3.10.23

Module II

Dual of ~~the~~ an LPP

consider an LPP,

maximise $z = CX$

st $AX \leq B$

with $X \geq 0$

Then the dual of this LPP is given by taking the dual variable

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

as maximise $z' = B^T . Y$

st $A^T Y \geq C^T$

with $Y \geq 0$

The given LPP is called primal LPP.

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Q: write the relationship b/w primal LPP & its dual.

A→

- Let max $z = \cancel{A}CX$

  st ~~with~~ $AX \leq B$

  with $X \geq 0$ be the given primal LPP.

  If the ~~opti~~ objective fn of the primal LPP is ~~to be~~ maximization, then that of the dual LPP ~~is~~ will be minimization & vice-versa.

- If the primal has 'm' constraints in 'n' variables then dual will have 'n' constraints in 'm' variables & vice-versa.

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~~Q: write the relationship b/w primal LPP & its dual.~~

- Transpose of coefficients of the variables in the objective fn of primal will be the constants ~~of~~ on the right of the constraints ~~of the~~ in the dual & vice-versa.
- If all the constraints in the primal are less than or equal to type then the constraints in the dual will be '$\geq$' type.
- Transpose of the coefficient matrix A of the primal is ~~are~~ the coefficient matrix of the dual & vice-versa.
- If the primal has an equality constraint the corresponding dual variable will un-restricted & vice-versa.

Q: Determine dual of the following LPP.

maximise $z = 4x + 4y$

st $x + 2y \leq 10$

$x + y \leq 6$

$x \leq 4$

with $x, y \geq 0$

A→ Since, there are 3 constraints in the given primal LPP, we take 3 dual variables,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The given LPP may be expressed in the form :-

maximise $z = CX$ → max $z = \begin{bmatrix} 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

st $AX \leq B$ → st $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}$

with $X \geq 0$ → with $X \geq 0$

$$\min \quad z' = B^T Y$$
$$\text{st} \quad A^T Y \geq C^T \quad \Rightarrow$$
$$\text{with} \quad Y \geq 0$$

The dual of the given LPP is $\Rightarrow$

$$\min \quad z' = \begin{bmatrix} 10 & 6 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\text{st} \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\text{with} \quad Y \geq 0$$

Q: Find the dual of the following ~~tot~~ LPP

$$\max \quad z = 4x_1 + 2x_2$$
$$\text{st} \quad \cancel{x_1 - x_2 \leq} \; x_1 + x_2 \geq 3$$
$$-x_1 + x_2 \geq -2$$
$$\text{with} \quad x_1, x_2 \geq 0$$

A) The given LPP may expressed in the form

$$\text{maximise} \quad z = 4x_1 + 2x_2$$
$$\text{st} \quad -x_1 - x_2 \leq -3$$
$$x_1 - x_2 \leq 2$$
$$\text{with } x_1, x_2 \geq 0$$

In matrix form,

$$\max \quad z = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{st} \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\text{with} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0.$$

Since there are 2 constraints in the given primal LPP, we introduce 2 dual LPP

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\therefore$ The dual of the given primal LPP is given by

$$\min \quad z' = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\text{st} \quad \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \cancel{\begin{bmatrix} 4 & 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\text{with} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0$$

Q: compute the dual of the following LPP

$$\text{maximise} \quad z = 4x_1 + 2x_2 + x_3$$
$$\text{st} \quad x_1 + x_2 \leq 10$$
$$3x_1 + x_2 + x_3 \geq 23$$
$$7x_1 - x_3 = 6 \rightarrow \begin{matrix} 7x_1 - x_3 \leq 6 \\ 7x_1 - x_3 \geq 6 \end{matrix}$$
$$\text{with} \quad x_1, x_2 \geq 0, \quad x_3 \text{ unrestricted.}$$

