sees that $L_0 v \geq \alpha\lambda$. So, one can ascertain that

$$L_0(u^+ + kv) \geq f + \alpha\lambda k > 0 \text{ in } \Omega^+,$$

provided k is chosen so that

$$k > \frac{1}{\alpha}\sup_{\overline{\Omega}} \frac{|f|}{\lambda}.$$

Now, the proof proceeds as previous ones. Suppose $w := u^+ + kv$ has a positive maximum in Ω^+ at a point y. In this situation we have

$$[D_{ij}w(y)] \leq 0 \Longrightarrow a^{ij}D_{ij}w(y) \leq 0, \;\; b^iD_iw(y) = 0,$$

which implies $Lw \leq 0$; a contradiction. So, this implies the second inequality of the following:

$$\begin{aligned} u + \frac{e^0}{\alpha}\sup_{\overline{\Omega}}\frac{|f|}{\lambda} &\leq w \\ &\leq \max_{\partial\Omega^+} w \\ &\leq \max_{\partial\Omega^+} u^+ + \frac{e^{\alpha d}}{\alpha}\cdot\sup_{\overline{\Omega}}\frac{|f|}{\lambda} \\ &\leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d}}{\alpha}\cdot\sup_{\overline{\Omega}}\frac{|f|}{\lambda}, \end{aligned}$$

again where $\alpha = \tilde{b} + 1$ and d is the breadth of Ω in the direction of ξ. From the above, we finally get

$$u \leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d} - 1}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}. \qquad \blacksquare$$

Remark: Given the above, one needs $\frac{|b|}{\lambda}$ and $\frac{|f|}{\lambda}$ bounded for the weak maximum principle to give a non-trivial result. Then one may apply the weak maximum principle to get uniqueness.

Theorem 7.9 (weak maximum principle, $c = 0$). *Ω open, bounded, $u \in C^2(\Omega) \cap C(\overline{\Omega})$. $\mathcal{L}$ uniformly elliptic (for divergence form) $\mathcal{L}u \leq 0$. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Analogously, if $\mathcal{L}u \geq 0$, then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Proof:

(1) $\mathcal{L}u < 0$ and u has a maximum at $x_0 \in \Omega$. $Du(x_0) = 0$ and

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \right)_{i,j} = D^2 u(x_0)$$

is negative semidefinite.

Fact from linear algebra: A, B symmetric and positive definite, then $A : B = \mathrm{tr}\,(AB) \geq 0$.
Q orthogonal, $QAQ^T = \Lambda$,
$\lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq 0$

$$\begin{aligned} \mathrm{tr}\,(AB) &= \mathrm{tr}\,(QABQ^T) \\ &= \mathrm{tr}\,(QAQ^T QBQ^T) \\ &= \mathrm{tr}\,(\Lambda \hat{B}) \end{aligned}$$

$\hat{B} = QBQ^T$. $\hat{B}$ symmetric and positive semidefinite $\Longrightarrow \hat{B}_{ii} \geq 0$.

$$\mathrm{tr}\,(\Lambda \hat{B}) = \sum_{i=1}^{n} \lambda_i \hat{B}_{ii} \geq 0$$

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{i=1}^n (AB)_{ii}\\ &= \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ji}\\ &= \sum_{i,j=1}^n A_{ij}B_{ij}\\ &= A : B\end{aligned}$$

$$\mathcal{L}u = -\sum_{i,j=1}^n a_{ij}\partial_{ij}u + \sum_{i=1}^n b_i\partial_i u$$
$$\mathcal{L}u(x_0) = -(a_{ij}) : D^2u(x_0) + 0$$

(a_{ij}) symmetric and positive definite. $-D^2u(x_0)$ symmetric and positive semidefinite. Conclusion:

$$0 \underbrace{>}_{\text{assump.}} \mathcal{L}u(x_0) \geq 0$$

Contradiction, u can not have a maximum at x_0.

(2) $\mathcal{L}u(x_0) = 0$.
Consider $u_\epsilon(x) = u(x) + \epsilon e^{rx_1}$,
$\epsilon > 0$, $r > 0$, $r = r(x)$ chosen later.

$$\begin{aligned}\mathcal{L}u_\epsilon &= \mathcal{L}u + \epsilon\mathcal{L}e^{rx_1} \leq \epsilon\mathcal{L}e^{rx_1}\\ &= \epsilon(-a_{11}r^2 + b_1 r)e^{rx_1}\end{aligned}$$

$a_{11} > 0$ (since $\sum a_{ij}\xi_i\xi_j \geq \lambda_0|\xi|^2$, take $\xi_1 = e_1 = (1, 0, \dots, 0)$, $a_{11} \geq \lambda_0 > 0$)

$b = b(x)$ by assumption in $C(\overline{\Omega})$.
$|b(x)| \leq B = \max_\Omega |b|$

$$\begin{aligned}\mathcal{L}u_\epsilon &\leq \epsilon(-a_{11}r^2 + Br)e^{rx_1}\\ &= \epsilon r(-a_{11}r + B)e^{rx_1} < 0\end{aligned}$$

if $r > 0$ is big enough.

By the previous argument,

$$\max_{\overline{\Omega}} u_\epsilon = \max_{\partial\Omega} u_\epsilon$$

As $\epsilon \to 0$, $u_\epsilon \to u$ and in the limit

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(In fact, r is an absolute constant, $r = r(\lambda_0, B)$). ■

Corollary 7.10 (Uniqueness). *There exists at most one solution of the BVP $\mathcal{L}u - f$ in Ω, $u = u_0$ in $\partial\Omega$ with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Proof: By contradiction with the maximum and minimum principle. ■

Theorem 7.11 (weak maximum principle, $c \geq 0$). *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and that $c \geq 0$.*

i.) If $\mathcal{L}u \leq 0$, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$$

ii.) If $\mathcal{L}u \geq 0$, then

$$\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-$$

where $u^+ = \max(u, 0)$, where $u^- = -\min(u, 0)$,

that is, $u = u^+ - u^-$.

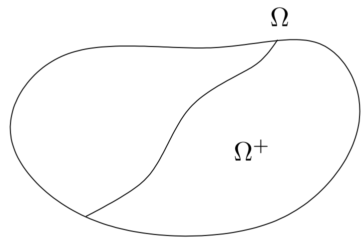
Proof: Proof for ii.) follows from i.) with $-u$.

i.) Let $\Omega^+ = \{\Omega,\ u(x) > 0\}$ and define $Ku = \mathcal{L}u - cu$. If $\Omega^+ = \emptyset$, then the assertion is obviously true.

On Ω^+, $Ku = \mathcal{L}u - cu \leq 0 - cu \leq 0$, i.e. on Ω^+, u is a subsolution for K and we may use the weak maximum principle for K.

If $\Omega^+ \neq \emptyset$, then use the weak maximum principle for K on Ω^+

$$\max_{\overline{\Omega^+}} u(x) \leq \max_{\partial\Omega^+} u(x) = \max_{\partial\Omega} u^+ \tag{7.7}$$



On $\partial\Omega^+ \cap \Omega$ we have $u = 0$.

Figure 7.4:

Since u is not less that or equal to zero on $\overline{\Omega}$,

$$\max_{\overline{\Omega}^+} u(x) = \max_{\partial\Omega^+} u(x) = \max_{\partial\Omega} u^+$$

where the second equality is due to (7.7). ■

Recall: <u>**Weak Maximum Principles**</u> $\mathcal{L}u \leq 0$

$$\begin{array}{lrcl} c = 0: & \max_{\overline{\Omega}} u & = & \max_{\partial\Omega} u \\ c \geq 0: & \max_{\overline{\Omega}} u & \leq & \max_{\partial\Omega} u^+ \end{array}$$

Now, we consider strong maximum principles.

7.2.4 Strong Maximum Principles

Theorem 7.12 (Hopf's maximum principle; Boundary point lemma). *Ω bounded, connected with smooth boundary. $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Subsolution $\mathcal{L}u \leq 0$ in Ω and suppose that $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for all $x \in \Omega$. Moreover, assume that Ω satisfies an interior ball condition at x_0, i.e. there exists a ball*

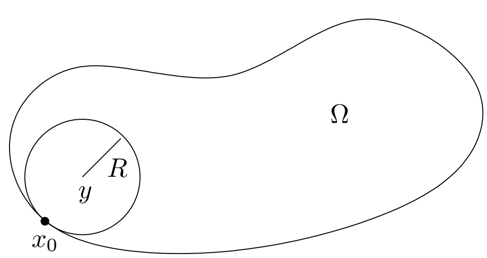


Figure 7.5:

$B(y, R) \subset \Omega$ such that $\overline{B(y, R)} \cap \overline{\Omega} = \{x_0\}$

i.) If $c = 0$, then

$$\frac{\partial u}{\partial \nu}(x_0)$$

strict > 0 follows from assumption.

ii.) If $c \geq 0$, then the same assertion holds if $u(x_0) \geq 0$

Proof:

Idea: Construct a barrier $v \geq 0$, $v > 0$ in Ω,
$w = u(x) - u(x_0) \mp \epsilon v(x) \leq 0$ in B.
$u(x) - u(x_0) \leq \epsilon v(x)$

Construction of $v(x)$: $v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}$ where $r = |x - y|$
$v(x) > 0$ in B
$v(x) = 0$ on ∂B

$$\begin{aligned}
\frac{\partial v}{\partial x_i} &= (-\alpha)2(x_i - y_i) \cdot e^{-\alpha r^2} \\
\mathcal{L}v &= -\sum a_{ij}\partial_{ij}v + \sum \partial_i v b_i + cu \\
&= -\sum 4\alpha^2(x_i - y_i)(x_j - y_i)e^{-\alpha r^2} a_{ij} \\
&\quad + \sum\{(a_{ii}2\alpha - 2\alpha b_i(x_i - y_i)\}e^{-\alpha r^2} + c(e^{-\alpha r^2} - e^{-\alpha R^2})
\end{aligned}$$

Ellipticity: $\sum a_{ij}\xi_i\xi_j \geq \lambda_0|\xi|^2 \iff -\sum a_{ij}\xi_i\xi_j \leq -\lambda_0|\xi|^2$

$$\begin{aligned}
&\leq -\lambda_0 4\alpha^2|x - y|^2 e^{-\alpha r^2} + \operatorname{tr}(a_{ij})2\alpha e^{-\alpha r^2} \\
&\quad +2\alpha|\vec{b}| \cdot |x - y|e^{-\alpha r^2} + c(e^{-\alpha r^2} - e^{-\alpha R^2})
\end{aligned}$$

$$\left[e^{-\alpha r^2} - e^{-\alpha R^2} = e^{-\alpha r^2}\underbrace{(1 - e^{-\alpha(R^2 - r^2)})}_{\leq 1}\right]$$

$$\leq e^{-\alpha r^2}\{-\lambda_0 4\alpha^2|x - y|^2 + 2\alpha \operatorname{tr}(a_{ij}) + 2\alpha|b| \cdot |x - y| + |c|\}$$

If $|x-y|$ bound from below, then $\mathcal{L}v \leq 0$ if α is big enough.

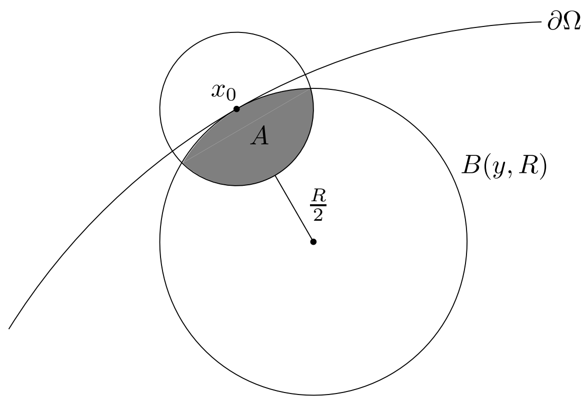


Figure 7.6:

$A = B(y,R) \cap B\left(x_0, \frac{R}{2}\right)$
Then $|x-y| \geq \frac{R}{2}$ in A and $\mathcal{L}v \leq 0$ for α big enough.
Define $w(x) = u(x) - u(x_0) + \epsilon v(x)$

$$\begin{aligned} \mathcal{L}w &= \underbrace{\mathcal{L}u}_{\leq 0} - \underbrace{\mathcal{L}u(x_0)}_{c(x_0)} + \epsilon \underbrace{\mathcal{L}v}_{\leq 0} \\ &\leq -cu(x_0) \\ &< 0 \end{aligned}$$

$\Longrightarrow w$ a subsolution.

w on ∂A: $\partial A \cap \partial B(y,R)$. $w \leq 0$
$\partial A \cap B(y,R)$: $u(x_0) > u(x)$ in Ω
$\Longrightarrow u(x) - u(x_0) \leq -\delta$, $\delta > 0$ by compactness.
$\Longrightarrow w \leq 0$ if ϵ is small enough.

$$\begin{aligned} \mathcal{L}w &\leq 0 \text{ in } B \\ w &\leq 0 \text{ on } \partial B \end{aligned}$$

Apply weak max principle to w.

$$\max_{\overline{B}} w \leq \max_{\partial\Omega} w^+ = 0$$

$\Longrightarrow w(x) \leq 0$ for all $x \in A$.

$$\begin{aligned} w(x) &= u(x) - u(x_0) + \epsilon v(x) \\ w(x_0) &= 0 \quad \text{and} \quad u(x) - u(x_0) \leq -\epsilon v(x) \end{aligned}$$

$$\begin{aligned}
\Longrightarrow \frac{\partial u}{\partial \nu}(x_0) &\geq -\epsilon \frac{\partial v}{\partial \nu}(x_0) \\
\frac{\partial v}{\partial x_i} &= -2\alpha(x_i - y_i)e^{-\alpha r^2} \\
\frac{\partial u}{\partial \nu}(x_0) &\geq -\epsilon(-2\alpha)(x_0 - y)e^{-\alpha R^2}\frac{x_0 - x}{|x_0 - y|} \\
&= 2\epsilon \cdot \alpha e^{-\alpha R^2} \underbrace{\frac{|x_0 - y|^2}{|x_0 - y|}}_{=R>0} \qquad \blacksquare
\end{aligned}$$

Theorem 7.13 (Strong maximum principle, $c = 0$). *Ω open and connected (not necessarily bounded), $c = 0$*

i.) If $\mathcal{L}u \leq 0$, and u attains max at an interior point, then $u \equiv$ constant.

ii.) $\mathcal{L}u \geq 0$, u has a minimum at an interior point, the u is constant.

Proof:

$$\begin{aligned}
M &= \max_{\overline{\Omega}} u, \quad u \neq M, \\
V &= \{x \in \Omega \mid u(x) < M\} \neq \emptyset \quad \text{(contradiction assump.)}
\end{aligned}$$

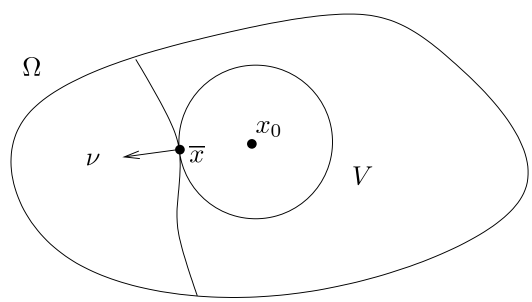


Figure 7.7:

Choose $x_0 \in V$ such that dist$(x_0, \partial V)$ <dist$(x, \partial\Omega)$
$\Longrightarrow B(x_0, |x - x_0|) \subset V$. We may assume that $\overline{B} \cap \partial V = \{\overline{x}\}$
$\Longrightarrow$ Satisfy assumptions in the boundary point lemma.
$\Longrightarrow \frac{\partial u}{\partial \nu}(\overline{x}) > 0$

Contradiction, since $\frac{\partial u}{\partial \nu}(\overline{x}) = Du(\overline{x}) \cdot \nu = 0$ since $u(\overline{x}) = M$, i.e. $\overline{x}$ is a maximum point of u. $\blacksquare$

Theorem 7.14 (Strong maximum principle, $c \geq 0$). *Ω open and connected (not necessarily bounded). $u \in C^2(\Omega)$, $c \geq 0$.*

i.) $\mathcal{L}u \leq 0$ in Ω and u attains a non-negative maximum, then $u =$ constant in Ω

ii.) $\mathcal{L}u \geq 0$ u has non-positive maximum then $u \equiv$ constant in Ω.

Proof: Hopf's Lemma for non-negative maximum point at the boundary. ■

Example: $c = 1$, $-u'' + u = 0$. $u = \sinh, \cosh$

7.3 Maximum Principles for Parabolic Equations

Definition 7.15. *$\mathcal{L}(x, t, D) = u_t + Au$,*

$$Au = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i\partial_i u + cu,$$

is said to be parabolic. A is (uniformly) elliptic

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \lambda_0|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n,\ \lambda_0 > 0.$$

Remark: Solutions of elliptic equations are stationary states of parabolic equations $\Longrightarrow$ Same assumptions concerning c.

Ω open, bounded, $\Omega_T = \Omega \times (0, T]$
$\Gamma_T = \partial\Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T)$

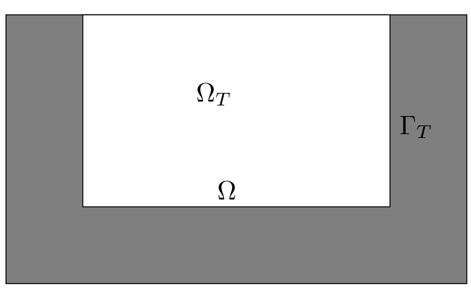


Figure 7.8:

7.3.1 Weak Maximum Principles

Theorem 7.16 (weak maximum principle, $c = 0$). *$u \in C_1^2(\Omega_T) \cap C(\overline{\Omega})$, $c = 0$ in Ω_T*

i.) $\mathcal{L}u = u_t + Au \leq 0$ in Ω_T, then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

ii.) $\mathcal{L}u \geq 0$,

$$\min_{\overline{\Omega}_T} u = \min_{\Gamma_T} u$$

Proof:

i.) Suppose $\mathcal{L}u < 0$ and that the max is attained at $x_0, t_0)$ in Ω_T. $0 > \mathcal{L}u = u_t + Au = 0 + (Au \geq 0) \geq 0$
since $Au \geq 0$ as in the elliptic case. contradiction.
If the max is attained at (x_0, T), $0 > \mathcal{L}u = u_t + Au \geq 0$ since $u_t(x_0, T) \geq 0$ if u is max at (x_0, T)

If $\mathcal{L}u \leq 0$, then take $u^\epsilon = u - \epsilon t$
$\mathcal{L}u^\epsilon = (\partial_t + A)(u - \epsilon t) = \mathcal{L}u - \epsilon \leq -\epsilon < 0$
u^ϵ satisfies $\mathcal{L}u^\epsilon < 0$, and then

$$\max_{\overline{\Omega}_T} u^\epsilon = \max_{\Gamma_T} u^\epsilon,$$

$\forall \epsilon > 0$, and the assertion follows as $\epsilon \searrow 0$. ∎

Theorem 7.17 (weak maximum principle, $c \geq 0$). *$u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$, $c \geq 0$ in Ω_T. $u = u^+ - u^-$, where $u^+ = \max(u, 0)$ $u^- = -\min(u, 0)$*

i.) If $\mathcal{L}u = u_t + Au \leq 0$,

$$\max_{\overline{\Omega}_T} u \leq \max_{\Gamma_T} u^+$$

ii.) $\mathcal{L}u \geq 0$, then

$$\min_{\overline{\Omega}_T} u \geq -\max_{\Gamma_T} u^-.$$

Proof: Suppose first $\mathcal{L}u < 0$ and that u has a positive maximum at (x_0, t_0) in Ω_T.

$$\begin{aligned} 0 &> \mathcal{L}u = (\partial_t + A)u \\ &\geq c(x_0, t_0)u(x_0, t_0) > 0 \qquad \text{contradiction} \end{aligned}$$

Similar for $t_0 = T$
General case: $u^\epsilon = u - \epsilon t$

$$\begin{aligned} \mathcal{L}u^\epsilon &= (\partial_t + A)(u - \epsilon t) \\ &\leq 0 - \epsilon - \epsilon \underbrace{c(x,t)}_{\geq 0} t \\ &\leq -\epsilon < 0 \end{aligned}$$

Previous sept applies. If u has a positive max, then u^ϵ has a positive max in Ω_T for ϵ small enough. ■

7.3.2 Strong Maximum Principles

Evans: Harnack inequality for parabolic equations
Protter, Weinberger: Maximum principles in differential equations, some form of Hopf's Lemma.

Theorem 7.18. *Ω bounded, open, smooth boundary and $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega}_T)$ with $\mathcal{L}u = u_t + Au \leq 0$. Suppose that either*

i.) $c = 0$

ii.) $c \geq 0$,

$$\max_{\overline{\Omega}_T} u = M \geq 0$$

The the strong maximum principle holds.

Idea of Proof: $\Omega = (\alpha, \beta)$. $n = 1$, $u_t - au_{xx} + bu_x + cu = \mathcal{L}$, $a = a(x,t) \geq \lambda_0 > 0$

Step 1: $B = B_\epsilon(\epsilon, \tau)$ open ball in Ω_T, $u < M$ in B_ϵ, $u(x_0, t_0) = M$, $(x_0, t_0) \in \partial B_\epsilon \Longrightarrow x_0 = \xi$

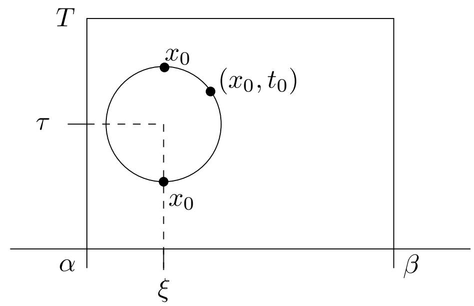


Figure 7.9:

The proof (by contradiction) uses the weak max principle for $w = u - M + \eta v$, where $v(x,t) = e^{-Ar^2} - e^{-A\epsilon^2}$, $A > 0$. Compute

$$\mathcal{L}v = e^{-Ar^2}\{-\alpha 4A^2(x-\xi)^2 + \dots\}$$

$\therefore r^2 = (x-\xi)^2 + (t-\tau)^2$

If the max is attained at a point (x, t_0) with $x_0 \neq \xi$, choose δ small enough such that $|x - \xi| > c(\delta) > 0 \; \forall x \in \overline{B_\delta(x_0)}$

w subsolution, $w < 0$ on ∂B_δ, $w(x_0, t_0) = 0$. Contradiction with weak max principle.

Step 2: Suppose $u(x_0, t_0) < M$ with $t_0 \in (0, T)$. Then $u(x, t_0) < M$ for all $x \in (\alpha, \beta)$.

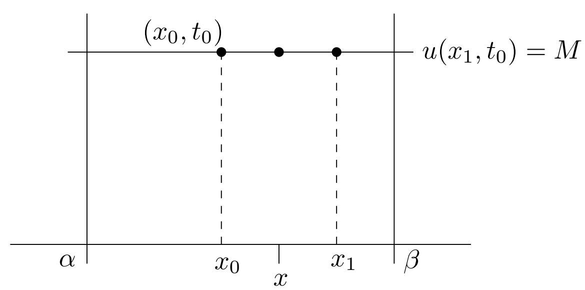


Figure 7.10:

Idea: Suppose $\exists\ x_1 \in (\alpha, \beta_0)$ with $u(x, t_0) = M$ and $u(x, t_0) < M$ for $x \in (x_0, x_1)$
$d(x)$ =distance to point (x, t_0)
from the set $\{u = M\}$ $d(x_1) = 0$, $d(x_0) > 0$

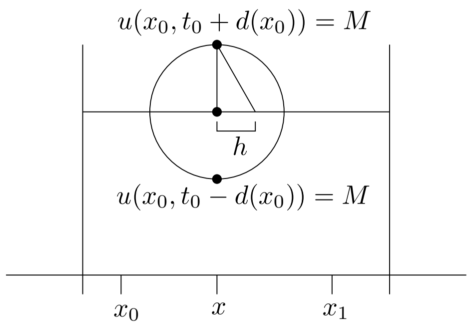


Figure 7.11:

$$\begin{aligned} d(x+h) &\leq \sqrt{d^2(x_0) + h^2} \\ d(x_0) &\leq \sqrt{d^2(x_0+h) + h^2} \end{aligned}$$

Suppose d is differentiable.
Rewrite these inequalities as

$$\begin{aligned} \sqrt{d^2(x) - h^2} &\leq d(x+h) \leq \sqrt{d^2(x) + h^2} \\ \frac{\sqrt{d^2(x) - h^2} - d(x)}{h} &\leq \frac{d(x+h) - d(x)}{h} \\ &\leq \frac{\sqrt{d^2(x) + h^2} - d(x)}{h} \end{aligned}$$

As $h \to 0$, $d'(x) = 0 \implies d = \text{constant}$
Contradiction, since $d(x_0) > 0$, $d(x_1) = 0$.

Step 3: Suppose $u(x,t) < M$ for $\alpha < x < \beta$, $0 < t < t_1$, $t_1 \leq T$, then $u(x, t_1) < M$ for $x \in (\alpha, \beta)$

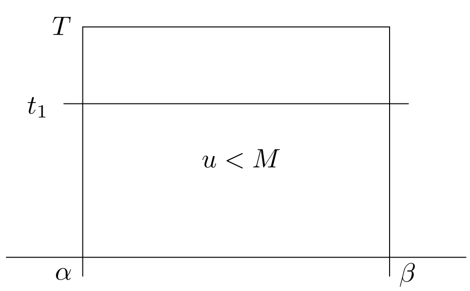


Figure 7.12:

Proof: Weak maximum principle for $w(x,t) = u - M + \eta v$, where $v(x,t) = \exp(-(x-x_1)^2 - A(t-t_1)) - 1$

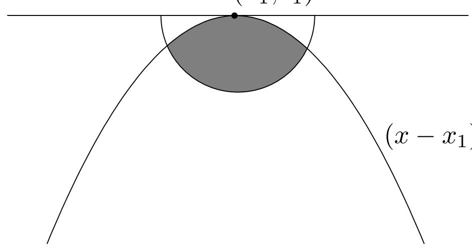


Figure 7.13:

To conclude that $u_t > 0$ $\mathcal{L}u \le 0$ subsolution.

$$\begin{aligned}\mathcal{L}u &= \underbrace{u_t}_{>0} - \underbrace{au_{xx}}_{\ge 0} + \underbrace{bu_x}_{=0} + \underbrace{\underbrace{cu}_{c(x,t)\cdot M}}_{\ge 0} \qquad \text{at } (x_i, t_1) \\ &> 0. \qquad \text{contradiction.}\end{aligned}$$

Conclusion. Suppose that $u(x_0, t_0) = M$, $(x_0, t_0) \in \Omega_T$

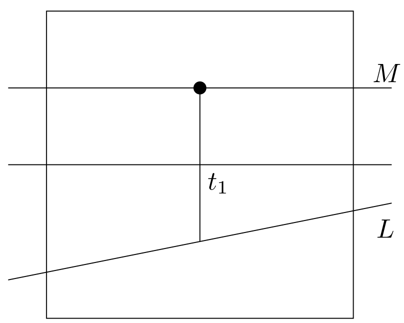


Figure 7.14:

To show if $u(x_0, t_0) = M = \max$, then $u \equiv M$ on Ω_{t_0}
We can not conclude that $u \equiv M$ on Ω_T.
Consider the line $L(x_0) - \{(x_0, t),\ 0 \le t < t_0\}$
Suppose there is a $t_1 \le 0$ such that $u(x_0, t) < M$ for $t < t_1$, $u(x_0, t_1) = M$
$\Longrightarrow u(x,t) < M$ for all (x,t) in $(\alpha, \beta) \times (0, t_1)$
$\Longrightarrow$ Step 3: $u(x_0, t) < 0$. Contradiction. ■

Modify the function v in n dimensions; the same argument works.

7.4 Exercises

7.1: [Qualifying exam 08/90] Consider the hyperbolic equation

$$\mathcal{L}u = u_{yy} - e^{2x} u_{xx} = 0 \quad \text{in } \mathbb{R}^2.$$

a) Find the characteristic curves through the point $\left(0, \frac{1}{2}\right)$.

b) Suppose that u is a smooth of the initial value problem

$$\begin{cases} u_{yy} - e^{2x} u_{xx} = 0 & \text{in} \quad \mathbb{R}^2, \\ u(x,0) = f(x) & \text{for} \quad x \in \mathbb{R}, \\ u_y(x,0) = g(x) & \text{for} \quad x \in \mathbb{R}. \end{cases}$$

Show that the characteristic curves bound the domain of dependence for u at the point $\left(0, \frac{1}{2}\right)$. To do this, prove that $u\left(0, \frac{1}{2}\right) = 0$ provided f and g are zero on the intersection of the domain of dependence with the x-axis. This can be done by defining the energy

$$e(y) = \frac{1}{2} \int_{g(y)}^{h(y)} (u_x^2 + e^{-2x} u_y^2) \, dx$$

where $(g(y), h(y))$ is the intersection of the domain of dependence with the parallel to the x-axis through the point $(0, y)$. Then show that $e'(y) \leq 0$. (See also proof for the domain of dependence for the wave equation with energy methods.)

7.2: [Evans 7.5 #6] Suppose that g is smooth and that u is a smooth solution of

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times [0, \infty), \\ u = g & \text{on} \quad \Omega \times \{t = 0\}, \end{cases}$$

and that the function c satisfies $c \geq \gamma > 0$. Prove the decay rate

$$|u(x,t)| \leq Ce^{-\gamma t} \quad \text{for } (x,t) \in \Omega \times (0, \infty).$$

Hint: The solution of an ODE can also be viewed as a solution of a PDE!

7.3: [Evans 7.5 #7] Suppose that g is smooth and that u is a smooth solution of

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times [0, \infty), \\ u = g & \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

Moreover, assume that c is bounded, but not necessarily nonnegative. Show that $u \geq 0$. Hint: What PDE does $v(x,t) = e^{\lambda t}u(x,t)$ solve?

7.4: [Qualifying exam 08/01] Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Use

a) a sharp version of the Maximum principle

b) and an energy method

to show that the only smooth solutions of the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \partial_n u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

(where ∂_n is the derivative in the outward normal direction) have the form $u =$ constant.

7.5: [Qualifying exam 01/02] Let B be the unit ball in $\mathbb{R}^n$ and let u be a smooth function satisfying

$$\begin{cases} u_t - \Delta u + u^{1/2} = 0, & 0 \leq u \leq M \text{ on } B \times (0,\infty), \\ u_r = 0, & \text{on } \partial B \times (0,\infty). \end{cases}$$

Here M is a positive number and u_r is the radial derivative on ∂B. Prove that there is a number T depending only on M such that $u = 0$ on $B \times (T,\infty)$.

Hint: Let v be the solution of the initial value problem

$$\frac{dv}{dt} + v^{1/2} = 0, \quad v(0) = M,$$

and consider the function $w = v - u$.

Part II

Functional Analytic Methods in PDE

Chapter 8

Facts from Functional Analysis and Measure Theory

8.1 Introduction

This chapter represents what, I believe, to be the most pedagogically challenging material in this book. In order to break into modern theory of PDE, a certain amount of functional analysis and measure theory must be understood; and that is what the material in this chapter is meant to convey. Even though I've made every effort to keep the following grounded in PDE via remarks and examples, it is understandable how the reader might regard some parts of the chapter as unmotivated. Unfortunately, I believe this to be inevitable as there is so much background material that needs to be "water over the back" to have a decent understanding of modern methods in PDE. In the next chapter we will start using the tools learned here to get some actual results in the context of linear elliptic and parabolic equations.

In regard to organization, I have tried to order the material in this chapter as sequentially as possible, i.e. material conveyed can be understood based off of material in the previous section. In addition important theorems and their consequence are presented throughout as soon as possible. I decided to focus on these two principles in hopes that the ordering or material will directly indicate to what generality theorems and ideas can be applied. For example, the validity of a given theorem in a Banach space as opposed to Hilbert spaces will hopefully be made clear by the order in which the material is presented. Lastly, for the sake of completeness, many proofs have been included in this chapter which can (and should) be skipped on the first reading of this book, as many of these dive into rather pedantic points of real and functional analysis.

8.2 Banach Spaces and Linear Operators

Philosophically, one of the major goals of modern PDE is to ascertain properties of solutions when equations are not able to be solved analytically. These properties precipitate out of classifying solutions by seeing whether or not they belong to certain abstract spaces of functions. These abstractions of vector spaces are called *Banach Spaces* (defined below), and they have many properties and characteristics that make them particularly useful for the study of PDE.

To begin, take X to be a *topological vector space* (TVS) over $\mathbb{R}$ ($\mathbb{C}$). Recalling that a TVS is a linear space, this means that X satisfies the typical vector space axioms pertaining to scalars in $\mathbb{R}$ ($\mathbb{C}$) and has a topology corresponding to the continuity of vector addition and scalar multiplication. If one is uncertain of this concept refer to [**?**] for more details.

Definition 8.1. *A norm is any mapping $X \to \mathbb{R}$, denoted $||x||$ for $x \in X$, having the following properties:*

i.) $||x|| \geq 0$, $\forall x \in X$;

ii.) $||\alpha x|| = |\alpha| \cdot ||x||$, $\forall \alpha \in \mathbb{R}, x \in X$;

iii.) $||x + y|| \leq ||x|| + ||y||$, $\forall x, y \in X$.

Remark: iii.) is usually referred to as the *triangle inequality*.

Definition 8.2. *A TVS, X, coupled with a given norm defined on X, namely $(X, ||\cdot||)$, is a normed linear space (NLS).*

Definition 8.3. *Given an NLS $(X, ||\cdot||)$, we define the following*

- *A metric is a mapping $\rho : X \times X \to \mathbb{R}$ defined by $\rho(x, y) = ||x - y||$.*
- *An open ball, denoted $B(x, r)$, is a subset of X characterized by $B(x, r) = \{y \in X : \rho(x, y) < r\}$.*
- *If $U \subset X$ is such that $\forall x \in U$, $\exists r > 0$ with $B(x, r) \subset U$, then U is open in X.*

Definition 8.4. *A NLS X is said to be complete, if every Cauchy sequence in X has a limit in X. Moreover,*

- *a complete NLS is called a Banach Space (BS);*
- *A BS is separable if it contains a countable dense subset.*

One can refer to either [?] or [?], if they are having a hard time understanding any of the above definitions. To solidify these definitions, let us consider a few examples.

Example 8.1. *i.) A classic example of a BS is $C^0([-1,1]) = \{f : [-1,1] \to \mathbb{R}$, continuous$\}$ with the norm $||f||_{C^0[-1,1]} := \max_{x\in[-1,1]} |f(x)|$. (Showing completeness of this space is a good exercise in real analysis. See [?] if you need help.) Of course, C^0 under this norm is complete over any compact subset of $\mathbb{R}^n$.*

ii.) A non-example of a BS is $X = \{f : [-1,1] \to \mathbb{R}, f$ polynomial$\}$. Indeed X is not complete as any function has an approximating sequence of Taylor polynomials that converges (in the sense of Cauchy) in all norms, hence X is not complete. One may try to argue that this statement is false considering the discrete map ($||f|| = 1$ if $f \not\equiv 0$ and $||f|| = 0$ otherwise), but the discrete map is not a valid norm as it violates Definition (8.1.ii).

iii.) Another classic example of a BS from real analysis is the space $l^p = \{\{\tilde{a}\} = (a_1, a_2, a_3, \ldots)\ \sum_{i=1}^{\infty} |a_i|^p < \infty\}$ $1 \le p < \infty$ with the norm

$$||\tilde{a}||_{l^p} = \sum_{i=1}^{\infty} |a_i|^p.$$

Again, showing this space is complete is another good exercise in real analysis.

Moving on, we now start to consider certain types of mapping on our newly defined spaces.

Definition 8.5. *Given X and Y are both NLS, $T : X \to Y$ is called a contraction if $\exists \theta < 1$ such that*

$$||Tx - Ty||_Y \le \theta ||x - y||_X, \quad \forall x, y \in X.$$

A special case of the above definition is when X and Y are the same NLS. If we go further and consider $T : X \to X$, where X is a BS, then we have the following extremely powerful theorem.

[Banach's fixed point theorem] If X is a BS and T a contraction on X, then T has a unique fixed point *in X; i.e. $\exists! x \in X$ such that $Tx = x$. Moreover, for any $y \in X$, $lim_{i\to\infty} T^i(y) = x$.*

Proof: Consider a given point $x_0 \in X$ and define the sequence $x_i = T^i(x_0)$. Then, with $\delta := ||x_0 - Tx_0||_X = ||x_0 - x_1||_X$, we have

$$\begin{aligned} ||x_0 - x_k||_X &\leq \sum_{i=1}^{k} ||x_{i-1} - x_i||_X \\ &\leq \sum_{i=1}^{\infty} \theta^{i-1} ||x_0 - x_1||_X \\ &= \delta \cdot \sum_{i=1}^{\infty} \theta^{i-1} \\ &= \frac{\delta}{1-\theta}. \end{aligned}$$

Also, for $m \geq n$,

$$\begin{aligned} ||x_n - x_m||_X &\leq \theta \cdot ||x_{n-1} - x_{m-1}||_X \leq \theta^2 \cdot ||x_{n-2} - x_{m-2}||_X \\ &\leq \cdots \leq \theta^n \cdot ||x_0 - x_{m-n}||_X \leq \delta \cdot \frac{\theta^n}{1-\theta} \end{aligned}$$

which tends to 0 as $n \to \infty$, since $\theta < 1$. Thus, $\{x_i\}$ is Cauchy. Put $\lim_{i\to\infty}(x_i) =: x \in X$, then we see that $Tx = \lim_{i\to\infty} Tx_i = \lim_{i\to\infty} x_{i+1} = x$.

To finish the proof at hand, we need show uniqueness. Assume that w is another fixed point of T, then

$$||x - w||_X = ||Tx - Tw||_X \leq \theta ||x - w||_X.$$

Since $\theta < 1$, this implies that $||x - w||_X = 0$ and hence $x = w$. ■

An immediate application of the above theorem is the proof of the Picard-Lindelof Theorem for ODE, which asserts the existence of first order ODE under a Lipschitz condition on any inhomogeneity. See [?] for the theorem and proof.

Continuing our exposition on properties of BS operators, we make the following definition.

Definition 8.7. *Take X and Y as NLS. $T : X \to Y$ is said to be* <u>*bounded*</u> *if*

$$||T|| = \sup_{x \in X, x \neq 0} \frac{||Tx||_Y}{||x||_X} < \infty. \tag{8.1}$$

From (8.1), we readily verify that the space of all bounded linear mapping from X to Y, denoted $L(X, Y)$, forms a vector space. Also, concerning the boundedness of BS operators we have the following theorem from functional analysis.

Theorem 8.8. *Take X and Y to be BS. If X is infinite dimensional, then a map T is linear and bounded if and only if it is continuous. On the other hand, if X is finite dimensional, then T is continuous if and only if T is linear.*

Proof: See [**?**].

We now are in a position to state another very powerful idea that is useful for proving existence of solutions for linear PDE.

Theorem 8.9 (Method of Continuity). *If X is a BS, Y a NLS, with L_0 and L_1 bounded linear operators mapping $X \to Y$ such that exists a $C > 0$ satisfying the* a priori *estimate*

$$||x||_X \leq C||L_t x||_Y \quad \forall t \in [0, 1], \tag{8.2}$$

where $L_t := (1-t)L_0 + tL_1$ and $t \in [0, 1]$, then L_1 maps X onto $Y \iff L_0$ map X onto Y.

Proof: Suppose that there exists an $s \in [0, 1]$ such that L_s is onto. (8.2) indicates that L_s is also one-to-one; and hence, L_s^{-1} exists with $||L_s^{-1}|| \leq C$, where C is the same constant in (8.2). Now fix an arbitrary $y \in Y$ and $t \in [0, 1]$. We now seek to prove that exists $x \in X$ such that $L_t(x) = y$, i.e. L_t is onto Y. It is clear that $L_t(x) = y$ if and only if

$$\begin{aligned} L_s(x) &= L_t x + L_s x - L_t x \\ &= y + (1-s)L_0 + sL_1 - [(1-t)L_0 + tL_1](x) \\ &= y + (t-s)[L_0 - L_1](x). \end{aligned}$$

Upon rewriting this equality, we see

$$x = \underbrace{L_s^{-1}y + (t-s)L_s^{-1}(L_0 - L_1)x}_{Tx}. \tag{8.3}$$

So, now we claim that $\exists x \in X$ such that (8.3) is satisfied. This is equivalent to showing T is a contraction mapping via Theorem 8.6. Taking $x_1, x_2 \in X$, we have

$$\begin{aligned} ||Tx_1 - Tx_2||_Y &= ||(t-s)L_s^{-1}(L_0 - L_1)(x_1 - x_2)||_Y \\ &\leq |t-s| \cdot \underbrace{||L_s^{-1}||}_{\leq C} \cdot \underbrace{||L_0 - L_1||}_{\leq 2C} \cdot ||x_1 - x_2||_X. \end{aligned}$$

Thus, if we consider t such that

$$|t-s| < \frac{1}{2C^2},$$

we get that T is a contraction; and hence, L_t is onto. Now, we can cover $[0,1]$ with open intervals of length C^{-2} to conclude that our original assumption (there existing an $s \in [0,1]$ where L_s is onto) leads to L_t being onto $\forall t \in [0,1]$. ■

The method of continuity essentially falls under the general category of *existence theorems*. While not particular good for explicit calculations, this theorem is very handy for proving, well, existence. The following is a nice exercise that demonstrates this.

Exercise: Prove the existence of solution of the equation $-(\phi u')' + u = f$.

Hints: Consider the mappings

$$\begin{aligned} L_0(u) &= -u'', \qquad L_0(u) = f \\ L_1(u) &= -(\phi \cdot u')' + u, \qquad L_1(u) = f, \end{aligned}$$

and note that L_0 is onto (just integrate the equation for L_0).

8.3 Fredholm Alternative

In this section, one of the most important tools in the study of PDE will be presented, namely the *Fredholm Alternative*. To begin the exposition, let us recall a fact from linear algebra.

Fact: If $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear, the mapping L is onto and one-to-one (and hence, invertible) if the kernel is trivial i.e. $\text{Ker}(L) = \{x : Lx = 0\} = \{0\}$.

It turns out in general that this is not true, unless you have a compact perturbation of the identity map. The Fredholm Alternative can be viewed as a BS analogue of the above fact; but before going on to that theorem, let us define what a compact operator is.

Definition 8.10. *Consider X and Y NLS with a bounded linear map $T : X \to Y$. T is said to be compact if T maps bounded sets in X into relatively compact sets in Y.*

Sequentially speaking, the above statement is equivalent to saying that T is compact if it maps bounded sequences in X into sequences in Y that contain converging subsequences.

Not only is the next lemma (due to Riesz) used in the proof of the Fredholm Alternative, but it is also a very important property of NLS in its own right.

Lemma 8.11 (Almost orthogonal element). *If X is a NLS, Y a closed subspace of X and $Y \neq X$. Then $\forall \theta < 1$, $\exists x_\theta \in X$ satisfying $||x_\theta|| = 1$, such that dist$(x_\theta, Y) \geq \theta$.*

Proof: First, we fix arbitary $x \in X \setminus Y$. Since Y is closed,

$$\text{dist}(x, Y) = \inf_{x \in Y} ||x - y|| = d > 0.$$

Thus, $\exists y_\theta \in Y$, such that

$$||x - y_\theta|| \leq \frac{d}{\theta}.$$

Now, we define

$$x_\theta = \frac{x - y_\theta}{||x - y_\theta||};$$

it is clear that $||x_\theta|| = 1$. Moreover, for any $y \in Y$,

$$\begin{aligned} ||x_\theta - y|| &= \frac{||x - [y_\theta - ||y_\theta - x||y]||}{||y_\theta - x||} \\ &\geq \frac{d}{||y_\theta - x||} \geq \theta, \end{aligned}$$

where the second inequality is based on the fact that $y_\theta - ||y_\theta - x||y \in Y$ since Y is a proper subspace of X. ■

Corollary 8.12. *If X is an infinite dimensional NLS, then $\overline{B_1(0)}$ is* not *precompact in X.*

Proof: The statement of the corollary is equivalent to saying there exists bounded sequences in $\overline{B_1(0)}$ that have no converging subsequences. To see this, we argue directly by making the following construction. For a given infinite dimensional BS X, pick an arbitrary element $x_0 \in \overline{B_1(0)} \subset X$. Since $\mathrm{Span}(x_0)$ is a proper subspace of X, we know exists by lemma 8.11 $x_1 \in X$ such that $\mathrm{dist}(x_1, \mathrm{Span}(x_0)) \geq \frac{1}{2}$ with $||x_1|| = 1 \implies x_1 \in \overline{B_1(0)}$. By the same argument, we pick $x_2 \in X$ such that $\mathrm{dist}(x_2, \mathrm{Span}(x_0, x_1)) \geq \frac{1}{2}$ with $x_2 \in \overline{B_1(0)}$. Repeating iteratively, we now have constructed a sequence $\{x_i\}$, for which there is clearly no converging subsequence in X (the distance between any two elements is greater than $\frac{1}{2}$). ■

The above corollary indicates that compactness will be a big consideration in PDE analysis: many of the arguments forthcoming are based on limit principles, which the above corollary presents a major obstacle. Thus, ideas of compactness of operators (and function space imbeddings; more later) play a major role in PDE theory, as compactness "doesn't come for free" just because we work in a BS.

Now, we are finally in a position to present the objective of this section.

Theorem 8.13. *[Fredholm Alternative] If X is a NLS and $T : X \to X$ is bounded, linear and compact, then*

i.) either the homogeneous equation has a non-trivial solution $x - Tx = 0$, i.e. Ker$(I - T)$ is non-trivial or

ii.) for each $y \in X$, the equation $x - Tx = y$ has a unique solution $x \in X$. Moreover, $(I - T)^{-1}$ is bounded.

Proof *: [?] For clarity the proof will be presented in four steps.

Step 1: *Claim:* Defining $S := I - T$, we claim there exists a constant C such that

$$\mathrm{dist}(x, \mathrm{Ker}(S)) \leq C||S(x)|| \qquad \forall x \in X. \tag{8.4}$$

*Can be omitted on first reading.

Proof of Claim: We will argue indirectly by contradiction. Suppose the claim is false, then there exists a sequence $\{x_n\} \subset X$ satisfying $||S(x_n)|| = 1$ and $d_n := \text{dist}(x_n, \text{Ker}(S)) \to \infty$. Upon extracting a subsequence we can insure that $d_n \leq d_{n+1}$. Choosing a sequence $\{y_n\} \subset \text{Ker}(S)$ such that $d_n \leq ||x_n - y_n|| \leq 2d_n$, we see that if we define

$$z_n := \frac{x_n - y_n}{||x_n - y_n||},$$

then we have $||z_n|| = 1$. Moreover,

$$\begin{aligned} ||S(z_n)|| &= \left|\left|\frac{S(x_n) - S(y_n)}{||x_n - y_n||}\right|\right| \\ &= \left|\left|\frac{1 - 0}{||x_n - y_n||}\right|\right| \\ &\leq \frac{1}{d_n}. \end{aligned}$$

Thus, the sequence $\{S(z_n)\} \to 0$ as $n \to \infty$. Now acknowledging that T is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{Tz_n\}$ converges to an element $w_0 \in X$. Since $z_n = (S + T)z_n$, we then also have

$$\begin{aligned} \lim_{n\to\infty} z_n &= \lim_{n\to\infty} (S + T)z_n \\ &= 0 + w_0 = w_0. \end{aligned}$$

From this, we see that $T(w_0) = w_0$, i.e. $z_n \to w_0 \in \text{Ker}(S)$. However this leads to a contradiction as

$$\begin{aligned} \text{dist}\,(z_n, \text{Ker}(S)) &= \inf_{y\in\text{Ker}(S)} ||z_n - y|| \\ &= ||x_n - y_n||^{-1} \inf_{y\in\text{Ker}(S)} ||x_n - y_n - ||x_n - y_n||y||||x_n - y_n|| \\ &= ||x_n - y_n||^{-1}\text{dist}\,(x_n, \text{Ker}(S)) \geq \frac{1}{2}. \qquad \blacksquare \end{aligned}$$

Step 2: We now wish to show that the range of S, $R(S) := S(X)$, is closed in X. This is equivalent to showing an arbitrary sequence in X, call it $\{x_n\}$, is mapped to another sequence with limit $S(x) = y \in X$, where x has to be shown to be in X.

Now consider the sequence $\{d_n\}$ defined by $d_n := \text{dist}(x_n, \text{Ker}(S))$. By (8.4) we know that $\{d_n\}$ is bounded. Choosing $y_n \in \text{Ker}(S)$ such that $d_n \leq ||x_n - y_n|| \leq 2d_n$, we see that the sequence defined by $w_n = x_n - y_n$ is also bounded. Thus, by the compactness of T, $T(w_n)$ converges to some $z_0 \in X$, upon passing to subsequence if necessary. We also have $S(w_n) = S(x_n) - S(y_n) = S(x_n) \to y$ by definition. These last two observations, combined with the fact that $w_n = (S+T)w_n$, leads us to notice that $w_n \to y + z_0$. Finally, we have that

$$\begin{aligned} \lim_{n\to\infty} S(w_n) &= \lim_{n\to\infty} S(x_n - y_n) \\ \Longrightarrow S(y+z_0) &= y, \end{aligned}$$

where the limit on the RHS is justified by the continuity of S (see theorem (8.13)). Since $y + z_0 \in X$, we conclude that $R(S)$ is indeed closed. ■

Step 3: We now will shot that if $\text{Ker}(S) = \{0\}$ then $R(S) = X$, i.e. if case i.) of the theorem does not hold then case ii.) is true.

To start we make the following. First we define the following series of spaces $S(X)_j := S^j(X)$. Now, it is easy to see that

$$S(X) \subset X \Longrightarrow S^2(X) \subset S(X) \Longrightarrow \cdots \Longrightarrow S^n(X) \subset S^{n-1}(X).$$

This combined with the conclusion of Step 2, indicates that $S(X)_j$ constitute a set of closed, non-expanding subspaces of X. Now, we make the following

Claim: There exists K such that for all $i, j > K$, $S(X)_i = S(X)_j$.

Proof of Claim: Suppose the claim is false, thus we can construct a sequence based on the following rule that $x_{n+1} \in S^{n+1}$ with $\text{dist}(x_{n+1}, S(X)_n) \geq \frac{1}{2}$, $||x_n|| = 1$ and $x_0 \in X$ is chosen arbitrarily. So, let us now take $n > m$ and consider

$$\begin{aligned} Tx_n - Tx_m &= x_n + (-x_m + S(x_m) - S(x_n)) \\ &= x_n + x, \end{aligned}$$

where $x \in S(X)_{m+1}$. From this equation, we see that

$$||Tx_n - Tx_m|| \geq \frac{1}{2} \tag{8.5}$$

by construction of our sequence. (8.5) contradicts the fact that T is a compact operator; our claim is proved. ■

As of yet, we still have not used the supposition that $\text{Ker}(S) = \{0\}$. Utilizing our claim, take $k > K$ so that $S(X)_k = S(X)_{k+1}$. So, taking arbitrary $y \in X$, we see that $S^k(y) \in S(X)_k = S(X)_{k+1}$. Thus, $\exists x \in S(X)_{k+1}$ with $S^k(y) = S^{k+1}(x)$. This leads to

$$S^k(S(x) - y) = 0 \Longrightarrow S(x) - y = S^{-k}(0);$$

but since $\text{Ker(S)} = \{0\}$, $S^{-k}(0) = 0$. We now can conclude that $S(x) = y$, and since $y \in X$ was arbitrary we get the result: $R(S) = X$. ■

Step 4: We lastly seek to prove that if $R(S) = X$ then $\text{Ker}(S) = \{0\}$.

Proof of Step 4: This time we define a non-decreasing sequence of closed subspaces $\{\text{Ker}(S)_j\}$ by setting $\text{Ker}(S)_j = S^{-j}(0)$. This is indeed a non-decreasing sequence as

$$\begin{aligned} &0 \in \text{Ker}(S) = S^{-1}(0) \\ \Longrightarrow &\quad S^{-1}(0) \subset S^{-2}(0) \\ \vdots &\qquad\qquad \vdots \\ \Longrightarrow &\quad S^{-k} \subset S^{-(k+1)}(0). \end{aligned}$$

Also, $\text{Ker}(S)_j$ are closed due to the continuity of S. By employing an analogous argument based on lemma 8.11 to that used in Step 3., we obtain that $\exists l$ such that $\text{Ker}(S)_j = \text{Ker}(S)_l$ for all $j \geq l$. Now, since $R(S) = X \Longrightarrow R(S^l) = X$, $\exists x \in X$ such that for any arbitrary element $y \in \text{Ker}(S)_l$, $y = S^l x$ is satisfied. Consequently $S^{2l}(x) = S^l(y) = 0$, which implies $x \in \text{Ker}(S)_{2l} = \text{Ker}(S)_l$ whence

$$y = S^l x = 0.$$

The result follows from seeing that $\text{Ker}(S) \subset \text{Ker}(S)_l = \{0\}$, which is indicated by the preceding equation and the non-decreasing nature of $\text{Ker}(S)_l$. ■

The boundedness of the operator $S^{-1} = (I - T)^{-1}$ in case ii.) follows from (8.4) with $\text{Ker}(S) = \{0\}$. This completes the entire proof, whew! ■

To conclude this section, we will consider a few examples of compact and non-compact operators.

If $T : X \to Y$ is continuous and Y finite dimensional, then T is compact. It is left to the reader to verify this, as it is a straight-forward calculation in real analysis.

Example 8.3. *If $X = C^1([0,1])$, $Y = C^0([0,1])$, and $T = \frac{d}{dx}$, then T* not *a compact operator. If we look at*

$$||Tu||_{C^0([0,1])} \leq C||u||_{C^0([0,1])}$$

We see that this can't hold $\forall u \in X$ and $\forall x \in [0,1]$. Just look at $u = \sqrt{x}$. Thus, T is not even bounded; and hence, it is not compact. This indicates why taking derivatives is a "bad" operation.

Example 8.4. *Take $X = C^0([0,1])$, $Y = C^0([0,1])$, and $Tu = \int_0^x u(s)\ ds$. Now, take u_j to be a bounded sequence in $C^0([0,1])$ (say it's bounded by a constant C). Thus, we have that $v_j = Tu_j$ is also bounded in C^0; in fact, it is easy to see that $v_j \leq C$. Now, we use the following to show T compact:*

Arzela-Ascoli Theorem*: If u_j is uniformly bounded and equicontinuous in a NLS X, then exists a converging subsequence in X. (We will come back to this theorem later.)*

So, we look at

$$|v_j(x) - v_j(y)| = \left| \int_x^y v_j'(s)\ ds \right| \leq |x-y| \cdot \sup |u_j| \leq C|x-y|$$

Thus, we see that v_j is equicontinuous. By applying Arzela-Ascoli, we find that v_j does indeed contain a converging subsequence in $C^0([0,1])$. We now conclude that T is compact.

8.4 Spectral Theorem for Compact Operators

In this section we will start investigations into the spectra of compact operators. First, for completness, we recall the following elementary definitions.

Definition 8.14. *Take X to be a NLS and $T \in L(X,X)$. The* <u>resolvent set</u>*, $\rho(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is one-to-one and onto. In addition, $R_\lambda = (\lambda I - T)^{-1}$ is the* <u>resolvent operator</u> *assocaited with T.*

From the Fredholm Alternative in the previous section, we see that R_λ is a bounded linear map for $\lambda \in \rho(T)$.

Definition 8.15. *Take X to be a NLS and $T \in L(X,X)$. $\lambda \in \mathbb{C}$ is called an eigenvalue of T if there exists a $x \neq 0$ such that $T(x_\lambda) = \lambda x_\lambda$. The x in this equation is called the eigenvector associated with λ. The set of eigenvalues of T is called the spectrum, denoted $\sigma(T)$. Also, we define*

- $N_\lambda = \{x : Tx = \lambda x\}$ *is the eigenspace.*
- $m_\lambda = dim(N_\lambda)$ *is the multiplicity of λ.*

Warning: It is not necessarily true that $[\rho(T)]^C = \sigma(T)$.

The next lemma is a simple result from real analysis that will be important in the proof of the main theorem of this section.

Lemma 8.16. *If a set $U \subset \mathbb{R}^n$ ($\mathbb{C}^n$) is bounded and has only a finite number of accumulation points, then U is at most countable.*

Proof *: Without lose of generality we consider $U \subset \mathbb{C}$ with only one accumulation point x_0 (the argument doesn't change for the other cases).

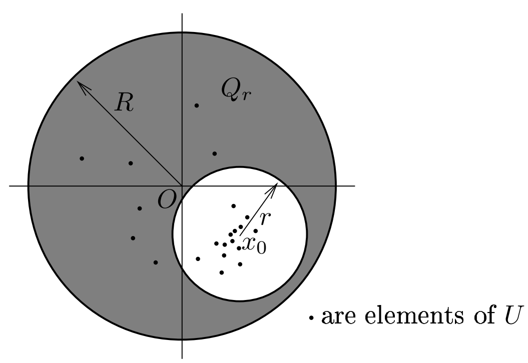


Figure 8.1: Layout for lemma 8.16, note that U is only approximated and x_0 is not necessarily contained in U.

Since U is bounded, there exists $0 < R < \infty$ such that $U \subset B_R(0)$. Since $B_R(0)$ is open, there exists $r > 0$ such that $B_r(x_0) \subset B_R(0)$; define $Q_r := B_R(0) \setminus B_r(x_0)$. To proceed, we make the following

*Can be omitted on first reading.

Claim: Q_r contains only a finite number of points of U.

Suppose not, then $Q_r \cap U$ is at least countably infinite. Since Q_r is closed and bounded, it is compact (Heine-Borel). Thus, we cover Q_r with a finite number of balls with radius 1. At least one of these balls, call it B_1^*, is such that $B_1^* \cap U$ is at least countably infinite. Consequently, we define $V_1 := \overline{B_1^*} \cap Q_r$. In a similar way, we cover V_1 with a finite number of balls with radius $\frac{1}{2}$, where $B_{1/2}^*$ is one of these balls with $B_{1/2}^* \cap U$ being at least countably infinite. In analogy with V_1, we define $V_2 := \overline{B_{1/2}^*} \cap V_1$. We thus iterativly construct a series of closed sets $V_n := \overline{B_{1/n}^*} \cap V_{n-1}$. By construction, we have $V_n \subset V_m$ for $m < n$ with $\operatorname{diam}(V_n) \leq \frac{1}{n}$. We now see that $\exists x \in U$ such that

$$x \in \bigcap_{n=1}^{\infty} V_n,$$

by the compactness of V_n; but this implies that x is an accumulation point of V, since every $V_n \cap U$ is at least countably infinite. Hence we have a contradiction, which proves the claim. ■

Now, it is clear that

$$B_R(0) = x_0 \cup \left[\bigcup_{n=1}^{\infty} Q_{r/n} \right].$$

Thus,

$$U \subset x_0 \cup \bigcup_{n=1}^{\infty} (U \cap Q_{r/n}), \tag{8.6}$$

since $U \subset B_R(0)$. Since the RHS of (8.6) is countable, so is U. ■

We now are in a position to state the main result of the section.

Theorem 8.17 (Spectral Theorem for Compact Operators). *If X is a NLS with $T \in L(X,X)$ compact, then $\sigma(T)$ is at most countably infinite with $\lambda = 0$ as the only possible accumulation point. Moreover, the multiplicity of each eigenvalue is finite.*

Proof: It is easily verified that eigenvectors associated with different eigenvalues are linearly independent. With that in mind, we can thus extract a series of $\{\lambda_j\}$ (not necessarily distinct) from $\sigma(T)$ with associated eigenvectors x_j such that $Tx_j = \lambda_j x_j$ and $x_1, \ldots, x_n$ is linearly independent $\forall n$. Now we define $M_n := \text{Span}\{x_1, \ldots, x_n\}$. Since $Tx = \lambda x$, we see that

$$||T|| \geq \frac{||Tx||}{||x||} = \lambda;$$

hence $\sigma(T)$ is bounded.

To prove that $\lambda = 0$ is the only possible accumulation point, we will argue indirectly by assuming that $\lambda \neq 0$ is an accumulation point of $\{\lambda_j\}$. To do this, we first utilize lemma (8.16) to construct a sequence $\{y_n\} \subset X$ such that $y_n \in M_n$, $||y_n|| = 1$ with $\text{dist}(y_n, M_{n-1}) \geq \frac{1}{2}$. We now wish to prove

$$\left|\left|\frac{1}{\lambda_n}Ty_n - \frac{1}{\lambda_m}Ty_m\right|\right| \geq \frac{1}{2} \tag{8.7}$$

for $n > m$. If this is indeed true and the accumulation point of $\{\lambda_j\}$ isn't 0, then we get a contradiction as $||y_n||$ is bounded, which implies Ty_n contains a convergent subsequence as T is compact (an impossibility in the face of (8.7)). So, let use prove (8.7).

Given $y_n \in M_n = \text{Span}\{x_1, \ldots, x_n\}$, we can write

$$y_n = \sum_{j=1}^{n} \beta_j x_j$$

for some $\beta_j \in \mathbb{C}$. Now, we calculate

$$\begin{aligned} y_n - \frac{1}{\lambda_n}Ty_n &= \sum_{j=1}^{n}\left(\beta_j x_j - \frac{1}{\lambda_n}\beta_j T x_j\right) \\ &= \sum_{j=1}^{n}\left(\beta_j x_j - \frac{\beta_j\lambda_j}{\lambda_n}x_j\right) \\ &= \sum_{j=1}^{n-1}\left(\beta_j x_j - \frac{\beta_j\lambda_j}{\lambda_n}x_j\right) \in M_{n-1}. \end{aligned}$$

Thus, $\frac{1}{\lambda_m}Ty_m \in M_{n-1}$ since $m < n$. It is convenient to now define

$$\sum_{j=1}^{n-1}\left(\beta_j x_j - \frac{\beta_j\lambda_j}{\lambda_n}x_j\right) - \frac{1}{\lambda_m}Ty_m =: z \in M_{n-1}.$$

Finally, looking at (8.7), we see

$$\begin{aligned}\left|\left|\frac{1}{\lambda_n}Ty_n - \frac{1}{\lambda_m}Ty_m\right|\right| &= ||y_n - z|| \\ &\geq \inf_{\omega\in M_{n-1}} ||y_n - \omega|| \\ &= \text{dist}(y_n, M_{n-1}) \\ &\geq \frac{1}{2}.\end{aligned}$$

This proves the claim; and hence, $\{\lambda_j\}$ can only have 0 as an accumulation point by the aforementioned contradiction. Applying this conclusion to lemma 8.16, we attain the full theorem. ■

8.5 Dual Spaces and Adjoint Operators

In addition to the rich structure that we have already seen in BS, it turns out that bounded linear mappings on BS also have some very nice properties that will be very useful in our studies of linear PDE. First, we make a formal definition.

Definition 8.18. *If X is a NLS, we define the dual space, X^*, as the set of all bounded linear mappings taking $X \to \mathbb{R}$.*

The following theorem is easily verified.

Theorem 8.19. *If X is a NLS, X^* is a BS with*

$$||f||_{X^*} = \sup_{x\neq 0}\frac{|f(x)|}{||x||_X}.$$

Notation: If $x \in X$ and $f \in X^*$, then by definition $f(x) \in \mathbb{R}$. Typically, since X^* won't always be a space of functions, we write $\langle f, x\rangle := f(x) \in \mathbb{R}$.

Since X^* is a BS in its own right, we state the following

Definition 8.20. *If X is a NLS, we define the bidual space of X as $X^{**} := (X^*)^*$. Moreover, there is a canonical imbedding $j : X \to X^{**}$ defined by $\langle j(x), f\rangle = \langle f, x\rangle$ for all $x \in X$ and $f \in X^*$. If the canonical imbedding is onto, then X is a reflexive BS.*

Linear operators between BS also have an analogy in dual spaces.

Definition 8.21. *If X, Y are BS and $T \in L(X,Y)$, the adjoint operator $T^* : Y^* \to X^*$ is defined by*

$$\langle T^* g, x\rangle = \langle g, Tx\rangle \qquad \forall x \in X, \forall g \in Y^*$$

From this it is easy to see that $T^* \in L(Y^*, X^*)$.

Before we can elucidate some nice properties of these adjoints, we require a little more "structure" beyond that stipulated in the definition of a BS. Thus, we move onto the next section.

8.6 Hilbert Spaces

Even though BS and their associated operators have a nice mathematical structure, they lack constructions that would enable one to speak of "geometry". As geometric structures yield useful results in Euclidean space (inequalities, angle relations, etc.), we are motivated to construct geometric analogies on the linear spaces (not necessarily BS) of previous sections. This nebulous reference to "geometry" on a linear space (LS) will be clarified shortly.

Definition 8.22. *Given X is an LS, a map $(\cdot,\cdot) : X \times X \to \mathbb{R}$ is called a scalar product if*

i.) $(x, y) = (y, x)$,

ii.) $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1(x_1, y) + \lambda_2(x_2, y)$, *and*

iii.) $(x, x) > 0 \qquad \forall x \neq 0$

for all $x_1, x_2, y \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Correspondingly, we now define

Definition 8.23. *A LS with a scalar product is called a Scalar Product Space (SPS). It is called a Hilbert Space (HS) if it is complete.*

Warning: Do not assume that a scalar product is the same as taking an element of a BS with it's dual, i.e. "$\langle\cdot,\cdot\rangle \neq (\cdot,\cdot)$". Actually, this comparison doesn't even make sense as a scalar product has both arguments in a given space whereas the composition $\langle\cdot,\cdot\rangle$ requires an element from the space and another from the associated dual space. That aside, the dual space of a BS may very well not be isomorphic to the BS itself. This *will* be the case in a

HS by the Riesz-representation theorem (stated below). With that theorem, one may characterize any functional on a HS via the scalar product. This does not go the other way around; i.e. on a BS, a dual space does not necessarily characterize a valid scalar product.

Also, one will notice that we only require that a SPS to be a LS (not a BS) with a scalar product; but, it is easily verified that $||x|| := \sqrt{(x,x)}$ satisfies the criteria stated in the definition of a norm. Thus, any HS (SP'S) is a BS (NLS). This is not to say that the only valid scalar product on a space is the one that generates the norm; case in point, $L^p(\Omega)$ spaces are frequently considered with the norm

$$\int_\Omega f \cdot g \, dx$$

where $f, g \in L^p(\Omega)$. It is easy to verify this satisfies definition 8.22 but does not generate the $L^p(\Omega)$ norm.

Scalar products add a great deal of structure to the spaces we have been considering previously. First, we can now start talking about the concept of "geometry" on a SPS. We have seen that for any SPS, there exists a norm which can be analogized to length in Euclidean space. Taking this analogy one step further, we can define the "angle", ϕ, between two elements x and y in a SPS, X, by $\cos\phi := \frac{(x,y)}{||x||\cdot||y||}$. This leads directly to the all-important orthogonality criterion $x \perp y \iff (x,y) = 0$. In addition to geometry, there are many useful inequalities that pertain to scalar products; here are a few.

- Cauchy Schwarz inequality: $|(x,y)| \leq ||x|| \cdot ||y||$
- Triangle inequality: $||x+y|| \leq ||x|| + ||y||$
- Parallelogram identity: $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$

Notation: From now on we will be referring to Hilbert Spaces, as it usually safe to assume that we are working in a complete SPS.

With these concepts, we can now analyze some key theorems that pertain to Hilbert Spaces.

Theorem 8.24 (Projection Theorem). *If H is a HS and M is a closed subspace of H, then each $x \in H$ has a unique decomposition $x = y + z$ with $y \in M$, $z \in M^\perp$ (i.e. is perpendicular to M, $(z, M) = 0$, $\forall m \in M$). Moreover dist$(x, M) = ||z||$.*

Proof: First, if $x \in M$ then as $x = x + 0$ we trivially get the conclusion. Thus, we consider $M \neq X$ with $x \notin M$. Defining, $d := \operatorname{dist}(x, M)$, we select a *minimizing sequence* $\{y_n\} \subset M$ such that $||x - y_n|| \to d$. Before proceeding, we need to show that $\{y_n\}$ is Cauchy. Utilizing the parallelogram identity, we write

$$\underbrace{4\left|\left|x - \frac{1}{2}(y_n + y_m)\right|\right|^2}_{\geq 4d^2} + ||y_n - y_m||^2 = 2(||x - y_n||^2 + ||x - y_m||^2) \to 4d^2,$$

where the first term on the LHS is $\geq 4d^2$ as $\frac{1}{2}(y_n + y_m) \in M$. Thus, we see that $||y_n - y_m|| \to 0$; the sequence is Cauchy. Since X is complete and M is closed, we ascertain that $\lim_{n\to\infty} y_n = y \in M$.

Writing $x = z + y$, where $z := x - y$, the conclusion of the proof will follow provided $z \in M^\perp$. Consider any $y^* \in M$ and $\alpha \in \mathbb{R}$, as $y + \alpha y^* \in M$, it is calculated

$$\begin{aligned} d^2 \leq ||x - y - \alpha y^*||^2 &= (z - \alpha y^*, z - \alpha y^*) \\ &= ||z||^2 - 2\alpha(y^*, z) + \alpha^2||y^*||^2 \\ &= d^2 - 2\alpha(y^*, z) + \alpha^2||y^*||^2; \end{aligned}$$

thus,

$$|(y^*, z)| \leq \frac{\alpha}{2}||y^*||^2 \quad \forall \alpha \in \mathbb{R}.$$

From this we conclude that $|(y^*, z)| = 0$; and since $y^* \in M$ was arbitrary, $z \in M^\perp$. ■

One of the nicest properties of Hilbert Spaces, call it H, is that one can characterize *any* bounded linear functional on H in terms of the scalar product. This is formalized by the following theorem.

Theorem 8.25 (Riesz-representation Theorem). *If F is a bounded linear function on a Hilbert space H, then there exists a unique $f \in H$ such that $F(x) = (x, f)$ and $||f|| = ||F||$.*

Proof: First, we consider F such that $\operatorname{Ker}(F) = H$; then, $F = 0$ and we correspondingly choose $f = 0$. If, on the other hand, we have $\operatorname{Ker}(F) \neq H$, then $\exists z \in H \setminus \operatorname{Ker}(F)$. By the projection theorem, we may assume that

$z \in \operatorname{Ker}(F)^{\perp}$ as it easily seen that $\operatorname{Ker}(F)$ is a proper subspace of H. Now, we see that

$$
\begin{aligned}
& F\left(x-\frac{F(x)}{F(z)} z\right)=0 \\
\Longrightarrow \quad & x-\frac{F(x)}{F(z)} z \in \operatorname{Ker}(F).
\end{aligned}
$$

Since $z \in \operatorname{Ker}(F)^{\perp}$, we have by orthogonality

$$
\left(x-\frac{F(x)}{F(z)} z, z\right)=0 \Longleftrightarrow (x, z)=\left(\frac{F(x)}{F(z)} z, z\right)=\frac{F(x)}{F(z)}||z||^{2}.
$$

Solving for $F(x)$ algebraically leads to

$$
F(x)=\left(x, \frac{F(z)}{||z||^{2}} z\right)=(x, f), \text{ where } f=\frac{F(z)}{||z||^{2}} z.
$$

Lastly, we need to show $||F||=||f||$. So, on one hand, we have

$$
||F|| \quad = \sup_{x \in H, x \neq 0} \frac{|F(x)|}{||x||}=\sup \frac{|(x, f)|}{||x||} \stackrel{\text{C-S}}{\leq} \sup \frac{||x|| \cdot||f||}{||x||}=||f||;
$$

but on the other, its calculated that

$$
||f||^{2} \quad = \quad (f, f)=F(f) \stackrel{\text{C-S}}{\leq}||F|| \cdot||f|| \Longrightarrow ||f|| \leq ||F||.
$$

Thus, we conclude that $||f||=||F||$. ■

Next, we will go over one of the key existence theorems pertaining to elliptic equations but first a quick definition.

Definition 8.26. *Give X a NLS, a mapping $B: X \times X \rightarrow \mathbb{R}$ is a bilinear form provided B is linear in one argument when the other argument is fixed. B is bounded if there exists a constant $C_0>0$ such that*

$$
|B(x, y)| \leq C_{0}||x|| \cdot||y||
$$

for all $x, y \in X$. Analogously, B is coercive if there exists a constant $C_1>0$ such that

$$
B(x, x) \geq C_{1}||x||^{2}
$$

for all $x \in X$.

Warning: B is not necessarily symmetric in its arguments.

Theorem 8.27 (Lax-Milgram Theorem). *If H is a HS and $B : H \times H \to \mathbb{R}$ is a bounded, coercive bilinear form, then for all $F \in H^*$ there exists a unique $f \in H$ such that*

$$B(x, f) = F(x).$$

Proof: First, fix $f \in H$ and define

$$G_f(x) := B(x, f) = (x, Tf), \tag{8.8}$$

where the last equality comes from the application of the Riesz representation theorem to $G_f(x)$. Given B is coercive, we calculate

$$C_1||f||^2 \leq B(f, f) = (f, Tf) \leq ||f|| \cdot ||Tf||,$$

where C_1 is the coercivity constant associated with B. Analogously, via the boundedness of B,

$$||Tf||^2 = (Tf, Tf) = B(Tf, f) \leq C_0||f|| \cdot ||Tf||,$$

where C_0 is the boundedness constant of B. From these last two equations, we derive that

$$C_1||f|| \leq ||Tf|| \leq C_0||f||, \tag{8.9}$$

which is tantamount to $R(T)$ being closed (right inequality) and T itself being one-to-one (left inequality). Now, we wish to show that T is onto, since this along with (8.9) would imply that T^{-1} is well-define.

If T is not onto, then by the projection theorem, there exists $z \in R^{\perp}(T)$ ($R(T)$ is easily noticed to be a proper subspace of H). Thus, $B(z, f) = (z, Tf) = 0$ for all $f \in H$. Choosing $f = z$, we calculate that $0 = B(z, z) \geq C_1||z||^2$ which implies $z = 0$, i.e. $R(T) = H$. So T^{-1} is indeed well-defined as we have now shown T to be onto.

The last step of the proof is to apply the Riesz representation theorem: for any $F \in H^*$, there exists a unique $g \in H$ such that $F(x) = (x, g)$ for all $x \in H$. With that, we get

$$F(x) = (x, g) = (x, TT^{-1}g) = B(x, T^{-1}g).$$

From this applied to (8.8), we see that $f = T^{-1}g$. The existence of f is thus guaranteed as we have shown T^{-1} to be well defined. ■

As was mentioned above, the Lax-Milgram theorem turns out to be the right tool for proving existence of solutions of many elliptic equation. That being said, one could be forgiven for thinking that the coercivity criterion is somewhat artificially induced, as it will be shown to correspond to the ellipticity criterion of a differential operator. It is true that this correspondence is very useful, but the Lax-Milgram theorem actually has a deeper implication than the existence of certain PDE.

Recalling the construction of Hilbert Spaces, one will remember that the properties of such a space are all dependent on the definition of the associated scalar product. Indeed, the norm and, hence, the topology of the space result from the particular definition of the scalar product (on top of the assumed ambient properties of a linear space). Philosophically speaking, Lax-Milgram indicates whether a given bilinear form acting as a scalar product will "replicate" the ***topological*** structure an already-established HS. If this is found to be the case, then one can view the conclusion of Lax-Milgram as a direct application of the Riesz-representation theorem on the space with the bilinear form as the scalar product. Now since topology is directly correlated to the norm of a HS, one can start to see the coercivity condition (along with the boundedness of the bilinear form) as a stipulation on the bilinear form that would guarantee the preservation of the topology of the HS if the scalar product were replaced by the bilinear form itself. Of course the geometry, as described in the beginning of the section, would be different; but this is not important, in this instance, as Lax-Milgram only refers two what elements are contained in a space (a topological attribute). Hopefully, this bit of insight will make the Lax-Milgram theorem (and existence theorems in general) a little more intuitive for the reader.

8.6.1 Adjoints Revisited and Fredholm Alternative in Hilbert Spaces

With the additional structure of a HS, we are now able to realize more properties of adjoint operators than was possible in BS.

Theorem 8.28. *If H is a HS with $T \in L(H,H)$, then T^* has the following properties:*

i.) $||T^*|| = ||T||$;

ii.) *If T is compact, then T^* is also compact;*

iii.) $\overline{R(T)} = Ker(T^*)^\perp$.

Proof:

i.) This is clear from the fact that

$$||T|| = \sup_{x \in H} \frac{||Tx||}{||x||}$$

(T^* has an analogous norm).

ii.) Let $\{x_n\}$ be a sequence in H satisfying $||x_n|| \leq M$. Then

$$\begin{aligned} ||T^*x_n||^2 = (T^*x_n, T^*x_n) &= (x_n, TT^*x_n) \\ &\leq ||x_n|| \cdot ||TT^*x_n|| \\ &\leq M||T|| \cdot ||T^*x_n|| \end{aligned}$$

so that $||T^*x_n|| \leq M||T||$; that is, the sequence $\{T^*x_n\}$ is also bounded. Hence, since T is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{TT^*x_n\}$ converges. But then

$$\begin{aligned} ||T^*(x_n - x_m)||^2 &= (T^*(x_n - x_m), T^*(x_n - x_m)) \\ &= (x_n - x_m, TT^*(x_n - x_m)) \\ &\leq 2M||TT^*(x_n - x_m)|| \to 0 \qquad \text{as } m, n \to \infty. \end{aligned}$$

Since H is complete, the sequence $\{T^*x_n\}$ is convergent and hence T^* is compact. ■

iii.) Fix arbitrary $x \in H$ and set $Tx =: y \in R(T)$. From this, it is seen that $(y, f) = (Tx, f) = (x, T^*f) = 0$ for all $f \in \text{Ker}(T^*)$. Thus, $R(T) \subset \text{Ker}(T^*)^\perp$; and since $\text{Ker}(T^*)$ is closed, $\overline{R(T)} \subset \text{Ker}(T^*)^\perp$. Now suppose that $y \in \text{Ker}(T^*)^\perp \setminus \overline{R(T)}$. By the projection theorem, $y = y_1 + y_2$ where $y_1 \in \overline{R(T)}$ and $y_2 \in \overline{R(T)}^\perp \setminus \{0\}$ Consequently, $0 = (y_2, Tx) = (T^*y_2, x)$ for all $x \in H$; hence, $y_2 \in \text{Ker}(T^*)$. Therefore, $(y_2, y) = (y_2, y_1) + ||y_2||^2 = ||y_2||^2$. Thus, we see that $(y_2, y) \neq 0$, i.e. $y \notin \text{Ker}(T^*)^\perp$. This contradiction on the assumption of $y \in \text{Ker}(T^*)^\perp \setminus \overline{R(T)}$ leads to the conclusion that $\overline{R(T)} = \text{Ker}(T^*)^\perp$. ■

Now, we can can combine the above properties of adjoints, the results pertaining to the spectra of compact operators and the Fredholm Alternative on BS to directly ascertain the following generalization.

Theorem 8.29 (Fredholm Alternative in Hilbert Spaces). *Consider a HS, H and a compact operator $T : H \to H$. In this case, there exists an at most countable set $\sigma(T)$ having no accumulation points except possibly $\lambda = 0$ such that the following holds: if $\lambda \neq 0$, $\lambda \notin \sigma(T)$ then the equations*

$$\lambda x - Tx = y \qquad \lambda x - T^* x = y \tag{8.10}$$

have a unique solution $x \in H$, for all $y \in H$. Moreover, the inverse mappings $(\lambda I - T)^{-1}$, $(\lambda I - T^*)^{-1}$ are bounded. If $\lambda \in \sigma(T)$, then $Ker(\lambda I - T)$ and $Ker(\lambda I - T^*)$ have positive and finite dimension, and we can solve the equation (8.10) if and only if y is orthogonal to $Ker(\lambda I - T^*)$ and $Ker(\lambda I - T)$ respectively for both cases.

There is a lot of information in this theorem; it's worth the effort to try and get one's mind around it. Also, keep in mind that $\text{Ker}(\lambda I - T)$ and $\text{Ker}(\lambda I - T^*)$ are just the eigenspaces T_λ and T^*_λ respectively.

8.7 Weak Convergence and Compactness; the Banach-Alaoglu Theorem

To begin this section we begin with some motivation. Consider a sequence of approximate solutions of $F(u, Du, D^2u) = 0$, i.e. a bounded sequence $\{u_k\} \in X$ such that $F(u_k, Du_k, D^2u_k) = \epsilon_k \to 0$. Now, there are two important questions that arise:

- Does bounded imply there is a convergent subsequence i.e. $u_{k_j} \to u$?
- Is this limit the solution?

As we have seen that bounded sequences in infinite dimensional BS do not necessarily contain convergent subsequences (i.e. in the strong sense), these questions are not readily answered with our current "set of tools". Given that such approximations as the one stated are frequently utilized in PDE, we now seek to define a new criterion for convergence so that we can indeed answer both these questions in the affirmative.

Definition 8.30. *Take X to be a BS, with X^* as its dual.*

i.) $\{x_j\} \subset X$ *converges weakly to $x \in X$ if*

$$\langle y, x_j \rangle \to \langle y, x \rangle \qquad \forall y \in X^*.$$

Notation: *u_k converging weakly to u is denoted by $u_k \rightharpoonup u$.*

ii.) $\{y_j\} \in X^*$ *converges weakly** *to* $y \in X^*$ *if*

$$\langle y_j, x\rangle \to \langle y, x\rangle \qquad \forall x \in X$$

It can be readily seen that weak convergence implies strong convergence (that is convergence in norm). Also, be the linearity of scalar products, it is also observed that weak limits are unique.

Now, we present the key theorem which essential would enable us to answer "yes" to the hypothetical situation presented at the beginning of the section.

Theorem 8.31 (Banach-Alaoglu). *If X is a TVS, with $\Omega \subset X$ bounded, open set containing the origin, then the unit ball in X^* is sequentially weak* compact, i.e. define*

$$K = \{F \in X^* : |F(x)| \le 1 \quad \forall x \in \Omega\},$$

then K is sequentially weak compact.*

Proof †: First, since Ω is open, bounded and contains the origin, there corresponds to each $x \in X$ a number $\gamma_x < \infty$ such that $x \in \gamma_x\Omega$, where $a\Omega := \left\{x \in X : \frac{x}{a} \in \Omega\right\}$ for any fixed $a \in \mathbb{R}$. Now for any $x \in X$, we know that $\frac{x}{\gamma_x} \in \Omega$, hence

$$|F(x)| \le \gamma(x) \qquad (x \in X, \Lambda \in K).$$

Let A_x be the set of all scalars α such that $|\alpha| \le \gamma_x$. Let τ be the product topology on P, the Cartesian product of all A_x, one for each $x \in X$. Indeed, P will almost always be a space with an uncountable dimension. By Tychonoff's theorem, P is compact since each E_x is compact. The elements of P are functions f on X (they may or may not be linear or even continuous) that satisfy

$$|f(x)| \le \gamma(x) \qquad (x \in X).$$

Thus $K \subset X^* \cap P$, as $K \subset X^*$ and $K \subset P$. It follows that K inherits two topologies: one from X^* (its weak*-topology, to which the conclusion of the theorem refers) and the other, τ, from P. Again, see [?] for the concept of inherited topologies.

We now claim that

i.) these two topologies coincide on K, and

ii.) K is closed subset of P.

If ii.) is proven, then we have that K is an uncountable Cartesian product of τ-compact sets (each one contained in some A_x), hence it is compact by Tychonoff's theorem. Moreover, if i.) is shown to be true then K is concluded to be weak*-compact from the previous statement.

So, we now see to prove our claims. To do this first fix $F_0 \in K$; choose $x_i \in X$, for $1 \leq i \leq n$; and fix $\delta > 0$. Defining

$$W_1 := \{F \in X^* : |\Lambda x_i - \Lambda_0 x_i| < \delta \quad \text{for} \quad 1 \leq i \leq n\}$$

and

$$W_2 := \{f \in P : |f(x_i) - \Lambda_0 x_i| < \delta \quad \text{for} \quad 1 \leq i \leq n\},$$

we let n, x_i, and δ range over all admissible values. The resulting sets $\tilde{W}_1$, then form a local base for the weak*-topology of X^* at Λ_0 and the sets $\tilde{W}_2$ form a local base for the product topology τ of P at Λ_0; i.e. any open set in X^* or P can be represented as a union of translates of the sets contained in $\tilde{W}_1$ or $\tilde{W}_2$ respectively. Since $K \subset P \cap X^*$, we have

$$\tilde{W}_1 \cap K = \tilde{W}_2 \cap K.$$

This proves (i).

Now, suppose f_0 is in the τ-closure of K. Next choose $x, y \in X$ with scalars α, β, and $\epsilon > 0$. The set of all $f \in P$ such that $|f - f_0| < \epsilon$ at x, at y, and at $\alpha x + \beta y$ is a τ-neighborhood of f_0. Therefore K contains such an f. Since this $f \in K \subset X^*$, and hence, linear, we have

$$\begin{aligned} f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) &= (f_0 - f)(\alpha x + \beta y) \\ &\quad + \alpha(f - f_0)(x) + \beta(f - f_0)(y). \end{aligned}$$

so that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\epsilon.$$

Since ϵ was arbitrary, ,we see that f_0 is linear. Finally, if $x \in \Omega$ and $\epsilon > 0$, the same argument shows that there is an $g \in K$ such that $|g(x) - f_0(x)| < \epsilon$. Since $|g(x)| \leq 1$, by the definition of K, if follows that $|f_0(x)| \leq 1$. We conclude that $f_0 \in K$. This proves ii.) and hence the theorem. ■

As an immediate consequence we have

Corollary 8.32. *If X is a reflexive BS, then the unit ball in X is sequentially weakly compact.*

From this corollary and the Riesz-representation theorem, we have that any bounded set in any HS is sequentially weakly compact (the Riesz-representation theory proves that all HS are reflexive BS). This is a key point that will be used frequently in the upcoming linear theory, and gives us the criterion under which we can answer "yes" to the questions posed at the beginning of the section.

Note: With this section I have broken the sequential structure of the chapter in that one only needs to have a TVS to speak of weak convergence, etc. The reason for presenting this material now, is that it can only be motivated once the student starts to realize how restrictive the notion of strong convergence in a BS/HS is. Thus, for motivation's sake I have decided to present this material now, late in the chapter.

8.8 Key Words from Measure Theory

This section is intended to be a whirlwind review of basic measure theory, i.e. one that is not already familiar with the following concepts should consult with [?] for a proper treatise of such. In reality, this section is a glorified appendix, in that most of the following results are not proven but are explained in relation to what has already been explained.

8.8.1 Basic Definitions

Let us recall the "typical" definition of a measure:

Definition 8.33. *Consider the set of all sets of real numbers, $\mathcal{P}(\mathbb{R})$. A mapping $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty)$ is a* <u>*measure*</u> *if the following properties hold:*

i.) for any interval l, $\mu l =$ the length of l;

ii.) if $\{E_n\}$ is a sequence of disjoint sets, then

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu E_n;$$

iii.) μ is translationally invariant; i.e., if E is a set for which μ is defined and if $E + y := \{x + y : x \in E\}$, then $\mu(E + y) = \mu E$ for all $y \in \mathbb{R}$.

Now, we say this is the "typical" definition as there are actually different ways to define a measure. To elaborate, in addition to the above three criterion, it would be nice to have μ defined for all subsets of $\mathbb{R}$; but no such mapping exists. Thus, one of these four criterion must be compromised to actually get a valid definition; obviously, this may be done in a number of ways. We take defintion 8.8.1 as a proper definition as the measures we will consider, Lebesgue (and more esoterically Hausdorff), obey the criterion contained in such.

The next definition is a necessary precursor to defining Lebesgue measure.

Definition 8.34. *The outer measure, denoted μ^* is defined for any $A \subseteq \mathbb{R}$ as*

$$\mu^* A = \inf_{A \subset \bigcup_n I_n} \sum_n l(I_n),$$

where I_n are intervals in $\mathbb{R}$ and $l(I_n)$ is the length of I_n.

Now, even though μ^* is defined for all subsets of $\mathbb{R}$ it does not satisfy condition iii.) of definition 8.8.1, which is undesirable for integration. Thus, we go about restricting the sets of consideration to guarantee that iii.) is satisfied by the following definition.

Definition 8.35. *A set $E \subset \mathbb{R}$ is Lebesgue measurable if*

$$\mu^* A = \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

for all $A \subset \mathbb{R}$. Moreover, the Lebesgue measure of E is just $\mu^* E$.

It is left to the reader that definition does indeed imply condition iii.) of definition ; again, see [**?**] if you need help.