A) Taking $x_3 = x_3' - x_3''$, we may express the given LPP in the following form

$$\max \quad z = 4x_1 + 2x_2 + x_3' - x_3''$$
$$\text{st} \quad x_1 + x_2 \leq 10 \qquad y_1$$
$$-3x_1 - x_2 - x_3' + x_3'' \leq -23 \qquad y_2$$
$$7x_1 - x_3' + x_3'' \leq 6 \qquad y_3'$$
$$-7x_1 + x_3' - x_3'' \leq -6 \qquad y_3''$$
$$\text{with} \quad x_1, x_2, x_3', x_3'' \geq 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -3 & -1 & -1 & 1 \\ 7 & 0 & -1 & 1 \\ -7 & 0 & 1 & -1 \end{bmatrix}$$

17.10.23 The dual of the LPP is given by,

$$\text{Min } z' = 10y_1 - 23y_2 + 6\cancel{3}y_3' - 6y_3''$$

$$\cancel{A} \begin{bmatrix} 1 & -3 & 7 & -7 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \geq \begin{bmatrix} 4 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

A^T

$$y_1 - 3y_2 + 7y_3' - 7y_3'' \geq 4$$
$$y_1 - y_2 \geq \cancel{4}\, 2$$
$$-y_2 - y_3' + y_3'' \geq 1$$
$$y_2 + y_3' - y_3'' \geq -1 \quad \text{with } y_1, y_2, y_3', y_3'' \geq 0.$$

Taking $y_3 = y_3' - y_3''$, we can write the dual.

$$\text{Min } z' = 10y_1 - 23y_2 + 6y_3$$

st $y_1 - 3y_2 + 7y_3 \geq 4$

$y_1 - y_2 \geq 2$

$-y_2 - y_3 \geq 1$

$y_2 + y_3 \geq -1$

$$\left.\begin{matrix} -y_2 - y_3 \geq 1 \\ -y_2 - y_3 \leq 1 \end{matrix}\right\} -y_2 - y_3 = 1$$

with y_1, y_2 unrestricted, $y_3 \geq 0$.

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dual (max) = $\leq$

Q: Find the dual of the following LPP

maximise, $z = 2x_1 + 3x_2 + x_3$

st $4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$3x_1 - 7x_2 + 9x_3 \leq 60$

with $x_1, x_2 \geq 0$ & x_3 ~~must~~ unrestricted

restricted

A: Since there are 3 constraints in the ~~given~~ primal LPP we introduce 3 dual variables y_1, y_2 and y_3

Dual is

$$\text{min } z' = 6y_1 + 4y_2 + 60y_3$$

st $4y_1 + 1y_2 + 3y_3 \geq 2$

$3y_1 + 2y_2 - 7y_3 \geq 3$

$y_1 + 5y_2 + 9y_3 = 1$

with $y_3 \geq 0$ and y_1, y_2 unrestricted.

Result :-

- Dual of the dual is primal.
- If the primal LPP has an unbounded solution then the sol of the dual is infeasible and vice-versa.
- If the dual problem has a degenerate soln, the primal problem will have alternate soln.

⇒ Fundamental theorem of duality :-

If both the primal & dual problem has feasible solns then both will have optimal soln. In this case, maximum of the primal equal minimum of the dual.

Q: compare of Primal LPP and Dual LPP :-

A>

	PRIMAL	DUAL
(i)	objective fn is to maximised.	· Objective fn is to be minimised.
(ii)	'n' variables & 'm' constraints.	· 'm' variables & 'n' constraint
(iii)	constraints are '$\leq$' type.	· constraints are '$\geq$' type
(iv)	c_j's are coefficient of the variables of the objective fn of the dual.	· B_j's are the coefficient of the variables of the objective fn of the dual.
(v)	Variables are non-negative.	· constraints are in-equality constraints.
(vi)	Variables are unrestricted.	· constraints are equality.
(vii)	Equality constraints.	· Unrestricted variables.
(viii)	Alternate ~~optima~~ optimum.	· Degenerate soln.
(ix)	It has Unbounded soln.	· Soln is infeasible.
(x)	Infeasible soln	· Unbounded soln.