Denoting the set of all measurable sets $\mathcal{B}$, let us review one of the properties of $\mathcal{B}$. To do this, we remember the following definition

Definition 8.36. *A group of sets $\mathcal{A}$ is a σ-ring on a space X if the following are true:*

- *$X \in \mathcal{A}$;*
- *if $E \in \mathcal{A}$, then $A^c := X \setminus A \in \mathcal{A}$;*

- *and if* $E_i \in \mathcal{A}$ *then*

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}.$$

Canonically extending the above concept of Lebesgue measure on $\mathbb{R}$ to $\mathbb{R}^n$ by replacing intervals with Cartesian products of intervals (and lengths with volumes), it is true that $\mathcal{B}$ is a σ-ring over $\mathbb{R}^n$.

Now, we move on to some important definitions pertaining to general measures.

Definition 8.37. *$N \in \mathcal{B}$ is called a <u>set of μ-measure zero</u> if $\mu(N) = 0$. This ie equivalent to saying there exists open sets $A_i \subset \mathcal{B}$ with $\bigcup A_i \subset N$ such that $\mu\left(\bigcup A_i\right) < \epsilon$.*

Definition 8.38. *A property hold <u>μ-a.e.</u> (μ almost everywhere) if the set of all x for which the property does not hold is a set of μ-measure zero.*

Lastly, we go over what it means for a *function* to be differentiable.

Definition 8.39. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{B}$ <u>measurable</u> if $\{x : f(x) \in U\} \in \mathcal{B}$ for all open sets U in $\mathbb{R}$.*

8.8.2 Integrability

For integrability studies, we first define a special clase of functions.

Definition 8.40. *A function f is <u>simple</u> if f is a finite non-zero constant on finitely many disjoint sets A_i which are measurable and zero elsewhere.*

A simple function f is said to be *integrable* if

$$\sum |f_i| \cdot \mu(A_i) < \infty;$$

and thus,

$$\int f \, d\mu = \sum f_i \cdot \chi_{A_i},$$

where

$$\chi_{A_i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

Note: Keep in mind that μ is a general measure; thus, the above criterion for integrability of a simple function is dependent on our choice of μ.Still considering a general measure, we now make the following generalization.

Definition 8.41. *Consider a general function g and a sequence of simple functions $\{f_n\}$ such that $g = \lim f_n$ a.e. g is said to be* integrable *(with respect to measure μ) if*

$$\lim_{n,k\to\infty} \int |f_n(x) - f_k(x)| \, d\lambda = 0.$$

Moreover, since the limit

$$\lim_{k\to\infty} \int f_k \, d\mu$$

thus exists, we see that the limit is independent of the choice of approximating sequence.

Now, we specify μ to be henceforth the Lebesgue measure. The corresponding Lebesgue integrals have nice convergence properties, i.e. there exists a good variety of conditions when one can interchange limits and integrals. We now rapidly go over these convergence theorems, referring the reader to [**?**] for proofs.

Theorem 8.42 (Lebesgue's Dominated Convergence). *If g is integrable with some $f_i \to f$ a.e., where f is measurable and $|f_i| \le g$ for all i, then f is integrable. Moreover,*

$$\int f_i \, d\mu \to \int f \, d\mu \qquad \left[\lim_{i\to\infty} \int f_i \, d\mu = \int \lim_{i\to\infty} f_i \, d\mu\right].$$

Theorem 8.43 (Monotone Convergence (Beppo-Levi)). *If f_i is a sequence of integrable functions such that $0 \le f_i \nearrow f$ monotonically and*

$$\int f_i \, d\mu \le C < \infty,$$

then f integrable and

$$\int f \, d\mu = \lim_{i\to\infty} \int f_i \, d\mu.$$

Lemma 8.44 (Fatou's). *If f_i is a sequence of integrable functions with $f \ge 0$ and*

$$\int f_i \, d\lambda \le C < \infty,$$

then $\liminf f_i$ is integrable and

$$\int \lim_{i\to\infty} f_i \, d\mu \le \lim_{i\to\infty} \inf \int f_i \, d\mu$$

Theorem 8.45 (Egoroff's). *Take f_i to be a sequence of functions such that $f_i \to f$ a.e. on a set E, with f measurable, where E is measurable and $\mu(E) < \infty$, then $\forall \epsilon > 0$, $\exists E_\epsilon$ measurable such that $\lambda(E \setminus E_\epsilon) < \epsilon$ and $f_i \to f$ uniformly on E_ϵ.*

The next theorem technically states that multi-dimensional integrals can indeed be computed via an iterative scheme.

Theorem 8.46 (Fubini's). *i.) If $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is measurable such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)|\ dxdy$$

exists, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y)\ dxdy = \int_{\mathbb{R}^n} g(y)\ dy,$$

where

$$g(y) = \int_{\mathbb{R}^m} f(x,y)\ dx$$

ii.) On the otherhand, if

$$\overline{g}(y) = \int_{\mathbb{R}^m} |f(x,y)|\ dx < \infty$$

for almost every y with

$$\int_{\mathbb{R}^n} \overline{g}(y)\ dy < \infty,$$

then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)|\ dxdy < \infty$$

8.8.3 Calderon-Zygmund Decomposition

[Cube Decomposition]

A useful construction for some measure theoretic arguments in the forthcoming chapters is that of cube decomposition. This is a recursive construction that enables one to ascertain estimates for the measure of a set imbedded in $\mathbb{R}^n$.

First, consider a cube $K_0 \in \mathbb{R}^n$, f a non-negative integrable function, and $t > 0$ all satisfying

$$\int_{K_0} f\,dx \leq t|K_0|,$$

where $|\cdot|$ indicates the Lebesgue measure of the set. Now, by bisecting the edges of K_0 we divide K_0 into 2^n congruent subcubes whose interiors are disjoint. The subcubes K which satisfy

$$(9.16) \int_K f\,dx \leq t|K|$$

are similarly divided and the process continues indefinately. Set K^* be the set of subcubes K thus obtained that satisfy

$$\int_K f\,dx > t|K|,$$

and for each $K \in K^*$, denote by $\tilde{K}$ the subcube whose subdivision gives K. It is clear that

$$\frac{|\tilde{K}|}{|K|} = 2^n$$

and that for any $K \in K^*$, we have

$$(9.17) t < \frac{1}{|K|} \int_K f\,dx \leq 2^n t.$$

Moreover, setting

$$F = \bigcup_{K \in K^*} K$$

and $G = K_0 \setminus F$, we have

$$(9.18) f \leq t \quad \text{a.e. in } G.$$

This inequality is a consequence of Lebesgue's differention theorem, as each point in G lies in a nested sequence of cubes satisfying (9.16) with the diameters of the sequence of cubes tending to zero.

For some pointwise estimates, we also need to consider the set

$$\tilde{F} = \bigcup_{K \in K^*} \tilde{K}$$

which satisfies by (9.16),

$$\int_{\tilde{F}} f\, dx \leq t|\tilde{F}|.$$

In particular, when f is the characteristic function χ_Γ of a measurable subset Γ of K_0, we obtain from (9.18) and (9.19) that

$$(9.20)|\Gamma| = |\Gamma \cap \tilde{F}| \leq t|\tilde{F}|$$

8.8.4 Distribution functions and a convenient lemma

Let f be a measurable function on a domain Ω (bounded or unbounded in $\mathbb{R}^n$. The *distribution function* $\mu = \mu_f$ of f is defined by

$$(9.21)\mu(t) = \mu_f(t) = |\{x \in \Omega | f(x) > t\}|$$

for $t > 0$, and measures the relative size of f. Note that μ is a decreasing function on $(0, \infty)$. The basic properities of the distribution function are embodied in the following lemma.

Lemma 8.47. *For any $p > 0$ and $|f|^p \in L^1(\Omega)$, we have*

$$(9.22)\mu(t) \leq t^{-p} \int_\Omega |f|^p\, dx$$

$$(9.23) \int_\Omega |f|^p\, dx = p \int_0^\infty t^{p-1} \mu(t)\, dt.$$

Proof: Clearly

$$\begin{aligned} \mu(t) \cdot t^p &\leq \int_{t \leq |f|} |f|^p\, dx \\ &\leq \int_\Omega |f|^p\, dx \end{aligned}$$

for all $t > 0$ and hence (9.22) follows. Next, suppose that $f \in L^1(\Omega)$. Then, by Fubini's Theorem

$$\begin{aligned} \int_\Omega |f|\,dx &= \int_\Omega \int_0^{|f(x)|} dt dx \\ &= \int_0^\infty \mu(t)\,dt, \end{aligned}$$

and the result (9.23) for general p follows by a change of variables. ■.

8.9 Exercises

8.1: Let $X = C^0([-1,1])$ be the space of all continuous function on the interval $[-1,1]$, $Y = C^1([-1,1])$ be the space of all continuously differentiable functions, and let $Z = C_0^1([-1,1])$ be the space of all function $u \in Y$ with boundary values $u(-1) = u(1) = 0$.

a) Define

$$(u,v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

On which of the spaces X, Y, and Z does $(\cdot,\cdot)$ define a scalar product? Now consider those spaces for which $(\cdot,\cdot)$ defines a scalar product and endow them with the norm $||\cdot|| = (\cdot,\cdot)^{1/2}$. Which of the spaces is a pre-Hilbert space and which is a Hilbert space (if any)?

b) Define

$$(u,v) = \int_{-1}^{1} u'(x)v'(x)\,dx.$$

On which of the spaces X, Y, and Z does $(\cdot,\cdot)$ define a scalar product? Now consider those spaces for which $(\cdot,\cdot)$ defines a scalar product and endow them with the norm $||\cdot|| = (\cdot,\cdot)^{1/2}$. Which of the spaces is a pre-Hilbert space and which is a Hilbert space (if any)?

8.2: Suppose that Ω is an open and bounded domain in $\mathbb{R}^n$. Let $k \in \mathbb{N}$, $k \geq 0$, and $\gamma \in (0,1]$. We define the norm $||\cdot||_{k,\gamma}$ in the following way. The γ^{th}-Holder seminorm on Ω is given by

$$[u]_{\gamma;\overline{\Omega}} = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma},$$

and we define the maximum norm by

$$|u|_{\infty;\overline{\Omega}} = \sup_{x\in\Omega} |u(x)|.$$

The Holder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\Omega)$ for which the norm

$$||u||_{k,\gamma;\overline{\Omega}} = \sum_{|\alpha|\leq k} ||D^\alpha u||_{\infty;\overline{\Omega}} + \sum_{|\alpha|=k} [D^\alpha u]_{\gamma;\overline{\Omega}}$$

is finite. Prove that $C^{k,\alpha}(\overline{\Omega})$ endowed with the norm $||\cdot||_{k,\gamma;\overline{\Omega}}$ is a Banach space.

Hint: You may prove the result for $C^{0,\gamma}([0,1])$ if you indicate what would change for the general result.

8.3: Consider the shift operator $T : l^2 \to l^2$ given by

$$(\mathbf{x})_i = x_{i+1}.$$

Prove the following assertion. It is true that

$$\mathbf{x} - T\mathbf{x} = 0 \iff \mathbf{x} = 0,$$

however, the equation

$$\mathbf{x} - T\mathbf{x} = \mathbf{y}$$

does not have a solution for all $\mathbf{y} \in l^2$. This implies that T is not compact and the Fredholm's alternative cannot be applied. Give an example of a bounded sequence $\mathbf{x}_j$ such that $T\mathbf{x}_j$ does not have a convergent subsequence.

8.4: The space l^2 is a Hilbert space with respect to the scalar product

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} x_i y_i.$$

Use general theorems to prove that the sequence of unit vectors in l^2 given by $(\mathbf{e}_i)_j = \delta_{ij}$ contains a weakly convergent subsequence and characterize the weak limit.

8.5: Find an example of a sequence of integrable functions f_i on the interval $(0,1)$ that converge to an integrable function f a.e. such that the identity

$$\lim_{i\to\infty} \int_0^1 f_i(x)\, dx = \int_0^1 \lim_{i\to\infty} f_i(x)\, dx$$

does not hold.

8.6: Let $a \in C^1([0,1])$, $a \geq 1$. Show that the ODE

$$-(au')' + u = f$$

has a solution $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ for all $f \in C^0([0,1])$.

Hint: Use the method of continuity and compare the equation with the problem $-u'' = f$. In order to prove the a-priori estimate derive first the estimate

$$\sup |tu_t| \leq \sup |f|$$

where u_t is the solution corresponding to the parameter $t \in [0,1]$.

8.7: Let $\lambda \in \mathbb{R}$. Prove the following alternative: Either (i) the homogeneous equation

$$-u'' - \lambda u = 0$$

has a nontrivial solution $u \in C^([0,1])$ with $u(0) = u(1) = 0$ or (ii) for all $f \in C^0([0,1])$, there exists a unique solution $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ of the equation

$$-u'' - \lambda u = f.$$

Moreover, the mapping $R_\lambda : f \mapsto u$ is bounded as a mapping from $C^0([0,1])$ into $C^2([0,1])$.

Hint: Use Arzela-Ascoli.

8.8: Let $1 < p \leq \infty$ and $\alpha = 1 - \frac{1}{p}$. Then there exists a constant C that depends only on a, b, and p such that for all $u \in C^1([a,b])$ and $x_0 \in [a,b]$ the inequality

$$||u||_{C^{0,\alpha}([a,b])} \leq |u(x_0)| + C||u'||_{L^p([a,b])}$$

holds.

8.9: Let $X = L^2(0,1)$ and

$$M = \left\{ f \in X : \int_0^1 f(x)\, dx = 0 \right\}.$$

Show that M is a closed subspace of X. Find $M^\perp$ with respect to the notion of orthogonality given by the L^2 scalar product

$$(\cdot,\cdot) : X \times X \to \mathbb{R}, \quad (u,v) = \int_0^1 u(x)v(x)\, dx.$$

According to the projection theorem, every $f \in X$ can be written as $f = f_1 + f_2$ with $f_1 \in M$ and $f_2 \in M^{\perp}$. Characterize f_1 and f_2.

8.10: Suppose that $p_i \in (1, \infty)$ for $i = 1, \dots, n$ with

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1.$$

Prove the following generalization of Holder's inequality: if $f_i \in L^{p_i}(\Omega)$ then

$$f = \prod_{i=1}^{n} f_i \in L^1(\Omega) \quad \text{and} \quad \left|\left|\prod_{i=1}^{n} f_i\right|\right|_{L^1(\Omega)} \leq \prod_{i=1}^{n} ||f_i||_{L^{p_i}(\Omega)}.$$

8.11: [Gilbarg-Trudinger, Exercise 7.1] Let Ω be a bounded domain in $\mathbb{R}^n$. If u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) = \left(\frac{1}{|\Omega|} \int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}.$$

Show that

$$\lim_{p \to \infty} \Phi_p(u) = \sup_\Omega |u|.$$

8.12: Suppose that the estimate

$$||u - u_\Omega||_{L^p(\Omega)} \leq C_\Omega ||Du||_{L^p(\Omega)}$$

holds for $\Omega = B(0,1) \subset \mathbb{R}^n$ with a constant $C_1 > 0$ for all $u \in W_0^{1,p}(B(0,1))$. Here u_Ω denotes the mean value of u on Ω, that is

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(z) \, dz.$$

Find the corresponding estimate for $\Omega = B(x_0, R)$, $R > 0$, using a suitable change of coordinates. What is $C_{B(x_0,R)}$ in terms of C_1 and R?

8.13: [Qualifying exam 08/99] Let $f \in L^1(0,1)$ and suppose that

$$\int_0^1 f\phi' \, dx = 0 \qquad \forall \phi \in C_0^\infty(0,1).$$

Show that f is constant.

Hint: Use convolution, i.e., show the assertion first for $f_\epsilon = \rho_\epsilon * f$.

8.14: Let $g \in L^1(a,b)$ and define f by

$$f(x) = \int_a^x g(y)\ dy.$$

Prove that $f \in W^{1,1}(a,b)$ with $Df = g$.

Hint: Use the fundamental theorem of Calculus for an approximation g_k of g. Show that the corresponding sequence f_k is a Cauchy sequence in $L^1(a,b)$.

8.15: [Evans 5.10 #8,9] Use integration by parts to prove the following interpolation inequalities.

a) If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\int_\Omega |Du|^2\ dx \le C \left(\int_\Omega u^2\ dx \right)^{\frac{1}{2}} \left(\int_\Omega |D^2 u|^2\ dx \right)^{\frac{1}{2}}$$

b) If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p \in [2,\infty)$, then

$$\int_\Omega |Du|^p\ dx \le C \left(\int_\Omega |u|^p\ dx \right)^{\frac{1}{2}} \left(\int_\Omega |D^2 u|^p\ dx \right)^{\frac{1}{2}}.$$

Hint: Assertion a) is a special case of b), but it might be useful to do a) first. For simplicity, you may assume that $u \in C_c^\infty(\Omega)$. Also note that

$$\int_\Omega |Du|^p\ dx = \sum_{i=1}^n \int_\Omega |D_i u|^2 |Du|^{p-2}\ dx.$$

8.16: [Qualifying exam 08/01] Suppose that $u \in H^1(\mathbb{R})$, and for simplicity suppose that u is continuously differentiable. Prove that

$$\sum_{n=-\infty}^{\infty} |u(n)|^2 < \infty.$$

8.17: Suppose that $n \ge 2$, that $\Omega = B(0,1)$, and that $x_0 \in \Omega$. Let $1 \le p < \infty$. Show that $u \in C^1(\Omega \setminus \{x_0\})$ belongs to $W^{1,p}(\Omega)$ if u and its classical derivative ∇u satisfy the following inequality,

$$\int_{\Omega \setminus \{x_0\}} (|u|^p + |\nabla u|^p)\ dx < \infty.$$

Note that the examples from the chapter show that the corresponding result does not hold for $n = 1$. This is related to the question of how big a set in $\mathbb{R}^n$ has to be in order to be "seen" by Sobolev functions.

Hint: Show the result first under the additional assumption that $u \in L^\infty(\Omega)$ and then consider $u_k = \phi_k(u)$ where ϕ_k is a smooth function which is close to

$$\psi_k(s) = \begin{cases} k & \text{if } s \in (k, \infty), \\ s & \text{if } s \in [-k, k], \\ -k & \text{if } s \in (-\infty, -k). \end{cases}$$

You could choose ϕ_k to be the convolution of ψ_k with a fixed kernel.

<u>8.18</u>: Give an example of an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(\Omega)$, such that u is *not* Lipschitz continuous on Ω.

Hint: Take Ω to be the open unit disk in $\mathbb{R}^2$, with a slit removed.

<u>8.19</u>: Let $\Omega = B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^n$, $n \geq 2$. Show that $\log \log \frac{1}{|x|} \in W^{1,n}(\Omega)$.

<u>8.20</u>: [Qualifying exam 01/00]

a) Show that the closed unit ball in the Hilbert space $H = L^2([0,1])$ is not compact.

b) Show that $g : [0,1] \to \mathbb{R}$ is a continuous function and define the operator $T : H \to H$ by

$$(Tu)(x) = g(x)u(x) \quad \text{for } x \in [0,1].$$

Prove that T is a compact operator if and only if the function g is identically zero.

<u>8.21</u>: [Qualifying exam 08/00] Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^3$. For any function $u : \overline{\Omega} \to \mathbb{R}$, define the trace Tu by restricting u to $\partial\Omega$, i.e., $Tu = u\big|_{\partial\Omega}$. Show that $T : L^2(\Omega) \to L^2(\partial\Omega)$ is not bounded.

<u>8.22</u>: [Qualifying exam 01/00] Let $H = H^1([0,1])$ and let $Tu = u(1)$.

a) Explain precisely how T is defined for all $u \in H$, and show that T is a bounded linear functional on H.

b) Let $(\cdot,\cdot)_H$ denote the standard inner product in H. By the Riesz representation theorem, there exists a unique $v \in H$ such that $Tu = (u,v)_H$ for all $u \in H$. Find v.

8.23: Let $\Omega = B(0,1) \subset \mathbb{R}^n$ with $n \geq 2$, $\alpha \in \mathbb{R}$, and $f(x) = |x|^\alpha$. For fixed α, determine for which values of p and q the function f belongs to $L^p(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Is there a relation between the values?

8.24: Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary. Prove that for $p > n$, the space $W^{1,p}(\Omega)$ is a Banach algebra with respect to the usual multiplication of functions, that is, if $f, g \in W^{1,p}(\Omega)$, then $fg \in W^{1,p}(\Omega)$.

8.25: [Evans 5.10 #11] Show by example that if we have $||D^h u||_{L^1(V)} \leq C$ for all $0 < |h| < \frac{1}{2}\text{dist}\,(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(\Omega)$.

8.26: [Qualifying exam 01/01] Let A be a compact and self-adjoint operator in a Hilbert space H. Let $\{u_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H consisting of eigenfunctions of A with associated eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Prove that if $\lambda \neq 0$ and λ is not an eigenvalue of A, then $A - \lambda I$ has a bounded inverse and

$$||(A-\lambda I)^{-1}|| \leq \frac{1}{\inf_{k\in\mathbb{N}} |\lambda_k - \lambda|} < \infty.$$

8.27: [Qualifying exam 01/03] Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary.

a) For $f \in L^2(\Omega)$ show that there exists a $u_f \in H_0^1(\Omega)$ such that

$$||f||^2_{H^{-1}(\Omega)} = ||u_f||^2_{H_0^1(\Omega)} = (f, u_f)_{L^2(\Omega)}.$$

b) Show that $L^2(\Omega) \subset\subset\hookrightarrow H^{-1}(\Omega)$. You may use the fact that $H_0^1(\Omega) \subset\subset\hookrightarrow L^2(\Omega)$.

8.28: [Qualifying exam 08/02] Let Ω be a bounded, connected and open set in $\mathbb{R}^n$ with smooth boundary. Let V be a closed subspace of $H^1(\Omega)$ that does *not* contain nonzero constant functions. Using $H^1(\Omega) \subset\subset\hookrightarrow L^2(\Omega)$, show that there exists a constant $C < \infty$, independent of u, such that for $u \in V$,

$$\int_\Omega |u|^2\, dx \leq C \int_\Omega |Du|^2\, dx.$$

Chapter 9

Function Spaces

9.1 Function Spaces: Introduction

A central goal to the theoretical study of PDE is to ascertain properties of solutions of PDE that are not directly attainable by direct analytic means. To this end, one usually seeks to classify solutions of PDE belonging to certain functions spaces whose elements have certain known properties. This is, by no means, trivial, as the process requires not only considering/defining appropriate function spaces but also elucidating the properties of such. In this chapter we will consider some of the most prolific spaces in theoretical PDE along with some of their respective properties.

9.2 Hölder Spaces

To start the discussion on function spaces, Hölder spaces will first be considered as they are easier to construct than Sobolev Spaces.

Consider open $\Omega \subset \mathbb{R}^n$ and $0 < \gamma \leq 1$. Recall that a Lipschitz continuous function $u : \Omega \to \mathbb{R}$ satisfies

$$|u(x) - u(y)| \leq C|x - y| \quad (x, y \in \Omega)$$

for some constant C. Of course, this estimates implies that u is continuous with a uniform modulus of continuity. A generalization of the Lipschitz condition is

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad (x, y \in \Omega)$$

for some constant C. If this estimate holds for a function u, that function is said to be *Hölder continuous with exponent* γ.

Definition 9.1. *(i) If* $u : \Omega \to \mathbb{R}$ *is bounded and continuous, we write*

$$||u||_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)|.$$

(ii) The γ^{th}*-Hölder seminorm of* $u : \Omega \to \mathbb{R}$ *is*

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\},$$

and the γ^{th}*-Hölder norm is*

$$||u||_{C^{0,\gamma}(\overline{\Omega})} := ||u||_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}$$

Definition 9.2. *The Hölder space*

$$C^{k,\gamma}(\overline{\Omega})$$

consists of all functions $u \in C^k(\overline{\Omega})$ *for which the norm*

$$||u||_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})}$$

is finite.

explain exactly what you mean by continuity of derivatives in terms of continuous extension... refer to your explanation in your AMSI summer notes.

So basically, $C^{k,\gamma}(\overline{\Omega})$ consists of those functions u which are k-times continuous differentiable with the k^{th}-partial derivatives Hölder continuous with exponent γ. As one can easily ascertain, these functions are very well-behaved, but these spaces also have a very good mathematical structure.

Theorem 9.3. *The function spaces* $C^{k,\gamma}(\overline{\Omega})$ *are Banach spaces.*

The proof of this is straight-forward and is left as an exercise for the reader.

9.3 Lebesgue Spaces

<u>Notation:</u> E is a measurable set, and Ω is an open set $\subset \mathbb{R}^n$.

Definition 9.4. *For $1 \leq p < \infty$ we define*

$$\begin{aligned}\mathcal{L}^p(E) &= \{f : E \to \mathbb{R},\ \textit{measurable},\ \int_E |f|^p < \infty\} \\ \mathcal{L}^\infty(E) &= \{f : E \to \mathbb{R},\ \textit{ess sup } |f| < \infty\}\end{aligned}$$

where ess sup$|f| = \inf\{K \geq 0, |f| \leq K \text{ a.e.}\}$.

Two functions are said to be equivalent if $f = g$ a.e. Then we define

$$L^p(E) = \mathcal{L}^p(E)/\sim$$

Theorem 9.5. *The spaces $L^p(E)$ are BS with*

$$\begin{aligned}||f||_p &= \left[\int_E |f|^p \, d\lambda\right]^{1/p} \qquad 1 \leq p < \infty \\ ||f||_\infty &= \textit{ess } \sup |f|\end{aligned}$$

- $L^p(E) = \{f : E \to \mathbb{R}, \int |f|^p \, dx < \infty\}$. These are indeed equivalence classes of functions (as stated in the last lecture). In general we can regard this as a space of functions. We do however, need to be careful sometimes. The main such point is that saying that $f \in L^p$ is continuous means: there exists a representative of f that is continuous. Sometimes referred to as "a version of f".
- $L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R}$ measurable. $\forall \Omega' \subset\subset \Omega,\ \int_{\Omega'} |f|^p \, dx < \infty\}$

Example: $L^1((0,1))$
$f(x) = \frac{1}{x} \notin L^1((0,1)),\ f \in L^1_{\text{loc}}((0,1))$

Theorem 9.6 (Young's Inequality). $1 < p < \infty, a, b \geq 0$

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'},$$

where p' is the dual exponent to p defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

General convention $(1)' = \infty$, $(\infty)' = 1$.

Proof: $a, b \neq 0$, ln is convex downward (concave). Thus,

$$\begin{aligned}\ln(\lambda x + (1-\lambda)x) &\geq \lambda \ln x + (1-\lambda) \ln y \\ &= \ln x^\lambda + \ln y^{1-\lambda}\end{aligned}$$

take exponential of both sides

$$\lambda x + (1-\lambda)y > x^{\lambda}y^{1-\lambda}$$

Define $\lambda = \frac{1}{p}$, $x^{\lambda} = a$, $y^{1-\lambda} = b \Longrightarrow$ Young's inequality.

$$1 - \lambda = 1 - \frac{1}{p} = \frac{1}{p'} \qquad \blacksquare$$

Theorem 9.7 (Holder's Inequality). *$f \in L^p(E), g \in L^{p'}(E)$, then $fg \in L^1(E)$ and*

$$\left| \int_E fg \right| \leq ||f||_p ||g||_{p'}$$

<u>Proof:</u> Young's inequality

$$\frac{|g|}{||g||_{p'}} \cdot \frac{|f|}{||f||_p} \leq \frac{1}{p}\frac{|f|^p}{||f||_p^p} + \frac{1}{p'}\frac{|g|^{p'}}{||g||_{p'}^{p'}}$$

Clear for $p = 1$ and $p = \infty$. Assume $1 < p < \infty$ to use Young's inequality. Integrate on E.

$$\begin{aligned} \int_E \frac{|g|}{||g||_{p'}} \cdot \frac{|f|}{||f||_p} &\leq \frac{1}{p}\int_E \frac{|f|^p}{||f||_p^p} + \frac{1}{p'}\int_E \frac{|g|^{p'}}{||g||_{p'}^{p'}} = \frac{1}{p} + \frac{1}{p'} = 1 \\ \Longrightarrow \int_E |f| \cdot |g| \, dx &\leq ||f||_p ||g||_{p'} \qquad \blacksquare \end{aligned}$$

<u>Remark:</u> Sometimes $f = 1$ is a good choice

$$\begin{aligned} \int_E 1 \cdot |g| \, dx &\leq \left(\int_E 1^p \right)^{\frac{1}{p}} \left(\int_E g^{p'} \right)^{\frac{1}{p'}} \\ &= |E|^{\frac{1}{p}} ||g||_{p'} \end{aligned}$$

You gain $|E|^{\frac{1}{p}}$.

Lemma 9.8 (Minkowski's inequality). *Let $1 \leq p \leq \infty$, then $||f+g||_p \leq ||f||_p + ||g||_p$.*

This is clear for $p = 1$ and $p = \infty$.

Assume $1 < p < \infty$. Convexity of x^p implies that

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p) \Longrightarrow f + g \in L^p$$

$$\begin{aligned}|f+g|^p &= |f+g||f+g|^{p-1}\\ &\leq \underbrace{|f|}_{L^p}\underbrace{|f+g|^{p-1}}_{L^{p'}}+\underbrace{|g|}_{L^p}\underbrace{|f+g|^{p-1}}_{L^{p'}}\end{aligned}$$

$p' = \frac{p}{p-1}$, $(|f+g|^{p-1})^{p'} = |f+g|^p$. Integrate and use Holder

$$\begin{aligned}\int |f+g|^p\ dx &\leq ||f||_p\left(\int |f+g|^{(p-1)p'}\right)^{\frac{1}{p'}} + ||g||_p\left(\int |f+g|^{(p-1)p'}\right)^{\frac{1}{p'}}\\ &= (||f||_p+||g||_p)\left(\int |f+g|^p\right)^{\frac{1}{p'}}\\ &= (||f||_p+||g||_p)\,||f+g||_p^{\frac{p}{p'}}\end{aligned}$$

Thus, we have

$$||f+g||_p^{p-\frac{p}{p'}} \leq ||f||_p + ||g||_p$$

Now $p-\frac{p}{p'} = p\left(1-\frac{1}{p'}\right) = p\frac{1}{p} = 1$. Thus we have the result.

Proof of 9.5(L^p is complete) $p=\infty$ ("simpler than $1 \leq p < \infty$").
If f_n is Cauchy, then $f_n(x)$ is Cauchy for a.e. x and you can define the limit $f(x)$ a.e.

Theorem (Riesz-Fischer) L^p complete for $1 \leq p < \infty$

Proof: Suppose that f_k is Cauchy, $\forall \epsilon > 0, \exists k(\epsilon)$ such that

$$||f_k - f_l|| < \epsilon \quad \text{if} \quad k, l \geq k(\epsilon)$$

Choose a subsequence that converges fast in the sense that

$$\sum_{i=1}^{\infty} ||f_{k_{i+1}} - f_{k_i}||_p < \infty$$

relabel the subsequence and call it again f_k. Now define

$$g_l = \sum_{k=1}^{l} |f_{k+1} - f_k|$$

then $g_l \geq 0$ and by Minkowski's inequality, $||g_l||_p \leq C$. By Fatou's Lemma

$$\begin{aligned}\int \lim_{l\to\infty} g_l^p \, dx &\leq \lim_{l\to\infty} \inf \int g_l^p \, dx \\ &\leq \left(\lim_{l\to\infty} \inf ||g_l||_p\right)^p < \infty\end{aligned}$$

Thus $\lim_{l\to\infty} g_l^p$ is finite for a.e. x, $\lim_{k\to\infty} g_l(x)$ exists $\Longrightarrow f_k(x)$ is a Cauchy sequence for a.e. x. Thus $\lim_{k\to\infty} f_k(x) = f(x)$ exists a.e. By Fatou

$$\begin{aligned}\int |f - f_l|^p \, dx &= \int \lim_{k\to\infty} |f_k(x) - f_l(x)|^p \, dx \\ &\leq \lim_{k\to\infty} \inf \int |f_k - f_l|^p \, dx \\ &\leq \left(\lim_{k\to\infty} \inf ||f_k - f_l||_p\right)^p \\ &\leq \left(\lim_{k\to\infty} \inf \sum_{k=l}^{\infty} ||f_k - f_{k+1}||_p\right)^p \to 0 \text{ as } l \to \infty \\ &\Longrightarrow ||f_l - f||_p \to 0\end{aligned}$$

We know f is in L^p by a simple application of Minkowski's inequality to the last result. ■

Sobolev Spaces $\Longleftrightarrow f \in L^p$ with derivatives.

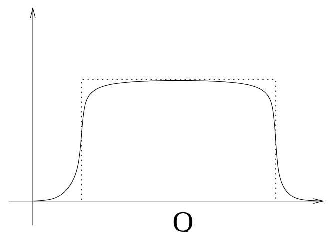
Calculus for smooth functions is linked to the calculus of Sobolev functions by ideas of approximation via convolution methods.

Theorem 9.9. *(see Royden for proof) Let $1 \leq p < \infty$. Then the space of continuous functions with compact support is dense in in L^p, i.e.*

$$C_c^0(\Omega) \subset L^p(\Omega) \text{ dense}$$

Idea of Proof:

1. Step functions $\displaystyle\sum_{i=1}^{\infty} a_i \chi_{E_i}$ are dense
2. Approximate χ_{E_i} by step functions on cubes.
3. Construct smooth approximation for χ_O, for cube O



9.3.1 Convolutions and Mollifiers

- Sobolev functions = L^p functions with derivatives
- Convolution = approximation of L^p functions by smooth function approximation of the Dirac δ
  - $\phi \in C_c^\infty(\mathbb{R}^n)$, $\text{supp}(\phi) \in B(0,1)$
  - $\phi \geq 0$, $\displaystyle\int_{\mathbb{R}^n} \phi(x)\ dx = 1$

Rescaling: $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right)$, $\displaystyle\int_{\mathbb{R}^n} \phi_\epsilon(x)\ dx = 1$. $\phi_\epsilon(x) \to 0$ as $\epsilon \to 0$ pointwise a.e. and $\displaystyle\int_{\mathbb{R}^n} \phi_\epsilon(x)\ dx = 1$.

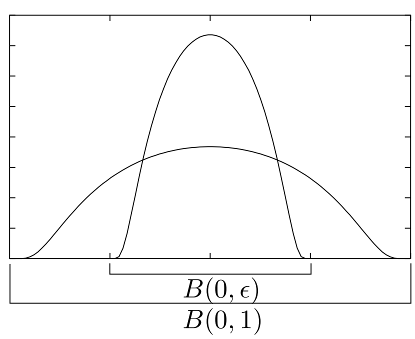


Figure 9.1:

Definition 9.10. *The convolution,*

$$\begin{aligned} f_\epsilon(x) &= (\phi_\epsilon * f)(x) \\ &= \int_{\mathbb{R}^n} \phi_\epsilon(x-y) f(y)\ dy \end{aligned}$$

$$= \int_{B(x,\epsilon)} \phi_\epsilon(x-y)f(y)\ dy$$

is the weighted average of f on a ball of radius $\epsilon > 0$.

Theorem 9.11. $f_\epsilon \in C^\infty$

Proof: Difference Quotients and Lebesgue's Dominated convergence theorem. ■

Now, we ask the question $f \in L^p$, $f_\epsilon \to f$ in L^p. If $p = \infty$ $f_\epsilon \to f$ implies $f \in C^0$ which implies $f \in L^p$. We know $f \in C^0$ since $f_\epsilon \in C^\infty \subset C^0$.

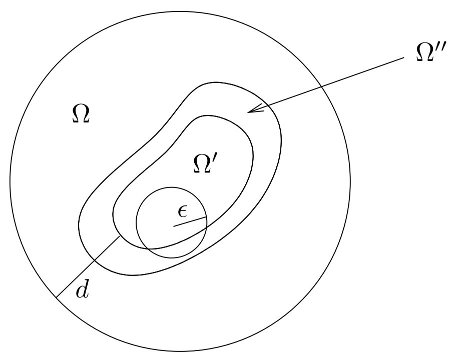


Figure 9.2:

Lemma 9.12. *$f \in C^0(\Omega)$, $\Omega'' \subset\subset \Omega$ compact subsets, then $f_\epsilon \to f$ uniformly on Ω''!*

Proof:

$$\begin{aligned} d &= \operatorname{dist}(\Omega', \partial\Omega) \\ &= \inf\{|x-y| : x \in \Omega', y \in \partial\Omega\} \end{aligned}$$

$\Omega'' = \{x \in \Omega, \operatorname{dist}(x, \Omega') \leq \frac{d}{2}\}$.
Let $\epsilon \leq \frac{d}{2}$, $x \in \Omega' \Longrightarrow B(x,\epsilon) \subset\subset \Omega''$, and $f_\epsilon(x)$ is well defined

$$\begin{aligned} |f_\epsilon(x) - f(x)| &= \left| \int_{\mathbb{R}^n} \phi_\epsilon(x-y)[f(y) - f(x)]\ dy \right| \\ &\leq \sup_{y \in B(x,\epsilon)} |f(y) - f(x)| \underbrace{\left| \int_{\mathbb{R}^n} \phi_\epsilon(x-y)\ dy \right|}_{=1} \end{aligned}$$

——

$f \in C^0(\Omega) \Longrightarrow f$ uniformly continuous on $\Omega'' \subset\subset \Omega$. Thus,

$$\omega(\epsilon) = \sup\{|f(x) - f(y)|,\ x, y \in \Omega'', |x - y| < \epsilon\} \to 0$$
as $\epsilon \to 0$

——

$$\leq \quad \omega(\epsilon) \to 0 \text{ uniformly on } \ \Omega'$$
as $\epsilon \to 0$ ■

Proposition 9.13. *If $u, v \in L^1(\mathbb{R}^n)$, then $u * v \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. This is a special case of the following: given $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty]$, then $f * g \in L^p(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$.*

Proof: We will prove the specialized case ($p = 1$) first.

$$\begin{aligned}
||f * g||_{L^1(\mathbb{R}^n)|} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\ dy \right|\ dx \\
&\leq \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x-y)|\ dxdy \\
&= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(z)|\ dzdy = ||f||_{L^1(\mathbb{R}^n)} ||g||_{L^1(\mathbb{R}^n)}
\end{aligned}$$

There is nothing to show for $p = \infty$. For $p \in (1, \infty)$, we use Holder's inequality to compute

$$\begin{aligned}
||f * g||^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y)g(x-y)\ dy \right|^p\ dx \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|^{\frac{1}{p'}} |f(y)|^{\frac{1}{p}} |g(x-y)|\ dy \right)^p\ dx \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|\ dy \right)^{\frac{p}{p'}} \left(\int_{\mathbb{R}^n} |f(y)| \cdot |g(x-y)|^p\ dy \right)\ dx \\
&= ||f||^{p/p'}_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)| \cdot |g(x-y)|^p\ dydx \\
&= ||f||^{p/p'}_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)|^p\ dydx \\
&= ||f||^{p/p'}_{L^1(\mathbb{R}^n)} ||f||_{L^1(\mathbb{R}^n)} ||g||^p_{L^p(\mathbb{R}^n)} = ||f||^p_{L^1(\mathbb{R}^n)} ||g||^p_{L^p(\mathbb{R}^n)}
\end{aligned}$$

which proves the assertion. ■

Lemma 9.14. *$f \in L^p(\Omega)$, $1 \le p < \infty$. Then $f_\epsilon \to f$ in $L^p(\Omega)$.*

Remarks:

- The same assertion holds in $L^p_{\text{loc}}(\Omega)$.
- $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Proof: Extend f by 0 to $\mathbb{R}^n$: assume $\Omega = \mathbb{R}^n$. By Proposition 8.31, we see that if $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, then $f*g \in L^p(\mathbb{R}^n)$ and $||f*g||_p \le ||f||_p||g||_1$. So, take

$$g = \phi_\epsilon : ||f_\epsilon||_p = ||f * \phi_\epsilon||_p \le ||f||_p \cdot ||\phi_\epsilon||_1 = ||f||_p \tag{9.1}$$

Need to show $\forall\delta > 0$, $\exists\epsilon(\delta) > 0$, $||f_\epsilon - f||_p < \delta$, $\forall\epsilon \le \epsilon(\delta)$.

By Theorem 8.29, $\exists f_1 \in C_c^0(\Omega)$ such that

$$f = f_1 + f_2,\ ||f_2||_p = ||f - f_1||_p < \frac{\delta}{3}$$

$$\begin{aligned} ||f_\epsilon - f|| &= ||(f_1 + f_2)_\epsilon + (f_1 + f_2)||_p \\ &\le ||f_{1\epsilon} - f_1||_p + ||f_{2\epsilon} - f_2||_p \\ &\le ||f_{1\epsilon} - f_1||_p + ||f_{2\epsilon}||_p + ||f||_p \\ &\le ||f_{1\epsilon} - f_1||_p + \frac{2\delta}{3} \qquad \text{by (9.1)} \end{aligned}$$

f_1 has compact support in Ω, $\text{supp}(f_1) \subset \Omega' \subset\subset \Omega$, $d = \text{dist}(\Omega', \partial\Omega)$, $\Omega'' = \left\{x \in \Omega', \text{dist}(x, \Omega') \le \frac{d}{2}\right\}$. Now suppose that $\epsilon < \frac{d}{2}$. By Lemma 8.30, $f_{1\epsilon} \to f_1$ uniformly in Ω'' and thus in Ω (since $f_1, f_{1\epsilon} = 0$ in $\Omega \setminus \Omega''$). Now choose ϵ_0 small enough such that $||f_{1\epsilon} - f||_p < \frac{\delta}{3}$
$\Longrightarrow ||f_\epsilon - f||_p \le \delta\ \forall\epsilon(\delta) \le \left\{\epsilon_0, \frac{\delta}{2}\right\}$. ■

9.3.2 The Dual Space of L^p

Explicit example of $T \in (L^p)^*$:

$$T_g : L^p \to \mathbb{R}, \qquad T_g f = \int fg\ dx$$

$T_g f$ is defined by Holder, and

$$|T_g f| \le ||f||_p ||g||_{p'}$$

Thus,

$$||Tg|| = \sup_{f\in L^p, f\neq 0} \frac{|T_g f|}{||f||_p} \le \sup_{f\in L^p, f\neq 0} \frac{||f||_p ||g||_{p'}}{||f||_p} = ||g||_{p'}$$

In fact $||Tg|| = ||g||_{p'}$. This follows by choosing $f = |g|^{p'-2}g$.

Theorem 9.15. *Let $1 \le p < \infty$,*
Then $J : L^{p'} \to (L^p)^*$, $J(g) = T_g$ is onto with $||J(g)|| = ||g||_{p'}$. In particular, L^p is reflexive for $1 < p < \infty$.

Remark: $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$.

9.3.3 Weak Convergence in L^p

- $1 \le p < \infty$, $f_j \rightharpoonup f$ weakly in L^p
$$\int f_j g \to \int fg \qquad \forall g \in L^{p'}$$
- $p = \infty$, $f_j \overset{*}{\rightharpoonup} f$ weakly* in L^∞
$$\int f_j g \to \int fg \qquad \forall g \in L^1$$

Examples: Obstructions to strong convergence. (1) Oscillations. (2) Concentrations.

- Oscillations: $\Omega = (0,1)$, $f_j(x) = \sin(jx)$. $f_j \overset{*}{\rightharpoonup} 0$ in $L^{*\infty}$, but $f_j \nrightarrow 0$ in L^∞. ($||0 - \sin(jx)||_\infty = 1$).
- Concentration: $g \in C_c^\infty(\mathbb{R}^n)$, $f_j(x) = j^{-n/p}g(j^{-1}x)$. So,
$$\int |f_j|^p \, dx = \int j^{-n} g^p(j^{-1}x)\, dx = \int |g|^p \, dx \qquad 1 < p < \infty$$
So we see that $f_j \rightharpoonup 0$ in L^p, but $f_j \nrightarrow 0$ in L^p.

Last Lecture: We showed $(L^p)^* = L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \le p < \infty$. With this we can talk about

- Weak-* convergence in L^∞.
- Weak convergence in L^p, $1 \le p < \infty$.

So, as an example take weak convergence in $L^2(\Omega)$, $u_j \rightharpoonup u$ in $L^2(\Omega)$ if

$$\int_\Omega u_j v \, dx \to \int_\Omega uv \, dx \qquad \forall v \in L^2(\Omega).$$

Obstructions to strong convergence which have weak convergence properties are oscillations and concentrations.

9.4 Morrey and Companato Spaces

Loosely speaking, Morrey Spaces and Companato Spaces characterize functions based on their "oscillations" in any given neighborhood in a set domain. This will be made clear by the definitions. Philosophically, Morrey spaces are a precursor of Companato Spaces which in turn are basically a modern re-characterization of the Hölder Spaces introduced earlier in the chapter. This new characterization is based on integration; and hence, is easier to utilize, in certain proofs, when compared against the standard notion of Hölder continuity.

It would have been possible to omit these spaces from this text altogether (and this is the convention in books at this level); but I have elected to discuss these for the two following reasons. First, the main theorem pertaining to the imbedding of Sobolev Spaces into Hölder spaces will be a natural consequence of the properties of Companato Spaces; this is much more elegant, in the author's opinion, than other proofs using standard properties of Hölder continuity. Second, we will utilize these spaces in the next chapter to derive the classical Schauder regularity results for elliptic systems of equations. Again, it is the authors opinion that these regularity results are proved much more intuitively via the use of Companato spaces as opposed to the more standard proofs of yore. For the following introductory exposition, we essentially follow [**?**].

Per usual, we consider Ω to be open and bounded in $\mathbb{R}^n$; but in addition to this, we will also assume for all $x \in \Omega$ and for all $\rho \leq \operatorname{diam}(\Omega)$ that

$$|B_\rho(x) \cap \Omega| \geq A\rho^n, \tag{9.2}$$

where $A > 0$ is a fixed constant. This is, indeed, a condition on the boundary of Ω that precludes domains with "sharp outward cusps"; geometrically, it can be seen that any Lipschitz domain will satisfy (9.2). With that we make our first definition.

Definition 9.16. *The* <u>*Morrey Space*</u> $L^{p,\lambda}(\Omega)$ *for all real* $p \geq 1$ *and* $\lambda \geq 0$ *is defined as*

$$L^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, ||u||_{L^{p,\lambda}(\Omega)} < \infty \right\},$$

where

$$||u||^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ 0 < \rho < diam(\Omega)}} \left(\rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p \, dx \right)$$

with $\Omega(x_0, \rho) = B(x_0, \rho) \cap \Omega$.

It is left up to the reader to verify that $L^{p,\lambda}(\Omega)$ is indeed a BS. One can easily see from the definition that $u \in L^{p,\lambda}(\Omega)$ if and only if there exists a constant $C > 0$ such that

$$\int_{\Omega(x_0,\rho)} |u|^p \, dx < C\rho^\lambda, \tag{9.3}$$

for all $x_0 \in \Omega$ with $0 < \rho < \rho_0$ for some fixed $\rho_0 > 0$. Sometimes, (9.3) is easier to use than the original definition of a Morrey Space. Also, since $x_0 \in \Omega$ is arbitrary in (9.3), it really doesn't matter what we set $\rho_0 > 0$, as Ω is assumed to be bounded (i.e. it can be covered by a finite number of balls of radius ρ_0).

In addition to this, we get that $L^{p,\lambda}(\Omega) \cong L^p(\Omega)$ for free from the definition. Now, we come to our first theorem concerning these spaces.

Theorem 9.17. *If* $\dfrac{n-\lambda}{p} \geq \dfrac{n-\mu}{q}$ *and* $p \leq q$, *then* $L^{q,\mu}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega)$.

Proof: First note that the index condition above can be rewritten as

$$\frac{\mu p}{q} + n - \frac{np}{q} \geq \lambda.$$

With this in mind, the rest of the proof is a straight-forward calculation using Hölder's inequality:

$$\begin{aligned}
\int_{\Omega(x_0,\rho)} |u|^p \, dx &\overset{\text{Hölder}}{\leq} \left(\int_{\Omega(x_0,\rho)} |u|^q \, dx \right)^{p/q} |\Omega(x_0,\rho)|^{1-p/q} \\
&\leq \alpha(n)^{1-p/q} \cdot \left(||u||^q_{L^{q,\mu}(\Omega)} \cdot \rho^\mu \right)^{p/q} \cdot \rho^{n(1-p/q)} \\
&\leq \alpha(n)^{1-p/q} \cdot ||u||^q_{L^{q,\mu}(\Omega)} \cdot \rho^{\mu p/q + n - np/q} \\
&\leq \alpha(n)^{1-p/q} \cdot ||u||^q_{L^{q,\mu}(\Omega)} \cdot \rho^\lambda.
\end{aligned}$$

Note in the third inequality utilizes the assumption $p \leq q$. The above calculation implies that

$$||u||^p_{L^{p,\lambda}(\Omega)} \leq C||u||^q_{L^{q,\mu}(\Omega)},$$

i.e. $L^{q,\mu}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega)$. ■

$L^{p,\lambda}(\Omega) \hookrightarrow L^p(\Omega)$ *for any* $\lambda > 0$.

Proof: This is a direct consequence of the previous theorem.

The next two theorems are more specific index relations for Morrey Spaces, but are very important properties nonetheless.

Theorem 9.19. *If dim*$(\Omega) = n$, *then* $L^{p,n}(\Omega) \cong L^\infty(\Omega)$.

Proof: First consider $u \in L^\infty(\Omega)$. In this case,

$$\begin{aligned} \int_{\Omega(x_0,\rho)} |u|^p \, dx &\overset{\text{Hölder}}{\leq} C \cdot |\Omega(x_0,\rho)| \cdot ||u||^p_{L^\infty(\Omega)} \\ &= C \cdot \alpha(n)\rho^n \cdot ||u||^p_{L^\infty(\Omega)} < \infty, \end{aligned}$$

i.e. $u \in L^{p,n}(\Omega)$. If, on the other hand, $u \in L^{p,n}(\Omega)$, then, considering for a.e. $x_0 \in \Omega$ we have

$$u(x_0) = \lim_{\rho \to 0} \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u \, dx$$

via Lebesgue's Theorem, it is calculated that

$$|u(x_0)| \leq \lim_{\rho \to 0} \fint_{\Omega(x_0,\rho)} |u| \, dx \leq \lim_{\rho \to 0} \left(\fint_{\Omega(x_0,\rho)} |u|^p \, dx \right)^{1/p} < \infty.$$

Thus, we conclude $u \in L^\infty(\Omega)$. ■

Theorem 9.20. *If dim*$(\Omega) = n$, *then* $L^{p,\lambda}(\Omega) = \{0\}$ *for* $\lambda > n$.

Proof: Fix an arbitrary $x_0 \in \Omega$, using Lebesgue's Theorem, we calculate

$$\begin{aligned}
|u(x_0)| &\leq \lim_{\rho \to 0} \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} |u|^p\,dx \\
&\overset{\text{Hölder}}{\leq} \lim_{\rho \to 0} \frac{|\Omega(x_0,\rho)|^{1-1/p}}{|\Omega(x_0,\rho)|^{1/p}} \left(\int_{\Omega(x_0,\rho)} |u|^p\,dx \right)^{1/p} \\
&\leq \lim_{\rho \to 0} \frac{\text{diam}(\Omega)^{1-1/p}}{|\Omega(x_0,\rho)|^{1/p}} \left(\int_{\Omega(x_0,\rho)} |u|^p\,dx \right)^{1/p} \\
&\leq \lim_{\rho \to 0} C \cdot \rho^{-n/p} \rho^{\lambda/p} \\
&= \lim_{\rho \to 0} C \cdot \rho^{\lambda - n/p} = 0,
\end{aligned}$$

since $\lambda - n > 0$. As $x_0 \in \Omega$ is arbitrary, we see that $u \equiv 0$. ■.

Now, that a few of the basic properties of Morrey Spaces have been elucidated, we will now consider a closely related group of spaces.

Definition 9.21. *The Companato Spaces, $\mathcal{L}^{p,\lambda}(\Omega)$, are defined as*

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, [u]_{\mathcal{L}^{p,\lambda}(\Omega)} < \infty \right\},$$

where

$$[u]^p_{\mathcal{L}^{p,\lambda}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ 0 < \rho < diam(\Omega)}} \rho^{-\lambda} \left(\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p\,dx \right)$$

with

$$u_{x_0,\rho} := \fint_{\Omega(x_0,\rho)} u\,dx.$$

First, one will notice that $[u]_{\mathcal{L}^{p,\lambda}(\Omega)}$ is a seminorm; this is seen from the fact that $[u]_{\mathcal{L}^{p,\lambda}(\Omega)} = 0$ if and only if u is a constant. Thus, we define

$$||\cdot||_{\mathcal{L}^{p,\lambda}(\Omega)} := ||\cdot||_{L^p(\Omega)} + [\,\cdot\,]^p_{\mathcal{L}^{p,\lambda}(\Omega)}$$

as the norm on $\mathcal{L}^{p,\lambda}(\Omega)$, making such a BS. The verification of this is left to the reader.

In analogy with theorem 9.17, we have the following.

Theorem 9.22. *If* $\dfrac{n-\lambda}{p} \geq \dfrac{n-\mu}{q}$ *and* $p \leq q$, *then* $\mathcal{L}^{q,\mu}(\Omega) \hookrightarrow \mathcal{L}^{p,\lambda}(\Omega)$

Proof: See proof of theorem 9.17.

Just from looking at the definitions, naive intuition suggests that there should be some correspondence between Morrey and Companato Spaces, but this correspondence is by no means obvious. The next two theorems will actually elucidate the correspondence between Morrey, Companato *and* Hölder Spaces, but we first prove a lemma that will subsequently be needed in the proofs of the aforementioned theorems.

Lemma 9.23. *If* Ω *is bounded and satisfies (9.2), then for any* $R > 0$,

$$|u_{x_0,R_k} - u_{x_0,R_h}| \leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R_k^{\lambda - n/p}, \tag{9.4}$$

where $R_i := 2^{-i} \cdot R$ *and* $0 < h < k$.

Proof: First we fix an arbitrary $R > 0$ and define r such that $0 < r < R$. Now, we calculate that

$$\begin{aligned}
&r^n |u_{x_0,R} - u_{x_0,r}|^p \\
&\quad \leq \frac{|\Omega(x_0,r)|}{\alpha(n)} |u_{x_0,R} - u_{x_0,r}|^p \\
&\quad \leq \frac{1}{\alpha(n)} \int_{\Omega(x_0,r)} |u_{x_0,R} - u_{x_0,r}|^p \, dx \\
&\quad \leq \frac{2^{p-1}}{\alpha(n)} \left[\int_{\Omega(x_0,r)} |u(x) - u_{x_0,R}|^p + |u(x) - u_{x_0,r}|^p \, dx \right] \\
&\quad \leq \frac{2^{p-1}}{\alpha(n)} \left[\int_{\Omega(x_0,R)} |u(x) - u_{x_0,R}|^p \, dx + \int_{\Omega(x_0,r)} |u(x) - u_{x_0,r}|^p \, dx \right] \\
&\quad \leq \frac{2^{p-1}}{\alpha(n)} \cdot [R^\lambda + r^\lambda] \cdot [u]^p_{\mathcal{L}^{p,\lambda}(\Omega)} \\
&\quad \leq C \cdot R^\lambda [u]^p_{\mathcal{L}^{p,\lambda}(\Omega)}.
\end{aligned}$$

Dividing by r^n and taking the p-th root of this result yields

$$|u_{x_0,R} - u_{x_0,r}| \leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} \cdot R^{\lambda/p} \cdot r^{-n/p}. \tag{9.5}$$

From here, we will utilize the definition of R_i and (9.5) to ascertain

$$\begin{aligned}|u_{x_0,R_i} - u_{x_0,R_{i+1}}| &\leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} \cdot 2^{i(n-\lambda)/p+n/p} \cdot R^{(\lambda-n)/p} \\ &\leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} \cdot 2^{i(n-\lambda)/p} \cdot R^{(\lambda-n)/p}\end{aligned}$$

In the last equality, the $2^{n/p}$ has been absorbed into the constant. Trudging on, we finish the calculation to discover

$$\begin{aligned}|u_{x_0,R} - u_{x_0,R_{h+1}}| &\leq \sum_{i=0}^{h} |u_{x_0,R_i} - u_{x_0,R_{i+1}}| \\ &\leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R^{(\lambda-n)/p} \sum_{i=0}^{h} 2^{i(n-\lambda)/p} \\ &= C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R^{(\lambda-n)/p} \left(\frac{2^{(h+1)(n-\lambda)/p} - 1}{2^{(n-\lambda)/p} - 1} \right) \\ &\leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R_{h+1}^{(\lambda-n)/p}. \qquad (9.6)\end{aligned}$$

The conclusion follows as (9.6) is valid for any $h \geq 0$ and $R > 0$ was chosen arbitrarily. ■

Theorem 9.24. *If dim*$(\Omega) = n$ *and* $0 \leq \lambda \leq n$, *then* $L^{p,\lambda}(\Omega) \cong \mathcal{L}^{p,\lambda}(\Omega)$.

Proof: First suppose $u \in L^{p,\lambda}(\Omega)$ and calculate

$$\begin{aligned}&\rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx \\ &\leq 2^{p-1} \rho^{-\lambda} \left[\int_{\Omega(x_0,\rho)} |u|^p + |u_{x_0,\rho}|^p \, dx \right] \\ &\leq 2^{p-1} \rho^{-\lambda} \left[\int_{\Omega(x_0,\rho)} |u|^p \, dx + |\Omega(x_0,\rho)| \cdot |u_{x_0,\rho}|^p \right] \\ &\overset{\text{Hölder}}{\leq} 2^{p-1} \rho^{-\lambda} \left[\int_{\Omega(x_0,\rho)} |u|^p \, dx + |\Omega(x_0,\rho)| \cdot \fint_{\Omega(x_0 \rho)} |u|^p \, dx \right] \\ &\leq 2^p \rho^{-\lambda} \int_{\Omega(x_0 \rho)} |u|^p \, dx < \infty.\end{aligned}$$

Thus, we see that $u \in \mathcal{L}^{p,\lambda}(\Omega)$.

Now assume $u \in \mathcal{L}^{p,\lambda}(\Omega)$. We find

$$\begin{aligned}
\rho^{-\lambda} &\int_{\Omega(x_0,\rho)} |u|^p \, dx \\
&= \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho} + u_{x_0,\rho}|^p \, dx \\
&\leq 2^{p-1} \rho^{-\lambda} \left[\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx + |\Omega(x_0,\rho)| \cdot |u_{x_0,\rho}|^p \right] \\
&\leq 2^{p-1} \left\{ \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx + C \cdot \rho^{n-\lambda} |u_{x_0,\rho}|^p \right\}.
\end{aligned} \tag{9.7}$$

To proceed with (9.7), we must endeavor to estimate estimate $|u_{x_0,\rho}|$. To do this, we consider $0 < r < R$ along with the calculation

$$|u_{x_0,\rho}|^p \leq 2^{p-1} \{ |u_{x_0,R}|^p + |u_{x_0,R} - u_{x_0,\rho}|^p \}. \tag{9.8}$$

Thus, we now seek to estimate the RHS. To do this, we utilize (9.6) with $\operatorname{diam}(\Omega) < R \leq 2 \cdot \operatorname{diam}(\Omega)$ chosen such that $R_h = \rho$ for some $h > 0$. Inserting this result into (9.8) and (9.8) in (9.7) yields

$$\begin{aligned}
\rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p \, dx \leq 2^{p-1} \Bigg[& \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx \\
& + C \cdot \rho^{n-\lambda} 2^{p-1} \cdot \left| ⨍_{\Omega(x_0,R)} u \, dx \right|^p + C \cdot 2^{p-1} \cdot [u]^p_{\mathcal{L}^{p,\lambda}(\Omega)} \Bigg] < \infty.
\end{aligned}$$

Thus, $u \in L^{p,\lambda}(\Omega)$. ■

Unlike $L^{p,\lambda}(\Omega)$, $L^\infty(\Omega) \subsetneq \mathcal{L}^{p,n}(\Omega)$. To show this we consider the following.

Example 9.1. *Let us look at consider* $\log x$, *where* $x \in (0,1)$. *On any given sub-interval* $[a,b] \subset (0,1)$, *we have*

$$\int_a^b |\log x| \, dx = a \cdot \log a - b \cdot \log b + (b - a).$$

Now, if we multiply by $\rho^{-n} = \frac{2}{b-a}$, *we have*

$$\frac{2}{b-a} \int_a^b |\log x| \, dx = 2 \left(\frac{a \cdot \log a - b \cdot \log b}{b - a} + 1 \right).$$

If we can show that the RHS is bounded for any $0 < a < b < 1$*, then we will have shown that* $\log x \in L^{1,1}((0,1))$*. But choosing* $b := e^{-i}$ *and* $a := e^{-(i+1)}$*, where* $i > 0$ *shows that*

$$\begin{aligned} \lim_{i\to\infty} \frac{2e^i}{1-e^{-1}} \int_{e^{-(i+1)}}^{e^{-i}} |\log x|\,dx &= 2 \cdot \lim_{i\to\infty} \left(\frac{i - e^{-1}(i+1)}{1-e^{-1}} + 1 \right) \\ &= 2 \cdot \lim_{i\to\infty} \left(\frac{(1-e^{-1})i - e^{-1}}{1-e^{-1}} + 1 \right) \\ &= \infty. \end{aligned}$$

Thus, $\log x \notin L^{1,1}(\Omega)$ *which is in agreement with theorem 9.19. Now considering the same interval* $[a,b]$*, it is also easily calculated that*

$$\frac{2}{b-a} \int_a^b |\log x - (\log x)_{(b+a)/2,(b-a)/2}|\,dx = 4,$$

i.e. $\log x \in \mathcal{L}^{1,1}(\Omega)$*. On the other hand, it is obvious that* $\log x \notin L^\infty((0,1))$*. Thus,* $L^\infty(\Omega) \subsetneq \mathcal{L}^{p,n}(\Omega)$*.*

As we have seen Companato spaces can be identified with Morrey spaces in the interval $0 \le \lambda < n$. For $\lambda > n$ we have the following

Theorem 9.25. *For* $n < \lambda \le n + p$ *holds*

$$\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\overline{\Omega}) \quad \text{with } \alpha = \frac{\lambda - n}{p}$$

whereas for $\lambda > n + p$

$$\mathcal{L}^{p,\lambda}(\Omega) = \{\text{constants}\}.$$

Proof: From lemma 9.23 and the assumption that $\lambda > n$ we observe that u_{x_0,R_k} is a Cauchy sequence for all $x_0 \in \Omega$. Thus by Lebesgue's theorem

$$u_{x_0,R_k} \to \tilde{u}(x_0) \quad (k \to \infty),$$

for all $x \in \Omega$. Thus we know $u_{x_0,R_k} < \infty$ exists. Next, we note that $u_{x,R_k} =: v_k(x) \in C(\overline{\Omega})$. This can proved from a standard continuity argument noting that $u \in L^1(\Omega)$. Taking $h \to \infty$ in (9.6) yields

$$|u_{x_0,R_k} - \tilde{u}(x_0)|_C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R_k^{\frac{\lambda-n}{p}}. \tag{9.9}$$

From this, we deduce that $v_k \to \tilde{u}$ is of uniform convergence in Ω; hence $\tilde{u}$ is continuous. Taking $\tilde{u}$ to represent u, we know go one to show that u is Hölder continuous. First, take $x, y \in \Omega$ and set $R = |x-y|$, and $\alpha = \dfrac{\lambda - n}{p}$.

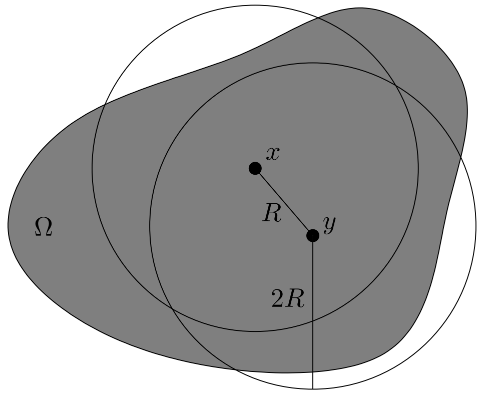


Figure 9.3: Layout for proof of theorem 9.25

We start first by approximating $|u(x) - u(y)|$:

$$|u(x) - u(y)| \leq \underbrace{|u_{x,2R} - u(x)|}_{\leq C \cdot R^\alpha} + |u_{x,2R} - u_{y,2R}| + \underbrace{|u_{y,2R} - u(y)|}_{\leq C \cdot R^\alpha};$$

the approximations in the underbraces come from (9.9). Next we figure

$$\begin{aligned}
&|\Omega(x, 2R) \cap \Omega(y, 2R)| \cdot |u_{x,2R} - u_{y,2R}| \\
&\qquad \leq \int_{\Omega(x,2R)} |u(z) - u_{x,2R}| \; dz + \int_{\Omega(y,2R)} |u(z) - u_{y,2R}| \; dz \\
&\qquad \leq \left(\int_{\Omega(x,2R)} |u(z) - u_{x,2R}|^p \; dz \right)^{1/p} |\Omega(x, 2R)|^{1-1/p} \\
&\qquad\qquad + \left(\int_{\Omega(y,2R)} |u(z) - u_{y,2R}|^p \; dz \right)^{1/p} |\Omega(y, 2R)|^{1-1/p} \\
&\qquad \leq \underbrace{2^{1+n+(n-\lambda)/p} \cdot \operatorname{diam}(\Omega)^n}_{C} \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R^{\lambda/p} R^{n(1-1/p)} \\
&\qquad = C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} R^\alpha.
\end{aligned} \tag{9.10}$$

As $R = |x - y|$, we have thus shown:

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)}. \tag{9.11}$$

We still aren't *quite* done. Remember, we are seeking our conclusion in the BS definition of $C^{0,\alpha}(\overline{\Omega})$ and $\mathcal{L}^{p,\lambda}(\Omega)$; the last calculation only pertains to the *seminorms*. Let us pick an arbitrary $x \in \Omega$ and define $v(x) := |u(x)| - \inf_\Omega |u(x)|$. Obviously, $v(x) > 0$ and exists $\{x_n\} \subset \Omega$ such that $v(x_n) \to 0$ as $n \to \infty$. Thus, it is clear that

$$\begin{aligned}
|v(x)| &= \lim_{n\to\infty} |v(x) - v(x_n)| \\
&\leq (\text{diam}(\Omega))^\alpha \frac{|v(x) - v(x_n)|}{|x - x_n|^\alpha} \\
&\leq C(\text{diam}(\Omega))^\alpha \cdot [v]_{\mathcal{L}^{p,\lambda}(\Omega)} \\
&= C(\text{diam}(\Omega))^\alpha \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)}.
\end{aligned}$$

The last equality comes from the fact that $|u(x)|$ and $|v(x)|$ differ only by a constant. With the last calculation in mind we finally realize

$$\begin{aligned}
|u(x)| &= |v(x)| + \inf_\Omega |u(x)| \\
&\leq C(\text{diam}(\Omega))^\alpha \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} + \frac{1}{|\Omega|}\int_\Omega |u|\,dx \\
&\overset{\text{Hölder}}{\leq} C(\text{diam}(\Omega))^\alpha \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega)} + |\Omega|^{-1/p}\left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Combining this result with (9.11), we conclude

$$\sup_{x\in\Omega} |u(x)| + [u]_{C^{0,\alpha}(\overline{\Omega})} \leq C\left(||u||_{L^p(\Omega)} + [u]_{\mathcal{L}^{p,\lambda}(\Omega)}\right),$$

completing the proof! ■

To conclude the section, two corollaries are put to the reader's attention. Both of these are proved by the exact same argument as the previous theorem.

Corollary 9.26. *If*

$$\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p\,dx \leq C\rho^\lambda$$

for all $\rho \leq \rho_0$ (instead of $\rho < diam(\Omega)$), then the conclusion of theorem (9.25) is true with Ω replaced by $\Omega(x_0, \rho_0)$, i.e. local version of (9.25).

Corollary 9.27. *If*

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \, dx \leq C\rho^\lambda$$

for all $x_0 \in \Omega$ and $\rho <$ dist$(x_0, \partial\Omega)$, then for $\lambda > n$, we conclude that u is Hölder continuous in every subdomain of Ω.

9.4.1 L^∞ Companato Spaces

Given the characterizations of Morrey and Companato spaces, one may ask if there is a generalization of these spaces. It turns out there is, but only parts (corresponding to ranges of indices) of these spaces are useful for our studies.

To construct our generalization, consider

$$\mathcal{P}_k := \left\{ T(x) \,|\, T(x) = \sum_{|\beta| \leq k} C_\beta \cdot x^\beta, \; C_\beta = \text{const.} \right\},$$

i.e. the set of all polynomials of degree k or less. Also, let us define $T_{k,x}u$ as the kth order Taylor expansion of u about x:

$$T_{k,x}u(y) = \sum_{|\beta| \leq k} \frac{D^\beta u(x)}{\beta!} \cdot (y - x)^\beta.$$

Now we define our generalization.

Definition 9.28. *The <u>L^p Companato Spaces</u>, $\mathcal{L}_k^{p,\lambda}(\Omega)$, are defined as*

$$\mathcal{L}_k^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, [u]_{\mathcal{L}_k^{p,\lambda}(\Omega)} < \infty \right\},$$

where

$$[u]_{\mathcal{L}_k^{p,\lambda}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ 0 < \rho < diam(\Omega)}} \rho^{-(k+\lambda)} \cdot \inf_{T \in \mathcal{P}_{||}} \left(\int_{\Omega(x_0,\rho)} |u - T|^p \, dx \right)$$

It is easy to notice that $\mathcal{L}_0^{p,\lambda}(\Omega)$ is just $\mathcal{L}_0^{p,\lambda}(\Omega)$. One could also fit Morrey spaces into this picture by defining $\mathcal{P}_* = \{0\}$ and denote the Morrey space $\mathcal{L}_*^{p,\lambda}(\Omega)$.

Now since $\mathcal{P}_k \subset \mathcal{P}_{k+1}$, we readily ascertain that $\mathcal{L}^{p,\lambda}_{k+1}(\Omega) \subset \mathcal{L}^{p,\lambda}_{k}(\Omega)$. At this point we are inclined to then ask is it useful to consider $k > 0$ as Companato spaces can already characterize Hölder continuous functions very well. In addition, if we did need a weaker criterion, we could always appeal to Sobolev spaces. We could counter act the weakening affect of increasing k by increasing p, but such index balancing isn't immediately useful within the scope of this book. What does turn out to be useful, is to consider the "other end of the index spectrum", i.e. take k arbitrary with $p = \infty$ (as opposed to taking p arbitrary with $k = 0$: the regular Companato space case). With $p = \infty$, it turns out that $\mathcal{L}^{\infty,\alpha}_{k}(\Omega) \cong C^{k,\alpha}(\overline{\Omega})$; but before we prove this, we first need the following lemma.

Warning: The following proof is somewhat technically involved and can be skipped without loosing continuity. That being said, the proof itself presents some clever tools that can be employed to bounding Hölder norms.

Lemma 9.29. *If $u \in C^{k,\alpha}(\overline{\Omega})$ with Ω bounded, then*

$$\rho_x^k \cdot \sup_{\Omega} |D^\beta u| \le C(\epsilon) \cdot \sup_{\Omega} |u| + \epsilon \cdot \rho_x^{\alpha+k} \cdot [u]_{C^{k,\alpha}(\Omega)},$$

for any $x \in \Omega$ and $R > 0$, where $\rho_x :=$ dist$(x, \partial\Omega)$ and $k = |\beta|$.

Proof: First we fix an arbitrary $x \in \Omega$ and consider $\frac{1}{2} \ge \lambda$, we define $R := \lambda \cdot \rho_x$; we will fix λ later in the proof. Upon selecting a coordinate system, we select a group of n line segments, denoted l_i, of length $2R$ with centers at x, which are parallel to the x_i axis. Let us denote the endpoints of each l_i by x_i' and x_i''. With these preliminaries, we proceed in four main steps.

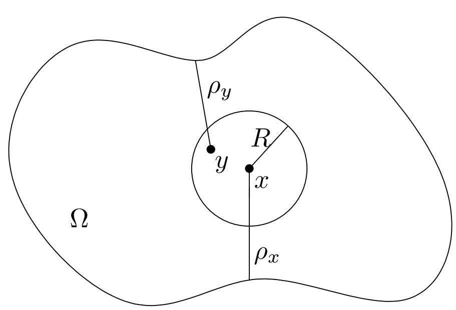


Figure 9.4: Layout for proof of lemma 9.29

Step 1: For the first step we claim that

$$\sup_{\Omega} |Du| \leq \frac{1}{R} \sup_{\Omega} |u|. \tag{9.12}$$

To prove this we first note that since $u \in C^{k,\alpha}(\overline{\Omega})$, we can utilize the mean value theorem to state $\exists x_i^* \in l_i$ such that

$$D_i u(x^*) = \frac{u(x_i'') - u(x_i')}{x'' - x'} \leq \frac{1}{R} \sup_{\Omega} |u|. \qquad \blacksquare$$

Step 2: Next we prove the claim that

$$\rho_x \cdot \sup_{\Omega} |D_i u| \leq \frac{1}{\lambda} \sup_{\Omega} |u| + 2^2 3^2 \lambda \cdot \rho_x^2 \cdot \sup_{B_R(x)} |D_{ij} u|.$$

We start with calculating

$$\begin{aligned} |D_i u(x)| &\leq |D_i u(x_j^*)| + |D_i u(x) - D_i u(x_j^*)| \\ &\leq |D_i u(x_j^*)| + \int_{x_j^*}^{x} |D_{ij} u(t)|\, dt \end{aligned}$$

Using (9.12) on the first term on the RHS and Hölder's inequality on the second we get

$$\begin{aligned} |D_i u(x)| &\leq \frac{1}{R} \sup_{\Omega} |u| + R \cdot \sup_{B_R(x)} |D_{ij} u| \\ &\leq \frac{1}{R} \sup_{\Omega} |u| + R \cdot \rho_y^{-2} \cdot \rho_y^2 \cdot \sup_{B_R(x)} |D_{ij} u|. \end{aligned}$$

We know that $\rho_y + R \geq \rho_x \implies \rho_y \geq \rho_x - R = (1-\lambda)\rho_x \geq \frac{\rho_x}{2}$. Using this the above becomes

$$|D_i u(x)| \leq \frac{1}{\lambda \rho_x} \cdot \sup_{\Omega} |u| + \frac{2^2 \lambda}{\rho_x} \cdot \rho_y^2 \cdot \sup_{B_R(x)} |D_{ij} u|.$$

Since we know that $\rho_y \leq R + \rho_x \implies \rho_y \leq 3\rho_x$. Thus,

$$\sup_{\Omega} |D_i u| \leq \frac{1}{\lambda \rho_x} \cdot \sup_{\Omega} |u| + 2^2 3^2 \lambda \cdot \rho_x \cdot \sup_{y \in B_R(x)} |D_{ij} u|.$$

Multiplying by ρ_x gets the conclusion of this step. $\blacksquare$

Step 3: Our last claim is that

$$\rho_x^2 \cdot \sup_{\Omega}(|D_{ij}u|) \leq \frac{\rho_x}{\lambda} \cdot \sup_{B_R(x)} |D_i u| + 2^2 3^{\alpha+2} \cdot \lambda^\alpha \cdot \rho_x^{\alpha+2}[u]_{C^{2,\alpha}(\Omega)}. \quad (9.13)$$

For this we start with

$$|D_{ij}u| \leq |D_{ij}u(x_j^*)| + |D_{ij}u(x) - D_{ij}u(x_j^*)|.$$

Again using (9.12) we get

$$\begin{aligned}
|D_{ij}u| &\leq \frac{1}{R}\sup_{\Omega}|D_i u| + R^\alpha \cdot \frac{|D_{ij}u(x) - D_{ij}u(x_j^*)|}{|x - x_j^*|^\alpha} \\
&\leq \frac{1}{R}\sup_{\Omega}|D_i u| + R^\alpha \cdot \rho_y^{-\alpha-2} \cdot \rho_y^{\alpha+2} \cdot [u]_{C^{2,\alpha}(\Omega)} \\
&\leq \frac{1}{\lambda\rho_x}\sup_{\Omega}|D_i u| + \frac{2^2\lambda^\alpha}{\rho_x^2}\rho_y^{\alpha+2} \cdot [u]_{C^{2,\alpha}(\Omega)} \\
&\leq \frac{1}{\lambda\rho_x}\sup_{\Omega}|D_i u| + 2^2 3^{\alpha+2}\lambda^\alpha \cdot \rho_x^\alpha \cdot [u]_{C^{2,\alpha}(\Omega)}
\end{aligned}$$

In the above, we again used the previously stated fact that $\frac{\rho_x}{2} \leq \rho_y \leq 3\rho_x$. As in the previous step, we take the supremum of the LHS and multiply both sides by ρ_x^2 to get the claim for Step 3. ■

Step 4: We are almost done. Next we realize that the proof in Step 3 can be easily adapted to also state

$$\rho_x \cdot \sup_{\Omega}|D_i u| \leq \frac{1}{\lambda} \cdot \sup_{\Omega}|u| + 2 \cdot 3^{\alpha+1}\lambda^\alpha \cdot \rho_x^{\alpha+1} \cdot [u]_{C^{1,\alpha}(\Omega)}. \quad (9.14)$$

Now $\lambda \leq \frac{1}{2}$ has not been fixed in any of these calculations. So for given $\epsilon > 0$ let us pick $\epsilon = 2^2 3^{\alpha+2}\lambda^\alpha$ in (9.13) to get

$$\rho_x^2 \cdot \sup_{\Omega}(|D_{ij}u|) \leq \frac{2^{2/\alpha}3^{1+2/\alpha}}{\epsilon^{1/\alpha}} \cdot \rho_x \cdot \sup_{B_R(x)} |D_i u| + \epsilon \cdot \rho_x^{\alpha+2} \cdot [u]_{C^{2,\alpha}(\Omega)}.$$

For (9.13) we choose λ so that $\epsilon = 2^2 3^2 \lambda$:

$$\rho_x \cdot \sup_{\Omega}|D_i u| \leq \frac{2^2 3^2}{\epsilon}\sup_{\Omega}|u| + \epsilon \cdot \rho_x^2 \cdot \sup_{y \in B_R(x)} |D_{ij}u|.$$

Combining these last two equations we get

$$\rho_x^2 \cdot \sup_\Omega |D_{ij}u| \leq \frac{2^{2+2/\alpha}3^{3+2/\alpha}}{\epsilon^{1+1/\alpha}} \sup_\Omega |u| + \frac{2^{2/\alpha}3^{1+2/\alpha}}{\epsilon^{1/\alpha-1}} \cdot \rho_x^2 \sup_{B_R(x)} |D_{ij}u(y)| + \epsilon \cdot \rho_x^{\alpha+2} \cdot [u]_{C^{2,\alpha}(\Omega)}.$$

We can immediately algebraically deduce that

$$\rho_x^2 \cdot \sup_\Omega |D_{ij}u| \leq \frac{\epsilon^{-2}2^{2+2/\alpha}3^{3+2/\alpha}}{\epsilon^{1/\alpha-1} - 2^{2/\alpha}3^{1+2/\alpha}} \sup_\Omega |u| + \frac{\epsilon^{1/\alpha}}{\epsilon^{1/\alpha-1} - 2^{2/\alpha}3^{1+2/\alpha}} \cdot \rho_x^{\alpha+2}[u]_{C^{2,\alpha}(\Omega)}.$$

Picking

$$\epsilon > 2^{\frac{2}{1-\alpha}} 3^{\frac{\alpha+2}{1-\alpha}}$$

guarantees everything stays positive. Taking this result with (9.14) we can apply an induction argument to ascertain the conclusion. ■

Theorem 9.30. *Given $\partial\Omega \in C^{k,\alpha}$ and $u \in C^{k,\alpha}(\overline{\Omega})$, there exists $0 \leq C_1 \leq C_2 < \infty$ such that*

$$C_1 \cdot [u]_{C^{k,\alpha}(\Omega)} \leq [u]_{\mathcal{L}_k^{\infty,\alpha}(\Omega)} \leq C_2 \cdot [u]_{C^{k,\alpha}(\Omega)},$$

i.e. $C^{k,\alpha}(\overline{\Omega}) \cong \mathcal{L}_k^{\infty,\alpha}(\Omega)$.

Proof: First, we claim that

$$||u - T_{x,k}u||_{L^\infty(\Omega(x,\rho))} \leq C_2 \cdot \rho^{k+\alpha} \cdot [u]_{C^{k,\alpha}(\Omega)},$$

for all $x \in \Omega$. So we first fix $y \in \Omega(x,\rho)$ to find

$$u(y) - T_{x,k}u(y) = \sum_{|\beta|=k} \frac{D^\beta u(z) - D^\beta u(x)}{\beta!} \cdot (z-x)^\beta, \tag{9.15}$$

where $z := tx + (1-t)y$ for some $t \in (0,1)$.

It needs to be noted that in order for the last equation to be valid, we may need to extend u to the convex hull of $\Omega(x,\rho)$. This is why we require the boundary regularity, to guarantee that such a Hölder continuous extension of u exists. We do not prove this here, but the idea of the proof is similar to that of extending Sobolev functions.

Continuing on, given our definition of z, we readily see $|z-x| \le |y-x| \le \rho$. This combined with (9.15) immediately indicates that

$$|u(y) - T_{x,k}u(y)| \le C_2 \cdot \rho^{k+\alpha} \cdot [u]_{C^{k,\alpha}(\Omega)},$$

proving the RHS of (9.15).

For the other inequality, we first notice that $\forall T \in \mathcal{P}_k$, $[u-T]_{C^{k,\alpha}(\Omega)} = [u]_{C^{k,\alpha}(\Omega)}$. We know this since $D^\beta T =$ const. when $|\beta| = k$. Now, from lemma 9.29 we have

$$\rho^k \cdot \sup_{\Omega(x,\rho)} |D^\beta(u-T)| \le C_1 \left(\rho^{\alpha+k} \cdot [u]_{C^{k,\alpha}(\Omega)} + \sup_{\Omega(x,\rho)} |D^\beta(u-T)| \right).$$

Taking the infimum of $T \in \mathcal{P}_k$ and dividing by $\rho^{k+\alpha}$ gives us

$$\rho^{-\alpha} \sup_{x,y\in\Omega(x,\rho)} |D^\beta u(x) - D^\beta u(y)| \le C_1 \left([u]_{C^{k,\alpha}(\Omega)} + [u]_{\mathcal{L}_k^{\infty,\alpha}(\Omega)} \right),$$

proving the left inequality, concluding the proof. ■

9.5 The spaces of bounded mean oscilation $= BMO$-spaces = John-Nirenberg-spaces

Let Q_0 be a cube in $\mathbb{R}^n$ and u a measurable function on Q_0. We define

$$u \in BMO(Q_0)$$

if and only if

$$|u|_{BMO(Q_0)} := \sup_Q \int_{Q\cap Q_0} \fint_{Q\cap Q_0} |u - u_{Q\cap Q_0}|\, dx < \infty.$$

It is easy to see that we can also take as definition

$$|u|_{BMO(Q_0)} = \sup_{Q\subset Q_0} \fint_Q |u - u_Q|\, dx$$

and that $u \in BMO(Q_0)$, if and only if $u \in \mathcal{L}^{1,n}(Q_0)$.

Theorem 9.31 (John/Nirenberg). *There exist constants $C_1 > 0$ and $C_2 > 0$ depending only on n, such that the following holds: If $u \in BMO(Q_0)$, then for all $Q \subset Q_0$*

$$\left|\{x \in Q_0 \,|\, (u - u_Q)(x) > t\}\right| \leq C_1 \cdot \exp\left(\frac{-C_2}{|u|_{BMO(Q_0)}}\right) |Q|.$$

Proof: It is sufficient to give the proof for $Q = Q_0$ (because $u \in BMO(Q_0) \Rightarrow u \in BMO(Q)$ for $Q \subset Q_0$). Moreover the inequality is homogenous and so we can assume that $|u|_{BMO(Q_0)} = 1$.

For $\alpha > 1 = \sup_{Q \subset Q_0} \fint_Q |u - u_Q|\, dx$ we use Calderon-Zygmund argument to get a sequence $\{Q_j^{(1)}\}$ with

i.) $\alpha < \fint_{Q_j^{(1)}} |u - u_{Q_0}|\, dx \leq 2^n \alpha$

ii.) $|u - u_{Q_0}| \leq \alpha$ on $Q_0 \setminus \bigcup_j Q_j^{(1)}$.

Then we also have

$$\left| u_{Q_j^{(1)}} - u_{Q_0} \right| \leq \fint_{Q_j^{(1)}} |u - u_{Q_0}|\, dx \leq 2^n \alpha$$

and

$$\sum_j |Q_j^{(1)}| \leq \frac{1}{\alpha} \int_{Q_0} |u - u_{Q_0}|\, dx.$$

Take *one* of the $Q_j^{(1)}$. Then as $|u|_{BMO(Q_0)} = 1$ we have

$$\fint_{Q_j^{(1)}} \left| u - u_{Q_j^{(1)}} \right| dx \leq 1 < \alpha$$

and we can apply Calderon-Zygmund argument again. Thus, there exists a sequence $\{Q_j^{(2)}\}$ with

$$|u(x) - u_{Q_0}| \leq \left| u(x) - u_{Q_j^{(1)}} \right| + |u_{Q_j^{(1)}} - u_{Q_0}| \leq 2 \cdot 2^n \alpha$$

for a.e. x on $Q_0 \setminus \bigcup Q_k^{(2)}$ and

$$\begin{aligned}
\sum_j |Q_j^{(2)}| &\leq \frac{1}{\alpha} \sum_j \int_{Q_j^{(1)}} \left| u - u_{Q_j^{(1)}} \right| dx \\
&\leq \frac{1}{\alpha} \sum_j |Q_j^{(1)}| \quad (u \in BMO \text{ and } |u|_{BMO} = 1) \\
&\leq \frac{1}{\alpha^2} \int_{Q_0} |u - u_{Q_0}|\, dx \\
&\leq \frac{1}{\alpha^2} |Q_0|.
\end{aligned}$$

Repeating this argument one gets by induction: For all $k \geq 1$ there exists sequence $\{Q_j^{(k)}\}$ such that

$$|u - u_{Q_0}| \leq k2^n\alpha \quad \text{a.e. on } Q_0 \setminus \bigcup_j Q_j^{(k)}$$

and

$$\sum_j |Q_j^{(k)}| \leq \frac{1}{\alpha^k} \int_{Q_0} |u - u_{Q_0}|\, dx.$$

Now for $t \in \mathbb{R}_+$ either $0 < t < 2^n\alpha$ or there exists $k \geq 1$ such that $2^n\alpha k < t < 2^n\alpha(k+1)$. In the first case

$$|\{x \in Q_0 \,|\, |u(x) - u_{Q_0}| > t\} \leq |Q_0| \leq |Q_0| e^{-At} e^{2^n A\alpha}$$

$(o \leq 2^n\alpha - t \Rightarrow 1 \leq e^{A(2^n\alpha - t)}$ for some $A > 0)$; in the secont case

$$\begin{aligned}
& |\{x \in Q_0 \,|\, |u(x) - u_{Q_0}| > t\} \\
\leq\ & |\{x \in Q_0 \,|\, |u(x) - u_{Q_0}| > 2^n\alpha k\} \\
\leq\ & \sum_j |Q_j^{(k)}| \leq \frac{1}{\alpha^k} \int_{Q_0} |u - u_{Q_0}|\, dx \\
\leq\ & e^{\left(1 - \frac{t}{2^n\alpha}\right)\ln\alpha} \int_{Q_0} |u - u_{Q_0}|u - u_{Q_0}|\, dx \leq \alpha e^{\frac{-\ln\alpha}{2^n\alpha}} |Q_0| \\
=:\ & \alpha e^{-At}|Q_0| \quad (\text{we need } -k \leq 1 - \frac{t}{2^n\alpha} \text{ and } \alpha^{-k} = e^{-k\ln\alpha}). \quad \blacksquare
\end{aligned}$$

Corollary 9.32. *If $u \in BMO(Q_0)$, then $u \in L^p(Q_0)$ for all $p > 1$. Moreover*

$$\sup_{Q \subset Q_0} \left(\int_Q |u - u_Q|^p\, dx \right)^{1/p} \leq C(n,p) \cdot |u|_{BMO(Q_0)}.$$

Proof:

$$\begin{aligned}\int_Q |u-u_Q|^p\,dx &= p\int_0^\infty t^{p-1}|\{x\in Q\,|\,|u-u_Q|>t\}|\,dt\\ &\le p\cdot C_1\int_0^\infty t^{p-1}\exp\left(-\frac{C_2}{|u|_{BMO(Q)}}t\right)|Q|\,dt\\ &= p\cdot C_1\left(\frac{C_2}{|u|_{BMO(Q)}}\right)^{-p}|Q|\int_0^\infty e^{-t}t^{p-1}\,dx \le C(n,p)|u|^p_{BMO(Q)}. \quad\blacksquare\end{aligned}$$

Finally we have

Theorem 9.33 (Characterization of BMO). *The following statements are equivalent:*

i.) $u\in BMO(Q_0)$

ii.) there exists constatns C_1, C_2 such that for all $Q\subset Q_0$ and all $s>0$

$$|\{x\in Q\,|\,|u(x)-u_Q|>s\}|\le C_1\exp(-C_2 s)|Q|$$

iii.) there exist constants C_3, C_4 such that for all $Q\subset Q_0$,

$$\int_Q(\exp(C_4|u-u_Q|)-1)\,dx\le C_3|Q|$$

iv.) if $v=\exp(C_4u)$, then

$$\sup_{Q\subset Q_0}\left(⨍_Q v\,dx\right)\left(⨍_Q\frac{1}{v}\,dx\right)\le C_5.$$

(We can take $C_1=C_3=C_5=C(n)$ and $C_2=2C_4=C(n)|u|^{-1}_{BMO(Q_0)}$.)