Q: Solve the following LPP using principle of duality :-

$$\max \quad z = \cancel{14x}\ 40x + 35y$$

st $2x + 3y \leq 60$

$4x + 3y \leq 96$

with $x, y \geq 0$

A> Introducing 2 variables y_1 & y_2, we get the dual

$$\min \quad z' = 60y_1 + 96y_2$$

st $2y_1 + 4y_2 \geq 40$

$3y_1 + 3y_2 \geq 35$

with $y_1, y_2 \geq 0$

Introducing surplus variables, s_1, s_2 & artifical variables A_1 & A_2 with penalty MA_1, MA_2

$$\min \quad z = 60y_1 + 96y_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

st $2y_1 + 4y_2 - s_1 + A_1 = 40$

$3y_1 + 3y_2 - s_2 + A_2 = 35$

with $y_1, y_2, s_1, s_2, A_1, A_2 \geq 0$

IBFS

$y_1 = y_2 = s_1 = s_2 = 0$

$A_1 = 40, A_2 = 35$

table-1

	c_j	60	90	0	0	M	M		
C_B	X_B	y_1	y_2	s_1	s_2	A_1	A_2	soln	ratio ~~result~~
M	A_1	2	$\boxed{4}$	-1	0	1	0	40	$\frac{40}{4} = 0$
M	A_2	3	3	0	-1	0	1	35	$\frac{35}{3} = 11.3$
	z_j	5M	7M	-M	-M	M	M	75M	
	$\Delta_j = c_j - z_j$	60-5M	96-7M ↑	M	M	0	0		

table-2

	c_j	60	90	0	0	M		
C_B	X_B	y_1	y_2	s_1	s_2	A_2	soln	ratio
96	y_2	1/2	1	-1/4	0	0	10	20
M	A_2	$\boxed{3/2}$	0	3/4	-1	1	5	$\frac{10}{3}$ →
	z_j	$\frac{3}{2}M + 48$	$\frac{3}{4}M - 24$		-M	M	5M+960	
	$\Delta_j = c_j - z_j$	$12 - \frac{3M}{2}$ ↑	0	$24 - \frac{3M}{4}$	M	0		

table-3

	c_j	60	90	0	0		
C_B	X_B	y_1	y_2	s_1	s_2	soln	ratio
96	y_2	0	1	-1/2	1/3	25/3	
60	y_1	1	0	1/2	-2/3	10/3	
	z_j	60	96	~~18~~ -18	-8		
	Δ_j	0	0	18	8		

Since, all Δ_j's are non-negative, this is optimal.

Optimium of dual

$y_1 = 10/3$, $y_2 = 25/3$

Min $z' = 96 \times \frac{25}{3} + 60 \times \frac{10}{3}$

$z' = 1000$

31-10-23

Q: Solve the following LPP using the principle of duality.

$$\max \; z = 4x_1 + 2x_2$$
$$\text{st} \quad x_1 + x_2 \geq 3$$
$$x_1 - x_2 \geq 2$$
$$\text{with} \quad x_1, x_2 \geq 0$$

A: Introducing 2 dual variables y_1 & y_2.

The given LPP can be written as,

$$\max \; z = 4x_1 + 2x_2$$
$$\text{st} \quad -x_1 - x_2 \leq -3$$
$$-x_1 + x_2 \leq -2$$
$$\text{with} \quad x_1, x_2 \geq 0$$

now, we find the dual as:

$$\min \; z' = -3y_1 - 2y_2$$
$$\text{st} \quad -y_1 - y_2 \geq 4$$
$$-y_1 + y_2 \geq 2$$
$$\text{with} \quad y_1, y_2 \geq 0$$

Since there are '$\geq$' type constraints, we use big M method to solve the dual LPP. Introduce surplus variables S_1, S_2 and artifical variables A_1, A_2 with a high +ve penalty MA_1, MA_2.

LPP becomes,

$$\min \; z' = -3y_1 - 2y_2 + 0S_1 + 0S_2 + MA_1 + MA_2$$
$$\text{st} \quad -y_1 - y_2 - S_1 + A_1 = 4$$
$$-y_1 + y_2 - S_2 + A_2 = 2$$
$$\text{with} \quad y_1, y_2, S_1, S_2, A_1, A_2 \geq 0$$

To find IBFS,

$y_1 = y_2 = S_1 = S_2 = 0$

$A_1 = 4$

$A_2 = 2$

table 1

	C_j	(-3	-2	0	0	M	M)		
C_B	X_B	y_1	y_2	S_1	S_2	A_1	A_2	soln	ratio
M	A_1	-1	-1	-1	0	1	0	4	-
M	A_2	-1	1	0	-1	0	1	2	2
	Z_j	-2M	0	-M	-M	M	M		
	$\Delta_j = C_j - Z_j$	-3+2M	-2	M	M	0	0		

$A_2 \rightarrow$ leaving variable

$y_2 \rightarrow$ entering variable

$1 \rightarrow$ key element

table 2

	C_j	-3	-2	0	0	M	M	
C_B	X_B	y_1	y_2	S_1	S_2	A_1	soln	ratio
M	A_1	-2	0	-1	-1	1	6	
-2	y_2	-1	1	0	-1	0	2	
	Z_j	-2M+2	-2	-M	-M+2	M		
	Δ_j	2M-5	0	M	M-2	0		

Since, all the Δ_j are non-negative, ~~then~~ this is the optimum table.

But this table has an artifical variable in the basis.

So, the dual LPP has infeasible solution (no feasible solution)

$\therefore$ Primal has unbounded solution.

—x—

- Apply graphical method to solve the LPP

$$\max \quad z = 4x_1 + 2x_2$$

$$\text{st} \quad x_1 + x_2 \geq 3$$

$$x_1 - x_2 \geq 2$$

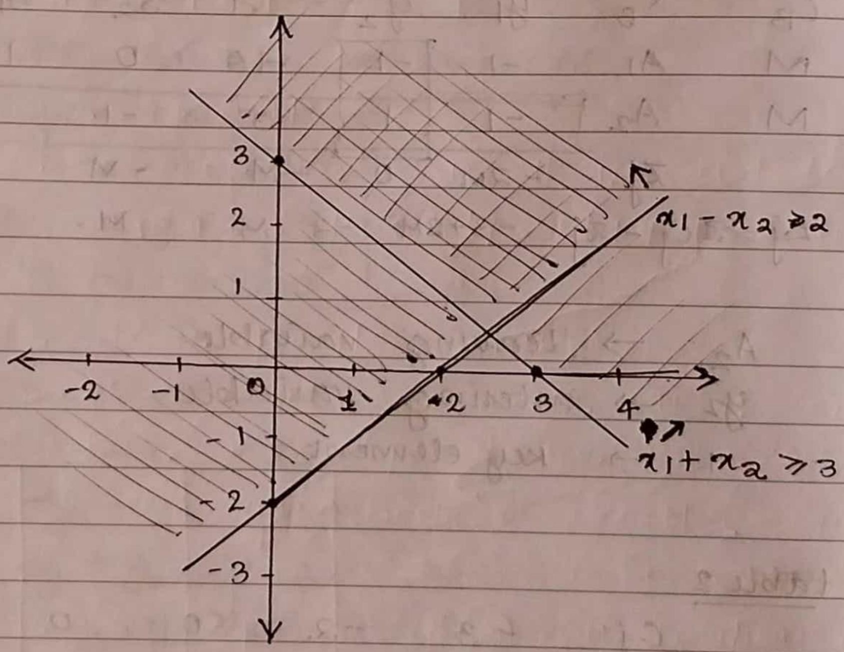
$$\text{with} \quad x_1, x_2 \geq 0$$

$\rightarrow x_1 + x_2 = 3$

x_1	0	3
x_2	3	0

$\rightarrow x_1 - x_2 = 2$

x_1	0	2
x_2	-2	0



ie., primal has unbounded solution proved.

—x—