9.6 Sobolev Spaces

Take $\Omega\subseteq\mathbb{R}^n$

Definition 9.34. *$f\in L^1(\Omega)$ is said to be weakly differentiable if $\exists g_1,\dots,g_n\in L^1(\Omega)$ such that*

$$\int_\Omega f\frac{\partial\phi}{\partial x_i}\,dx=-\int_\Omega g_i\phi\,dx\qquad\forall\phi\in C_c^\infty(\Omega)$$

Notation: $Df = (g_1, \dots, g_n)$, $g_i = D_i f = \frac{\partial f}{\partial x_i}$. If $f \in C^1(\Omega)$, then $Df = \nabla f$.

Definition 9.35. *The Sobolev space $W^{1,p}(\Omega)$ is*

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega),\ Df \text{ exists},\ D_i f \in L^p(\Omega),\ i = 1, \dots, n\}$$

Proposition 9.36. *The space $W^{1,p}(\Omega)$ is a Banach space with the norm*

$$||f||_{1,p} = \left(\int_\Omega |f|^p + |Df|^p \, dx \right)^{\frac{1}{p}} \qquad 1 \le p < \infty,$$

where

$$|Df| = \left(\sum_{i=1}^n |D_i f|^2 \right)^{\frac{1}{2}}.$$

In the $p = \infty$ case, we define

$$||f||_{1,\infty} = ||f||_\infty + ||Df||_\infty$$

Proof: The triangle inequality follows from Minkowski's inequality.

Completeness of $W^{1,p}(\Omega)$: Take f_j Cauchy in $W^{1,p}(\Omega) \implies f_j$ and $D_i f_j$ are both Cauchy in $L^p(\Omega)$. By the completeness of L^p, $f_j \to f \in L^p(\Omega)$ and $D_i f_j \to g_i \in L^p(\Omega)$.

Need: $Df = g$,

Call $D_i f_j = (g_j)_i \to g_i \in L^p(\Omega)$. By definition

$$\begin{array}{lrcll} & \displaystyle\int_\Omega f_j \frac{\partial \phi}{\partial x_i} \, dx & = & \displaystyle -\int_\Omega (g_j)_i \phi \, dx & \forall \phi \in C_c^\infty(\Omega) \\ \underline{\text{Cauchy}} & \downarrow & & \downarrow & \\ & \displaystyle\int_\Omega f \frac{\partial \phi}{\partial x_i} \, dx & = & \displaystyle -\int_\Omega g_i \phi \, dx & \blacksquare \end{array}$$

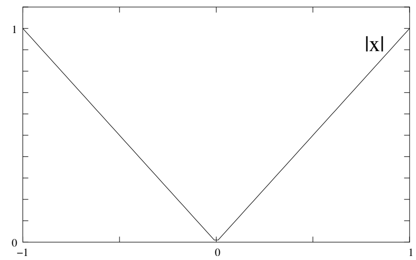
Remark: $W^{1,p}(\Omega)$ is a Hilbert space with

$$(u, v) = \int_\Omega (uv + Du \cdot Dv) \, dx$$

Thus, we can use the Riesz-Representation Theorem and Lax-Milgram in this space.

Examples:

- $f(x) = |x|$ on $(-1, 1)$



We will prove $f \in W^{1,1}(-1, 1)$ and

$$g = Df = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in (0, 1) \end{cases}$$

To show this, look at

$$\begin{aligned} \int_{-1}^{1} f\phi' \, dx = -\int_{-1}^{1} g\phi \, dx &= -\left[\int_{-1}^{0} \phi \, dx + \int_{0}^{1} \phi \, dx\right] \\ &= \int_{-1}^{0} \phi \, dx - \int_{0}^{1} \phi \, dx \end{aligned}$$

But we also have

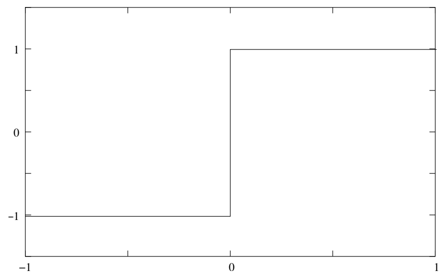
$$\begin{aligned} \int_{-1}^{1} f\phi' \, dx &= -\int_{-1}^{0} x\phi' \, dx + \int_{0}^{1} x\phi' \, dx \\ &= \int_{-1}^{0} \phi \, dx + 0 - \int_{0}^{1} \phi \, dx + 0 \qquad \surd \end{aligned}$$

Thus, g is the weak derivative of f.

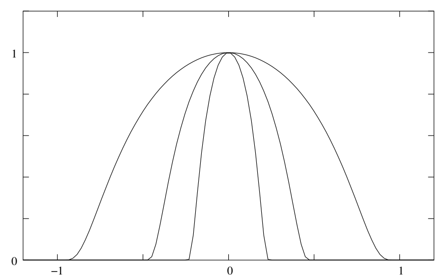
- $$f(x) = \operatorname{sgn}(x) = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in (0, 1) \end{cases}$$

Our intuition says that $f' = 2\delta_0 \notin L^2(-1, 1)$. Thus, we see to prove f is not weakly integrable. So, suppose otherwise, $g \in L^1(-1, 1)$, $Df = g$.

$$\begin{aligned} \int_{-1}^{1} g\phi \, dx &= -\int_{-1}^{1} f\phi' \, dx = \int_{-1}^{0} \phi' \, dx - \int_{0}^{1} \phi' \, dx \\ &= \phi(0) - \phi(-1) - [\phi(1) - \phi(0)] \\ &= 2\phi(0) \end{aligned}$$



Now choose ϕ_i such that $\phi \geq 0$, $\phi \leq 1$, $\phi_j(0) = 1$, and $\phi_j(x) \to 0$ a.e.



With this choice of ϕ_j we have

$$\begin{aligned} 2\phi_j(0) &= \int_{-1}^{1} g\phi_j \ dx \\ \Longrightarrow 2 &= \lim_{j\to\infty} 2\phi_j(0) = \lim_{j\to\infty} \int_{-1}^{1} g\phi_j \ dx \\ &= 0 \qquad \text{contradiction} \end{aligned}$$

Where the last equality is a result of Lebesgue's dominated convergence theorem.

For higher derivatives, α is a multi-index. $g = D^\alpha f$ if

$$\int_\Omega f D^\alpha \phi \ dx = (-1)^{|\alpha|} \int_\Omega g\phi \ dx \qquad \forall \phi \in C_c^\infty(\Omega)$$

Definition 9.37.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega),\ D^\alpha f \ \textit{exists},\ D^\alpha f \in L^p(\Omega),\ |\alpha| \leq k\}$$

Sobolev functions are like classical functions. We will address the following in the next few lectures

- Approximation
- Calculus. Specifically product and chain rules.
- Boundary values
- Embedding Theorems. In the HW we showed that

$$||u||_{C^{0,\alpha}(\mathbb{R})} \leq |u(x_0)| + C||u'||_{L^p(\mathbb{R})}$$

which gives us the intuition that

$$W^{1,p}(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R}).$$

Recall: $W^{1,p}(\Omega)$ means we have weak derivatives in $L^p(\Omega)$, $g = D^\alpha f$

$$\int_\Omega f D^\alpha \phi \; dx = (-1)^{|\alpha|} \int_\Omega g\phi \; dx \qquad \forall \phi \in C_c^\infty(\Omega)$$

- $\underline{n = 1}$ $W^{1,p} \subset C^{0,\alpha}$ with $\alpha = 1 - \frac{1}{p}$.
- $\underline{n \geq 2}$ $W^{n,p}$ not holder continuous.

Lemma 9.38. *$f \in L^1_{loc}(\Omega)$, α is a multi-index, and $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$ with usual mollification.*
If $D^\alpha f$ exists, then $D^\alpha(\phi_\epsilon * f)(x) = (\phi_\epsilon * D^\alpha f)(x)$ $\forall x$ with $dist(x, \partial\Omega) > \epsilon$.

Proof: Without loss of generality take $|\alpha| = 1$, $D^\alpha = \frac{\partial}{\partial x_i}$.

$$\begin{aligned}
\frac{\partial}{\partial x_i}(\phi_\epsilon * f)(x) &= \frac{\partial}{\partial x_i}\int_{\mathbb{R}^n} \phi_\epsilon(x-y) f(y) \; dy \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\phi_\epsilon(x-y) f(y) \; dy \\
&= -\int_{\mathbb{R}^n} \frac{\partial}{\partial y_i}\phi_\epsilon(x-y) f(y) \; dy
\end{aligned}$$

Now since $\phi(x-y) \in C_c^\infty(\Omega)$, we apply the definition of the weak partial derivative to get

---

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \phi_\epsilon(x-y) \frac{\partial}{\partial y_i} f(y) \; dy \\
&= \left(\phi_\epsilon * \frac{\partial}{\partial x_i} f\right) \qquad \blacksquare
\end{aligned}$$

Proposition 9.39. *$f, g \in L^1_{loc}(\Omega)$. Then $g = D^\alpha f \iff \exists f_m \in C^\infty(\Omega)$ such that $f_m \to f$ in $L^1_{loc}(\Omega)$ and $D^\alpha f_m \to g$ in $L^1_{loc}(\Omega)$.*

Proof:
($\Longleftarrow$) $f_m \to f$ and $D^\alpha f_m \to g$

$$\begin{array}{rcl} \displaystyle\int_\Omega f_m \cdot D^\alpha \phi \, dx & = & (-1)^{|\alpha|} \displaystyle\int_\Omega D^\alpha f_m \cdot \phi \, dx \\ \text{(see following aside)} \downarrow & & \qquad\qquad \downarrow \\ \displaystyle\int_\Omega f \cdot D^\alpha \phi \, dx & = & (-1)^{|\alpha|} \displaystyle\int_\Omega g \phi \, dx \end{array}$$

$$\begin{aligned} \left| \int_\Omega f_m \cdot D^\alpha \phi \, dx - \int_\Omega f \cdot D^\alpha \phi \, dx \right| &\le \int_\Omega |f_m - f| \underbrace{|D^\alpha \phi|}_{C} \, dx \\ &\le C \int_{\text{supp}(\phi)} |f_m - f| \, dx \to 0 \text{ given} \end{aligned}$$

($\Longrightarrow$) Use convolution and lemma 8.37. ■

9.6.1 Examples and Applications: Calculus w/ Sobolev Functions

Proposition 9.40 (Product Rule). *$f \in W^{1,p}(\Omega)$, $g \in W^{1,p'}(\Omega)$, then $fg \in W^{1,1}(\Omega)$ and $D(fg) = Df \cdot g + f \cdot Dg$.*

Proof: By Holder, fg, $Df \cdot g$, and $f \cdot Dg$ are in $L^1(\Omega)$. Explicitly, this is

$$\int_\Omega |fg| \, dx \le ||f||_p ||g||_{p'}$$

Now suppose $p < \infty$ (by symmetry) and f_ϵ is the convolution of f defined by $f_\epsilon = \phi_\epsilon * f$. $f_\epsilon \to f$ in $L^1_{\text{loc}}(\Omega)$. Let Ω' =supp(ϕ), ϵ <dist($\Omega', \partial\Omega'$).

$$\begin{aligned} \int_\Omega f_\epsilon g \cdot D_i \phi \, dx &= \int_\Omega g \cdot D_i (f_\epsilon \phi) \, dx - \int_\Omega g \cdot D_i f_\epsilon \cdot \phi \, dx \\ \text{(def. of weak deriv. of } g\text{)} \quad &= -\int_\Omega D_i g \cdot f_\epsilon \phi \, dx - \int_\Omega g \cdot D_i f_\epsilon \cdot \phi \, dx \end{aligned}$$

$$\Longrightarrow \int_\Omega f_\epsilon g \cdot D_i \phi \, dx = -\int_\Omega (D_i f_\epsilon \cdot g + f_\epsilon \cdot D_i g) \phi \, dx$$

Now $f_\epsilon \to f$ in L^1_{loc} and by Lemma 8.37 $D_i f_\epsilon \to D_i f$ in L^1_{loc} as $\epsilon \to 0$. Thus,

$$\int_\Omega fg \cdot D_i\phi \, dx = -\int_\Omega (D_i f \cdot g + f \cdot D_i g)\phi \, dx$$

Now by applying the definition of weak derivatives, we get

$$D_i(fg) = D_i f \cdot g + f \cdot D_i g \qquad \blacksquare$$

Proposition 9.41 (Chain Rule). *$f \in C^1$, f bounded and $g \in W^{1,p}$, then $(f \circ g) \in W^{1,p}(\Omega)$ and*

$$D(f \circ g) = (f' \circ g)Dg$$

Remark: The same assertion holds if f is continuous and piecewise C^1 with bounded derivative.

Proof: $g_\epsilon = \phi_\epsilon * g$, $g_\epsilon \to g$ in L^p_{loc}, $Dg_\epsilon \to Dg$ in L^p_{loc}. (Assume $p < \infty$, even $p = 1$ would be sufficient).

$$D(f \circ g_\epsilon) = (f' \circ g_\epsilon)Dg_\epsilon \to (f' \circ g)Dg \text{ in } L^1_{\text{loc}}$$

Take $\Omega' \subset\subset \Omega$, ϵ small enough.

$$\begin{aligned} &\int_\Omega |(f' \circ g_\epsilon)Dg_\epsilon - (f' \circ g)Dg| \, dx \\ \le \; &\int_\Omega |(f' \circ g_\epsilon)(Dg_\epsilon - Dg)| \, dx + \int_\Omega |(f' \circ g - f' \circ g_\epsilon)Dg| \, dx \end{aligned}$$

Now $f' \circ g_\epsilon$ is bounded since f' is bounded and $Dg_\epsilon - Dg \to 0$ in L^1_{loc}. Thus the first term on the RHS goes to 0 as $\epsilon \to 0$. Since $g_\epsilon \to g$ in L^1_{loc}, $g_\epsilon \to g$ in measure and thus exists a subsequence such that $g_\epsilon \to g$ a.e. Thus we can apply Lebesgue's dominated convergence theorem to see the second term in the RHS goes to 0 as $\epsilon \to 0$. Thus we have

$$\int_\Omega |(f' \circ g_\epsilon)Dg_\epsilon - (f' \circ g)Dg| \, dx = 0$$

and by the application of the definition of a weak derivative, we attain the result. $\blacksquare$

Lemma 9.42. *Let $f^+ = \max\{f, 0\}$, $f^- = \min\{f, 0\}$, $f = f^+ + f^-$, if $f \in W^{1,p}$, then f^+ and f^- are in $W^{1,p}$,*

$$\begin{aligned} Df^+ &= \begin{cases} Df & \text{on } f > 0 \\ 0 & \text{on } f \le 0 \end{cases} \\ D|f| &= \begin{cases} Df & \text{on } f > 0 \\ -Df & \text{on } f < 0 \\ 0 & f = 0 \end{cases} \end{aligned}$$

Corollary 9.43. *Let $c \in \mathbb{R}$, $E = \{x \in \Omega, f(x) = c\}$. Then $Df = 0$ a.e. in E.*

Theorem 9.44. *$1 \leq p < \infty$, then $W^{k,p}(\Omega) \cap C^{\infty}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof: Choose $\Omega_1 \Subset \Omega_2 \Subset \Omega_2 \Subset \cdots \Subset \Omega$, e.g. $\Omega_i = \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \frac{1}{i}\right\}$. Define the following open sets: $U_i = \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$

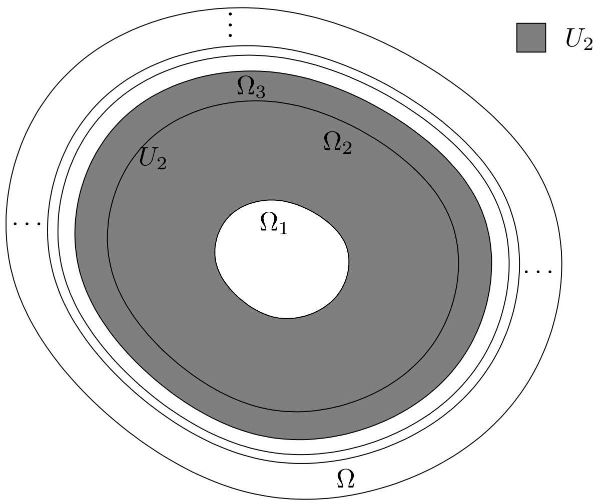


Figure 9.5:

Then $\bigcup_i U_i = \Omega$, and thus U_i is an open cover of Ω. This cover is said to be locally finite, i.e., $\forall x \in \Omega$ there exists $\epsilon > 0$, such that $B(x, \epsilon) \cap U_i \neq 0$ only for finitely many i.

———

Theorem 9.45 (Partition of Unity). *$\{U_i\}$ is an open and locally finite cover of Ω. Then there exists $\eta_i \in C_c^{\infty}(U_i)$, $\eta_i \geq 0$, and*

$$\sum_{i=1}^{\infty} \eta_i = 1 \text{ in } \Omega$$

Intuitive Picture in 1D:

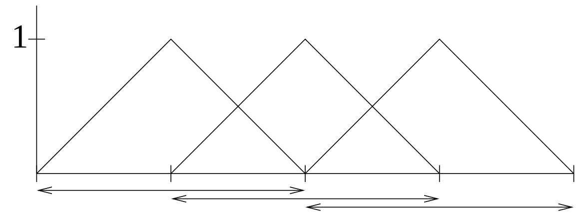


Figure 9.6:

——

Proof of 8.43 cont.:
η_i is a partition of unity corresponding to U_i, $f_i = \eta_i f \in W^{k,p}(\Omega)$.
$\text{supp}(f_i) \subset U_i = \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$.
Choose $\epsilon_i > 0$, such that $\phi_{\epsilon_i} * f_i =: g_i$ satisfies

- $\text{supp}(g_i) \subset \Omega_{i+2} \setminus \overline{\Omega_{i-2}}$
- $||g_i - f_i||_{W^{k,p}(\Omega)} \leq \frac{\delta}{2^i}$, $\delta > 0$ fixed.
- $g_i \in C_c^\infty(\Omega)$

Now define

$$g(x) = \sum_{i=1}^{\infty} g_i(x)$$

(this sum is well-defined since $g_i(x) \neq 0$ only for finitely many i).

$$\begin{aligned} ||g - f||_{W^{k,p}(\Omega)} &= \left|\left| \sum_{i=1}^{\infty} g_i - \sum_{i=1}^{\infty} \eta_i f \right|\right|_{W^{k,p}(\Omega)} \\ &\leq \sum_{i=1}^{\infty} ||g_i - \eta_i f||_{W^{k,p}(\Omega)} \\ &\leq \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta \qquad \blacksquare \end{aligned}$$

Remark: The radius of the ball on which you average to get $g(x)$ is proportional to $\text{dist}(x, \partial\Omega)$.

Theorem 9.46 (Fundamental Theorem of Calculus). *i.) $g \in L^1(a,b)$, $f(x) = \int_a^x g(t)\ dt$ then $f \in W^{1,1}(a,b)$ and $Df = g$.*

ii.) If $f \in W^{1,1}(a,b)$ and $g = Df$, then $f(x) = C + \int_a^x g(t)\ dt$ for almost all $x \in (a,b)$, where C is a constant.

Remark: In particular, there exists a representative of f that is continuous.

Proof of ii.): Choose $f_\epsilon \in W^{1,1}(a,b) \cap C^\infty(a,b)$ such that $f_\epsilon \to f$ in $W^{1,1}(\Omega)$, i.e., $f_\epsilon \to f$ in $L^1(a,b)$ and $Df_\epsilon \to Df = g$ in $L^1(a,b)$. There exists a subsequence such that $f_\epsilon(x) \to f(x)$ a.e.
Fundemental theorem for f_ϵ

$$\begin{array}{rcl} f_\epsilon(x) - f_\epsilon(a) & = & \displaystyle\int_a^x f_\epsilon'(t)\ dt \\ a.e. \downarrow \qquad \downarrow & & \quad \downarrow \\ f(x) - L & = & \displaystyle\int_a^x g(t)\ dt \end{array}$$

where L is finite and $f(a)$ is chosen such that $L = \lim_{x \to a} =: f(a)$ (Only changes f on a set of measure zero). ■

9.6.2 Boundary Values, Boundary Integrals, Traces

Definition 9.47. $1 \leq p < \infty$

1. *$W_0^{1,p}(\Omega)$ = closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$*

2. *$p = \infty$: $W_0^{1,\infty}(\Omega)$ is given by*

$$\begin{aligned} W_0^{1,\infty}(\Omega) \quad = \quad & \big\{ f \in W^{1,\infty}(\Omega) : \exists f_i \in C_c^\infty(\Omega) \text{ s.t.} \\ & f_j \to f \text{ in } W_{loc}^{1,1}(\Omega),\ \sup ||f_j||_{W^{1,\infty}(\Omega)} \leq C < \infty \big\} \end{aligned}$$

3. *If $f_0 \in W^{1,p}(\Omega)$. Then $f = f_0$ on $\partial\Omega$ if*

$$f - f_0 \in W_0^{1,p}(\Omega)$$

Remark: It's not always possible to define boundary values.

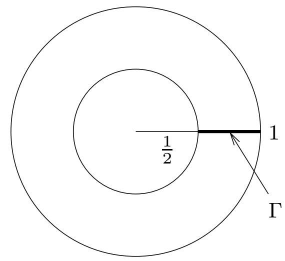


Figure 9.7:

$\Omega \subseteq \mathbb{R}^2$, $\Omega = B(0,1) \setminus \overline{B}\left(0, \frac{1}{2}\right) \setminus \Gamma$, where $\Gamma = \left\{x = (x_1, 0), \frac{1}{2} < x_1 < 1\right\}$. $u(x_1, x_2) = \phi =$ polar angle. Ω not "on one side" of Γ.

Definition 9.48. *Lipschitz boundary*
f is Lipschitz continuous if $f \in C^{0,1}$, i.e.

$$|f(x) - f(y)| \leq L|x - y|,$$

for some constant L. $\partial\Omega$ is said to be Lipschitz, if $\exists U_i$, $i = 1, \ldots, m$ open such that $\partial\Omega \subset \bigcup_{i=1}^m U_i$, where in any U_i, we can find a local coordinate system such that $\partial\Omega \cap U_i$ is the graph of a Lipschitz function and $\Omega \cap U_i$ lies on one side of the graph.

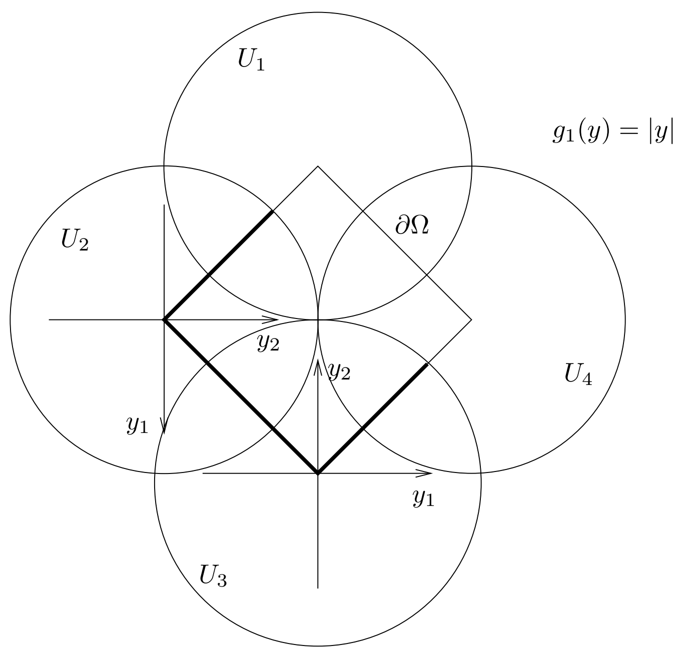


Figure 9.8:

Remark: Analogous definition with $C^{k,\alpha}$ boundary. $\partial\Omega \cup U_i$ = graph of a $C^{k,\alpha}$ functions.

The following figure should make clear the concept of graphs of boundaries.

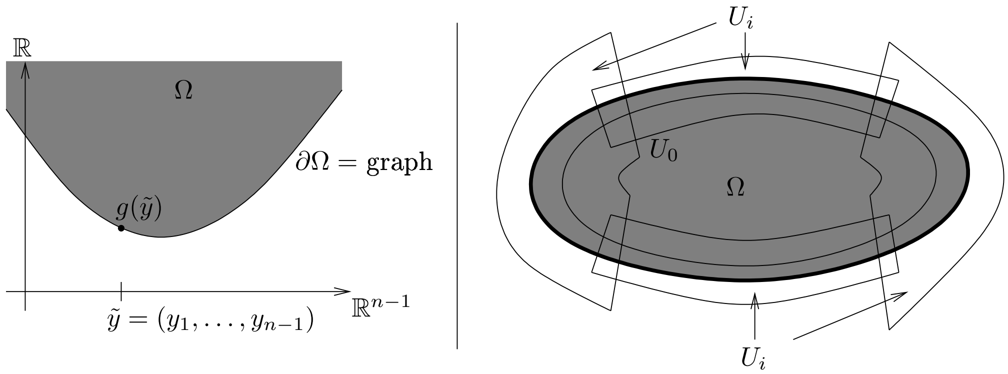


Figure 9.9:

Definition 9.49. $f : \partial\Omega \to \mathbb{R}$. *$f$ is measurable/integrable if $f(g(\tilde{y}))$ measurable integrable.*

Boundary integral: Choose U_0 such that $\overline{\Omega} \subset \bigcup_{i=0}^m U_i$. Now pick a partition of unity, η_i corresponding to these U_i such that $\sum_{i=1}^m \eta_i = 1$ on $\overline{\Omega}$

$$\int_{\partial\Omega} f \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} f \, dS = \sum_{i=1}^{m} \int_{\partial\Omega} \eta_i f \, dS$$

To make it clear what the integral in the last equality means, assume f has support in one U_i, then in that U_i with coordinates properly redefined,

$$\int_{\partial\Omega} f \, dS = \int_{\mathbb{R}^{n-1}} f(\tilde{y}, g(\tilde{y}))\sqrt{1 + |Dg(\tilde{y})|^2} \, d\tilde{y}.$$

So, it's clear how to extend this to general f since $\eta_i f$ does have support in U_i.

Remark: $\sqrt{1 + |Dg(\tilde{y})|^2} \, d\tilde{y}$ can be defined for g Lipschitz (derivatives well defined up except for a countable number of "kinks"). We will assume $\partial\Omega$ is C^1.

Definition 9.50.

$$\begin{aligned} L^p(\partial\Omega) &= \left\{ f : \partial\Omega \to \mathbb{R}, \int_{\partial\Omega} |f|^p \, dS < \infty \right\} \\ L^\infty(\partial\Omega) &= \left\{ f : \partial\Omega \to \mathbb{R}, \text{ ess} \sup_{\partial\Omega} |f| < \infty \right\} \end{aligned}$$

9.6.3 Extension of Sobolev Functions

Theorem 9.51 (Extension). *$\Omega \subseteq \mathbb{R}^n$, bounded, $\partial\Omega$ Lipschitz. Suppose Ω' open neighborhood of $\overline{\Omega}$, i.e. $\overline{\Omega} \Subset \Omega'$, then exists a bounded and linear operator*

$$E: \; W^{1,p}(\Omega) \to W_0^{1,p}(\Omega') \text{ such that } Ef\Big|_{\Omega} = f$$

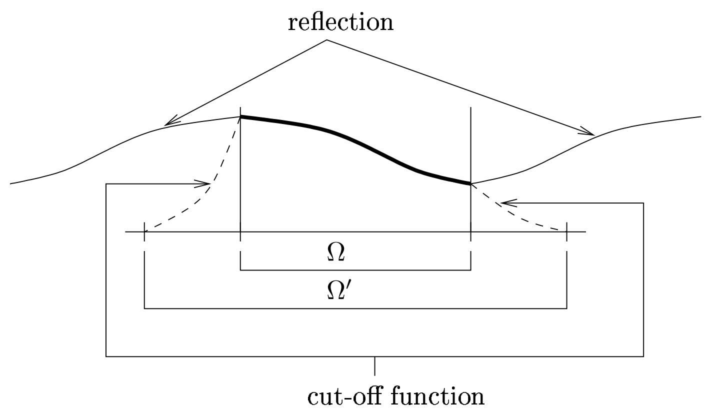


Figure 9.10:

Outline of the Proof: Reflect and multiply by cut-off functions.
C^1 boundary: $\exists$ transformation that maps $\partial\Omega$ locally to a hyperplane "straightening or flattening the boundary".

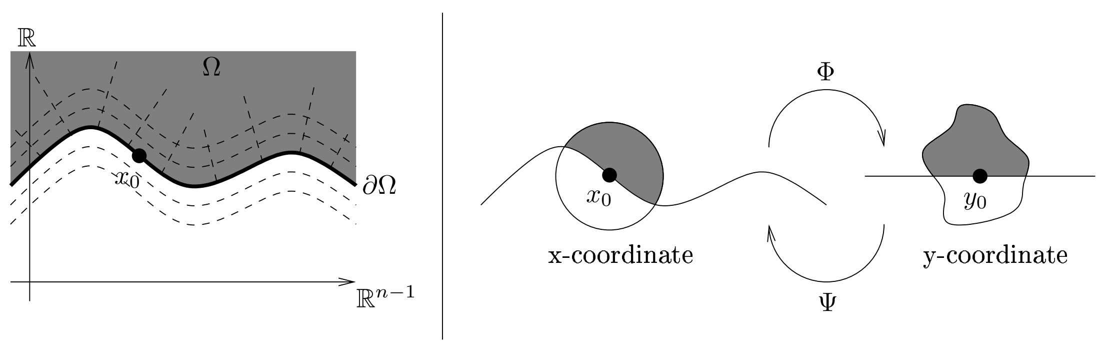


Figure 9.11:

Define Φ by

$$\begin{aligned} y_i &= x_i = \Phi^i(x),\ i = 1, \ldots, n-1 \\ y_n &= x_n - g(\tilde{x}) = \Phi^n(x),\ \tilde{x} = (x_1, \ldots, x_{n-1}) \\ &\implies \det(D\Phi(x)) = 1 \end{aligned}$$

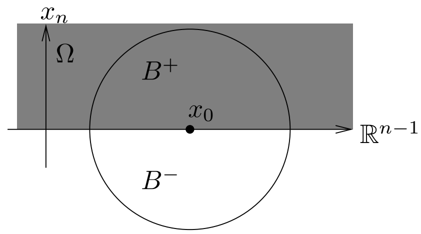


Figure 9.12:

1. $\partial\Omega$ flat near x_0, $\partial\Omega \subset \{x_n = 0\}$
   $B(x_0, R)$ such that $B(x_0, R) \cap \partial\Omega \subset \{x_n = 0\}$

$$\begin{aligned} B^+ &= B \cap \{x_n \geq 0\} \subset \overline{\Omega} \\ B^- &= B \cap \{x_n \leq 0\} \subset \mathbb{R}^n \setminus \Omega \end{aligned}$$

2. Approximate: suppose first that $u \in C^\infty(\overline{\Omega})$ extend u by higher order reflection:

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \quad := u^+ \\ -3u(\tilde{x}, -x_n) + 4u\left(\tilde{x}, -\frac{x_n}{2}\right) & \text{if } x \in B^- \quad := u^- \end{cases}$$

3. Check this is C^1 across the boundary.

   - $\overline{u}$ continuous $\{x_n = 0\}$
   - all tangential derivatives are continuous across $\{x_n = 0\}$

$$\begin{aligned} \frac{\partial \overline{u}}{\partial x_n}(x) &= 3\frac{\partial u}{\partial x_n}(\tilde{x} - x_n) - 2\frac{\partial u}{\partial x_n}\left(\tilde{x}, -\frac{x_n}{2}\right) \\ \left.\frac{\partial u}{\partial x_n}\right|_{\{x_n=0\}} &= \frac{\partial u}{\partial x_n}(\tilde{x}, 0) \end{aligned}$$

4. Calculate

$$\begin{aligned} ||\overline{u}||_{L^p(B)} &\leq C||u||_{L^p(B^+)} \\ ||\overline{Du}||_{L^p(B)} &\leq C||Du||_{L^p(B^+)} \end{aligned}$$

5. Straightening out of the boundary:

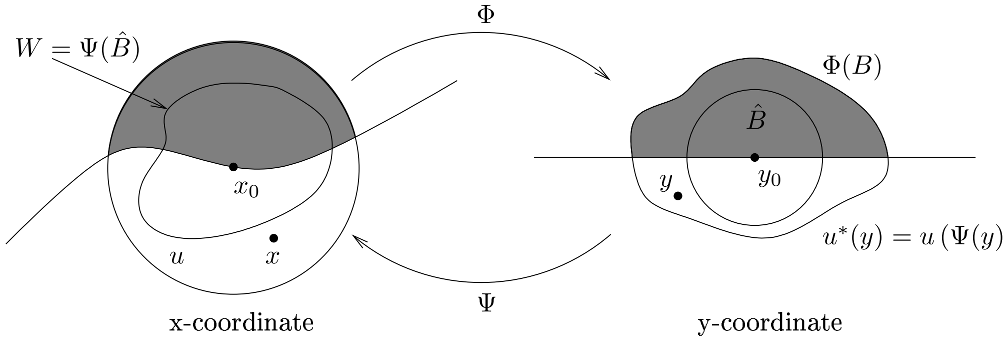


Figure 9.13:

Choose $\hat{B}(y_0, R_0) \subset \Phi(B)$. Construction for u^* (see above figure). Now extend $\overline{u}^*$ such that

$$\begin{aligned} ||\overline{u}^*||_{W^{1,p}(\hat{B})} &\leq C||\overline{u}^*||_{W^{1,p}(\hat{B}^+)} \\ &= C||u^*||_{W^{1,p}(\hat{B}^+)} \end{aligned}$$

Change coordinates from y back to x to get an extension $\overline{u}$ on W that satisfies

$$\begin{aligned} ||\overline{u}||_{W^{1,p}(W)} &\leq C||\overline{u}||_{W^{1,p}(W\cap\Omega)} \\ &\leq ||u||_{W^{1,p}(W)} \end{aligned}$$

<u>Remark:</u> The constant C depends (by the chain rule) on $||\Phi||_\infty$, $||\Psi||_\infty$, $||D\Phi||_\infty$, and $||D\Psi||_\infty$ but <u>not</u> on u.

6. $\forall x_0 \in \partial\Omega$, $W(x_0)$, $\overline{u}$ extension, $\partial\Omega$ is compact. Thus cover $\partial\Omega$ by finitely many sets $W_1, \ldots, W_m$ (with corresponding $\overline{u}_i$ extensions), choose W_0 such that $\overline{\Omega} \subseteq \bigcup_{i=0}^m W_i$, $u_0 = u$ and η_i the corresponding partition of unity.

$$\overline{u}(x) = \sum_{i=0}^{m} \eta_i \overline{u}_i$$

Making $W(x_0)$ smaller if necessary, we may assume that $W(x_0) \subset \Omega'$, thus $\overline{u} = 0$ outside Ω'. By Minkowski

$$||\overline{u}||_{W^{1,p}(\Omega')} \leq C||u||_{W^{1,p}(\Omega)}$$

7. Define $Eu = \overline{u}$, bounded and linear on $C^\infty(\overline{\Omega})$

8. Approximation: Refinement of approximation result.
   Find $u_m \in C^\infty(\overline{\Omega})$ such that $u_m \to u$ in $W^{1,p}(\Omega)$

$$\begin{aligned} ||\overline{u}_m - \overline{u}_n||_{W^{1,p}(\Omega')} &= ||Eu_m - Eu_n||_{W^{1,p}(\Omega')} \\ &\leq C||u_m - u_n||_{W^{1,p}(\Omega)} \to 0 \end{aligned}$$

   $\Longrightarrow Eu_m$ is Cauchy in $W^{1,p}(\Omega')$, and thus has a limit in this space, call it Eu. ■

9.6.4 Traces

Theorem 9.52. *Ω bounded domain in $\mathbb{R}^n$, Lipschitz boundary. $1 \leq p < \infty$. Then exists a unique linear bounded operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

$$Tu = u\Big|_{\partial\Omega} \qquad \textit{if } u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$$

Proof: $u \in C^1(\overline{\Omega})$, $x_0 \in \partial\Omega$, $\partial\Omega$ flat near x_0. $\exists B(x_0, R)$ such that $\partial\Omega \cap B(x_0, R) \subseteq \{x_n = 0\}$. Define $\hat{B} = B\left(x_0, \frac{R}{2}\right)$ and let $\Gamma = \partial\Omega \cap \hat{B}$.

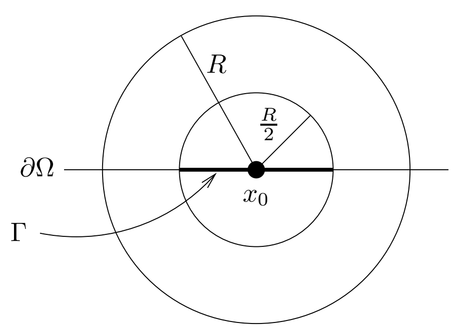


Figure 9.14:

$\tilde{x} = (x_1, \ldots, x_{n-1}) \in \{x_n = 0\}$
Choose $\xi \in C_c^\infty(B(x_0, R))$, $\xi \geq 0$, $\xi \equiv 1$ on $\hat{B}$

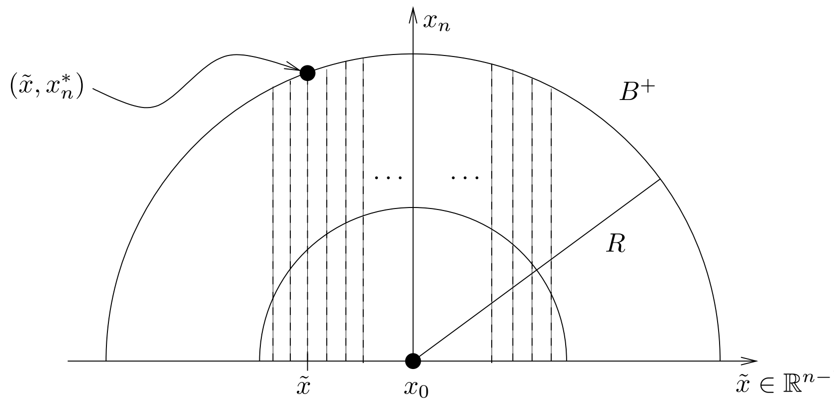


Figure 9.15:

<u>Note:</u> From the above figure we see

$$
\begin{aligned}
\xi|u|^p(\tilde{x}, 0) &= (\xi|u|^p)(\tilde{x}, 0) - (\xi|u|^p)(\tilde{x}, x_n^*) \\
&= \int_{x_n^*}^{0} (\xi|u|^p)_{x_n} \, dx_n
\end{aligned}
\tag{9.16}
$$

So we have for the main calculation:

$$
\begin{aligned}
\int_\Gamma |u(\tilde{x}, 0)|^p \, d\tilde{x} &\leq \int_{\mathbb{R}^{n-1}} \xi|u|^p \, d\tilde{x} \\
&= -\int_{B^+} (\xi|u|^p)_{x_n} \, dx \\
&= -\int_{B^+} \left(|u|^p \xi_{x_n} + \underbrace{\xi|u|^{p-1}}_{L^{p'}} \operatorname{sgn}(u) \underbrace{u_{x_n}}_{L^p} \right) dx \\
&\overset{\text{Hölder}}{\leq} -\int_{B^+} |u|^p \xi_{x_n} \, dx + C \left(\int_{B^+} |u|^p \, dx \right)^{\frac{1}{p'}} \left(\int_{B^+} |u_{x_n}|^p \, dx \right)^{\frac{1}{p}} \\
&\overset{\text{Young's}}{\leq} C \int_{B^+} (|u|^p + |Du|^p) \, dx
\end{aligned}
$$

General situation: non-linear change of coordinates just as before

$$
\int_\Gamma |u|^p \, dS \leq C \int_C (|u|^p + |Du|^p) \, dx
$$

where $\Gamma \subset \partial\Omega$ contains the points x_0.

Now, $\partial\Omega$ compact, choose finitely many open sets $\Gamma_i \subset \partial\Omega$ such that $\partial\Omega \subset \bigcup_{i=1}^m \Gamma_i$ with

$$\int_{\Gamma_i} |u|^p \, dS \quad \leq \quad C \int_C (|u|^p + |Du|^p) \, dx$$

$\Longrightarrow$ If we define for $u \in C^1(\overline{\Omega})$, $Tu = u\Big|_{\partial\Omega}$
$\Longrightarrow$ $||u||_{L^p(\partial\Omega)} \leq C||u||_{W^{1,p}(\Omega)}$

Approximation: $u \in W^{1,p}(\Omega)$ choose $u_k \in C^\infty(\overline{\Omega})$ such that

$$u_k \to u \quad \text{in } W^{1,p}(\Omega)$$

Estimate

$$||Tu_m - Tu_n||_{L^p(\Omega)} \leq C||u_m - u_n||_{W^{1,p}(\Omega)}$$

$\Longrightarrow$ Tu_m is a Cauchy sequence in $L^p(\partial\Omega)$.

Define

$$Tu = \lim_{k\to\infty} Tu_k \qquad \text{in } L^p(\partial\Omega)$$

If $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ then the approximation u_k converges uniformly on $\overline{\Omega}$ to u and $Tu = u\Big|_{\partial\Omega}$. ■

Remark: $W^{1,\infty}(\Omega)$ is space of all functions that have Lipschitz continuous representations.

Theorem 9.53 (Trace-zero functions in $W^{1,p}$). *Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose furthermore that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \iff Tu = 0$$

Proof: ($\Longrightarrow$) We are given $u \in W_0^{1,p}(\Omega)$. Then by the definition, $\exists u_m \in C_c^\infty(\Omega)$ such that

$$u_m \to u \qquad \text{in } W^{1,p}(\Omega).$$

clearly $Tu_m = 0$ on $\partial\Omega$. Thus,

$$\sup_{\partial\Omega} |Tu - Tu_m| \quad \leq \quad C \sup_{\partial\Omega} |u - u_m| = 0$$

Thus, $Tu = \lim_{m\to\infty} Tu_m = 0$.

($\Longleftarrow$) Assume $Tu = 0$ on $\partial\Omega$. As usual assume that $\partial\Omega$ is flat around a point x_0 and using standard partitions of unity we may assume for the following calculations that

$$\begin{aligned}&u \in W^{1,p}(\mathbb{R}^n_+),\ \operatorname{supp}(u) \Subset \overline{R}^n_+,\\ &Tu = 0,\ \text{on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}\end{aligned}$$

Since $Tu = 0$ on $\mathbb{R}^{n-1}$, there exists $u_m \in C^1(\overline{\mathbb{R}}^n_+)$ such that

$$u_m \to u \qquad \text{in } W^{1,p}(\mathbb{R}^n_+) \tag{9.17}$$

and

$$Tu_m = u_m\Big|_{\mathbb{R}^{n-1}} \to 0 \qquad \text{in } L^p(\mathbb{R}^{n-1}) \tag{9.18}$$

Now, if $\tilde{x} \in \mathbb{R}^{n-1}$, $x_n \geq 0$, we have by the fundamental theorem of calculus

$$\begin{aligned}|u_m(\tilde{x}, x_n)| &= |u_m(\tilde{x}, 0) + u_m(\tilde{x}, x_n) - u_m(\tilde{x}, 0)|\\ &\leq |u_m(\tilde{x}, 0)| + \int_0^{x_n} |u_{m,x_n}(\tilde{x}, t)|\ dt\\ &\overset{\text{Hölder}}{\leq} |u_m(\tilde{x}, 0)| + x_n^{\frac{p-1}{p}} \left(\int_0^{x_n} |u_{m,x_n}(\tilde{x}, t)|^p\ dt\right)^{\frac{1}{p}}\\ &\leq |u_m(\tilde{x}, 0)| + x_n^{\frac{p-1}{p}} \left(\int_0^{x_n} |Du_m(\tilde{x}, t)|^p\ dt\right)^{\frac{1}{p}}\end{aligned}$$

From this, we have

$$\int_{\mathbb{R}^{n-1}} |u_m(\tilde{x}, x_n)|^p\ d\tilde{x} \leq 2^{p-1} \left(\int_{\mathbb{R}^{n-1}} |u_m(\tilde{x}, 0)|^p\ d\tilde{x} + x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du_m(\tilde{x}, t)|^p\ d\tilde{x}dt\right)$$

Letting $m \to \infty$ and using (9.17) and (9.18), we have

$$\int_{\mathbb{R}^{n-1}} |u(\tilde{x}, x_n)|^p\ d\tilde{x} \leq 2^{p-1} x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du(\tilde{x}, t)|^p\ d\tilde{x}dt \tag{9.19}$$

Approximation: Next, let $\xi \in C^\infty(\mathbb{R})$. We require that $\xi \equiv 1$ on $[0, 1]$ and $\xi \equiv 0$ on $\mathbb{R} \setminus [0, 2]$ with $0 \leq \xi \leq 1$ and we write

$$\begin{cases} \xi_m(x := \xi(mx_n)\ (x \in \mathbb{R}^n_+)\\ w_m := u(x)(1 - \xi_m(x))\end{cases}$$

Then we have

$$\begin{cases} w_{m,x_n} &= u_{x_n}(1-\xi_m) - mu\xi' \\ D_{\tilde{x}} w_m &= D_{\tilde{x}} u(1-\xi_m) \end{cases}$$

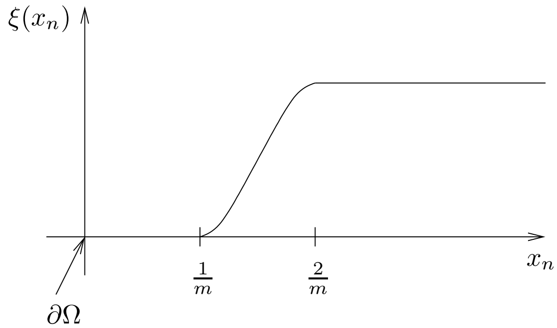


Figure 9.16:

A possible ξ

Consequently, we have

$$\begin{aligned} \int_{\mathbb{R}^n_+} |Dw_m - Du|^p \, dx &= \int_{\mathbb{R}^n_+} |-\xi_m Du - mu\xi'|^p \, dx \\ &\leq \int_{\mathbb{R}^n_+} |\xi_m|^p |Du|^p \, dx + Cm^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p \, d\tilde{x} dx_n \\ &=: A + B \end{aligned}$$

Note: In deriving B, C is picked to bound $|\xi'|^p$. Also the integral is restricted since $\xi' = 0$ if $x_n > 2/m$ by construction.

Now

$$\lim_{m\to\infty} A = 0$$

since $\xi_m \neq 0$ only if $0 \leq x_n \leq 2/m$. To estimate the term B, we use (9.19)

$$\begin{aligned} B &\leq C \cdot 2^{p-1} m^p \left(\frac{2}{m}\right)^p \left(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p \, d\tilde{x} dx_n\right) \\ &\leq C \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p \, d\tilde{x} dx_n \to 0 \text{ as } m \to \infty \end{aligned}$$

Thus, we deduce $Dw_m \to Du$ in $L^p(\mathbb{R}^n_+)$. Since clearly $w_m \to u$ in $L^p(\mathbb{R}^+_n)$ (is clear from construction of w_m). We can now conclude

$$w_m \to u \qquad \text{in } W^{1,p}(\mathbb{R}^n_+)$$

But $w_m = 0$ if $0 < x_n < 1/m$. We can therefore mollify the w_m to produce function $u_m \in C_c^\infty(\mathbb{R}^n_+)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$. Hence $w \in W_0^{1,p}(\mathbb{R}^n_+)$. ■.

9.6.5 Sobolev Inequalities

Theorem 9.54 (Sobolev-Gagliardo-Nirenberg Inequality). *Ω is a bounded set in $\mathbb{R}^n$. If $u \in W^{1,1}(\mathbb{R}^n)$, then*

$$||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |Du| \ dx.$$

Remark: In the above theorem means that $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and $||u||_{L^{p^*}(\Omega)} \leq C||u||_{W_0^{1,p}(\Omega)}$.

Proof: $C_c^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$.
First estimate for smooth functions, then pass to the limit in the estimate.
It thus suffices to prove

$$||f||_{L^{p^*}(\Omega)} \leq C||Df||_{L^p(\Omega)} \qquad \forall f \in C_c^\infty(\Omega)$$

Special case: $p = 1$, $p^* = \frac{n}{n-1}$

$n = 2$: $p^* = 2$

$$f(x) = f(x_1, x_2) = \int_{-\infty}^{x_1} D_1 f(t, x_2) \ dt$$
$$\implies \quad |f(x)| \leq \int_{-\infty}^{x_1} |D_1 f(t, x_2)| \ dt \leq \int_{-\infty}^{\infty} |D_1 f(t, x_2)| \ dt = g_1(x_2)$$

Same idea

$$\begin{aligned} |f(x)| &\leq \int_{-\infty}^{\infty} |D_2 f| \ dx_2 = g_2(x_1) \\ |f(x)|^2 &\leq g_1(x_2) g_2(x_1) \end{aligned}$$

$$\begin{aligned}
||f||_2^2 = \int_{\mathbb{R}^2} |f(x)|^2 \, dx &= \int_{\mathbb{R}^2} g_1(x_2) g_2(x_1) \, dx_1 dx_2 \\
&= \int_{\mathbb{R}} g_1(x_2) \, dx_2 \int_{\mathbb{R}} g_2(x_1) \, dx_1 \\
&= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_1 f| \, dx_1 dx_2 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 f| \, dx_2 dx_1 \right) \\
&\leq \left(\int_{\mathbb{R}^2} |Df| \, dx \right)^2
\end{aligned}$$

$\implies ||f||_{L^2(\mathbb{R}^2)} \leq ||Df||_{L^1(\mathbb{R}^2)}$.

<u>General Case:</u>

$$|f(x)| \leq g_i(x) = \int_{-\infty}^{\infty} |D_i f| \, dx_i$$

We have

$$\begin{aligned}
|f(x)|^{\frac{n}{n-1}} &\leq \left(\prod_{i=1}^n g_i \right)^{\frac{1}{n-1}} \\
\int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}} \, dx_1 &\leq \int_{\mathbb{R}} \left(\prod_{i=1}^n g_i \right)^{\frac{1}{n-1}} \, dx_1
\end{aligned}$$

Now taking Holder for $n-1$ terms with $p_i = n-1$

$$[g_1(x)]^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\prod_{i=2}^n g_i \right)^{\frac{1}{n-1}} \, dx_1 \leq [g_1(x)]^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{\mathbb{R}} g_i(x) \, dx_1 \right)^{\frac{1}{n-1}}$$

integrate with respect to x_2 to get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}} \, dx_1 dx_2 \\
&\qquad \leq \int_{\mathbb{R}} [g_1(x)]^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{\mathbb{R}} g_i(x) \, dx_1 \right)^{\frac{1}{n-1}} \, dx_2 \\
&\qquad = \left(\int_{\mathbb{R}} g_2(x) \, dx_1 \right)^{\frac{1}{n-1}} \left[\int_{\mathbb{R}} [g_1(x)]^{\frac{1}{n-1}} \prod_{i=3}^n \left(\int_{\mathbb{R}} g_i(x) \, dx_1 \right)^{\frac{1}{n-1}} \, dx_2 \right] \\
&\qquad \stackrel{\text{Hölder}}{\leq} \left(\int_{\mathbb{R}} g_2(x) \, dx_1 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} g_1(x) \, dx_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^n \left(\int_{\mathbb{R}} \int_{\mathbb{R}} g_i(x) \, dx_1 dx_2 \right)^{\frac{1}{n-1}}
\end{aligned}$$

Integrate $\int dx_3 \cdots \int dx_n$ + Holder

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}}\ dx &\leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} g_i(x)\ dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \right)^{\frac{1}{n-1}} \\ \text{(def. of } g_i) &= \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i f|\ dx \right)^{\frac{1}{n-1}}\end{aligned}$$

So this implies that

$$||f||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i f|\ dx \right)^{\frac{1}{n}} = \int_{\mathbb{R}^n} |Df|\ dx = ||Df||_{L^1(\mathbb{R}^n)}$$

We can do a little better than this by observing

$$\left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

(which is just a generalized Young's inequality). So we have

$$||f||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} |D_i f|\ dx \leq \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |Df|\ dx$$

Since

$$\sum_{i=1}^{n} a_i = \sqrt{n} \sqrt{\sum_{i=1}^{n} a_i^2}$$

In the general case use: $|f|^{\gamma}$ with γ suitably chosen. ■

Theorem 9.55 (Poincare's Inequality). *$\Omega \subseteq \mathbb{R}^n$ bounded, then*

$$||u||_{L^p(\Omega)} \leq C||Du||_{L^p(\Omega)} \quad \forall v \in W_0^{1,2}(\Omega), 1 \leq p \leq \infty$$

If $\partial\Omega$ is Lipschitz, then

$$||u - \overline{u}||_{L^p(\Omega)} \leq C||Du||_{L^p(\Omega)} \quad \forall v \in W_0^{1,2}(\Omega), 1 \leq p \leq \infty$$

where

$$\overline{u} = u_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u(x)\ dx = \fint_{\Omega} u(x)\ dx$$

Remark:

1. We need $u - \overline{u}$ to exclude constant functions
2. C depends on Ω, see HW for $\Omega = B(x_0, R)$

This will be an indirect proof with no explicit scaling given. Use additional scaling argument to find scaling.

Proof: Sobolev-Gagliardo-Nirenberg:

$$||u||_{L^{p^*}(\Omega)} \leq C_S ||Du||_{L^p(\Omega)}$$

Now we apply Holder under the assumption that Ω is bounded:

$$||u||_{L^p(\Omega)} \leq C(\Omega)||u||_{L^{p^*}(\Omega)} \leq C(\Omega)C_S||Du||_{L^p(\Omega)}$$

Proof of $||u - \overline{u}||_{L^p(\Omega)} \leq C||Du||_{L^p(\Omega)}$. Without loss of generality take $\overline{u} = 0$, other wise apply inequality to $v = u - \overline{u}$. Now we prove
$\underline{Dv = Du}$.
Suppose otherwise $\exists u_j \in W^{1,p}(\Omega)$, $\overline{u_j} = 0$, such that

$$||u_j||_{L^p(\Omega)} \geq j||Du_j||_{L^p(\Omega)}$$

May suppose, $||u_j||_{L^p(\Omega)} = 1$

$$\Longrightarrow \frac{1}{j} \geq ||Du_j||_{L^p(\Omega)} \Longrightarrow Du_j \to 0 \text{ in } L^p(\Omega)$$

and

$$||u_j||_{W^{1,p}(\Omega)} = ||u_j||_{L^p(\Omega)} + ||Du_j||_{L^p(\Omega)} \leq 1 + \frac{1}{j} \leq 2$$

Compact Sobolev embedding (See Section 9.7): $\exists$ subsequence such that $u_{jk} \in u$ in $L^p(\Omega)$ with $Du = 0 \Longrightarrow u$ is constant, $0 = \int_\Omega u_j \, dx \to \int_\Omega u \, dx$ $\Longrightarrow u = 0 \Longrightarrow 1 = ||u_{jk}||_{W^{1,p}(\Omega)} \to 0$. Contradiction. ■

9.6.6 The Dual Space of $H_0^1 = W_0^{1,2}$

Definition 9.56. $\Omega \subset \mathbb{R}^n$ *bounded,* $H^{-1}(\Omega) = (H_0^1)^* = (W_0^{1,2})^*$

Theorem 9.57. *Suppose that $T \in H^{-1}(\Omega)$. Then there exists functions $f_0, f_1, \dots, f_n \in L^2(\Omega)$ such that*

$$T(v) = \int_\Omega \left(f_0 v + \sum_{i=1}^n f_i D_i v \right) \, dx \qquad \forall v = H_0^1(\Omega) \tag{9.20}$$

Moreover,

$$||T||^2_{H^{-1}(\Omega)} = \inf \left\{ \int_\Omega \sum_{i=0}^n |f_i|^2 \, dx, \ f_i \in L^2(\Omega), \ (9.20) \ \textit{holds} \right\}$$

Remark: One frequently works formally with, $T = f_0 - \sum_{i=1}^n D_i f_i$

Proof: $u, v \in H_0^1(\Omega)$,

$$(u, v) = \int_\Omega (uv + Du \cdot Dv) \, dx$$

if $T \in H^{-1}(\Omega)$, then by Riesz, $\exists! u_0$ such that

$$\begin{aligned} T(u) &= (v, u_0) \qquad \forall v \in H_0^1(\Omega) \\ &= \int_\Omega (v u_0 + Dv \cdot Du_0) \, dx \end{aligned} \tag{9.21}$$

i.e. (9.20) with $f_0 = u_0$ and $f_i = D_i u_0$.

Suppose $g_0, g_1, \dots, g_n$ satisfy

$$T(v) = \int_\Omega (v g_0 + D_i v \cdot g_i) \, dx$$

take $v = u_0$ in (9.21)

$$\begin{aligned} ||u_0||^2_{H_0^1(\Omega)} = (u_0, u_0) &= T(u_0) = \int_\Omega \left(u_0 g_0 + \sum_{i=1}^n D_i u_0 g_i \right) \, dx \\ &\overset{\text{Hölder}}{\leq} ||u_0||_{L^2(\Omega)} ||g_0||_{L^2(\Omega)} + \sum_{i=1}^n ||D_i u_0||_{L^2(\Omega)} ||g_i||_{L^2(\Omega)} \\ &\overset{\text{C-S}}{\leq} ||u_0||_{H_0^1(\Omega)} \left(\int_\Omega |g_0|^2 + |g_1|^2 + \cdots + |g_n|^2 \right) \, dx \end{aligned}$$

$$(9.20) \Longrightarrow \int_\Omega (u_0 v + D_i u_0 \cdot D_i v) \, dx \leq ||v||_{H_0^1(\Omega)} ||u_0||_{H_0^1(\Omega)}$$

Now, we see that

$$||T||_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)} \frac{|T(v)|}{||v||_{H_0^1(\Omega)}} \leq ||u_0||_{H_0^1(\Omega)}$$

On the other hand, choose $v = \frac{u_0}{||u_0||_{H_0^1(\Omega)}}$

$$T(v) = ||u_0||_{H_0^1(\Omega)} \Longrightarrow ||T||_{H^{-1}(\Omega)} = ||u_0||_{H_0^1(\Omega)} \qquad \blacksquare$$

9.7 Sobolev Imbeddings

Theorem 9.58. *Ω is a bounded set in $\mathbb{R}^n$. Then the following embeddings are continuous*

i.) $1 \leq p < n$, $p^* = \frac{np}{n-p}$, $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$

ii.) $p > n$, $\alpha = 1 - \frac{n}{p}$, $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$

Remark:

1. i.) in the above theorem means that $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and $||u||_{L^{p^*}(\Omega)} \leq C||u||_{W_0^{1,p}(\Omega)}$
2. "Sobolev number": $W^{m,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, then $m - \frac{n}{p}$ is the Sobolev number.
   $L^q(\Omega) = W^{0,q}(\Omega)$, $-\frac{n}{q}$ is its Sobolev number.
   Philosophy: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $1 - \frac{n}{p} \geq -\frac{n}{q}$, where maximal q corresponds to equality.
   $$1 - \frac{n}{p} = -\frac{n}{q} \iff \frac{p-n}{p} = -\frac{n}{q} \iff \frac{n-p}{p} = \frac{n}{q} \iff q = \frac{np}{n-p} = p^*$$
3. Analogously $C^{k,\alpha}(\Omega)$ has a Sobolev number $k + \alpha$
   $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ if $1 - \frac{n}{p} \geq 0 + \alpha = \alpha$, optimal: $\alpha = 1 - \frac{n}{p}$.
4. More generally: $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $m - \frac{n}{p} \geq -\frac{n}{q}$ and the optimal q satisfies $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$.
   $W^{m,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ if $m - \frac{n}{p} \geq 0 + \alpha = \alpha$

Proof: $C_c^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$.
First estimate for smooth functions, then pass to the limit in the estimate.
It thus suffices to prove

$$||f||_{L^{p^*}(\Omega)} \leq C||Df||_{L^p(\Omega)} \qquad \forall f \in C_c^\infty(\Omega)$$

Special case: $p = 1$, $p^* = \frac{n}{n-1}$

<u>$n = 2$:</u> $p^* = 2$

$$f(x) = f(x_1, x_2) = \int_{-\infty}^{x_1} D_1 f(t, x_2)\ dt$$

$$\implies \quad |f(x)| \leq \int_{-\infty}^{x_1} |D_1 f(t, x_2)|\ dt \leq \int_{-\infty}^{\infty} |D_1 f(t, x_2)|\ dt = g_1(x_2)$$

Same idea

$$\begin{aligned} |f(x)| &\leq \int_{-\infty}^{\infty} |D_2 f|\ dx_2 = g_2(x_1) \\ |f(x)|^2 &\leq g_1(x_2) g_2(x_1) \end{aligned}$$

$$\begin{aligned} ||f||_2^2 = \int_{\mathbb{R}^2} |f(x)|^2\ dx &= \int_{\mathbb{R}^2} g_1(x_2) g_2(x_1)\ dx_1 dx_2 \\ &= \int_{\mathbb{R}} g_1(x_2)\ dx_2 \int_{\mathbb{R}} g_2(x_1)\ dx_1 \\ &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_1 f|\ dx_1 dx_2 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 f|\ dx_2 dx_1 \right) \\ &\leq \left(\int_{\mathbb{R}^2} |Df|\ dx \right)^2 \end{aligned}$$

$\implies ||f||_{L^2(\mathbb{R}^2)} \leq ||Df||_{L^1(\mathbb{R}^2)}$.

<u>General Case:</u>

$$|f(x)| \leq g_i(x) = \int_{-\infty}^{\infty} |D_i f|\ dx_i$$

We have

$$\begin{aligned} |f(x)|^{\frac{n}{n-1}} &\leq \left(\prod_{i=1}^n g_i \right)^{\frac{1}{n-1}} \\ \int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}}\ dx_1 &\leq \int_{\mathbb{R}} \left(\prod_{i=1}^n g_i \right)^{\frac{1}{n-1}}\ dx_1 \end{aligned}$$

Now taking Holder for $n-1$ terms with $p_i = n-1$

$$[g_1(x)]^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\prod_{i=2}^n g_i \right)^{\frac{1}{n-1}}\ dx_1 \leq [g_1(x)]^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{\mathbb{R}} g_i(x)\ dx_1 \right)^{\frac{1}{n-1}}$$

integrate with respect to x_2 to get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}}\ dx_1dx_2 \\
&\qquad \leq \int_{\mathbb{R}} [g_1(x)]^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} g_i(x)\ dx_1\right)^{\frac{1}{n-1}}\ dx_2 \\
&\qquad = \left(\int_{\mathbb{R}} g_2(x)\ dx_1\right)^{\frac{1}{n-1}} \left[\int_{\mathbb{R}} [g_1(x)]^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}} g_i(x)\ dx_1\right)^{\frac{1}{n-1}}\ dx_2\right] \\
&\qquad \overset{\text{Hölder}}{\leq} \left(\int_{\mathbb{R}} g_2(x)\ dx_1\right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} g_1(x)\ dx_2\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} g_i(x)\ dx_1dx_2\right)^{\frac{1}{n-1}}
\end{aligned}$$

Integrate $\int dx_3 \cdots \int dx_n$ + Holder

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}}\ dx &\leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} g_i(x)\ dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n\right)^{\frac{1}{n-1}} \\
(\text{def. of } g_i) &= \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_if|\ dx\right)^{\frac{1}{n-1}}
\end{aligned}$$

So this implies that

$$||f||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_if|\ dx\right)^{\frac{1}{n}} = \int_{\mathbb{R}^n} |Df|\ dx = ||Df||_{L^1(\mathbb{R}^n)}$$

We can do a little better than this by observing

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

(which is just a generalized Young's inequality). So we have

$$||f||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n}\int_{\mathbb{R}^n} \sum_{i=1}^{n} |D_if|\ dx \leq \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n} |Du|\ dx$$

Since

$$\sum_{i=1}^{n} a_i = \sqrt{n}\sqrt{\sum_{i=1}^{n} a_i^2}$$

In the general case use: $|f|^\gamma$ with γ suitably chosen. This result is referred to as the Sobolev-Gagliardo-Nirenberg inequality. ■

Recall: Sobolev-Gagliardo-Nirenberg:

$$||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{\sqrt{n}}||Du||_{L^1(\mathbb{R}^n)} \qquad u \in W_0^{1,1}(\mathbb{R}^n)$$

This was for $p = 1$, now consider

General Case: $1 \le p < n$, Use $p = 1$ for $|u|^\gamma$

$$\begin{aligned} ||\ |u|^\gamma\ ||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{\sqrt{n}}||\ D|u|^\gamma\ ||_{L^1(\mathbb{R}^n)} &\le \frac{1}{\sqrt{n}}||\ \gamma|u|^{\gamma-1}D|u|\ ||_{L^1(\mathbb{R}^n)} \\ \text{Holder} &\le \frac{\gamma}{\sqrt{n}}||\ |u|^{\gamma-1}\ ||_{L^{p'}(\mathbb{R}^n)} \cdot ||Du||_{L^p(\mathbb{R}^n)} \end{aligned}$$

Choose γ such that

$$\begin{aligned} & \frac{\gamma n}{n-1} = (\gamma-1)p' = (\gamma-1)\frac{p}{p-1} \\ \iff\quad & \gamma\left(\frac{n}{n-1} - \frac{p}{p-1}\right) = \frac{-p}{p-1} \\ \iff\quad & \gamma\frac{np-n-np+p}{(n-1)(p-1)} = -\frac{p}{p-1} \\ \iff\quad & \gamma\frac{p-n}{(n-1)(p-1)} = \frac{p}{p-1} \\ \iff\quad & \gamma = \frac{p(n-1)}{n-p} > 0 \end{aligned}$$

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\ dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}} \le \frac{(n-1)p}{n-p}\frac{1}{\sqrt{n}}||Du||_{L^p(\mathbb{R}^n)}$$

Performing some algebra on the exponent, we see

$$\begin{aligned} & \frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np} = \frac{1}{p^*} \\ & \Longrightarrow ||u||_{L^{p^*}(\mathbb{R}^n)} \le \frac{(n-1)p}{n-p}\frac{1}{\sqrt{n}}||Du||_{L^p(\mathbb{R}^n)} = C_S||Du||_{L^p(\mathbb{R}^n)} \end{aligned}$$

Remark: The estimate "explodes" as $p \to n$.

This is a proof for $W_0^{1,p}(\Omega)$

General case with Lipschitz boundary. Extension theorem:
$\Omega \Subset \Omega' \Longrightarrow$ extension $\overline{u} \in W_0^{1,p}(\Omega')$ such that $||u||_{W^{1,p}(\Omega')} \leq C||u||_{W^{1,p}(\Omega)}$.
Use estimate for $\overline{u}$ on Ω'

$$||u||_{L^{p^*}(\Omega)} \leq ||\overline{u}||_{L^{p^*}(\Omega')} \leq C_S||\overline{u}||_{W^{1,p}(\Omega')} \leq C_S C_E ||u||_{W^{1,p}(\Omega)} \quad \blacksquare$$

9.7.1 Companato Imbeddings

Theorem 9.59 (Morrey's Theorem on the growth of the Dirichlet integral). *Let u be in $H^{1,p}_{loc}(B_1)$ and suppost that*

$$\int_{B_\rho(x)} |Du|^p\,dx \leq \rho^{n-p+p\alpha} \qquad \forall \rho < dist(x, B_1)$$

then $u \in C^{0,\alpha}_{loc}(B_1)$.

Proof: We use Poincare's inequality:

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^p\,dx \leq C\rho^p \int_{B_\rho(x_0)} |Du|^p\,dx \leq C \cdot \rho^{n+p\alpha}$$

i.e. $u \in \mathcal{L}^{p,\lambda}_{\text{loc}}(B_1)$ and therefore by 1) $u \in C^{0,\alpha}_{\text{loc}}(B_1)$. $\blacksquare$

Theorem 9.60 (Global Version of (9.59)). *If $u \in H^{1,p}(\Omega)$, $p > n$, then $u \in \mathcal{L}^{1,n+(1-n/p)}(\Omega) \Longrightarrow u \in C^{0,1-n/p}(\Omega)$ via Companato's Theorem.*

Proof: For all $\rho < \rho_0$ and a $u \in H^{1,p}(\Omega)$ we have

$$\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p\,dx \leq C\rho^p \int_{\Omega(x_0,\rho)} |Du|^p\,dx,$$

where C only depends on the geometry of Ω; then we fine

$$\int_{\Omega(x_0,\rho)} |Du|\,dx \leq \left(\int_{\Omega(x_0,\rho)} |Du|^p\,dx\right)^{1/p} |\Omega(x_0,\rho)|^{1-1/p} \leq C \cdot ||u||_{H^{1,p}(\Omega)}\rho^{n-n/p}.$$

Again by Poincare's inequality:

$$\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|\,dx_\rho \int_{\Omega(x_0,\rho)} |Du|\,dx \leq C\rho^{n+1-n/p}||u||_{H^{1,p}(\Omega)} \quad \blacksquare$$

Corollary 9.61 (Morrey's theorem). *For $p > n$, the imbedding*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-n/p}(\overline{\Omega})$$

is a continous operator, i.e.

$$||u||_{C^{0,1-n/p}(\overline{\Omega})} \leq C||u||_{W^{1,p}(\Omega)}$$

Remark: So, from the above we see

$$\begin{aligned} Du \in L^{p,\lambda}(\Omega) &\implies u \in \mathcal{L}^{p,\lambda+p}(\Omega) \\ (Du \in L^{2,n-2+2\alpha}(\Omega) &\implies u \in \mathcal{L}^{2,n+2\alpha}(\Omega)). \end{aligned}$$

Theorem 9.62 (Estimates for $W^{1,p}$, $n < p \leq \infty$). *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega$ being C^1. Assume $n < p \leq \infty$, and $u \in W^{1,p}(\Omega)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{\Omega})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$||u^*||_{C^{0,\gamma}(\overline{\Omega})} \leq C||u||_{W^{1,p}(\Omega)}.$$

The constant C depends only on p, n and Ω.

Proof: Since $\partial\Omega$ is C^1, there exists an extension $Eu = \overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \overline{u} = u \text{ in } \Omega, \\ \overline{u} \text{ has compact support, and} \\ ||\overline{u}||_{W^{1,p}(\Omega)}. \end{cases}$$

Since $\overline{u}$ has compact support, we know that there exists $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to \overline{u} \text{ in } W^{1,p}(\mathbb{R}^n)$$

Now from the previous theorem, $||u_m - u_l||_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C||u_m - u_l||_{W^{1,p}(\mathbb{R}^n)}$ for all $l, m \geq 1$, thus there exists a function $u^* \in C^{0,1-n/p}(\mathbb{R}^n)$ such that

$$u_m \to u^* \quad C^{0,1-n/p}(\mathbb{R}^n)$$

So we see from the last two limits, $u^* = u$ a.e. on Ω; such that u^* is a version of u. The last theorem also implies $||u_m||_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C||u_m||_{W^{1,p}(\mathbb{R}^n)}$ this combined with the last two limits, we get

$$||u^*||_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C||\overline{u}||_{W^{1,p}(\mathbb{R}^n)} \qquad \blacksquare$$

9.7.2 Compactness of Sobolev/Morrey Imbeddings

$1 \leq p < n$, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact if $q < p^*$ i.e. if $\{u_j\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ then $\{u_j\}$ is precompact in $L^q(\Omega)$, i.e., contains a subsequence that converges in $L^q(\Omega)$.

Proposition 9.63. *X is a BS*

i.) $A \subset X$ is precompact $\iff$ $\forall \epsilon > 0$ there exists finitely many balls $B(x_i, \epsilon)$ such that $A \subset \bigcup_{i=1}^{m_\epsilon} B(x_i, \epsilon)$

ii.) A is precompact $\iff$ every bounded sequence in A contains a converging subsequence (with limit in X).

Theorem 9.64 (Arzela-Ascoli). *$S \subseteq \mathbb{R}^n$ compact, $A \subseteq C^0(S)$ is precompact $\iff$ A is bounded and equicontinuous.*

Proof: ($\Longrightarrow$) $\forall \epsilon > 0$ $\exists B(f_i^\epsilon, \epsilon)$ such that $A \subseteq \bigcup_{i=1}^{m_\epsilon} B(f_i^\epsilon, \epsilon)$

$$\begin{aligned} ||f||_\infty &\leq ||f - f_i^\epsilon|| + ||f_i^\epsilon||_\infty, \ f \in B(f_i^\epsilon, \epsilon) \\ &\leq \epsilon + \max_{i=1,\ldots,m_\epsilon} ||f_i^\epsilon|| < \infty \end{aligned}$$

$$\begin{aligned} |f(x) - f(y)| &\leq 2\epsilon + |f_\epsilon^i(x) - f_\epsilon^i(y)| \\ &\leq 2\epsilon + \max_{i=1,\ldots,m_\epsilon} |f_\epsilon^i(x) - f_\epsilon^i(y)| \end{aligned}$$

so if $|x - y|$ tends to zero $\Longrightarrow$ equicontinuity.

($\Longleftarrow$) $\forall \epsilon > 0$, $||f||_\infty \leq C$ $\forall f \in A$

Choose $\xi_i \in \mathbb{R}$, $i = 1, \ldots, k$ such that $B(0, C) = [-C, C] \subset \bigcup_{i=1}^k B(\xi_i, \epsilon)$.
$x_j \in \mathbb{R}^n$, $j = 1, \ldots, m$ such that $S \subset \bigcup_{i=1}^m B(x_j, \epsilon)$. Basically we are covering the range and domain.

Let $\pi : \{1, \ldots, m\} \to \{1, \ldots, k\}$
Let $A_\pi = \{f \in A : |f(x_j) - \xi_{\pi(j)}| < \epsilon, j = 1, \ldots, m\}$

Choose $f_\pi \in A_\pi$ if $A_\pi \neq 0$. If $x \in S$ then $x \in B(x_j, \epsilon)$ for some j and $f \in A_\pi$.

Claim: $A \subset \bigcup_\pi B(f_\pi, \epsilon)$, i.e. the (finitely many) functions f_π define the centers of the balls. To show this, we see that

$$\begin{aligned} |f(x) - f_\pi(x)| &\leq |f(x) - f(x_j)| + \underbrace{|f(x_j) - \xi_{\pi(j)}|}_{\leq \epsilon} \\ &\quad + \underbrace{|\xi_{\pi(j)} - f_\pi(x_j)|}_{\leq \epsilon} + |f_\pi(x_j) - f_\pi(x)| \\ &\leq 2\epsilon + \sup_{|x-y|\leq\epsilon} \sup_{g\in A} |g(x) - g(y)| := r_\epsilon \end{aligned}$$

$r_\epsilon \to 0$ as $\epsilon \to 0$ since A is equicontinuous.
$\Longrightarrow f \in B(f_\pi, r_\epsilon)$. Suppose you want to find a finite cover $A \subset \bigcup B(f_\pi, \delta)$, $\delta > 0$ then choose ϵ small enough such that $r_\epsilon \leq \delta$. ■

Remark: If X, Y, and Z are Banach Spaces and we have $X \hookrightarrow Y \subset\subset\hookrightarrow Z$ or $X \subset\subset\hookrightarrow Y \hookrightarrow Z$, then $X \subset\subset\hookrightarrow Z$.

Theorem 9.65 (Compactness of Sobolev Imbeddings). *If $\Omega \subset \mathbb{R}^n$ bounded, then the following Imbedding is true*

i.) $1 \leq p < n$, $q < p^* = \frac{np}{n-p}$

$$W_0^{1,p}(\Omega) \subset\subset\hookrightarrow L^q(\Omega)$$

Moreover, if $\partial\Omega$ is Lipschitz, then the same is true for $W^{1,p}(\Omega)$

ii.) If $\Omega \subset \mathbb{R}^n$ bounded and satisfies condition () with $p > n$, $1 - \frac{n}{p} > \alpha$,

$$W_0^{1,p}(\Omega) \subset\subset\hookrightarrow C^{0,\alpha}(\overline{\Omega})$$

Remark: A special case of the above is $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$; This is called **Rellich's Theorem**.

Proof:

i.) $q = 1$ $\{f_j\}$ bounded in $W_0^{1,p}(\Omega)$ need to prove that $\{f_j\}$ is precompact in L^1.

Extend by zero to $\mathbb{R}^n$. Without loss of generality, take f_j to be smooth with $\tilde{f}_j \in C_c^\infty(\mathbb{R}^n)$, $(||f_j - \tilde{f}_j|| \leq \epsilon)$.

1. $A_\epsilon = \{\phi_\epsilon * f_j\}$, where ϕ_ϵ is the standard mollifier, $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$.
   Then A_ϵ precompact in L^1 $\forall \epsilon \geq 0$. We now show this.
   <u>Idea:</u> Use Arzela-Ascoli

$$\begin{aligned}
|\phi_\epsilon * f_j|(x) &= \left|\int_{\mathbb{R}^n} \phi_\epsilon(x-y)f(y)\ dy\right| \\
&\leq \sup|\phi_\epsilon| \int_{\mathbb{R}^n} |f(y)|\ dy \\
&= \sup|\phi_\epsilon| \cdot ||f_j||_{L^1(\mathbb{R}^n)} \\
&\leq \sup|\phi_\epsilon| \cdot ||f_j||_{L^p(\mathbb{R}^n)} < C_\epsilon < \infty
\end{aligned}$$

   Equicontinuity follows if $D(\phi_\epsilon * f_i)$ uniformly bounded,

$$|D\phi_\epsilon * f_j|(x) \leq \sup|D\phi_\epsilon| \cdot ||f_j||_{L^p(\mathbb{R}^n)} < C_\epsilon < \infty$$

   $\Longrightarrow$ $(\phi_\epsilon * f_j)$ uniformly bounded hence equicontinuous.
   $\Longrightarrow$ precompact.

   Step 2: $||\phi_\epsilon * f_j - f_j||_{L^1(\mathbb{R}^n)} \leq C\epsilon \quad \forall j$

2. (Proof of Step 2)

$$\begin{aligned}
(\phi_\epsilon * f_j - f_j)(x) &= \int_{\mathbb{R}^n = B(x,\epsilon)} \phi_\epsilon(x-y)(f_j(y) - f_j(x))\ dy \\
(\text{sub. } y = x - \epsilon z) &= \int_{\mathbb{R}^n = B(0,1)} \phi(z)\underbrace{(f_j(x-\epsilon z) - f_j(x))}_{\int_0^1 \frac{d}{dt} f_j(x - t\epsilon z)\ dt}\ dy
\end{aligned}$$

   Integrate in x

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\phi_\epsilon * f_j - f_j|\ dx \\
\leq\ & \int_{\mathbb{R}^n}\int_{B(0,1)} \phi(z) \int_0^1 |Df_j(x - \epsilon t z)| \cdot |\epsilon z|\ dt dz dx \\
(\text{Fubini}) =\ & \epsilon \int_{B(0,1)} \phi(z) \int_0^1 \underbrace{\int_{\mathbb{R}^n} |Df_j(x - t\epsilon z)| \cdot \underbrace{|z|}_{\leq 1}\ dx}_{= \int_{\mathbb{R}^n} |Df_j|(\tilde{x})\ d\tilde{x}}\ dt dz \\
\leq\ & \epsilon \left(\int_{\mathbb{R}^n} |Df_j|(\tilde{x})\ d\tilde{x}\right) \int_{B(0,1)} \phi(z)\ dz \leq \epsilon ||f_j||_{L^1(\mathbb{R}^n)} \leq \epsilon ||f_j||_{L^p(\mathbb{R}^n)} \\
\leq\ & \epsilon C
\end{aligned}$$

$\Longrightarrow ||\phi_\epsilon * f_j - f_j||_{L^1(\mathbb{R}^n)} \leq \epsilon C$ uniformly $\forall j$

3. Suppose $\delta > 0$ given, choose ϵ small enough such that $\epsilon C < \frac{\delta}{2}$

   $A_\epsilon = \{\phi_\epsilon * f_j\}$ is precompact in $L^1(\mathbb{R}^n)$, for $\frac{\delta}{2}$ we find balls $B\left(g_i, \frac{\delta}{2}\right)$ such that

$$A_\epsilon \subseteq \bigcup_{i=1}^{m_\epsilon} B\left(g_i, \frac{\delta}{2}\right) \Longrightarrow \{f_j\} \subseteq \bigcup_{i=1}^{m_\delta} B(g_i, \delta) \Longrightarrow f_j$$

   precompact in $L^1(\mathbb{R}^n)$.

   Remark: Evans constructs a converging subsequence.

4. general $q < p^*$.
   Trick is to use the "interpolation inequality":

$$||g||_{L^q(\mathbb{R}^n)} \leq ||g||^\theta_{L^1(\mathbb{R}^n)} ||g||^{1-\theta}_{L^{p^*}(\mathbb{R}^n)}$$

   where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. So if $f_j \to f$ in $L^1(\mathbb{R}^n)$ then

$$||f_j - f||_{L^q(\mathbb{R}^n)} \leq \underbrace{||f_j - f||^\theta_{L^1(\mathbb{R}^n)}}_{\to 0} \underbrace{||f_j - f||^{1-\theta}_{L^{p^*}(\mathbb{R}^n)}}_{\leq C}$$

   So, we just need to prove the interpolation inequality.
   Idea: Use Holder with $r = \frac{1}{q\theta}$ with the corresponding conjugate index

$$f' = \frac{r}{r-1} = \frac{\frac{1}{q\theta}}{\frac{1}{q\theta} - 1} = \frac{1}{1 - q\theta}$$

   Thus

$$\frac{(1-\theta)q}{1 - q\theta} = p^*$$

   So,

$$\begin{aligned} \int g^q \, dx &= \int g^{q\theta} g^{(1-\theta)q} \, dx \\ &\overset{\text{Hölder}}{\leq} \left(\int g \, dx\right)^{q\theta} \left(\int g^{p^*} \, dx\right)^{1-q\theta} \end{aligned}$$

Now take the qth root

$$\begin{aligned} ||g||_{L^q(\mathbb{R}^n)} &\le ||g||^{\theta}_{L^1(\mathbb{R}^n)} ||g||^{(1-q\theta)p^*/q}_{L^{p^*}(\mathbb{R}^n)} \\ &= ||g||^{\theta}_{L^1(\mathbb{R}^n)} ||g||^{1-\theta}_{L^{p^*}(\mathbb{R}^n)} \quad \blacksquare \end{aligned}$$

ii.) General Idea: We will prove that $W^{1,p}(\Omega) \hookrightarrow C^{0,\mu}(\overline{\Omega}) \subset\subset\hookrightarrow C^{0,\lambda}(\overline{\Omega})$, where $0 < \mu < \lambda < 1 - \frac{n}{p}$ and then we will use the remark following the statement of the Arzela-Ascolli theorem to conclude $W^{1,p}(\Omega) \subset\subset\hookrightarrow C^{0,\lambda}(\overline{\Omega})$.

Since $0 < \lambda < 1 - \frac{n}{p}$, we know $\exists \mu$ with $0 < \mu < \lambda < 1 - \frac{n}{p}$; and by Morrey's theorem we know $W^{1,p}(\Omega) \hookrightarrow C^{0,\mu}(\overline{\Omega})$. We now only need to show $C^{0,\mu}(\overline{\Omega}) \subset\subset\hookrightarrow C^{0,\lambda}(\overline{\Omega})$. It is obvious that $C^{0,\mu}(\overline{\Omega}) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ is continuous, so our proof is reduced to showing the compactness of this imbedding. We do this in two steps

Step 1. Claim $C^{0,\nu}(\overline{\Omega}) \subset\subset\hookrightarrow C(\overline{\Omega})$. Again, the continuity of the imbedding is obvious; and compactness is the only thing needed to be proven. If A is bounded in $C^{0,\nu}(\overline{\Omega})$, then we know $||\phi||_{C^{0,\nu}(\overline{\Omega})} \le M$ for all $\phi \in A$. This obviously implies that $|\phi(x)| \le M$ in addition to $|\phi(x) - \phi(y)| \le M|x-y|^{\nu}$ for all $x, y \in \overline{\Omega}$. Thus, A is precompact via the Arzela-Ascoli theorem.

Step 2. Consider

$$\begin{aligned} \frac{|\phi(x) - \phi(y)|}{|x-y|^{\lambda}} = \left(\frac{|\phi(x) - \phi(y)|}{|x-y|^{\mu}} \right)^{\lambda/\mu} |\phi(x) - \phi(y)|^{1-\lambda/\mu} \\ \le \quad M \cdot |\phi(x) - \phi(y)|^{1-\lambda/\mu} \end{aligned}$$

for all $\phi \in A$ for which A is bounded in $C^{0,\lambda}(\overline{\Omega})$. We now use the compactness of the imbedding in Step 1. by extracting a subsequence of this bounded sequence converging in $C(\overline{\Omega})$. With this new sequence () indicates this sequence is Cauchy and so converges in $C^{0,\lambda}(\overline{\Omega})$

This completes the proof. $\blacksquare$

9.7.3 General Sobolev Inequalities

Theorem 9.66 (General Sobolev inequalities). . *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary.*

i.) If

$$k < \frac{n}{p}, \tag{9.22}$$

then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n},$$

with an imbedding constant depending on k, p, n *and* Ω.

ii.) If

$$k > \frac{n}{p}, \tag{9.23}$$

then $C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega}) \hookrightarrow W^{k,p}(\Omega)$, *where*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

with an imbedding constant depending on k, p, n, γ *and* Ω.

Proof. Step 1: Assume (9.22). Then since $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| = k$, the Sobolev-Nirenberg-Gagliardo inequality implies

$$||D^\beta u||_{L^{p^*}(\Omega)} \le C||u||_{W^{k,p}(\Omega)} \quad \text{if } |\beta| = k-1,$$

and so $u \in W^{k-1,p^*}(\Omega)$. Similarly, we find $u \in W^{k-2,p^{**}}(\Omega)$, where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$. Continuing, we eventually discover after k steps that $u \in W^{0,q}(\Omega) = L^q(\Omega)$, for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. The corresponding continuity estimate follows from multiplying the relevant estimates at each stage of the above argument.

Step 2: Assume now condition (**??**) holds, and $\frac{n}{p}$ is not an integer. Then as above we see

$$u \in W^{k-l,r}(\Omega), \tag{9.24}$$

for

$$\frac{1}{r} = \frac{1}{p} - \frac{l}{n}, \tag{9.25}$$

provided $lp < n$. We choose the integer l so that

$$l < \frac{n}{p} < l + 1; \tag{9.26}$$

that is, we set $l = \left[\frac{n}{p}\right]$. Consequently (9.25) and (9.26) imply $\frac{pn}{n-pl} > n$. Hence (9.24) and Morrey's inequality imply that $D^\alpha u \in C^{0,1-\frac{n}{r}}(\overline{\Omega})$ for all $|\alpha| < k - l - 1$. Observe also that $1 - \frac{n}{r} = 1 - \frac{n}{p} + l = \left[\frac{n}{p}\right] + 1 - \frac{n}{p}$. Thus $u \in C^{k-\left[\frac{n}{p}\right]-1,\left[\frac{n}{p}\right]+1-\frac{n}{p}}(\overline{\Omega})$, and the stated estimate follows.

Step 3: Finally, suppose (9.25) holds, with $\frac{n}{p}$ an integer. Set $l = \left[\frac{n}{p}\right] - 1 = \frac{n}{p} - 1$. Consequently, we have as above $u \in W^{k-l,r}(\Omega)$ for $r = \frac{pn}{n-pl} = n$. Hence the Sobolev-Nirenberg-Gagliardo inequality shows $D^\alpha u \in L^q(\Omega)$ for all $n \le q < \infty$ and all $|\alpha| \le k - l - 1 = k - \left[\frac{n}{p}\right]$. Therefore Morrey's inequality further implies $D^\alpha u \in C^{0,1-\frac{n}{q}}(\overline{\Omega})$ for all $n < q < \infty$ and all $|\alpha| \le k - \left[\frac{n}{p}\right] - 1$. Consequently $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega})$ for each $0 < \gamma < 1$. As before, the stated continuity estimate follows as well.

□

9.8 Difference Quotients

In this section we develop a tool that will enable us to determine the existence of derivatives of certain solutions to various PDE. More succinctly, studies that pertain to this pursuit are referred to collectively as regularity theory. We will go into this more in later chapters.

Recalling 1D calculus, we know that f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. In nD with $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we analogize to say that f is differentiable at x_0 in the direction of s if nD: $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$,

$$\lim_{h \to 0} \Delta_h^k f(x) = \lim_{h \to 0} \frac{f(x + he_s) - f(x)}{h}, \; e_k = (0, \dots, 1, \dots, 0)$$

exists, where

$$e_s = (0, \dots, \underbrace{1}_{s\text{th place}}, \dots, 0);$$

i.e., to have the s partial derivative, $\lim_{h\to 0} \Delta_h^k f(x)$ must exist.

The power of using difference quotients is conveniently summed up in the following proposition.

Proposition 9.67. *If $1 \leq p < \infty$, $f \in W^{1,p}(\Omega)$, $\Omega' \Subset \Omega$, and h small enough with $h < dist(\Omega', \partial\Omega)$, then*

i.)

$$||\Delta_h^k f||_{L^p(\Omega')} \leq C||D_k f||_{L^p(\Omega)}, \qquad k = 1, \ldots, n.$$

ii.) Moreover, if $1 < p < \infty$ and

$$||\Delta_h^k f||_{L^p(\Omega')} \leq K \qquad \forall h < dist(\Omega', \partial\Omega),$$

then $f \in W^{1,p}_{loc}(\Omega)$ and $||D_k f||_{L^p(\Omega')} \leq K$ with $\Delta_h^k f \to D_k f$ strongly in $L^p(\Omega')$.

Proof: First, consider $f \in W^{1,p}(\Omega) \cap C^1(\Omega)$. We start off by calculating

$$\begin{aligned} |\Delta_h^k f(x)| &= \left| \frac{f(x + he_k) - f(x)}{h} \right| \\ &\leq \frac{1}{h} \int_0^h |D_k f(x + te_k)| \; dt \\ &\overset{\text{Hölder}}{\leq} h^{-1/p} \left(\int_0^h |D_k f(x + te_k)|^p \; dt \right)^{\frac{1}{p}}. \end{aligned}$$

Taking the pth power, we obtain

$$|\Delta_h^k f(x)|^p \leq \frac{1}{h} \left(\int_0^h |D_k f(x + te_k)|^p \; dt \right).$$

Next we integrate both sides over Ω':

$$\begin{aligned} \int_{\Omega'} |\Delta_h^k f(x)|^p \; dx &\leq \frac{1}{h} \int_0^h \int_{\Omega'} |D_k f(x + te_k)|^p \; dxdt \\ &\leq \frac{1}{h} \int_0^h \int_{\Omega' + te_k} |D_k f(x)|^p \; dxdt \\ &\leq \frac{1}{h} \int_0^h \int_{\Omega} |D_k f(x)|^p \; dxdt \\ &= \int_{\Omega} |D_k f|^p \; dx \leq K. \end{aligned}$$

Now, we need to verify the last calculation for general f; we do this by the standard argument that $W^{1,p}(\Omega) \cap C^1(\Omega)$ is dense in $W^{1,p}(\Omega)$. Indeed, approximate $f \in W^{1,p}(\Omega)$ by $f_n \in W^{1,p}(\Omega) \cap C^1(\Omega)$ such that $f_n \to f$ in $W^{1,p}(\Omega)$ and use the above estimate for f_n to ascertain

$$\begin{array}{ccc} \displaystyle\int_{\Omega'} |\Delta_h^k f_n|^p \, dx & \leq & \displaystyle\int_{\Omega} |D_k f_n|^p \, dx \\ \downarrow \ L^p(\Omega) & & \downarrow \ L^p(\Omega) \\ \displaystyle\int_{\Omega'} |\Delta_h^k f|^p \, dx & \leq & \displaystyle\int_{\Omega} |D_k f|^p \, dx \quad \forall f \in W^{1,p}(\Omega), \end{array}$$

for h fixed. This proves i.).

Now suppose that the last conclusion holds for $h < \text{dist}(\Omega', \partial\Omega)$; i.e. since $f \in W^{1,p}(\Omega)$ we see that $\Delta_h^k f$ is uniformly bounded in $L^p(\Omega')$. Thus, we have established the stipulations of part ii.). Next, we construct a sequence $\{\Delta_{h_n}^k f\}$ by considering $h_n \searrow 0$. Taking the boundedness $\Delta_h^k f$ in $L^p(\Omega')$ we apply the Banach-Alaoglu theorem to establish the existence of a weakly convergent subsequence of $\{\Delta_{h_n}^k f\}$. Upon relabeling such a subsequence, we define g_k as the weak limit, i.e. $\Delta_{h_n}^k f \rightharpoonup g_k$ in $L^p(\Omega')$. Now, we claim that $g_k = D_k f$ on Ω'. Claim: $g_k = D_k f$ on Ω', i.e.,

$$\int_{\Omega'} f \cdot D_k \phi \, dx = -\int_{\Omega'} g_k \phi \, dx \qquad \forall \phi \in C_c^\infty(\Omega').$$

To this end, we calculate

$$\begin{aligned} \int_{\Omega'} \Delta_h^k f \cdot \phi \, dx &= \int_{\Omega'} \frac{f(x+he_k) - f(x)}{h} \phi(x) \, dx \\ &= \frac{1}{h} \int_{\Omega'} f(x+he_k)\phi(x) \, dx - \frac{1}{h} \int_{\Omega'} f(x)\phi(x) \, dx \\ &= \frac{1}{h} \int_{\Omega' + he_k} f(x)\phi(x-he_k) \, dx - \frac{1}{h} \int_{\Omega'} f(x)\phi(x) \, dx \\ &= \int_{\Omega} f(x) \left[-\frac{\phi(x-he_k) - \phi(x)}{-h} \right] \, dx \\ &= -\int_{\Omega} f(x)(\Delta_{-h}^k \phi(x)) \, dx \end{aligned}$$

Now as $D_k\phi, \Delta_{-h}^k \phi(x) \in C_c^\infty(\Omega)$, we can use a simple real-analysis argument to say that $\Delta_{-h}^k \phi \to D_k \phi$ uniformly in Ω'. Thus, we have shown

$$\int_{\Omega'} g_k \phi \, dx = -\int_{\Omega'} f \cdot D_k \phi \, dx \qquad \forall \phi \in C_c^\infty(\Omega'),$$

proving the claim. Moreover, this last equality clearly indicate that $g_k \in L^p(\Omega')$, as $f \in W^{1,p}(\Omega)$. Now, we just have to show that $\Delta_h^k f \to D_k f$ strongly in $L^p(\Omega')$. As we did for part i.), first consider $f \in W^{1,p}(\Omega) \cap C^1(\Omega)$. Calculating we see

$$\begin{aligned}
\int_{\Omega'} |\Delta_h^k f - D_k f|\, dx &= \int_{\Omega'} \left| \frac{1}{h} \int_0^h (D_k f(x + te_k) - D_k f(x))\, dt \right|^p dx \\
&= \int_{\Omega'} \frac{1}{h} \int_0^h |D_k f(x + te_k) - D_k f(x))|^p\, dt dx \\
&= \frac{1}{h} \int_0^h \int_{\Omega'} |D_k f(x + te_k) - D_k f(x))|^p\, dt dx
\end{aligned}$$

Now, since $f \in C^1(\Omega)$, the integrand on the RHS is strictly bounded, i.e. we can apply Lebesgue's dominated convergence theorem (to the inner integral) to state

$$\lim_{h \to 0} \int_{\Omega'} |\Delta_h^k f - D_k f|\, dx = 0.$$

For general $f \in W^{1,p}(\Omega')$, we do as before by picking $f_n \in W^{1,p}(\Omega) \cap C^1(\Omega)$ such that $f_n \to f$ in $W^{1,p}(\Omega')$ to conclude that the above limit still holds. ■

9.9 A Note about Banach Algebras

The following theorem is sometimes useful in the manipulation of weak formulations of PDE.

Theorem 9.68. *Let $\Omega \subset \mathbb{R}^n$ be an open and bonded domain with smooth boundary. If $p > n$, then the space $W^{1,p}(\Omega)$ is a Banach algebra with respect to the usual multiplication of functions, that is, if $f, g \in W^{1,p}(\Omega)$, then $f \cdot g \in W^{1,p}(\Omega)$.*

Proof: The Sobolev embeddings imply that

$$f, g \in C^{0,\alpha}(\Omega) \quad \text{with } \alpha = 1 - \frac{n}{p} > 0$$

and that the estimates

$$||f||_{C^{0,\alpha}(\Omega)} \le C_0 ||f||_{W^{1,p}(\Omega)}, \;\; ||g||_{C^{0,\alpha}(\Omega)} \le C_0 ||g||_{W^{1,p}(\Omega)}$$

hold. For $n \geq 2$ we have $p' < p$ and we may use the usual product rule to deduce that the weak derivative of fg is given by $f \cdot Dg + Df \cdot g$. Then

$$\begin{aligned} ||D(fg)||^p_{L^p(\Omega)} &\leq C \int_\Omega (|f \cdot Dg|^p + |g \cdot Df|^p)\, dx \\ &\leq B(||f||^p_{L^\infty(\Omega)} ||Dg||^p_{W^{1,p}(\Omega)} + ||g||^p_{L^\infty(\Omega)} ||Df||^p_{W^{1,p}(\Omega)}) \\ &\leq 2BC_1 ||f||^p_{W^{1,p}(\Omega)} ||g||^p_{W^{1,p}(\Omega)}. \end{aligned}$$

The same arguments work for $n = 1$ and $p \geq 2$. If $p \in (1, 2)$, then we choose $f_\epsilon = \phi_\epsilon * f$ where ϕ_ϵ is a standard mollifier. Suppose that $\Omega = (a, b)$. Then $f_\epsilon \to f$ in $W^{1,1}_{\text{loc}}(a, b)$ and locally uniformly on (a, b) since f is continuous. It follows that

$$\int_a^b f_\epsilon g \phi'\, dx = -\int_a^b (Dg \cdot f_\epsilon + g \cdot Df_\epsilon)\phi\, dx.$$

As $\epsilon \to 0$, we conclude

$$\int_a^b fg\phi'\, dx = -\int_a^b (Dg \cdot f + g \cdot Df)\phi\, dx.$$

Thus $fg \in W^{1,1}(\Omega)$ with $D(fg) = Df \cdot g + g \cdot Df$. The p-integrability of the derivative follows from the embedding theorems as before. ■

9.10 Exercises

8.1: Let $X = C^0([-1,1])$ be the space of all continuous function on the interval $[-1,1]$, $Y = C^1([-1,1])$ be the space of all continuously differentiable functions, and let $Z = C_0^1([-1,1])$ be the space of all function $u \in Y$ with boundary values $u(-1) = u(1) = 0$.

a) Define

$$(u,v) = \int_{-1}^{1} u(x)v(x)\ dx.$$

On which of the spaces X, Y, and Z does $(\cdot,\cdot)$ define a scalar product? Now consider those spaces for which $(\cdot,\cdot)$ defines a scalar product and endow them with the norm $||\cdot|| = (\cdot,\cdot)^{1/2}$. Which of the spaces is a pre-Hilbert space and which is a Hilbert space (if any)?

b) Define

$$(u,v) = \int_{-1}^{1} u'(x)v'(x)\ dx.$$

On which of the spaces X, Y, and Z does $(\cdot,\cdot)$ define a scalar product? Now consider those spaces for which $(\cdot,\cdot)$ defines a scalar product and endow them with the norm $||\cdot|| = (\cdot,\cdot)^{1/2}$. Which of the spaces is a pre-Hilbert space and which is a Hilbert space (if any)?

8.2: Suppose that Ω is an open and bounded domain in $\mathbb{R}^n$. Let $k \in \mathbb{N}$, $k \geq 0$, and $\gamma \in (0,1]$. We define the norm $||\cdot||_{k,\gamma}$ in the following way. The γ^{th}-Holder seminorm on Ω is given by

$$[u]_{\gamma;\overline{\Omega}} = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}},$$

and we define the maximum norm by

$$|u|_{\infty;\overline{\Omega}} = \sup_{x\in\Omega} |u(x)|.$$

The Holder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\Omega)$ for which the norm

$$||u||_{k,\gamma;\overline{\Omega}} = \sum_{|\alpha|\leq k} ||D^{\alpha}u||_{\infty;\overline{\Omega}} + \sum_{|\alpha|=k} [D^{\alpha}u]_{\gamma;\overline{\Omega}}$$

is finite. Prove that $C^{k,\alpha}(\overline{\Omega})$ endowed with the norm $||\cdot||_{k,\gamma;\overline{\Omega}}$ is a Banach space.

Hint: You may prove the result for $C^{0,\gamma}([0,1])$ if you indicate what would change for the general result.

8.3: Consider the shift operator $T : l^2 \to l^2$ given by

$$(\mathbf{x})_i = x_{i+1}.$$

Prove the following assertion. It is true that

$$\mathbf{x} - T\mathbf{x} = 0 \iff \mathbf{x} = 0,$$

however, the equation

$$\mathbf{x} - T\mathbf{x} = \mathbf{y}$$

does not have a solution for all $\mathbf{y} \in l^2$. This implies that T is not compact and the Fredholm's alternative cannot be applied. Give an example of a bounded sequence $\mathbf{x}_j$ such that $T\mathbf{x}_j$ does not have a convergent subsequence.

8.4: The space l^2 is a Hilbert space with respect to the scalar product

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} x_i y_i.$$

Use general theorems to prove that the sequence of unit vectors in l^2 given by $(\mathbf{e}_i)_j = \delta_{ij}$ contains a weakly convergent subsequence and characterize the weak limit.

8.5: Find an example of a sequence of integrable functions f_i on the interval $(0,1)$ that converge to an integrable function f a.e. such that the identity

$$\lim_{i\to\infty} \int_0^1 f_i(x)\, dx = \int_0^1 \lim_{i\to\infty} f_i(x)\, dx$$

does not hold.

8.6: Let $a \in C^1([0,1])$, $a \geq 1$. Show that the ODE

$$-(au')' + u = f$$

has a solution $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ for all $f \in C^0([0,1])$.

Hint: Use the method of continuity and compare the equation with the problem $-u'' = f$. In order to prove the a-priori estimate derive first the estimate

$$\sup |tu_t| \leq \sup |f|$$

where u_t is the solution corresponding to the parameter $t \in [0,1]$.

8.7: Let $\lambda \in \mathbb{R}$. Prove the following alternative: Either (i) the homogeneous equation

$$-u'' - \lambda u = 0$$

has a nontrivial solution $u \in C^{(}[0,1])$ with $u(0) = u(1) = 0$ or (ii) for all $f \in C^0([0,1])$, there exists a unique solution $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ of the equation

$$-u'' - \lambda u = f.$$

Moreover, the mapping $R_\lambda : f \mapsto u$ is bounded as a mapping from $C^0([0,1])$ into $C^2([0,1])$.

Hint: Use Arzela-Ascoli.

8.8: Let $1 < p \leq \infty$ and $\alpha = 1 - \frac{1}{p}$. Then there exists a constant C that depends only on a, b, and p such that for all $u \in C^1([a,b])$ and $x_0 \in [a,b]$ the inequality

$$||u||_{C^{0,\alpha}([a,b])} \leq |u(x_0)| + C||u'||_{L^p([a,b])}$$

holds.

8.9: Let $X = L^2(0,1)$ and

$$M = \left\{ f \in X : \int_0^1 f(x)\ dx = 0 \right\}.$$

Show that M is a closed subspace of X. Find $M^\perp$ with respect to the notion of orthogonality given by the L^2 scalar product

$$(\cdot,\cdot) : X \times X \to \mathbb{R}, \quad (u,v) = \int_0^1 u(x)v(x)\ dx.$$

According to the projection theorem, every $f \in X$ can be written as $f = f_1 + f_2$ with $f_1 \in M$ and $f_2 \in M^\perp$. Characterize f_1 and f_2.

8.10: Suppose that $p_i \in (1, \infty)$ for $i = 1, \ldots, n$ with

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1.$$

Prove the following generalization of Holder's inequality: if $f_i \in L^{p_i}(\Omega)$ then

$$f = \prod_{i=1}^n f_i \in L^1(\Omega) \quad \text{and} \quad \left|\left|\prod_{i=1}^n f_i\right|\right|_{L^1(\Omega)} \leq \prod_{i=1}^n ||f_i||_{L^{p_i}(\Omega)}.$$

8.11: [Gilbarg-Trudinger, Exercise 7.1] Let Ω be a bounded domain in $\mathbb{R}^n$. If u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) = \left(\frac{1}{|\Omega|}\int_\Omega |u|^p \ dx\right)^{\frac{1}{p}}.$$

Show that

$$\lim_{p\to\infty} \Phi_p(u) = \sup_\Omega |u|.$$

8.12: Suppose that the estimate

$$||u - u_\Omega||_{L^p(\Omega)} \leq C_\Omega ||Du||_{L^p(\Omega)}$$

holds for $\Omega = B(0,1) \subset \mathbb{R}^n$ with a constant $C_1 > 0$ for all $u \in W_0^{1,p}(B(0,1))$. Here u_Ω denotes the mean value of u on Ω, that is

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(z) \ dz.$$

Find the corresponding estimate for $\Omega = B(x_0, R)$, $R > 0$, using a suitable change of coordinates. What is $C_{B(x_0,R)}$ in terms of C_1 and R?

8.13: [Qualifying exam 08/99] Let $f \in L^1(0,1)$ and suppose that

$$\int_0^1 f\phi' \ dx = 0 \qquad \forall \phi \in C_0^\infty(0,1).$$

Show that f is constant.

Hint: Use convolution, i.e., show the assertion first for $f_\epsilon = \rho_\epsilon * f$.

8.14: Let $g \in L^1(a,b)$ and define f by

$$f(x) = \int_a^x g(y)\, dy.$$

Prove that $f \in W^{1,1}(a,b)$ with $Df = g$.

Hint: Use the fundamental theorem of Calculus for an approximation g_k of g. Show that the corresponding sequence f_k is a Cauchy sequence in $L^1(a,b)$.

8.15: [Evans 5.10 #8,9] Use integration by parts to prove the following interpolation inequalities.

a) If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\int_\Omega |Du|^2\, dx \le C\left(\int_\Omega u^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |D^2u|^2\, dx\right)^{\frac{1}{2}}$$

b) If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p \in [2,\infty)$, then

$$\int_\Omega |Du|^p\, dx \le C\left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |D^2u|^p\, dx\right)^{\frac{1}{2}}.$$

Hint: Assertion a) is a special case of b), but it might be useful to do a) first. For simplicity, you may assume that $u \in C_c^\infty(\Omega)$. Also note that

$$\int_\Omega |Du|^p\, dx = \sum_{i=1}^n \int_\Omega |D_iu|^2 |Du|^{p-2}\, dx.$$

8.16: [Qualifying exam 08/01] Suppose that $u \in H^1(\mathbb{R})$, and for simplicity suppose that u is continuously differentiable. Prove that

$$\sum_{n=-\infty}^{\infty} |u(n)|^2 < \infty.$$

8.17: Suppose that $n \ge 2$, that $\Omega = B(0,1)$, and that $x_0 \in \Omega$. Let $1 \le p < \infty$. Show that $u \in C^1(\Omega \setminus \{x_0\})$ belongs to $W^{1,p}(\Omega)$ if u and its classical derivative ∇u satisfy the following inequality,

$$\int_{\Omega\setminus\{x_0\}} (|u|^p + |\nabla u|^p)\, dx < \infty.$$

Note that the examples from the chapter show that the corresponding result does not hold for $n = 1$. This is related to the question of how big a set in $\mathbb{R}^n$ has to be in order to be "seen" by Sobolev functions.

Hint: Show the result first under the additional assumption that $u \in L^\infty(\Omega)$ and then consider $u_k = \phi_k(u)$ where ϕ_k is a smooth function which is close to

$$\psi_k(s) = \begin{cases} k & \text{if } s \in (k, \infty), \\ s & \text{if } s \in [-k, k], \\ -k & \text{if } s \in (-\infty, -k). \end{cases}$$

You could choose ϕ_k to be the convolution of ψ_k with a fixed kernel.

<u>8.18</u>: Give an example of an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(\Omega)$, such that u is *not* Lipschitz continuous on Ω.

Hint: Take Ω to be the open unit disk in $\mathbb{R}^2$, with a slit removed.

<u>8.19</u>: Let $\Omega = B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^n$, $n \geq 2$. Show that $\log\log\frac{1}{|x|} \in W^{1,n}(\Omega)$.

<u>8.20</u>: [Qualifying exam 01/00]

a) Show that the closed unit ball in the Hilbert space $H = L^2([0,1])$ is not compact.

b) Show that $g : [0,1] \to \mathbb{R}$ is a continuous function and define the operator $T : H \to H$ by

$$(Tu)(x) = g(x)u(x) \quad \text{for } x \in [0,1].$$

Prove that T is a compact operator if and only if the function g is identically zero.

<u>8.21</u>: [Qualifying exam 08/00] Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^3$. For any function $u : \overline{\Omega} \to \mathbb{R}$, define the trace Tu by restricting u to $\partial\Omega$, i.e., $Tu = u\big|_{\partial\Omega}$. Show that $T : L^2(\Omega) \to L^2(\partial\Omega)$ is not bounded.

<u>8.22</u>: [Qualifying exam 01/00] Let $H = H^1([0,1])$ and let $Tu = u(1)$.

a) Explain precisely how T is defined for all $u \in H$, and show that T is a bounded linear functional on H.

b) Let $(\cdot,\cdot)_H$ denote the standard inner product in H. By the Riesz representation theorem, there exists a unique $v \in H$ such that $Tu = (u,v)_H$ for all $u \in H$. Find v.

8.23: Let $\Omega = B(0,1) \subset \mathbb{R}^n$ with $n \geq 2$, $\alpha \in \mathbb{R}$, and $f(x) = |x|^\alpha$. For fixed α, determine for which values of p and q the function f belongs to $L^p(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Is there a relation between the values?

8.24: Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary. Prove that for $p > n$, the space $W^{1,p}(\Omega)$ is a Banach algebra with respect to the usual multiplication of functions, that is, if $f, g \in W^{1,p}(\Omega)$, then $fg \in W^{1,p}(\Omega)$.

8.25: [Evans 5.10 #11] Show by example that if we have $||D^h u||_{L^1(V)} \leq C$ for all $0 < |h| < \frac{1}{2}\text{dist}\,(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(\Omega)$.

8.26: [Qualifying exam 01/01] Let A be a compact and self-adjoint operator in a Hilbert space H. Let $\{u_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H consisting of eigenfunctions of A with associated eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Prove that if $\lambda \neq 0$ and λ is not an eigenvalue of A, then $A - \lambda I$ has a bounded inverse and

$$||(A-\lambda I)^{-1}|| \leq \frac{1}{\inf_{k\in\mathbb{N}} |\lambda_k - \lambda|} < \infty.$$

8.27: [Qualifying exam 01/03] Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary.

a) For $f \in L^2(\Omega)$ show that there exists a $u_f \in H_0^1(\Omega)$ such that

$$||f||^2_{H^{-1}(\Omega)} = ||u_f||^2_{H_0^1(\Omega)} = (f, u_f)_{L^2(\Omega)}.$$

b) Show that $L^2(\Omega) \subset\subset\hookrightarrow H^{-1}(\Omega)$. You may use the fact that $H_0^1(\Omega) \subset\subset\hookrightarrow L^2(\Omega)$.

8.28: [Qualifying exam 08/02] Let Ω be a bounded, connected and open set in $\mathbb{R}^n$ with smooth boundary. Let V be a closed subspace of $H^1(\Omega)$ that does *not* contain nonzero constant functions. Using $H^1(\Omega) \subset\subset\hookrightarrow L^2(\Omega)$, show that there exists a constant $C < \infty$, independent of u, such that for $u \in V$,

$$\int_\Omega |u|^2 \, dx \leq C \int_\Omega |Du|^2 \, dx.$$

Chapter 10

Linear Elliptic PDE

General assumption $\Omega \subset \mathbb{R}^n$ open and bounded

10.1 Existence of Weak Solutions for Laplace's Equation

Want to solve:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{10.1}$$

multiply by $v \in W_0^{1,2}(\Omega)$, integrate by parts:

$$\int_\Omega Du \cdot Dv \; dx = \int_\Omega fv \; dx$$

Definition 10.1. *Let $f \in L^2(\Omega)$. A function $u \in W_0^{1,2}(\Omega)$ is called a* *weak solution* *of (10.1) if*

$$\int_\Omega Du \cdot Dv \; dx = \int_\Omega fv \; dx \quad \forall v \in W_0^{1,2}(\Omega)$$

Theorem 10.2. *The equation (10.1) has a unique weak solution in $W_0^{1,2}(\Omega)$ and we have $||u||_{W^{1,2}(\Omega)} \leq C||f||_{L^2(\Omega)}$*

Notation: $u = (-\Delta)^{-1}f$ *implicitly understood* $u = 0$ *on* $\partial\Omega$

Proof: $H_0^1 = W_0^{1,2}(\Omega)$ is a Hilbert space, $F : H_0^1 \to \mathbb{R}$,

$$F(v) = \int_\Omega fv \; dx$$

Then F is linear and bounded

$$\begin{aligned} |F(v)| &= \left|\int_\Omega fv\ dx\right| \\ &\leq \int_\Omega |fv|\ dx \\ &\overset{\text{Hölder}}{\leq} ||f||_{L^2(\Omega)}||v||_{L^2(\Omega)} \\ &\leq ||f||_{L^2(\Omega)}||v||_{W^{1,2}(\Omega)} \end{aligned}$$

Define $B : H \times H \to \mathbb{R}$, $B(u,v) = \int_\Omega Du \cdot Dv\ dx$. B is bilinear. B is also bounded:

$$|B(u,v)| \leq ||Du||_{L^2(\Omega)}||Dv||_{L^2(\Omega)} \leq ||u||_{W^{1,2}(\Omega)}||v||_{W^{1,2}(\Omega)}$$

We can thus use Lax-Milgram if B is also coercive, i.e.,

$$B(u,u) \geq \nu||u||^2_{W^{1,2}(\Omega)},\ B(u,u) = \int_\Omega |Du|^2\ dx$$

So, to show this we use Poincare:

$$\begin{aligned} & ||u||^2_{L^2(\Omega)} \leq C_p||Du||^2_{L^2(\Omega)} \\ \iff\quad & ||u||^2_{W^{1,2}(\Omega)} = ||u||^2_{L^2(\Omega)} + ||Du||^2_{L^2(\Omega)} \leq (C_p^2+1)||Du||^2_{L^2(\Omega)} \\ \implies\quad & ||Du||^2_{L^2(\Omega)} \geq \frac{1}{C_p^2+1}||u||^2_{W^{1,2}(\Omega)} \end{aligned}$$

Thus,

$$B(u,u) \geq \frac{1}{C_p^2+1}||u||^2_{W^{1,2}(\Omega)},\ \text{i.e. } B \text{ is coercive}$$

Now Lax-Milgram says $\exists u \in H^1_0(\Omega) = W^{1,2}_0(\Omega)$ such that

$$B(u,v) = F(v) \qquad \forall v \in H^1_0(\Omega) \iff \int_\Omega Du \cdot Dv\ dx = \int_\Omega fv\ dx \qquad \forall v \in H^1_0(\Omega)$$

i.e. u is a unique weak solution. The estimate $||u||_{W^{1,2}(\Omega)} \leq C||f||_{L^2(\Omega)}$, follows from the estimate in Lax-Milgram. ■

Proposition 10.3. *The operator*

$$(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega) \text{ *is compact*}$$

Proof: Since $||u||_{W^{1,2}(\Omega)} \leq C||f||_{L^2(\Omega)}$ we know that $(-\Delta)^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega)$ is bounded. From compact embedding, we thus have

$$(-\Delta)^{-1} : L^2(\Omega) \xrightarrow{\text{bounded}} W^{1,2}(\Omega) \subset\subset\!\!\longrightarrow L^2(\Omega)$$

$(-\Delta)^{-1}$ maps bounded sets in $L^2(\Omega)$ into precompact sets in $L^2(\Omega)$, i.e., is compact. ■

Theorem 10.4. *Either*

i.) $\forall f \in L^2(\Omega)$ *the equations*

$$\begin{cases} -\Delta u = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{10.2}$$

has a unique weak solution, or

ii.) there exists a nontrivial solution of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{10.3}$$

iii.) If ii.) holds, then the space N *of solutions of (10.3) is finite dimensional*

iv.) the equation (10.2) has a solution $\Longleftrightarrow$

$$(f, v)_{L^2(\Omega)} = 0 \qquad \forall v \in N$$

Remark: In general iv.) is true if $f \perp N^*$, where N^* is the null-space of the dual operator.

Proof: Almost all assertions follow from Fredholm's-alternative in Hilbert Spaces.

Suppose that $K = K*$
(10.2) is equivalent to

$$u = (-\Delta)^{-1}u + (-\Delta)^{-1}f$$

take $K = (-\Delta)^{-1}$ compact. $\tilde{f} = (-\Delta)^{-1}f$. Without loss of generality, $\lambda \neq 0$, $(I - \lambda K)u = f$.

i.) has a solution if $\tilde{f}$ is perpendicular to the null-space of $I - \lambda K$, i.e. $(\tilde{f}, v) = 0$ $\forall v$ with $v - \lambda K v = 0$.
$\Longrightarrow (Kf, v) = 0 \iff (f, K^* v) = 0 \iff \frac{1}{\lambda}(f, v) = 0$
$\Longrightarrow (f, v) = 0$, $\forall v \in N$ since $\lambda \neq 0$.

Finally, we just need to prove K is self-adjoint, i.e., $K = K^*$, $f \in L^2(\Omega)$, $h = Kf$
$\iff$ h is a weak solution of $-\Delta h = f$, $f \in W_0^{1,2}(\Omega)$

$$\begin{aligned}
\iff \quad & \int_\Omega Dh \cdot Dv \; dx = \int_\Omega fv \; dx \qquad \forall v \in W_0^{1,2}(\Omega) \\
\iff \quad & \underbrace{\int_\Omega D(Kf) \cdot Dv \; dx = \int_\Omega fv \; dx}_{v = Kg \ (*)} \qquad \forall v \in W_0^{1,2}(\Omega)
\end{aligned}$$

So with this definition of g, we see

$$\begin{aligned}
(K^* f, g) &= (f, Kg) \\
\text{(by (*))} \quad &= \big(D(Kf), D(Kg)\big) \\
\text{(by symmetry)} \quad &= \big(D(Kg), D(Kf)\big) \\
&= (g, Kf) \\
&= (Kf, g)
\end{aligned}$$

$\Longrightarrow$ this holds $\forall g \in L^2$
$\Longrightarrow K^* f = Kf \; \forall f$
$\Longrightarrow K^* = K.$ ■

———

$$-\Delta u = f, \qquad L = -\Delta, f \rightsquigarrow F \in H^{-1}(\Omega)$$

$$B(u,v) = \mathcal{L}(u,v) = \int_\Omega Du \cdot Dv \; dx$$

Weak solution:

$$\mathcal{L}(u,v) = \int_\Omega fv \; dx = \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega)$$

Existence and uniqueness: Lax-Milgram
More generally, we can solve

$$-\Delta u = \underbrace{g + \sum_{i=1}^{n} D_i f_i}_{\in H^{-1}(\Omega)}, \qquad g, f_i \in L^2(\Omega)$$

Weak solution:

$$\mathcal{L}(u,v)\left\langle g + \sum_{i=1}^{n} D_i f_i, v \right\rangle = \int_\Omega \left(gv - \sum_{i=1}^{n} f_i g_i \right)\, dx$$

<u>Notation:</u> Summation convention. Frequently drop the summation symbol, sum over repeated indexes.

General boundary conditions: $u_0 \in W^{1,2}(\Omega)$, want to solve

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

Define $\tilde{u} = u - u_0 \in W_0^{1,2}(\Omega)$, and find equation for $\tilde{u}$.

$$\begin{aligned} & \int_\Omega Du \cdot Dv\, dx = 0 \qquad \forall v \in H_0^1(\Omega) \\ \iff \quad & \int_\Omega ((Du - Du_0) + Du_0) \cdot Dv\, dx = 0 \\ \iff \quad & \int_\Omega D\tilde{u} \cdot Dv\, dx = -\int_\Omega Du_0 \cdot Dv\, dx \end{aligned}$$

Equation for $\tilde{u}$ with general $H^{-1}(\Omega)$ on the RHS.

10.2 Existence for General Elliptic Equations of 2nd Order

$$Lu = -D_i(a_{ij} D_j u + b_i u) + c_i D_i u + du$$

with $a_{ij}, b_i, c_i, d \in L^\infty(\Omega)$. General elliptic operator in divergence form, $-\text{div}(A \cdot Du)$ is natural to work with in the framework of weak solutions..

Non-divergence form

$$Lu = a_{ij}D_iD_ju + b_iD_iu + cu$$

If coefficients smooth, then non-divergence is equivalent to divergence form.

Corresponding bilinear form (multiply by v, integrate by parts)

$$\mathcal{L}(u,v) = \int_\Omega (a_{ij}D_ju \cdot D_iv + b_iuD_iv + cD_iu \cdot v + duv)\ dx$$

want to solve $Lu = g + D_if_i \in H^{-1}(\Omega)$, i.e.,

$$\mathcal{L}(u,v) = \int_\Omega (gv + f_iD_iv)\ dx \qquad \forall v \in H_0^1(\Omega) \tag{10.4}$$

Definition 10.5. *$u \in H_0^1(\Omega)$ is a weak solution of $\mathcal{L}u = g + D_if_i$ if (10.4) holds.*

Definition 10.6. *$\mathcal{L}$ is strictly elliptic if $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$, $\forall \xi \in \mathbb{R}^n, \lambda > 0$.*

Definition 10.7. *Let $\mathcal{L}$ strictly elliptic, $\sigma > 0$ sufficiently large and let $L_\sigma u = Lu + \sigma u$, then*

$$\begin{cases} L_\sigma u = g + D_if_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and $||u||_{W^{1,2}(\Omega)} \leq C(\sigma)\left(||g||_{L^2(\Omega)} + ||\tilde{f}||_{L^2(\Omega)}\right)$, $\tilde{f} = (f_1, \ldots, f_n)$

Proof: Lax-Milgram in $H_0^1(\Omega)$,

$$F(v) = \int_\Omega (gv - f_iD_iv)\ dx \leq \left(||g||_{L^2(\Omega)} + ||\tilde{f}||_{L^2(\Omega)}\right)||v||_{W^{1,2}(\Omega)}$$

Bilinear form:

$$\mathcal{L}(u,v) = \int_\Omega (a_{ij}D_ju \cdot D_iv + b_iuD_iv + c_iD_iu \cdot v + duv)\ dx$$

$\mathcal{L}$ is bounded since

$$\mathcal{L}(u,v) \overset{\text{Hölder}}{\leq} C(a_{ij}, b_i, c_i, d)||u||_{W^{1,2}(\Omega)}||v||_{W^{1,2}(\Omega)}$$

Crucial: check whether $\mathcal{L}$ is coercive

$$\begin{aligned}
\mathcal{L}(u,u) &= \int_\Omega \left(\underbrace{a_{ij}D_ju \cdot D_iu}_{\text{ellipticity: } \geq \lambda |Du|^2} + \underbrace{b_iuD_iu \cdot u + du^2}_{\leq C(b_i,c_i,a)\|u\|_{L^2(\Omega)}\|u\|_{W^{1,2}(\Omega)}} \right) dx \\
&\geq \lambda \int_\Omega |Du|^2 \, dx - C\|u\|_{L^2(\Omega)}\|u\|_{W^{1,2}(\Omega)}
\end{aligned}$$

$$\begin{aligned}
\text{Poincare:} \qquad & \|u\|^2_{L^2(\Omega)} \leq C_p^2 \|Du\|^2_{L^2(\Omega)} \\
\implies & \|u\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)} \leq (C_p^2+1)\|Du\|^2_{L^2(\Omega)} \\
\implies & \|Du\|^2_{L^2(\Omega)} \geq \frac{1}{C_p^2+1}\|u\|^2_{W^{1,2}(\Omega)}
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{Poincaré}}{\geq} \quad \frac{\lambda}{C_p^2+1}\|u\|^2_{W^{1,2}(\Omega)} - C\|u\|_{L^2(\Omega)}\|u\|_{W^{1,2}(\Omega)} \\
&\overset{\text{Young's}}{\geq} \quad \lambda'\|u\|^2_{W^{1,2}(\Omega)} - \frac{\lambda'}{2}\|u\|^2_{W^{1,2}(\Omega)} - C\|u\|^2_{L^2(\Omega)} \\
&= \quad \frac{\lambda'}{2}\|u\|^2_{W^{1,2}(\Omega)} - C\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

In general, $\mathcal{L}$ is not coercive, but if we take $L_\sigma = Lu + \sigma u$,

$$\begin{aligned}
\mathcal{L}_\sigma(u,v) &= \mathcal{L}(u,v) + \sigma \int_\Omega uv \, dx, \text{ then} \\
\mathcal{L}_\sigma(u,u) &\geq \frac{\lambda'}{2}\|u\|^2_{W^{1,2}(\Omega)} + \sigma\|u\|^2_{L^2(\Omega)} - \sigma\|u\|^2_{L^2(\Omega)} - C\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

and L_σ is coercive as soon as $\sigma > C(b_i, c_i, d)$.

<u>Question:</u> Under what conditions can we solve

$$Lu = F \qquad \forall F \in H^{-1}(\Omega)?$$

<u>Answer:</u> Applications of Fredholm's alternative.

$$Lu = -D_i(a_{ij}D_ju + b_iu) + c_iD_iu + du$$

Define:

$$\begin{aligned}
\langle Lu, v\rangle &= \mathcal{L}(u,v) \\
&= \int_\Omega (a_{ij}D_ju \cdot D_iv + b_iuD_iv + c_iD_iu\cdot v + duv)\ dx
\end{aligned}$$

Formally adjoint operator L^*

$$\begin{aligned}
\langle L^*u, v\rangle &= \mathcal{L}^*(u,v) = \mathcal{L}(v,u) \\
&= \int_\Omega (\underbrace{a_{ij}D_jv \cdot D_iu}_{a_{ji}D_iv\cdot D_ju} + b_ivD_iu + c_iD_iv\cdot u + duv)\ dx \\
&= \int_\Omega (-D_i(a_{ji}D_iu + C_iu) + B_iD_iu + du)v\ dx \\
&= \langle L^*u, v\rangle,\ \text{i.e.} \\
L^* &= -D_i(a_{ji}D_ju + c_iu) + b_iD_iu + du
\end{aligned}$$

10.2.1 Application of the Fredholm Alternative

Theorem 10.8 (Fredholm's Alternative). *L strictly elliptic. Then either*

i.) $\forall g,\ f_i \in L^2(\Omega)$ *the equation*

$$\begin{cases} Lu = g + D_if_i & \text{in } \Omega \\ \qquad\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution, or

ii.) there exists a non-trivial, weak solution of the homogeneous equation

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \ \ u = 0 & \text{on } \partial\Omega \end{cases}$$

iii.) If ii.) holds, then $Lu = g + D_if_i = F$ *has a solution* $\iff$

$$\langle F, v\rangle = \int_\Omega (gv - f_iD_iv)\ dx = 0$$

$\forall v \in N(L^*)$, *i.e. for all weak solutions of the adjoint equation* $L^*v = 0$.

Proof: Setting up Fredholm

$$\begin{aligned}
& Lu = F \in H^{-1}(\Omega) \\
\iff\quad & L_\sigma u - \sigma u = F,
\end{aligned}$$

σ large enough such that $L_\sigma u = F$ has a unique solution $\forall F$.
Formally:

$$I_1 : H_0^1(\Omega) \hookrightarrow L^2, \; I_0 : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

So, $I_2 = I_0 I_1 : H_0^1(\Omega) \to H^{-1}(\Omega)$. Interpret our equation as $L_\sigma u - I_2 u = F$ is $H^{-1}(\Omega)$.

———

$$(I_0 u)(v) = \int_\Omega uv \, dx \; I_0 u \text{ bounded and linear}$$

———

$$u - \sigma L_\sigma^{-1} I_2 u = L_\sigma^{-1} F \quad \text{in } H_0^1(\Omega) \tag{10.5}$$

One option: use Fredholm for this equation (done in Gilbarg and Trudinger). We will actually shift into $L^2(\Omega)$ first.

Interpret this as an equation in $L^2(\Omega)$.

$$\begin{aligned} & I_1 u - \sigma I_1 L_\sigma^{-1} I_2 u = I_1 L_\sigma^{-1} F \\ \iff \quad & (Id - \sigma I_1 L_\sigma^{-1} I_0) I_1 u = I_1 L_\sigma^{-1} F \\ \iff \quad & (Id - \sigma I_1 L_\sigma^{-1} I_0)\tilde{u} = I_1 L_\sigma^{-1} F \text{ in } L^2(\Omega) \end{aligned} \tag{10.6}$$

where $\tilde{u} = I_1 u \in L^2(\Omega)$.

Remark: If $\tilde{u}$ is a solution of (10.6), then

$$u = L_\sigma^{-1}(\sigma I_0 \tilde{u} + F) \tag{10.7}$$

is a solution of (10.5).

Now we come back to the Fredholm's alternative:

$$K = \sigma \underbrace{I_1}_{\text{cmpct}} \cdot \underbrace{L_\sigma^{-1} I_0}_{\text{bnded}} : L^2(\Omega) \to L^2(\Omega)$$

$\Longrightarrow K$ is compact.
$\Longrightarrow$ either (10.6) has a unique solution for all RHS in $L^2(\Omega)$ or 2., the homogeneous equation has non-trivial solutions.

- What i.) means:
  Take RHS $\tilde{f} = I_1 L_\sigma^{-1} F$
  $\Longrightarrow \exists!$ solution $\tilde{u} \in L^2(\Omega)$
  $\Longrightarrow \exists!$ u of (10.5) by (10.7), that is

$$\begin{aligned} & u - \sigma L_\sigma^{-1} I_2 u = L_0^{-1} F \\ \Longleftrightarrow \quad & L_\sigma u - \sigma I_2 u = F \iff Lu = F \end{aligned}$$

- What ii.) means:
  $(Id - K)\tilde{u} = 0$ which implies the this a non-trivial weak solution of

$$\begin{aligned} & u - \sigma L_\sigma^{-1} I_2 u = 0 \\ \Longleftrightarrow \quad & L_\sigma u - \sigma I_2 u = F \iff Lu = 0 \end{aligned}$$

Proof of iii.): $K^* = \sigma I_1 (L_\sigma^*)^{-1} I_0$

By definition (also see below aside): $(K^* g, h)_{L^2} \quad = \quad (g, Kh)$

——

$$\begin{aligned} Lu = f \iff \mathcal{L}(u,v) &= (f,v)\ \forall v, \qquad L_\sigma u = f \iff \mathcal{L}_\sigma(u,v) = (f,v) = \mathcal{L}_\sigma(L_\sigma^{-1} f, v) \\ L^* u = f \iff \mathcal{L}^*(u,v) &= (f,v)\ \forall v, \qquad L_\sigma^* u = f \iff \mathcal{L}_\sigma^*(u,v) = (f,v) = \mathcal{L}_\sigma((L_\sigma^*)^{-1} f, v) \\ \therefore \mathcal{L}_\sigma^*(v,u) &= \mathcal{L}_\sigma(u,v) \end{aligned}$$

——

$$\begin{aligned} (\text{dual eqn } v = Kh) \quad &= \quad \mathcal{L}_\sigma^*((L_\sigma^*)^{-1} g, Kh) \\ &= \quad \mathcal{L}_\sigma(Kh, (L_\sigma^*)^{-1} g) \\ (\text{definition of } K) \quad &= \quad \sigma \mathcal{L}_\sigma(\underbrace{(L_\sigma^{-1})h}_{H_0^1(\Omega)}, (L_\sigma^*)^{-1} g) \end{aligned}$$

So,

$$\begin{aligned} \text{original equation} \quad &= \quad \sigma(h, (L_\sigma^*)^{-1} g) \\ &= \quad \sigma((L_\sigma^*)^{-1} g, h) \Longrightarrow (\underbrace{(K^* - \sigma(L_\sigma^*)^{-1})g}_{=0}, h) = 0\ \forall h \in L^2, \forall g \in L^2 \qquad \blacksquare \end{aligned}$$

Solvability of Original Equation

$$Lu = F \in H^{-1} \iff (I - K)\tilde{u} = I_1 L_\sigma^{-1} F$$

Fredholm in L^2: If you don't have a unique solution, then we can solve 2. if the RHS is perpendicular to the null-space of the adjoint operator $I - K^*$, i.e.,

$$(I_1 L_\sigma^{-1} F, v) = 0, \ \forall v \in N(I - K^*)$$

$$\begin{aligned} v \in N(I - K^*) &\iff (I - \sigma(L_\sigma^*)^{-1})v = 0 \iff L_\sigma^* v - \sigma v = 0 \\ &\iff L^* v = 0, \text{ i.e. } v \in N(L^*) \end{aligned}$$

Solvability condition $\iff (I_1 L_\sigma^{-1} F, v) = 0, \ \forall v \in N(L^*)$. Multiplying by σ

$$\begin{aligned} &\iff && (I_1 L_\sigma^{-1} F, \sigma v) = 0 = (I_1 L_\sigma^{-1} F, L_\sigma^* v) = \\ & && \langle L_\sigma^{-1} F, L_\sigma^* v \rangle \\ &\iff && \langle L_\sigma^{-1} F, L_\sigma^* v \rangle = 0 \iff \mathcal{L}_\sigma^*(v, L_\sigma^{-1} F) = 0 \\ &\iff && L\sigma(L_\sigma^{-1} F, v) = \langle F, v \rangle = 0 \quad \forall v \in N(L^*) \end{aligned}$$

Remark: $\Delta u = f$ in the space of Periodic functions $W^{1,2}_{\text{per}}$. Obviously, $u \equiv$constant is periodic with $\Delta u = 0$. Solvability condition comes from Fredholm's alternative, it turns out that we need $\int_\Omega f \, dx = 0$.

10.2.2 Spectrum of Elliptic Operators

Theorem 10.9 (Spectrum of elliptic operators). *L strictly elliptic, then there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the following holds*

1. *if $\lambda \in \Sigma$, then*

$$\begin{cases} Lu = \lambda u + g + D_i f_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

2. *if $\lambda \in \Sigma$, then the eigenspace $N(\lambda) = \{u : Lu = \lambda u\}$ is non-trivial and has finite dimension.*

3. *If Σ is infinite, then $\Sigma = \{\lambda_k, k \in \mathbb{N}\}$, and $\lambda_k \to \infty$ as $k \to \infty$.*

Remark: In 1D, $-u'' = \lambda u$ on $[0, \pi]$ $u(0) = u(\pi) = 0$ $u_k(x) = \sin(kx)$, i.e. $\lambda_k = k^2 \to \infty$

Proof: Essentially a consequence of the spectral Theorem of compact operators 7.10

Fredholm: uniqueness fails $\iff$ $Lu = \lambda u$ has a non-trivial solution. $Lu =$

$\lambda u \iff Lu + \sigma u = (\lambda+\sigma)u \iff L_\sigma u = (\lambda+\sigma)u$.
$u = (\lambda+\sigma)L_\sigma^{-1} = \frac{\lambda+\sigma}{\sigma}Ku$, K the compact operator as before, $\iff Ku = \frac{\sigma}{\lambda+\sigma}u \implies$ spectrum is at most countable, eigenspace finite dimension, zero only possible accumulation point $\implies \lambda_k \to \infty$ if Σ not finite. ■

General Elliptic Equation: $Lu = f$

$$\begin{aligned} Lu &= -D_i(u_{ij}D_j u + b_i D_i u) + c_i D_i u + du \\ &\updownarrow \\ \mathcal{L}(u,v) &= \langle Lu, v\rangle, \text{ existence works, } \mathcal{L}(u,u) \geq \lambda ||u||^2_{W^{1,2}(\Omega)} \end{aligned}$$

e.g. $L_\sigma u = Lu + \sigma u \implies \exists$ of solutions, Fredholm.
If e.g., $b_i, c_i, d = 0$, then $\sigma = 0$ will work.

10.3 Elliptic Regularity I: Sobolev Estimates

Elliptic regularity is one of the most challenging aspects of theoretical PDE. Regularity is basically the studies "how good" a weak solution is; given that weak formulations of PDE are not unique by any means, regularity tends to be very diverse in terms of the type of calculations it requires.

The next two sections are devoted to the "classical" regularity results pertaining to linear elliptic PDE. "Classical" is quoted because we will actually be employing more modern methods pertaining to Companato Spaces for derivation of Schauder estimates in the next section.

Aside: One will notice that the coming expositions focus heavily on Hölder estimates as opposed to approximations in Sobolev Spaces. Philosophically speaking, Sobolev Spaces is the tool to use for existence and are hence the starting assumption in many regularity arguments; but the ideal goal of regularity is to prove that a weak solution is indeed a classical one. Thus, in regularity we usually are aiming for continuous derivatives, i.e. solutions in Hölder Spaces (at least).

To get the ball moving, let us consider a brief example. This will hopefully help convince the reader that regularity is something that really does need to be proven as opposed to a pedantic technicality that is always true.

Example 10.1. *Consider the following problem in* $\mathbb{R}$*:*

$$\begin{cases} -u'' = f \text{ in } (0,1) \\ u(0) = 0 \end{cases}.$$

Using integration by parts to attain a weak formulation of our solution we see that

$$u(x) = \int_0^x \int_0^t f(s)\ ds\ dt$$

will solve the original equation, but the harder question of regularity u persist. Now if $f \in L^2([0,1])$ it can be shown that $u \in H^2([0,1])$.

In higher dimensions, if u is a weak solution of $-\Delta u = f$, then $f \in L^2(\Omega)$ usually implies $u \in H^2_{loc}(\Omega)$ (provided $\partial\Omega$ has decent geometry, etc.). On the other hand, if $f \in C^0(\Omega)$ is is generally not *the case that $u \in C^2(\Omega)$! There is hope though; we will prove (along with the previous statements) that if $f \in C^{0,\alpha}(\Omega)$ $(\alpha > 0)$, then $u \in C^2(\overline{\Omega})$ in general.*

The next proposition concludes in one of the most important inequalities in regularity studies of elliptic PDE.

10.3.1 Caccioppoli Inequalities

In this first subsection, we will develop the essential *Caccioppoli inequalities.* Even though these inequalities ultimately provide a means of bounding first derivatives of solutions of elliptic equations, the conditions under which the inequality itself holds vary. Thus, we first consider the most intuitive example of the Laplace's equation; following this, a more general situation will be presented.

Proposition 10.10 (Caccioppoli's inequality for the $-\Delta u = 0$). *If $u \in H^1(B(x_0, R))$ is a weak solution of $-\Delta u = 0$, i.e.*

$$\int_\Omega Du \cdot D\phi\, dx = 0 \quad \forall \phi \in H^1_0(\Omega),$$

then

$$\int_{B(x_0,r)} |Du|^2\ dx \le \frac{C}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u-\lambda|^2\ dx$$

for all $\lambda \in \mathbb{R}$ and $0 < r < R$.

Proof: The proof of this proposition is based on the idea of picking a "good" test function in $H^1_0(\Omega)$ to manipulate our weak formulation with.

With the purpose of finding such a function, let us consider a function η such that i.) $\eta \in C_c^\infty(B(x_0, R))$,

ii.) $0 \le \eta \le 1$,

iii.) $\eta \equiv 1$ on $B(x_0, r)$, and

iv.) $|D\eta| \le \dfrac{C}{R-r}$.

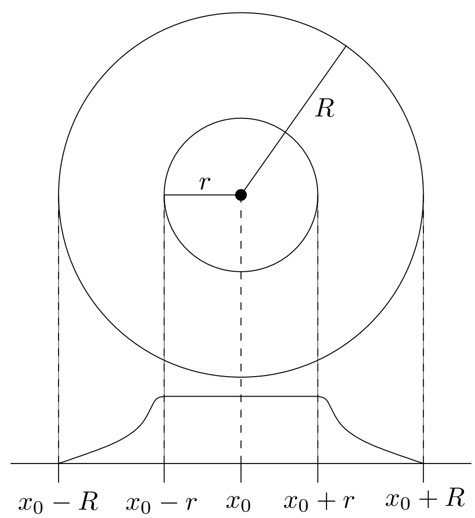


Figure 10.1: A potential form of η in $\mathbb{R}^2$

See figure (10.1) for a visualization of a potential η in $\mathbb{R}^2$.

Next, fix $\lambda \in \mathbb{R}$ and define $\phi = \eta^2(u-\lambda) \in W_0^{1,2}(B(x_0,R))$. Calculating

$$D\phi = 2\eta D\eta(u-\lambda) + \eta^2 Du$$

and plugging into the weak formulation, we ascertain

$$0 = \int_{B(x_0,R)} Du \cdot D\phi \; dx = \int_{B(x_0,R)} 2\eta D\eta \cdot Du(u-\lambda) + \eta^2 |Du|^2 \; dx.$$

Algebraic manipulation of the above equality shows that

$$\begin{aligned}
&\int_{B(x_0,R)} \eta^2|Du|^2\ dx \\
&= -\int_{B(x_0,R)} 2\eta D\eta\cdot Du(u-\lambda)\ dx \\
&\le \int_{B(x_0,R)} 2\eta|D\eta|\cdot|Du|\cdot|u-\lambda|\ dx \\
&\overset{\text{Hölder}}{\le} 2\left(\int_{B(x_0,R)} \eta^2|Du|^2\ dx\right)^{\frac{1}{2}}\left(\int_{B(x_0,R)} |D\eta|^2|u-\lambda|^2\ dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Now we divide through by the first integral on the RHS and square the result to find that

$$\int_{B(x_0,R)} \eta^2|Du|^2\ dx \le 4\int_{B(x_0,R\setminus B(x_0,r)} \underbrace{|D\eta|^2}_{\frac{C}{(R-r)^2}}|u-\lambda|^2\ dx.$$

This leads immediately to the conclusion that

$$\int_{B(x_0,r)} |Du|^2\ dx \le \frac{C}{(R-r)^2}\int_{B(x_0,R)\setminus B(x_0,r)} |u-\lambda|^2\ dx \qquad \blacksquare$$

Before moving onto Caccioppoli's inequalities for elliptic systems, we have the following special cases of proposition 10.10:

Case 1: $r=\frac{R}{2}$, $\lambda=0$.

$$\begin{aligned}
\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2\ dx &\le \frac{C}{R^2}\int_{B(x_0,R)\setminus B(x_0,R/2)} |u|^2\ dx \\
&\le \frac{C}{R^2}\int_{B(x_0,R)} |u|^2\ dx
\end{aligned}$$

Case 2: $r=\frac{R}{2}$, $\lambda=u_{B(x_0,R)}$.

$$\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2\ dx \le \frac{C}{R^2}\int_{B(x_0,R)} |u-u_{B(x_0,R)}|^2\ dx$$

Case 3: $r = \frac{R}{2}$, $\lambda = u_{B(x_0,R)\setminus B(x_0,R/2)}$.

$$\begin{aligned}
&\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 \, dx \\
&\qquad \leq \frac{C}{R^2} \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left| u - u_{B(x_0,R)\setminus B(x_0,R/2)} \right|^2 \, dx \\
&\qquad \overset{\text{Poincaré}}{\leq} C \int_{B(x_0,R)\setminus B(x_0,R/2)} |Du|^2 \, dx.
\end{aligned}$$

We can "plug the hole" in the area of integration on the RHS by adding C times the LHS of the above to both sides of the inequality to get

$$(1+C) \int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 \, dx \quad \leq \quad C \int_{B(x_0,R)} |Du|^2 \, dx,$$

i.e.

$$\begin{aligned}
\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 \, dx &\leq \frac{C}{C+1} \int_{B(x_0,R)} |Du|^2 \, dx \\
&= \Theta \int_{B(x_0,R)} |Du|^2 \, dx, \qquad (10.8)
\end{aligned}$$

where $0 < \Theta < 1$ is constant!

Note: The general Caccioppoli inequalities do not requires the integrations to be over concentric balls. It is easy to see that as long as $B_{R/2}(y_0) \subset B_R(x_0)$, the inequality still holds (just appropriately modify η in the proofs). This generalization will be assumed in some of the more heuristic arguments to come.

A direct consequence of this last case is

Corollary 10.11 (Lioville's Theorem). *If $u \in H^1(\mathbb{R}^n)$ is a weak solution of $-\Delta u = 0$, then u is constant.*

Proof: Take limit $R \to \infty$ of both sides of (10.8) to see

$$\int_{\mathbb{R}^n} |Du|^2 \, dx \leq \Theta \int_{\mathbb{R}^n} |Du|^2 \, dx,$$

which implies Du is constant a.e. $0 < \Theta < 1$; but as $u \in H^1(\mathbb{R}^n)$, we conclude that Du constant everywhere in $\mathbb{R}$. ■

Now we move on to consider elliptic systems of equations (which may or may not be linear).

Proposition 10.12 (Caccioppoli's inequality). *Consider the elliptic system*

$$-D_\alpha\left(A_{ij}^{\alpha\beta}(x)D_\beta u^j\right) = g, \text{ for } i = 1,\dots,m, \tag{10.9}$$

where $g \in H^{-1}(\Omega)$. Suppose we choose $f_i, f_i^\alpha \in L^2(\Omega)$ such that $g = f_i + D_\alpha f_i^\alpha$ and consider a weak solution $u \in H^1_{loc}(\Omega)$ of (10.9), e.g.

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i \, dx = \int_\Omega (f_i\phi^i + f_i^\alpha D_\alpha \phi^i)\, dx. \quad \text{for } \forall \phi \in H_0^1(\Omega).$$

If we ascertain that

$$\begin{cases} i.) & A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \\ ii.) & A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \ge \nu|\xi|^2, \nu > 0 \end{cases}$$

or

$$\begin{cases} i.) & A_{ij}^{\alpha\beta} = \text{constant} \\ ii.) & A_{ij}^{\alpha\beta}\xi^i\xi^j\eta_\alpha\eta_\beta \ge \nu|\xi|^2|\eta|^2, \nu > 0\ \forall \xi \in \mathbb{R}^m \text{ and } \eta \in \mathbb{R}^n. \\ & \text{(This is called the Legendre-Hadamard condition)} \end{cases}$$

or

$$\begin{cases} i.) & A_{ij}^{\alpha\beta} \in C^0(\Omega) \\ ii.) & \text{Legendre-Hadamard condition holds} \\ iii.) & R \text{ sufficiently small} \end{cases},$$

then the following Caccioppoli inequality is true

$$\int_{B(x_0,R/2)} |Du|^2\, dx \le C(n)\left\{\frac{1}{R^2}\int_{B(x_0,R)} |u-\lambda|^2\, dx + R^2\int_{B(x_0,R)}\sum_i f_i^2\, dx + \int_{B(x_0,R)}\sum_{i,\alpha}(f_i^\alpha)^2\, dx\right\}$$

for all $\lambda \in \mathbb{R}$.

Proof: * Here we will prove the third case; it's not hard to adapt the following estimates to prove cases i.) and ii.). From our weak formulation we derive

$$\begin{aligned}\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0) D_\alpha u^i D_\beta \phi^j \, dx \quad &= \quad \int_{B_R(x_0)} \left[A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\right] D_\alpha u^i D_\beta \phi^j \, dx \\ &\quad + \int_{B_R(x_0)} f_i \phi^i \, dx + \int_{B_R(x_0)} f_i^\alpha D_\alpha \phi^i \, dx\end{aligned}$$

for all $\phi \in H_0^1(B_R(x_0))$. So, we consider the above again with $\phi = (u-\lambda)\eta^2$. Now we calculate:

$$\begin{aligned}&\int_{B_R(x_0)} |D[(u-\lambda)\eta]|^2 \, dx \\ &\quad \leq \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0) D_\alpha[(u^i - \lambda^i)\eta] D_\beta[(u^j - \lambda^j)\eta] \, dx \\ &\quad = \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0) D_\alpha u^i D_\beta[(u^j - \lambda^j)\eta^2] \\ &\qquad + \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0)(u^i - \lambda^i)(u^j - \lambda^j) D_\alpha \eta D_\beta \eta \, dx \\ &\quad = \int_{B_R(x_0)} \left[A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\right] D_\alpha u^i D_\beta \left[(u^j - \lambda^j)\eta^2\right] \, dx \\ &\qquad + \int_{B_R(x_0)} f_i (u^i - \lambda^i)\eta^2 \, dx + \int_{B_R(x_0)} f_i^\alpha D_\alpha \left[(u^i - \lambda^i)\eta^2\right] \, dx \\ &\qquad + \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0)(u^i - \lambda^i)(u^j - \lambda^j) D_\alpha \eta D_\beta \eta \, dx \\ &\quad \leq \delta \int_{B_R(x_0)} D_\alpha u^i D_\beta u^j \eta^2 \, dx + 2\delta \int_{B_R(x_0)} D_\alpha u^i \cdot D_\beta \eta \cdot (u^j - \lambda^j)\eta \, dx \\ &\qquad + \int_{B_R(x_0)} f_i \eta (u^i - \lambda^i) \, dx + \int_{B_R(x_0)} f_i^\alpha D_\alpha \cdot u^i \eta^2 \, dx \\ &\qquad + 2 \int_{B_R(x_0)} \sum_\alpha f_i^\alpha (u^i - \lambda^i)\eta D\eta \, dx + C \int_{B_R(x_0)} |u-\lambda|^2 |D\eta|^2 \, dx \\ &\quad \overset{\text{Young's}}{\leq} \delta \int_{B_R(x_0)} |Du|^2 \eta^2 \, dx + 2\delta\nu \int_{B_R(x_0)} |Du|^2 \cdot \eta^2 \, dx \\ &\qquad + \frac{2\delta}{\nu} \int_{B_R(x_0)} |u-\lambda|^2 |D\eta|^2 \, dx + \frac{1}{R^2} \int_{B_R(x_0)} |u-\lambda|^2 \, dx\end{aligned}$$

*Can be omitted on the first reading.

$$
\begin{aligned}
&+\int_{B_R(x_0)} \sum_i f_i^2\,dx + \frac{1}{\nu}\int_{B_R(x_0)} \sum_{i,\alpha} f_i^\alpha\,dx \\
&+\nu\int_{B_R(x_0)} |Du|^2\eta^2\,dx + \frac{2}{\nu}\int_{B_R(x_0)} \sum_{i,\alpha}(f_i^\alpha)^2\,dx \\
&+2\nu\int_{B_R(x_0)} |u-\lambda|^2|D\eta|^2\,dx + C\int_{B_R(x_0)} |u-\lambda|^2|D\eta|^2\,dx \\
\le{}& (\delta+2\delta\nu+\nu)\int_{B_R(x_0)} |Du|^2\eta^2\,dx + \frac{3}{\nu}\int_{B_R(x_0)} \sum_{i,\alpha}(f_i^\alpha)^2\,dx \\
&+\frac{1+2\nu^{-1}\delta+2\nu+C}{R^2}\int_{B_R(x_0)} |u-\lambda|^2\,dx \\
&+R^2\int_{B_R(x_0)} \sum_i f_i^2\,dx.
\end{aligned} \tag{10.10}
$$

In case it is unclear, δ is the standard free constant of continuity; and C bounds all $A_{ij}^{\alpha\beta}$ on $B_R(x_0)$. Also, ν and some R^2 terms arise via Young's inequality. Next, choose a specific η such that $\eta\equiv 1$ on $R/2$. Now, we consider

$$
\begin{aligned}
&\int_{B_{R/2}(x_0)} |Du|^2\,dx \\
&\quad\le \int_{B_{R/2}(x_0)} |Du|^2\eta^2\,dx \\
&\quad= \int_{B_{R/2}(x_0)} (D[(u-\lambda)\eta])^2 - (u-\lambda)^2(D\eta)^2 - 2DuD\eta(u-\lambda)\eta\,dx \\
&\quad\overset{\text{Young's}}{\le} \int_{B_{R/2}(x_0)} |D[(u-\lambda)\eta]|^2\,dx + 2\nu\int_{B_{R/2}(x_0)} |Du|^2\eta^2\,dx \\
&\qquad+\left(1+\frac{2}{\nu}\right)\int_{B_{R/2}(x_0)} |u-\lambda|^2|D\eta|^2\,dx.
\end{aligned}
$$

Solving this last inequality algebraically and utilizing the properties of η, we get

$$
(1-2\nu)\int_{B_{R/2}(x_0)} |Du|^2\,dx \quad\le\quad \int_{B_{R/2}(x_0)} |D[(u-\lambda)\eta]|^2\,dx
$$

$$+\frac{\nu+2}{\nu R^2}\int_{B_{R/2}(x_0)}|u-\lambda|^2\,dx.$$

Marching on, we put in estimate (10.10) into the RHS of the above (remembering that we now put in $R/2$ instead of R into the estimate) to ascertain

$$\begin{aligned}
&(1-2\nu)\int_{B_{R/2}(x_0)}|Du|^2\,dx \\
&\quad\le(\delta+2\delta\nu+\nu)\int_{B_{R/2}(x_0)}|Du|^2\eta^2\,dx+\frac{3}{\nu}\int_{B_{R/2}(x_0)}\sum_{i,\alpha}(f_i^\alpha)^2\,dx \\
&\qquad+\frac{2\nu^2+(C+2)\nu+2\delta+3}{R^2}\int_{B_{R/2}(x_0)}|u-\lambda|^2\,dx \\
&\qquad+R^2\int_{B_{R/2}(x_0)}\sum_i f_i^2\,dx
\end{aligned}$$

Finally solving algebraically we conclude that

$$\begin{aligned}
&(1-3\nu-\delta(1-2\nu))\int_{B_{R/2}(x_0)}|Du|^2\,dx \\
&\quad\le\frac{3}{\nu}\int_{B_R(x_0)}\sum_{i,\alpha}(f_i^\alpha)^2\,dx+\frac{2\nu^2+(C+2)\nu+2\delta+3}{R^2}\int_{B_R(x_0)}|u-\lambda|^2\,dx \\
&\qquad+R^2\int_{B_R(x_0)}\sum_i f_i^2\,dx.
\end{aligned}$$

In the above we have used the properties of η and after solving algebraically, expanded the RHS integrals to balls of radius R (obviously this can be done as we are integrating over strictly positive quantities). This last inequality brings us to our conclusion as ν, δ, and C are independent of u, along with the fact that δ and ν can be chosen small enough so the LHS of the above remains positive. ■

10.3.2 Interior Estimates

We will now start exploring the classical theory of linear elliptic regularity. To begin, we will go over estimation of elliptic solutions via bounding Sobolev norms, utilizing difference quotients. The first two theorems of this subsection deal with interior estimates, while the last theorem treats the issue of boundary regularity.

Changing the system under consideration, we will now work with an elliptic system of the form

$$-D_\alpha\left(A_{ij}^{\alpha\beta}(x)D_\beta u^j\right) = f_i + D_\alpha f_i^\alpha, \text{ for } i = 1,\dots,m,$$

where $f_i \in L^2(\Omega)$ and $f_i^\alpha \in H^1_{\text{loc}}(\Omega)$. Basically, we will no longer assume that the inhomogenous term in the system is an element in $H^{-1}(\Omega)$. Indeed, we will actually need true weak differentiability of the inhomogenaity (i.e. a bounded Sobolev norm) to attain the full regularity theory of these linear systems. With that, we consider a weak solution $u \in H^1_{\text{loc}}(\Omega)$ of the elliptic system:

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i\,dx = \int_\Omega (f_i\phi^i + f_i^\alpha D_\alpha\phi^i)\,dx \quad \forall \phi \in H^1_0(\Omega); \quad (10.11)$$

this particular weak formulation is derived by the usual method of integration by parts. Now, we go onto our first theorem.

Theorem 10.13. *Suppose that $A_{ij}^{\alpha\beta} \in Lip(\Omega)$, that (10.11) is elliptic with $f_i \in L^2(\Omega)$ and $f_i^\alpha \in H^1(\Omega)$. If $u \in H^1_{loc}(\Omega)$ is a weak solution of (10.11), then $u \in H^2_{loc}(\Omega)$.*

Proof: We start off by recalling the notation from section §9.8: $u_h(x) := u(x + he_s)$, where

$$e_s := (0,\dots,\underbrace{1}_{s\text{th place}},\dots,0).$$

With this we consider $\Omega' \Subset \Omega$ with h small enough so that $u_h(x)$ is defined for all $x \in \Omega'$. Now, we can rewrite our weak formulation as

$$\int_{\Omega'} A_{ij}^{\alpha\beta}(x+he_s)D_\beta u_h^j D_\alpha\phi^i\,dx \\ = \int_{\Omega'} (f_i(x+he_s)\phi^i(x) + f_i^\alpha(x+he_s)D_\alpha\phi^i(x))\,dx.$$

Subtracting the original weak formulation from this, we have

$$\int_{\Omega'} A_{ij}^{\alpha\beta}\left[\frac{D_\beta u_h^j - D_\beta u^j}{h}\right] D_\alpha\phi^i\,dx + \left[\frac{A_{ij}^{\alpha\beta}(x+he_s) - A_{ij}^{\alpha\beta}(x)}{h}\right] D_\beta u^j D_\alpha\phi^i\,dx \\ = \int_{\Omega'}\left[\frac{f_i(x+he_s) - f(x)_i}{h}\right]\phi^i(x)\,dx + \int_{\Omega'}\left[\frac{f_i^\alpha(x+he_s) - f_i^\alpha(x)}{h}\right] D_\alpha\phi^i\,dx.$$

Using notation from section §9.8 and commuting the difference and differential operator on u, we obtain

$$\int_{\Omega'} A_{ij}^{\alpha\beta} \cdot D_\beta(\Delta_h^s u^i) \cdot D_\alpha \phi^j + \Delta_h^s A_{ij}^{\alpha\beta} \cdot D_\beta u^j \cdot D_\alpha \phi^i \, dx \\ = \int_{\Omega'} \Delta_h^s f_i \cdot \phi^i + \Delta_h^s f_i^\alpha \cdot D_\alpha \phi^i \, dx.$$

We can algebraically manipulate this to get

$$\int_{\Omega'} A_{ij}^{\alpha\beta} \cdot D_\beta(\Delta_h^s u^j) \cdot D_\alpha \phi^i \\ = \int_{\Omega'} \Delta_h^s f_i \cdot \phi^i + (\Delta_h^s f_i^\alpha - \Delta_h^s A_{ij}^{\alpha\beta} \cdot D_\beta u^j) D_\alpha \phi^i \, dx. \quad (10.12)$$

At this point, we would like to apply Caccioppoli's inequality to the above, but the first term on the RHS is not necessarily bounded; in order to proceed we now move to prove it is indeed bounded. So, let us look at this term with $\phi^i := \eta^2 \Delta_h^s u^i$, where η has the standard definition. After some algebraic tweaking around, we see that

$$\int_{\Omega'} f_i \cdot \Delta_{-h}^s(\eta^2 \cdot \Delta_h^s u^i) \, dx \\ = \int_{\Omega'} f_i \eta \cdot \Delta_h^s u^i \cdot \Delta_{-h}^s \eta \, dx + \int_{\Omega'} f_i \cdot \eta(x - he_s) \cdot \Delta_{-h}^s(\eta \cdot \Delta_h^s u^i) \, dx.$$

We now estimate the second term on the RHS by applying Young's inequality, a derivative product rule, and difference quotients bounds with derivatives. The ultimate result is:

$$\begin{aligned} &\int_{\Omega'} f_i \cdot \Delta_{-h}^s(\eta^2 \cdot \Delta_h^s u^i) \, dx \\ &\overset{\text{Young's}}{\leq} \int_{\Omega'} f_i \eta \cdot \Delta_h^s u^i \cdot \Delta_{-h}^s \eta \, dx \\ &\quad + \frac{1}{\nu} \int_{\Omega'} |f_i \eta|^2 \, dx + \nu \left[\int_{\Omega'} \eta^2 |D(\Delta_h^s u^i)|^2 \, dx + \int_{\Omega'} |D\eta|^2 |\Delta_h^s u^i|^2 \, dx \right] \\ &\leq \int_{\Omega'} f_i \eta \cdot \Delta_h^s u^i \cdot \Delta_{-h}^s \eta \, dx \\ &\quad + C(R) \left(\sup_i ||f_i||_{L^2(\Omega')} + \sup_i ||u^i||_{H^1(\Omega')} \right. \\ &\quad \left. + \sup_i \int_{\Omega'} |D(\Delta_h^s u^i)|^2 \, dx \right). \end{aligned}$$

$$\overset{\text{Hölder}}{\leq} C(R)\Bigg(\left[\sup_i ||f_i||_{L^2(\Omega')} + \sup_i ||u^i||_{H^1(\Omega')} + 1\right]^2 + \sup_i \int_{\Omega'} |D(\Delta_h^s u^i)|^2\, dx\Bigg).$$

Next, this estimate is applied to (10.12) yields

$$\begin{aligned}
&\int_{\Omega'} A_{ij}^{\alpha\beta} \cdot D_\beta(\Delta_h^s u^j) \cdot D_\alpha \phi^i \\
&\leq \int_{\Omega'} f_i \eta \cdot \Delta_h^s u^i \cdot \Delta_{-h}^s \eta + (\Delta_h^s f_j^\beta - \Delta_h^s A_{ij}^{\alpha\beta} \cdot D_\alpha u^i) D_\beta \phi^j\, dx \\
&\qquad + C(R)\Bigg(\left[\sup_i ||f_i||_{L^2(\Omega')} + \sup_i ||u^i||_{H^1(\Omega')} + 1\right]^2 \\
&\qquad + \sup_i \int_{\Omega'} |D(\Delta_h^s u^i)|^2\, dx\Bigg).
\end{aligned}$$

Finally, we now can apply Caccioppoli's inequality to this despite the fact that the $C(R)$ term is not accounted for. Basically, we keep the $C(R)$ term as is; and reflecting on the proof of Caccioppoli's inequality it is obvious that we can indeed say

$$\begin{aligned}
&\int_{B_{R/2}(x_0)} |D(\Delta_h^s u^i)|^2 \\
&\leq C(R)\Bigg(\frac{1}{R^2}\int_{B_R(x_0)} |\Delta_h^s u^i|^2\, dx + \int_{B_R(x_0)} \sum_{j,\alpha} |\Delta_h^s A_{ij}^{\alpha\beta}|^2 |D_\beta u^i|^2\, dx \\
&\qquad + \int_{B_R(x_0)} \sum_\alpha |\Delta_h^s f_i^\alpha|^2\, dx + \left[\sup_i ||f_i||_{L^2(R)} + \sup_i ||u^i||_{H^1(R)} + 1\right]^2\Bigg).
\end{aligned}$$

Thus, we have finally bounded the difference of the derivative of u. By proposition 9.67 we have shown $u \in H^2_{\text{loc}}(\Omega)$. ■

Now, we move onto the next theorem which generalizes the first.

Theorem 10.14 (Interior Sobolev Regularity). *Suppose that* $A_{ij}^{\alpha\beta} \in C^{k,1}(\Omega)$*, that (10.11) is elliptic with* $f_i \in H^k(\Omega)$ *and* $f_i^\alpha \in H^{k+1}(\Omega)$*. If* $u \in H^1_{loc}(\Omega)$ *is a weak solution of*

$$-D_\alpha \left(A_{ij}^{\alpha\beta}(x) D_\beta u^j \right) = f_i - D_\alpha f_i^\alpha, \ \textit{for } i = 1, \ldots, m,$$

then $u \in H^{k+2}_{loc}(\Omega)$*.*

Proof: We append the proof of theorem (10.13), i.e. we have shown that $u \in H^2_{\text{loc}}(\Omega)$. Now, let us set $\phi^i := D_s\psi^i$, where $\psi^i \in C_0^\infty(\Omega)$. Upon integrating by parts the weak formulation we ascertain

$$-\int_\Omega D_s(A_{ij}^{\alpha\beta} D_\beta u^j) D_\alpha \psi^i \, dx = \int_\Omega f_i D_s \psi^i - \int_\Omega D_s(f_i^\alpha) D_\alpha \psi^i \, dx.$$

Upon carrying out the D_s derivative on the RHS and algebraically manipulating we conclude

$$\begin{aligned}
\int_\Omega A_{ij}^{\alpha\beta} D_\beta (D_s u^j) D_\alpha \psi^i \, dx &= -\int_\Omega D_s(A_{ij}^{\alpha\beta}) D_\beta u^j D_\alpha \psi^i \, dx \\
&\quad - \int_\Omega f_i D_s(\psi^i) \, dx + \int_\Omega D_s(f_i^\alpha) D_\alpha \psi^i \, dx \\
&= \int_\Omega [-D_s(A_{ij}^{\alpha\beta}) D_\beta u^j - f_i - D_s f_i^\alpha] D_\alpha \psi^i \, dx
\end{aligned}$$

Now, in the case of the theorem of $k = 1$, we have the assumptions that $D_s A_{ij}^{\alpha\beta} \in \text{Lip}(\Omega)$, $f_i \in H^1_{\text{loc}}(\Omega)$ and $f_i^\alpha \in H^2_{\text{loc}}(\Omega)$, i.e. we can integrate the LHS of the last equation to ascertain that $D_s u^i \in H^2_{\text{loc}}(\Omega)$ or $u \in W^{3,2}_{\text{loc}}(\Omega)$. We get our conclusion by applying a standard induction argument to the above method. ■

From this last theorem, it is clear that the regularity of the homogeneity and coefficients determine the regularity of our weak solution. We note particularly that if $A_{ij}^{\alpha\beta}, f_i, f_i^\alpha \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$. More specifically, if $A_{ij}^{\alpha\beta} = \text{const.}$, $f_i \equiv 0$, and $f_i^\alpha \equiv 0$, then for any $B_R \Subset \Omega$ and for any k, it can be seen from the above proofs that

$$||u||_{H^k(B_{R/2})} \leq C(k, R) ||u||_{L^2(B_R)}. \tag{10.13}$$

10.3.3 Boundary/Global Estimates

Now we move our discussion to the boundary of Ω. Fortunately, we have done most of the hard work already for this situation.

Theorem 10.15. *Suppose that $\partial\Omega \in C^{k+1}$, $A_{ij}^{\alpha\beta} \in C^k(\Omega)$, $f_i \in H^k(\Omega)$ and $f_i^\alpha \in H^{k+1}(\Omega)$. If u is a weak solution of the Dirichlet-problem, ie. $u \in H^1_{loc}(\Omega)$ solves*

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i \, dx = \int_\Omega (f_i \phi^i + f_i^\alpha D_\alpha \phi^i) \, dx$$

$\forall \phi \in H^1_0(\Omega)$ with $\quad u = u_0$ on $\partial\Omega$, then $u \in H^{k+1}(\Omega)$ and moreover

$$||u||_{H^{k+1}(\Omega)} \leq C(\Omega) \left[||f||_{W^{k-1,2}(\Omega)} + \sum_{i,\alpha} ||f_i^\alpha||_{H^k(\Omega)} + ||u||_{L^2(\Omega)} \right].$$

*Proof:*Setting $v = u - u_0$, it is clear that v satisfies the same system with a homogeneous boundary condition. So, without loss of generality, we assume $u_0 = 0$ in our problem. Since $\partial\Omega \in C^{k+1}$, we know for any $x_0 \in \partial\Omega$, there exists a diffeomorphism

$$\Phi : B_R^+ \to W \cap \Omega$$

of class C^{k+1} which maps Γ_R onto $M \cap \partial\Omega$.

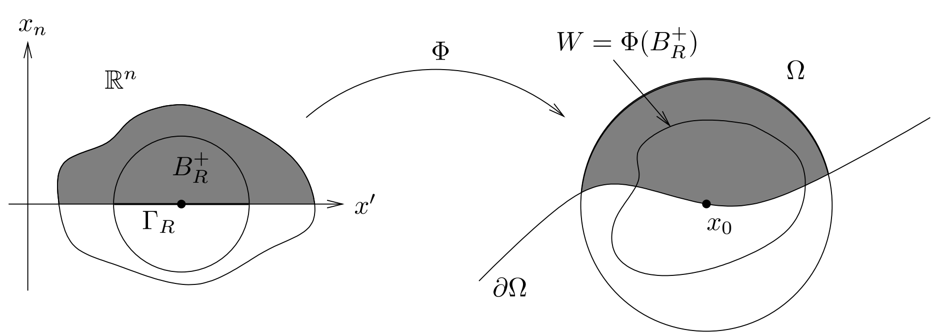


Figure 10.2: Boundary Diffeomorphism

The diffeomorphism Φ induces a map Φ^* defined by $(\Phi^* u)(y) := u(\Phi(y))$ with $y \in B_R^+$. Since Φ is of class C^{k+1}, the chain-rule dictates that Φ^* is an isomorphism between $H^{k+1}(B_R^+)$ and $H^{k+1}(\Omega \cap W)$. Upon defining

$u_* := \Phi^* u$, the chain-rule again says that $Du(\Phi(y)) = J_\Phi^{-1} Du_*(y)$, where J_Φ is the Jacobian matrix of Φ. Recalling multi-dimensional calculus, we see that u_* satisfies

$$\int_{B_R^+} \tilde{A}_{ij}^{\alpha\beta} D_\beta u_*^i D_\alpha \phi^j \, dy = \int_{B_R^+} \tilde{f}_i \phi^i \, dy + \int_{B_R^+} \tilde{f}_i^\alpha D_\alpha \phi^i \, dy, \tag{10.14}$$

where

$$\begin{aligned} \tilde{A}_{ij}^{\alpha\beta} &= |\det(J_\Phi)| J_\Phi^{-1} A_{ij}^{\alpha\beta} J_\Phi \\ \tilde{f}_i^\alpha &= |\det(J_\Phi)| J_\Phi^{-1} \Phi^* f_i^\alpha \\ \tilde{f}_i &= |\det(J_\Phi)| \Phi^* f_i. \end{aligned}$$

From this we see that $\tilde{A}_{ij}^{\alpha\beta}$ is a positive multiple of a similarity transformation of $A_{ij}^{\alpha\beta}$, i.e. $\tilde{A}_{ij}^{\alpha\beta}$ is essentially a coordinate rotation of the original coefficients and thus are still elliptic.

After all that, we have reduced the problem to that of $u_* = 0$ on Γ_R. Now, for ever direction but x_n we can use the previous proof to bound the difference quotient. Specifically, we take $\Psi^i = \Delta_{-h}^s(\eta^2 \Delta_h^s u_*^i)$ for $s = 1, \ldots, n-1$ and $\eta \in C_c^\infty(B_R(x_0))$ to ascertain

$$||\Delta_h^s(D_\alpha u_*^i)||_{L^2(B_{R/2}^+)} \le C \left(||u_*^i||_{H^1(B_R^+)} + \sup_i ||f_i||_{L^2(B_R^+)} + \sup_{\alpha,i} ||f_i^\alpha||_{H^1(B_R^+)} \right).$$

precisely as we did for the interior case. This only leaves $D_{nn}u_*$ left to be handled. Looking at (10.14) and writing the whole thing out, we obtain

$$\begin{aligned} &\int_{B_R^+} \tilde{A}_{ij}^{nn} D_n u_*^j D_n \phi^i \, dy \\ &\qquad = -\sum_{\alpha,\beta=1}^{n-1} \int_{B_R^+} \tilde{A}_{ij}^{\alpha\beta} D_\beta u_*^j D_\alpha \phi^i \, dy + \int_{B_R^+} \tilde{f}_i \phi^i \, dy + \int_{B_R^+} \tilde{f}_i^\alpha D_\alpha \phi^i \, dy. \end{aligned}$$

This is an explicit representation of $D_{nn}u_*$ in the weak sense. Since the RHS is known to exist, we thus conclude that so must $D_{nn}u_*$. Now, we can apply the induction and covering arguments employed in earlier proofs of this section to obtain the result. ■

10.4 Elliptic Regularity II: Schauder Estimates

In the last section we considered regularity results that pertained to estimates of Sobolev norms. As was eluded to at the beginning of the last section, these estimates provide a starting point for more useful estimates on linear elliptic systems. Namely, we now have enough tools to derive the classical Schauder estimates of linear elliptic systems. The power of these estimates is that they give us bounds on the Hölder norms of solutions with relatively light criteria placed upon the weak solution being considered and the coefficients of the system. We will go through a gradual series of results, starting with the relatively specialized case of systems with constant coefficient working up to results pertaining to non-divergence form systems with Hölder continuous coefficients.

Notation: For some of the following proofs, the indices have been ommitted when they have no affect on the logic or idea of the corresponding proof. Such pedantic notation is simply too distracting when it is not needed for clarity.

To start off this section we will first go over a very useful lemma that will be used in several of the following proofs.

Lemma 10.16. *Consider $\Psi(\rho)$ non-negative and non-decreasing. If*

$$\Psi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \epsilon\right]\Psi(R) + BR^{\beta} \qquad \forall \rho < R \le R_0, \tag{10.15}$$

with A,B,α, and β non-negative constants and $\alpha > \beta$, then $\exists \epsilon_0(\alpha,\beta,A)$ such that

$$\Psi(\rho) \le C(\alpha,\beta,A)\left[\left(\frac{\rho}{R}\right)^{\alpha}\Psi(R) + B\rho^{\beta}\right]$$

is true if (10.15) holds for some $\epsilon < \epsilon_0$.

Proof: First, consider set $0 < \tau < 1$ and set $\rho = \tau R$. Plugging into (10.15) gives

$$\Psi(\tau R) \le \tau^{\alpha} A[1 + \epsilon\tau^{-\alpha}]\Psi(R) + BR^{\beta}.$$

Next fix $\gamma \in (\beta,\alpha)$ and choose τ such that $2A\tau^{\alpha} = \tau^{\gamma}$. Now, pick ϵ_0 such that $\epsilon_0\tau^{-\alpha} < 1$. Our assumption now becomes

$$\begin{aligned}\Psi(\tau R) &\le 2A\tau^{\alpha}\cdot\Psi(R) + BR^{\beta}\\ &\le \tau^{\gamma}\Psi(R) + BR^{\beta}.\end{aligned}$$

Now, we can use this last inequality as a scheme for iteration:

$$\begin{aligned}\Psi(\tau^{k+1}R) &\leq \tau^\gamma \Psi(\tau^k R) + B(\tau^k R)^\beta \\ &\leq \tau^\gamma[\tau^\gamma \Psi(\tau^{k-1}R) + B(\tau^{k-1}R)^\beta] + B\tau^{\beta k}R^\beta;\end{aligned}$$

and after doing this iteration $k+1$ times, we ascertain

$$\Psi(\tau^{k+1}R) \leq \tau^{(k+1)\gamma}\Psi(R) + B\tau^{\beta k}R^\beta \sum_{j=0}^{k} \tau^{j(\gamma-\beta)}.$$

Since $\rho < R$ there exists k such that $\tau^{k+1}R \leq \rho < \tau^k R$. Consequent upon this choice of k:

$$\begin{aligned}\Psi(\rho) \leq \Psi(\tau^{k+1}R) &\leq \tau^{(k+1)\gamma}\Psi(R) + C \cdot B(\tau^k R)^\beta \\ &\leq \left(\frac{\rho}{R}\right)^\gamma \Psi(R) + C \cdot B \left(\frac{\rho}{\tau}\right)^\beta. \qquad \blacksquare\end{aligned}$$

Remark: The above proof might seem to be grossly unintuitive at first, but in reality given so many free constants, the proof just systemattically selects the constants to apply the iterative step that sets the exponents correctly.

The value of the above lemma is that it allows us to replace the BR^β term with a $B\rho^\beta$, in addition to letting us drop the ϵ term. Thus Ψ can be bounded by a linear combination of ρ^α and ρ^β via this lemma; a very useful tool for reconciling Companato semi-norms (as one can intuitively guess from definitions).

10.4.1 Systems with Constant Coefficients

Moving on to actual regularity results, we first consider the simplest system where the coefficients are constant. After going through the regularity results, we will briefly explore the very interesting consequences of such.

Theorem 10.17. *If $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = 0, \tag{10.16}$$

where $A^{\alpha\beta}_{ij}$ = const. and satisfy the L-H condition, then for some fixed R_0, we have that for any $\rho < R < R_0$,

i.)

$$\int_{B_\rho(x_0)} |u|^2\,dx \leq C \cdot \left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2\,dx. \tag{10.17}$$

ii.) Moreover,

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \, dx \le C \cdot \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 \, dx. \tag{10.18}$$

Proof (10.17): First, if $\rho \geq R/2$ we can clearly pick C such that both of the inequalities above are true. So we consider now when $\rho \leq R/2$. Since u is a solution of (10.16), we recall the interior Sobolev estimate from the last section (theorem 10.14) to write

$$||u||_{H^k\left(B_{R/2}(x_0)\right)} \le C(k, R) \cdot ||u||_{L^2(B_R)}$$

Thus for $\rho < R/2$ we fine

$$\begin{aligned}
\int_{B_\rho(x_0)} |u|^2 \, dx &\overset{\text{Hölder}}{\le} C(n) \cdot \rho^n \cdot \sup_{B_\rho(x_0)} |u|^2 \\
&\le C(n) \cdot \rho^n \cdot ||u||_{H^k\left(B_{R/2}(x_0)\right)} \\
&\le C(n, k, R) \cdot \rho^n \cdot \int_{B_R(x_0)} |u|^2 \, dx
\end{aligned}$$

The second inequality comes from the fact that $H^k(\Omega) \hookrightarrow C^1(\overline{\Omega}) \subset L^\infty(\Omega)$ for k choosen such that $k > \frac{n}{p}$ (see §9.7). Also, we can employ a simple rescaling argument to see that $C(n, k, R) = C(k)R^{-n}$.

(10.18): This essentially is a consequence of (10.17) used in conjunction with Poincaré and Caccioppoli inequalities:

$$\begin{aligned}
\int_{B_\rho(x_0)} |u - u_{\rho,x_0}|^2 \, dx &\overset{\text{Poincaré}}{\le} C \cdot \rho^2 \int_{B_\rho(x_0)} |Du|^2 \, dx \\
&\le C \cdot \rho^2 \left(\frac{\rho}{R}\right)^n \int_{B_{R/2}(x_0)} |Du|^2 \, dx \\
&= C \cdot \left(\frac{\rho}{R}\right)^{n+2} R^2 \int_{B_{R/2}(x_0)} |Du|^2 \, dx \\
&\overset{\text{Caccioppoli}}{\le} C \cdot \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 \, dx.
\end{aligned}$$

Specifically, we have used (10.17) applied to Du to justify the second inequality. We can do this since Du also solves (10.16) since the coefficients are constant.

□

The above theorem has some very interesting consequences, that we will now go into. First, we note that theorem (10.17) holds for all derivatives $D^\alpha u$, as derivatives of a solution of a system with constant coefficients is itself a solution. With that in mind, let us consider the polynomial $\mathcal{P}_{m-1}$ such that

$$\int_{B_\rho(x_0)} D^\alpha(u - \mathcal{P}_{m-1})\, dx = 0$$

for all $|\alpha| \leq m-1$. With this particular choice of polynomial, we can now apply Poincaré's inequality m times to obtain

$$\int_{B_\rho(x_0)} |u - \mathcal{P}_{m-1}|^2\, dx \overset{\text{Poincaré}}{\leq} C\rho^{2m} \int_{B_\rho(x_0)} |D^m u|^2\, dx.$$

Since $D^m u$ also solves (10.16) we apply theorem (10.17) to get

$$\int_{B_\rho(x_0)} |u - \mathcal{P}_{m-1}|^2\, dx \leq \left(\frac{\rho}{R}\right)^{n+2m} \int_{B_R(x_0)} |u|^2\, dx.$$

Now, if we place the additional stipulation on u that $|u(x)| \leq A|x|^\sigma$, $\sigma = [m]+1$ in addition to its solving (10.16), then by letting $R \to \infty$ in the last equation, we get

$$\int_{B_\rho(x_0)} |u - \mathcal{P}_{m-1}|^2\, dx = 0.$$

Thus,

If the growth of u is polynomial, then u actually is a polynomial!

This is a very interesting result in that we are actually able to determine the form of u merely from asymptotics!

We now look at *non-homogenous* elliptic systems with constant coefficients. For this, suppose $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a solution of

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha f^\alpha_i, \quad i = 1, \ldots, m \tag{10.19}$$

where the $A^{\alpha\beta}_{ij}$ are again constant and satisfy a L-H Condition. In this situation, we have the following result:

Theorem 10.18. *Suppose $f \in \mathcal{L}^{2,\lambda}_{loc}(\Omega)$, then $Du \in \mathcal{L}^{2,\lambda}_{loc}(\Omega)$ for $0 \leq \lambda < n+2$.*

Proof: We start by splitting u in $B_R \subset \Omega$ such that $u = v+(u-v) = v+w$ where v is the soluton (which exists by Lax-Milgram, see section §10.2) of the Dirichlet BVP:

$$\begin{cases} D_\alpha(A^{\alpha\beta} D_\beta v^j) = 0 \quad i = 1, \ldots, m & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases}$$

Since this system has constant coefficients, Dv also solves this system locally for all $\rho < R$. Thus, by theorem 10.17.ii), we have

$$\int_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho}|^2\, dx \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Dv - (Dv)_{x_0,\rho}|^2\, dx.$$

Using this, we start our calculations:

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx \\
&\quad = \int_{B_\rho(x_0)} |Dv - Dw - (Dv)_{x_0,\rho} + (Dw)_{x_0,\rho}|^2\, dx \\
&\quad \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Dv-(Dv)_{x_0,\rho}|^2\, dx + \int_{B_\rho(x_0)} |Dw-(Dw)_{x_0,\rho}|^2\, dx.
\end{aligned}$$

Next, we use the fact that $v = u - w$.

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx \\
&\quad \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - Dw - (Du)_{x_0,\rho} + (Dw)_{x_0,\rho}|^2\, dx \\
&\qquad + \int_{B_r} |D(w) - (Dw)_{x_0,\rho}|^2\, dx \\
&\quad \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx \\
&\qquad + 2\int_{B_\rho(x_0)} |D(w) - (Dw)_{x_0,\rho}|^2\, dx \\
&\quad \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx + 2\int_{B_\rho(x_0)} |D(u-v)|^2\, dx \\
&\qquad + 2\int_{B_\rho(x_0)} |(Dw)_{x_0,\rho}|^2\, dx.
\end{aligned} \tag{10.20}$$

Looking at the last term on the RHS, we see

$$\begin{aligned}
\int_{B_\rho(x_0)} |(Dw)_{x_0,\rho}|^2\,dx &\leq \frac{1}{\rho}\left(\int_{B_\rho(x_0)} |Dw|\,dx\right)^2 \\
&\overset{\text{Hölder}}{\leq} \int_{B_\rho(x_0)} |Dw|^2\,dx \\
&= \int_{B_\rho(x_0)} |D(u-v)|^2\,dx
\end{aligned}$$

This inequality combined with (10.24), we see that

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2\,dx \\
&\quad\leq C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R(x_0)} |Du-(Du)_{x_0,\rho}|^2\,dx + C\int_{B_R(x_0)} |D(u-v)|^2\,dx.
\end{aligned} \tag{10.21}$$

To bound the second term on the RHS, we use the L-H condition to ascertain

$$\begin{aligned}
\int_{B_R(x_0)} |D(u-v)|^2\,dx &\leq \int_{B_R(x_0)} A\cdot D(u-v)\cdot D(u-v)\,dx \\
&\leq C\int_{B_R(x_0)} (f-f_{x_0,R})D(u-v)\,dx.
\end{aligned}$$

The second equality above comes from noticing that since u solves (10.19) in the weak sense, $u-v$ solves

$$D_\alpha(A_{ij}^{\alpha\beta}D_\beta(u-v)^j) = -D_\alpha(f_\alpha^i - f_{x_0,R}),\quad i=1,\ldots,m$$

also in the weak sense (also we have taken $u-v\in W^{1,p}_{\text{loc}}(\Omega)$ to be the test function). From this, we see that

$$\int_{B_R(x_0)} |D(u-v)|^2\,dx \leq \int_{B_R(x_0)} |f-f_{x_0,R}|^2\,dx,$$

i.e., upon combining this with (12.16),

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2\,dx \\
&\qquad\qquad\leq C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R(x_0)} |Du-(Du)_{x_0,R}|^2\,dx + C[f]^2_{\mathcal{L}^{2,\lambda}(\Omega)}R^\lambda.
\end{aligned}$$

To attain our result, we now apply lemma 10.16 to the above to justify replacing the second term on the RHS by ρ^λ. ■

Looking closer at the above proof, it is clear that we have proved even more than the result stated in the theorem. Specifically, we have shown the bound

$$[Du]_{\mathcal{L}^{2,\lambda}(\overline{\Omega})} \leq C(\Omega,\Omega')\left[||u||_{H^1(\Omega)} + [f]_{\mathcal{L}^{2,\lambda}(\Omega)}\right] \quad \text{with } \Omega' \Subset \Omega$$

to be true. An immediate consequence of theorem 10.18 and Companato's theorem is

Corollary 10.19. *If $f \in C^{0,\mu}_{loc}(\Omega)$, then $Du \in C^{0,\mu}_{loc}(\Omega)$, where $\mu \in (0,1)$.*

10.4.2 Systems with Variable Coefficients

Theorem 10.20. *Suppose $u \in W^{1,p}_{loc}(\Omega)$ is a solution of (10.19 with $A^{\alpha\beta}_{ij} \in C^0(\Omega)$ which also satisfy a L-H condtion. If $f^i_\alpha \in L^{2,\lambda}(\Omega)$ ($\cong \mathcal{L}^{2,\lambda}(\Omega)$ for $0 \leq \lambda \leq n$), then $Du \in L^{2,\lambda}_{loc}(\Omega)$.*

Proof: We first rewrite the elliptic system as

$$D_\alpha\left(A^{\alpha\beta}_{ij}(x_0)D_\beta u^j\right) = D_\alpha\left\{\left(A^{\alpha\beta}_{ij}(x_0) - A^{\alpha\beta}_{ij}(x)\right)D_\beta u^j + f^\alpha_i\right\} =: D_\alpha F^\alpha_i,$$

where we have defined a new inhomogeneity. Now, we essential have the situation in theorem (10.18) with $D_\alpha F^\alpha_i$ as the new inhomogenity. Thus, using the exact method used in that theorem we ascertain

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du|^2\,dx \\
&\qquad \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\,dx + C\int_{B_R(x_0)} |D(u-v)|^2\,dx \\
&\qquad \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\,dx + C\int_{B_R(x_0)} |F|^2\,dx \\
&\qquad \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\,dx + C\int_{B_R(x_0)} |[A(x) - A(x_0)]Du|^2\,dx \\
&\qquad\qquad + C\cdot\int_{B_R(x_0)} |f|^2\,dx \\
&\qquad \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\,dx + C\cdot\omega(R)\int_{B_R(x_0)} |Du|^2\,dx
\end{aligned}$$

$$+ C \cdot \int_{B_R(x_0)} |f|^2 \, dx$$

Now $\int_{B_R(x_0)} |f|^2 \, dx \leq R^\lambda$ as $f \in L^{2,\lambda}(\Omega)$. With that in mind, if we choose R_0 sufficiently small with $\rho < R < R_0$, then $\omega(R)$ is small and again we can apply lemma 10.16 with $\omega(R)$ corresponding to the ϵ term; this gives us the result. ■

Theorem 10.21. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of*

$$D_\alpha \left(A^{\alpha\beta}_{ij}(x) D_\beta u^j \right) = D_\alpha f^\alpha_i \qquad j = 1, \ldots, m$$

with $A^{\alpha\beta}_{ij} \in C^{0,\mu}(\overline{\Omega})$ which satisfy a L-H condition. If $f^i_\alpha \in C^{0,\mu}(\overline{\Omega})$, then $Du \in C^{0,\mu}_{loc}(\overline{\Omega})$ and moreover

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \leq C \left\{ ||u||_{H^1(\Omega)} + [f]_{C^{0,\mu}(\overline{\Omega})} \right\}.$$

Proof: Again, we use the exact same idea as in the previous proof to see that

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du - (Du)_\rho|^2 \, dx \\
&\quad \leq C \left(\frac{\rho}{R} \right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,r}|^2 \, dx \\
&\qquad + \int_{B_R(x_0)} |f - f_R|^2 \, dx + C \int_{B_R(x_0)} |[A(x) - A(x_0)] Du|^2 \, dx.
\end{aligned} \tag{10.22}$$

To bound this we start out by looking at the first term on the RHS of the above:

$$\begin{aligned}
\int_{B_R(x_0)} |f - f_R|^2 \, dx &\leq \int_{B_R(x_0)} |x - x_0|^{2\mu} \cdot [f]^2_{C^{0,\mu}(B_R(x_0))} \, dx \\
&\leq CR^{2\mu+n}.
\end{aligned}$$

Now we look at the second term on the RHS of (10.22) to notice

$$
\begin{aligned}
C \int_{B_R(x_0)} &|[A(x) - A(x_0)] \cdot Du|^2\, dx \\
&\le C \int_{B_R(x_0)} ||x - x_0|^\mu \cdot [A]_{C^{0,\mu}(B_R(x_0))} \cdot Du|^2\, dx \\
&\le CR^{2\mu} \cdot \int_{B_R(x_0)} |Du|^2\, dx \\
&\overset{\text{Hölder}}{\le} CR^{2\mu+n} ||u||_{W^{1,2}(B_R(x_0))} \\
&\le CR^{2\mu+n}
\end{aligned}
$$

Putting these last two estimates into (10.22) yields

$$
\begin{aligned}
\int_{B_\rho(x_0)} &|Du - (Du)_\rho|^2\, dx \\
&\le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,r}|^2\, dx + CR^{2\mu+n}.
\end{aligned}
$$

Per usual this fits the form required by lemma 10.16 and consequently this leads us to our conclusion. ■

10.4.3 Interior Schauder Estimates for Elliptic Systems in Non-divergence Form

In this section, we consider with systems in non-divergence form. Obviously, if A is continuously differentiable, there is no difference between the divergent and non-divergent case; but in this subsection we will consider the situation where A is only continuous. I.e. we consider

$$
A_{ij}^{\alpha\beta} D_\beta D_\alpha u^j = f_i \quad j = 1, \ldots, m. \tag{10.23}
$$

with $u \in H^2_{\text{loc}}(\Omega)$ and $A_{ij}^{\alpha\beta} \in C^{0,\mu}(\overline{\Omega})$. In this situation, we have the following theorem.

Theorem 10.22. *If $f^i \in C^{0,\mu}(\overline{\Omega})$ then $D_\beta D_\alpha u \in C^{0,\mu}(\overline{\Omega})$; moreover,*

$$
[D_\beta D_\alpha u]_{C^{0,\mu}(\overline{\Omega})} \le C \left[||D_\beta D_\alpha u||_{L^2(\Omega)} + [f]_{C^{0,\mu}(\overline{\Omega})} \right].
$$

Proof: Essentially, we follow the progression of proofs of the last subsection, each of which remain almost entirely unchanged. Corresponding to the progression of the previous section we consider the proof in three main steps.

Step 1: Suppose $A =$ const. and $f = 0$, then we are in a divergence form situation and therefore

$$\int_{B_\rho(x_0)} |D_\beta D_\alpha u - (D_\beta D_\alpha u)_{x_0,\rho}|^2\,dx \leq C \cdot \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |D_\beta D_\alpha u - (D_\beta D_\alpha u)_{x_0,\rho}|^2\,dx.$$

Step 2: Suppose $A =$ const and $f \neq 0$. Again we seek to get this into a divergence form; to this end we set

$$z^i(x) := u^i(x) - \frac{(D_\beta D_\alpha u^i)_{x_0,R}}{2} \cdot x_\beta \cdot x_\alpha.$$

Thus,

$$D_\sigma D_\gamma z^i = D_\sigma D_\gamma u^i - (D_\sigma D_\gamma u^i)_{x_0,R},$$

plugging this into (10.23) yields

$$A_{ij}^{\alpha\beta} D_\beta D_\alpha z^j = f_i - (f_i)_{x_0,R}.$$

As in the previous section, we now split $z = v + (z - v)$ where v is a solution of of the homogenous equation:

$$\begin{cases} A^{\alpha\beta} D_\beta D_\alpha v^j = 0 & \text{in} \quad B_R(x_0) \\ v^j = z^j & \text{on} \quad \partial B_R(x_0) \end{cases}.$$

With that, (10.23) indicates

$$\begin{cases} A_{ij}^{\alpha\beta} D_\beta D_\alpha (z^j - v^j) = f_i - (f_i)_{x_0,R} & \text{in} \quad B_R(x_0) \\ z^j - v^j = 0 & \text{on} \quad \partial B_R(x_0) \end{cases}$$

As in the divergence form case then, we have the estimate

$$\int_{B_R(x_0)} |D^2(z-v)|^2\,dx \leq \int_{B_R(x_0)} |f - f_{x_0,R}|^2\,dx.$$

The spatial indices have been dropped above under the assumption that the supremum of i is taken on the RHS to bound all second order derivatives (as stated at the beginning of this section, writing this triviality out serves merely as a distraction at this point). Using this estimate we ascertain

$$\begin{aligned}\int_{B_\rho(x_0)} &|D_\beta D_\alpha z - (D_\beta D_\alpha z)_{x_0,\rho}|^2\,dx \\ &\leq C\cdot\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R(x_0)}|D_\beta D_\alpha z-(D_\beta D_\alpha z)_{x_0,R}|^2\,dx \\ &\quad + C\int_{B_R(x_0)}|f-f_{x_0,R}|^2\,dx,\end{aligned}$$

i.e.

$$\begin{aligned}\int_{B_\rho(x_0)} &|D_\beta D_\alpha u - (D_\beta D_\alpha u)_{\rho}|^2\,dx \\ &\leq C\cdot\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R(x_0)}|D_\beta D_\alpha u-(D_\beta D_\alpha u)_{x_0,R}|^2\,dx \\ &\quad + C\int_{B_R(x_0)}|f-f_{x_0,R}|^2\,dx\end{aligned}$$

Step 3: The proof now continues as in the divergence form case by writing the equation

$$A_{ij}^{\alpha\beta}(x)D_\beta D_\alpha u^j = f_i$$

as

$$A_{ij}^{\alpha\beta}(x_0)D_\beta D_\alpha u^j = [A_{ij}^{\alpha\beta}(x_0)-A_{ij}^{\alpha\beta}(x)]D_\beta D_\alpha u^j + f_i =: F_i$$

and concludes by using Step 2 with F_i substituted by the RHS of (10.23). ■

10.4.4 Regularity Up to the Boundary: Dirichlet BVP

In this section, we consider solutions of the Dirichlet BVP

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta}(x)D_\beta u^j\right) = D_\alpha f_i^\alpha & \text{in} \quad \Omega \\ u^j = g^j & \text{on} \quad \partial\Omega\in C^{1,\mu}\end{cases}$$

with $f_i^\alpha \in C^{0,\mu}(\Omega)$ and $g \in C^{1,\mu}(\overline{\Omega})$ and we shall extend the local regularity result §10.4.2. Again, we have already seen many of the techniques in previous proofs that we will require here. So, we will simply sketch out these two proofs pointing out the differences that should be considered.

Theorem 10.23. *Let $u \in W^{1,p}(\Omega)$ be a solution of*

$$D_\alpha \left(A_{ij}^{\alpha\beta}(x) D_\beta u^j \right) = D_\alpha f_i^\alpha \qquad j = 1, \dots, m$$

with $A_{ij}^{\alpha\beta} \in C^{0,\mu}(\overline{\Omega})$ which satisfy a L-H condition. If $f_\alpha^i \in C^{0,\mu}(\overline{\Omega})$, then $Du \in C^{0,\mu}(\overline{\Omega})$ and moreover

$$[Du]_{C^{0,\mu}(\overline{\Omega})} \leq C \left\{ ||u||_{H^1(\Omega)} + [f]_{C^{0,\mu}(\overline{\Omega})} \right\}.$$

Proof: First, we associate u with its trivial extension outside of Ω. Given the interior result of theorem 10.21, we just have to prove boundary regularity to get the global result. So, as in the Sobolev estimate case, we consider a zero BVP without loss of generality as the transformation $u \mapsto u - g$ does not change our system. Also, in the same vein as the global Sobolev proof, we know that the assumption $\partial\Omega \in C^{1,\mu}$ says that (locally) there exists a diffeomorphism

$$\Psi : U \to B_R^+$$

which flattens the boundary i.e. maps $\partial\Omega$ onto Γ_R (see figure (10.3)).

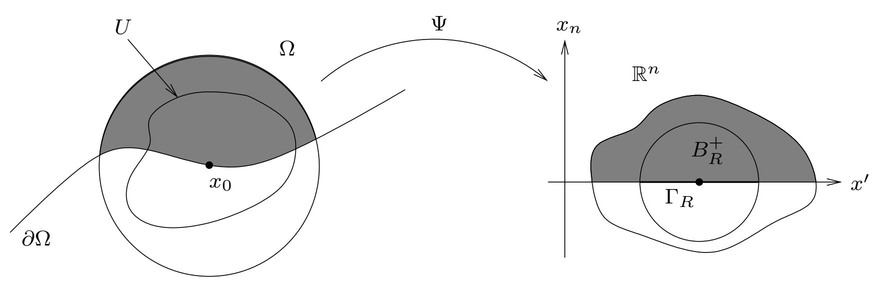


Figure 10.3: Boundary diffeomorphism

In this proof we will be considering B_R^+ with the goal of finding estimates on $B_{R'}^+$, where $R' = \mu R$ for some fixed constant $\mu < 1$. Then the global estimate will follow by using a standard covering argument.

So, with that in mind, we may suppose that $\Omega = B_R^+$ and $\partial\Omega = \Gamma_R$. Generally speaking, we seek the analogies of (10.17) and (10.18); once these are established the completion of the proof mimics the progression of the proofs of interior estimates. To prove these estimates we proceed in a series of steps.

Step 1: First, suppose $A_{ij}^{\alpha\beta} =$ const. and $f_i^\alpha = 0$. For $\rho < R/2$, we have

$$\begin{aligned}
\int_{B_\rho^+} |Du|^2\,dx &\overset{\text{Hölder}}{\leq} C\rho^n \sup_{B_\rho^+} |Du|^2 \\
&\leq C(k)\cdot\rho^n ||u||_{H^k(B_{R/2}^+)} \\
&\leq C(k)\cdot\rho^n ||u||_{H^k(B^*)} \\
&\leq D(R)\cdot\rho^n \int_{B^*} |u|^2\,dx \\
&\leq D(R)\cdot\rho^n \int_{B_R^+} |u|^2\,dx \\
&\overset{\text{Caccioppoli}}{\leq} C(R)\cdot\rho^n \int_{B_R^+} |u_{x_n}|^2\,dx,
\end{aligned}$$

where $B_{R/2}^+ \subset B^* \subset B_R^+$ and ∂B^* is C^∞. This result corresponds to (10.17). The fourth inequality in the above calculation comes from the interior Sobolev regularity estimate of the previous section; hence B^* was introduced into the above calculation to satisfy the boundary requirement required for the global Sobolev regularity. Also, as with the derivation of (10.17), the second inequality comes from the continuous imbedding $H^k(B_{R/2}^+) \hookrightarrow C^{k-1,1-n/2}(B_{R/2}^+) \subset L^\infty(B_{R/2}^+)$.

The counterpart of (10.18) is

$$\begin{aligned}
\int_{B_\rho^+} |Du - (Du)_\rho|^2\,dx &\overset{\text{Poincaré}}{\leq} C\rho^2 \int_{B_\rho^+} |D^2u|^2\,dx \\
&\overset{\text{Caccioppoli}}{\leq} C\rho^{n+2} \int_{B_R^+} |u_{x_n}|^2\,dx.
\end{aligned}$$

To proceed, we note that the same calculation can be carried out for $u - \lambda x_n$ (which is also a solution with zero boundary value). For this, we have to replace u_{x_n} by $u_{x_n} - \lambda$ ($\lambda =$ constant) in the above estimate.

Step 2: Now, that we have a way to bound $I(B_\rho^+)$ (the integral over B_ρ^+) with $\rho < R/2$, we now need to bound $I(B_\rho(z))$ for $z \in B_{R'}(x_0)$ with $\rho < R/2$. To do this, we need to consider two cases for balls centered at z.

Case 1: We first consider when $\operatorname{dist}(z,\Gamma) = r \le \mu R$ with $B_\rho(z) \subset B_R^+(x_0)$.

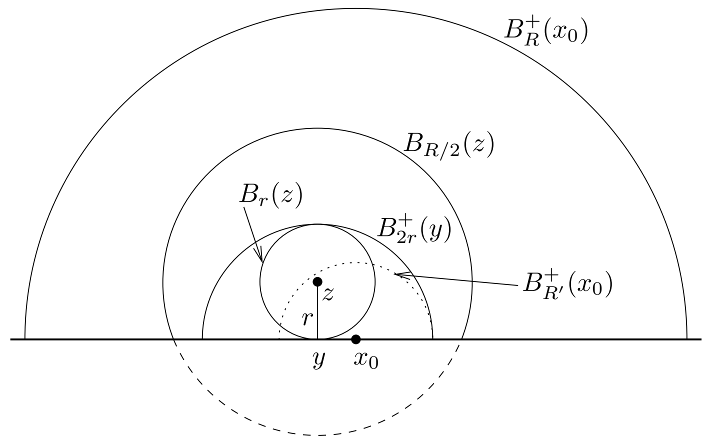


Figure 10.4: Case 1 for proof of theorem 10.4

For this, we first estimate $I(B_\rho(z))$ by $I(B_r(z))$, then $I(B_r(z))$ by $I(B_{2r}^+(y))$ and finally $I(B_{2r}^+(y))$ by $I(B_{R/2}(z))$:

$$\begin{aligned}
&\int_{B_\rho(z)} |u - u_{z,\rho}|^2\,dx \\
&\quad\le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(z)} |u - u_{z,r}|^2\,dx \\
&\quad\le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{2r}^+(y)} |u - u_{z,2r}|^2\,dx \\
&\quad\le D\left(\frac{\rho}{r}\right)^{n+2} \left(\frac{2r}{\sqrt{R^2/4 - r^2}}\right)^{n+2} \int_{B^+_{\sqrt{R^2/4-r^2}}(y)} |u - u_{z,R/2}|^2\,dx \\
&\quad\le D\left(\frac{4}{\frac{1}{4} - \mu^2}\right)^{n+2} \left(\frac{\rho}{r}\right)^{n+2} \left(\frac{r}{R}\right)^{n+2} \int_{B_{R/2}(z)} |u - u_{z,R/2}|^2\,dx.
\end{aligned}$$

In the third inequality we use the analogy of the result form Step 1 for u (the proof is the same).

Case 2: In the second case we have $\operatorname{dist}(z,\Gamma) = r < \mu R$ with $B_\rho(z) \not\subseteq B_R^+(x_0)$. First we remember that we really only want to find a bound for $I(B_\rho(z) \cap B_R^+(x_0))$ as we are considering a trivial extension of u outside Ω. Heuristically, we first estimate $I(B_\rho(z) \cap$

$B_R^+(x_0))$ by $I(B_{2\rho}^+(y))$ and finally $I(B_{2\rho}^+(y))$ by $I(B_{R/2}(z))$. Again, the calculation follows that of Case 1.

It is a strait-forward geometrical calculation that μ can be taken to be $\frac{1}{2\sqrt{17}}$ in order to ensure $B_{2r}^+(y) \subset B_{R/2}(z)$, that Step 1 can be applied in the above cases, and that all the above integrations are well defined (i.e. the domains of integration only extend outside of $B_R^+(x_0)$ over Γ where the trivial extension of u is still well defined). Since μ is independent of the original boundary point (x_0), we can thus apply the standard covering argument to conclude the analogies of (10.17) and (10.18) do indeed hold on the boundary. ■

The nonvariational case

We look at a solution of

$$\begin{cases} A^{\alpha\beta}(x)u^j_{x_\alpha,x_\beta} &= f^i \in C^{2,\mu} \quad \text{in} \quad \Omega \\ u &= g \in C^{2,\mu} \quad \text{on} \quad \partial\Omega \in C^{2,\mu} \end{cases}$$

As in 9.8.1 we find the equivalent of (10.17):

$$\int_{B_\rho^+} |D_\beta D_\alpha u|^2\,dx \le C\left(\frac{\rho}{R}\right)^n \int_{B_R^+} |D_\beta D_\alpha u|^2\,dx$$

$$\int_{B_\rho^+} |D_\beta D_\alpha u - (D_\beta D_\alpha u)_\rho|^2\,dx \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |D_\beta D_\alpha u - (D_\beta D_\alpha u)_\rho|^2\,dx \tag{10.24}$$

moreover we have

$$||u||_{C^{2,\mu}(\overline{\Omega})} \le C\left\{||f||_{C^{0,\mu}(\overline{\Omega})} + ||D_\beta D_\alpha u||_{L^2(\Omega)}\right\}.$$

The proof is the same as in 9.8.1 with the only exception that in the scond step instead of $z = u - 1/2(u_{x_\alpha,x_\beta})_{x_0,R} \cdot x_\alpha \cdot x_\beta$, one takes

$$z = u - \frac{1}{2}(a_{ij}^{mn})^{-1}(f^j)_{x_0,R} \cdot x_\beta^2,$$

then there appears a term $\int_{B_R(x_0)} (f - f_{x_0,R})^2\,dx$ on the RHS of (10.19). ■

10.4.5 Existence and Uniqueness of Elliptic Systems

We finally show how the above a prior estimate implies existence. The idea is to use the *continuation method*.
Consider

$$\begin{cases} L_t := (1-t)\Delta u + t \cdot Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where L is the given elliptic differential operator.

We assume that for L_t ($t \in [0,1]$) and boundary values zero, we have uniqueness. Set

$$\Sigma := \{t \in [0,1] | L_t \text{ is uniquely solvable}\}.$$

Since $t = 0 \in \Sigma$, $\Sigma \neq \emptyset$.

We shall show that Σ is open and closed and therefore $\Sigma = [0,1]$. Especially we conclude (for $t = 1$) that

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has a unique solution.

From the regularity theorem, we know that if $A^{\alpha\beta} \in C^{0,\mu}$ and $f \in C^{0,\mu}$ then

$$||D_\beta D_\alpha u||_{C^{0,\mu}(\overline{\Omega})} \leq C \left\{ ||f||_{C^{0,\mu}(\overline{\Omega})} + ||D_\beta D_\alpha u||_{L^2(\Omega)} \right\}.$$

Now the first step is to show the

Theorem 10.24. *Suppose for the elliptic operator $Lu = A^{\alpha\beta}_{ij} u^i_{x_\alpha, x_\beta}$ we have*

$$\begin{cases} Lu = 0 & \textit{in} \quad \Omega \\ u = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

has only the zero solution. Then if

$$\begin{cases} Lu = f & \textit{in} \quad \Omega \\ u = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

we have

$$||u||_{H^{2,2}(\Omega)} \leq C_1 ||f||_{C^{0,\mu}(\overline{\Omega})}.$$

Proof: Suppose that the theorem is not true, then there exists $A_{ij}^{\alpha\beta(k)}$, $f^{(k)}$ such that for all k

$$A_{ij}^{\alpha\beta(k)}\xi_\alpha\xi_\beta \geq \nu|\xi|^2; \quad ||A^{(k)}||_{C^{0,\mu}(\overline{\Omega})} \leq M$$

and $||f^{(k)}||_{C^{0,\mu}(\Omega)} \to 0$ as $k \to \infty$.

If $u^{(k)}$ is a solution with $f^{(k)}$ and $||u^(k)||_{H^{2,2}(\Omega)} = 1$ then $u^{(k)} \to u$ as $k \to \infty$ and $Lu = 0$ on Ω and $u = 0$ on $\partial\Omega$ i.e. $u = 0$, hence a contradiction. ■

Remark: Uniqueness of second order equations follows from Hopf's maximum principle.

Suppose u is a solution of

$$\begin{cases} A^{\alpha\beta}u_{x_\alpha,x_\beta} = 0 & \text{on} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

If u has a maximum point in $x_0 \in \Omega$, then the Hessian $\{u_{x_\alpha,x_\beta}\} \leq 0$ and $v := u + \epsilon|x - x_0|^2$ has still a maximum point in x_0 (for ϵ sufficiently small). But then $A^{\alpha\beta}v_{x_\alpha,x_\beta} = A^{\alpha\beta}u_{x_\alpha,x_\beta} + \epsilon$ const. > 0 and that is a contradiction. One concludes that $u = 0$.

As $0 = t \in \Sigma$, we have

$$||D_\beta D_\alpha u||_{L^2(\Omega)} \leq C||f||_{C^{0,\mu}(\overline{\Omega})}$$

and therefore

$$||D_\beta D_\alpha u||_{C^{0,\mu}(\overline{\Omega})} \leq C||f||_{C^{0,\mu}(\overline{\Omega})}$$

by the a-priori estimate above.

We show that Σ is closed:

Let $t_k \in \Sigma$ and $t_k \to t$ as $k \to \infty$ with

$$\begin{cases} L_{t_k}u^{(k)} = f & \text{in} \quad \Omega \\ u^{(k)} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

because $||u_{xx}^{(k)}||_{C^{0,\mu}(\overline{\Omega})} \leq C||f||_{C^{0,\mu}(\overline{\Omega})}$, we have $u^{(k)} \to u$ in $C^2(\overline{\Omega})$ and

$$\begin{cases} L_t u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

therefore $t \in \Sigma$.

Now we show that Σ is also open:

Let $t_0 \in \Sigma$. Then there exists u such that

$$\begin{cases} L_{t_0} u = f & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

For $w \in C^{2,\mu}(\overline{\Omega})$ there exists a unique u_w such that

$$\begin{cases} L_{t_0} u_w &= (L_{t_0} - L_t)w + f \quad \text{in} \quad \Omega \\ \quad u_w &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases}$$

We conclude that

$$||u_w||_{C^{2,\mu}(\overline{\Omega})} \leq C|t - t_0| \cdot ||w||_{C^{2,\mu}(\overline{\Omega})} + C \cdot ||f||_{C^{2,\mu}(\overline{\Omega})}$$

and

$$||u_{w_1} - u_{w_2}||_{C^{2,\mu}(\overline{\Omega})} \leq C|t - t_0| \cdot ||w_1 - w_2||_{C^{2,\mu}(\overline{\Omega})}$$

If we choose $t - t_0| =: \delta < 1/c$, we see that the operator

$$\begin{aligned} T: \quad & C^{0,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega}) \\ & w \mapsto u_w \end{aligned}$$

is a contradiction and therefore has a fixpoint which just means that $(t_0 - \delta, t_0 + \delta) \subset \Sigma$, i.e that Σ is open. ■

10.5 Estimates for Higher Derivatives

$u \in W^{1,2}(\Omega)$ is a weak solution of $-\Delta u = 0$.

<u>interior</u>

interior estimates: $B(x_0, R) \Subset \Omega$
boundary estimates: $x_0 \in \partial\Omega$, transformations $\Longrightarrow$ $\partial\Omega$ flat near x_0

$$-\Delta u = 0, \ \frac{\partial}{\partial s} \Longrightarrow -\Delta u D_s u = 0, \ v_s := D_s u \text{ is a solution}$$

Caccioppoli with v_s

$$\begin{aligned}
\int_{B\left(x_0,\frac{R}{2}\right)} |Dv_s|^2\ dx &\leq \frac{C}{R^2}\int_{B(x_0,R)} |v_s|^2\ dx \\
\int_{B\left(x_0,\frac{R}{2}\right)} |DD_s u|^2\ dx &\leq \frac{C}{R^2}\int_{B(x_0,R)} |D_s u|^2\ dx
\end{aligned}$$

$\Longrightarrow$ all 2nd derivatives are in $L^2(\Omega) \Longrightarrow u \in W^{2,2}(\Omega)$

To fully justify this, we replace derivatives with difference quotients. Define $u_h = u(x+he_s)$, $e_s = (0,\ldots,1,\ldots,0)$. Suppose $\Omega' \Subset \Omega$, with h small enough so that u_h is defined in Ω'. Take $\phi \in W_0^{1,2}(\Omega')$ and compute

$$\begin{aligned}
\int_{\Omega'} Du_h(x)\cdot D\phi(x)\ dx &= \int_{\Omega'} Du(x+he_s)\cdot D\phi(x)\ dx \\
&= \int_{he_s+\Omega'\subseteq\Omega} Du(x)\cdot D\phi(x-he_s)\ dx \\
&= \int_{\Omega} Du(x)\cdot D\psi(x)\ dx,
\end{aligned}$$

where $\psi(x) = \phi(x-he_s) \in W_0^{1,2}(\Omega)$. If u is a weak solution of $-\Delta u = 0$ in Ω, then u_h is a weak solution in Ω!

Define difference quotient $\Delta_h^s u = \frac{u(x+he_s)-u(x)}{h}$

$$\begin{aligned}
\int_{\Omega'} Du_h(x)\cdot D\phi(x)\ dx &= 0, \\
\int_{\Omega} Du(x)\cdot D\phi(x)\ dx &= 0
\end{aligned}$$

$$\Longrightarrow \int_{\Omega} \frac{Du_h(x)-Du(x)}{h} D\phi(x)\ dx = 0 \qquad \forall \phi \in W_0^{1,2}(\Omega)$$

$\Longrightarrow \Delta_h^s u = \frac{u_h - u}{h}$ is a weak solution of $-\Delta u = 0$ in Ω'

Suppose that $B(x_0, 2R) \subset \Omega'$, $h < R$, Caccioppoli inequality for $\Delta_s^h u$:

$$\begin{aligned}
\int_{B\left(x_0,\frac{R}{2}\right)} |D\Delta_h^s u|^2 \, dx &\leq \frac{C}{R^2} \int_{B(x_0,R)} |\Delta_h^s u|^2 \, dx \\
\text{(by Theorem 8.59)} \quad &\leq \frac{C}{R^2} \int_{B(x_0,R+h)} |D_s u|^2 \, dx \\
&\leq \frac{C}{R^2} \int_{B(x_0,2R)} |D_s u|^2 \, dx
\end{aligned}$$

By Theorem 8.56, $D_s u$ exists, and we can pass to the limit to get

$$\int_{B\left(x_0,\frac{R}{2}\right)} |DD_s u|^2 \, dx \leq \frac{C}{R^2} \int_{B(x_0,2R)} |D_s u|^2 \, dx$$

Taking $s = 1, \ldots, n$, we get

$$\begin{aligned}
&\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 \, dx \leq \frac{C}{R^2} \int_{B(x_0,2R)} |Du|^2 \, dx \\
&\Longrightarrow W^{1,2}_{\text{loc}}(\Omega)
\end{aligned}$$

Now you can "boot-strap"; apply this argument to $v = D_s u$ and get $v \in W^{1,2}_{\text{loc}}(\Omega) \Longrightarrow u \in W^{3,2}_{\text{loc}}(\Omega) \Longrightarrow \cdot \Longrightarrow u \in W^{k,2}_{\text{loc}}(\Omega)\ \forall k \Longrightarrow u \in C^\infty(\Omega)$ by Sobolev embedding.

In general, the regularity of the coefficients determine how often you can differentiate your equation.

In the boundary situation, this works in all tangential derivatives, i.e. $DD_s u \in L^2(\Omega)$, $s = 1, \ldots, n-1$ (after reorienting the axises).

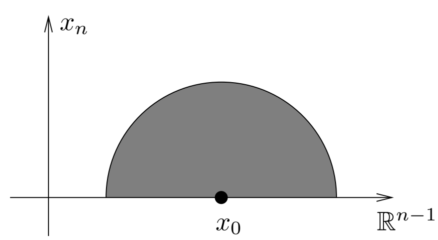


Figure 10.5:

The only derivative missing is $D_{nn}u$. But from the original PDE we have

equation: $$D_{nn}u = -\sum_{i=1}^{n-1} D_{ii}u \in L^2(\Omega)$$

10.6 Eigenfunction Expansions for Elliptic Operators

$A : \mathbb{R}^n \to \mathbb{R}^n$ (A is a $n \times n$ matrix)
A symmetric $\iff (Ax, y) = (x, Ay)\ \forall x, y \in \mathbb{R}^n$
$\implies \exists$ an orthonormal basis of eigenvectors, $\exists \lambda_i \in \mathbb{R}$ such that $Av_i = \lambda_i v_i$, where $v_i \in \mathbb{R}^n \setminus \{0\}$.

Unique expansion:

$$\begin{aligned} \nu &= \sum_i \alpha_i v_i,\ \alpha_i = (v, v_i) \\ |\nu|^2 &= \sum_i \alpha_i^2 \end{aligned}$$

and

$$Av = A\left(\sum_i \alpha_i v_i\right) = \sum_i \alpha_i \lambda_i v_i$$

A diagonal in the eigenbasis.

Application: Solve the ODE $\dot{y} = Au$, i.e., $y(t) \in \mathbb{R}^n$ with $y(0) = y_0$. Explicit representation: $y(t) = \sum_i \alpha_i(t)\nu_i$,

$$\implies \sum_i \dot{\alpha}_i(t)\nu_i = \sum_i \alpha_i(t)\lambda_i\nu_i \qquad \sum_i \alpha_i^0 \nu_i$$

$$\implies \begin{cases} \dot{\alpha}_i(t) = \lambda_i \alpha_i(t),\ i = 1, \ldots, n \\ \alpha_i(0) = \alpha_i^0 \end{cases}$$

Natural Generalization: H is a Hilbert Space, $K : H \to H$ compact and symmetric, i.e., $(Kx, y) = (x, Ky)\ \forall x, y \in H$.

Theorem 10.25 (Eigenvector expansion). *H is a Hilbert Space, K symmetric, compact, and $K \neq 0$. Then there exists an orthonormal system of eigenvectors e_i, $i \in N \subseteq \mathbb{N}$ and corresponding eigenvalues $\lambda_i \in \mathbb{R}$ such that*

$$Kx = \sum_{i \in N} \lambda_i (x, e_i) e_i$$

If N is infinite, then $\lambda_k \to 0$ as $k \to \infty$. Moreover,

$$H = N(k) \oplus \overline{span\{e_i,\ i \in N\}}$$

Definition 10.26. *The elliptic operator L is said to be symmetric if $L = L^*$, i.e.*

$$a_{ij} = a_{ji}, \; b_i = c_i \qquad a.e. \; x \in \Omega$$

Theorem 10.27 (Eigenfunction expansion for elliptic operators). *Suppose L is symmetric and strictly elliptic. There there exists an orthonormal system of eigenfunctions, $u_i \in L^2(\Omega)$, and eigenvalues, λ_i, with $\lambda_i \le \lambda_{i+1}$, with $\lambda_i \to \infty$ such that*

$$\begin{cases} Lu_i = \lambda_i u_i & in \; \Omega \\ u_i = 0 & on \; \Omega \end{cases}$$

$$L^2(\Omega) = \overline{span\{e_i, \; i \in N\}}$$

Remark: $Lu_i = \lambda_i u_i \in L^2(\Omega) \Longrightarrow u_i \in H^2(\Omega)$ by regularity theory if the coefficients are smooth, then $Lu_i = \lambda_i u_i \in H^2(\Omega) \Longrightarrow u_i \in H^4(\Omega), \ldots$, i.e. $u_i \in C^\infty(\Omega)$ if the coefficients are smooth.

Proof: As before L_σ is given by $L_\sigma = Lu + \sigma u$ is invertible for σ large enough, L_σ^{-1} is compact as a mapping from $L^2(\Omega) \to L^2(\Omega)$. Moreover, $L_\sigma^{-1} u = 0 \iff u = L_\sigma 0 = 0$, i.e. $N(L_\sigma^{-1}) = \{0\}$.

By Theorem 9.13, $\exists u_i \in L^2(\Omega)$ such that $L^2(\Omega) = \overline{\text{span}\{u_i, \; i \in n\}}$. Moreover,

$$L_\sigma^{-1} u = \sum_{i \in N} \mu_i (u, u_i) u_i$$

Choose u_j such that

$$L_\sigma^{-1} u_j = \sum_{i \in N} \mu_i \underbrace{(u_j, u_i)}_{\delta_{ij}} u_i = \mu_j u_j, \; \mu_j \neq 0$$

$\Longrightarrow L_\sigma u_j = \frac{1}{\mu_j} u_j$

$$L_\sigma u_j = \frac{1}{\mu_j} u_j \iff Lu_j + \sigma u_j = \frac{1}{\mu_j} u_j \iff Lu_j = \underbrace{\left(\frac{1}{\mu_j} - \sigma \right)}_{\lambda_j} u_j$$

$$L_\sigma u_j = \frac{1}{\mu_j} u_j \iff 0 \le \mathcal{L}(u_j, u_j) = \frac{1}{\mu_j} (u_j, u_j) = \frac{1}{\mu_j}$$

$\Longrightarrow \mu_j > 0$
$\Longrightarrow \mu_j$ has zero as its only possible accumulation point.
$\Longrightarrow \lambda_j \to +\infty$ as $j \to \infty$. ■

Proposition 10.28 (Weak lower semicontinuity of the norm). *X is a Banach Space, $x_j \rightharpoonup x$ in X, then*

$$||x|| \leq \lim_{j\to\infty} \inf ||x_j||$$

Proof: Take $y \in X^*$, then

$$\begin{aligned} |\langle y, x\rangle| \leftarrow |\langle y, x_j\rangle| &\overset{\text{Hölder}}{\leq} ||y||_{X^*}||x_j||_X \\ \implies \quad |\langle y, x\rangle| &\leq \left(\lim_{j\to\infty} \inf ||x_j||_X\right) ||y||_{X^*} \end{aligned}$$

Hahn-Banach: For every $x_0 \in X$, $\exists y \in X^*$ such that $|\langle y, x_0\rangle| = ||x_0||_X$ and $\langle y, z\rangle \leq ||z||_X$, $\forall z \in X$ (see Rudin 3.3 and corollary). This obviously implies that $||y||_{X^*} = 1$. So, we pick $y^* \in X^*$ such that $|\langle y^*, x\rangle| = ||x||_X$. Thus,

$$\implies ||x|| = \langle y^*, x\rangle \leq \left(\lim_{j\to\infty} \inf ||x_j||_X\right) \underbrace{||y^*||_{X^*}}_{=1}$$

Theorem 10.29. *L strictly elliptic, λ_i are eigenvalues. Then,*

$$\begin{aligned} \lambda_1 &= \min\{\mathcal{L}(u,u) : u \in H_0^1(\Omega),\ ||u||_{L^2(\Omega)} = 1\} \\ \lambda_{k+1} &= \min\{\mathcal{L}(u,u) : u \in H_0^1(\Omega),\ ||u||_{L^2(\Omega)} = 1,\ (u, u_i) = 0,\ i = 1, \ldots, k\} \end{aligned}$$

Define $\mu := \inf\{\mathcal{L}(u,u) : ||u||_{L^2(\Omega)} = 1\}$

Proof:

1. $\inf = \min$. Choosing a minimizing sequence $u_j \in H_0^1(\Omega)$, $||u_j||_{L^2(\Omega)} = 1$ such that

$$\mathcal{L}(u_j, u_j) \to \mu$$

   We know from the proof of Fredholm's alternative

$$\begin{aligned} \mathcal{L}(u_j, u_j) &\geq \frac{\lambda}{2}\int_\Omega |Du_j|^2\ dx - C\underbrace{\int_\Omega |u_j|\ dx}_{=1} \\ \implies \quad & \frac{\lambda}{2}\int_\Omega |Du_j|^2\ dx \leq \underbrace{\mathcal{L}(u_j, u_j)}_{\to\mu} + C < \infty \\ \implies \quad & u_j \in W_0^{1,2}(\Omega) \text{ bounded} \end{aligned}$$

$W^{1,2}(\Omega)$ reflexive $\implies \exists$ a subsequence u_{jk} such that $u_{jk} \rightharpoonup \overline{u}$ in $W^{1,2}(\Omega)$ (Banach-(Banach-Alaoglu).

Since $u_{jk} \rightharpoonup \overline{u}$ in $L^2(\Omega) \xrightarrow{\text{Sobolev}} u_{jk} \to \overline{u}$ strongly in $L^2(\Omega)$
(More precisely, u_{jk} bounded in $L^2(\Omega) \implies u_{jk}$ bounded in $W^{1,2}(\Omega)$ $\implies u_{jk}$ has a subsequence converging strongly to some $\overline{u}^*$ in $L^2(\Omega)$ by Sobolev embedding. Clearly, $\overline{u}^*$ is also a weak limit of u_{jk}. Thus, by the uniqueness of weak limits, we have $\overline{u}^* = \overline{u}$ a.e.)

$Du_{jk} \rightharpoonup D\overline{u}$ in $L^2(\Omega)$ weakly and $||\overline{u}||_{L^2(\Omega)} = 1$ from strong convergence.

2. Show that $\mathcal{L}(\overline{u}, \overline{u}) \le \mu$ ($\implies \mathcal{L}(\overline{u}, \overline{u}) = \mu$, $\overline{u}$ minimizer, inf $=$ min). As before $\mathcal{L}_\sigma(u, u) \ge \gamma ||u||^2_{W^{1,2}(\Omega)}$ for σ sufficiently large, i.e.

$$|||u|||^2 = \mathcal{L}_\sigma(u, u)$$

$\mathcal{L}_\sigma(u, u)$ defines a scalar product equivalent to the standard $H^1(\Omega)$ scalar product, $|||\cdot|||$ defines a norm.

weak lower semicontinuity of the norm:
$u_{jk} \rightharpoonup \overline{u}$ in $W^{1,2}(\Omega)$

$$\begin{aligned} \mathcal{L}_\sigma(\overline{u}, \overline{u}) &= |||\overline{u}|||^2 \le \lim_{k\to\infty} \inf |||u_{jk}|||^2 \\ &= \lim_{k\to\infty} \inf(\underbrace{\mathcal{L}(u_{jk}, u_{jk})}_{\to\mu} + \sigma \underbrace{||u_{jk}||^2_{L^2(\Omega)}}_{=1}) \\ &= \mu + \sigma \end{aligned}$$

$\implies \mathcal{L}(\overline{u}, \overline{u}) \le \mu \implies \overline{u}$ is the minimizer

3. μ is an eigenvalue, $L\overline{u} = \mu\overline{u}$

Consider the variation:

$$u_t = \frac{\overline{u} + tv}{||\overline{u} + tv||_{L^2(\Omega)}}$$

Then $t \mapsto \mathcal{L}(u_t, u_t)$ has a minimum at $t = 0$.

$$\mathcal{L}(u_t, u_t) = \frac{1}{||\overline{u} + tv||^2_{L^2(\Omega)}} (\mathcal{L}(\overline{u}, \overline{u}) + \underbrace{t\mathcal{L}(\overline{u}, v) + t\mathcal{L}(v, \overline{u})}_{2t\mathcal{L}(\overline{u}, v) \text{ by sym.}} + t^2\mathcal{L}(v, v))$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(u_t, u_t) = 2\mathcal{L}(\overline{u}, v) - 2\left(\int_\Omega (\overline{u+0})v\ dx\right) \underbrace{\mathcal{L}(\overline{u}, \overline{u})}_{=\mu} = 0$$

where we have used

$$\begin{aligned} \frac{1}{\int_\Omega (\overline{u} + tv)^2\ dx} &= \underbrace{\left[\int_\Omega (\overline{u} + tv)^2\ dx\right]^{-1}}_{=1\ @\ t=0} \\ \frac{d}{dt}\frac{1}{\int_\Omega (\overline{u} + tv)^2\ dx} &= -1 \cdot \underbrace{\left[\int_\Omega (\overline{u} + tv)^2\ dx\right]^{-2}}_{=1\ @\ t=0} \cdot 2 \cdot \int_\Omega (\overline{u} + tv)v\ dx \end{aligned}$$

$\Longrightarrow \mathcal{L}(\overline{u}, v) = \mu(\overline{u}, v) \qquad \forall v \in H_0^1(\Omega)$
$\Longleftrightarrow\ L\overline{u} = \mu\overline{u}$, i.e., μ is an eigenvalue.

4. $\mu = \lambda_1$
   Suppose $\mu > \lambda_1$, $\exists u_1 \in H_0^1(\Omega)$ with $||u_1||_{L^2(\Omega)} = 1$ such that

$$Lu_1 = \lambda_1 u_1$$

$$\begin{aligned} \Longleftrightarrow\ \mathcal{L}(u_1, u_1) &= \lambda_1(u_1, u_1) \\ &= \lambda_1 < \mu = \inf_{||u||_{L^2(\Omega)}=1} \mathcal{L}(u, u) \end{aligned}$$

   Thus, we have a contradiction. ■

Simple situation: a_{ij} smooth, $b_i, c_i, d = 0$.

Proposition 10.30. *L strictly elliptic, then the smallest eigenvalue λ_1 (also called the principle eigenvalue) is simple, i.e., the dimension of the corresponding eigenspace is one.*

Proof: Following the last theorem: variational characterization of λ_1,

$$\min \mathcal{L}(u, u) = \lambda_1,\ \mathcal{L}(\overline{u}, \overline{u}) = \lambda_1 \iff Lu_1 = \lambda_1 u_1$$

Key Step: Suppose that u solves

$$\begin{cases} Lu = \lambda_1 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then either $u > 0$ or $u < 0$ in Ω.
Suppose that $||u||_{L^2(\Omega)} = 1$ and define $u^+ := \max(u, 0)$, and $u^- := -\min(u, 0)$, i.e., $u = u^+ - u^-$. Then $u^+, u^- \in W_0^{1,2}(\Omega)$, and

$$\begin{aligned} Du^+ &= \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{else} \end{cases} \\ Du^- &= \begin{cases} -Du & \text{a.e. on } \{u < 0\} \\ 0 & \text{else} \end{cases} \end{aligned}$$

Thus, we see that $\mathcal{L}(u^+, u^-) = 0$ since $\int_\Omega u^+u^-\ dx = 0, \int_\Omega Du^+ \cdot Du^-\ dx = 0, \ldots$.

$$\begin{aligned} \Longrightarrow \lambda_1 = \mathcal{L}(u, u) &= \mathcal{L}(u^+ - u^-, u^+ - u^-) \\ &= \mathcal{L}(u^+, u^+) - \mathcal{L}(u^+, u^-) - \mathcal{L} + \mathcal{L}(u^-, u^-) \\ &= \mathcal{L}(u^+, u^+) + \mathcal{L}(u^-, u^-) \\ &\geq \lambda_1 ||u^+||^2_{L^2(\Omega)} + \lambda_1 ||u^-||^2_{L^2(\Omega)} = \lambda_1 ||u||^2_{L^2(\Omega)} = \lambda_1 \end{aligned}$$

$\Longrightarrow$ so we have equality throughout,

$$\mathcal{L}(u^+, u^+) = \lambda_1 ||u^+||^2_{L^2(\Omega)},\ \ \mathcal{L}(u^-, u^-) = \lambda_1 ||u^-||^2_{L^2(\Omega)}$$

$\Longrightarrow Lu^\pm = \lambda_1 u^\pm,\ ,u^\pm \in W_0^{1,2}(\Omega)$ since every minimizer is a solution of the eigen-problem. Coefficients being smooth $\Longrightarrow u^\pm$ smooth, i.e. $u^\pm \in C^\infty(\Omega)$. In particular, $Lu^+ = \lambda_1 u^+ \geq 0$ $(\lambda_1 \geq 0)$.

$\Longrightarrow u^+$ is a supersolution of the elliptic equation.

The strong maximum principle says that either $u^+ \equiv 0$ or $u^+ > 0$ in Ω. Analogously, either $u^- \equiv 0$, or $u^- < 0$ in Ω.

$\Longrightarrow$ either $u = u^+$ or $u = u^-$, i.e. either $u > 0$ or $u < 0$ in Ω.

Suppose now that u and $\tilde{u}$ are solutions of $Lu = \lambda_1 u$ in Ω. $u, \tilde{u} \in W_0^{1,2}(\Omega)$ $\Longrightarrow u$ and $\tilde{u}$ have a sign, and we can choose γ such that

$$\int_\Omega (u + \gamma\tilde{u})\ dx = 0$$

Since the original equation is linear, $u + \gamma\tilde{u}$ is also a solution $\Longrightarrow u + \gamma\tilde{u} = 0$ (otherwise $u + \gamma\tilde{u}$ would have a sign) $\Longrightarrow u = -\gamma\tilde{u} \Longrightarrow$ all solutions are multiples of $u \Longrightarrow \dim(\mathrm{Eig}(\lambda_1)) = 1$. ■

Other argument: If $\dim(\mathrm{Eig}(\lambda_1)) > 1$, $\exists\, u_1, u_2$ in the eigenspace such that

$$\int_\Omega u_1 u_2 \, dx = 0$$

which is a contradiction.

10.7 Applications to Parabolic Equations of 2nd Order

General assumption, $\Omega \subset \mathbb{R}^n$ open and bounded, $\Omega_T = \Omega \times (0, T]$ solve the initial/boundary value problem for $u = u(x,t)$

$$\begin{cases} u_t + Lu = f & \text{on } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{t = 0\} \end{cases}$$

L strictly elliptic differential operator, $Lu = -D_i(a_{ij}D_j u) + b_i D_i u + cu$ with $a_{ij}, b_i, c \in L^\infty(\Omega)$.

Definition 10.31. *The operator $\partial_t + L$ is uniformly parabolic, if $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ $\forall \xi \in \mathbb{R}^n$, $\forall (x,t) \in \Omega_T$, with $\lambda > 0$.*

Standard example: Heat equation, $L = -\Delta$, $u_t - \Delta u = f$.

Suppose that $f, g \in L^2(\Omega)$, and define

$$\mathcal{L}(u, v; t) = \int_\Omega a_{ij} D_j u D_i v + b_i D_i u \cdot v + uv \, dx$$

$\forall v \in H_0^1(\Omega)$, $t \in [0, T]$.

New point of view: parabolic PDE $\leftrightarrow$ ODE in Hilbert Space.

$$\begin{aligned} &u(x,t) \rightsquigarrow U : [0,T] \to H_0^1(\Omega) \\ &(U(t))(x) = u(x,t) \qquad x \in \Omega,\ t \in (0,T] \end{aligned}$$

analogously: $$\begin{aligned} &f(x,t) \rightsquigarrow F : [0,T] \to L^2(\Omega) \\ &(F(t))(x) = f(x,t) \qquad x \in \Omega,\ t \in (0,T] \end{aligned}$$

Suppose we have a smooth solution of the PDE, multiply by $v \in W_0^{1,2}(\Omega)$ and integrate parts in $\int_\Omega dx$

$$\int_\Omega U'(t) \cdot v \; dx + \mathcal{L}(U, v; t) = \int_\Omega F(t) \cdot v \; dx$$

What is u_t? PDE $u_t + Lu = f$ says that

$$u_t = f - Lu = \underbrace{f}_{L^2(\Omega)} + D_i \underbrace{(a_{ij} D_j u)}_{L^2(\Omega)} - \underbrace{b_i D_i u}_{L^2(\Omega)} - \underbrace{cu}_{L^2(\Omega)}$$

$$\Longrightarrow u_t = g_0 + D_i g_i \in H^{-1}(\Omega),$$

where $g_0 = f - b_i D_i u - cu$ and $g_i = a_{ij} D_j u$.

This suggests that $\int_\Omega U'(t) \cdot v \; dx$ has to be interpreted as

$$\langle U'(t), v \rangle = \text{duality in } \left(H^{-1}(\Omega), H_0^1(\Omega) \right)$$

for $y(t) = \alpha_i(t) v_i$.

———-

$Lu = -D_i(a_{ij} D_j u) + b_i D_i u + cu$
equation:

$$\begin{cases} u_t + Lu = f & \text{in } \Omega \times (0, T] \\ u = 0 & \text{on } \partial\Omega \times [0, t] \\ u = g & \text{on } \Omega \times \{t = 0\} \end{cases}$$

parabolic if L elliptic: $\sum a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$, $\lambda > 0$.

$$u_t = f - Lu = f + D_i(a_{ij} D_j u) - b_i D_i u - cu \in H^{-1}(\Omega)$$

(u_t, v) duality between H^{-1} and H_0^1

parabolic PDE $\leftrightarrow$ ODE with BS-valued functions

$$\begin{aligned} U(t)(x) &= u(x, t) \qquad \forall x \in \Omega, \; t \in [0, T] \\ \mathcal{L}(u, v; t) &= \int_\Omega (a_{ij} D_j u \cdot D_i v + b_i D_i u + cuv) \; dx \end{aligned}$$

10.7.1 Outline of Evan's Approach

Definition 10.32. *$u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H^{-1})$ is a weak solution of the IBVP if*

i.) $\langle U', v\rangle + \mathcal{L}(U,v;t) = (F,v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$, a.e. $t \in (0,T)$

ii.) $U(0) = g$ in Ω

Remark: $u \in L^2(0,T;H_0^1(\Omega))$, $u' \in L^2(0,T;H^{-1})$ then $C^0(0,T;L^2)$ and thus $U(0)$ is well defined.

General framework: Calculus in abstract spaces.

Definition 10.33. *1) $L^p(0,T;X)$ is the space of all measurable functions $u : [0,1] \to X$ with*

$$||u||_{L^p(0,T;X)} = \left(\int_0^T ||u(t)||_X^p\right)^{\frac{1}{p}} < \infty,\ 1 \le p < \infty$$

and

$$||u||_{L^\infty(0,T;X)} = ess \sup_{0\le t\le T} ||u(t)||_X < \infty$$

2) $C^0([0,T];X)$ is the space of all continuous functions $u : [0,T] \to X$ with

$$||u||_{C^0([0,T];X)} = \max_{0\le t\le T} ||u(t)||_X < \infty$$

3) $u \in L^1(0,T;X)$, then $v \in L^1(0,T;X)$ is the weak derivative of u

$$\int_0^T \phi'(t)u(t)\ dt = -\int_0^T \phi(t)v(t)\ dt \qquad \forall \phi \in C_c^\infty(0,T)$$

4) The Sobolev space $W^{1,p}(0,T;X)$ is the space of all u such that $u, u' \in L^p(0,T;X)$

$$||u||_{W^{1,p}(0,T;X)} = \left(\int_0^T ||u(t)||_X^p + ||u'(t)||_X^p\ dt\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and

$$||u||_{w^{1,\infty}(0,T;X)} = ess \sup_{0\le t\le T} (||u(t)||_X + ||u'(t)||_X)$$

Theorem 10.34. *$u \in W^{1,p}(0,T;X)$, $1 \leq p \leq \infty$. Then*

i.) $u \in C^0([0,T];X)$ i.e. there exists a continuous representative.

ii.)

$$u(t) = u(s) + \int_x^t u'(\tau)\ d\tau \qquad 0 \leq s \leq t \leq T$$

iii.)

$$\max_{0\leq t\leq T} ||u(t))|| = C \cdot ||u||_{W^{1,p}(0,T;X)}$$

$X = H_0^1$, $X = H^{-1}$

10.7.2 Evan's Approach to Existence

Galerkin method: Solve your equation in a finite dimensional space, typically the span of the first k eigenfunctions of the elliptic operator; solution u_k.
Key: energy (a priori) estimates of the form

$$\max_{0\leq t\leq T} ||u_k(t)||_{L^2(\Omega)} + ||u_k||_{L^2(0,T;H_0^1)} + ||u_k'||_{L^2(0,T;H^{-1})}$$
$$\leq C\left(||f||_{L^2(0,T;L^2)} + ||g||_{L^2(\Omega)}\right)$$

with C independent of k

Crucial: (weak) compactness bounded sequences in $L^2(0,T;H_0^1)$ and $L^2(0,T;H^{-1})$ contain weakly convergent subsequences: $u_{k_l} \rightharpoonup u$, $u_{k_l}' \rightharpoonup v$
Then prove that $v = u'$, and that u is a weak solution of the PDE.

Galerkin method: More formal construction of solutions. $\dot{y} = Ay$, $y(0) = y_0$.
A symmetric and positive definite $\Longrightarrow$ exists basis of eigenvectors u_i with eigenvalues λ_i

$$\text{express} \qquad y(t) = \sum_{i=1}^n \alpha_i(t)u_i,\ y_0 = \sum_{i=1}^n \alpha_i^0 u_i$$
$$\Longrightarrow \sum_{i=1}^n \dot{\alpha}_i(t)u_i = \sum_{i=1}^n \lambda_i\alpha_i(t)u_i \qquad \Bigg|\ \cdot u_j$$
$$\Longrightarrow \dot{\alpha}_i(t) = \lambda_i\alpha_i(t),\ \alpha_i(0) = \alpha_i^0,\ \alpha_i(t) = \alpha_i^0 e^{\lambda_1 t}$$
$$y(t) = \sum_{i=1}^n \alpha_i^0 e^{\lambda_i t}$$

Same idea for $A = \Delta = -L$, model for elliptic operators L.

Want to solve:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega \end{cases}$$

Let u_i be the orthonormal basis of eigenfunctions

$$-\Delta u_i = \lambda_i u_i$$

and try to find

$$u(x,t) = \sum_{i=1}^{\infty} \alpha_i(t) u_i(t)$$

$$u_t = \Delta u : \qquad \sum_{i=1}^{\infty} \dot{\alpha}_i(t) u_i(x) = \sum_{i=1}^{\infty} (-\lambda_i)\alpha_i(t) u_i(x) \quad \Big| \cdot u_j, \int dx$$

$$\Longrightarrow \dot{\alpha}_i(t) = -\lambda_i \alpha_i(t),\ \alpha_i(0) = \alpha_i^0,\ \sum_{i=1}^{\infty} \alpha_i^0 u_i(x) = u_0(x)$$

Formal solution:

$$u(x,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \alpha_i^0 u_i(x)$$

Justification via weak solutions:
Multiply by test function, integrate by parts, choose $\phi \in C_c^\infty([0,T) \times \Omega)$

$$\begin{aligned} -\int_0^T u \cdot \phi_t \, dxdt - \int_\Omega u_0(x)\phi(x,0)\, dx &= -\int_0^T \int_\Omega Du \cdot D\phi \, dxdt \\ &= \int_0^T \int_\Omega u \cdot \Delta\phi \, dxdt \end{aligned}$$

Expand the initial data

$$u_0(x) = \sum_{i=1}^{\infty} \alpha_i^0 u_i(x)$$

and define

$$u_j^{(k)}(x) = \sum_{i=1}^{k} \alpha_i^0 u_u(x)$$

Then $u_0^{(k)} \to u_0$ in L^2.

Then

$$u^{(k)}(x) = \sum_{i=1}^{k} \alpha_i^0 e^{-\lambda_i t} u_i(x) \quad \text{is a weak solution}$$

of the IBVP with u_0 replaced by $u_0^{(k)}$. Galerkin method: approximate by $u^{(k)} \in$span$\{u_1, \ldots, u_k\}$

Moreover:

$$\begin{aligned} ||u^{(k)}(t) - u(t)||_{L^2(\Omega)} &= \left|\left|\sum_{k+1}^{\infty} \alpha_i^0 \underbrace{e^{-\lambda_i t}}_{\leq 1} u_i\right|\right|_{L^2(\Omega)} \\ &\leq \left(\sum_{i=k+1}^{\infty} (\alpha_i^0)^2\right)^{\frac{1}{2}} \to 0 \text{ as } k \to \infty \end{aligned}$$

$\Longrightarrow u_k(t) \to u(t)$ uniformly in t as $k \to \infty$

$$-\int_0^T \int_\Omega u^{(k)} \phi_t \ dxdt - \int_\Omega u_0^{(k)} \phi(x,0) \ dx = \int_0^T \int_\Omega u^{(k)} \Delta\phi \ dxdt$$

$\Longrightarrow u$ is a weak solution (as $k \to \infty$)

Representation:

$$u(x,t) = \sum_{i=1}^{\infty} \alpha_i^0 e^{-\lambda_i t} u_i(x)$$

gives good estimates.

$$\begin{aligned}
||u(t)||_{L^2(\Omega)} &= \left|\left|\sum_{i=1}^{\infty} \alpha_i^0 e^{-\lambda_i t} u_i(x)\right|\right|_{L^2(\Omega)} \\
&= \left(\sum_{i=1}^{\infty} (\alpha_i^0)^2 \left(e^{-\lambda_i t}\right)^2\right)^{\frac{1}{2}} \\
&\leq e^{-\lambda_1 t}\left(\sum_{i=1}^{\infty} (\alpha_i^0)^2\right)^{\frac{1}{2}} \\
&= e^{-\lambda_1 t} ||u_0||_{L^2(\Omega)}
\end{aligned}$$

$\Longrightarrow ||u(t)||_{L^2(\Omega)} \leq ||u_0||_{L^2(\Omega)} e^{-\lambda_1 t}$

$$\begin{aligned}
||\Delta u(t)||_{L^2(\Omega)} &= \left|\left|\sum_{i=1}^{\infty} (\alpha_i^0 e^{-\lambda_i t}(-\lambda_i) u_i)\right|\right|_{L^2(\Omega)} \\
&= \frac{1}{t}\left(\sum_{i=1}^{\infty} (\alpha_i^0 \underbrace{t \cdot \lambda_i}_{x} e^{-\lambda_i t})^2\right)^{\frac{1}{2}} \qquad (xe^{-x} \leq C, x \geq 0) \\
&\leq \frac{C}{t}\left(\sum_{i=1}^{\infty} (\alpha_i^0)^2\right)^{\frac{1}{2}} \leq \frac{C}{t} ||u_0||_{L^2(\Omega)}
\end{aligned}$$

Theorem 10.35. *L symmetric and strictly elliptic, then the IBVP*

$$\begin{cases} u_t + Lu = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \end{cases}$$

has for all $u_0 \in L^2(\Omega)$ the solution

$$u(x,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \alpha_i^0 u_i(x)$$

where $\{u_i\}$ is a basis of orthonormal eigenfunctions of L in $L^2(\Omega)$ with corresponding eigenvalue $\lambda_1 \leq \lambda_2 \leq \cdots$

10.8 Neumann Boundary Conditions

Neumann $\leftrightarrow$ no flux
$F(x,t) =$ heat flux $= -\kappa(x)Du(x,t)$ Fourier's law of cooling
$Du \cdot \nu = 0 \iff F(x,t) \cdot \nu = 0$ no heat flux out of Ω

Simple model problem:

$$\begin{cases} -\operatorname{div}(a(x)Du) + c(x)u = f & \text{in } \Omega \\ (a(x)Du) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

a and c are smooth, $f \in L^2$ ($f \in H^{-1}$)

Weak formulation of the problem: multiply by $v \in C^\infty(\Omega) \cap H^1(\Omega)$ and integrate in Ω,

$$\int_\Omega [-\operatorname{div}(a(x)Du) + c(x)u]\, v\, dx = \int_\Omega fv\, dx \tag{10.25}$$

$$= \int_\Omega \underbrace{(a(x)Du)}_{=0} \cdot \nu v\, dx + \int_\Omega a(x)Du \cdot Dv + c(x)uv\, dx$$

$$\implies \int_\Omega [a(x)Du \cdot Dv + c(x)uv]\, dx = \int_\Omega fv\, dx \qquad \forall v \in H^1 \tag{10.26}$$

Definition 10.36. *A function $u \in H^(\Omega)$ is said to be a weak solution of the Neumann problem if (10.26) holds.*

Remark: $(a(x)Du) \cdot \nu = 0$ on $\partial\Omega$
Subtle issue: $a(x)Du$ seems to be only an L^2 function and is not immediately clear how to define Du on $\partial\Omega$. However: $-\operatorname{div}(a(x)Du) = -c(x) + f \in L^2$. Turns out that $a(x)Du \in L^2(div)$

Proposition 10.37. *Suppose that u is a weak solution and suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then u is a solution of the Neumann problem.*

Proof: Use (10.26) first for $V \in C_c^\infty(\Omega)$. Reverse the integration by parts (10.25) to get

$$\int_\Omega [-\operatorname{div}(a(x)Du) + c(x)u - f]v\, dx = 0 \qquad \forall v \in C_c^\infty(\Omega)$$

$\implies -\operatorname{div}(a(x)Du) + cu = f$ in Ω, i.e. the PDE Holds.

Now take arbitrary $v \in C^\infty(\Omega)$

$$\begin{aligned}
&\int_\Omega [a(x)Du \cdot Dv + c(x)v)\ dx = \int_\Omega fv\ dx \\
&= \int_{\partial\Omega} [a(x)Du \cdot \nu)v\ dS + \underbrace{\int_\Omega [-\operatorname{div}(a(x)Du) + c(x)u]v\ dx}_{\int_\Omega fv\ dx} \\
\implies &\int_{\partial\Omega} [a(x)Du \cdot \nu)v\ dS = 0 \qquad \forall v \in C^\infty(\Omega) \\
\implies &a(x)Du \cdot \nu = 0 \text{ on } \partial\Omega \quad \blacksquare
\end{aligned}$$

Remark: The Neumann BC is a "natural" BC in the sense that it follows from the weak formulation. It's not enforced as e.g. the Dirichlet condition $u = 0$ on $\partial\Omega$ by seeking $u \in H_0^1(\Omega)$.

Theorem 10.38. *Suppose $a, c \in C(\overline{\Omega})$ and that there exists $a_1, a_2, c_1, c_2 > 0$ such that $c_1 \le c(x) \le c_2$, $a_1 \le a(x) \le a_2$ (i.e., the equation is elliptic). Then there exists for all $f \in L^2(\Omega)$ a unique weak solution u of*

$$\begin{cases} -\mathit{div}\,(a(x)Du) + cu = f & \text{in } \Omega \\ (a(x)Du) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

and $||u||_{H^1(\Omega)} \le C||f||_{L^2(\Omega)}$.

Remark: The same assertion holds with f a linear function on $H^1(\Omega)$, i.e. $f \in H^{-1}$.

Proof: Weak solution meant

$$\int_\Omega [a(x)Du \cdot Dv + c(x)uv]\ dx = \int_\Omega fv\ dx \qquad \forall v \in H^1(\Omega)$$

linear functional $F(v) = \int_\Omega fv\ dx$.
Bilinear form:

$$B(u,v) = \int_\Omega [a(x)Du \cdot Dv + c(x)uv]\ dx$$

our Hilbert space is clearly $H^1(\Omega)$. The assertion of the theorem follows from Lax-Milgram if:

$$\begin{aligned}
B(u,v) &= \int_\Omega [a(x)Du \cdot Dv + c(x)uv] \ dx \\
&\leq a_2 \int_\Omega |Du| \cdot |Dv| \ dx + c_2 \int_\Omega |u| \cdot |v| \ dx \\
&\leq a_2 ||Du||_{L^2(\Omega)} ||Dv||_{L^2(\Omega)} + c_2 ||u||_{L^2(\Omega)} ||v||_{L^2(\Omega)} \\
&\leq C ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}
\end{aligned}$$

$\Longrightarrow B$ is bounded.

$$\begin{aligned}
B(u,u) &= \int_\Omega (a(x)|Du|^2 + c(x)|u|^2) \ dx \\
&\geq \min(a_1, c_1) ||u||^2_{h^1(\Omega)}
\end{aligned}$$

i.e., B is coercive. ■

Remarks:

1) We need $c_1 > 0$ to get the L^2-norm in the estimate, there is no Poincare-inequality in H^1

$$\text{Poincare:} \quad \int_\Omega |u|^2 \, dx \le C_p \int_\Omega |Du|^2 \, dx$$

and this fails automatically if the space contains non-zero constants.

2) This theorem does not allow us to solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ Du \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned} \int_\Omega f \, dx &= \int_\Omega -\Delta u \, dx \\ &\overset{\text{I.P.}}{=} -\int_{\partial\Omega} Du \cdot \nu \, dx = 0 \end{aligned}$$

So, we have the constraint that $\int_\Omega f \, dx = 0$.

10.8.1 Fredholm's Alternative and Weak Solutions of the Neumann Problem

$$\begin{cases} -\Delta u + u + \lambda u = f & \text{in } \Omega \\ Du \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

(i.e., $a(x) = 1$, $c(x) = 1$, $-\Delta u + u = f$ has a unique solution with Neumann BCs)

Operator equation:

$$\begin{aligned} & (-\Delta + I + \lambda \cdot I)u = f \quad \Big| \cdot (-\Delta + I)^{-1} \\ \iff \quad & [I + (-\Delta + I)^{-1}\lambda \cdot I]u = (-\Delta + I)^{-1}(\lambda \cdot I) \\ \iff \quad & (I - T)u = f \text{ in } H^1(\Omega), \; T = -(-\Delta + I)^{-1}(\lambda \cdot I) \end{aligned}$$

Fredholm applies to compact perturbations of I. We want to show $T = (-\Delta + I)^{-1}(\lambda \cdot I)$ is compact.

$(-\Delta + I)^{-1} : H^{-1} \to H^1$
$\lambda \cdot I : H^1 \to H^{-1}$, $I = I_1 I_0$
where $I_0 : H^1(\Omega) \subset\subset\!\!\to L^2(\Omega)$ for $\partial\Omega$ smooth
$\quad I_1 : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$
$\Longrightarrow I$ compact $\Longrightarrow T$ compact.

Fredholm alternative: Either unique solution for all g or non-trivial solution of the homogeneous equation, $(I - T)u = 0$

$$\begin{aligned} &\Longleftrightarrow \quad [I + (-\Delta + I)^{-1}\lambda \cdot I]u = 0 \\ &\Longleftrightarrow \quad [(-\Delta + I) + \lambda \cdot I]u = 0 \\ &\Longleftrightarrow \quad [-\Delta + (\lambda + 1)I]u = 0 \end{aligned}$$

Special case $\lambda = -1$ has non-trivial solutions, e.g. $u \equiv$ constant.

Solvability criterion: We can solve if we in $N^{\perp}((I - T)^*) = N^{\perp}(I - T)$ since $T = T^*$.
Non-trivial solutions for $\lambda = -1$ satisfy

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \text{ (in the weak sense)} \\ Du \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned} \Longleftrightarrow \quad & \int_\Omega Du \cdot Dv \; dx = 0 \qquad \forall v \in H^1(\Omega) \\ & u = v : \quad \int_\Omega |Du|^2 \; dx = 0 \end{aligned}$$

$\Longrightarrow u \equiv$constant (if Ω is connected)
$\Longrightarrow$ Null-space of adjoint operator is $\{u \equiv \text{ constant}\}$
$\Longrightarrow$ perpendicular to this means $\int_\Omega f \; dx = 0$.

There is only uniqueness only up to constants.

Remark: Eigenfunctions of $-\Delta$ with Neumann BC, i.e. solutions of

$$-\Delta u = \lambda u, \; Du \cdot \nu = 0$$

then $\lambda = 0$ is the smallest eigenvalue with a 1D eigenspace, the space of constant functions.

10.8.2 Fredholm Alternative Review

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ Du \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{array}{l} H^1 \overset{I_1}{\hookrightarrow} L^2 \overset{I_0}{\hookrightarrow} H^{-1} \\ I_2 = I_0 I_1 \end{array}$$

$L = -\Delta u + u$, $-\Delta u + u + \lambda u = f$

$$\begin{aligned} \mathcal{L}(u,v) &= \int_\Omega (Du \cdot Dv + vu)\ dx \\ L^* : \langle L^* u, v\rangle &= \mathcal{L}^*(u,v) = \mathcal{L}(v,u) = \langle Lu, v\rangle \end{aligned}$$

Here $L = L^*$, $(Lu = (D_i a_{ij} D_j u)$ L^* has coefficients $a_{ij})$

$$\text{Operator equation:} \quad u + L^{-1}(\lambda I_2 u) = L^{-1}F \text{ in } H^1$$

As for Dirichlet conditions, it's convenient to write the equation for L^2:

$$\begin{aligned} & I_1 u + I_1 L^{-1}(\lambda I_2) u = I_1 L^{-1} F \qquad (\tilde{u} = I_1 u,\ \tilde{F} = I_1 L^{-1} F) \\ \iff \quad & \tilde{u} + \underbrace{(I_1 L^{-1} \lambda I_0)}_{\text{cmpct. since } I_1 \text{ is cmpct.}} \tilde{u} = \tilde{F} \end{aligned}$$

This last equation is the same equation as for the Dirichlet problem, but L^{-1} has Neumann conditions.

Application of Fredholm Alternative:

i.) Solution for all right hand sides.

ii.) Non-trivial solutions of the homogeneous equations.

iii.) If ii.) holds, solutions exists if $\tilde{F}$ is perpendicular to the kernel of the adjoint.

What happens in ii.)? Check: if $\tilde{u}$ is a solves $\tilde{u} + (I_1 L^{-1} \lambda I_0)\tilde{u} = 0$ then $u \in L^{-1}(I_0 \tilde{u}) \in H^1$ solves $u + L^{-1}(\lambda I_2) u = 0$ in $H^1 \iff -\Delta u + u + \lambda u = 0$.

Now consider T to be our compact operator, defined by $T = -I_1 L^{-1} \lambda I_0$.

Solvability of the original problem

(1) Check that $T^* = -\lambda I_1 (L^*)^{-1} I_0 = -\lambda I_1 L^{-1} I_0$

(2) $\tilde{F}$ has to be orthogonal to $N((I-T)^*) = N(I-T)$ $\tilde{u}$ non-trivial solution of $\tilde{u}+(I_1L^{-1}\lambda I_0)\tilde{u} = 0 \Longrightarrow u = L^{-1}I_0\tilde{u}$ solves $u+(L^{-1}\lambda I_2)u = 0 \iff u =$ constant (in our special case $\lambda = -1$) $\iff \tilde{u} =$ constant.

(3) Solvability: then if $\tilde{F} = I_1L^{-1}F \perp \{\text{constants}\}$ (this again is for the special case of $\lambda = -1$).

$$\begin{aligned} &\iff \quad (\tilde{F}, v)_{L^2} = 0 \qquad \forall v \in \{\text{constant}\} \\ &\iff \quad \cdots \iff \langle F, v\rangle = 0 \qquad \forall v \in \{\text{constant}\} \end{aligned}$$

If $F(v) = \int_\Omega fv\ dx$, if we solve $-\Delta u = f$ in L^2, then

$$\begin{aligned} &\int_\Omega fv\ dx = 0 \qquad \forall v \in \{\text{constant} \\ \Longrightarrow \quad &\int_\Omega f\ dx = 0 \end{aligned}$$

So we conclude that

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ Du \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

has a solutions

$$\iff \int_\Omega f\ dx = 0$$

Books: D. Henry (Springer lecture notes)
Pazy (Springer)

10.9 Semigroup Theory

Idea: PDE $\leftrightarrow$ ODE in a BS $\dot{y} = ay$ which we know to have a solution of $y(t) = y_0e^{at}$. Now consider $\dot{y} = Ay$, where A is a $n \times n$ matrix which is also symmetric

$$y(t) = e^{At}y_0, \qquad e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = Qe^{t\Lambda}Q^T$$

if $A = Q\Lambda Q^T$, Λ diagonal

Goal: Solve $u_t = Au$ $(= \Delta u)$ in a similar way by saying

$$u(t) = S(t)u_0 \qquad (S(t) = e^{At})$$

where $S(t) : X \to X$ is a bounded, linear operator over a suitable Banach space, with

- $S(0)u = u$, $\forall u \in X$
- $S(t+s)u = S(t)S(s)u$
- $t \mapsto S(t)u$ continuous $\forall u \in X$, $t \in [0, \infty)$

Definition 10.39. *i.) A is a family of bounded and linear maps* $\{S(t)\}_{t\geq 0}$ *with* $S(t) : X \to X$, X *a BS, is called a semigroup (meaning that* S *has a group structure but no inverses)*

(1) $S(0)u = u$, $\forall u \in X$

(2) $S(t+s)u = S(t)S(s)u = S(s)S(t)u$, $\forall s, t > 0$, $u \in X$

(3) $t \mapsto S(t)u$ *continuous as a mapping* $[0, \infty) \to X$, $\forall u \in X$

ii.) $\{S(t)\}_{t\geq 0}$ *is called a contraction semigroup if* $||S(t)||_X \leq 1$, $\forall t \in [0, \infty)$

Remark: It becomes clear that S^{-1} *may not exist if one consider the reverse heat equation. Solutions with smooth initial data will become very bad in finite time.*

Definition 10.40. $\{S(t)\}_{t\geq 0}$ *semigroup, then define*

$$\mathcal{D}(A) = \left\{ u \in X : \lim_{t\searrow 0^+} \frac{S(t)u - u}{t} \text{ exists in } X \right\},$$

where

$$Au = \lim_{t\searrow 0^+} \frac{S(t)u - u}{t} \qquad u \in \mathcal{D}(A).$$

$A : \mathcal{D}(A) \to X$ *is called the infinitesimal generator of* $\{S(t)\}_{t\geq 0}$, $\mathcal{D}$ *is called its domain.*

General assumption: $\{S(t)\}_{t\geq 0}$ is a contraction semigroup

$$e^{tA} = Qe^{t\Lambda}Q^T, \qquad \Lambda \text{ diagonal}$$

$$e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix} \qquad \text{with all } \lambda_i \leq 0,$$

where we recall that

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad \frac{d}{dt} e^{tA} = A e^{tA}$$

Theorem 10.41. *Let $u \in \mathcal{D}(A)$*

i.) $S(t)u \in \mathcal{D}(A)$, $t \geq 0$

ii.) $A \cdot S(t)u = S(t)Au$

iii.) $t \mapsto S(t)u$ *is differentiable for* $t > 0$

iv.) $\frac{d}{dt} S(t)u = A \cdot S(t)u = S(t)Au$ *where the last equality is due to ii.).*

Proof:

$$\begin{aligned} v \in \mathcal{D}(A) \quad &\iff \quad \lim_{s \searrow 0^+} \tfrac{S(s)u-u}{s} \text{ exists} \\ S(t)u \in \mathcal{D}(A) \quad &\iff \quad \lim_{s \searrow 0^+} \tfrac{S(s)S(t)u - S(t)u}{s} \text{ exists} \\ = \quad & \lim_{s \searrow 0^+} \frac{S(t)S(s)u - S(t)u}{s} \quad \text{(by (2) in the def. of semigroup)} \\ = \quad & S(t) \cdot \lim_{s \searrow 0^+} \frac{S(s)u - u}{s} \\ = \quad & S(t)Au \quad \text{(by def. of } A\text{)} \end{aligned}$$

$\Longrightarrow$ the limit exists, i.e. $S(t) \in \mathcal{D}(A)$, and thus

$$\lim_{s \searrow 0^+} \frac{S(t)S(s)u - S(t)u}{s} = A \cdot S(t)u = S(t)Au$$

$\Longrightarrow$ i.) and ii.) are proved.

iii.) indicates that backward and forward difference quotients exists. So take $u \in \mathcal{D}(A)$, $h > 0$, $t > 0$

$$\begin{aligned}
& \lim_{h \searrow 0^+} \left[\frac{S(t)u - S(t-h)u}{h} - S(t)Au \right] \\
= & \lim_{h \searrow 0^+} \left[\frac{S(t-h)S(h)u - S(t-h)u}{h} - S(t)Au \right] \\
= & \lim_{h \searrow 0^+} \left[S(t-h)\frac{S(h)u - u}{h} - S(t)Au \right] \\
= & \lim_{h \searrow 0^+} \Bigg[\underbrace{\underbrace{\underbrace{S(t-h)}_{||S(t-h)||_X \leq 1} \left(\frac{S(h)u-u}{h} - Au \right)}_{||\cdot||_X \leq \left|\left| \frac{S(h)u-u}{u} - Au \right|\right|_X}}_{\to 0 \text{ since } u \in \mathcal{D}(A)} + \underbrace{\underbrace{(S(t-h)Au - S(t)Au)}_{=(S(t-h)-S(t))Au}}_{\to 0 \text{ by contin. of } t \mapsto S(t)u} \Bigg]
\end{aligned}$$

$\Longrightarrow$ limit exists and

$$\lim_{h \searrow 0^+} \frac{S(t)u - S(t-h)u}{h} = S(t)Au$$

Analogous argument for forwards quotient; the limit is the same and

$$\begin{aligned}
\frac{d}{dt}S(t)u &= S(t)Au \\
&= A \cdot S(t)u \qquad \blacksquare
\end{aligned}$$

<u>Remark:</u> $S(t)$ being continuous $\Longrightarrow$ $\frac{d}{dt}S(t)$ is continuous. So, $S(t)$ is continuously differentiable.

Theorem 10.42 (Properties of infinitesimal generators). *i.) For each $u \in X$,*

$$\lim_{t \searrow 0^+} \frac{1}{t} \int_0^t G(s)u \, ds = u.$$

ii.) $\mathcal{D}(A)$ is dense in X

iii.) A is a closed operator, i.e. if $u_k \in \mathcal{D}(A)$ with $u_k \to u$ and $Au_k \to v$, then $u \in \mathcal{D}(A)$ and $v = Au$

<u>Proof:</u>

i.): Take $u \in X$. By the continuity of $G(\cdot)u$ we have $\lim_{t\searrow 0^+} ||G(t)u - u||_X = \lim_{t\searrow 0^+} ||G(t)u - G(0)u||_X = 0$. Conclusion i.) follows since $u = \dfrac{1}{t}\displaystyle\int_0^t u\,ds$.

ii.): Take $u \in X$ and define

$$u^t = \int_0^t S(s)u\ ds,$$

then

$$\frac{u^t}{t} = \frac{1}{t}\int_0^t S(s)u\ ds \to S(0)u = u \text{ as } t \searrow 0^+.$$

ii.) follows if $u^t \in \mathcal{D}(A)$, i.e.

$$\begin{aligned}
\frac{S(r)u^t - u^t}{r} &= \frac{1}{r}\left\{\int_0^t S(r)S(s)u\ ds - \int_0^t S(s)u\ ds\right\} \\
\text{(semigroup prop.)} \quad &= \frac{1}{r}\int_0^t S(r+s)u\ ds - \frac{1}{r}\int_0^t S(s)u\ ds \\
&= \frac{1}{r}\int_r^{t+r} S(s)u\ ds - \frac{1}{r}\int_0^t S(s)u\ ds \\
&= \frac{1}{r}\int_t^{t+r} S(s)u\ ds \pm \frac{1}{r}\int_r^t S(s)u\ ds \\
&\quad -\frac{1}{r}\int_0^r S(s)u\ ds \mp \frac{1}{r}\int_r^t S(s)u\ ds \\
&= \frac{1}{r}\int_t^{t+r} S(s)u\ ds - \frac{1}{r}\int_0^r S(s)u\ ds
\end{aligned}$$

taking the limit of both sides

$$\lim_{r\searrow 0^+} \frac{S(r)u^t - u^t}{r} = S(t)u - u \qquad \text{by continuity of } S(t)$$

$\Longrightarrow u^t \in \mathcal{D}(A)$, $Au^t = S(t)u - u$.

iii.): (proof of A being closed).
Take $u_k \in \mathcal{D}(A)$ with $u_k \to u$ and $Au_k \to v$ in X. We need to show that $u \in \mathcal{D}(A)$ and $v = Au$. So consider,

$$\begin{aligned}
S(t)u_k - u_k &= \int_0^t \frac{d}{dt'}S(t')u_k\ dt' \\
&= \int_0^t S(s)\underbrace{Au_k}_{\to v}\ ds.
\end{aligned}$$

Taking $k \to \infty$, we have

$$\begin{aligned} S(t)u - u &= \int_0^t S(s)v \; ds \\ \implies \lim_{t \searrow 0^+} \frac{S(t)u - u}{t} &= \lim_{t \searrow 0^+} \frac{1}{t} \int_0^t S(s)v \; ds = v \end{aligned}$$

$\implies Au = v$ with $u \in \mathcal{D}(A)$. ■

Definition 10.43. *i.) We say $\lambda \in \rho(A)$, the* <u>*resolvent set*</u> *of A, if the operator $\lambda I - A : \mathcal{D}(A) \to X$ is one-to-one and onto.*

ii.) If $\lambda \in \rho(a)$, then the <u>*resolvent operator*</u> *$R_\lambda : X \to X$ is defined by $R_\lambda = (\lambda I - A)^{-1}$*

<u>Remark:</u> The closed graph theorem says that if $A : X \to Y$ is a closed and linear operator, then A is bounded. It turns out that, R_λ is closed, thus R_λ is therefore bounded.

Theorem 10.44 (Properties of the resolvent). *i.) If $\lambda, \mu \in \rho(A)$, then*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \qquad \textit{(resolvent identity)}$$

which immediately implies

$$R_\lambda R_\mu = R_\mu R_\lambda$$

ii.) If $\lambda > 0$, then $\lambda \in \rho(A)$ and

$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \; dt$$

and hence

$$||R_\lambda||_X = \frac{1}{\lambda} \qquad \textit{(from estimating the integral)}$$

<u>Proof:</u>

i.):

$$\begin{aligned} R_\lambda &= R_\lambda(\mu I - A) R_\mu \\ &= R_\lambda((\mu - \lambda)I + (\lambda I - A))R_\mu \\ &= (\mu - \lambda) R_\lambda R_\mu + \underbrace{R_\lambda(\lambda I - A)}_{I} R_\mu \end{aligned}$$

$\implies R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$
$\implies R_\lambda R_\mu = R_\mu R_\lambda$

ii.): $S(t)$ contraction semigroup, $||S(t)||_X \leq 1$. Define

$$\tilde{R}_\lambda = \int_0^\infty e^{-\lambda t} S(t)\ dt$$

which clearly exists. Now fix $h > 0$, $u \in X$ and compute

$$\begin{aligned}
\frac{S(h)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{h} &= \frac{1}{h}\int_0^\infty [e^{-\lambda t} S(t+h)u - S(t)u]\ dt \\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)} S(t)u\ dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t)u\ dt \\
&= -\frac{1}{h}\int_0^h e^{-\lambda(t-h)} S(t)u\ dt + \frac{1}{h}\int_0^\infty (e^{-\lambda(t-h)} - e^{-\lambda t}) S(t)u\ dt \\
&= \underbrace{-e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda t} S(t)u\ dt}_{\to -u} + \underbrace{\frac{e^{\lambda h}-1}{h}}_{\to\lambda} \underbrace{\int_0^\infty e^{-\lambda t} S(t)u\ dt}_{\tilde{R}_\lambda u}.
\end{aligned}$$

Collecting terms we have

$$\lim_{h \searrow 0^+} \frac{S(h)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{h} = -u + \lambda \tilde{R}_\lambda u$$

$\Longrightarrow \tilde{R}_\lambda u \in \mathcal{D}(A)$, and

$$\begin{aligned}
& A\tilde{R}_\lambda u = -u + \lambda \tilde{R}_\lambda u \\
\Longleftrightarrow\quad & u = (\lambda I - A)\tilde{R}_\lambda u \qquad \forall x
\end{aligned} \tag{10.27}$$

If $u \in \mathcal{D}(A)$, then

$$\begin{aligned}
A\tilde{R}_\lambda u &= A\int_0^\infty e^{-\lambda t} S(t)\ dt \\
\text{(exer. in Evans)} \qquad &= \int_0^\infty e^{-\lambda t} A \cdot S(t)u\ dt \\
&= \int_0^\infty e^{-\lambda t} S(t) \underbrace{Au}\ dt \\
&= \tilde{R}_\lambda(Au)
\end{aligned}$$

$$\begin{aligned}
\Longleftrightarrow \tilde{R}_\lambda(\lambda I - A)u &= \lambda\tilde{R}_\lambda u - \tilde{R}_\lambda Au = \lambda\tilde{R}_\lambda u - A\tilde{R}_\lambda u \\
&= (\lambda I - A)\tilde{R}_\lambda u \\
&= u \quad \text{(by (10.27))}
\end{aligned}$$

Now we need to prove the following.
Claim: $\lambda I - A$ is one-to-one and onto

- one-to-one: Assume $(\lambda I - A)u = (\lambda I - A)v$, $u \neq v$

$$\begin{aligned} & (\lambda I - A)(u-v) = 0 \\ \Longrightarrow \quad & \tilde{R}_\lambda(\lambda I - A)(u-v) = 0, \end{aligned}$$

  $u, v \in \mathcal{D}(A) \Longrightarrow u - v \in \mathcal{D}(A)$
  $\Longrightarrow \tilde{R}_\lambda(\lambda I - A)(u-v) = u - v \neq 0$ a contradiction.
- onto: $(\lambda I - A)\tilde{R}_\lambda u = u$. $\tilde{R}_\lambda u = w$ solves $(\lambda I - A)w = u \iff (\lambda I - A) : \mathcal{D}(A) \to X$ is onto.

So we have $(\lambda I - A)$ is one-to-one and onto if $\lambda \in \rho(A)$ and $\lambda > 0$. We also have $\tilde{R}_\lambda = (\lambda I - A)^{-1} = R_\lambda$. ■

Theorem 10.45 (Hille, Yoshida). *If A closed and densely defined, then A is infinitesimal generator of a contraction semigroup $\iff$ $(0,\infty) \subset \rho(A)$ and $||R_\lambda||_X \leq \frac{1}{\lambda}$, $\lambda > 0$.*

Proof:

$\Longrightarrow$) This is immediately obvious from the last part of Theorem 10.6.

$\Longleftarrow$) We must build a contraction semigroup with A as its generator. For this fix $\lambda > 0$ and devine

$$A_\lambda := -\lambda I + \lambda^2 R_\lambda = \lambda A R_\lambda. \tag{10.28}$$

The operator A_λ is a kind of regularized approximation to A.

Step 1. Claim:

$$A_\lambda u \to Au \quad \text{as } \lambda \to \infty \ (u \in D(A)). \tag{10.29}$$

Indeed, since $\lambda R_\lambda u - u = AR_\lambda u = R_\lambda Au$,

$$||\lambda R_\lambda u - u|| \leq ||R_\lambda|| \cdot ||Au|| \leq \frac{1}{\lambda}||Au|| \to 0.$$

Thus $\lambda R_\lambda u \to u$ as $\lambda \to \infty$ if $u \in D(A)$. But since $||\lambda R_\lambda|| \leq 1$ and $D(A)$ is dense, we deduce then as well

$$\lambda R_\lambda u \to u \quad \text{as } \lambda \to \infty, \ \forall u \in X. \tag{10.30}$$

Now if $u \in D(A)$, then

$$A_\lambda u = \lambda A R_\lambda u = \lambda R_\lambda A u.$$

Step 2. Next, define

$$S_\lambda(t) := e^{tA_\lambda} = e^{-\lambda t} e^{\lambda^2 t R_\lambda} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^k}{k!} R_\lambda^k.$$

Observe that since $||R_\lambda|| \leq \lambda^{-1}$,

$$||S_\lambda(t)|| \leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\lambda^{2k} t^k}{k!} ||R_\lambda||^k \leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} = 1.$$

Consequently $\{S_\lambda(t)\}_{t\geq 0}$ is a contrction semigroup, and it is easy to check its generator is A_λ, with $D(A_\lambda) = X$.

Step 3. Let $\lambda, \mu > 0$. Since $R_\lambda R_\mu = R_\mu R_\lambda$, we see $A_\lambda A_\mu = A_\mu A_\lambda$, and so

$$A_\mu S_\lambda(t) = S_\lambda(t) A_\mu \quad \text{for each } t > 0.$$

Tus if $u \in D(A)$, we can compute

$$\begin{aligned} S_\lambda(t)u - S_\mu(t)u &= \int_0^t \frac{d}{ds}[S_\mu(t-s)S_\lambda(s)u]\, ds \\ &= \int_0^t S_\mu(t-s)S_\lambda(s)(A_\lambda u - A_\mu u)\, ds, \end{aligned}$$

because $\frac{d}{dt}S_\lambda(t)u = A_\lambda S_\lambda(t)u = S_\lambda(t)A_\lambda u$. Consequently, $||S_\lambda(t)u - S_\mu(t)u|| \leq t||A_\lambda u - A_\mu u|| \to 0$ as $\lambda, \mu \to \infty$. Hence

$$S(t)u := \lim_{\lambda\to\infty} S_\lambda(t)u \quad \text{exists for each } t \geq 0, u \in D(A). \tag{10.31}$$

As $||S_\lambda(t)|| \leq 1$, the limit (10.31) in fact exists for all $u \in X$, uniformly for t in compact subsets of $[0, \infty)$. It is now straightforward to verify $\{S(t)\}_{t\geq 0}$ is a contraction semigroup on X.

Step 4. It remains to show A is the generator of $\{S(t)\}_{t\geq 0}$. Write B to denote this generator. Now

$$S_\lambda(t)u - u = \int_0^t S_\lambda(s)A_\lambda u\, ds. \tag{10.32}$$

In addition

$$||S_\lambda(s)A_\lambda u - S(s)Au|| \leq ||S_\lambda(s)|| \cdot ||A_\lambda u - Au|| + ||(S_\lambda(s) - S(s))Au|| \to 0$$

as $\lambda \to \infty$, if $u \in D(A)$. Passing therefore to limits in (10.32), we deduce

$$S(t)u - u = \int_0^t S(s)Au\,ds$$

if $u \in D(A)$. Thus $D(A) \subseteq D(B)$ and

$$Bu = \lim_{t\to 0^+} \frac{S(t)u-u}{t} = Au \quad (u \in D(A)).$$

Now if $\lambda > 0$, $\lambda \in \rho(A) \cap \rho(B)$. Also $(\lambda I - B)(D(A)) = (\lambda I - A)(D(A)) = X$, according to the assumptions of the theorem. Hence $(\lambda I - B)\big|_{D(A)}$ is one-to-one and onto; whence $D(A) = D(B)$. Therefore $A = B$, and so A is indeed the generator of $\{S(t)\}_{t\geq 0}$. ■

Recall: In a contraction semigroup, we have $||S(t)||_X \leq 1$
Take A to be a $n \times n$ matrix, $A = QAQ^T$

$$||S(t)||_X = ||e^{At}||_X = ||Qe^{\Lambda t}Q^T||_X \leq 1$$

Since

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix},$$

we need all $\lambda_i \leq 0$. In general, we can only expect that

$$||S(t)||_X \leq Ce^{\lambda t}, \qquad \lambda > 0.$$

Remark: $\{S(t)\}_{t\geq 0}$ is called an ω-contraction semigroup if

$$||S(t)||_X \leq e^{\omega t}, \qquad t \geq 0$$

variant of the Hille-Yoshida Theorem:
If A closed and densely defined, then A is a generator of an ω-contraction semigroup $\iff$ $(\omega, \infty) \subset \rho(A)$, $||R_\lambda||_X \leq \frac{1}{\lambda - \omega}$ and $\lambda > \omega$

10.10 Exercises

9.1: [Qualifying exam 08/00] Let $\Omega \subset \mathbb{R}^n$ be open and suppose that the operator L, formally defined by

$$Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j}$$

has smooth coefficients $a^{ij} : \overline{Omega} \to \mathbb{R}$. Suppose that L is uniformly elliptic, i.e., for some $c > 0$,

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq c|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

uniformly in Ω. Show that if $\lambda > 0$ is sufficiently large, then whenever $Lu = 0$, we have

$$L(|Du|^2 + \lambda|u|^2) \leq 0.$$

9.2: [Qualifying exam 01/02] Let B be the unit ball in $\mathbb{R}^2$. Prove that there is a constant C such that

$$||u||_{L^2(B)} \leq C(||Du||_{L^2(B)} + ||u||_{L^2(B)}) \quad \forall u \in C^1(\overline{B}).$$

9.3: [Qualifying exam 01/02] Let Ω be the unit ball in $\mathbb{R}^2$.

a) Find the weak formulation of the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u \pm \partial_n u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

b) Decide for which choice of the sign this problem has a unique weak solution $u \in H^1(\Omega)$ for all $f \in L^2(\Omega)$. For this purpose you may use without proof the estimate

$$||u||_{L^2(\Omega)} \leq \gamma\left(||Du||_{L^2(\Omega)} + ||u||_{L^2(\partial\Omega)}\right) \quad \forall u \in H^1(\Omega).$$

c) For the sign that does not allow a unique weak solution, prove non-uniqueness by finding two smooth solutions of the boundary value problem with $f = 0$.

9.4: [John, Chapter 6, Problem 1] Let $n = 3$ and $\Omega = B(0, \pi)$ be the ball centered at zero with radius π. Let $f \in L^2(\Omega)$. Show that a solution u of the boundary value problem

$$\begin{aligned} \Delta u + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

can only exists if

$$\int_\Omega f(x)\frac{\sin|x|}{|x|}\, dx = 0.$$

9.5: Let $f \in L^2(\mathbb{R}^n)$ and let $u \in W^{1,2}(\mathbb{R}^n)$ be a weak solution of

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^n,$$

that is,

$$\int_{\mathbb{R}^n} (Du \cdot D\phi + u\phi)\, dx = \int_{\mathbb{R}^n} f\phi\, dx \qquad \forall \phi \in W^{1,2}(\mathbb{R}^n).$$

Show that $u \in W^{2,2}(\mathbb{R}^n)$.

Hint: It suffices to prove that $D_k u \in W^{1,2}(\mathbb{R}^n)$. In order to show this, check first that $u_h = u(\cdot + he_k)$ is a weak solution of the equation

$$-\Delta u_h + u_h = f_h \quad \text{in } \mathbb{R}^n,$$

with $f_h = f(\cdot + he_k)$. Then derive an equation for $(u_h - u)/h$ and use $u_h - u$ as a test function.

9.6: [Qualifying exam 08/00] Let Ω be the unit ball in $\mathbb{R}^2$. Suppose that $u, v : \overline{\Omega} \to \mathbb{R}$ are smooth and satisfy

$$\begin{cases} \Delta u = \Delta v & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

a) Prove that there is a constant C independent of u and v such that

$$||u||_{H^1(\Omega)} \le C||v||_{H^1(\Omega)}.$$

b) Show there is *no* constant C independent of u and v such that

$$||u||_{L^2(\Omega)} \le C||v||_{L^2(\Omega)}.$$

Hint: Consider sequences satisfying $u_n - v_n = 1$.

9.7: [Qualifying exam 08/02] Let Ω be a bounded and open set in $\mathbb{R}^n$ with smooth boundary Γ. Let $q(x)$, $f(x) \in L^2(\Omega)$ and suppose that

$$\int_\Omega q(x)\, dx = 0.$$

Let $u(x,t)$ be the solution of the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = q & x \in \Omega, t > 0, \\ u(x,0) = f(x) & x \in \Omega, \\ Du \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

Show that $||u(\cdot,t) - v||_{W^{1,2}(\Omega)} \to 0$ exponentially as $t \to \infty$, where v is the solution of the boundary value problem

$$\begin{cases} -\Delta v = q & x \in \Omega, \\ Dv \cdot \nu = 0 & \text{on}\Gamma, \end{cases}$$

that satisfies

$$\int_\Omega v(x)\, dx = \int_\Omega f(x)\, dx.$$

9.8: [Qualifying exam 01/97] Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, and let $u(x,t)$ be the solution of the initial-boundary value problem

$$\begin{cases} u_{tt} - c^2 \Delta u = q(x,t) & \text{for} \quad x \in \Omega,\ t > 0, \\ u = 0 & \text{for} \quad x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_t(x,0) = 0 & \text{for} \quad x \in \Omega. \end{cases}$$

Let $\{\phi_n(x)\}_{n=1}^\infty$ be a complete orthonormal set of eigenfunctions for the Laplace operator Δ in Ω with Dirichlet boundary conditions, so $-\Delta\phi_n = \lambda_n \phi_n$. Set $\omega_n = c\sqrt{\lambda_n}$. Suppose

$$q(x,t) = \sum_{n=1}^\infty q_n(t)\phi_n(x),$$

where each $q_n(t)$ is continuous, $\sum q_n^2(t) < \infty$ and $|q_n(t)| \le Ce^{-nt}$. Show that there are sequences $\{a_n\}$ and $\{b_n\}$ such that $\sum(a_n^2 + b_n^2) < \infty$ and if

$$v(x,t) = \sum_{n=1}^\infty [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)]\phi_n(x),$$

then

$$||u(\cdot,t) - v(\cdot,t)||_{L^2(\Omega)} \to 0$$

at an exponential rate as $t \to \infty$.

9.9: [Qualifying exam 08/99] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $a^{ij} \in C(\overline{\Omega})$ for $i,j = 1,\dots,n$, with $\sum_{i,j} a^{ij}(x)\xi_i\xi_j \ge |\xi|^2$ for all $\xi \in \mathbb{R}^n$, $x \in \Omega$. Suppose $u \in H^1(\Omega) \cap C(\overline{\Omega})$ is a weak solution of

$$-\sum_{i,j} \partial_j(a^{ij}\partial_i u) = 0 \quad \text{in } \Omega.$$

Set $w = u^2$. Show that $w \in H^1(\Omega)$, and show that w is a weak subsolution, i.e., prove that

$$B(w,v) := \int_\Omega \sum_{i,j} a^{ij} \partial_j w \partial_i v \, dx \leq 0$$

for all $v \in H_0^1(\Omega)$ such that $v \geq 0$.

9.10: [Qualifying exam 01/00] Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^n$. Using iteration and the maximum principle (and other result with proper justification), prove that there exists some solution to the boundary value problem

$$\begin{cases} -\Delta u = \arctan(u+1) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

9.11: [Qualifying exam 01/97] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, nonnegative and has compact support. Consider the initial value problem

$$\begin{cases} u_t - \Delta u = u^2 & \text{for} \quad x \in \mathbb{R}^n, \ 0 < t < T, \\ u(x,0) = f(x) & \text{for} \quad x \in \mathbb{R}^n. \end{cases}$$

Using Duhamel's principle, reformulate the initial value problem as a fixed point problem for an integral equation. Prove that if $T > 0$ is sufficiently small, the integral equation has a *nonnegative* solution in $C(\mathbb{R}^n \times [0,T])$.

9.12: Let $X = L^2(\mathbb{R})$ and define for $t > 0$ the operator $S(t0 : X \to X$ by

$$(S(t)u)(x) = u(x+t).$$

Show that $S(t)$ is a semigroup. Find $\mathcal{D}(A)$ and A, the infinitesimal generator of the semigroup.

Hint: You may use without proof that for $f \in L^p(\mathbb{R})$

$$||f(\cdot + h) - f(\cdot)||_{L^p(\mathbb{R})} \to 0 \quad \text{as } |h| \to 0$$

for $p \in [1, \infty)$.

Chapter 11

Distributions and Fourier Transforms

[Renardy and Rogers *PDE*, Springer]

11.1 Distributions

11.1.1 Distributions on C^∞ Functions

Definition 11.1. *Suppose Ω non-empty, open set in $\mathbb{R}^n$. f is a* <u>*test function*</u> *if $f \in C_c^\infty(\Omega)$, i.e., f is infinitely many time differentiable, supp$(f) \subset K \Subset \Omega$, K compact. The set of all test functions f is denoted by $\mathcal{D}(\Omega)$.*

Definition 11.2. *$\{\phi_n\}_{n\in\mathbb{N}}$, $\phi \in \mathcal{D}(\Omega)$. Then $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ if*

i.) There exists a compact set $K \Subset \Omega$, such that supp$(\phi_n) \subseteq K$ $\forall n$, supp$(\phi) \subset K$.

ii.) ϕ_n and arbitrary derivatives of ϕ_n converge uniformly to ϕ and the corresponding derivatives of ϕ.

Definition 11.3. *A* <u>*distribution*</u> *or* <u>*generalized function*</u> *is a linear mapping $\phi \mapsto (f, \phi)$ from $\mathcal{D}(\Omega)$ into $\mathbb{R}$ which is continuous in the following sense: If $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ then $(f, \phi_n) \to (f, \phi)$. The set of all distributions is denoted $\mathcal{D}'(\Omega)$.*

<u>**Example:**</u> If $f \in C(\Omega)$, then f gives a distribution T_f defined by

$$(T_f, \phi) = \int_\Omega f(x)\phi(x)\, dx$$

Not all distributions can be identified with functions. If $\Omega = D(0,1)$, we can define δ_0 by $(\delta_0, \phi) = \phi(0)$.

Lemma 11.4. *$f \in \mathcal{D}'(\Omega)$, $K \Subset \Omega$. Then there exists a $n_K \in \mathbb{N}$ and a constant C_K with the following property:*

$$|(f,\phi)| \leq C_K \sum_{|\alpha| \leq n_K} \max_{x \in K} |D^\alpha \phi(x)|$$

for all test functions ϕ with supp$(\phi) \subseteq K$.

Remarks:

1) The examples above satisfy this with $n = 0$.

2) If n_K can be chosen independent of K, then the smallest n with this property is called the order of the distribution.

3) There are distributions with $n = \infty$

Proof: Suppose the contrary. Then there exists a compact set $K \Subset \Omega$ and a sequence of functions ψ_n with supp$(\psi_n) \subseteq K$ and

$$|(f,\psi_n)| \geq n \sum_{|\alpha| \leq n} \max_{x \in K} |D^\alpha \psi_n(x)| > 0$$

Define $\phi_n = \frac{\psi_n}{(f,\psi_n)}$. Then $\phi_n \to 0$ in $\mathcal{D}(\Omega)$

- supp$(\phi_n) \subseteq K \Subset \Omega$.
- $|\alpha| = k$, $n \geq k$. Then

$$\begin{aligned} |D^\alpha \phi_n(x)| &= \frac{|D^\alpha \psi_n(y)|}{(f,\psi_n)} \leq \frac{D^\alpha \psi_n(y)}{n \sum_{|\beta| \leq n} \max_{x \in K} |D^\alpha \psi_n(x)|} \\ &\leq \frac{1}{n} \end{aligned}$$

$\implies D^\alpha \phi_n \to 0$ uniformly as $n \to \infty$.
However,

$$(f,\phi_n) = \frac{(f,\psi_n)}{(f,\psi_n)} \text{ contradiction.} \qquad \blacksquare$$

Example: We can define a distribution f by

$$\begin{array}{lcl} (f,\phi) & = & -\phi'(0) \\ \| & & \\ (\delta_0)' & & \end{array}$$

$|(f,\phi)| \leq \sup_{x\in K} |\phi'(x)|$, f is a distribution of order one.

Localization: $G \subset \Omega$, G open, then $\mathcal{D}(G) \subseteq \mathcal{D}(\Omega)$. $f \in \mathcal{D}'(\Omega)$ defines a distribution in $\mathcal{D}'(G)$ simply by restriction. $f \in \mathcal{D}'(\Omega)$ vanishes on the open set G if $(f,\phi) = 0$ for all $\phi \in \mathcal{D}(G)$. With this in mind, we say that $f = g$ on G if $f - g$ vanishes on G. If $f \in \mathcal{D}'(\Omega)$, then one can show that there exists a largest open set $G \subset \Omega$, such that f vanishes on G (this can be the empty set). The complement of this set in Ω is called the support of f.

Example: $\Omega = B(0,1)$, $f = \delta_0$, $\text{supp}(\delta_0) = \{0\}$.

Definition 11.5. *A sequence $f_n \in \mathcal{D}'(\Omega)$ converges to $f \in \mathcal{D}'(\Omega)$ if $(f_n,\phi) \to (f,\phi)$ for all $\phi \in \mathcal{D}(\Omega)$.*

Example: $\Omega = \mathbb{R}$

$$f_n(x) = \begin{cases} n & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} (f_n,\phi) &= \int_{\mathbb{R}} f_n(x)\ dx \\ &= n\int_0^{\frac{1}{n}} \phi(x)\ dx = \frac{1}{\frac{1}{n}}\int_0^{\frac{1}{n}} \phi(x)\ dx = \phi(0) = (\delta_0,\phi) \end{aligned}$$

$\implies f_n \to \delta_0$ in $\mathcal{D}'(\Omega)$

Analogously: Standard mollifiers, ϕ

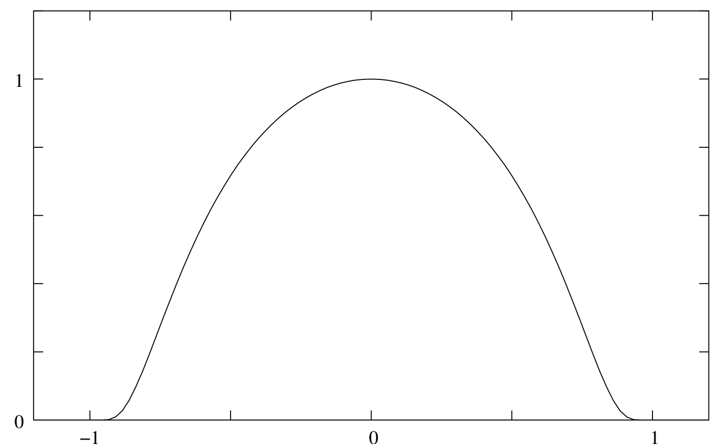


Figure 11.1:

$\mathrm{supp}(\phi) \subseteq [-1, 1]$, $\int_{\mathbb{R}} \phi(x)\ dx = 1$
$\phi_\epsilon = \epsilon^{-1}\phi(\epsilon^{-1}x)$ then $\phi_\epsilon \to \delta_0$ in $\mathcal{D}'(\Omega)$ as $\epsilon \to 0$.

Theorem 11.6. *$f_n \in \mathcal{D}'(\Omega)$ such that (f_n, ϕ) converges for all test function ϕ, then there $\exists f \in \mathcal{D}'(\Omega)$ such that $f_n \to f$.*

Proof: (See Renardy and Rogers) Define f by

$$(f, \phi) = \lim_{n\to\infty} (f_n, \phi).$$

Then f is clearly linear, and it remains to check the required continuity property of f.

Definition 11.7.
- *$\mathcal{S}(\mathbb{R}^n)$ is the space of all $C^\infty(\mathbb{R}^n)$ functions such that*

$$|x|^k |D^\alpha \phi(x)|$$

is bounded for every $k \in \mathbb{N}$, for every multi-index α.

- *We say that $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ if the derivatives of ϕ_n of all orders converge uniformly to the derivatives of ϕ, and if the constants $C_{k,\alpha}$ in the estimate*

$$|x|^k |D^\alpha \phi_n(x)| \le C_{k,\alpha}$$

are independent of n.

- *$\mathcal{S}(\mathbb{R}^n)$ = space of rapidly decreasing function or the Schwarz space.*

Example: $\phi(x) = e^{-|x|^2}$

11.1.2 Tempered Distributions

Definition 11.8. • *A tempered distribution is a linear mapping $\phi \mapsto (f, \phi)$ from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{R}$ which is continuous in the sense that $(f, \phi_n) \to (f, \phi)$ if $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$.*

- *The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$*

11.1.3 Distributional Derivatives

Definition 11.9. *Let $f \in \mathcal{D}'(\Omega)$, then the derivative of f with respect to x_i is defined by*

$$\left(\frac{\partial f}{\partial x_i}, \phi\right) = -\left(f, \frac{\partial \phi}{\partial x_i}\right) \tag{11.1}$$

Remarks:

1) If $f \in C(\Omega)$, then the distributional derivative is the classical derivative, and (11.1) is just the formula for integration by parts.

2) Analogously, definition for higher order derivatives: $\alpha \in \mathbb{R}^n$ is a multi-index

$$(D^\alpha f, \phi) = (-1)^{|\alpha|}(f, D^\alpha \phi)$$

3) Taking derivatives of distributions is continuous in the following sense

$$\begin{aligned} &\text{if} \quad f_n \to f \text{ in } \mathcal{D}'(\Omega), \text{ then} \\ &\qquad \frac{\partial f_n}{\partial x_i} \to \frac{\partial f}{\partial x_i} \text{ in } \mathcal{D}'(\Omega) \end{aligned}$$

4) We can define a "translated" distribution $f(x + he_i)$ by

$$(f(x + he_i), \phi) = (f, \phi(x - he_i)).$$

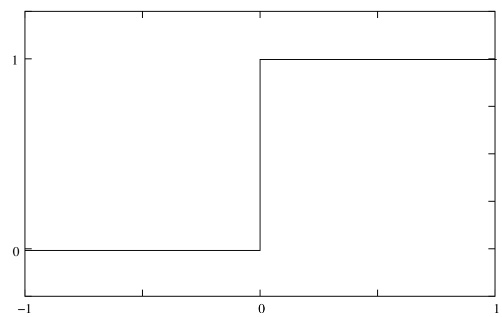
Then, we see that

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}$$

is a limit of difference quotients.

Example:

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R})$$



this is not weakly diffentiable

Figure 11.2:

$\frac{dH}{dx}$ in the sense of distributions:

$$\begin{aligned} \left(\frac{dH}{dx}, \phi\right) &= -\left(H, \frac{d\phi}{dx}\right) = -\int_{\mathbb{R}} H(x)\frac{d\phi}{dx}\, dx \\ &= -\int_0^\infty \frac{d\phi}{dx}\, dx = -\phi \Big|_0^\infty = \phi(0) = (\delta_0, \phi) \end{aligned}$$

Recall:

- $\mathcal{D}(\Omega)$ = test functions, $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ means that $\text{supp}(\phi_n) \subset K \Subset \Omega$ and $D^\alpha \phi_n \to D^\alpha \phi$ uniformly.

- $\mathcal{D}'(\Omega)$ = distributions on Ω = linear functionals of $\mathcal{D}(\Omega)$, $(f, \phi_n) \to (f, \phi)$ if $\phi_n \to \phi$

- $\mathcal{S}(\mathbb{R}^n)$ = rapidly decreasing test functions, $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ means that $D^\alpha \phi_n \to D^\alpha \phi$ uniformly $\forall \alpha$.

$$|x|^k |D^\alpha \phi_n| \le C_{k,\alpha} \quad \underline{\text{independent}} \text{ of } n$$

- $\mathcal{S}'(\mathbb{R}^n)$ = tempered distributions = linear functionals of $\mathcal{S}(\mathbb{R}^n)$

- $f \in \mathcal{D}'(\Omega)$, $(D^\alpha f, \phi) = (-1)^{|\alpha|}(f, D^\alpha \phi)$

Example: $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary, $f \in C^1(\overline{\Omega})$. f defines a distribution in $\mathcal{D}'(\Omega)$

$$\begin{aligned}\left(\frac{\partial f}{\partial x_j}, \phi\right) &= -\left(f, \frac{\partial \phi}{\partial x_j}\right) = -\int_{\mathbb{R}^n} f(x)\frac{\partial \phi}{\partial x_j}(x)\ dx = -\int_{\Omega} f(x)\frac{\partial \phi}{\partial x_j}(x)\ dx \\ &= -\int_{\partial\Omega} f(x)\phi(x)\nu_j\ dS + \int_{\Omega}\frac{\partial f}{\partial x_j}(x)\phi(x)\ dx\end{aligned}$$

distributional derivatives have two terms: a volume term and a boundary term.

Proposition 11.10. *Ω open, bounded and connected. $u \in \mathcal{D}'(\Omega)$ such that $\nabla u = 0$ (in the sense of distributions), then $u \equiv$ constant.*

Proof: We do this only for $n = 1$; only a suitable induction is needed to prove the general case.

$\Omega = I = (a, b)$; by assumption $u' = 0$ or $0 = (u', \phi) = -(u, \phi')$
$\Longrightarrow (u, \psi) = 0$ whenever $\psi = \phi'$, $\phi \in \mathcal{D}(I)$;
But $\psi = \phi' \iff \int_I \psi(s)\ ds = 0$ and $\psi \in C_c^\infty(I)$.

Choose any $\phi_0 \in \mathcal{D}(I)$ with $\int_a^b \phi_0(s)\ ds = 1$

$$\phi \in \mathcal{D}(I), \quad \phi(x) = \underbrace{[\phi(x) - \overline{\phi}\cdot\phi_0(x)]}_{\substack{\int[\phi(x)-\overline{\phi}\cdot\phi_0(x)]\ dx = \\ \overline{\phi}-\overline{\phi}\underbrace{\int_a^b \phi_0(x)\ dx}_{=1} = 0}} + \overline{\phi}\cdot\phi_0(x),$$

where $\overline{\phi} = \int_a^b \phi(s)\ ds$.

$$\begin{aligned}&\Longrightarrow \quad \phi(x) = \psi'(x) + \overline{\phi}\cdot\phi_0(x) \\ &\Longrightarrow \quad (u, \phi) = \underbrace{(u, \psi')}_{=0} + (u, \overline{\phi}\cdot\phi_0(x)) = (u, \phi_0)\int_a^b \phi(x)\ dx \\ &\Longrightarrow \quad u = (u, \phi_0) = \text{ constant.} \qquad \blacksquare\end{aligned}$$

Recall:

- $\mathcal{D}(\Omega) =$ test functions, $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ means that $\text{supp}(\phi_n) \subset K \Subset \Omega$ and $D^\alpha\phi_n \to D^\alpha\phi$ uniformly.

- $\mathcal{D}'(\Omega)$ = distributions on Ω = linear functionals of $\mathcal{D}(\Omega)$, $(f, \phi_n) \to (f, \phi)$ if $\phi_n \to \phi$
- $\mathcal{S}(\mathbb{R}^n)$ = rapidly decreasing test functions, $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ means that $D^\alpha \phi_n \to D^\alpha \phi$ uniformly $\forall \alpha$.

$$|x|^k |D^\alpha \phi_n| \le C_{k,\alpha} \quad \underline{\text{independent}} \text{ of } n$$

- $\mathcal{S}'(\mathbb{R}^n)$ = tempered distributions = linear functionals of $\mathcal{S}(\mathbb{R}^n)$
- $f \in \mathcal{D}'(\Omega)$, $(D^\alpha f, \phi) = (-1)^{|\alpha|}(f, D^\alpha \phi)$

Example: $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary, $f \in C^1(\overline{\Omega})$. f defines a distribution in $\mathcal{D}'(\Omega)$

$$\begin{aligned}
\left(\frac{\partial f}{\partial x_j}, \phi\right) &= -\left(f, \frac{\partial \phi}{\partial x_j}\right) = -\int_{\mathbb{R}^n} f(x)\frac{\partial \phi}{\partial x_j}(x)\, dx = -\int_\Omega f(x)\frac{\partial \phi}{\partial x_j}(x)\, dx \\
&= -\int_{\partial\Omega} f(x)\phi(x)\nu_j \, dS + \int_\Omega \frac{\partial f}{\partial x_j}(x)\phi(x)\, dx
\end{aligned}$$

distributional derivatives have two terms: a volume term and a boundary term.

11.2 Products & Convolutions of Distributions; Fundamental Solutions

Difficulty: No good definition of $f \cdot g$ for $f, g \in \mathcal{D}'(\Omega)$.

For example: $f, g \in L^1(\Omega)$, but $f \cdot g \notin L^1_{\text{loc}}(\Omega)$, then $f \cdot g \notin \mathcal{D}'(\Omega)$.

However, if $f \in \mathcal{D}'(\mathbb{R}^p)$, $g \in \mathcal{D}'(\mathbb{R}^q)$, then we can define $f \cdot g \in \mathcal{D}'(\mathbb{R}^{p+q})$ in the following way:

Definition 11.11. *If $f \in \mathcal{D}'(\mathbb{R}^p)$, $g \in \mathcal{D}'(\mathbb{R}^q)$, then we define $f(x)g(x) \in \mathcal{D}'(\mathbb{R}^{p+q})$ by*

$$(f(x)g(y), \phi(x,y)) = (f(x), \underbrace{(g(y), \phi(x,y))}_{\text{consider } x \text{ as a parameter}})$$

Check:

- $(g(y), \phi(x,y))$ is a test function.

- Continuity: $(f(x)g(y), \phi_n(x,y)) \to (f(x)g(y), \phi(x,y))$.

- Commutative operation: $f(x)g(y) = g(y)f(x)$. To check this, use that test functions of the form $\phi(x,y) = \phi_1(x)\phi_2(y)$ are dense

Example:

$$\begin{aligned}
(\delta_0(x)\delta_0(y), \phi(x,y)) &= (\delta_0(x), (\delta_0(y), \phi(x,y))) \\
&= (\delta_0(x), \phi(x,0)) \\
&= \phi(0,0) = \delta_0(x,y)
\end{aligned}$$

11.2.1 Convolutions of Distributions

Suppose $f, g \in \mathcal{S}(\mathbb{R}^n) \qquad (\mathcal{D}(\mathbb{R}^n))$

$$\begin{aligned}
(f * g, \phi) &= \int_{\mathbb{R}^n} (f*g)(x)\phi(x)\ dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)\phi(x)\ dydx \\
&= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x-y)\phi(x)\ dx\right] g(y)\ dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\phi(x+y)\ dxdy \\
&= (f(x)g(y), \phi(x+y))
\end{aligned}$$

Suggestion: If $f, g \in \mathcal{S}'(\mathbb{R}^n)$, then

$$(f * g, \phi) = (f(x)g(y), \phi(x+y))$$

Difficulty: $\phi(x+y)$ does not have compact support even if ϕ does.

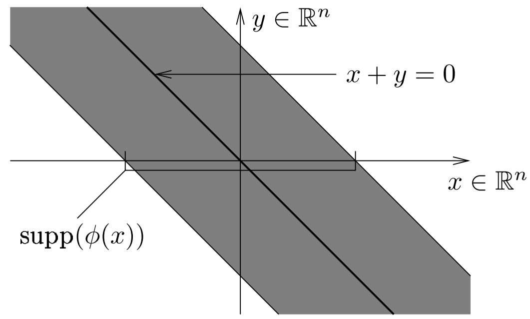


Figure 11.3:

Sufficient condition: We can define $f * g$ the following way: $f(x)g(y)$ has compact support, e.g., f or g has compact support. We then modify $\phi(x+y)$ outside the support to be compactly supported/rapidly decreasing.

Example:

1)

$$\begin{aligned}
(\delta_0 * f, \phi) &= (\delta_0(x)f(y), \phi(x+y)) \\
\text{(commutativity)} \quad &= (f(y)\delta_0(x), \phi(x+y)) \\
&= (f(y), (\delta_0(x), \phi(x+y))) \\
&= (f(y), \phi(y)) \\
&= (f, \phi)
\end{aligned}$$

$\Longrightarrow \underline{\delta_0 * f = f}$

2) $f \in \mathcal{D}'(\mathbb{R}^n)$, $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}
(f * \psi, \phi) &= (f(x)\psi(y), \phi(x+y)) \\
&= (f(x), (\psi(y), \phi(x+y))) \\
&= \left(f(x), \int_{\mathbb{R}^n} \psi(y)\phi(x+y)\ dy \right) \\
&= \left(f(x), \int_{\mathbb{R}^n} \psi(y-x)\phi(y)\ dy \right) \\
\text{(justified by Riemann Sums)} \quad &= \int_{\mathbb{R}^n} (f(x), \psi(x-y))\phi(y)\ dy
\end{aligned}$$

Therefore, we see that $f*\psi$ can be identified with the function $(f(x), \psi(x-y)) \in C^\infty(\mathbb{R}^n)$ (this function being C^∞ needs to be checked).

Remark: Suppose that $f_n \to f$ in $\mathcal{D}'(\mathbb{R}^n)$ and that either $\exists K$ compact, $K \subset \mathbb{R}^n$ with $\text{supp}(f_n) \subseteq K$, or that $g \in \mathcal{D}'(\mathbb{R}^n)$ has compact support, then

$$f_n * g \to f * g \text{ in } \mathcal{D}'(\mathbb{R}^n)$$

Theorem 11.12. *$\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$.*

Example: $\delta_0 \leftrightarrow \phi_\epsilon$, where $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$, $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$.

Idea of Proof:

(1) Distributions with compact support are dense in $\mathcal{D}'(\mathbb{R}^n)$. To show this, choose test functions $\phi_n \in \mathcal{D}(\mathbb{R}^n)$ with $\phi_n \equiv 1$ on $B_n(0)$. Now define $\phi_n \cdot f \in \mathcal{D}'(\mathbb{R}^n)$, $(\phi_n \cdot f, \phi) = (f, \phi_n \cdot \phi)$ so that we have $\phi_n \cdot f \to f$ in $\mathcal{D}'(\mathbb{R}^n)$ and $\phi_n \cdot f$ has compact support.

(2) Distributions with compact support are limits of test functions. To show this choose $\phi_\epsilon = \epsilon^{-n}\phi(\epsilon^{-1}x)$, $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$ standard mollifier. Then $\phi_n * f$ is a test function and by the above remark,

$$\phi_n * f \to \delta_0 * f = f \quad \blacksquare$$

11.2.2 Derivatives of Convolutions

$$\begin{aligned}
(D^\alpha(f * g), \phi) &= (-1)^{|\alpha|}(f * g, D^\alpha \phi) \\
&= (-1)^{|\alpha|}(f(x)g(y), D^\alpha \phi(x + y)) \\
&= (-1)^{|\alpha|}(g(y), (f(x), D^\alpha \phi(x + y))) \\
&= (g(y), (D^\alpha f(x), \phi(x + y))) \\
&= (g(y) D^\alpha f(x), \phi(x - y)) \\
&= (D^\alpha f * g, \phi) = (f * D^\alpha g, \phi)
\end{aligned}$$

11.2.3 Distributions as Fundemental Solutions

$L(D)$ is a differential operator

$$\Delta = L(D) = \sum_{j=1}^{n} a_j \cdot D_j^2$$

$$u_t - \Delta u = L(D) = b \cdot D_t - \sum_{j=1}^{n} a_j \cdot D_j^2$$

Definition 11.13. *Suppose that $L(D)$ is a differential operator with constant coefficients. A* <u>*fundamental solution*</u> *for L is a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ satisfying $L(D)G = \delta_0$.*

<u>Remark:</u> Why is this a good definition?

$$L(D)(G * f) = (L(D)G) * f = \delta_0 * f = f$$

The first equality of the above uses the result obtained for differentiating convolutions. So if $G * f$ is defined (e.g. f has compact support), then $u = G * f$ is a solution of $L(D)u = f$

<u>**Example:**</u> $f(x) = \frac{1}{|x|}$ in $\mathbb{R}^3$
Check: f is up to a constant a fundamental solution for the Laplace operator.

$$\begin{aligned}
(\Delta f, \phi) &= (f, \Delta\phi) \\
&= \int_{\mathbb{R}^3} f(x)\Delta\phi(x)\ dx \\
&= \lim_{\epsilon\to 0} \int_{\mathbb{R}^3\setminus B_\epsilon(0)} \frac{1}{|x|}\Delta\phi(x)\ dx \\
&= \lim_{\epsilon\to 0} \left[-\int_{\mathbb{R}^3\setminus B_\epsilon(0)} \left(D\frac{1}{|x|} \right) \cdot D\phi(x)\ dx - \int_{\partial B_\epsilon(0)} \frac{1}{r} D\phi(x) \cdot \nu\ dS \right] \\
&= \lim_{\epsilon\to 0} \left[\int_{\mathbb{R}^3\setminus B_\epsilon(0)} \underbrace{\Delta\frac{1}{|x|}}_{=0} \phi(x)\ dx + \underbrace{\int_{\partial B_\epsilon(0)} D\frac{1}{|x|} \cdot \nu\phi(x)\ dx}_{:=I} \right. \\
&\qquad \left. - \underbrace{\int_{\partial B_\epsilon(0)} \frac{1}{r} D\phi(x) \cdot \nu\ dS}_{\sim C\epsilon\to 0} \right]
\end{aligned}$$

So, from PDE I we know

$$\begin{aligned}
I &= -4\pi \fint_{B_\epsilon(0)} \phi(x)\ dS \to -4\pi\phi(0) \\
&= -4\pi(\delta_0, \phi).
\end{aligned}$$

This gives us the correct notion of what "$\Delta G = \delta_0$" really means.

11.3 Fourier Transform of $\mathcal{S}(\mathbb{R}^n)$

Definition 11.14. *If $f \in \mathcal{S}(\mathbb{R}^n)$ then we define*

$$\begin{aligned} \mathcal{F}f(\xi) &= \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\ dx \\ \mathcal{F}^{-1}f(x) &= \tilde{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\ d\xi \end{aligned}$$

$\mathcal{F}$ is the Fourier transform
$\mathcal{F}^{-1}$ is the inverse Fourier transform.

Two important formulas

1. $D^\alpha(\hat{f}(\xi) = \mathcal{F}[(-ix)^\alpha f](\xi)$
2. $(i\xi)^\beta \hat{f}(\xi) = \mathcal{F}[D^\beta f](\xi)$

Proof:

1.

$$\begin{aligned} D^\alpha_\xi \hat{f}(\xi) &= D^\alpha_\xi \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\ dx \right] \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} D^\alpha_\xi [e^{-ix\cdot\xi} f(x)]\ dx = \mathcal{F}[(-ix)^\alpha f](\xi) \end{aligned}$$

2.

$$\begin{aligned} (i\xi)^\beta \hat{f}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (i\xi)^\beta e^{-ix\cdot\xi} f(x)\ dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (-1)^{|\beta|} (-i\xi)^\beta e^{-ix\cdot\xi} f(x)\ dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (-1)^{|\beta|} D^\beta_x e^{-ix\cdot\xi} \cdot f(x)\ dx \\ &\stackrel{\text{I.P.}}{=} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \cdot D^\beta f(x)\ dx = \mathcal{F}[D^\beta f](\xi). \end{aligned}$$

Proposition 11.15. *$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear and continuous.*

Proposition 11.16. *$\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear and continuous.*

Proof:

$$(-1)^{|\beta|+|\alpha|}\xi^{\beta} \cdot D_{\xi}^{\alpha} \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \underbrace{D^{\beta}(x^{\alpha} f(x))}_{\substack{\text{finite sum of} \\ \text{rapidly decreas-} \\ \text{ing functions}}} \, dx$$

$\Longrightarrow \hat{f}(\xi)$ is rapidly decreasing, $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$

continuous: $f_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ then $\hat{f}_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$. Suppose that $f_n \to 0$, show that $D^{\alpha}\hat{f}_n \to 0$ uniformly $\forall \alpha \in \mathbb{N}^n$. It will be sufficient to show this for $\alpha = 0$,

$$\begin{aligned}
\hat{f}_n(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f_n(x) \, dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{B_R(0)} e^{-ix\cdot\xi} f(x) \, dx + \int_{\mathbb{R}^n \setminus B_R(0)} e^{-ix\cdot\xi} \underbrace{f(x)}_{\leq \frac{1}{|x|^k}} \, dx
\end{aligned}$$

($f(x) \leq \frac{1}{|x|^k}$ since $f \in \mathcal{S}(\mathbb{R}^n)$).

say $k = n+2$

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_R(0)} |f_n(x)| \, dx &\leq \int_{\mathbb{R}^n \setminus B_R(0)} \frac{C}{|x|^{n+2}} \, dx = C \int_R^{\infty} \frac{\rho^{n-1}}{\rho^{n+2}} \, d\rho \\
&= C \int_R^{\infty} \frac{1}{\rho^3} \, d\rho = C \frac{1}{R^2}
\end{aligned}$$

2nd integral as small as we want if R large enough.
1st integral as small as we want if n is large enough since $f_n \to 0$ uniformly.
This analogously for higher derivatives. ∎

Theorem 11.17. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathcal{F}^{-1}(\mathcal{F}f) = f, \quad \mathcal{F}(\mathcal{F}^{-1}f) = f$$

i.e., $\mathcal{F}^{-1}$is the inverse of the Fourier transform.

Fourier Transform

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx$$

$$(\mathcal{F}^{-1}f)(\xi) = \tilde{f}(\xi) = \check{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x) \, dx$$

Example:

$$f(x) = e^{-|x|^2/2}, \hat{f}(y) = e^{-|y|^2/2}$$

$$\hat{f}(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-|x|^2/2}\, dx \stackrel{!}{=} e^{-|y|^2/2}$$

It is sufficient to show this for $n = 1$.

$$\begin{aligned} \underbrace{(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-t^2/2} e^{-it\cdot u}\, dt} &= e^{-u^2/2} \\ \int_{\mathbb{R}} e^{-t^2/2 - it\cdot u}\, dx &= e^{-u^2/2} \int_{\mathbb{R}} e^{-(t+iu)^2/2}\, dt \quad \text{(completing the square)} \end{aligned}$$

Once way to evaluate the integral is to realize that $e^{-z^2/2}$ is analytic, thus by Cauchy we know that integrating this on a closed contour will yield a value of 0.

$\gamma_1(s) = s,\ s \in (-\lambda, \lambda)$
$\gamma_2(s) = \lambda + is,\ s \in (0, u)$
$\gamma_3(s) = -s + iu,\ s \in (-\lambda, \lambda)$
$\gamma_4(s) = -\lambda + i(u - s),\ s \in (0, u)$

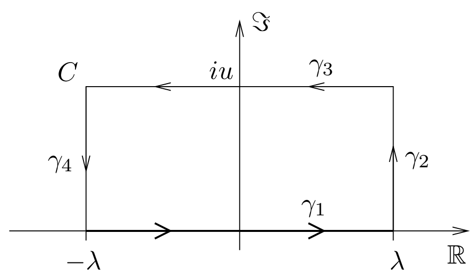


Figure 11.4:

$$\begin{aligned} 0 &= \int_C e^{-z^2/2}\, ds \\ &= \int_{-\lambda}^{\lambda} e^{-s^2/2}\, ds + \underbrace{\int_0^u e^{-(\lambda+is)^2/2}\, i ds}_{:=I_2} + \int_{-\lambda}^{\lambda} e^{-(-s+iu)^2/2}\, (-1) ds \\ &\quad + \int_0^u e^{-(-\lambda + i(u-s))^2/2}\, (-i) ds \end{aligned}$$

Looking at the second integral:

$$I_2 = \int_0^u \underbrace{e^{-\lambda^2/2}}_{\substack{\to\ 0 \text{ as} \\ \lambda \to \infty}} \underbrace{e^{-i\lambda s} e^{s^2/2}}_{\leq C}\, ds \to 0 \quad \text{as} \quad \lambda \to \infty$$

Similarly, $I_4 \to 0$ as $\lambda \to \infty$. Thus, in the limit we have

$$\begin{aligned}
\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} e^{-s^2/2}\,ds &= \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} e^{-(-s+iu)^2/2}\,ds \\
\Longrightarrow \underbrace{\int_{-\infty}^{\infty} e^{-s^2/2}\,ds}_{=\sqrt{2\pi}} &= \int_{-\infty}^{\infty} e^{-(t+iu)^2/2}\,dt = \int_{-\infty}^{\infty} e^{-t^2/2}e^{-iut}e^{u^2/2}\,dt \qquad \blacksquare
\end{aligned}$$

Remark: The generalization of this is the following: if $f(x) = e^{-\epsilon|x|^2}$, then $\hat{f}(\xi) = (2\epsilon)^{-n/2}e^{-|\xi|^2/4\epsilon}$. This is basically the same calculation as the previous example.

Theorem 11.18. *$f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}^{-1}\mathcal{F}f = f$, $\mathcal{F}\mathcal{F}^{-1}f = f$.*

Proof: $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\int_{\mathbb{R}^n} g(\xi)\tilde{f}(\xi)e^{ix\cdot\xi}\,dx &= \int_{\mathbb{R}^n} g(\xi)\left[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}f(y)\,dy\right]e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} e^{-i\xi(y-x)}g(\xi)\,d\xi\right]f(y)\,dy \\
&= \int_{\mathbb{R}^n}\hat{g}(y-x)f(y)\,dy = \int_{\mathbb{R}^n}\hat{g}(y)f(x+y)\,dy
\end{aligned}$$

Replace $g(\xi)$ by $g(\epsilon\cdot\xi)$. Need to computer $\widehat{g(\epsilon\cdot)}$:

$$\begin{aligned}
(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(\epsilon\cdot\xi)\,d\xi &= (2\pi)^{-\frac{n}{2}}\frac{1}{\epsilon^n}\int_{\mathbb{R}^n} e^{-iy\cdot\frac{z}{\epsilon}}g(z)\,dz \\
&= \frac{1}{\epsilon^n}\hat{g}\left(\frac{y}{\epsilon}\right)
\end{aligned}$$

So, we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} g(\epsilon\cdot\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi &= \int_{\mathbb{R}^n}\widehat{g(\epsilon\cdot)}(y)\cdot f(x+y)\,dy \\
&= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\widehat{g}\left(\frac{y}{\epsilon}\right)f(x+y)\,dy \\
&= \int_{\mathbb{R}^n}\hat{g}(y)f(x+\epsilon y)\,dy
\end{aligned}$$

Choose $g(x) = e^{-|x|^2/2}$ and take the limit $\epsilon \to 0$.

$$\begin{aligned}
g(0)\int_{\mathbb{R}^n}\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi &= f(x)\int_{\mathbb{R}^n}\hat{g}(y)\,dy \\
&= f(x)\int_{\mathbb{R}^n} e^{-|y|^2/2}\,dy = (\sqrt{2\pi})^{-\frac{n}{2}}f(x)
\end{aligned}$$

$$\therefore \quad f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi = ((\mathcal{F}^{-1}\mathcal{F})f)(x)$$

The proof of $(\mathcal{F}\mathcal{F}^{-1}f)(x) = f(x)$ is analogous. ■

Remark: Suppose $L(D)$ is a differential operator with constant coefficients, i.e.,

$$L(D) = \sum_{i=1}^{n} D_i^2 = -\Delta.$$

Then $\mathcal{F}(L(D)u) = L(i\xi)\hat{u}$, where $L(i\xi)$ is just the symbol of the operator.

Example: Suppose we can extend Fourier transforms to $L^2(\mathbb{R}^n)$ (see below), then with $L(D) = -\Delta$ and $L(i\xi) = \sum \xi_i^2 = |\xi|^2$, we can find the solution of $-\Delta u = f \in L^2(\mathbb{R}^n)$ by Fourier Transform:

$$\begin{aligned} & \widehat{(-\Delta u)} = \tilde{f} \\ \iff \quad & |\xi|^2 \hat{u}(\xi) = \tilde{f}(\xi) \\ \iff \quad & \hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2} \\ \iff \quad & u(x) = \mathcal{F}^{-1}\left(\frac{\hat{f}(\xi)}{|\xi|^2} f\right)(x) \end{aligned}$$

Theorem 11.19. *$f, g \in \mathcal{S}(\mathbb{R}^n)$, then $(f,g) = (\hat{f}, \hat{g})$, i.e., the Fourier Transform preserves the inner-product*

$$(f,g) = \int_{\mathbb{R}^n} \overline{f} \cdot g \, dx$$

Remark: Using a Fourier Transform, we have to use complex valued functions, $|z|^2 = z \cdot \overline{z}$.

Proof: Computer. $g(x) = (\mathcal{F}^{-1}\mathcal{F}g)(x) = (\mathbb{R}^{-1}\hat{g})(x)$

$$\begin{aligned} \int_{\mathbb{R}^n} \overline{f}(x) g(x)\, dx &= \int_{\mathbb{R}^n} \overline{f}(x) \cdot (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{g}(\xi)\, d\xi dx \\ &= \int_{\mathbb{R}^n} \hat{g}(\xi) \cdot (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{f}(x) e^{ix\cdot\xi}\, dx d\xi \\ &= \int_{\mathbb{R}^n} \hat{g}(\xi) \cdot (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{f(x) e^{-ix\cdot\xi}}\, dx d\xi \\ &= \int_{\mathbb{R}^n} \hat{g}(\xi) \overline{\hat{f}}(\xi)\, d\xi = (\hat{f}, \hat{g}) \end{aligned}$$ ■

11.4 Definition of Fourier Transform on $L^2(\mathbb{R}^n)$

Proposition 11.20. *If $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{u*v} = (2\pi)^{n/2}\hat{u}\cdot\hat{v}$.*

Proof:

$$\begin{aligned}
\widehat{u*v}(y) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot y}(u*v)(x)\ dx \\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot y}\left[\int_{\mathbb{R}^n} u(z)v(x-z)\ dz\right] dx \\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-iz\cdot y}u(z)\int_{\mathbb{R}^n} e^{-iy(x-z)}v(\underbrace{x-z}_{\text{sub.}w})\ dxdz \\
&= \int_{\mathbb{R}^n} e^{-izy}u(z)\hat{v}(y)\ dz \\
&= (2\pi)^{\frac{n}{2}}\hat{u}(y)\hat{v}(y) \qquad \blacksquare
\end{aligned}$$

Theorem 11.21 (Plancherel's Theorem). *Suppose that $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{u} \in L^2(\mathbb{R}^n)$ and*

$$||\hat{u}||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)}$$

Proof: Let $v(x) = \overline{u}(-x)$. Define $w = u * v$. So, we have $\hat{w}(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{v}(\xi)$. Now $w \in L^1(\mathbb{R}^n)$ and w is continuous (using Proposition 8.31). Now,

$$\hat{v}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\overline{u}(-x)\cdot e^{-ix\cdot\xi}\ dx.$$

Let $y = -x$, so

$$\hat{v}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\overline{u}(y)\cdot e^{-iy\cdot\xi}\ dy = (2\pi)^{-\frac{n}{2}}\overline{\int_{\mathbb{R}^n} u(y)\cdot e^{iy\cdot\xi}\ dy} = \overline{\hat{u}}(y)$$

Now we claim for any $v, w \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} v(x)\cdot\hat{w}(x)\ dx = \int_{\mathbb{R}^n}\hat{v}(\xi)\cdot w(\xi)\ d\xi.$$

Check:

$$\begin{aligned}
\int_{\mathbb{R}^n} v(x)\cdot\hat{w}(x)\ dx &= \int_{\mathbb{R}^n} v(x)\cdot\left[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} w(\xi)e^{-ix\cdot\xi}\ d\xi\right]\ dx \\
&= \int_{\mathbb{R}^n}\left[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} v(x)e^{-ix\cdot\xi}\ dx\right]\cdot w(\xi)\ d\xi \\
&= \int_{\mathbb{R}^n}\hat{v}(\xi)\cdot w(\xi)\ d\xi
\end{aligned}$$

Now set $f(\xi) = e^{-\epsilon|\xi|^2}$ so $\hat{f}(x) = (2\epsilon)^{-n/2}e^{-|\xi|^2/4\epsilon}$ (as shown in the example earlier). Then,

$$\int_{\mathbb{R}^n} \hat{w}(\xi)e^{-\epsilon|\xi|^2}\, d\xi = (2\epsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} w(x)e^{-\epsilon|x|^2/4\epsilon}\, dx.$$

Now, take the limit as $\epsilon \to 0$ to get

$$\int_{\mathbb{R}^n} \hat{w}(\xi)\, d\xi = (2\pi)^{\frac{n}{2}} w(0)$$

($w(0)$ because the integral was a limit representation of the dirac distribution). But we know

$$w = u * v \quad \text{and} \quad \hat{w}(\xi) = (2\pi)^{\frac{n}{2}}\hat{u}\cdot\hat{v}(\xi) = (2\pi)^{\frac{n}{2}}\hat{u}\cdot\overline{\hat{u}}(\xi) = (2\pi)^{\frac{n}{2}}|\hat{u}|^2,$$

so

$$(2\pi)^{\frac{n}{2}} w(0) = \int_{\mathbb{R}^n} \hat{w}(\xi)\, d\xi = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} |\hat{u}|^2\, d\xi \Longrightarrow w(0) = \int_{\mathbb{R}^n} |\hat{u}|^2\, d\xi.$$

We also know

$$w = u * v \Longrightarrow w(0) = \int_{\mathbb{R}^n} u(y)\cdot v(-y)\, dy = \int_{\mathbb{R}^n} u(y)\cdot \overline{u}(y)\, dy = \int_{\mathbb{R}^n} |u|^2\, dy.$$

Thus,

$$\int_{\mathbb{R}^n} |\hat{u}|^2\, d\xi = w(0) = \int_{\mathbb{R}^n} |u|^2\, dy \Longrightarrow ||\hat{u}||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)} \qquad \blacksquare$$

Definition 11.22 (Fourier Transform in $L^2(\mathbb{R}^n)$). *If $u \in L^2(\mathbb{R}^n)$, choose $u_k \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $u_k \to u$ in $L^2(\mathbb{R}^n)$. By Plancherel's Theorem:*

$$\begin{aligned} ||\hat{u}_k - \hat{u}_l||_{L^2(\mathbb{R}^n)} &= ||\widehat{u_k - u_l}||_{L^2(\mathbb{R}^n)} \\ &= ||u_k - u_l||_{L^2(\mathbb{R}^n)} \to 0 \quad \textit{as} \quad k, l \to \infty \end{aligned}$$

$\Longrightarrow$ *$\hat{u}_k$ is Cauchy in $L^2(\mathbb{R}^n)$. The limit is defined to be $\hat{u} \in L^2(\mathbb{R}^n)$.*

<u>Note:</u> This definition does not depend on the approximating sequence.

<u>Remarks:</u>

1) We have

$$||u||_{L^2(\mathbb{R}^n)} = ||\hat{u}||_{L^2(\mathbb{R}^n)}$$

if $u \in L^2(\mathbb{R}^n)$ by approximation of u in $L^2(\mathbb{R}^n)$ by functions $u_k \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$; Plancherel's Theorem.

2) You can use this to find integrals of functions. Suppose u, $\hat{u}$, then

$$\int_{\mathbb{R}} |\hat{u}|^2 \ dx = \int_{\mathbb{R}} |u|^2 \ dx$$

Theorem 11.23. *If $u, v \in L^2(\mathbb{R}^n)$, then*

i.)

$$(u,v) = (\hat{u}, \hat{v}), \ i.e., \ \int_{\mathbb{R}^n} \overline{u} \cdot v \ dx = \int_{\mathbb{R}^n} \hat{u} \cdot \overline{\hat{v}} \ dx$$

ii.) $(\widehat{D^\alpha u})(y) = (iy)^\alpha \hat{u}(y), \ D^\alpha u \in L^2(\mathbb{R}^n)$

iii.) $(\widehat{u * v}) = (2\pi)^{n/2} \hat{u} \cdot \hat{v}$

iv.) $u = \tilde{\hat{u}} = \mathcal{F}^{-1}[\mathcal{F}(u)]$

Proofs: See Evans.

11.5 Applications of FTs: Solutions of PDEs

(1) Solve $-\Delta u + u = f$ in $\mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$.
The Fourier Transform converts the PDE into an algebraic expression.

$$\begin{aligned} & |y|^2 \hat{u} + \hat{u} = \hat{f} \\ \iff \quad & \hat{u} = \frac{\hat{f}}{|y|^2 + 1} \\ \iff \quad & u = \mathcal{F}^{-1}\left(\frac{\hat{f}}{|y|^2+1}\right) = \mathcal{F}^{-1}(\hat{f} \cdot \widehat{B}), \end{aligned}$$

where $\widehat{B} = \frac{1}{|y|^2+1}$. Since $\widehat{u * v} = (2\pi)^{n/2} \hat{u} \cdot \hat{v}$,

$$\begin{aligned} \hat{u} &= \hat{f} \cdot \widehat{B} = (2\pi)^{-\frac{n}{2}} (\widehat{f * B}) \\ u &= (2\pi)^{-\frac{n}{2}} (f * B). \end{aligned}$$

Now we need to find B:

$$\underline{\text{Trick:}} \qquad \frac{1}{a} = \int_0^\infty e^{-ta}\ dt$$
$$\Longrightarrow \qquad \frac{1}{|y|^2+1} = \int_0^\infty e^{-t(|y|^2+1)}\ dt.$$

So,

$$\begin{aligned}
B(x) &= (\mathcal{F}^{-1}\widehat{B})(x) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot y} \int_0^\infty e^{-t(|y|^2+1)}\ dtdy \\
&= (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-t} \int_{\mathbb{R}^n} e^{ix\cdot y} e^{-t|y|^2}\ dydt \\
\text{(change var.)} \quad &= \int_0^\infty e^{-t}(2t)^{-\frac{n}{2}} e^{-|x|^2/4t}\ dt.
\end{aligned}$$

This last integral is called the Bessel Potential.

So, we at last have an explicit representation of our solution:

$$\begin{aligned}
u(x) &= (2\pi)^{-\frac{n}{2}}(B * f) \\
&= (4\pi)^{-\frac{n}{2}} \int_0^\infty \int_{\mathbb{R}^n} t^{-\frac{n}{2}} e^{-t-|x-y|^2/4t} f(y)\ dydt
\end{aligned}$$

(2) Heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Take the Fourier transform in the spatial variables:

$$\begin{cases} \hat{u}_t(y) - |y|^2\hat{u}(y) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \hat{u}(y) = \hat{g}(y) & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Now, we can view this as an ODE in t:

$$\begin{aligned}
\hat{u}(y,t) &= \hat{g}(y)e^{-|y|^2t} \\
\Longrightarrow u(x,t) &= \mathcal{F}^{-1}[\hat{g}(y)e^{-|y|^2t}](x) \\
&= (2\pi)^{-\frac{n}{2}}(g * F),
\end{aligned}$$

where

$$\widehat{F} = e^{-|y|^2t}.$$

Changing variables as before,

$$\begin{aligned} F(x) &= (2t)^{-\frac{n}{2}} e^{-|x|^2/4t} \\ \Longrightarrow u(x,t) &= (2\pi)^{-\frac{n}{2}} (g * F)(x,t) \\ &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(y) e^{-|x-y|^2/4t} \, dy \end{aligned}$$

This is just the representation formula from the last term!

11.6 Fourier Transform of Tempered Distributions

Definition 11.24 (Fourier Transform of tempered distributions). *If $f \in \mathcal{S}'(\mathbb{R}^n)$, then we define*

$$(\mathcal{F}(f), \phi) = (f, \mathcal{F}^{-1}(\phi))$$

Check that the following holds:

$$\begin{aligned} D^\alpha \hat{f} &= \mathcal{F}((-ix)^\alpha f) \\ (i\xi)\hat{f} &= \mathcal{F}(D^\alpha f) \end{aligned}$$

Examples:

(1) $f = \delta_0$,

$$\begin{aligned} \langle \mathcal{F}(\delta_0), \phi \rangle &= \langle \delta_0, \mathcal{F}^{-1}\phi \rangle = (\mathcal{F}^{-1}\phi)(0) \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \phi(x) \, dx \\ &= \left\langle (2\pi)^{-\frac{n}{2}}, \phi \right\rangle \\ &\Longrightarrow \mathcal{F}(\delta_0) = (2\pi)^{-\frac{n}{2}} \end{aligned}$$

(2) $\mathcal{F}(1)$, $1 =$ constant function.

$$\begin{aligned} \langle \mathcal{F}(1), \phi \rangle &= \langle 1, \mathcal{F}^{-1}\phi \rangle \\ &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\phi)(x) \, dx \\ &= (2\pi)^{\frac{n}{2}} \mathcal{F}\mathcal{F}^{-1}(\phi)(0) \\ &= (2\pi)^{\frac{n}{2}} \phi(0) \\ &= \left\langle (2\pi)^{\frac{n}{2}} \delta_0, \phi \right\rangle \end{aligned}$$

11.7 Exercises

11.1: Show that the Dirac delta distribution cannot be identified with any continuous function.

11.2: Let $\Omega = (0,1)$. Find an example of a distribution $f \in \mathcal{D}'(\Omega)$ of infinite order.

11.3: Let $a \in \mathbb{R}$, $a \neq 0$. Find a fundamental solution for $L = \frac{d}{dx} - a$ on $\mathbb{R}$.

11.4: Find a sequence of test functions that converges in the sense of distributions in $\mathbb{R}$ to δ'.

11.5: Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^2$, and let $\mathbf{w} \in W^{1,p}(B(0,1);\mathbb{R}^2)$ be a vector valued function, that is, $\mathbf{w}(x) = (u(x), v(x))$ with $u, v \in W^{1,p}(\Omega)$. Recall that $D\mathbf{w}$, the gradient of $\mathbf{w}$, is given by

$$D\mathbf{w} = \begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix}.$$

a) Suppose $\mathbf{w}$ is smooth. Show that

$$\partial_1(u \cdot \partial_2 v) - \partial_2(u \cdot \partial_1 v) = \det D\mathbf{w}.$$

b) Let $\mathbf{w} \in W^{1,p}(B(0,1);\mathbb{R}^2)$. We want to define a distribution $\operatorname{Det} D\mathbf{w} \in \mathcal{D}'(\Omega)$ by

$$\langle \operatorname{Det} D\mathbf{w}, \phi \rangle = -\int_\Omega (u \cdot \partial_2 v \cdot \partial_1 \phi - u \cdot \partial_1 v \cdot \partial_2 \phi)\, dx.$$

Find the smallest $p \geq 1$ such that $\operatorname{Det} D\mathbf{w}$ defines a distribution, that is, the integral on the right hand side exists.

c) It is in general not true that $\operatorname{Det} D\mathbf{w} = \det D\mathbf{w}$. Here is an example; let $\mathbf{w}(x) = x/|x|$. Show that there exists a constant c such that

$$\operatorname{Det} D\mathbf{w} = c\delta_0,$$

but that $\det D\mathbf{w}(x) = 0$ for a.e. $x \in \Omega$.

11.6: [Qualifying exam 08/98] Assume $u : \mathbb{R}^n \to \mathbb{R}$ is continuous and its first-order distribution derivatives Du are bounded functions, with $|Du(x)| \leq$

L for all x. By approximating u by mollification, prove that u is Lipschitz, with

$$|u(x) - u(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^n.$$

11.7: [Qualifying exam 01/03] Consider the operator

$$A : \mathcal{D}(A) \to L^2(\mathbb{R}^n)$$

where $\mathcal{D}(A) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and

$$Au = \Delta u \quad \text{for } u \in \mathcal{D}(A).$$

Show that

a) the resolvent set $\rho(A)$ contains the interval $(0, \infty)$, i.e., $(0, \infty) \subset \rho(A)$;

b) the estimate

$$||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0$$

holds.

11.8: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin x & \text{if } x \in (-\pi, \pi) \\ 0 & \text{else.} \end{cases}$$

Show that

$$\hat{f}(\xi) = i\frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin \pi\xi}{\xi^2 - 1}.$$

11.9: We define for real-valued functions f and $s \geq 0$

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}.$$

a) Show that

$$\overline{\hat{f}}(\xi) = \hat{f}(-\xi).$$

b) Let $f \in H^s(\mathbb{R}^n)$ with $s \geq 1$. Show that $g(x) = \mathcal{F}^{-1}(i\xi \hat{f}(\xi))(x)$ is a (vector-valued) function with values in the real numbers.

c) Prove that $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$. (This identity is in fact true in any space dimension for $s \in \mathbb{N}$).

d) Show that $\hat{f} \in C^0(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$. Analogously, if $\hat{f} \in L^1(\mathbb{R}^n)$, then $f \in C^0(\mathbb{R}^n)$ (you need not show the latter assertion).

e) Show that $H^s(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ if $s > \frac{n}{2}$. Hint: Use Holder's inequality.

11.10:

a) Find the Fourier transform of $\chi_{[-1,1]}$, the characteristic function of the interval $[-1,1]$.

b) Find the Fourier transform of the function

$$f(x) = \begin{cases} 2+x & \text{for } x \in [-2,0], \\ 2-x & \text{for } x \in [0,2], \\ 0 & \text{else.} \end{cases}$$

Hint: What is $\chi_{[-1,1]} * \chi_{[-1,1]}$?

c) Find

$$\int_{\mathbb{R}} \frac{\sin^2 \xi}{\xi^2}\, d\xi.$$

d) Show that $\chi_{[-1,1]} \in H^{\frac{1}{2}-\epsilon}(\mathbb{R})$ for all $\epsilon \in \left(0, \frac{1}{2}\right]$.

Chapter 12

Variational Methods

12.1 Direct Method of Calculus of Variations

We wish to solve $-\Delta u = 0$, by using minimizers

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2 \; dx$$

Crucial Question: Can we decide whether there exists a minimizer?
General Question: If X a BS, $||\cdot||$, and $F : X \to \mathbb{R}$, is there a minima of F?

The answer will be yes, if we have the following:

- F bounded from below

$$\text{let } M = \inf_{x\in X} F(x) \; dx > -\infty$$

- Can choose a minimizing sequence $x_n \in X$ such that

$$F(x_n) \to M \text{ as } n \to \infty$$

- Bounded sets are compact.
  Suppose (subsequence) $x_k \to x$

- Continuity property:

$$F(x_n) \to F(x) = M, \; x \text{ a minimizer}$$

  Turns out the lower-semicontinuity will be enough:

$$\begin{aligned} & M = \lim_{k\to\infty} \inf F(x_k) \geq F(x) \geq M \\ \Longrightarrow \quad & F(x) = M, \; x \text{ minimizer} \end{aligned}$$

There is a give and take between convergence and continuity.

	Strong Convergence	Weak Convergence
Compactness	difficult	easy
Lower Semicontinuity	easy	difficult
		Our Choice

Definition 12.1. $1 < p < \infty$. *A function F is said to be (sequentially) weakly lower semicontinuous (swls) on $W^{1,p}(\Omega)$ if $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$ implies that*

$$F(u) \leq \lim_{k\to\infty} \inf F(u_k)$$

Example: Dirichlet integral

$$F(u) = \int_\Omega \frac{1}{2}|Du|^2 \, dx$$

and the generalization

$$F(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 + fu\right) \, dx$$

with $f \in L^2(\Omega)$ in $X = H_0^1(\Omega)$

Goal: $\exists!$ minimizer in $H_0^1(\Omega)$ which solves $-\Delta u = f$ in the weak sense.

(1) Lower bound on F:

$$\begin{aligned} F(u) &= \int_\Omega \left(\frac{1}{2}|Du|^2 - fu\right) \, dx \\ &\overset{\text{Hölder}}{\geq} \int_\Omega \frac{1}{2}|Du|^2 \, dx - ||f||_{L^2(\Omega)}||u||_{L^2(\Omega)} \end{aligned}$$

———

Now we use Young's inequality with $\left(\sqrt{\epsilon}a - \frac{b}{2\sqrt{\epsilon}}\right)^2 \geq 0 \implies ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$

———

$$\overset{\text{Young's}}{\geq} \int_\Omega \frac{1}{2}|Du|^2 \, dx - \epsilon||u||^2_{L^2(\Omega)} - \frac{1}{4\epsilon}||f||^2_{L^2(\Omega)}$$

———

Recall Poincare's inequality in $H_0^1(\Omega)$: $\exists C_p$ such that $||u||^2_{L^2(\Omega)} \leq$

$\underline{C_p||Du||^2_{L^2(\Omega)}}$

$$\overset{\text{Poincaré}}{\geq} \int_\Omega \frac{1}{2}|Du|^2\,dx - \epsilon\cdot C_p\int_\Omega |Du|^2\,dx - \frac{1}{4\epsilon}||f||^2_{L^2(\Omega)}$$

Now, choose $\epsilon = \frac{1}{4C_p}$,

$$\implies \quad F(u) \geq \underbrace{\frac{1}{4}\int_\Omega |Du|^2\,dx}_{\geq 0} - \underbrace{C_p\int_\Omega |f|^2\,dx}_{\substack{\text{fixed number and}\\ \text{finite since } f\in\\ L^2(\Omega)}} \geq C > -\infty$$

Recall:
Goal: Want to solve $-\Delta u = f$ by variational methods, i.e., we need to prove the existence of a minimizer of the functional

$$F(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - fu\right)\,dx.$$

We can do this via the direct method of the calculus of variations. This entails

- Showing F is bounded from below.
- choosing a minimizing sequence x_n.
- Show that this minimizing sequence is bounded, so we can use weak compactness of bounded sets to show $x_n \rightharpoonup x$.
- Prove the lower semicontinuity of F

$$M \leq F(X) \leq \lim_{n\to\infty}\inf F(x_n) = M$$

So we start where we left off:

(2)

$$\frac{1}{4}\int_\Omega |Du|^2\,dx \underbrace{-C_p\cdot\int_\Omega |f|^2\,dx}_{-C_p\cdot||f||^2_{L^2(\Omega)}} \leq F(u) \tag{12.1}$$

$$\begin{aligned} &\implies \quad F(u) \geq -C_p\cdot||f||^2_{L^2(\Omega)} > -\infty \\ &\implies \quad F \text{ is bounded from below.}\end{aligned}$$

(3) Define

$$M = \inf_{u \in H_0^1(\Omega)} F(u) > -\infty.$$

Choose $u_n \in H_0^1(\Omega)$ such that $F(u_n) \to M$ as $n \to \infty$

(4) Boundedness of $\{u_n\}$: choose $n \geq n_0$ large enough such that $F(u_n) \leq M + 1$ for $n \geq n_0$. (12.1) tells us that

$$\begin{aligned} \frac{1}{4}\int_\Omega |Du_n|^2 \, dx &\leq C_p \int_\Omega |f|^2 \, dx + F(u_n) \\ &\leq C_p \int_\Omega |f|^2 \, dx + M + 1 \text{ for } n \geq n_0 \end{aligned}$$

——

$$\text{Poincare:} \quad ||u_n||^2_{L^2(\Omega)} \leq C_p ||Du_n||^2_{L^2(\Omega)}$$

——

$\Longrightarrow$ $\{u_n\}$ is bounded in $H_0^1(\Omega)$.
$\Longrightarrow$ weak compactness: $\exists$ a subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u$ in $H_0^1(\Omega)$ as $j \to \infty$, and
compact Sobolev embedding: $\exists$ a subsequence of $\{u_{n_j}\}$ such that $u_{n_{j_k}} \to u$ in $L^2(\Omega)$.

Call $\{u_k\}_{k \in \mathbb{N}}$ this subsequence.

(5) weak lower semicontinuity: Need

$$\begin{aligned} F(u) &\leq \liminf_{k \to \infty} F(u_k) \\ &= \liminf_{k \to \infty} \int_\Omega \left(\frac{1}{2}|Du_k|^2 - f u_k \right) \, dx \end{aligned}$$

Now since $u_k \to u$ in $L^2(\Omega)$, we know

$$\int_\Omega f u_k \, dx \to \int_\Omega f u \, dx.$$

A question remains. Does the following relation hold?

$$\frac{1}{2}\int_\Omega |Du|^2 \, dx \leq \liminf_{k \to \infty} \frac{1}{2} \int_\Omega |Du_k|^2 \, dx$$

To see this we utilize the following:

Note:

$$\begin{aligned}\frac{1}{2}|p|^2 &= \frac{1}{2}|q|^2 + (q, p-q) + \frac{1}{2}|p-q|^2 \\ &= -\frac{1}{2}|q|^2 + p \cdot q\end{aligned}$$

$$\begin{aligned}\iff \quad & \frac{1}{2}|p|^2 + \frac{1}{2}|q|^2 - p \cdot q = \frac{1}{2}|p-q|^2 \\ \iff \quad & \frac{1}{2}[|p|^2 - 2p \cdot q + |q|^2] = \frac{1}{2}|p-q|^2\end{aligned}$$

Choose $p = Du_k$ and $q = Du$.

$$\begin{aligned}\int_\Omega \frac{1}{2}|Du_k|^2\ dx &= \int_\Omega \left[\frac{1}{2}|Du|^2 + (Du, Du_k - Du) + \underbrace{\frac{1}{2}|Du_k - Du|^2}_{\geq 0}\right]\ dx \\ &\geq \int_\Omega \frac{1}{2}|Du|^2\ dx + \underbrace{\int_\Omega Du \cdot \underbrace{(Du_k - Du)}_{\rightharpoonup 0 \text{ in } L^2(\Omega)}\ dx}_{\to 0 \text{ as } k \to 0}\end{aligned}$$

$$\begin{aligned}\implies \liminf_{k\to\infty} \int_\Omega \frac{1}{2}|Du_k|^2\ dx &\geq \int_\Omega \frac{1}{2}|Du|^2\ dx + \liminf_{k\to\infty} \int_\Omega Du \cdot (Du_k - Du)\ dx \\ &= \int_\Omega \frac{1}{2}|Du|^2\ dx\end{aligned}$$

Trick: The inequality

$$\frac{1}{2}|p|^2 \geq \frac{1}{2}|q|^2 + (q, p-q)$$

allows us to replace the quadratic term in Du_k by a term that is linear in Du_k.

$\implies u$ is a minimizer!

(6) Uniqueness: Use again that

$$\frac{1}{2}|p|^2 = \frac{1}{2}|q|^2 + (q, p-q) + \frac{1}{2}|p-q|^2$$

Suppose we have two minimizers u, v. Define $w = \frac{1}{2}(u+v)$. Now we choose $p = Du$, $q = \frac{1}{2}(Du + Dv)$, $p - q = Du - \frac{1}{2}(Du+Dv) = \frac{1}{2}(Du - Dv)$.

$$\implies \int_\Omega \frac{1}{2}|Du|^2\,dx \quad = \quad \int_\Omega \left\{ \frac{1}{2}|Dw|^2 + \frac{1}{2}\int_\Omega Dw\cdot(Du - Dv)\,dx + \frac{1}{2}\left|\frac{1}{2}(Du-Dv)\right|^2 \right\}\,dx$$

We apply the same inequality with $p = Dv$, $q = Dw$ and $p - q = Dv - \frac{1}{2}(Du+Dv) = \frac{1}{2}(Dv - Du)$.

$$\implies \int_\Omega \frac{1}{2}|Dv|^2\,dx = \int_\Omega \left\{ \frac{1}{2}|Dw|^2 + \frac{1}{2}Dw\cdot(Dv - Du) + \frac{1}{2}\left|\frac{1}{2}(Dv-Du)\right|^2 \right\}\,dx$$

Now we sum these inequalities.

$$\frac{1}{2}\int_\Omega |Du|^2\,dx + \frac{1}{2}\int_\Omega |Dv|^2\,dx = \int_\Omega |Dw|^2\,dx + \frac{1}{4}\int_\Omega |Du - Dv|^2\,dx.$$

Now we remember that u and v are minimizers of

$$F(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - fu\right)\,dx.$$

So, we calculate

$$\begin{aligned}
\int_\Omega \left(\frac{1}{2}|Du|^2 - fu\right)\,dx &+ \int_\Omega \frac{1}{2}(|Dv|^2 - fv)\,dx \\
&= \int_\Omega |Dw|^2 - \underbrace{(u+v)f}_{=2wf}\,dx + \frac{1}{4}\int_\Omega |Du-Dv|^2\,dx \\
&= 2\int_\Omega \frac{1}{2}|Dw|^2 - wf\,dx + \frac{1}{4}\int_\Omega |Du-Dv|^2\,dx
\end{aligned}$$

$$\begin{aligned}
&\implies \quad M + M \geq 2M + \frac{1}{4}\int_\Omega |Du - Dv|^2\,dx \\
&\implies \quad 0 \geq \frac{1}{4}\int_\Omega |Du-Dv|^2\,dx
\end{aligned}$$

$\implies Du = Dv$, $u, v \in H_0^1(\Omega) \implies u = v \implies$ uniqueness!

Recall: To solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases},$$

we looked for a minimizer in $H_0^1(\Omega)$ of the functional

$$F(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - fu \right) \, dx$$

Already done: Existence and uniqueness of minimizer.

(7) Euler-Lagrange equation: We have the following necessary condition. If $v \in H_0^1(\Omega)$, $\epsilon \mapsto F(u + \epsilon v)$, then this function has a minimum at $\epsilon = 0$ and

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(u + \epsilon v) = 0.$$

So for our Dirichlet integral we have

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_\Omega \left[\frac{1}{2}|Du + \epsilon \cdot Dv|^2 - f(u + \epsilon v) \right] \, dx = \int_\Omega (Du \cdot Dv - fv) \, dx = 0$$

i.e. $$\int_\Omega (Du \cdot Dv - fv) \, dx = 0 \qquad \forall v \in H_0^1(\Omega)$$

i.e., u is a weak solution of Laplace's equation.

Remarks:

1) Laplace's equation has a variational structure, i.e., its weak formulation is the Euler-Lagrange equation of a variational problem.

2) $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous if $x_n \to x$ and

$$f(x) \le \lim_{n \to \infty} \inf f(x_n).$$

f continuous $\Longrightarrow$ f lower semicontinuous.

If f discontinuous at x, we have the following situations:

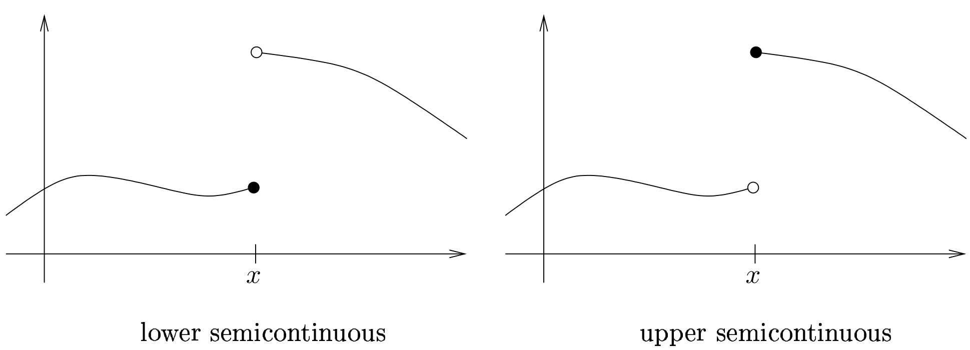


Figure 12.1:

3) Weak convergence is <u>not</u> compatible with non-linear functions: if $f_n \rightharpoonup f$ in $L^2(\Omega)$, and if g is a non-linear functions, then $g(f_n) \not\rightharpoonup g(f)$.

Example: $f_n = \sin(nx)$ on $(0,\pi)$, $f_n \rightharpoonup 0$ in $L^2((0,\pi))$.

<u>Need:</u> $$\int_0^\pi f_n g \; dx \to 0 \qquad \forall g \in L^2((0,\pi)).$$

It will be sufficient to show this for $\phi \in C_c^\infty((0,\pi))$.

$$\begin{aligned}
\int_0^\pi \sin(nx)\phi(x)\; dx &= \int_0^\pi \frac{d}{dx}\left(-\frac{1}{n}\cos(nx)\right)\phi(x)\; dx \\
&= \int_0^\pi \underbrace{\cos(nx)}_{|\cdot|\le 1}\phi'(x)\; dx \\
&\le \frac{1}{n}\cdot \max|\phi'| \int_0^\pi dx \\
&= \frac{\pi}{n}\cdot \max|\phi'| \to 0 \text{ as } n\to\infty
\end{aligned}$$

Now choose $g(x) = x^2$, $g(f_n) \rightharpoonup g(f)$? <u>NO</u>

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\pi \sin^2(nx)\phi(x)\; dx &= \lim_{n\to\infty}\int_0^\pi \frac{d}{dx}\left[\frac{x}{2} - \frac{1}{4n}\sin(2nx)\right]\phi(x)\; dx \\
&= \lim_{n\to\infty}\int_0^\pi \frac{1}{2}\phi(x)\; dx + \underbrace{\lim_{n\to\infty}\frac{1}{4n}\int_0^\pi \sin(2nx)\phi'(x)\; dx}_{=0} \\
&= \int_0^\pi \frac{1}{2}\phi(x)\; dx.
\end{aligned}$$

So, $\sin^2(nx) \rightharpoonup \frac{1}{2}$ in $L^2((0,\pi))$

So, we need lower semicontinuity since $F(u)$ will in general not be linear (for our Dirichlet integral $|Du|^2$ is not linear), and thus passing a weak limit within $F(u)$ isn't justified. Lower semicontinuity saves this by trapping the weak sequence as follows

$$\inf_{u\in H_0^1(\Omega)} F(u) \le F(u) \le \lim_{k\to\infty}\inf F(u_k) \to \inf_{v\in H_0^1(\Omega)} F(v)$$

$$\Longrightarrow F(u) = \inf_{v\in H_0^1(\Omega)} F(v)$$

4) Crucial was

$$\frac{1}{2}|p|^2 \geq \frac{1}{2}|q|^2 + (q, p-q)$$

Define $F(p) = \frac{1}{2}|p|^2 \Longrightarrow F'(p) = p$.
We need: $F(p) \geq F(q) + (F'(q), p-q)$ with q fixed, $p \in \mathbb{R}^n$, i.e., F is convex.

Proposition 12.2. *i.) Let f be in $C^1(\Omega)$, then f is convex, i.e., $f(\lambda p + (1-\lambda)q) \leq \lambda f(p) + (1-\lambda) f(q)$ for all $p, q \in \mathbb{R}^n$ and $\lambda \in [0,1]$ $\iff$*

$$f(p) \geq f(q) + (f'(q), p-q) \qquad \forall p, q \in \mathbb{R}^n$$

ii.) If $f \in C^2(\Omega)$, then f is convex $\iff$ $D^2 f$ is positive semidefinite, i.e.,

$$\sum_{i,j=1}^{n} D_{ij} f(p) \xi_i \xi_j \geq 0, \qquad \forall p \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n$$

We needed for uniqueness

$$\frac{1}{2}|p|^2 = \frac{1}{2}|q|^2 + (q, p-q) + \frac{1}{2}|p-q|^2.$$

The argument would work if

$$\frac{1}{2}|p|^2 \geq \frac{1}{2}|q|^2 + (q, p-q) + \lambda |p-q|^2 \qquad \lambda > 0.$$

Now take $F(p) = \frac{1}{2}|p|^2$.

$$F(p) = F(q) + (F'(q), p-q) + \frac{1}{2}(F''(q)(p-q), p-q)$$

We know $F'(p) = p$ and $F''(p) = I$. So,

$$(F''(q)(p-q), (p-q)) = |p-q|^2.$$

For general F, this argument works if $D^2 F(q)$ is uniformly positive definite, i.e., $\exists \lambda > 0$ such that

$$(D^2 F(q)\xi, \xi) \geq \lambda |\xi|^2 \qquad \forall q \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n \tag{12.2}$$

Definition 12.3. *$f \in C^2(\Omega)$ is uniformly convex if (12.2) holds.*

Theorem 12.4. *Suppose that $f \in C^2(\Omega)$ and f is uniformly convex, i.e., there exists constants $\alpha > 0, \beta \geq 0$ such that*

$$f(p) \geq \alpha |p|^2 - \beta,$$

then the variational problem:

$$\textit{minimize } F(u) \textit{ in } H_0^1(\Omega)$$

with

$$F(u) = \int_\Omega (f(Du) - gu)\ dx$$

has a unique minimizer which solves the Euler-Lagrange equation $-div(f'(Du)) = g$ in Ω and $u = 0$ on $\partial\Omega$.

Proof:

- $F(u)$ bounded from below since

$$\begin{aligned} F(u) &= \int_\Omega (f(Du) - gu)\ dx \\ &\geq \int_\Omega (\alpha |Du|^2 - \beta - gu)\ dx \\ &\geq \frac{\alpha}{2} \int_\Omega |Du|^2\ dx - C > -\infty \end{aligned}$$

by Holder, Young and Poincare inequalities.

- weak lower semicontinuity since f is convex.
- uniqueness since f uniformly convex.
- Euler-Lagrange equations:

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} & \int_\Omega \{f(Du + \epsilon \cdot Dv) - g(u + \epsilon v)\}\ dx \\ &= \int_\Omega \left(\frac{\partial f}{\partial p_i}(Du) \cdot \frac{\partial v}{\partial x_i} - gv \right)\ dx \\ &= \int_\Omega \big(-\text{div}\, f'(Du) - g\big)\, v\ dx \qquad \forall v \in H_0^1(\Omega) \quad \blacksquare \end{aligned}$$

Recall: Given a functional

$$F(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - fu \right)\, dx,$$

we have shown the following:

- If we have weak lower semicontinuity AND
- convexity, we have existence of a minimizer.
- If we have uniform convexity, we get uniqueness of a minimizer.

i.e., given

$$F(u) = \int_\Omega \mathcal{F}(Du, u, x)\, dx,$$

$p \mapsto \mathcal{F}(p, z, x)$ is convex.

——

Euler-Lagrange equation:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(u + \epsilon v) = 0 \qquad \forall v \in H_0^1(\Omega)$$

We calculate

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(u+\epsilon v) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_\Omega \mathcal{F}(Du + \epsilon \cdot Dv, u + \epsilon v, x)\, dx \\ &= \left.\int_\Omega \left[\sum_{j=1}^n \frac{\partial \mathcal{F}}{\partial p_j}(Du, u, x) \frac{\partial v}{\partial x_j} + \frac{\partial \mathcal{F}}{\partial z}(Du, u, x) \right] v\, dx \right|_{\epsilon=0} = 0 \quad \forall v \in H_0^1(\Omega) \end{aligned}$$

(This last equality is the weak formulation of our original PDE.)

$$\begin{aligned} &\iff \int_\Omega \left[\sum_{j=1}^n -\frac{\partial}{\partial x_j} \frac{\partial}{\partial p_j}(Du, u, x) \cdot v + \frac{\partial \mathcal{F}}{\partial z}(Du, u, x) \cdot v \right] = 0 \\ &\iff \int_\Omega \left[-\mathrm{div}\, D_p \mathcal{F}(Du, u, x) + \frac{\partial \mathcal{F}}{\partial z}(Du, u, x) \right] v = 0 \\ &\iff -\mathrm{div}\, D_p F(Du, u, x) + \frac{\partial \mathcal{F}}{\partial z}(Du, u, x) = 0 \end{aligned}$$

This last line represents the strong formulation of our PDE.

If a PDE has variational form, i.e., it is the Euler-Lagrange equation of a variational problem, then we can prove the existence of a weak solution by proving that there exists a minimizer.

12.2 Integral Constraints

In this section we shall expand upon our usage of the calculus of variations to include problems with integral constraints. Before going into this, a digression will be made to prove the implicit function theorem, which will subsequently be used to prove the resulting Euler-Lagrange equation that arises from problems with constraints.

12.2.1 The Implicit Function Theorem

[Bredon] In this subsection we will prove the implicit function theorem with the goal of giving the reader a better general intuition regarding this theorem and its use. As this section is self-contained, it may be skipped if the reader feels they are already familiar with this theorem.

Theorem 12.5 (The Mean Value Theorem). *Let $f \in C^1(\mathbb{R}^n)$ with $x, \overline{x} \in \mathbb{R}^n$, then*

$$f(x) - f(\overline{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\overline{\overline{x}})(x_i - \overline{x}_i)$$

Proof: Apply the Mean Value Theorem to the function $g(t) = f(tx + (1-t)\overline{x})$ and use the Chain Rule:

$$\left.\frac{dg}{dt}\right|_{t=t_0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\overline{\overline{x}} \left.\frac{d(tx_i + (1-t)\overline{x}_i)}{dt}\right|_{t=t_0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\overline{\overline{x}})(x_i - \overline{x}_i),$$

where $\overline{\overline{x}} = t_0 x + (1-t_0)\overline{x}$. ■

Corollary 12.6. *Let $f \in C^1(\mathbb{R}^k \times \mathbb{R}^m)$ with $x \in \mathbb{R}^k$ and $y, \overline{y} \in \mathbb{R}^m$, then*

$$f(x,y) - f(x,\overline{y}) = \sum_{i=1}^{m} \frac{\partial f}{\partial y_i}(x, \overline{\overline{y}})(y_i - \overline{y}_i)$$

for some $\overline{\overline{y}}$ on the line segment between y and $\overline{y}$.

Lemma 12.7. *Let $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ be given, and let $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^m)$. Assume that $f(\xi, \eta) = \eta$ and that*

$$\left.\frac{\partial f_i}{\partial y_j}\right|_{x=\xi, y=\eta} = 0, \quad \forall i, j,$$

where x and y are the coordinates of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then there exists numbers $a > 0$ and $b > 0$ such that there exists a unique function $\phi : A \to B$, where $A = \big\{x \in \mathbb{R}^n \big| ||x - \xi|| \le a\big\}$ and $B = \big\{y \in \mathbb{R}^m \big| ||y - \eta|| \le b\big\}$, such that $\phi(\xi) = \eta$ and $\phi(x) = f(x, \phi(x))$ for all $x \in A$. Moreover, ϕ is continuous.

Proof: WLOG, assume $\xi = \eta = 0$. Applying Corollary 12.6 to each coordinate of f and using the assumption that $\partial f_i/\partial y_j = 0$ at the origin, and hence small in a neighborhood of 0, we can find $a > 0$ and $b > 0$ such that for any given constant $0 < K < 1$,

$$||f(x, y) - f(x, \overline{y})|| \le K||y - \overline{y}|| \tag{12.3}$$

for $||x|| < a$, $||y|| < b$, and $||\overline{y}|| \le b$. Moreover, we can take a to be even smaller so that we also have for all $||x|| \le a$,

$$Kb + ||f(x, 0)|| \le b \tag{12.4}$$

Now consider the set F of all functions $\phi : A \to B$ with $\phi(0) = 0$. Define the norm on F to be $||\phi - \psi||_F = \sup\big\{||\phi(x) - \psi(x)|| \big| x \in A\big\}$. This clearly generates a complete metric space. Now define $T : F \to F$ by putting $(T\phi)(x) = f(x, \phi(x))$. To prove the theorem, we must check

i) $(T\phi)(0) = 0$ and

ii) $(T\phi)(x) \in B$ for $x \in A$.

For i), we see $(T\phi)(0) = f(0, \phi(0)) = f(0, 0) = 0$. Next, we calculate

$$\begin{aligned} ||(T\phi)(x)|| &= ||f(x, \phi(x))|| \le ||f(x, \phi(x)) - f(x, 0)|| + ||f(x, 0)|| \\ &\le K \cdot ||\phi(x)|| + ||f(x, 0)|| \quad \text{by (12.3)} \\ &\le Kb + ||f(x, 0) \le b \quad \text{by (12.4)}, \end{aligned}$$

which proves ii). Now, we claim T is a contraction by computing

$$\begin{aligned} ||T\phi - T\psi||_F &= \sup_{x\in A}(||(T\phi)(x) - (T\psi)(x)||) \\ &= \sup_{x\in A}(||f(x, \phi(x)) - f(x, \psi(x))||) \\ &\le \sup_{x\in A} \theta||\phi(x) - \psi(x)|| \quad \text{by (12.3)} \\ &= \theta \cdot ||\phi - \psi||_F. \end{aligned}$$

Thus by the Banach fixed point theorem (Theorem 8.3), we see that there is a unique $\phi : A \to B$ with $\phi(0) = 0$ and $T\phi = \phi$; i.e. $f(x, \phi(x)) = \phi(x)$ for all $x \in A$. Theorem 8.3 also states that $\phi = \lim_{i\to\infty} \phi_i$, where ϕ_0 is arbitrary and $\phi_{i+1} = T\phi_i$. Put $\phi_0(x) \equiv 0$. Then ϕ_i is continuous $\forall i$. Thus, we see that ϕ is the uniform limit of continuous functions which implies ϕ is itself continuous. ■.

Theorem 12.8 (The Implicit Function Theorem). *Let $g \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ and let $\xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m$ be given with $g(\xi, \eta) = 0$. (g only needs to be defined in a neighborhood of (ξ, η).) Assume that the differential of the composition*

$$\begin{aligned} &\mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m, \\ &y \mapsto (\xi, y) \mapsto g(\xi, y), \end{aligned}$$

is onto at η. (This is a fancy way of saying the Jacobian with respect to the second argument of g, $J_y(g)$, is non-singular or $\det[J_y(g)] \neq 0$.) Then there are numbers $a > 0$ and $b > 0$ such that there exists a unique function $\phi : A \to B$ (with A, B as in Lemma 12.7), with $\phi(\xi) = \eta$, such that

$$g(x, \phi(x)) = 0 \qquad \forall x \in A.$$

(i.e. ϕ is "solves" or the implicit relation $g(x, y) = 0$). Moreover, if g is C^p, then so is ϕ (including the case $p = \infty$).

<u>Proof:</u> The differential referred to is the linear map $L : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$L_i(y) = \sum_{j=1}^{m} \frac{\partial g_i}{\partial y_i}(\xi, \eta) y_j,$$

That is, it is the linear map represented by the Jacobian matrix $(\partial g_i/\partial y_i)$ at (ξ, η). The hypothesis says that L is non-singular, and hence has a linear inverse $L^{-1} : \mathbb{R}^m \to \mathbb{R}^m$. Let

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

be defined by

$$f(x, y) = y - L^{-1}(g(x, y)).$$

Then $f(\xi, \eta) = \eta - L^{-1}(0) = \eta$. Also, computer differentials at η of $y \mapsto f(\xi, y)$ gives

$$I - L^{-1}L = I - I = 0.$$

Explicitly, this computation is as follows: Let $L = (a_{i,j})$ so that $a_{i,j} = (\partial g_i/\partial y_j)(\xi,\eta)$ and let $L^{-1} = (b_{i,j})$ so that $\sum b_{i,k}a_{k,i} = \delta_{i,j}$. Then

$$\begin{aligned}\frac{\partial f_i}{\partial y_j}(\xi,\eta) &= \delta_{i,j} - \frac{\partial}{\partial y_j}\left[\sum_{k=1}^m b_{i,k}g_k(x,y)\right]_{(\xi,\eta)} = \delta_{i,j} - \sum_{k=1}^m b_{i,k}\frac{\partial g_k}{\partial y_j}(\xi,\eta) \\ &= \delta_{i,j} - \sum_{k=1}^m b_{i,k}a_{k,j} = 0.\end{aligned}$$

Applying Lemma 12.7 to get a, b, and ϕ with $\phi(\xi) = \eta$ and $f(x,\phi(x)) = \phi(x)$, we see that

$$\phi(x) = f(x,\phi(x)) = \phi(x) - L^{-1}(g(x,\phi(x))),$$

which is equivalent to $g(x,\phi(x)) = 0$.

Now we must show that ϕ is differentiable. Since the determinant of the Jacobian $J_y(g) \neq 0$ at (ξ,η), it is nonzero in a neighborhood, say $A\times B$. To show that ϕ is differentiable at a point $x \in A$, we can assume $x = \xi = \eta = 0$ WLOG. With this assumption, we apply the Mean Value Theorem (Theorem 12.5) to $g(x,y)$, $g(0,0) = 0$:

$$\begin{aligned} 0 &= g_i(x,\phi(x)) = g_i(x,\phi(x)) - g_i(0,0) \\ &= \sum_{j=1}^n \frac{\partial g_i}{\partial x_j}(p_i,q_i)\cdot x_j + \sum_{k=1}^m \frac{\partial g_i}{\partial y_k}(p_i,q_i)\cdot \phi_k(x),\end{aligned}$$

where (p_i,q_i) is some point on the line segment between $(0,0)$ and $(x,\phi(x))$. Let $h^{(j)}$ denote the point $(0,0,\ldots,h,0,\ldots,0)$, with the h in the jth place, in $\mathbb{R}^n$. Then, putting $h^{(j)}$ in place of x in the above equation and dividing by h, we get

$$0 = \frac{\partial g_i}{\partial x_j}(p_i,q_i) + \sum_{k=1}^m \frac{\partial g_i}{\partial y_k}(p_i,q_i)\frac{\phi_k(h^{(j)})}{h}.$$

For j fixed, $i = 1,\ldots,m$ and $k = 1,\ldots,m$ these are m linear equations for the m unknowns

$$\frac{\phi_k(h^{(j)})}{h} = \frac{\phi_k(h^{(j)}) - \phi_k(0)}{h}$$

and they can be solved since the determinant of the Jacobian matrix is nonzero in $A\times B$. The solution (Cramer's Rule) has a limit as $h \to 0$. Thus,

$(\partial\phi_k/\partial x_j)(0)$ exists and equals this limit.

We know that ϕ is differentiable (once) and in a neighborhood $A \times B$ of (ξ, η) and thus, we can apply standard calculus to compute the derivative of the equations $g(x, \phi(x)) = 0$. The Chain Rule gives

$$0 = \frac{\partial g_i}{\partial x_j}(x, \phi(x)) + \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(x, \phi(x)) \frac{\phi_k}{x_j}(x).$$

Again, these are linear equations with non-zero determinant near (ξ, η) and hence has, by Cramer's Rule, a solution of the form

$$\frac{\partial \phi_k}{\partial x_j}(x) = F_{k,j}(x, \phi(x)),$$

where $F_{k,j}$ is C^{p-1} when g is C^p. If ϕ is C^r for $r < p$, then the RHS of this equation is also C^r. Thus, the LHS $\partial\phi_k/\partial x_j$ is C^r and hence the ϕ_k are C^{r+1}. By induction, the ϕ_k are C^p. Consequently, ϕ is C^p when g is C^p, as claimed. ■

12.2.2 Nonlinear Eigenvalue Problems

[Evans] Now we are ready to investigate problems with *integral constraints*. As in the previous section, we will first consider a concrete example. So, let us look at the problem of minimizing the energy functional

$$I(w) = \frac{1}{2} \int_\Omega |Dw|^2 \, dx$$

over all functions w with, say, $w = 0$ on $\partial\Omega$, but subject now also to the side condition that

$$J(w) = \int_\Omega G(w) \, dx = 0,$$

where $G : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.

We will henceforth write $g = G'$. Assume now

$$|g(z)| \leq C(|z| + 1)$$

and so

$$|G(z)| \leq C(|z|^2 + 1) \qquad (\forall z \in \mathbb{R}) \tag{12.5}$$

for some constant C.

Now let us introduce an appropriate admissible space

$$\mathcal{A} = \left\{ w \in H_0^1(\Omega) \middle| J(w) = 0 \right\}.$$

We also make the standard assumption that the open set Ω is bounded, connected and has a smooth boundary.

Theorem 12.9 (Existence of constrained minimizer). *Assume the admissible space $\mathcal{A}$ is nonempty. Then $\exists u \in \mathcal{A}$ satisfying*

$$I(u) = \min_{w \in \mathcal{A}} I(w)$$

Proof: As usual choose a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset \mathcal{A}$ with

$$I(u_k) \to m = \inf_{w \in \mathcal{A}} I(w).$$

Then as in the previous section, we can extract a weakly converging subsequence

$$u_{k_j} \rightharpoonup u \quad \text{in } H_0^1(\Omega),$$

with $I(u) \le m$. All that we need to show now is

$$J(u) = 0,$$

which shows that $u \in \mathcal{A}$. Utilizing the fact that $H_0^1(\Omega) \subset\subset\hookrightarrow L^2(\Omega)$, we extract another subsequence of u_{k_j} such that

$$u_{k_j} \to u \quad \text{in } L^2(\Omega),$$

where we have relabeled u_{k_j} to be this new subsequence. Consequently,

$$\begin{aligned} |J(u)| &= |J(u) - J(u_k)| \le \int_\Omega |G(u) - G(u_k)|\, dx \\ &\le C \int_\Omega |u - u_k|(1 + |u| + |u_k|)\, dx \quad \text{by (12.5)} \\ &\to 0 \text{ as } k \to \infty. \qquad \blacksquare \end{aligned}$$

A more interesting topic than the existence of the constrained minimizers is an examination of the corresponding Euler-Lagrange equation.

Theorem 12.10 (Lagrange multiplier). *Let $u \in \mathcal{A}$ satisfy*

$$I(u) = \min_{w \in \dashv} I(w).$$

Then $\exists \lambda \in \mathbb{R}$ such that

$$\int_\Omega Du \cdot Dv \, dx = \lambda \int_\Omega g(u) \cdot v \, dx \tag{12.6}$$

$\forall v \in H_0^1(\Omega)$.

Remark: Thus u is a weak solution of the nonlinear BVP

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda g(u) & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{12.7}$$

where λ is the *Lagrange Multiplier* corresponding to the integral constraint

$$J(u) = 0.$$

A problem of the form (12.7) for the unknowns (u, λ), with $u \not\equiv 0$, is a *nonlinear eigenvalue problem.*

Proof of 12.10:

Step 1) Fix any function $v \in H_0^1(\Omega)$. Assume first

$$g(u) \neq 0 \text{ a.e. in } \Omega. \tag{12.8}$$

Choose then any function $w \in H_0^1(\Omega)$ with

$$\int_\Omega g(u) \cdot w \, dx \neq 0; \tag{12.9}$$

this is possible because of (12.8). Now write

$$\begin{aligned} j(\tau, \sigma) &= J(u + \tau v + \sigma w) \\ &= \int_\Omega G(u + \tau v + \sigma w) \, dx \quad (\tau, \sigma \in \mathbb{R}). \end{aligned}$$

Clearly

$$j(0,0) = \int_\Omega G(u) \, dx = 0.$$

In addition, j is C^1 and

$$\frac{\partial j}{\partial \tau}(\tau, \sigma) = \int_\Omega g(u + \tau v + \sigma w) \cdot v \, dx, \tag{12.10}$$

$$\frac{\partial j}{\partial \sigma}(\tau, \sigma) = \int_\Omega g(u + \tau v + \sigma w) \cdot w \, dx, \tag{12.11}$$

Consequently, (12.9) implies

$$\frac{\partial j}{\partial \sigma}(0, 0) \neq 0.$$

According to the Implicit Function Theorem, $\exists \phi \in C^1(\mathbb{R})$ such that

$$\phi(0) = 0$$

and

$$j(\tau, \phi(\tau)) = 0 \tag{12.12}$$

for all sufficiently small τ, say $|\tau| \leq \tau_0$. Differentiating, we discover

$$\frac{\partial j}{\partial \tau}(\tau, \phi(\tau)) \tfrac{\partial j}{\partial \sigma}(\tau, \phi(\tau))\phi'(\tau) = 0;$$

whence (12.10) and (12.11) yield

$$\phi'(0) = -\frac{\displaystyle\int_\Omega g(u) \cdot v \, dx}{\displaystyle\int_\Omega g(u) \cdot w \, dx}. \tag{12.13}$$

Step 2) Now set

$$w(\tau) = \tau v + \phi(\tau) w \quad (|\tau| \leq \tau_0)$$

and write

$$i(\tau) = I(u + w(\tau)).$$

Since (12.12) implies $J(u + w(\tau)) = 0$, we see that $u + w(\tau) \in \mathcal{A}$. So the C^1 function $i(\,\cdot\,)$ has a minimum at 0. Thus

$$\begin{aligned} 0 = i'(0) &= \left. \int_\Omega (Du + \tau Dv + \phi(\tau) Dw) \cdot (Dv + \phi'(\tau) Dw) \, dx \right|_{\tau=0} \\ &= \int_\Omega Du \cdot (Dv + \phi'(0) Dw) \, dx. \end{aligned} \tag{12.14}$$

Recall now (12.13) and *define*

$$\lambda = \frac{\int_\Omega Du \cdot Dw\, dx}{\int_\Omega g(u) \cdot w\, dx},$$

to deduce from (12.14) the desired equality

$$\int_\Omega Du \cdot Dv\, dx = \lambda \int_\Omega g(u) \cdot v\, dx \quad \forall v \in H_0^1(\Omega).$$

Step 3) Suppose now instead (12.8) that

$$g(u) = 0 \quad \text{a.e. in } \Omega.$$

Approximating g by bounded functions, we deduce $DG(u) = g(u) \cdot Du = 0$ a.e. Hence, since Ω is connected, $G(u)$ is constant a.e. It follows that $G(u) = 0$ a.e., because

$$J(u) = \int_\Omega G(u)\, dx = 0.$$

As $u = 0$ on $\partial\Omega$ in the trace sense, it follows that $G(0) = 0$.

But then $u = 0$ a.e. as otherwise $I(u) > I(0) = 0$. Since $g(u) = 0$ a.e., the identity (12.6) is trivially valid in this case, for any λ. ■

12.3 Uniqueness of Solutions for the PDE

Warning: In general, uniqueness of minimizers does not imply uniqueness of weak solutions for the Euler-Lagrange equation. However, if the joint mapping $(p, z) \mapsto \mathcal{F}(p, z, x)$ is convex for $x \in \Omega$, then uniqueness of minimizers implies uniqueness of weak solutions.
Reason: In this case, weak solutions are automatically minimizers. Suppose u is a weak solution of

$$-\operatorname{div} \frac{\partial \mathcal{F}}{\partial p} + \frac{\partial \mathcal{F}}{\partial z} = 0 \qquad u = g \text{ on } \partial\Omega$$

$(p, z) \mapsto \mathcal{F}(p, z, x)$ is convex $\forall x \in \Omega$), i.e.,

$$\mathcal{F}(q, y, x) \geq \mathcal{F}(p, z, x) + D_p\mathcal{F}(p, z, x) \cdot (q - p) + D_z\mathcal{F}(p, z, x) \cdot (y - z)$$

(the graph of $\mathcal{F}$ lies above the tangent plane).
Suppose that $u = g$ on $\partial\Omega$. Choose $p = Du$, $z = u$, $q = Dw$ and $y = w$.

$$\mathcal{F}(Dw, w, x) \geq \mathcal{F}(du, u, x) + D_p\mathcal{F}(Du, u, x) \cdot (Dw - Du) + D_z\mathcal{F}(Du, u, x) \cdot (w - u)$$

Integrate in Ω:

$$\begin{aligned}\int_\Omega \mathcal{F}(Dw, w, x)\ dx \quad &\geq \quad \int_\Omega \mathcal{F}(Du, u, x)\ dx \\ &\qquad + \underbrace{\int_\Omega [D_p\mathcal{F}(Du, u, x) \cdot (Dw - Du) + D_z\mathcal{F}(Du, u, x) \cdot (w - u)]\ dx}_{\substack{= 0 \text{ since } w - v = 0 \text{ on } \partial\Omega \text{ and } u \text{ is a} \\ \text{weak sol. of the Euler Lagrange eq.}}}\end{aligned}$$

$$\Longrightarrow F(w) = \int_\Omega \mathcal{F}(Dw, w, x)\ dx \geq \int_\Omega \mathcal{F}(Du, u, x)\ dx = F(u)$$

12.4 Energies

<u>Ideas:</u> Multiply by u, u_t, integrate in space/space-time and try to find non-negative quantities. Typical terms in energies:

$$\int_\Omega |Du|^2\ dx, \ \int_\Omega |u|^2\ dx, \ \int_\Omega |u_t|^2\ dx.$$

If you take $\frac{dE}{dt}$, you should find a term that allows you to use the original PDE:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u_t(x,t)|^2\ dx = \int_\Omega u_t(x,t) \cdot u_{tt}(x,t)\ dx$$

is good for hperbolic equations, and

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u|^2\ dx = \int_\Omega u(x,t) \cdot u_t(x,t)\ dx$$

is good for parabolic equations.

12.5 Weak Formulations

Typically: integral identities

- test functions

- contain less derivatives than the original PDE in the strong form.
- multiply strong form by a test function and integrate by parts

Examples:

1) $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

$$\int_\Omega -(\Delta u)\ dx = \int_\Omega fv\ dx \iff \int_\Omega Du \cdot Dv\ dx = \int_\Omega fv\ dx \quad \forall v \in H_0^1(\Omega).$$

So, we should have:

Weak formulation + Regularity $\implies$ Strong formulation, e.g.,
$u \in H_0^1(\Omega) \cap H^2(\Omega)$ + weak solution $\implies -\Delta u = f$ in Ω.

– Dirichlet conditions: Enforced in the sense that we require u to have correct boundary data, e.g.,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = g & \text{on } \partial\Omega \end{cases}$$

seek $u \in H^1(\Omega)$ with $u = g$ on $\partial\Omega$.

– Neumann conditions: Natural boundary conditions, they should arise as a consequence of the weak formulations.

Weak formulation:

$$\int_\Omega Du \cdot Dv\ dx = \int_\Omega fv\ dx \qquad \forall v \in H^1(\Omega).$$

Now suppose $u \in H_0^1(\Omega) \subset H^1(\Omega) \implies -\Delta u = f$ in Ω.
Now take $u \in H^1(\Omega)$

$$\int_\Omega Du \cdot Dv\ dx = \int_\Omega fv\ dx.$$

Integrate by parts:

$$\begin{aligned} & -\int_\Omega \Delta u \cdot v\ dx + \int_{\partial\Omega} (Du \cdot \nu)v\ dS = \int_\Omega fv\ dx \\ \implies & \int_{\partial\Omega} (Du \cdot \nu)v\ dS = 0 \qquad \forall v \in H^1(\Omega) \\ \implies & Du \cdot \nu = 0 \text{ on } \partial\Omega \end{aligned}$$

2) Now we consider a problem with split BCs.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ Du \cdot \nu = g & \text{on } \Gamma_1 \end{cases}$$

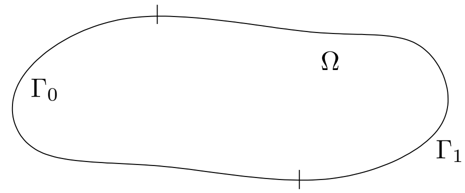


Figure 12.2:

Ω smooth, bounded, connected, Γ_0 and Γ_1 have positive surface measure and $\partial\Omega = \Gamma_0 \cup \Gamma_1$.

Define $H(\Omega) = \{\phi \in H^1(\Omega), \text{ s.t. } \phi = 0 \text{ on } \Gamma_0\}$.

$$\int_\Omega -\Delta u \cdot \phi \, dx = \int_\Omega f\phi \, dx \qquad \forall \phi \in H(\Omega)$$

$$\|$$

$$\int_\Omega Du \cdot D\phi \, dx - \int_{\partial\Omega = \Gamma_0 \cup \Gamma_1} (Du \cdot \nu) \underbrace{\phi}_{\substack{= \ 0 \\ \text{on } \Gamma_0}} dS$$

$$= \int_\Omega Du \cdot D\phi \, dx - \int_{\Gamma_1} g\phi \, dS$$

So for the weak formulation, we seek $u \in H(\Omega)$ such that

$$\int_\Omega Du \cdot D\phi \, dx = \int_{\Gamma_1} g\phi \, dS + \int_\Omega f\phi \, dx \qquad \forall \phi \in H(\Omega)$$

<u>Check:</u> Weak solution + Regularity $\Longrightarrow$ Strong solution.

Dirichlet data okay, $u \in H(\Omega) \Longrightarrow u = 0$ on Γ_0.

$$\begin{aligned} \int_\Omega Du \cdot D\phi \, dx &= -\int_\Omega \Delta u \cdot \phi \, dx + \int_{\partial\Omega} (Du \cdot \nu)\phi \, dS \\ &= \int_{\Gamma_1} g\phi \, dS + \int_\Omega f\phi \, dx \qquad \forall \phi \in H(\Omega) \end{aligned}$$

Now, taking $\phi \in H_0^1(\Omega) \subset H(\Omega)$, we see

$$\int_\Omega (-\Delta u - f)\phi \, dx = 0 \qquad \forall \phi \in H_0^1(\Omega)$$

$$\Longrightarrow \quad -\Delta u - f = 0 \text{ in } \Omega \iff -\Delta u = f \Longrightarrow \text{ PDE Holds.}$$

The volume terms cancel, and we get

$$\int_{\partial\Omega}(Du\cdot\nu)\phi\;dS = \int_{\Gamma_1} g\phi\;dS \qquad \forall\phi\in H(\Omega)$$

$$\|$$

$$\int_{\Gamma_1}(Du\cdot\nu)\phi\;dS = \int_{\Gamma_1} g\phi\;dS \Longrightarrow Du\cdot\nu = g.$$

Where we have thus recovered the Neumann BC.

12.6 A General Existence Theorem

Goal: Existence of a minimizer for

$$F(u) = \int_\Omega L(Du,u,x)\;dx$$

with $L:\mathbb{R}^n\times\mathbb{R}\times\Omega\to\mathbb{R}$ smooth and bounded from below.

Note:

$$L(p,z,x) = \frac{1}{2}|p|^2 - f(x)\cdot z \qquad f\in L^2(\Omega)$$

doesn't fit into this framework.

Theorem 12.11. *Suppose that $p\mapsto L(p,z,x)$ is convex. Then $F(\cdot)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$, $1<p<\infty$.*

Idea of Proof: Have to show: $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$, then $\liminf_{k\to\infty} F(u_k)\geq F(u)$. True if $\liminf_{k\to\infty} F(u_k)=\infty$.

Suppose $M=\liminf_{k\to\infty} F(u_k)<\infty$, and that $\liminf = \lim$ (subsequence, not relabeled).

$$u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega) \Longrightarrow ||u_k||_{W^{1,p}(\Omega)} \text{ bounded}$$

Compact Sobolev embedding: without loss of generality $u_k\to u$ in $L^p(\Omega)$ $\Longrightarrow$ convergence in measure $\Longrightarrow$ further subsequence: $u_k\to u$ pointwise a.e. Egoroff: exists E_ϵ such that $u_k\to u$ uniformly on E_ϵ, $|\Omega\setminus E_\epsilon|<\epsilon$.

Define F_ϵ by

$$F_\epsilon = \left\{x\in\Omega : |u(x)|+|Du(x)|\leq\frac{1}{\epsilon}\right\}$$

Claim: $|\Omega \setminus F_\epsilon| \to 0$. Fix $K \left(= \frac{1}{\epsilon}\right)$. Then,

$$\begin{aligned}
|\{|u(x)| > K\}| &\leq \frac{1}{K}\int_\Omega |u(x)|\ dx \leq \frac{1}{K}\int_\Omega |u(x)|\ dx \\
&\stackrel{\text{Hölder}}{\leq} \frac{1}{K}\left(\int_\Omega 1\ dx\right)^{\frac{1}{p'}}\left(\int_\Omega |u(x)|^p\ dx\right)^{\frac{1}{p}} \\
&\leq \frac{C}{K}
\end{aligned}$$

Define good points $G_\epsilon = F_\epsilon \cap E_\epsilon$, then $|\Omega \setminus G_\epsilon| \to 0$ as $\epsilon \to 0$.

By assumption, L bounded from below $\Longrightarrow$ may assume that $L \geq 0$.

$$\begin{aligned}
F(u_k) &= \int_\Omega L(Du_k, u_k, x)\ dx \\
&\geq \int_{G_\epsilon} L(Du_k, u_k, x)\ dx \\
\text{(by convex. of } L\text{)} \quad &\geq \underbrace{\int_{G_\epsilon} L(Du, u_k, x)\ dx}_{:=I_k} + \underbrace{\int_{G_\epsilon} D_pL(Du, u_k, x)\cdot(Du_k - Du)\ dx}_{:=J_k}
\end{aligned}$$

Since $L(q,z,x) \geq L(p,z,x) + L_p(p,z,x)\cdot(p-q)$. Now

$$I_k \to \int_{G_\epsilon} L(Du, u, x)\ dx \text{ uniformly}$$

and

$$J_k = \int_{G_\epsilon} \underbrace{\underbrace{(D_pL(Du, u_k, x) - D_pL(Du, u, x))}_{\to 0 \text{ uniformly}}\cdot(Du_k - Du)\ dx}_{\to 0} + \underbrace{\int_{G_\epsilon} D_pL(Du, u, x)\cdot(Du_k - Du)\ dx}_{\to 0}.$$

So, we have

$$\lim_{k\to\infty}\inf F(u_k) \geq \int_{G_\epsilon} L(Du, u, x)\ dx.$$

Now, we use monotone convergence for $\epsilon \searrow 0$ to get

$$\lim_{k\to\infty}\inf F(u_k) \geq \int_\Omega L(Du, u, x)\ dx = F(u) \qquad \blacksquare$$

Recall the assumption:

$$F(u) = \int_\Omega L(Du, u, x)\ dx, \qquad p \mapsto L(p,z,x) \text{ is convex.}$$

Theorem 12.12. *Suppose that L satisfies the coercivity assumption $L(p,z,x) \geq \alpha|p|^q - \beta$ with $\alpha > 0$, $\beta \geq 0$ and $1 < q < \infty$. Suppose in addition that L is convex in p for all z, x. Also, suppose that the class of admissable functions*

$$\mathcal{A} = \{w \in W^{1,q}(\Omega), w(x) = g(x) \text{ on } \partial\Omega\}$$

is not empty. Lastly, assume that $g \in W^{1,q}(\Omega)$. If these conditions are met, then there exists a minimizer of the variational problem:

$$\text{Minimize} \quad F(u) = \int_\Omega L(Du, u, x)\, dx \text{ in } \mathcal{A}$$

——

Remark: It's reasonable to assume that there exists a $w \in \mathcal{A}$ with $F(w) < \infty$. Typically one assumes in addition that L satisfies a growth condition of the form

$$L(p,z,x) \leq C(|p|^q + 1)$$

This ensures that F is finite on $\mathcal{A}$.

——

Idea of Proof: If $F(w) = +\infty$ $\forall w \in \mathcal{A}$, then there is nothing to show. Thus, suppose $F(w) < \infty$ for at least on $w \in \mathcal{A}$.

Given $L(p,z,x) \geq \alpha|p|^q - \beta$, we see that if $F(w) < \infty$, then

$$\begin{aligned} \int_\Omega L(Du,u,x)\, dx &\geq \int_\Omega (\alpha|Du|^q - \beta)\, dx \\ &\geq \alpha \int_\Omega |Du|^q\, dx - \beta \cdot |\Omega| \end{aligned} \tag{12.15}$$

$\Longrightarrow F(w) \geq -\beta \cdot |\Omega|$ for $w \in \mathcal{A} \Longrightarrow F$ is bounded from below:

$$-\infty < M = \inf_{w \in \mathcal{A}} F(w) < \infty.$$

Choose a minimizing sequence w_k such that $F(w_k) \to M$ as $k \to \infty$. Then by (12.15):

$$\alpha \int_\Omega |Dw_k|^q\, dx \leq F(w_k) + \beta \cdot |\Omega| \leq M + 1 + \beta \cdot |\Omega|$$

if k is large enough.

Now since $w_k \in \mathcal{A}$, we know $w_k = g$ on $\partial\Omega$. Thus,

$$\begin{aligned}
||w_k||_{L^q(\Omega)} &= ||w_k - g + g||_{L^q(\Omega)} \\
&\leq ||w_k - g||_{L^q(\Omega)} + ||g||_{L^q(\Omega)} \\
&\overset{\text{Poincaré}}{\leq} C_p||Dw_k - Dg||_{L^q(\Omega)} + ||g||_{L^q(\Omega)} \\
&\leq C_p||Dw_k||_{L^q(\Omega)} + C_p||Dg||_{L^q(\Omega)} + ||g||_{L^q(\Omega)} \\
&\leq C < \infty \qquad \forall k
\end{aligned}$$

$\Longrightarrow w_k$ is a bounded sequence in $W^{1,q}(\Omega)$
$\Longrightarrow w_k$ contains a weakly converging subsequence (not relabeled), $w_k \rightharpoonup w$ in $W^{1,q}(\Omega)$.
Weak lower semicontinuity result: Theorem 12.5 applies and

$$M \leq F(w) \leq \lim_{k\to\infty} \inf F(w_k) \to M = \inf_{v\in\mathcal{A}} F(v).$$

If $w \in \mathcal{A}$ then

$$F(w) = M = \inf_{v\in\mathcal{A}} F(v)$$

$\Longrightarrow w$ is a minimizer.
To show this, we apply the following theorem:

Theorem 12.13 (Mazur's Theorem). *X is a BS, and suppose that $x_n \rightharpoonup x$ in X. Then for $\epsilon > 0$ given, there exists a convex combination $\sum_{j=1}^k \lambda_j x_j$ of elements $x_j \in \{x_n\}_{n\in\mathbb{N}}$, $\lambda_j \geq 0$, $\sum_{j=1}^k \lambda_j = 1$, such that*

$$\left|\left|x - \sum_{j=1}^k \lambda_j x_j\right|\right| < \epsilon.$$

In particular, there exists a sequence of convex combinations that converges strongly to x.

Proof: [Yoshida] Consider the totality M_T of all convex combinations of x_n. We can assume $0 \in M_T$ by replacing x with $x - x_1$ and x_j with $x_j - x_1$.

We go for a proof by contradiction. Suppose the contrary of the theorem, i.e., $||x - u||_X > \epsilon$ for all $u \in M_T$. Now consider the set,

$$M = \left\{ v \in X;\ ||v - u||_X \leq \frac{\epsilon}{2} \text{ for some } u \in M_T \right\},$$

which is a convex (balanced and also absorbing) neighborhood of 0 in X and $||x - v||_X > \epsilon/2$ for all $v \in M$.

Recall the Minkowski functional p_A for given set A:

$$p_A(x) = \inf_{\alpha > 0,\ \alpha^{-1} x \in A} \alpha.$$

Now, since $x \notin M$, there exists $u_0 \in M$ such that $x = \beta^{-1} u_0$ with $0 < \beta < 1$. Thus, $p_M(x)\beta^{-1} > 1$.

Consider now the real linear subspace of X:

$$X_1 = \{x \in X;\ x = \gamma \cdot u_0,\ -\infty < \gamma < \infty\}$$

and define $f_1(x) = \gamma$ for $x = \gamma \cdot u_0 \in X_1$. This real linear functional f_1 on X_1 satisfies $f_1(x) \leq p_M(x)$. Thus, by the Hahn-Banach Extension theorem, there exists a real linear extension f of f_1 defined on the real linear space X and such that $f(x) \leq p_M(x)$ on X. Since M is a neighborhood of 0, the Minkowski functional $p_M(x)$ is continuous in x. From this we see that f must also be a continuous real linear functional defined on X. Thus, we see that

$$\sup_{x \in M_T} f(x) \leq \sup_{x \in M} f(x) \leq \sup_{x \in M} p_M(x) = 1 < \beta^{-1} = f(\beta^{-1} u_0) = f(x).$$

Clearly we see that x can not be a weak accumulation point of M_1, this is a contradiction since we assumed that $x_n \rightharpoonup x$. ■

Corollary 12.14. *If Y is a strongly closed, convex set in a BS X, then Y is weakly closed.*

Proof: $x_n \rightharpoonup x \Longrightarrow$ exists a sequence of convex combinations $y_k \in Y$ such that $y_k \to x$. Y closed $\Longrightarrow x \in Y$. ■

Application to proof of 12.6: $\mathcal{A}$ is a convex set. We see that $\mathcal{A}$ is closed by the following argument. If $w_k \to w$ in $W^{1,q}(\Omega)$, then by the Trace theorem, we have

$$||w_k - w||_{L^q(\partial\Omega)} \le C_{\mathrm{Tr}}||w_k - w||_{W^{1,q}(\Omega)} \to 0$$

since $w_k = g$ on $\partial\Omega \implies w = g$ on $\partial\Omega$.

So, we apply Mazur's Theorem, to see that $\mathcal{A}$ is also weakly closed, i.e., if $w_k \rightharpoonup w$ in $W^{1,q}(\Omega)$, then $w \in \mathcal{A}$. Thus, this establishes existence by the arguments stated earlier in the proof. ■

Uniqueness of this minimizer is true if L is uniformly convex (see Evans for proof).

12.7 A Few Simple Applications to Semilinear Equations

Here we want to show how the technique used in the linear case extends to some nonlinear cases. First we show that "lower-order terms" are irrelevant for the regulairty. For example consider

$$\int_\Omega Du \cdot D\phi\, dx = \int_\Omega a(u) \cdot D\phi\, dx \quad \forall \phi \in H_0^1(\Omega) \tag{12.16}$$

If $\displaystyle\int_{B_R(x_0)} Dv \cdot D\phi\, dx = 0$ for $\forall \phi \in H_0^1(\Omega)$ and $v = u$ on ∂B_R then

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2\, dx &\le C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\, dx + C\int_{B_R(x_0)} |D(u-v)|^2\, dx \\ &\le C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2\, dx + C\int_{B_R(x_0)} |a(u)|^2\, dx.\end{aligned}$$

Now if $|a(u)| \le |u|^\alpha$, then the last integral is estimated by

$$\int_{B_R(x_0)} |a(u)|^2\, dx \le \left(\int)B_R(x_0)|u|^{2n/(n-2)}\, dx\right)^{\alpha(1-2/n)} R^{n(1-\alpha(1-2/n))}$$

so that u is Hölder continuous provided $\alpha < \frac{2}{n-2}$.

The same result holds if we have instead of (12.16)

$$\int_\Omega Du \cdot D\phi\, dx = \int_\Omega a(u) \cdot D\phi\, dx + \int_\Omega a(Du) \cdot \phi\, dx$$

with $a(Du) \leq |Du|^{1-2/n}$ or $a(Du) \leq |Du|^{2-\epsilon}$ and $|u| \leq M = \text{const}$.

The above idea works also if we study small perturbations of a system with constant coefficients:

For

$$\int_\Omega A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta \phi^j \, dx = 0$$

with

$$A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \quad \text{and} \quad \left| A_{ij}^{\alpha\beta}(v) - \delta_{\alpha\beta}\delta_{ij} \right| \leq \epsilon_0$$

one finds

$$\int_{B_\rho(x_0)} |Du|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2 \, dx + C \int_{B_R(x_0)} \underbrace{|\delta_1\delta_2 - A|}_{\leq \epsilon_0} \cdot Du^2 \, dx$$

because

$$\int_\Omega D_\alpha u^i D_\beta \phi^j \, dx = \int_\Omega (\delta_{\alpha\beta}\delta_{ij} - A_{ij}^{\alpha\beta}) D_\alpha u^i D_\beta \phi^j \, dx.$$

So we can apply the lemma from the very begining of 9.8 and conclude that Du is Hölder continuous. Finally we mention the

Theorem 12.15. *Let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ be a minimizer of*

$$\int_\Omega F(Du) \, dx$$

and let us assume that

$$\frac{F(tp)}{t^2} \to |p|^2 \quad \textit{for } t \to \infty$$

then $u \in C^{0,\mu}(\Omega)$.

<u>Proof:</u> We consider the problem

$$\int_{B_R(x_0)} |DH|^2 \, dx \to \text{minimize}, \quad H = 0 \quad \text{on } \partial B_R$$

(H stands for harmonic function!) then

$$\int_{B_\rho(x_0)} |Du|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2 \, dx + \int_{B_R(x_0)} |Du - DH|^2 \, dx.$$

Now

$$\begin{aligned}
\int_{B_R(x_0)} D(u-H)\cdot D(u-H)\,dx &= \int_{B_R(x_0)} Du\cdot D(u-H)\,dx - \int_{B_R(x_0)} DH\cdot D(u-H)\,dx \\
&= \int_{B_R(x_0)} |Du|^2\,dx - \int_{B_R(x_0)} Du\cdot DH\,dx \\
&= \int_{B_R(x_0)} |Du|^2\,dx - \int_{B_R(x_0)} (D(u-H)+DH)DH\,dx \\
&= \int_{B_R(x_0)} |Du|^2\,dx - \int_{B_R(x_0)} |DH|^2\,dx \\
&= \int_{B_R(x_0)} |Du|^2\,dx + \underbrace{\int_{B_R(x_0)} F(Du)\,dx - \int_{B_R(x_0)} F(DH)\,dx}_{\leq 0 (u \text{ is a minimizer})} \\
&\quad - \int_{B_R(x_0)} F(Du)\,dx + \int_{B_R(x_0)} F(DH)\,dx - \int_{B_R(x_0)} |DH|^2\,dx.
\end{aligned}$$

By the growth assumption on F and because

$$|p|^2 - F(p) = |p|^2\left(1 - \frac{F(p)}{|p|^2}\right)$$

we find

$$\int_{B_R(x_0)} |D(u-H)|^2\,dx \leq \epsilon \int_{B_R(x_0)} |Du|^2\,dx + C\cdot R^n.$$

So we can apply the lemma at the beginning of the section and conclude that Du is Hölder continuous.

12.8 Regularity

We discuss in this section the smoothness of minimizers to our energy functionals. This is generally a quite difficult topic, and so we will make a number of strong simplifying assumptions, most notably that L depends only on p. Thus we henceforth assume our functions $I[\cdot]$ to have the form

$$I[w] := \int_\Omega L(Dw) - wf\,dx, \tag{12.17}$$

for $f \in L^2(\Omega)$. We will also take $q = 2$, and suppose as well the growth condition

$$|D_pL(p)| \leq C(|p|+1) \quad (p \in \mathbb{R}^n). \tag{12.18}$$

Then any minimizer $u \in \mathcal{A}$ is a weak solution of the Euler-Lagrange PDE

$$-\sum_{i=1}^{n}(L_{p_i}(Du))_{x_i} = f \quad \text{in } \Omega; \tag{12.19}$$

that is,

$$\int_\Omega \sum_{i=1}^{n} L_{p_i}(Du)v_{x_i}\,dx = \int_\Omega fv\,dx \tag{12.20}$$

for all $v \in H_0^1(\Omega)$.

12.8.1 DeGiorgi-Nash Theorem

[Moser-Iteration Technique] Let u be a subsolution for the elliptic operator $-D_\beta(a_{\alpha\beta}D_\alpha)$, i.e.

$$\int_\Omega a^{\alpha\beta}D_\alpha u D_\beta \phi\,dx \le 0 \quad \forall \phi \in H_0^{1,2}(\Omega),\ \phi \ge 0$$

where $a^{\alpha\beta} \in L^\infty(\Omega)$ and satisfy an ellipticity-condition. The idea is to use as test-function $\phi := (u^+)^p\eta^2$ with $p > 1$ and η the cut-off-function we defined on page 20. For the sake of simplicity we assume that $u \ge 0$. Then

$$\begin{aligned} p\int_{B_R} a^{\alpha\beta}D_\alpha u D_\beta u u^{p-1}\eta^2\,dx + \int_{B_R} a^{\alpha\beta}D_\alpha u u^p \eta D\eta\,dx &\le 0 \\ \int_{B_R}|Du|^2 u^{p-1}\eta^2\,dx \le \frac{c}{p}\int_{B_R}|Du|u^{\frac{p-1}{2}}u^{\frac{p+1}{2}}\eta|D\eta|\,dx \\ \int_{B_R}|Du|^2u^{p-1}\eta^2\,dx \le \frac{c}{p^2}\int_{B_R}u^{p+1}|D\eta|^2\,dx. \end{aligned}$$

As

$$\left|Du^{\frac{p+1}{2}}\right|^2 = \left(\frac{p+1}{2}\right)^2 u^{p-1}|Du|^2$$

we get

$$\int_{B_R}\left|Du^{\frac{p+1}{2}}\right|^2\eta^2\,dx \le c\left(\frac{p+1}{p}\right)^2\int_{B_R}u^{p+1}|D\eta|^2\,dx$$

and together with

$$\left|D(\eta u^{\frac{p+1}{2}})\right|^2 \le c\left|Du^{\frac{p+1}{2}}\right|^2\eta^2 + u^{p+1}|D\eta|^2$$

follows

$$\begin{aligned}\int_{B_R}\left|Du^{\frac{p+1}{2}}\right|^2 dx &\le c\left(1+\left(\frac{p+1}{2}\right)^2\right)\int_{B_R}u^{p+1}|D\eta|^2\,dx \quad (p>1)\\ &\le c(p+1)^2\left(1+\frac{1}{p^2}\right)\int_{B_R}u^{p+1}|D\eta|^2\,dx.\end{aligned}$$

By Sobolev's inequality and the properties of η we finally have:

$$\left(\int_{B_R}(u^{\frac{p+1}{2}}\eta)^{2*}\,dx\right)^{2/2*}\le c(p+1)^2\left(1+\frac{1}{p^2}\right)\frac{1}{(R-\rho)^2}\int_{B_R}u^{p+1}\,dx.$$

For $p=1$, this is just Caccioppoli's inequality. Set $\lambda := 2*/2 = \frac{n}{n-2}$ and $q := p+1 > 2$, then

$$\left(\int_{B_R}u^{\lambda q}\,dx\right)^{1/\lambda}\le c\frac{(1+q)^2}{(R-\rho)^2}\int_{B_R}u^q\,dx.$$

Now we choose

$$\begin{aligned}q_i &:= 2\lambda^i=\lambda q_{i-1}\quad (q_0=2)\\ R_i &:= \frac{R}{2}+\frac{R}{2^{i+1}}\quad (R_0=R)\end{aligned}$$

and we find that

$$\begin{aligned}\left(\int_{B_{R_{i+1}}}u^{q_{i+1}}\,dx\right)^{\frac{1}{\lambda^{i+1}}} &= \left(\int_{B_{R_{i+1}}}u^{\lambda q_i}\,dx\right)^{\frac{1}{\lambda}\frac{1}{\lambda^i}}\\ &\le \left[\frac{c(1+q_i)^2}{2^{-2(i+1)}R^2}\right]^{1/\lambda^i}\left(\int_{B_{R_i}}u^{q_i}\,dx\right)^{1/\lambda^i}\\ &\le \prod_{k=0}^{i}\left[\frac{c(1+q_k)^2}{R^2 4^{-k-1}}\right]^{1/\lambda^k}\left(\int_{B_R}u^2\,dx\right)^{1/2}.\end{aligned}$$

We have to estimate the above product:

$$\begin{aligned}\prod_{k=0}^{\infty}\left[\frac{c(1+q_k)^2}{R^2 4^{-k-1}}\right]^{1/\lambda^k} &= \exp\left(\ln\prod_{k=0}^{\infty}\left[\frac{c(1+q_k)^2}{R^2}4^{-k-1}\right]^{1/\lambda^k}\right)\\ &= \exp\sum_{k=0}^{\infty}\frac{2}{\lambda^k}\ln\hat{c}\frac{(1+q_k)}{R2^{-k-1}}\\ &= \exp(\hat{c}-n\ln R)\left(q_k=2\lambda^k,\sum_{k=0}^{\infty}\lambda^{-j}=\frac{n}{2}\right)\end{aligned}$$

thus $\forall k$

$$\left(\fint_{B_{R_k}} u^{q_k}\,dx\right)^{1/q_k} \leq \hat{c}\left(\fint_{B_R} u^2\,dx\right)^{1/2}$$

from which immediately follows that

$$\sup_{x\in B_{R/2}} u(x) \leq \hat{c}\left(\fint_{B_R} |u|^2\,dx\right)^{1/2}.$$

Remark: Instead of 2, one can take any expoenent p>0 in the above formulas.

If we start with a supersolution and $u > 0$, i.e.

$$\int_\Omega a^{\alpha\beta}(x)D_\alpha u D_\beta \phi\,dx \geq 0 \quad \forall \phi \in H_0^{1,2}(\Omega), \quad \phi \geq 0$$

then we can take $p < -1$. Exactly as before we then get for $\lambda := \frac{2*}{2} > 1$ and $q := p + 1 < 0$.

$$\left(\int_{B_\rho} u^{\lambda q}\,dx\right)^{1/\lambda} \leq c(1+q^2)\left(1+\frac{1}{(q-1)^2}\right)\frac{1}{(R-\rho)^2}\int_{B_R} u^q\,dx.$$

As $|q-1| > 1$ we have

$$\left(\int_{B_\rho} u^{\lambda q}\,dx\right)^{1/\lambda} \leq c\frac{(1+q)^2}{(R-\rho)^2}\int_{B_R} u^q\,dx.$$

Now we set $R_i := \frac{R}{2} + \frac{R}{2^{i+1}}$ as before and $q_i := q_0\lambda^i$ but $q_0 < 0$. Then follows

$$\inf_{x\in B_{R/2}} u(x) \leq c\left(\fint_{B_R} u^{q_0}\,dx\right)^{1/q_0} \qquad (q_0 < 0).$$

Up to now we have the following two estimates for a positive solution u: For any $p > 0$ and any $q < 0$ holds:

$$\begin{aligned}\sup_{x\in B_{R/2}} u(x) &\leq k_1\left(\fint_{B_R} u^p\,dx\right)^{1/p}\\ \inf_{x\in B_{R/2}} u(x) &\geq k_2\left(\fint_{B_R} u^q\,dx\right)^{1/q},\end{aligned}$$

for some constants k_1 and k_2, $q < 0$. The main point now is that the John-Nirenberg-lemma allows us to combine the two inequalities to get

Theorem 12.16 (Harnack's Inequality). *If u is a solution and $u > 0$, then*

$$\inf_{x \in B_{R/2}} u(x) \geq C \sup_{x \in B_{R/2}} u(x).$$

Proof: We have

$$\int_\Omega a^{\alpha\beta} D_\alpha u D_\beta \phi \, dx = 0 \quad \forall \phi \in H_0^{1,2}(\Omega) \quad \phi > 0.$$

As $u > 0$, we can choose $\phi := \frac{1}{u}\eta^2$ and get

$$-\int_\Omega a^{\alpha\beta} D_\alpha u D_\beta u \frac{1}{u^2}\eta^2 \, dx + \int_\Omega a^{\alpha\beta} D_\alpha u \frac{1}{u} \eta D\eta \, dx = 0$$

thus

$$\int_\Omega \frac{|Du|^2}{u^2}\eta^2 \, dx \leq \int_\Omega |D\eta|^2 \, dx,$$

i.e.

$$\int_{B_R} |D \ln u|^2 \, dx \leq C R^{n-2}$$

which implies by the Poincaré's inequality that $\ln u \in BMO$. Now we use the characerization of BMO by which there exists a constant α, such that for $v := e^{\alpha \ln u}$, we have

$$\int_{B_R} v \, dx \leq C \left(\int_{B_R} \frac{1}{v} \, dx \right)^{-1}.$$

Hence $(p = 1, q = -1)$,

$$\inf_{B_{R/2}} u \geq K_2 \left(\fint_{B_R} u^{-a} \, dx \right)^{-1/\alpha} \geq K_2 C^{-1} \left(\fint_{B_R} u^\alpha \, dx \right)^{1/\alpha} \geq \frac{K_2}{K_1 C} \sup_{B_{R/2}} u. \blacksquare$$

Remark: It follows from Harnack's inequality that a solution of the equation at the beginning of this section is Hölder-continuous: $M(2R) - u \geq 0$ is a supersolution, thus

$$M(2R) - m(R) \leq C(M(2R) - M(R))$$

$u - m(2R) \geq 0$ is also a supersolution, thus

$$M(R) - m(2R) \leq C(m(R) - m(2R))$$

and from this we get

$$\omega(2R) + \omega(R) \leq C(\omega(2R) - \omega(R))$$

hence $\omega(R) \leq \frac{C+1}{C-1}\omega(2R)$ and u is Hölder-continuous.

12.8.2 Second Derivative Estimates

We now intend to show if $u \in H^1(\Omega)$ is a weak solution of the nonlinear PDE (12.19), then in fact $u \in H^2_{\text{loc}}(\Omega)$. But to establish this, we will need to strengthen our growth condition on L. Let us first of all suppose

$$|D^2L(p)| \leq C \quad (p \in \mathbb{R}^n). \tag{12.21}$$

In addition, let us assume that L is uniformly convex, and so there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} L_{p_ip_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \quad (p, \xi \in \mathbb{R}^n). \tag{12.22}$$

Clearly, this is some sort of nonlinear analogue of our uniform ellipticity condition for linear PDE. The idea will therefore be to try to utilize, or at least mimic, some of the calculations from that chapter.

Theorem 12.17 (Second derivatives for minimizers). *(i) Let $u \in H^1(\Omega)$ be a weak solution of the nonlinar PDE (12.19), where L statisfies (12.21), (12.22). Then*

$$u \in H^2_{loc}(\Omega).$$

(ii) If in addition $u \in H^1_0(\Omega)$ and $\partial\Omega$ is C^2, then

$$u \in H^2(\Omega),$$

with the estimate

$$||u||_{H^2(\Omega)} \leq C \cdot ||f||_{L^2(\Omega)}$$

Proof:

1. We will largely follow the proof of blah blah, the corresponding assertion of local H^2 regularity for solutions of linear second-order elliptic PDE.

   Fix any open set $V \Subset \Omega$ and choose then an open set W so that $V \Subset W \Subset \Omega$. Select a smooth cutoff function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } V, \ \zeta \equiv 0 \text{ in } \mathbb{R}^n \setminus W \\ 0 \leq \zeta \leq 1 \end{cases}$$

Let $|h| > 0$ be small, choose $k \in \{1, \dots, n\}$, and substitute

$$v := -D_k^{-h}(\zeta^2 D_k^h u)$$

into (12.20). We are employing her the notion from the difference quotients section:

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \quad (x \in W).$$

Using the identity $\int_\Omega u \cdot D_k^{-h} v \, dx = -\int_\Omega v \cdot D_k^h u \, dx$, we deduce

$$\sum_{i=1}^n \int_\Omega D_k^h(L_{p_i}(Du)) \cdot (\zeta^2 D_k^h u)_{x_i} \, dx = -\int_\Omega f \cdot D_k^{-h}(\zeta^2 D_k^h u) \, dx \quad (12.23)$$

Now

$$\begin{aligned} D_k^h L_{p_i}(Du(x)) &= \frac{L_{p_i}(Du(x + he_k)) - L_{p_i}(Du(x))}{h} \\ &= \frac{1}{h} \int_0^1 \frac{d}{ds} L_{p_i}(s \cdot Du(x + he_k) + (1 - s) \cdot Du(x)) \, ds \\ &\qquad (u_{x_j}(x + he_k) - u_{x_j}(x)) \\ &= \sum_{j=1}^n a_{ij}^h(x) \cdot D_k^h u_{x_j}(x), \qquad (12.24) \end{aligned}$$

for

$$a_{ij}^h(x) := \int_0^1 L_{p_i p_j}(s \cdot Du(x + he_k) + (1 - s) \cdot Du(x)) \, dx \quad (i, j = 1, \dots, n) \quad (12.25)$$

We substitute (12.24) into (12.23) and perform simplce calculations, to arrive at the identity:

$$\begin{aligned} A_1 + A_2 &:= \sum_{i,j=1}^n \int_\Omega \zeta^2 a_{ij}^h D_k^h u_{x_j} D_k^h u_{x_i} \, dx \\ &\qquad + \sum_{i,j=1}^n \int_\Omega a_{ij}^h D_k^h u_{x_j} D_k^h u \cdot 2\zeta \zeta_{x_i} \, dx \\ &= -\int_\Omega f \cdot D_k^{-h}(\zeta^2 D_k^h u) \, dx =: B. \end{aligned}$$

Now the uniform convexity condition (12.22) implies

$$A_1 \geq \theta \int_\Omega \zeta^2 |d_k^h Du|^2 \, dx.$$

Furthermore we see from (12.21) that

$$\begin{aligned} |A_2| &\leq C \int_W \zeta |D_k^h Du| \cdot |D_k^h u| \, dx \\ &\leq \epsilon \int_W \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\epsilon} \int_W |D_k^h u|^2 \, dx. \end{aligned}$$

Furthermore, as in the proof of interior H^2 regularity of elliptic PDE, we have

$$|B| \leq \epsilon \int_\Omega \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\epsilon} \int_\Omega f^2 + |Du|^2 \, dx.$$

We select $\epsilon = \frac{\theta}{4}$, to deduce from the foregoing bounds on A_1, A_2, B, the estimate

$$\int_\Omega \zeta^2 |D_k^h Du|^2 \, dx \leq C \int_W f^2 + |D_k^h u|^2 \, dx \leq C \int_\Omega f^2 + |Du|^2 \, dx,$$

the last inequaltiy valid from the properties of difference quotients.

2. Since $\zeta \equiv 1$ on V, we find

$$\int_V |D_k^h Du|^2 \, dx \leq C \int_\Omega f^2 + |Du|^2 \, dx$$

for $k = 1, \ldots, n$ and all sufficiently small $|h| > 0$. Consequently prop of diff. quotients imply $Du \in H^1(V)$, and so $u \in H^2(V)$. This is true for each $V \Subset U$; thus $u \in H^2_{\text{loc}}(\Omega)$.

3. If $u \in H^1_0(\Omega)$ is a weak solution of (12.19) and ∂U is C^2, we can then mimic the proof of the boundary regularity theorem 4 of elliptic pde to prove $u \in H^2(\Omega)$, with the estimate

$$||u||_{H^2(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||u||_{H^1(\Omega)});$$

details are left to the reader. Now from (12.22) follows the inequality

$$(DL(p) - DL(0)) \cdot p \geq \theta |p|^2 \quad (p \in \mathbb{R}^n).$$

If we then put $v = u$ in (12.20), we can employ this estimate to derive the bound

$$||u||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)},$$

and so finish the proof. ∎

12.8.3 Remarks on Higher Regularity

We would next like to show that if L is infinitely differentiable, then so is u. By analogy with the regularity theory developed for second-order linear elliptic PDE in 6.3, it may seem natural to try to extend the H^2_{loc} estimate from teh previous section to obtain further estimates in higher order Sobolev spaces $H^k_{\text{loc}}(\Omega)$ for $k = 3, 4, \ldots$

This mehtod will not work for the nonlinear PDE (12.19) however. The reason is this. For linear equations we could, roughly speaking, differntiate the equation many times and still obtain a linear PDE of the same general form as that we began with. See for instance the proof of theorem 2 6.3.1. But if we differentiate a nonlinear differential equaiton many times, the resulting increasingly complicated expressions quickly becvome impossible to handle. Much deeper ideas are called for, the full development of which is beyond the scope of this book. We will nevertheless at least outline the basic plan.

To start with, choose a test function $w \in C^\infty_c(\Omega)$, select $k \in \{1, \ldots, n\}$, and set $v = -w_{x_k}$ in the identity (12.20), where for simplicity we now take $f \equiv 0$. Since we now know $u \in H^2_{\text{loc}}(\Omega)$, we can integrate by parts to find

$$\int_\Omega \sum_{i,j=1}^n L_{p_i p_j}(Du) u_{x_k x_j} w_{x_i}\, dx = 0. \tag{12.26}$$

Next write

$$\tilde{u} := u_{x_k} \tag{12.27}$$

and

$$a^{ij} := L_{p_i p_j}(Du) \quad (i, j = 1, \ldots, n). \tag{12.28}$$

Fix also any $V \Subset \Omega$. Then after an approximation we find from (12.26)-(12.28) that

$$\int_\Omega \sum_{i,j=1}^n a^{ij}(x) \tilde{u}_{x_j} w_{x_i}\, dx = 0 \tag{12.29}$$

for all $w \in H^1_0(V)$. This is to say that $\tilde{u} \in H^1(V)$ is a weak solution of the linear, second order elliptic PDE

$$-\sum_{i,j=1}^n (a^{ij} \tilde{u}_{x_j})_{x_i} = 0 \quad \text{in } V. \tag{12.30}$$

But we can not just apply our regularity theory from 6.3 to conclude from (12.30) that $\tilde{u}$ is smooth, the reason being that we can deduce from (12.21) and (12.28) only that

$$a^{ij} \in L^\infty(V) \quad (i,j = 1,\dots,n).$$

However DeGiorgi's theorem asserts that any weak solution of (12.30) must in fact be locally Hölder continuous for some exponent $\gamma > 0$. Thus if $W \Subset V$ we have $\tilde{u} \in C^{0,\gamma}(W)$, and so

$$u \in C^{1,\gamma}_{\text{loc}}(\Omega).$$

Return to the definition (12.28). If L is smooth, we now know $a^{ij} \in C^{0,\gamma}_{\text{loc}}(\Omega)$ $(i,j = 1,\dots,n)$. Then (12.19) and the Schauder estimates from Chapter 8 assert that in fact

$$u \in C^{2,\gamma}_{\text{loc}}(\Omega).$$

But then $a^{ij} \in C^{1,\gamma}_{\text{loc}}(\Omega)$; and so another version of Schauder's estimate implies

$$u \in C^{3,\gamma}_{\text{loc}}(\Omega).$$

We can continue this so called "bootstrap" argument, eventually to deduce u is $C^{k,\gamma}_{\text{loc}}(\Omega)$ for $k = 1,\dots,$ and so $u \in C^\infty(\Omega)$. ■

12.9 Exercises

<u>12.1</u>: [Qualifying exam 08/98] Let $\Omega \subset \mathbb{R}^n$ be open (perhaps not bounded, nor with smooth boundary). Show that if there exists a function $u \in C^2(\overline{\Omega})$ vanishing on $\partial\Omega$ for which the quotient

$$I(u) = \frac{\displaystyle\int_\Omega |Du|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}$$

reaches its infimum λ, then u is an eigenfunction for the eigenvalue λ, so that $\Delta u + \lambda u = 0$ in Ω.

<u>12.2</u>: [Qualifying exam 08/02] Let Ω be a bounded, open set in $\mathbb{R}^n$ with smooth boundary Γ. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are nonempty, disjoint, smooth $(n-1)$-dimensional surfaces in $\mathbb{R}^n$. Suppose that $f \in L^2(\Omega)$.

a) Find the weak formulation of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ Du \cdot \nu = 0 & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_0. \end{cases}$$

b) Prove that for $f \in L^2(\Omega)$ there is a unique weak solution $u \in H^1(\Omega)$.

12.3: [Qualifying exam 08/02] Let Ω be a bounded domain in the plane with smooth boundary. Let f be a positive, strictly convex function over the reals. Let $w \in L^2(\Omega)$. Show that there exists a unique $u \in H^1(\Omega)$ that is a weak solution to the equation

$$\begin{cases} -\Delta u + f'(u) = w & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Do you need additional assumptions on f?

Part III

Modern Methods for Fully Nonlinear Elliptic PDE

Chapter 13

Non-Variational Methods in Nonlinear PDE

13.1 Introduction

In the final part of this book, the the material will be somewhat more focused. Up to this point, the theory presented has enjoyed a somewhat general applicability. In the realm of fully Nonlinear PDE, no such generalities exist; indeed theory almost always has to been 'hand-made' for a particular fully-nonlinear PDE. Actually, it is often worse than this in that many fully Nonlinear PDE require different theory for various boundary conditions. In this chapter, the definitions of sub-harmonic and super-harmonic functions will be progressively generalized from that of what was presented in the first part of this book. This will lead into the formal definition of 'viscosity solutions' whose formulation has been key in developing theory for many nonlinear elliptic PDE (Pierre Lions recieved the Fields Medal for his pioneering work in this area). With that, the maxima principle of Alexandrov-Bakelman will be presented, which is the key to proving Hölder estimates for the Monge-Ampere equation. The last sections of the book will be dedicated to presenting the full proof of the existence of solutions for the Monge-Ampere equation. While this material is somewhat esoteric, it is an excellent example of the types of calculations that are needed in modern elliptic PDE theory.

13.2 Viscosity Solutions

13.2.1 Alternative Definitions of Subharmonic Functions

The problem with the Perron definition of subharmonic and superharmonic functions is that they relied on a classical notion of a function being harmonic. For a general differential operator, call it F, there might not be an analogy of a harmonic function as the equation $F[u] = 0$ may or may not have a solution in the classical sense. This is opposed to solutions of $F[u] \geq 0$ of $F[u] \leq 0$, these solutions can readily be constructed for differential operators. With this in mind, we make the following alternative definition:

Definition 13.1. *A function $u \in C^0(\Omega)$ is* subharmonic(superharmonic) *if for every ball $B \Subset \Omega$ and superharmonic(subharmonic) function $v \in C^2(B) \cap C^0(\overline{B})$ (i.e. $\Delta v \leq (\geq)\, 0$) which is $\geq u$ on ∂B one has $v \geq (\leq)\, u$ in B.*

It's not hard to understand the above definition is equivalent to the Perron definition.

Remarks:

i.) By replacing Δ by other operators, denoted F, we then get general definitions of the statement "$F[u] \geq (\leq)\, 0$" for $u \in C^0(\Omega)$.

ii.) One can define a solution of "$F[u] = 0$" as a function satisfying both $F[u] \geq 0$ and $F[u] \leq 0$.

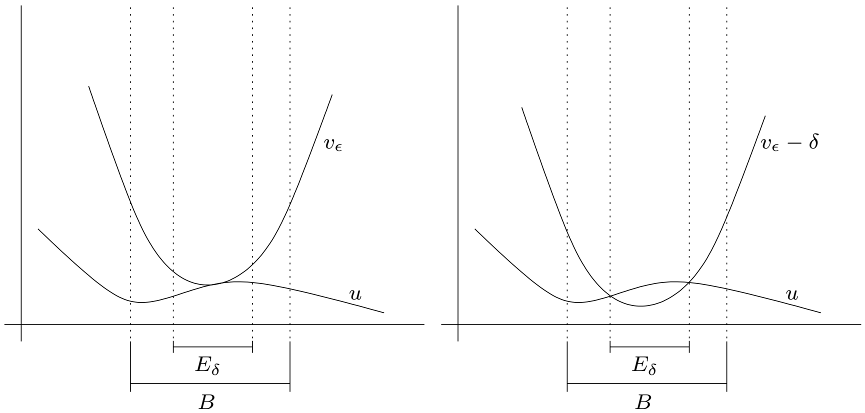
iii.) The solution of $F[u] = 0$ in this sense are referred to as *viscosity solutions.*

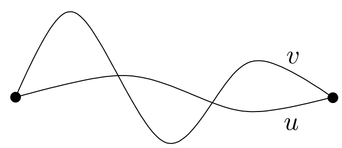
We can still further generalize the above definition still further:

Definition 13.2. *A function $u \in C^0(\Omega)$ is* subharmonic(superharmonic) *(with respect to Δ) if for every point $y \in \Omega$ and function $v \in C^2(\Omega)$ such that $u - v$ is maximized(minimized) at y, we have $\Delta v(y) \geq (\leq)\, 0$.*

Proposition 13.3. *Definitions (13.1) and (13.2) are equivalent.*

Proof: (13.1) $\Longrightarrow$ (13.2): Let u be subharmonic. Suppose now there exists $y \in \Omega$ such that $\Delta v(y) < 0$. Since $v \in C^2(\Omega)$ there exists a ball B containing y such that $\Delta v(y) \leq 0$ on B. Now, define $v_\epsilon = v + \epsilon|x - y|^4$ which implies that y corresponds to a strict maximum of $u - v_\epsilon$. With this we know there exists a $\delta > 0$ such that $v_\epsilon - \delta > u$ on ∂B with $v_\epsilon - \delta < u$ on some set





$E_\delta \subset B$; a contradiction.

(13.2) $\Longrightarrow$ (13.1): Suppose that the maximum of $u - v$ is attained at $y \in \Omega$ implies that $\Delta v(y) \geq 0$ for any $v \in C^2(\Omega)$. Next, suppose that u is not subharmonic; i.e. there exists a ball B and superharmonic v such that $v \geq u$ on ∂B but $v \leq u$ in a set $E_\delta \subset B$. Another way to put this is to say there exists a point $y \in B$ where $u - v$ attains a positive max. Now, define $v_\epsilon := v + \epsilon(p^2 - |x-y|^2)$; clearly $\Delta v_\epsilon(y) < 0$. For ϵ small enough $u - v_\epsilon$ will have a positive max in B (the constant p can be adjusted so that $v_\epsilon \geq u$ on $\partial\Omega$). This yields a contradiction as $\Delta v_\epsilon(y) < 0$ by construction. ■

Now, we can make the final generalization of our definition. First, we redefine the idea of ellipticity. Consider the operator

$$F[u](x) := F(x, u, Du, D^2u), \tag{13.1}$$

where $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n)$. Note that $\mathbb{S}^n$ represents the space of all symmetric $n \times n$ matrices.

Definition 13.4. *An operator* $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n)$ *is* degenerate elliptic *if*

$$\frac{\partial F}{\partial r^{ij}} \geq 0$$

and

$$\frac{\partial F}{\partial z} \leq 0,$$

where the arguments of F are understood as $F(x, z, p, [r^{ij}])$.

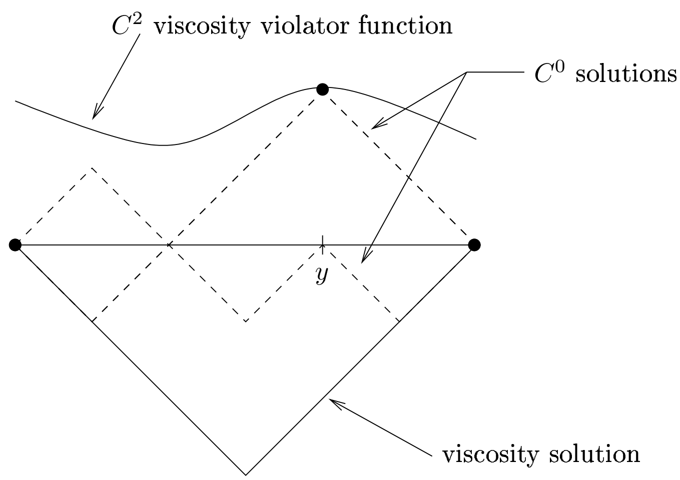
Finally, we make our most general definition for subharmonic and superharmonic functions.

Definition 13.5. *If u is upper(lower)-semicontinuous on Ω, we say $F[u] \geq (\leq)\, 0$ in the viscosity sense if for every $y \in \Omega$ and $v \in C^2(\Omega)$ such that $u - v$ is maximized(minimized) at y, one has $F[v](y) \geq (\leq)\, 0$.*

Remark: A viscosity solution is a function that satisfies both $F[u] \geq 0$ and $F[u] \leq 0$ in the viscosity sense.

This final definition for viscosity solutions even has meaning in first order equations:

Example 13.1. *Consider $|Du|^2 = 1$ in $\mathbb{R}$. It is obvious that generalized solutions will have a "saw tooth" form and are not unique for specified boundary values. That being understood, it is easily verified that the viscosity solution to this equation is unique.*



To see this, the above figure illustrates a non-viscosity solution with an accompanying C^2 function which violates the viscosity definition. At point y, it is clear that $u - v$ is maximized but that $\Delta v(y) < 0$. Basically, any "peak" in a C^0 solution will violate the viscosity criterion with a similar C^2 example.

The only solution with no "peaks" is the viscosity solution depicted in the figure.

Aside: *A nice way to derive this viscosity solution is to consider solutions u_ϵ of the equation:*

$$F_\epsilon[u] := \epsilon \Delta u_\epsilon + |Du_\epsilon|^2 - 1 = 0.$$

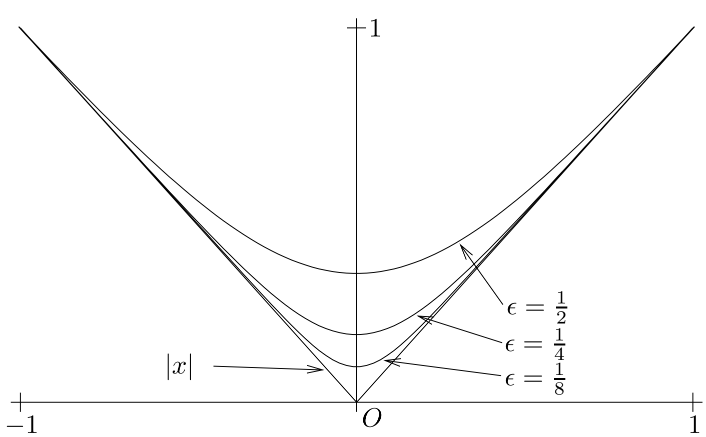
Clearly, $F_\epsilon \to F$, where $F[u] := |Du|^2 - 1$ as $\epsilon \to 0$. It is not clear whether or not u_ϵ converges. Even if u_ϵ converged to some limit, say $\tilde{u}$, it's not clear whether or not $F[\tilde{u}] = 0$ in any sense. Fortunately, in this particular example, all this can be ascertained through direct calculation. Since we are in $\mathbb{R}$, we have

$$F_\epsilon[u] = \epsilon \cdot u_{\epsilon xx} + |u_{\epsilon x}|^2 - 1 = 0.$$

To solve this ODE take $v_\epsilon := u_{\epsilon x}$; this makes the ODE separable. After integrating, one gets

$$u_\epsilon = \epsilon \cdot \ln\left(\frac{e^{\frac{2x}{\epsilon}} + 1}{e^{-\frac{2}{\epsilon}} + 1}\right) - x,$$

where the constants of integration have been set so $u(-1) = u(1) = 1$. The following figure is a graph of these functions. It is graphically clear that



$\lim_{\epsilon \to 0} u_\epsilon = |x|$; *i.e. we have u_ϵ converging (uniformly) to the viscosity solution of $|Du|^2 = 1$. This limit is easy to verify algebraically via the use of L'Hopital's rule.*

13.3 Alexandrov-Bakel'man Maximum Principle

Let $u \in C^0(\overline{\Omega})$ and consider the graph u given by $x_{n+1} = u(x)$ as well as the hyperplane which contacts the graph from below. For $y \in \Omega$, set

$$\chi_u(y) := \{p \in \mathbb{R}^n \,\big|\, u(x) \geq u(y) + (x-y)\cdot p \ \forall x \in \Omega\}.$$

Define the lower contact set Γ^- or Γ_u^- by

$$\begin{aligned} \Gamma^- &:= \big\{p \in \mathbb{R}^n \,\big|\, u(x) \geq u(y) + (x-y)\cdot p \ \forall x \in \Omega \\ &\qquad \text{for some } p = p(y) \in \mathbb{R}^n\big\} \\ &= \{\text{points of convexity of graph of } u\}. \end{aligned}$$

Proposition 13.6. *Let $u \in C^0(\overline{\Omega})$. We have*

i.) if u is differentiable at $y \in \Gamma^-$ then $\chi_u(y) = Du(y)$,

ii.) if u is convex in Ω then $\Gamma^- = \Omega$ and χ_u is a subgradient of u,

iii.) if u is twice differentiable at $y \in \Gamma^-$ then $D^2u(y) \geq 0$

Investigate the relation between χ_u, $\max u$, and $\min u$. Let $p_0 \notin \chi_u(\Omega) := \bigcup_{y\in\Omega} \chi_u(y)$. By definition, there exists a point $y \in \partial\Omega$ such that

$$u(x) \geq u(y) + p_0 \cdot (x-y) \ \forall x \in \Omega. \tag{13.2}$$

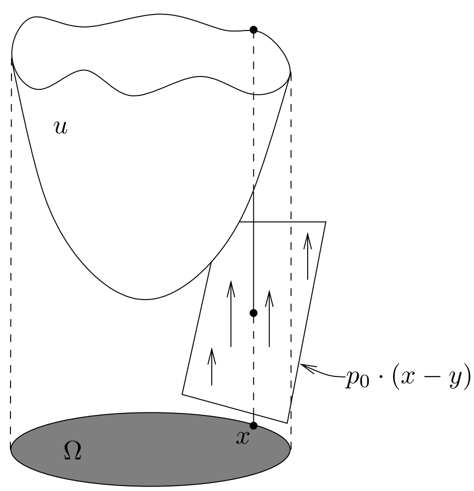


Figure 13.1:

You can see 13.2) by considering the hyperplane of the type $x_{n+1} = p_0 \cdot x + C$, first placing far from below the graph u and tending it upward.(13.2) yields an estimate of u; i.e.,

$$u \geq \min_{\partial\Omega} u - |p_0| d, \tag{13.3}$$

where $d = \operatorname{diam} \Omega$ as before.

The relation with differential operator is as follows. Suppose $u \in C^2(\Omega)$, then $\chi_u(y) = Du(y)$ for $y \in \Gamma^-$ and the Jacobian matrix of χ_u is $D^2 u \geq 0$ in Γ^-. Thus, the n-dimensional Lebesgue measure of the normal image of Ω is given by

$$|\chi(\Omega)| = |\chi(\Gamma^-)| = |Du(\Gamma^-)| \leq \int_{\Gamma^-} |\det D^2 u| \, dx \tag{13.4}$$

since $D^2 u \geq 0$ on Γ^-. (13.4) can be realised as a consequence of the classical change of variables formula by considering, for positive ϵ, the mapping $\chi_\epsilon = \chi + \epsilon I$, whose Jacobian matrix $D^2 u + \epsilon I$ is then strictly positive in the neighbourhood of Γ^-, and by subsequently letting $\epsilon \to 0$. If we can show that χ_ϵ is 1-to-1, then equality will in fact hold in (13.4). To show this, suppose not, i.e. there exists $x \neq y$ such that $\chi_\epsilon(x) = \chi_\epsilon(y)$. In this case we have

$$\chi(x) - \chi(y) = \epsilon(y - x).$$

But, we also have (after using abusive notation), that

$$\begin{aligned} u(x) &\geq u(y) + (x - y) \cdot \chi(y) \\ u(y) &\geq u(x) + (y - x) \cdot \chi(x) \end{aligned}$$

Adding these we have

$$0 \geq (\chi(x) - \chi(y))(y - x) = \epsilon(y - x)^2 \geq 0,$$

which leads us to conclude $y = x$, a contradiction.

More generally, if $h \in C^0(\Omega)$ is positive on $\mathbb{R}^n$, then we have

$$\int_{\chi_u(\Omega)} \frac{dp}{h(p)} = \int_{\Gamma^-} \frac{\det D^2 u}{h(Du)} \, dx. \tag{13.5}$$

Remark: It is trivial to extend this to $u \in (C^2(\Omega) \cap C^0(\overline{\Omega})) \cup W^{2,n}(\Omega)$ by using approximating functions and Lebesgue dominated convergence. Also note that even thous $C^2(\overline{\Omega}) \subset W^{2,n}(\Omega)$, $C^2(\Omega) \cap C^0(\overline{\Omega}) \not\subseteq W^{2,n}(\Omega)$. Not sure if further generalization is possible.

Theorem 13.7. *Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies*

$$\det D^2 u \le f(x)h(Du) \text{ on } \Gamma^- \tag{13.6}$$

for some positive function $h \in C^2(\mathbb{R}^n)$. Assume moreover,

$$\int_{\Gamma^-} f\,dx < \int_{\mathbb{R}^n} \frac{dp}{h(p)}.$$

Then there holds an estimate of the type

$$\min_{\Omega} u \ge \min_{\partial\Omega} u - Cd,$$

where C denotes some constant independent of u.

Proof. Let $B_R(0)$ be a ball of radius R centered at 0 such that

$$\int_{\chi_u(\Omega)} \frac{dp}{h(p)} \le \int_{\Gamma^-} f\,dx = \int_{B_R(0)} \frac{dp}{h(p)} < \int_{\mathbb{R}^n} \frac{dp}{h(p)}.$$

Thus, for $R' > R$, there exists $p \notin \chi_u(\Omega)$ such that $|p| < R'$. So (13.3 now implies that

$$\min_{\Omega} u \ge \min_{\partial\Omega} u - Rd,$$

which completes the proof. □

13.3.1 Applications

Linear equations

First we consider the case $h \equiv 1$ in theorem 13.7. The result is simply stated as follows: The differential inequality

$$\det D^2 u \le f \text{ in } \Gamma^-$$

implies an estimate

$$\min_{\Omega} u \ge \min_{\partial\Omega} - \left(\frac{1}{\omega_n}\int_{\Gamma^-} f\,dx\right)^{1/n} \cdot d.$$

Next, suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$Lu := a^{ij} D_{ij} u \le g \text{ in } \Omega,$$

where $\mathcal{A} := [a^{ij}] > 0$ and g a given bounded function. We derive

$$\det \mathcal{A} \cdot \det D^2 u \leq \left(\frac{|g|}{n}\right)^n \text{ on } \Gamma^-,$$

by virtue of an elementary matrix inequality for any symmetric $n \times n$ matrices $A, B \geq 0$:

$$\det A \cdot \det B \leq \left(\frac{\mathrm{Tr}(AB)}{n}\right)^n,$$

whose proof is left as an exercise.

——

As $\det(AB) = \det A \cdot \det B$, we really only need to show that $\det A \leq \left(\frac{\mathrm{Tr}\, A}{n}\right)^n$. Since the matrices are non-singular we assume the matrix A is diagonalized wlog. So,

$$\det A = \prod_{i=1}^n \lambda_i = \exp\left(n \cdot \sum_{i=1}^n \frac{\ln \lambda_i}{n}\right) \leq \left(\frac{1}{n}\sum_{i=1}^n \lambda_i\right)^n = \left(\frac{\mathrm{Tr}\, A}{n}\right)^n$$

The inequality of the of the above comes from the convexity of exponential functions.

——

Therefore it follows that

$$\min_{\Omega} u \geq \min_{\partial\Omega} u - \frac{d}{n\omega_n^{1/n}} \left|\left| \frac{g}{(\det \mathcal{A})^{1/n}} \right|\right|_{L^n(\Gamma^-)}.$$

We can find an estimate for $\max_\Omega u$ similarly; if $Lu \geq g$ in Ω, thenwe have

$$\max_{\Omega} u \leq \max_{\partial\Omega} u - \frac{d}{n\omega_n^{1/n}} \left|\left| \frac{g}{(\det \mathcal{A})^{1/n}} \right|\right|_{L^n(\Gamma^+)},$$

where $\Gamma_u^+ = \Gamma_{-u}^-$.

Monge-Ampére Equation

Suppose a convex function $u \in C^2(\Omega)$ solves

$$\det D^2 u \leq f(x) \cdot h(Du),$$

for some functions f, h which are assumed to satisfy

$$\int_\Omega f(x)\, dx < \int_{\mathbb{R}^n} \frac{dp}{h(p)}$$

Then we have an estimate

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u - Rd,$$

where R is defined by

$$\int_{B_R(0)} \frac{dp}{h(p)} = \int_{\Omega} f(x)\, dx.$$

This result is a consequence of theorem 13.7. If Ω is convex, then we infer $\sup_{\Omega} u \leq \sup_{\partial\Omega} u$, taking into account the convexity of u.

Remark: If $\det D^2 u = f(x) \cdot h(Du)$ in Ω, then

$$\int_{\Omega} \frac{\det D^2 u}{h(Du)} = \int_{\Omega} f(x)\, dx.$$

Thus, the inequality

$$\int_{\Omega} f(x)\, dx \leq \int_{\mathbb{R}^n} \frac{dp}{h(p)}$$

is a necessary condition for solvability. It is also sufficient, by the work of Urbas. The inequality is strict if Du is bounded.

General linear equations

Consider a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of

$$Lu := a^{ij} D_{ij} u + b^i D_i u \leq f.$$

We rewrite this in the form

$$a^{ij} D_{ij} u \leq -b^i D_i u + f.$$

Define

$$h(p) = (|p|^{\frac{n}{n-1}} + \mu^{\frac{n}{n-1}})^{n-1}$$

for $\mu := \left|\left|\frac{f}{\mathcal{D}^{1/n}}\right|\right|_{L^n(\Gamma^+)} \neq 0$ and $\mathcal{D} := \det \mathcal{A}$. We infer the following estimate as before:

$$\max_{\Omega} u \leq \max_{\partial\Omega} u + C \left|\left|\frac{f}{\mathcal{D}^{1/n}}\right|\right|_{L^n(\Gamma^+)},$$

where C depedends on n, d, and $\left|\left|\frac{f}{\mathcal{D}^{1/n}}\right|\right|_{L^n(\Gamma^+)}$.

Oblique boundary value problems

The above estimates can all be extended to boundary conditions of the form

$$\beta \cdot Du + \gamma \cdot u = g \text{ on } \partial\Omega$$

where $\beta \cdot \nu > 0$ on $\partial\Omega$, $\gamma > 0$ on Ω and g is a given function. Here ν denotes the outer normal on $\partial\Omega$. We obtain the more general estimate

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} \frac{g}{\gamma} - C\left(\sup_{\partial\Omega} \frac{|\beta|}{\gamma} + d\right)$$

where $d = \operatorname{diam}\Omega$.

13.4 Local Estimates for Linear Equations

13.4.1 Preliminaries

Consider the linear operator $Lu := a^{ij}D_{ij}u$ with strict ellipticity condition:

$$\lambda I \leq [a^{ij}] \leq \Lambda I \text{ for} 0 < \lambda \leq \Lambda, \;\; \lambda, \Lambda \in \mathbb{R}.$$

To see the reason why we discuss the linear operator Lu, we deal ith the following nonlinear equation of simple case:

$$F(D^2u) = 0. \tag{13.7}$$

Differentiate (13.8) with respect to a non-zero vector γ, to obtain

$$\frac{\partial F}{\partial r_{ij}}(D^2u)D_{ij\gamma}u = 0.$$

If we set $L := \frac{\partial F}{\partial r_{ij}}D_{ij}$, then we have $LD_\gamma u = 0$. Differentiate again and you find

$$\frac{\partial F}{\partial r_{ij}}D_{ij\gamma\gamma}u + \frac{\partial^2 F}{\partial r_{ij}\partial r_{kl}}D_{ij\gamma}u \cdot D_{kl\gamma}u = 0.$$

If F is concave with respect to r, which most of the important examples satisfy, then we derive

$$\frac{\partial F}{\partial r_{ij}}(D^2u)D_{ij}D_{\gamma\gamma}u \geq 0,$$
$$i.e., \;\; L(D_{\gamma\gamma}u) \geq 0.$$

This is the linear differential inequality and justifies our investigation of the linear operator $Lu = a^{ij}D_{ij}u$.

13.4.2 Local Maximum Principle

Theorem 13.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let $B_R := B_R(y) \subset \Omega$ be a ball. Suppose $u \in C^2(\Omega)$ fulfuills $Lu \geq f$ in Ω for some given function f. Then for any $0 < \sigma < 1$, $p > 0$, we have*

$$\sup_{B_{\sigma R}} u \leq C\left\{\left(\frac{1}{R^n}\int_{B_R}(u^+)^p\,dx\right)^{1/p} + \frac{R}{\lambda}||f||_{L^n(B_R)}\right\},$$

where C depends on n, p, σ and Λ/λ. $u^+ = \max\{u, 0\}$ and $B_{\sigma R}(y)$ denotes the concentric subball.

Proof. We may take $R = 1$ in view of the scaling $x \to x/R$, $f \to R^2 f$. Let

$$\eta = \begin{cases} (1-|x|^2)^\beta,\ \beta > 1 & \text{in } B_1 \\ 0 & \text{in } \Omega \setminus B_1. \end{cases}$$

It is easy to verify that

$$|D\eta| \leq C\eta^{1-1/\beta},\ |D^2\eta| \leq C\eta^{1-2/\beta}, \tag{13.8}$$

for some constant C.

Introduce the test function $v := \eta(u^+)^2$ and compute

$$\begin{aligned} Lv &= a^{ij}D_{ij}\eta(u^+)^2 + 2a^{ij}D_i\eta D_j((u^+)^2) + \eta a^{ij}D_{ij}((u^+)^2) \\ &\geq -C\Lambda\eta^{1-2/\beta}(u^+)^2 + 4u^+a^{ij}D_i\eta D_j u^+ \\ &\quad +2\eta a^{ij}D_i u^+ \cdot D_j u^+ + 2\eta u^+ a^{ij}D_{ij}u^+ \\ &\geq -C\Lambda\eta^{1-2/\beta}(u^+)^2 + 2\eta u^+ f. \end{aligned}$$

here the use of (13.8) and the next inequality was made:

$$|a^{ij}D_i\eta D_j u^+| \leq (a^{ij}D_i\eta D_j\eta)^{1/2}(a^{ij}D_i u^+ D_j u^+)^{1/2}.$$

Apply the Alexandrov maximum principle 2.3.1 to find

$$\begin{aligned} \sup_{B_1} v &\leq \frac{C(n)}{\lambda}||\Lambda\eta^{1-2/\beta}(u^+)^2 + \eta u^+ f||_{L^n(B_1)} \\ &\leq \frac{C(n)}{\lambda}||\Lambda v^{1-2/\beta}(u^+)^{4/\beta} + v^{1/2}\eta^{1/2}f||_{L^n(B_1)} \\ &\leq C\left\{\frac{\Lambda}{\lambda}||v^{1-2/\beta}(u^+)^{4/\beta}||_{L^n(B_1)} + \frac{1}{\lambda}||v^{1/2}\eta^{1/2}f||_{L^n(B_1)}\right\} \\ &\leq C\left\{\frac{\Lambda}{\lambda}\left(\sup_{B_1} v\right)^{1-2/\beta}||(u^+)^{4/\beta}||_{L^n(B_1)} + \frac{(\sup_{B_1} v)^{1/2}}{\lambda}||f||_{L^n(B_1)}\right\}. \end{aligned}$$

Choosing $\beta = 4n/p$, we deduce

$$\sup_{B_1} v^{1/2} \leq C\left\{\frac{\Lambda}{\lambda}\left(\sup_{B_1} v\right)^{1/2-2/\beta}\left(\int_{B_1}(u^+)^p\,dx\right)^{1/n} + \frac{1}{\lambda}||f||_{L^n(B_1)}\right\},$$

from which we conclude, invoking the Young inequality, that

$$\sup_{B_1} v^{1/2} \leq C\left(\frac{\Lambda}{\lambda}, n, p\right)\cdot\left\{\left(\int_{B_1}(u^+)^p\,dx\right)^{1/p} + \frac{1}{\lambda}||f||_{L^n(B_1)}\right\}.$$

This completes the proof. □

13.4.3 Weak Harnack inequality

Lemma 13.9. *Let $u \in C^2(\Omega)$ satisfies $Lu \leq f$, $u \geq 0$ in $B_R := B_R(y) \subset \Omega$, for some function f. For $0 < \sigma < \tau < 1$, we have*

$$\inf_{B_{\sigma R}} u \leq C\left\{\inf_{B_{\tau R}} u + \frac{R}{\lambda}||f||_{L^n(B_R)}\right\},$$

where $C = C(n, \sigma, \tau, \Lambda/\lambda)$.

Proof. Set for $k > 0$

$$w := \frac{|x/R|^{-k} - 1}{\sigma^{-k} - 1}\inf_{B_{\sigma R}} u.$$

There holds $w \leq u$ on $\partial B_{\sigma R}$ and ∂B_R. Now, compute

$$\begin{aligned}
L|x|^{-k} &= a^{ij}D_{ij}|x|^{-k} \\
&= a^{ij}(-k\delta_{ij}|x|^{-k-2} + k(k+2)x_ix_j|x|^{-k-4}) \\
&= -k|x|^{-k-2}\left(\operatorname{Tr}\mathcal{A} - (k+2)\frac{a^{ij}x_ix_j}{|x|^2}\right) \\
&\geq -k|x|^{-k-2}(\operatorname{Tr}\mathcal{A} - (k+2)\lambda) \\
&\geq 0,
\end{aligned}$$

if we choose $k > \frac{\operatorname{Tr}\mathcal{A}}{\lambda} - 2$. Thus

$$\begin{aligned}
Lw &\geq 0, \\
L(u-w) &\leq f.
\end{aligned}$$

Applying the Alexandrov maximum principle again, we obtain

$$u \geq w - \frac{CR}{\lambda}||f||_{L^n(B_R)}.$$

Take the infimum and you get the desired result. □

Theorem 13.10. *Let $u \in C^2(\Omega)$ satisfies $Lu \le f$, $u \ge 0$ in $B_R := B_R(y) \subset \Omega$, for some function f. Then for any $0 < \sigma, \tau < 1$, there exists $p > 0$ such that*

$$\left(\frac{1}{R^n}\int_{B_{\sigma R}} u^p\,dx\right)^{1/p} \le C\left\{\inf_{B_{\tau R}} u + \frac{R}{\lambda}||f||_{L^n(B_R)}\right\},$$

where p, C depend on n, σ, τ and Λ/λ.

Proof. WLOG thake $y = 0$ so that $B_R(0) = B_R$. Consider

$$\eta(x) = 1 - \frac{|x|^2}{R^2};$$

then $\eta - u \le 0$ on ∂B_R with $L(\eta - u) = L\eta - Lu \le -2\frac{\operatorname{Tr} A}{R^2} - g$ the above lemma now indicates

$$\begin{aligned}
\eta - u &\le \frac{d}{n\omega_n^{1/n}}\left|\left|-2\frac{\operatorname{Tr} A}{\det A^{1/n}\cdot R^2} - \frac{g}{\det A^{1/n}}\right|\right|_{L^n(\{x\in B_R\,|\,\eta-u\ge 0\})} \\
&\le \frac{d}{\lambda n\omega_n^{1/n}}\left|\left|-2\frac{\operatorname{Tr} A}{R^2} - g\right|\right|_{L^n(\{x\in B_R\,|\,\eta-u\ge 0\})} \\
&\le \frac{\Lambda}{\lambda}\frac{d}{n\omega_n^{1/n}}\left|\left|-\frac{2}{R^2} - \frac{g}{\operatorname{Tr} A}\right|\right|_{L^n(\{x\in B_R\,|\,\eta-u\ge 0\})} \\
&\le C\frac{\Lambda}{\lambda}|\{x \in B_R\,|\,\eta \ge u\}| \le \frac{1}{2}\eta,
\end{aligned}$$

if

$$\frac{|\{u \le 1 \cap B_R\}|}{|B_R|}$$

is suffciently small. Thus, we have $u \ge C$ on $B_{\sigma R}$ and apply the previous lemma to ascertain $u \ge C$ on $B_{\tau R}$. *Remarks:*

i.) This ascertion contains all the 'information' for the PDE.

ii.) We essentially shrink the ball after the calculation and 3.5 to then expand the ball back out (perterbing by measure).

At this point, in order to simplify a later measure theoretic argument, we change our basis domain from balls to cubes. Also take $f = 0$ and drop the bar for simplicity.

Let $K := K_R(y)$ be an open cube, parallel to the coordinate axis with center y and side length $2R$. The corresponding estimate of (**??**) is as follows: if

$$|\{u(x) \geq 1\} \cap K| \geq \theta |K| \text{ and } K_{3R} \subset B := B_1, \tag{13.9}$$

then $u \geq \gamma$ on K_R. Now consider any fixed cube K_0 in $\mathbb{R}^n$ with $\Gamma \subset K_0$ a measurable subset and $0 < \theta < 1$. Defining

$$\tilde{\Gamma} := \bigcup \left\{ K_{3R}(y) \cap K_0 \,\middle|\, |K_R(y) \cap \Gamma| \geq \theta |K_R(y)| \right\},$$

we have either $\tilde{\Gamma} = K_0$ or $|\tilde{\Gamma}| \geq \theta^{-1}|\Gamma|$ by the cube decomposition at the beginning of the notes. With this in mind, we make the following claim:

$$\inf_K u \geq C \left(\frac{|\Gamma|}{|K|} \right)^{\ln \gamma / \ln \theta}. \tag{13.10}$$

Proof of Claim We use induction. So if we take $\Gamma := \{x \in K \mid u \geq \gamma^0 = 1\}$, then we have $u \geq \gamma$ on $\tilde{\Gamma}$ with $|\tilde{\Gamma}| \geq \theta^{-1}|\Gamma|$. This is the $m = 1$ step and is the result of the previous argument in the proof. Now, we assume the condition $|\Gamma| > \theta^m |K|$, $m \in \mathbb{N}$, implies the estimate

$$u \geq \gamma^{m+0} = \gamma^m \text{ on } K.$$

The representation of $\gamma^0 = 1$ may seem absurd at this point, but it will clarify the final inductive step immensely. Now, to proceed to $m+1$, we first note that given the inductive hypothesis Γ also satisfies the weaker condition $|\Gamma| \geq \theta^{m+1}|K|$. Recalling the $m = 1$ case, we also have from (13.9) that

$$u \geq \gamma \text{ on } \tilde{\Gamma}.$$

Now cube decomposition indicates that $|\tilde{\Gamma}| \geq \theta^{-1}|\Gamma| \geq \theta^m |K|$. Thus, we ascertain

$$u \geq \gamma^{m+1} \text{ on } K.$$

This argument is tricky in a sense that γ is not only apart of the inductive result, it's a paramater of Γ and $\tilde{\Gamma}$ (hence the γ^0) notation. Once this is clear, we get the claim by picking γ and C correctly. ■

Returning to the ball, we infer that

$$\inf_B u \geq C \left(\frac{|\Gamma \cap B|}{|B|} \right)^{\kappa}, \tag{13.11}$$

where C and κ are positive constants depending only on n and Λ/λ. Now define

$$\Gamma_t := \{x \in B \,|\, u(x) \geq t\}.$$

Replacing u by u/t in (13.12) yields

$$\inf_B u \geq Ct \left(\frac{|\Gamma \cap B|}{|B|} \right)^{\kappa}.$$

To obtain the weak Harnack inequality, we normalize $\inf_B u = 1$ and hence

$$|\Gamma_t| \leq C \cdot t^{-1/k}.$$

Choosing $p = (2\kappa)^{-1}$, we have

$$\begin{aligned}
\int_B u^p \, dx &= \int_B \int_0^{u(x)} pt^{p-1} \, dtdx \\
&= \int_0^{\infty} pt^{p-1} \cdot |\{x \in B \,|\, u \geq t\}| \, dt \\
&\leq \int_0^{\infty} pt^{p-1} |\Gamma_t| \, dt \\
&\leq C.
\end{aligned}$$

the second inequality is the coarea formula from geometric measure theory. Intuitivly, this can be thought of as slicing the $B \times [0, u(x)]$ by hyperplanes $w(x,t) = t$, and integrating these. Thus, we have obtained our result for B. The completion of the proof follows from a covering argument.

$$\int_{\Omega} g(x) \cdot |J_m f(x)| \, dx = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} g(x) \, d\mathcal{H}^{n-m}(x) dy$$

$\square$

Theorems (13.8) and (13.10) easily give

Theorem 13.11. *Suppose $Lu = f$, $u \geq 0$ in $B_R := B_R(y) \subset \Omega$. Then for any $0 < \sigma < 1$, we have*

$$\sup_{B_{\sigma R}} u \leq C \left(\inf_{B_R} u + \frac{R}{\lambda} ||f||_{L^n(B)} \right),$$

where $C = C(n, \sigma, \Lambda/\lambda)$.

13.4.4 Hölder Estimates

Lemma 13.12. *Suppose a non-decreasing function ω on $[0, R_0]$ satisfies*

$$\omega(\sigma R) \leq \sigma \cdot \omega(R) + R^\delta$$

for all $R \leq R_0$. Here $\sigma < 1$, $\delta > 0$ are given constants. Then we have

$$\omega(R) \leq C \left(\frac{R}{R_0}\right)^\alpha w(R_0),$$

where C, α depend on σ and δ.

Proof. We may take $\delta < 1$. Replace ω by

$$\overline{\omega}(R) := \omega(R) - AR^\delta,$$

and we obtain $\overline{\omega}(\sigma R) \leq \sigma \cdot \overline{\omega}(R)$, selecting A suitably. Thus, we have by iteration

$$\overline{\omega}(\sigma^\nu R) \leq \sigma^\nu \cdot \overline{\omega}(R_0).$$

Choosing ν such that $\sigma^{\nu+1}R_0 < R \leq \sigma^\nu R_0$ gives the result upon noting that ω is non-decreasing. □

Now we present the next pointwise estimate, which is due to Krylov ans Safonov.

Theorem 13.13. *Suppose u satsifies $Lu = f$ in $\Omega \subset \mathbb{R}^n$ for some $f \in L^n(\Omega)$. Let $B_R := B_R(y) \subset \Omega$ be a ball and $0 < \sigma < 1$. Then we have*

$$\operatorname*{osc}_{B_{\sigma R}} u \leq C\sigma^\alpha \left\{ \operatorname*{osc}_{B_R} u + \frac{R}{\lambda}||f||_{L^n(B_R)} \right\}, \tag{13.12}$$

where $\alpha = \alpha(n, \Lambda/\lambda) > 0$ and $C = C(n, \Lambda/\lambda)$.

This is analogous to the famous De Giorgi-Nash estimate for the divergence equation $D_i(a^{ij}(x)D_j u) = u$.

Proof. Take $B_R = B_R(y) \subset \Omega$ as in the theorem and define

$$M_R := \sup_{B_R} u, \quad m_R := \inf_{B_R} u.$$

There holds

$$\operatorname*{osc}_{B_R} u = M_R - m_r =: \omega(R).$$

Applying the weak Harnack inequality to $M_R - u$ and $m_r - u$ respectively, we obtain

$$\begin{aligned}\left(\frac{1}{R^n}\int_{B_{\sigma R}}(M_R-u)^p\,dx\right)^{1/p} &\le C\left\{M_R - M_{\sigma R} + \frac{R}{\lambda}||f||_{L^n(B_R)}\right\}\\ \left(\frac{1}{R^n}\int_{B_{\sigma R}}(u-m_R)^p\,dx\right)^{1/p} &\le C\left\{m_{\sigma R} - M_R + \frac{R}{\lambda}||f||_{L^n(B_R)}\right\}.\end{aligned}$$

Add these two inequalities and you get

$$\omega(R) \le C\{\omega(R) - \omega(\sigma R) + R\}, \tag{13.13}$$

where C depends on $||f||_{L^n(B_R)}$. Here we have used the inequality:

$$\left(\int |f+g|^p\,dx\right)^{1/p} \le 2^{1/p}\left\{\left(\int |f|^p\,dx\right)^{1/p} + \left(\int |g|^p\,dx\right)^{1/p}\right\}.$$

(13.13) can be rewritten in the form

$$\omega(\sigma R) \le (1 - 1/C)\omega(R) + R.$$

Applying lemma 13.12 now yields the desired result. □

13.4.5 Boundary Gradient Estimates

Now we turn our attention to the boundary gradient estimate, due to Krylov. The situation is as follows: Let B_R^+ be a half ball of radius R centered at the origin:

$$B_R^+ := B_R(0) \cap \{x_n > 0\}$$

Suppose u satisfies

$$\begin{cases} Lu := a^{ij}D_{ij}u = f & \text{in } B_R^+,\\ u = 0 & \text{on } B_R(0)\cap\{x_n = 0\}.\end{cases} \tag{13.14}$$

Then we have

Theorem 13.14. *Let u fulfil (13.14) and let $0 < \sigma < 1$. Then there holds*

$$\operatorname*{osc}_{B_{\sigma R}^+}\frac{u}{x_n} \le C\sigma^\alpha\left\{\operatorname*{osc}_{B_R^+}\frac{u}{x_n} + \frac{R}{\lambda}\sup_{B_R^+}|f|\right\}.$$

where $\alpha > 0$,$C > 0$ depend on n and Λ/λ.

From this theorem, we infer the Hölder estimate for the gradient on the boundary $x_n = 0$. In fact

$$\operatorname*{osc}_{|x'|<\sigma R} Du(x',0) \le C\sigma^\alpha \left\{ \sup_{B_R^+} |Du| + \frac{R}{\lambda} \sup_{B_R^+} |f| \right\}$$

holds, where $x' = \{x_1, \ldots, x_{n-1}\}$. *Proof:* Define $v := \frac{u}{x_n}$. We may assume that v is bounded in B_R^+. Write for some $\delta > 0$,

$$B_{R,\delta} := \{|x'| < R, 0 < x_n < \delta R\}.$$

Under the assumption that $u \ge 0$ in B_R^+, we want to prove the next claim. *Claim:* There exists a $\delta = \delta(n, \Lambda/\lambda)$ such that

$$\inf_{\substack{|x'| \,<\, R \\ x_n \,=\, \delta R}} v \le 2 \left(\inf_{B_{\frac{R}{2},\delta}} v + \frac{R}{\lambda} \sup_{B_R^+} |f| \right)$$

for any $R \le R_0$.
Proof of Claim: We may normalize as before to take $\lambda = 1$, $R = 1$ and

$$\inf_{\substack{|x'| \,<\, R \\ x_n \,=\, \delta R}} v = 1.$$

Define the comparison function w in $B_{1,\delta}$ as

$$w(x) := \left\{ 1 - |x'|^2 + (1 + \sup |f|) \frac{x_n - \delta}{\sqrt{\delta}} \right\} x_n.$$

It is easy to check that $w \le u$ on $\partial B_{1,\delta}$.

Compute

$$\begin{aligned} Lw &= -2\sum_{i=1}^{n-1} a_{ii} x_n + 2(1 + \sup |f|) \frac{a_{nn}}{\sqrt{\delta}} - 2\sum_{i=1}^{n-1} a_{in} x_i \\ &\ge f \quad \text{in } B_{1,\delta} \end{aligned}$$

if we take δ sufficiently small. Thus the maximum principle implies that on $B_{1/2,\delta}$

$$\begin{aligned} v &\ge 1 - |x'|^2 + (1 + \sup |f|) \frac{x_n - \delta}{\sqrt{\delta}} \\ &\ge \frac{1}{2} - \sup |f| \end{aligned}$$

again for sufficiently small δ. Normalizing back we obtain the claim. ■

To resume the proof of the theorem, we apply the Harnack inequality in the region $\left\{|x'| < R, \frac{\delta R}{2} < x_n < \frac{3\delta R}{2}\right\}$ to obtain

$$\begin{aligned}
\sup_{\substack{|x'| < R \\ \frac{\delta R}{2} < x_n < \frac{3\delta R}{2}}} v &\leq C\left(\inf_{\substack{|x'| < \frac{R}{2} \\ \frac{\delta R}{2} < x_n < \frac{3\delta R}{2}}} v + \frac{R}{\lambda}\sup|f|\right) \qquad (13.15)\\
&\leq C\left(\inf_{\substack{|x'| < R \\ x_n = \delta R}} v + \frac{R}{\lambda}\sup|f|\right)\\
&\leq C\left(\inf_{B_{\frac{R}{2},\delta}} v + \frac{R}{\lambda}\sup|f|\right).
\end{aligned}$$

The last inequality comes from the claim.

Now we set up the oscillation estimate: Define

$$\begin{aligned}
M_R &:= \sup_{B_{R,\delta}} v, \quad m_R := \inf_{B_{R,\delta}} v\\
\omega(R) &:= M_R - m_R = \operatorname*{osc}_{B_{R,\delta}} v.
\end{aligned}$$

Consider $M_{2R} - v$, $v - m_{2R}$ and invoke (13.15). We find

$$\begin{aligned}
M_{2R} - \inf_{\substack{|x'| < R \\ \frac{\delta R}{2} < x_n < \frac{3\delta R}{2}}} v &\leq C\left(M_{2R} - \sup_{B_{\frac{R}{2},\delta}} v + \frac{R}{\lambda}\sup|f|\right)\\
\inf_{\substack{|x'| < R \\ \frac{\delta R}{2} < x_n < \frac{3\delta R}{2}}} v - m_{2R} &\leq C\left(\inf_{B_{\frac{R}{2},\delta}} v - m_{2R} + \frac{R}{\lambda}\sup|f|\right)
\end{aligned}$$

Adding each terms tells you

$$\omega(2R) \leq C\left(\omega(2R) - \omega(R/2) + \frac{R}{\lambda}\sup|f|\right).$$

Lemma 13.12 now implies our desired result:

$$\omega(R) \leq C\left(\frac{R}{R_0}\right)^{\alpha}\left\{\omega(R_0) + \frac{R_0}{\lambda}\sup|f|\right\}. \quad ■$$

13.5 Hölder Estimates for Second Derivatives

In this section, we wish to derive the second derivative estimates for general nonlinear equation of simple case

$$F(D^2u) = \psi \quad \text{in } \Omega, \tag{13.16}$$

where $\Omega \subset \mathbb{R}^n$ is a domain. General cases can be handled by perterbation arguments. See Safonov [Sa2].

We begin with the interior estimates of Evans [E1] and Krylov [K2]. For further study, suitable function spaces are introduced. Various interpolation inequalities are recalled. Finally, we show the global bound for second derivates, due to Krylov. For the interior estimate, we follow the treatment of [GT] but with an idea of Safonov [Sa2] being used to avoid the Motzkin-Wasco lemma. The corresponding boundary estimate will be derived from Theorem 13.14

13.5.1 Interior Estimates

Concerning (13.16), it is assumed to satisfy $F \in C^2(\mathbb{R}^{n\times n})$, $u \in C^4(\Omega)$, $\psi \in C^2(\Omega)$ and

i.) $\lambda I \le F_r(D^2u) \le \Lambda I$, where $0 < \lambda \le \Lambda$ and $F_r := [\partial F/\partial r_{ij}]$;

ii.) F is concave on some open set $\mathcal{U} \subset \mathbb{R}^{n\times n}$ including the range of D^2u.

Under these assumptions, our first result is the next

Theorem 13.15. *Let $B_R = B_R(y) \subset \Omega$ and let $0 < \sigma < 1$. Then we have*

$$\operatorname*{osc}_{B_{\sigma R}} D^2u \le C\sigma^\alpha \left(\operatorname*{osc}_{B_R} D^2u + \frac{R}{\lambda}\sup|D\psi| + \frac{R^2}{\lambda}\sup|D^2\psi| \right),$$

where $C, \alpha > 0$ depend only on $n, \Lambda/\lambda$.

Proof: Recall the connection between (13.16) and the linear differential inequalities stated at 3.1.

Differentiating (13.16) with respect to γ twice, we obtain

$$F_{ij}D_{ij}D_{\gamma\gamma}u \ge D_{\gamma\gamma}\psi, \tag{13.17}$$

in view of the assumption ii.), where $F_{ij} := F_{r_{ij}}$. If we set

$$a^{ij}(x) := F_{ij}(D^2u(x)), \quad L := a^{ij}D_{ij},$$

then there holds

$$Lw \geq D_{\gamma\gamma}\psi \quad \text{for } w = w(x,\gamma) := D_{\gamma\gamma}u$$

and $\lambda I \leq [a^{ij}] \leq \Lambda I$.

Take a ball $B_{2R} \subset \Omega$ and define as before

$$\begin{aligned} M_2(\gamma) &:= \sup_{B_{2R}} w(\cdot,\gamma), \qquad & M_1(\gamma) &:= \sup_{B_R} w(\cdot,\gamma), \\ m_2(\gamma) &:= \inf_{B_{2R}} w(\cdot,\gamma), \qquad & m_1(\gamma) &:= \inf_{B_R} w(\cdot,\gamma). \end{aligned}$$

Applying the weak Harnack inequality, Theorem to $M_2 - w$ on B_R, we fine

$$\left\{R^{-n}\int_{B_R}(M_2(\gamma)-w(x,\gamma))^p\,dx\right\}^{1/p} \leq C\left\{M_2(\gamma)-M_1(\gamma)+\frac{R^2}{\lambda}\sup|D^2\psi|\right\} \tag{13.18}$$

To establish the Hölder estimat, we need a bound for $-w$. At present, however, we only have the inequality (13.17). Thus we return to the equation

$$F(D^2u(x)) - F(D^2(y)) = \psi(x) - \psi(y),$$

for $x \in B_R$, $y \in B_{2R}$. Since F is concave:

$$\begin{aligned} \psi(y) - \psi(x) &\leq F_{ij}(D^2u(x))(D_{ij}u(y) - D_{ij}u(x)) \qquad (13.19)\\ &= a^{ij}(x)(D_{ij}u(y) - D_{ij}u(x)) \\ &=: \alpha^i(x)\mu_i, \end{aligned}$$

where $\mu_1 \leq \cdots \leq \mu_n$ are the eigenvalues of the matrix $D^2u(y) - D^2u(x)$ and $\lambda \leq \alpha^i \leq \Lambda$.

Setting

$$\begin{aligned} \omega(2R) &:= \max_{|\gamma|=1}(M_2(\gamma) - m_2(\gamma)), \qquad & \omega(R) &:= \max_{|\gamma|=1}(M_1(\gamma) - m_1(\gamma)), \\ \omega^*(2R) &:= \int_{|\gamma|=1}(M_2(\gamma) - m_2(\gamma))\,d\gamma, \qquad & \omega^*(R) &:= \int_{|\gamma|=1}(M_1(\gamma) - m_1(\gamma))\,d\gamma, \end{aligned}$$

if follows by integration of (13.18) that

$$\begin{aligned} &\left\{R^{-n}\int_{|\gamma|=1}\int_{B_R}(M_2(\gamma)-w(x,\gamma))^p\,dxd\gamma\right\}^{1/p} \qquad (13.20)\\ &\leq C\left\{\omega^*(2R)-\omega^*(R)+\frac{R^2}{\lambda}\sup|D^2\psi|\right\}. \end{aligned}$$

For any fixed $x \in B_R$, one of the inequalities

$$\max_{|\gamma|=1}(M_1(\gamma) - w(x,\gamma)), \max_{|\gamma|=1}(w(x,\gamma) - w(y,\gamma)) \geq \frac{1}{2}\omega(R)$$

must be valid. Suppose it is the second one and choose $y \in \overline{B_R}$ so that also

$$\max_{|\gamma|=1}(w(x,\gamma) - w(y,\gamma)) \geq \frac{1}{2}\omega(R).$$

The from (13.19) we have

$$\begin{aligned} \mu_n = \max_{|\gamma|=1}(w(y,\gamma) - w(x,\gamma)) &\geq -\frac{\lambda}{(n-1)\Lambda}\mu_1 - \frac{2R}{(n-1)\Lambda}\sup|D\psi| \\ &\geq \frac{\lambda}{2(n-1)\Lambda}\omega(R) - \frac{2R}{(n-1)\Lambda}\sup|D\psi|. \end{aligned}$$

Consequently, in all cases we obtain

$$\max_{|\gamma|=1}(M_1(\gamma) - w(x,\gamma)) \geq \delta\left(\omega(R) - \frac{4R}{\lambda}\sup|D\psi|\right),$$

for some positive constant δ depending on n, Λ/λ.But then, because w is a quadratic form in γ, we must have

$$\int_{|\gamma|=1}(M_1(\gamma) - w(x,\gamma))\,d\gamma \geq \delta^*\left(\omega(R) - \frac{4R}{\lambda}\sup|D\psi|\right)$$

for a further $\delta^* = \delta^*(n, \Lambda/\lambda)$, so that from (13.20), we obtain

$$\omega(R) \leq C\left\{\omega^*(2R) - \omega^*(R) + \frac{R}{\lambda}\sup|D\psi| + \frac{R^2}{\lambda}\sup|D^2\psi|\right\}$$

and the result follows by Lemma 13.12 and the following lemma of Safonov [Sa2].

Lemma 13.16. *We have*

$$\omega(R) \leq C_1\omega^*(R) \leq C_2\omega(R),$$

for some positive constants $C_1 \leq C_2$.

Proof: Since the right hand side inequality is obvious, it sufficies to prove

$$M_1(\gamma) - m_1(\gamma) \le C_1\omega^*(R)$$

for any $|\gamma| = 1$, so that without loss of generality, we can assume $\gamma = e_1$. Writing

$$2D_{11}u(x) = w(x, \gamma + 2e_1) - 2w(x, \gamma + e_1) + w(x, \gamma),$$

we obtain by integration of $|\gamma| < 1$, for any $x, y \in B_R$,

$$\begin{aligned} 2\omega_n(D_{11}u(y) - D_{11}u(x)) &= \left(\int_{B_1(2e_1)} -2\int_{B_1(e_1)} + \int_{B_1(0)}\right)(w(y,\gamma) - w(x,\gamma)) \\ &\le 2\int_{B_3(0)} (M_1(\gamma) - m_1(\gamma))\, d\gamma \\ &= 2\int_0^3 r^{n+1}\, dr \int_{|\gamma|=1} (M_1(\gamma) - m_1(\gamma))\, d\gamma \end{aligned}$$

and hence the result follows with $C_1 = \frac{3^{n+2}}{(n+2)\omega_n}$. ■

The proof of the theorem is now complete. ■

13.5.2 Function Spaces

Here we introduce some suitable function spaces for further study on the classical theory of second order nonlinear elliptic equations. The notations are compatible with those of [GT].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose $u \in C^0(\Omega)$. Define seminorms:

$$[u]_{0,\alpha;\Omega} := \sup_{x,y\in\Omega, x\ne y} \frac{|u(x) - u(y)|}{|x-y|^\alpha},\ 0 < \alpha \le 1,$$

$$[u]_{k;\Omega} := \sup_{|\beta|=k, x\in\Omega} |D^\beta u(x)|, \quad k = 0, 1, 2, \dots,$$

and the space

$$C^{k,\alpha}(\overline{\Omega}) := \left\{ u \in C^k(\Omega) \,\middle|\, ||u||_{k,\alpha;\Omega} := \sum_{j=0}^k [u]_{j;\Omega} + \sum_{|\beta|=k} [D^\beta u]_{\alpha;\Omega} < \infty \right\}.$$

$C^{k,\alpha}(\overline{\Omega})$ is a Banach space under the norm $||\cdot||_{k,\alpha;\Omega}$. Moreover, we see

$$C^k(\overline{\Omega}) = C^{k,0}(\overline{\Omega})$$

with the norm $||u||_{k;\Omega} := \sum_{j=0}^{k}[u]_{j;\Omega}$.

Recall

$$C^{k,\alpha}(\Omega) = \{u \in C^k(\Omega) \,\big|\, u\big|_{\Omega'} \in C^{k,\alpha}(\overline{\Omega}') \text{ for any } \Omega' \Subset \Omega\},$$

and define the interior seminorms:

$$[u]^*_{\alpha;\Omega} := \sup_{\Omega' \Subset \Omega} \operatorname{dist}^{\alpha}_{\Omega'}[u]_{\alpha;\Omega}, \quad 0 < \alpha \leq 1,$$

where $u \in C^{0,\alpha}(\Omega)$, $\operatorname{dist}_{\Omega'} := \operatorname{dist}(\Omega', \partial\Omega)$. Similarly

$$[u]^*_{k;\Omega} := \sup_{\Omega' \Subset \Omega} \operatorname{dist}^{k}_{\Omega'}[u]_{k;\Omega'}, \quad k = 0, 1, 2, \ldots,$$

$$[u]^*_{k,\alpha;\Omega} := \sup_{\Omega' \Subset \Omega, |\beta| = k} \operatorname{dist}^{k+\alpha}_{\Omega'}[D^\beta u]_{k;\Omega'},$$

and

$$C^{k,\alpha}_*(\Omega) := \left\{ u \in C^k(\Omega) \,\Bigg|\, ||u||^*_{k,\alpha;\Omega} := \sum_{j=0}^{k}[u]^*_{j;\Omega} + [u]^*_{k,\alpha;\Omega} < \infty \right\}.$$

$C^{k,\alpha}_*(\Omega)$ is a Banach space under the norm $||\cdot||^*_{k,\alpha;\Omega}$.

13.5.3 Interpolation inequalities

For the aboe defined spaces, there hold various interpolation inequalities.

Lemma 13.17. *Suppose $j + \beta < k + \alpha$, where $j, k = 0, 1, 2, \ldots$, and $0 \leq \alpha, \beta \leq 1$. Let Ω be an open subset of $\mathbb{R}^n_+$ with a boundary portion T on $x_n = 0$, and assume $u \in C^{k,\alpha}(\Omega \cup T)$. Then for any $\epsilon > 0$ and some constant $C = C(\epsilon, j, k)$ we have*

$$\begin{aligned} [u]^*_{j,\beta;\Omega\cup T} &\leq C|u|_{0;\Omega} + \epsilon[u]^*_{k,\alpha;\Omega\cup T'}, \\ |u|^*_{j,\beta;\Omega\cup T} &\leq C|u|_{0;\Omega} + \epsilon[u]^*_{k,\alpha;\Omega\cup T}. \end{aligned} \tag{13.21}$$

Proof: Again we suppose that the right members are finite. The proof is patterned after (the interior global result you did earlier) and we emphasize only the details in which the proofs differ. In the following we omit the subscript $\Omega \cup T$, which will be implicitly understood.

We consider first the cases $1 \le j \le k \le 2$, $\beta = 0$, $\alpha \ge 0$, starting with the inequality

$$[u]_2^* \le C(\epsilon)|u|_0 + \epsilon[u]_{2,\alpha}^*, \quad \alpha > 0. \tag{13.22}$$

Let x be any point in Ω, $\overline{d_x}$ its distance from $\partial\Omega - T$, and $d = \mu\overline{d_x}$ where $\mu \le \frac{1}{4}$ is a constant to be specificed later. If $\text{dist}(x, T) \ge d$, then the ball $B_d(x) \subset \Omega$ and the argument proceeds as in (your interior proof) leading to the inequality

$$\overline{d}_x^2|D_{il}u(x)| \le C(\epsilon)[u]_1^* + \epsilon[u]_{2,\alpha}^*$$

provided $\mu = \mu(\epsilon)$ is chosen sufficiently small. If $\text{dist}(x, T) < d$ we consider the ball $B = B_d(x_0) \subset \Omega$, where x_0 is on the perpendicular to T passing through x and $\text{dist}(x, x_0) = d$. Let x', x'' be the endpoints of the diamter of B parallel to the x_l axis. Then we have for some $\overline{x}$ on this diamater

$$\begin{aligned}|D_{il}u(\overline{x})| &= \frac{|D_iu(x') - D_iu(x'')|}{2d} \le \frac{1}{d}\sup_B |D_iu| \le \frac{2}{\mu}\overline{d}_x^{-2} \sup_{y\in B} \overline{d}_y|D_iu(y)| \\ &\le \frac{2}{\mu}\overline{d}_x^{-2}[u]_1^*, \quad \text{since } \overline{d}_y > \overline{d}_x/2 \ \forall y \in B;\end{aligned}$$

and

$$\begin{aligned}|D_{il}u(x)| &\le |D_{il}u(\overline{x})| + |D_{il}u(x) - D_{il}u(\overline{x})| \\ &\le \frac{2}{\mu}\overline{d}_x^{-2}[u]_1^* + 2d^\alpha \sup_{y\in B} \overline{d}_{x,y}^{-2-\alpha} \sup_{y\in B} \overline{d}_{x,y}^{2+\alpha} \frac{|D_{il}u(x) - D_{il}u(y)|}{|x-y|^\alpha};\end{aligned}$$

hence

$$\begin{aligned}\overline{d}_x^2|D_{il}u(x)| &\le \frac{2}{\mu}[u]_1^* + 16\mu^\alpha[u]_{2,\alpha}^* \\ &\le C[u]_1^* + \epsilon[u]_{2,\alpha}^*\end{aligned}$$

provided $16\mu^\alpha \le \epsilon$, $C = 2/\mu$. Choosing the smaller value of μ, corresponding to the two cases $\text{dist}(x, T) \ge d$ and $\text{dist}(x, T) < d$, and taking the supremem over all $x \in \Omega$ and $i, l = 1, \ldots, n$, we obtain (the first equation in this proof) for $j = k = 1$, we proceed as in (the proof you did for the interior result) with the modifications suggested by the above proof of (the first equation of the proof). Together with the preceding cases, this gives us (the equation in the statement of the lemma) for $1 \leq j \leq k \leq 2$, $\beta = 0, \alpha \geq 0$.

The proof of (the statement eq) for $\beta > 0$ follows closely that in cases (iii) and (iv) of (the interior proof you did before). The principal difference is that the argument for $\beta > 0$, $\alpha = 0$ now requires application of the theorm of the mean in the truncated ball $B_d(x) \cap \Omega$ for points x such that $\text{dist}(x, T) < d$. ■

Lemma 13.18. *Suppose $k_1 + \alpha_1 < k_2 + \alpha_2 < k_3 + \alpha_3$ with k_1, k_2, $k_3 = 0, 1, 2, \ldots$, and $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$. Suppose further $\partial\Omega \in C^{k_3,\alpha_3}$. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon = O(\epsilon^{-k})$ for some positive k such that*

$$||u||_{k_2,\alpha_2;\Omega} \leq \epsilon||u||_{k_2,\alpha_3;\Omega} + C_\epsilon||u||_{k_1,\alpha;\Omega}$$

holds for any $u \in C^{k_3,\alpha_3}(\overline{\Omega})$.

Proof: The proof is based on a reduction to lemma 6.34 by means of an argument very similar to that in Lemma 6.5. As in that lemma, at each point $x_0 \in \partial\Omega$ let $B_\rho(x_0)$ be a ball and ψ be a $C^{k,\alpha}$ diffeomorphism that straightens the boundary in a neighborhood containing $B' = B_\rho(x_0) \cap \Omega$ and $T = B_\rho(x_0) \cap \partial\Omega$. Let $\psi(B') = D' \subset \mathbb{R}^n_+$, $\psi(T) = T' \subset \partial\mathbb{R}^n_+$. Since T' is a hyperplane portion of $\partial D'$, we may apply the interpolation inequality from previous lemma in D' to the function $\tilde{u} = u \circ \psi^{-1}$ to obtain

$$[\tilde{u}]_{j,\beta;D'\cup T'} \leq C(\epsilon)|\tilde{u}|_{0;D'} + \epsilon|\tilde{u}|^*_{k,\alpha;D'\cup T'}.$$

from

Let Ω be a bounded domain with $C^{k,\alpha}$ boundary portion $T, k \geq 1, 0 \leq \alpha \leq 1$. Suppose that $\Omega \Subset D$, where D is a domain that is mapped by a $C^{k,\alpha}$ diffeomorphism ψ onto D'. Letting $\psi(\Omega) = \Omega'$ and $\psi(T) = T'$, we can define the quantities in (definitions of spaces) with respect to Ω' and T'. If $x' = \psi(x)$, $y' = \psi(y)$, one sees that

$$K^{-1}|x - y| \leq |x' - y'| \leq K|x - y|$$

for all points $x, y \in \Omega$, where K is a constant depending on ψ and Ω. Letting $u(x) \to \tilde{u}(x')$ under the mapping $x \to x'$, we find after a calculation using the above equation: for $0 \leq j \leq k$,$0 \leq \beta \leq 1$, $j + \beta \leq k + \alpha$,

$$[u]_{j,\beta;B'\cup T} \leq C(\epsilon)|u|_{0;D'} + \epsilon|u|^*_{k,\alpha;B'\cup T}.$$

In these inequalities K denotes constants depending on the mapping ψ and the domain Ω.

---

if follows that

$$[u]_{j,\beta;B'\cup T} \leq C(\epsilon)|u|_{0;D'} + \epsilon|u|^*_{k,\alpha;B'\cup T}.$$

(we recall that the same notation $C(\epsilon)$ is being used for different functions of ϵ.) Letting $B'' = B_{\rho/2}(x_0) \cap \Omega$, we infer from

We note that $|u|^*_{k;\Omega}$ and $|u|^*_{k,\alpha;\Omega}$ are norms on the subspaces of $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ respectively for which they are finite. If Ω is bounded and $d = \operatorname{diam}\Omega$, then obviously these interior norms and the global norms (standard defs) are related by

$$|u|^*_{k,\alpha;\Omega} \leq \max(1, d^{k+\alpha})|u|_{k,\alpha;\Omega}.$$

If $\Omega' \Subset \Omega$ and $\sigma = \operatorname{dist}(\Omega, \partial\Omega)$, then

$$\min(1, \sigma^{k+\alpha})|u|_{k,\alpha;\Omega'} \leq |u|^*_{k,\alpha;\Omega}$$

$$\begin{aligned} |u|_{j,\beta;B''} &\leq C(\epsilon)|u|_{0,B'} + \epsilon|u|_{k,\alpha;B'} \\ &\leq C(\epsilon)|u|_{0;\Omega} + \epsilon|u|_{k,\alpha;\Omega}. \end{aligned} \tag{13.23}$$

Let $B_{\rho_i/4}(x_i)$, $x_i \in \partial\Omega$, $i = 1, \ldots, N$, be a finite collection of balls covering $\partial\Omega$, such that the intequality (2nd eq prev lemma) holds in each set $B_i'' = B_{\rho_i/2}(x_i) \cap \Omega$, with a constant $C_i(\epsilon)$. Let $\delta = \min \rho_i/4$ and $C = C(\epsilon) = \max C_i(\epsilon)$. Then at every point $x_0 \in \partial\Omega$, we have $B = B_\delta(x_0) \subset B_{\rho_i/2}(x_i)$ for some i and hence

$$|u|_{j,\beta;B\cap\Omega} \leq C|u|_{0;\Omega} + \epsilon|u|_{k,\alpha;\Omega} \tag{13.24}$$

The remainder of the argument is analogous to that in Theorem 6.6 and is left to the reader. ■

13.5.4 Global Estimates

Returning to the interior estimate, we can write the second derivative estimate Theorem 4.1 in the form:

$$[u]^*_{2,\alpha;\Omega} \leq C([u]^*_{2;\Omega} + 1),$$

where C depends on $\lambda, \Lambda, n, ||\psi||_{2;\Omega}$ and $\operatorname{diam}\Omega$. Therefore if λ, Λ are independent of u, we obtain by interpolation

$$||u||^*_{2,\alpha;\Omega} \leq C\left(\sup_\Omega |u| + 1\right).$$

Now we want to prove the next global second derivative Hölder estimate due to essentially Krylov [K6].

Theorem 13.19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^3$. Suppose $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ fulfils*

$$\begin{cases} F(D^2u) = \psi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13.25}$$

where $\psi \in C^2(\overline{\Omega})$. $F \in C^2(\mathbb{S}^n)$ is assumed to satisfy i.) and ii.) in the first theorem of this section. Then we have an estimate

$$|u|_{2,\alpha;\Omega} \leq C(|u|_{2;\Omega} + |\psi|_{2;\Omega}),$$

where $\alpha = \alpha(n, \Lambda/\lambda) > 0$, $C = C(n, \Omega, \Lambda/\lambda)$.

Proof: First we straighten the boundary.

Let $x_0 \in \partial\Omega$. By (Defintion of boundary reg.), there exists a ball $B = B(x_0) \subset \mathbb{R}^n$ and a one-to-one mapping $\chi : B \to D \subset \mathbb{R}^n$ such that

i.) $\chi(B \cap \Omega) \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$

ii.) $\chi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$

iii.) $\chi \in C^3(B)$, $\chi^{-1} \in C^3(\Omega)$.

Let us write $y := \chi(x)$. An elementary calculation shows

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \frac{\partial \chi^k}{\partial x_i}(x) \cdot \frac{\partial u}{\partial y_k} \\ \frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial \chi^k}{\partial x_i}\frac{\partial \chi^l}{\partial x_j} \cdot \frac{\partial^2 u}{\partial y_k \partial y_l} + \frac{\partial}{\partial x_j}\left(\frac{\partial \chi^k}{\partial x_i}\right) \cdot \frac{\partial u}{\partial y_k}. \end{aligned}$$

Thus, the ellipticity condition is preserved under the transformation χ with

$$\tilde{\lambda} I \leq [J\, F_r\, J^T] \leq \tilde{\Lambda} I, \quad J := D_\chi,$$

where $0 < \tilde{\lambda} \leq \tilde{\Lambda} < \infty$. Moreover, χ also preserves the Hölder spaces and transforms the equation (**??**) into the more general form:

$$F(x, u, Du, D^2u) = 0.$$

Therefore it suffices to deal with the following situation:

Suppose u satisfies

$$\begin{cases} F(x, u, Du, D^u) = 0 & \text{in } B := B_R^+ = \{x \in \mathbb{R}^n_+ \ |x| < R\} \\ u = 0 & \text{on } T := \{x \in B \,|\, x_n = 0\}, \end{cases}$$

where $F \in C^3(\overline{B} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n)$.

Differentiation with respectto x_k, $1 \le k \le n-1$ leads

$$\frac{\partial F}{\partial r_{ij}} D_{ijk}u + \frac{\partial F}{\partial p_i} D_{ik}u + \frac{\partial F}{\partial u} D_k u + \frac{\partial F}{\partial x_k} = 0,$$

which can be summarized in the form

$$\begin{cases} Lv = f & \text{in } B \\ v = 0 & \text{on } T. \end{cases} \tag{13.26}$$

Here we have set $v := D_k u$, $1 \le k \le n-1$ and

$$\begin{aligned} L &= a^{ij} D_{ij} := \frac{\partial F}{\partial r_{ij}} D_{ij} \\ -f &:= \frac{\partial F}{\partial p_i} D_{ij}u + \frac{\partial F}{\partial u} D_k u + \frac{\partial F}{\partial x_k}, \end{aligned}$$

with $\lambda I \le [a^{ij}] \le \Lambda I$.

You can apply the Krylov boundary estimate (previous boundary result) to obtain

$$\begin{aligned} \operatorname*{osc}_{B_\delta^+} \frac{v}{x_n} &\le C\left(\frac{\delta}{R}\right)^\alpha \left\{ \operatorname*{osc}_{B_R^+} \frac{v}{x_n} + \frac{R}{\lambda} |f|_{0;B_R^+} \right\} \\ &\le C\left(\frac{\delta}{R}\right)^\alpha \left\{ |u|_{2;B_R^+} + \frac{R}{\lambda} |f|_{0;B_R^+} \right\}. \end{aligned}$$

where α, C depend on n and Λ/λ. By subtracting a linear function of x_n from v if necessar, we can assume that $v(x) = O(|x_n|^2)$ as $x \to 0$. Thus, fixing R and taking $\delta = x_n$, we find

$$\frac{v(x)}{x_n^{1+\alpha}} \le C(|u|_{2;B_R^+} + |f|_{0;B_R^+}). \tag{13.27}$$

This is the boundary estimate to start with.

For the interior estimate, Theorem 4.1 gives for any $B_r \subset B_R^+$

$$\begin{aligned} \operatorname*{osc}_{B_{\sigma r}} &\le C\sigma^\alpha \left(\sup_{B_r} |D^2 u| + 1 \right) \\ &\le C\sigma^\alpha \left(\sup_{B_r} |DD'u| + 1 \right), \end{aligned} \tag{13.28}$$

where $D'u = (D_1 u, \ldots, D_{n-1}u)$, $\alpha = \alpha(n, \Lambda/\lambda) > 0$ and C depend furthermore on f. The second inequality in (13.28) comes from the equation itself.

In fact, $F(x, u, Du, D^2u) = 0$ can be solved so that $D_{nn}u$ = function of $(x, u, Du, DD'u)$ and hence

$$|D_{nn}u| \leq C(\sup |DD'u| + 1).$$

(13.28) implies in particular

$$\underset{B_{\sigma R}}{\operatorname{osc}} DD'u \leq C\sigma^{\alpha} \left(\sup_{B_r} |DD'u| + 1 \right).$$

By interpolating the interior norm, we get from (13.27)

$$\begin{aligned} [DD_k u]_{\alpha;B_r} &\leq C\{\operatorname{dist}(B_r, T)^{-1-\alpha}|D'u|_{0;B_r} + 1\} \\ &\leq C(|u|_{2;B_R^+} + |f|_{0;B_R^+}), \end{aligned} \tag{13.29}$$

for $1 \leq k \leq n-1$.

The equation itself again shows that (13.29) holds true for $k = n$.

In summary, we finally obtain

$$[D^2u]_{\alpha;B_R^+C}(|u|_{2;B_R^+} + |\psi|_{2;B_R^+}).$$

Usual covering argument now yields the full global estimate. This completes the proof. ■

Corollary 13.20. *Under the assumptions of the previous theorem, we have by interpolation*

$$\begin{aligned} |u|_{2,\alpha;\Omega} &\leq C(|u|_{0;\Omega} + |\psi|_{2;\Omega}) \\ &\leq C|\psi|_{2;\Omega}. \end{aligned}$$

13.5.5 Generalizations

The method of proof of theorem above provides more general results. We present one of them without proof.

Theorem 13.21. *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves the linar partial differential equation of the type (13.26):*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^2$, $f \in L^{\infty}(\Omega)$ and

$$\underset{B_{\sigma R}}{\operatorname{osc}} Du \leq C_0\sigma^{\alpha} \left(\underset{B_R}{\operatorname{osc}} Du + |f|_{0;\Omega} + 1 \right),$$

for some $\delta = \delta(n, \alpha, \Lambda/\lambda)$, $C = C(n, \alpha, C_0, \Omega, \Lambda/\lambda$.

The form of the interior gradient estimate can be weakened; (see [T3], Theorem 4).

13.6 Existence Theorems

After establishing various estimates, we are now able to investigate the problem of the existence of solutions. This section is devoted to the solvability for equations of the form

$$F(D^2u) = \psi, \tag{13.30}$$

Two cases are principally discussed; (i) uniformly elliptic case and (ii) Monge-Ampere equation.

We begin with recalling the method of continuity. Then (i)(ii) are studied.

13.6.1 Method of Continuity

First we quickly review the usual existence method of continuity.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and let $F : \mathcal{U} \to \mathcal{B}_2$ be a mapping, where $\mathcal{B} \subset \mathcal{B}_1$ is an open set. We say that F is differentiable at $u \in \mathcal{U}$ if there exists a bounded linear mapping $L : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\frac{||F[u+h] - F[u] - Lh||_{\mathcal{B}_2}}{||h||_{\mathcal{B}_1}} \to 0 \quad \text{as } h \to 0 \text{ in } \mathcal{B}_1. \tag{13.31}$$

We write $L = F_u$ and call it the Fréchet derivative F at u. When $\mathcal{B}_1$, $\mathcal{B}_2$ are Euclidean spaces, $\mathbb{R}^n$, $\mathbb{R}^m$, the Fréchet derivative coincides with the usual notion of differentia, and , moreover, the basic theory for the infinite dimensional cas can be modelled on that for the finite dimensional case as usually treated in advanced calculus. In particular, it is evident from (the above limit) that the Fréchet differentiablity of F at u implies that F is continuous at u and that the Fréchet derivative F_u is determined uniquely by (the limit def above). We call F *continuously differentiable* at u if F is Fréchet differentiable in a neighbourhood of u and the resulting mapping

$$v \mapsto F_v \in E(\mathcal{B}_1, \mathcal{B}_2)$$

is continuous at u. Here $E(\mathcal{B}_1, \mathcal{B}_2)$ denotes the Banach space of bounded linear mapping from $\mathcal{B}_1$ to $\mathcal{B}_2$ with the norm given by

$$||L|| = \sup_{\substack{v \in \mathcal{B}_1 \\ v \neq 0}} \frac{||Lv||_{\mathcal{B}_2}}{||v||_{\mathcal{B}_1}}.$$

The chain rule holds for Fréchet differentiation, that is, if $F : \mathcal{B}_1 \to \mathcal{B}_2$, $G : \mathcal{B}_2 \to \mathcal{B}_3$ are Fréchet differentiable at $u \in \mathcal{B}_1$, $F[u] \in \mathcal{B}_2$, respectively, then the composite mapping $G \circ F$ is differentiable at $u \in \mathcal{B}_1$ and

$$(G \circ F)_u = G_{F[u]} \circ F_u. \tag{13.32}$$

The theorem of the mean also holds in the sense that if $u, v \in \mathcal{B}_1$, $F : \mathcal{B}_1 \to \mathcal{B}_2$ is differentiable on the closed line segment γ joining u and v in $\mathcal{B}_1$, then

$$||F[u] - F[v]||_{\mathcal{B}_2} \leq K||u - v||_{\mathcal{B}_1}, \tag{13.33}$$

where

$$K = \sup_{w \in \gamma} ||F_w||.$$

With the aid of these basic properties we may deduce an *implicit fuinction theorem* for Fréchet differentiable mappings. Suppose that $\mathcal{B}_1$, $\mathcal{B}_2$ and X are Banach spaces and that $G : \mathcal{B}_1 \times X \to \mathcal{B}_2$ is Fréchet differentiable at a point (u, σ), where $u \in \mathcal{B}_1$ and $\sigma \in X$. The *partial Fréchet derivatives*, $G^1_{(u,\sigma)}$, $G^2_{(u,\sigma)}$ at (u, σ) are the bounded linear mappings from $\mathcal{B}_1$, X, respectively, into $\mathcal{B}_2$ defined by

$$G_{(u,\sigma)}(h, k) = G^1_{(u,\sigma)}(h) + G^2_{(u,\sigma)}(k)$$

for $h \in \mathcal{B}_1$, $k \in X$. We state the implicit function theorem in the following form.

Theorem 13.22. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ and X be Banach spaces and G a mapping from an open subset of $\mathcal{B}_1 \times X$ into $\mathcal{B}_2$. Let (u_0, σ_0) be a point in $\mathcal{B}_1 \times X$ satisfying:*

i.) $G[u_0, \sigma_0] = 0$;

ii.) G is continuously differentiable at (u_0, σ_0);

iii.) the partial Fréchet derivative $L = G^1_{(u_0,\sigma_0)}$ is invertible.

Then there exists a neighborhood $\mathcal{N}$ of σ_0 in X such that the equation $G[u, \sigma] = 0$, is solvable for each $\sigma \in \mathcal{N}$, with solution $u = u_\sigma \in \mathcal{B}_1$.

Sketch of proof: This may be proved by reduction to the contraction mapping principle. Indeed the equation $G[u, \sigma] = 0$, is equivalent to the equation

$$u = Tu \equiv u - L^{-1}G[u, \sigma],$$

and the operator T will, by virtue (13.31) and (13.32), be a contraction mapping in a closed ball $\overline{B} = \overline{B}_\delta(u_0)$ in $\mathcal{B}_1$ for δ sufficiently small and σ sufficiently close to σ_0 in X. One can further show that the mapping $F : X \to \mathcal{B}_1$ defined by $\sigma \to u_\sigma$ for $\sigma \in \mathcal{N}$, $u_\sigma \in \overline{B}$, $G[u_\sigma, \sigma] = 0$ is differentiable at σ_0 with Fréchet derivative

$$F_{\sigma_0} = -L^{-1} G^2_{(u_0,\sigma_0)}.$$

In order to apply the above theorem we suppost that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces with F a mapping from an open subset $\mathcal{U} \subset \mathcal{B}_1$ into $\mathcal{B}_2$. Let ψ be a fixed element in $\mathcal{U}$ and define for $u \in \mathcal{U}$, $t \in \mathbb{R}$ the mapping $G : \mathcal{U} \times \mathbb{R} \to \mathcal{B}_2$ by

$$G[u,t] = F[u] - tF[\psi].$$

Let $\mathcal{S}$ and $\mathcal{E}$ be the subsets of $[0,1]$ and $\mathcal{B}_1$ defined by

$$\begin{aligned} \mathcal{S} &:= \{t \in [0,1] \,|\, F[u] = tF[\psi] \text{ for some } u \in \mathcal{U}\} \\ \mathcal{E} &:= \{u \in \mathcal{U} \,|\, F[u] = tF[\psi] \text{ for some } t \in [0,1]\}. \end{aligned}$$

Clearly $1 \in \mathcal{S}$, $\psi \in \mathcal{E}$ so that $\mathcal{S}$ and $\mathcal{E}$ are not empty. Let us next suppose that the mapping F is continuously differentiable on $\mathcal{E}$ with invertible Fréchet derivative F_u. It follows then from the implicit function theorem (previous theorem) that the set $\mathcal{S}$ is open in $[0.1]$. Consequently we obtain the following version of the method of continuity.

Theorem 13.23. *Suppose F is continuously Fréchet differentiablewith the invertable Fréchet derivative F_u. Fix $\psi \in \mathcal{U}$ and set*

$$\begin{aligned} \mathcal{S} &:= \{t \in [0,1] \,|\, F[u] = tF[\psi] \textit{ for some } u \in \mathcal{U}\} \\ \mathcal{E} &:= \{u \in \mathcal{U} \,|\, F[u] = tF[\psi] \textit{ for some } t \in [0,1]\}. \end{aligned}$$

Then the equation $F[u] = 0$ is solvable for $u \in \mathcal{U}$ if $\mathcal{S}$ is closed.

The aim is to apply this to the Dirichlet problem. Let us define

$$\begin{aligned} \mathcal{B}_1 &:= \{u \in C^{2,\alpha}(\overline{\Omega}) \,|\, u = 0 \,\text{on}\, \partial\Omega\} \\ \mathcal{B}_2 &:= C^\alpha(\overline{\Omega}), \end{aligned}$$

and let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be given by

$$F[u] := F(x, u, Du, D^2u).$$

Assume $F \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n)$. Then F is continuously Fréchet differentiable in $\mathcal{B}_1$ with

$$\begin{aligned} F_v[u] \quad = \quad & \frac{\partial F}{\partial r_{ij}}(x, v(x), Dv(x), D^2 v(x)) D_{ij} u \\ & + \frac{\partial F}{\partial p_i}(x, v(x), Dv(x), D^2 v(x)) D_i u + \frac{\partial F}{\partial z}(x, v(x), Dv(x), D^2 v(x)) u. \end{aligned}$$

Check this formular and observe that $F \in C^{2,\alpha}$ is more than enough.

To proceed further, we recall the result of Classical Schauder theory proven in Chapter 9(?) (put here for convenience:

Theorem 13.24. *Consider the linear operator*

$$Lu := a^{ij} D_{ij} u + b^i D_i u + cu,$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. Assume

$$\begin{aligned} & a^{ij}, b^i, c \in C^{\alpha}(\overline{\Omega}), \quad c \leq 0 \\ & \lambda I \leq [a^{ij}] \leq \Lambda I, \quad 0 < \lambda \leq \Lambda I, \quad 0 < \lambda \leq \Lambda < \infty. \end{aligned}$$

Then for any function $f \in C^{\alpha}(\overline{\Omega})$, there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ and $|u|_{2,\alpha;\Omega} \leq C|f|_{\alpha;\Omega}$, where C depends on n, λ, $|a^{ij}, b^i, c|_{\alpha;\Omega}$ and Ω.

Theorem 13.25. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{2,\alpha}$ for some $0 < \alpha < 1$. Let $\mathcal{U} \subset C^{2,\alpha}(\overline{\Omega})$ be open and let $\psi \in \mathcal{U}$. Define*

$$\mathcal{E} := \{u \in \mathcal{U} \,|\, F[u] = \sigma F[\psi] \text{ for some } \sigma \in [0,1] \text{ and } u = \psi \text{ on } \partial\Omega\},$$

where $F \in C^{2,\alpha}(\overline{\Gamma})$ with $\Gamma := \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$. Suppose further

i.) F is elliptic on $\overline{\Omega}$ with respect to $u \in \mathcal{U}$;

ii.) $F_z \leq 0$ on $\mathcal{U}$; (or $F_z \geq 0$ on U?)

iii.) $\mathcal{E}$ is bounded in $C^{2,\alpha}(\overline{\Omega})$;

iv.) $\overline{\mathcal{E}} \subset \mathcal{U}$, where the closure is taken in an appropriate sense.

Then there exists a unique solution $u \in \mathcal{U}$ of the following Dirichlet problem:

$$\begin{cases} F[u] = 0 & \text{in } \Omega \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

Proof: We can reduce to the case of zero boundary values by replacing u with $u - \psi$. The mapping $G : \mathcal{B}_1 \times \mathbb{R} \to \mathcal{B}_2$ is then defined by taking $\psi \equiv 0$ so that

$$G[u, \sigma] = F[u + \psi] - \sigma F[\psi].$$

It then follows from the previous theorem and the remarks proceeding the statement of this theorem, that the given Dirichlet problem is solvable provided the set $\mathcal{S}$ is closed. However the closure of $\mathcal{S}$ (and also of $\mathcal{E}$) follows readily from the boundedness of $\mathcal{E}$ (and hypothesis iv.)) by virtue of the Arzela-Ascoli theorem. ■

Observe that by this theorem, the solvability of the Dirichlet problem depends upon the estimates in $C^{2,\alpha}(\overline{\Omega})$ independent of σ (this is why we need Schauder theory).

13.6.2 Uniformly Elliptic Case

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \subset C^3$. Suppose $F \in C^2(\mathbb{S}^n)$ satisfying the following

i.) $\lambda I \leq F_r(s) \leq \Lambda I$ for all $s \in \mathbb{S}^n$ with some fixed constants $0 < \lambda \leq \Lambda$.

ii.) F is concave.

We also assume without loss of generality that $F(0) = 0$.

Theorem 13.26 (Krylov [K2]. .] *Under the above assumptions, the Dirichlet problem*

$$\begin{cases} F(D^2u) = \psi & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases} \tag{13.34}$$

is uniquely solvable for any $\psi \in C^2(\overline{\Omega})$. The solution u belongs to $C^{2,\alpha}(\overline{\Omega})$ for each $0 < \alpha < 1$.

Proof: Recall the a-priori estimate (global result prev. section)

$$[u]_{2,\alpha\Omega} \leq C(|u|_{0;\Omega} + |\psi|_{2;\Omega}),$$

where $\alpha = \alpha(n, \Lambda/\lambda) > 0$, $C = C(n, \partial\Omega, \Lambda/\lambda)$.

Since we can write

$$F(D^2u) = \left(\int_0^1 F_{r_{ij}}(\theta D^2 u)\, d\theta \right) D_{ij}u, \tag{13.35}$$

by virture of $F(0) = 0$, the classical maximum principle (find this) gives

$$|u|_{0;\Omega} \le \frac{|\psi|_{0;\Omega}}{\lambda} (\operatorname{diam} \Omega)^2.$$

It is easy to see that the corresponding estimates for $F(D^2u) = \sigma\psi$, $\sigma \in [0,1]$ are independent of σ. Thus the result follows from the method of continuity. ■

Remark: Once you get $u \in C^2(\overline{\Omega})$, then the linear theory ensures $u \in C^{2,\alpha}(\overline{\Omega})$ for all $\alpha < 1$.

The above proof owes its validity to the regularity of F. If you want to examine a non-smooth F, you need to mollify F; replace F by

not the most obvious argument actually;explain it

$$F_\epsilon(r) = \int_{\mathbb{S}^n} \rho\left(\frac{r-s}{\epsilon}\right) F(s)\, ds,$$

where $\rho \in C_0^\infty(\mathbb{S}^n)$ is defined by

$$\rho \ge 0, \qquad \int_{\mathbb{S}^n} \rho(s)\, ds = 1.$$

Observe that F_ϵ satisfies the assumptions (i)(ii). Then solve the problem

$$\begin{cases} F_\epsilon(D^2u_\epsilon) = \psi & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases},$$

and let $\epsilon \to 0$. The examples include the following Bellman equation:

$$F(D^2u) = \inf_{k=1,\dots,n} a_k^{ij} D_{ij}u.$$

Next, we wish to modify the argument to include the non-zero boundary value problems, especially with continuous boundary values:

$$\begin{cases} F(D^2u) = \psi & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{13.36}$$

where $g \in C^0(\partial\Omega)$.

As a first step,we approximate g by $\{g_n\} \subset C^2(\overline{\Omega})$ such that $g_n \to g$ uniformly on $\partial\Omega$ as $n \to \infty$. Secondly, in view of $\partial\Omega \in C^3$, you can construct barriers; for any $y \in \partial\Omega$, there exists a neighbourhood $\mathcal{N}_y$ of y and a function $w \in C^2(\mathcal{N}_y)$ such that i.) $w > 0$ in $\overline{\Omega \cap \mathcal{N}_y} - \{y\}$

ii.) $w(y) = 0$,

iii.) and $F(D^2 w) \leq \psi$ in $\Omega \cap \mathcal{N}_y$.

(refer to peron process for proofs about barries).

Suppose $\{u_n\} \subset C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ are given by the solutions of

$$\begin{cases} F(D^2 u_n) = \psi & \text{in } \Omega \\ u_n = g_n & \text{on } \partial\Omega \end{cases}$$

Then we can appeal to the linear theory to conclude that $\{u_n\}$ are equicontinuous on $\overline{\Omega}$, taking into account (13.34) and the existence of barriers.

On the other hand, the Schauder estimate (put the theorem here) implies the equicontinuity of Du, $D^2 u$ on any compact subset of Ω. Therefore, by letting $n \to \infty$, we obtain $u \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ which enjoys (13.36).

13.6.3 Monge-Ampére equation

The equation we wish to examine here is

$$\det D^2 u = \psi \quad \text{in } \Omega, \tag{13.37}$$

where $\Omega \subset \mathbb{R}^n$ is a iniformly convex domain with $\partial\Omega \in C^3$.

(13.37) is elliptic with respect to u provided $D^2 u > 0$. Thus the set $\mathcal{U}$ of admissable functions is defined to be

$$\mathcal{U} := \{u \in C^{2,\alpha}(\overline{\Omega}) \mid D^2 u > 0\}.$$

Now our main result is the next

Theorem 13.27 (Krylov [K4,5). *, Caffarelli, Nirenberg and Spruck [CNS1].] Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex domain with $\partial\Omega \in C^3$ and let $\psi \in C^2(\overline{\Omega})$ be such that $\psi > 0$ in $\overline{\Omega}$. Then there exists a unique solution $u \in \mathcal{U}$ of*

$$\begin{cases} \det D^2 u = \psi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.38}$$

For the rest of this subsection, we prove this theorem, employing apriori estimates step-by-step *Proof:*

Step 1: $|u|_{2;\Omega}$ estimate implies $|u|_{2,\alpha;\Omega}$ estimate.

First, we show the concavity of $F(r) := (\det r)^{1/n}$ for $r > 0$. Recalling,

$$(\det D^2 u)^{1/n} (\det \mathcal{A})^{1/n} \leq \frac{1}{n}(a^{ij} D_{ij} u),$$

we have

$$(\det D^2u)^{1/n} = \inf_{\det \mathcal{A}\geq 1} \frac{1}{n}(a^{ij}D_{ij}u).$$

This proves the concavity of F, since the RHS is a concavity function of D^2u.

Next recall $\det D^2u = \prod_{i=1}^n \lambda_i$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalue of D^2u.

Thus we have

$$\lambda_i = \frac{\psi\lambda_i}{\prod_{j=1}^n \lambda_j} \geq \frac{\inf\psi}{(\sup|D^2u|)^{n-1}},$$

which means that the C^2 boundedness of u ensures the uniform ellipticity of the equation.

The (globals $C^{2,\alpha}$ estimate theorem thingy) then implies

$$|u|_{2,\alpha;\Omega} \leq C,$$

where $C = C(|u|_{2;\Omega}, |\psi|_{2;\Omega}, \partial\Omega)$. This proves Step 1.

Step 2: $|u|_{0;\Omega}$ and $|u|_{1;\Omega}$ estimates.

Since u is convex, we have $u \leq 0$ in Ω. Let $w := A|x|^2 - B$ and compute

$$\det D^2w = (2nA)^n.$$

Taking A, B large enough, we find

$$\begin{aligned} \det D^2w \geq \psi \qquad & \text{in } \Omega \\ w \leq u \qquad & \text{on } \partial\Omega, \end{aligned}$$

from which we obtain $|u|_{0;\Omega} \leq C$ by applying the maximum principle.

Next we want to give a bound for $|u|_{1;\Omega}$. The convexity implies

$$\sup_{\Omega}|Du| \leq \sup_{\partial\Omega}|Du|.$$

Thus it sufficies to estimate Du on the boundary $\partial\Omega$. More precisely, we with to estimate u_ν from above, where u_ν denotes the exterior normal derivative of u on $\partial\Omega$. This is because $u_\nu \geq 0$ on $\partial\Omega$ and the tangential derivatives are zero.

We exploit the uniform convexity of the domain Ω; for any $y \in \partial\Omega$, there exists a C^2 function w defined on a ball $\Omega \subset B_y$ such that $w \leq 0$ in $\overline{\Omega}$,

$w(y) = 0$ and $\det D^2 w \geq \psi$ in Ω. This can be seen by taking $\Omega \subset B_y$ with $\partial B_y \cap \overline{\Omega} = \{y\}$ and considering the function of the type $A|x - x_0|^2 - B$ as before.

Therefore, we infer

$$w \leq u \quad \text{in } \Omega$$

and in particular $w_\nu \geq u_\nu$ at y. Since $\partial\Omega$ is compact, we obtain $|u|_{1;\partial\Omega}$ estimate.

Step 3: $\sup_{\partial\Omega} |D^2 u|$ *estimate implies* $\sup_\Omega |D^2 u|$ *estimate.*

We use the concavity of

$$F(D^2 u) := (\det D^2 u)^{1/n} = \psi^{1/n}.$$

as in (the linearization first part), we deduce

$$F_{ij} D_{ij} D_{\gamma\gamma} u \geq D_{\gamma\gamma}(\psi^{1/n}) \quad \text{for } |\gamma| = 1.$$

Then the result of (the linear app of AB max prin) leads to

$$D_{\gamma\gamma} u \leq \sup_{\partial\Omega} D_{\gamma\gamma} u + C \cdot \sup_\Omega \frac{|D_{\gamma\gamma}(\psi^{1/n})|}{\mathrm{Tr}(F_{ij})}.$$

On the other hand, we have

$$F_{ij} = \frac{1}{n} (\det D^2 u)^{-1+1/n} \cdot u^{ij},$$

where $[u^{ij}]$ denotes the cofactor matrix of $[u_i j]$. Compute

$$\begin{aligned} \det F_{ij} &= \frac{1}{n^n} \\ \frac{\mathrm{Tr}\, F_{ij}}{n} &\geq (\det F_{ij})^{1/n} = \frac{1}{n}. \end{aligned}$$

Thus, there holds

$$D_{\gamma\gamma} u \leq \sup_{\partial\Omega} D_{\gamma\gamma} u + C,$$

where C depends only on the know quantitites. This proves Step 3.

Step 4: $\sup_{\partial\Omega} |D^2 u|$ *estimate.*

Take any $y \in \partial\Omega$. Without loss of generality, we may suppose that y is the origin and the x_n-axis points the interior normal direction.

Near the origin, $\partial\Omega$ is represented by

$$x_n = \omega(x') := \frac{1}{2}\Omega_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3).$$

Here $x' := (x_1, \ldots, x_{n-1})$ and in the summation, Greek letters go from 1 to $n-1$. The uniform convexity of $\partial\Omega$ is equivalent to the positivity of $\Omega_{\alpha\beta}$. For more details, see (section after next).

Our aim is an estimation for $|u_{ij}(0)|$. First, we see that the boundary condition $u = 0$ on $\partial\Omega$ implies

$$(D_\beta + \omega_\alpha D_n)(D_\alpha + \omega_\beta D_n)u = 0$$

on $\partial\Omega$ near the origin; in particular,

$$|D_{\alpha\beta}u(0)| \leq C$$

at the origin.Next,differential the equation (13.43) to obtain

$$u^{ij}D_{ij}(D_\alpha u + \omega_\alpha D_n u) = D_\alpha\psi + \omega_\alpha D_n\psi + 2u^{ij}\omega_{\alpha i}D_{jn}u + u^{ij}D_n u D_{ij}\omega.$$

By virtue of $u^{ij}D_{jn}u = \delta_{in}\det D^2u$,there holds

$$|u^{ij}D_{ij}(D_\alpha + \omega_\alpha D_n u)| \leq C(1 + \operatorname{Tr} u^{ij}),$$

in $\Omega \cap \mathcal{N}$ denotes a suitable neighbourhood of the origin.

On the other hand, introduce a barrier function of the type

$$w(x) := -A|x|^2 + Bx_n,$$

and compute

$$u^{ij}D_{ij}w = -2A\operatorname{Tr} u^{ij}.$$

Choosing A and B large enough, we conclude that

$$\begin{aligned} |u^{ij}D_{ij}(D_\alpha u + \omega_\alpha D_n u)| + u^{ij}D_{ij}w &\leq 0 \qquad \text{in } \Omega \cap \mathcal{N} \\ |D_\alpha u + \omega_\alpha D_n u| &\leq w \qquad \text{on } \partial(\Omega \cap \mathcal{N}). \end{aligned}$$

By the maximum principle, we now find

$$|D_\alpha u + \omega_\alpha D_n u| \leq w \quad \text{in}\Omega \cap \mathcal{N}$$

and hence

$$|D_{n\alpha}u(0)| \le B.$$

To summarize, we have so far derived

$$|D_{kl}u(0)| \le C \quad \text{if } k+l < 2n \tag{13.39}$$

Finally, we want to estimate $|D_{nn}u(0)|$. Since $u = 0$ on $\partial\Omega$, the unit outer normal ν on $\partial\Omega$ is given by

$$\nu = \frac{Du}{|Du|}$$

and so

$$D^2u(0) = \begin{pmatrix} |Du|D'\nu & D_{in}u \\ D_{ni}u & D_{nn}u \end{pmatrix}(x), \tag{13.40}$$

where $D' := (\partial/\partial x_1, \ldots, \partial/\partial x_{n-1})$. The choice of our coordinates gives

$$D'\nu(0) = \operatorname{diag}[\kappa'_1, \ldots, \kappa'_{n-1}](0)$$

and $|Du(0)| = |D_n u(0)|$. Here $\kappa'_1, \ldots, \kappa'_{n-1}$ denote the principal curvatures of $\partial\Omega$ at 0. (see section after next). By our regularity assumption on $\partial\Omega$, for any $y \in \partial\Omega$, there exists an interior ball $B_y \subset \Omega$ with $\partial B_y \cap \partial\Omega = \{y\}$. Thus the comparison argument as before yields

$$|Du| \ge \delta > 0 \tag{13.41}$$

for some $\delta > 0$.

In view of (13.40), we have

$$\begin{aligned} \psi(0) &= \det D^2u(0) \\ &= |D_n u|^{n-2} \prod_{i=1}^{n-1} \kappa'_i \left\{ |D_n u| D_{nn}u - \sum_{i=1}^{n-1} \frac{(D_{in}u)^2}{\kappa'_i} \right\}(0), \end{aligned}$$

from which we infer

$$0 < D_{nn}u(0) \le C, \tag{13.42}$$

taking into account (13.41) and (13.39) and the uniform convexity $\kappa'_i > 0$. This proves our claim.

Now we have established an estimate for $|u|_{2,\alpha;\Omega}$ and the method of continuity can be applied to compete the proof. ■

13.6.4 Degenerate Monge-Ampere Equation

Here we want to generalize (the previous theorem) so that ψ merely enjoys $\psi \geq 0$ in Ω.

In the proof of (the above theorem), the steps 2 and 3 are not altred. The step 4 is also as before except for the estimation of $D_n u$ away from zero (see (13.41), which follows from the convexity of u and being able to assume $\psi(x_0) > 0$ for some $x_0 \in \Omega$. The step 1 does not hold true in general.

To deal with these points, we approximate ψ by $\psi+\epsilon$ for $\epsilon > 0$ sufficiently small. Theorem (above) is applicable and we obtain a convex $C^{2,\alpha}$ solution u_ϵ. We with to make $\epsilon \to 0$. However, an estimation independent of ϵ can be obtained up to $|u|_{1;\Omega}$.

In summarly, by virtue of the convexity of u_ϵ, we derive

Theorem 13.28. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex domain with $\partial\Omega \in C^3$ and let $\psi \in C^2(\overline{\Omega})$ be non-negative in $\overline{\Omega}$. Then there exists a unique solution $u \in C^{1,1}(\overline{\Omega})$ of*

$$\begin{cases} \det D^2 u = \psi^n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.43}$$

13.6.5 Bondary Curvatures

Here we collect some properties of boundary curvatures and the distance function, which are taken frpm the Appendix in Ch14 of[GT].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with its boundary $\partial\Omega \in C^2$. Take any rotation of coordinate if necessary, the x_n axis lies in the direction of the inner normal to $\partial\Omega$ at the origin.Also, there exists a neighbourhood $\mathcal{N}$ of the origin such that $\partial\Omega$ is given by the height function $x_n = \omega(x')$ whre $x' := (x_1, \ldots, x_{n-1})$. Moreover, we can assume that h is represented by

$$\omega(x') = \frac{1}{2}\Omega_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3), \quad \Omega_{\alpha\beta} = D_{\alpha\beta}\omega(0),$$

in $\mathcal{N} \cap T$ where T denotes the tangent hyperplane to $\partial\Omega$ at the origin.

The principal curvatures of $\partial\Omega$ at the origin are now defined as the eigenvalues of $[h_{ij}]$ and we denote the $\kappa_1', \ldots, \kappa_{n-1}'$. The corresponding eigenvectors are called the principal directions.

We define the tangential gradient ∂ on $\partial\Omega$ as

$$\partial := D - \nu(\nu \cdot D),$$

where ν denotes the unit outer normal to $\partial\Omega$. ν can be extended smoothly in a neighbourhood of $\partial\Omega$. In terms of ∂, we infer that $\{\kappa'_1, \dots, \kappa'_{n-1}, 0\}$ are given by the eigenvalues of $[\partial\nu]$.

Let $d(x) := \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Then we find $d \in C^{0,1}(\overline{\Omega})$ and $d \in C^2\{0 < d < a\}$ where $a := 1/max_{\partial\Omega}|\kappa'|$. We call $d(x)$ the distance function. Remark that $\nu(x) = -Dd(x)$ for $x \in \partial\Omega$.

A principal coordinate system at $y \in \partial\Omega$ consists of the x_n axis being along the unit normal and others along the principal directions. For $x \in \{x \in \overline{\Omega} \,|\, 0 < d(x) < a\}$, let $y = y(x) \in \partial\Omega$ be such that $|y(x) - x| = d(x)$. Then with respect to a principal coordinate system at $y = y(x)$, we have

$$[D^2 d(x)] = \operatorname{diag}\left[\frac{-\kappa'_1}{1 - d\kappa'_1}, \dots, \frac{-\kappa'_{n-1}}{1 - d\kappa'_{n-1}}, 0\right],$$

and for $\alpha, \beta = 1, 2, \dots, n-1$,

$$\begin{aligned} D_{\alpha\beta} u &= \partial_\alpha \partial_\beta u - D_n u \cdot \kappa'_\alpha \delta_{\alpha\beta} \\ &= \partial_\alpha \partial_\beta u - D_n u \cdot \Omega_{\alpha\beta}. \end{aligned}$$

If $u = 0$ on $\partial\Omega$, then $D_{\alpha\beta} u = -D_n u \cdot \Omega_{\alpha\beta}$ for $\alpha, \beta = 1, 2, \dots, n-1$ at $y = y(x)$.

13.6.6 General Boundary Values

The estimations in (ma equation) extend automatically to the case of general Dirichlet boundary values given by a function $\phi \in C^3(\overline{\Omega})$, except for the double normal second derivative on the boundary. Here the reultant estimate (for $\phi \in C^{3,1}(\overline{\Omega})$) is due to Ivochkina [Iv4]. We indicate here a simple proof of this estimate using a recent trick. introduced by Trudinger in [T14], in the study of Hessian and curvature equations. Accordingly, les us suppose that $u \in C^3(\overline{\Omega})$ is a convex solution of the Dirichlet problem,

$$\begin{cases} \det D^2 u = \psi & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{13.44}$$

with $\psi \in C^2(\overline{\Omega})$, $\inf_\Omega \psi > 0$ as before, and $\phi \in C^4(\overline{\Omega})$, $\partial\Omega \in C^4$. Fix a point $y \in \partial\Omega$ and a principal coordinate system at y. Then the estimation of $D_{nn}u(y)$ is equivalent to estimation away from zero of the minimum eigenvalue λ' of the matrix

$$D_{\alpha\beta} u = \partial_\alpha \partial_\beta - D_n u \cdot \Omega_{\alpha\beta}, \quad 1 \le \alpha, \beta \le n-1.$$

Nor for $\xi = (\xi_1, \ldots, \xi_{n-1})$, $|\xi| = 1$, the function

$$w = D_n u - \frac{\partial_\alpha \partial_\beta \phi \xi_\alpha \xi_\beta}{\Omega_{\alpha\beta}\xi_\alpha\xi_\beta} - A|x'|^2$$

will take a maximum value at a point $z \in \mathcal{N} \cap \partial\Omega$, for sufficiently large A, depending on $\mathcal{N}$, $\max|Du|$, $\max|D^2\phi|$ and $\partial\Omega$. But then an estimate

$$D_{nn}u(z) \leq C$$

follows by the same barrier considerations as before, whence $\lambda'(z)$ is estimated away from zero and consequently $\lambda'(y)$ as required. Note that this argument does not extend to the degenerate case. We thus establish the following extension of (non-deg monge-ampere exist theorem) to general boundary values [CNS1]

Theorem 13.29. *Let Ω be a uniformly convex domain with $\partial\Omega \in C^4$, $\phi \in C^4(\overline{\Omega})$ and let $\psi \in C^2(\overline{\Omega})$ be such that $\psi > 0$ in $\overline{\Omega}$. Then there exists a unique solution $u \in \mathcal{U}$ of the Dirichlet problem:*

$$\begin{cases} \det D^2 u = \psi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} .$$

Appendix A: Notation

For derivatives $D = \nabla$ and $\vec{Du} = \left(\frac{\partial}{\partial x_1}u, \ldots, \frac{\partial}{\partial x_n}u\right)$.

Bibliography

List of Figures

Index