

1.11.23

Module IIITRANSPORTATION PROBLEM

Transportation problem is a special class of allocation problem in which the objective is to transport various quantities of a single homogenous commodity that are kept at different origins to a number of destinations such as shops, markets etc.

Let  $C_{ij}$  be the cost of transporting 1 unit of commodity from  $m$  origins  $O_1, O_2, O_3, \dots, O_m$  to the different shops / destinations  $d_1, d_2, \dots, d_n$ . Then the cost matrix of a TP can be expressed as a table.

	$D_1$	$D_2$	$D_3$	$\dots$	$D_n$	Available
$O_1$	$C_{11}$	$C_{12}$	$C_{13}$	$\dots$	$C_{1n}$	$a_1$
$O_2$	$C_{21}$	$C_{22}$	$C_{23}$	$\dots$	$C_{2n}$	$a_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$O_m$	$C_{m1}$	$C_{m2}$	$C_{m3}$	$\dots$	$C_{mn}$	$a_m$

~~to~~

where  $a_i$ 's are the quantity of resource available in each of the origins  $O_i = 1, 2, 3, \dots, m$ .

and  $b_j$ 's are the requirement, demand of the shops.

Let  $x_{ij}$  be the no. of unit of ~~commodity~~ commodity transported from the  $i^{\text{th}}$  origin to  $j^{\text{th}}$  ~~origin~~ destination. Then the mathematical form~~the~~ of TP may be expressed as

$$\min \quad Z = \sum_i \sum_j C_{ij} x_{ij}$$

st

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, 3, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, 3, \dots, n$$

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j, x_{ij} \geq 0 \quad x_{ij} \geq 0$$

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→ Balanced and Unbalanced Transportation Problem  
A TP is said to be balanced if the demand of the shops = sum of the resources.

A feasible solution of a TP is a set of non-ve individual allocation which satisfy the row & column sum restrictions.

- Basic feasible solutions:-

A feasible solution of an  $m \times n$  TP is said to be basic feasible soln if the total no. of non-ve allocation =  $m+n-1$ .

- Optimal solution of a TP:-

A feasible soln of an  $m \times n$  TP is said to be optimal if it minimizes the total transportation cost.

- Non degenerate basic feasible solution:-

A feasible solution of an  $m \times n$  TP is said to be non-degenerate basic feasible solution if

• The no. of non-zero allocations is exactly equal to  $m+n-1$ .

• These allocations are at independent positions:

- A set of allocations of a TP are said to be at independent positions if it is not possible to form a polygon joining occupied cells.

- Methods of finding IBFS:-

3 methods:-

- North-West corner rule
- Matrix minimum / minimum cost method
- VAM (Vogel's approximation method)

Q. Find an initial feasible soln to the following TP using

- North-West corner rule
- Matrix - <sup>minimum</sup> multiplication
- Vogel's approximation method.

-> a) North-West Corner Rule:-

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	Availability
O <sub>1</sub>	5	2	4	0
O <sub>2</sub>	3	3	1	0
O <sub>3</sub>	5	4	7	0
O <sub>4</sub>	1	6	2	14
	7	9	18	
	2	3	14	

no. of allocations  
=  $m+n-1$   
=  $4+3-1$   
= 6 allocations

North-west corner rule:-

Allocate the min. of 5 and 7 to the (1,1) cell at the north-west corner of the table. Then, the resource of polygon origin O<sub>1</sub> is fully utilized. Then strike of the 1st row. Then allocate the min of (3, 7-5) to the cell (2,1) which is at the n-w corner position of D<sub>1</sub> is fully satisfied. Strike of the first column. Continue this procedure until all demands are fulfilled.



utilized.

b) Matrix - min minimum method :-

	D1	D2	D3	availability
O1	2	2	4	5 0
O2	3	3	1	8 0
O3	5	4	7	4 7 0
O4	1	6	2	14 7 0
demand	7	9	18	
	0	7	10	

$$\text{cost} = 5 \times 7 + 8 \times 1 + 4 \times 4 + 3 \times 7 + 7 \times 1 + 7 \times 2$$

$$= 76$$

c) VAM method :-

	D1	D2	D3		difference of min. of lowest & next lowest
O1	2	2	4	5(0)	5(0) -
O2	3	3	1	8(2)	-
O3	5	4	7	7(1)	7(1) + (1) 7(1) 3
O4	1	6	2	14(1)	14(1) 14(1) 14(1) 4(5)
	7(1)	9(1)	18(1)		
	7(1)	9(2)	10(2)		
	7(4)	4(2)	10(5)		
	7(4)	4(2)	10(5)		

$$\Rightarrow 5 \times 2 + 8 \times 1 + 3 \times 5 + 4 \times 4 + 4 \times 1 + 10 \times 2$$

$$= 73$$

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8. Find an initial feasible solution for the following TP using north-west corner rule & then use matrix modified - distribution method to find the optimal solution, which will minimize the total cost of transportation.

	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	w <sub>4</sub>	w <sub>5</sub>	Available res.	
F <sub>1</sub>	20	2	4	6	8	9	20
F <sub>2</sub>	20	6	4	1	5	8	30 10 4 0
F <sub>3</sub>	4	11	4	11	4	3	15 11 0
F <sub>4</sub>	2	1	9	3	14	6	16 13 6
demand	40	6	8	18	6		

check: total demand = total availability

$$40 + 6 + 8 + 18 + 6 = 20 + 30 + 15 + 13$$

$$78 = 78$$

(if shortage, add row or column)

check  $\rightarrow$  no. of non-zero allocation if  $m+n-1 = 8 \Rightarrow 4+5-1 = 8$

$\therefore$  no. of non-zero allocation  $= m+n-1$

MODI method :-

the no. of

### MODI method

$$V_1=2, V_2=10, V_3=1, V_4=21, V_5=28$$

$u_1=0$	20		(-6)	(5)	(-13)	(-14)
	2	4	6	8	9	
$u_2=0$	20	(6)	4		(-16)	(-19)
	2	10	1	9	8	
$u_3=19$	7	(-18)	4	11	40	(-37)
	7	11	20	40	3	
$u_4=-7$	(7)	(2)	9	(15)	7	6
	2	1	9	14	6	16

$$m+n=11$$

variables are introduced

$$u_i + v_j = C_{ij}$$

$$m+n-1 \text{ no. of equations}$$

$$u_i + v_j = C_{ij}$$

$$u_1 + v_1 = 2$$

$$u_2 + v_1 = 2$$

$$u_2 + v_2 = 10$$

$$u_2 + v_3 = 1$$

$$u_3 + v_3 = 20$$

$$u_3 + v_4 = 40$$

$$u_4 + v_4 = 14$$

$$u_4 + v_5 = 16$$

assign values to  $u_i$ 's &  $v_j$ 's.  
(assuming putting  $u_1=0$ )

cell evaluation / net evaluation

corresponding to each unoccupied cell

$$\text{calculate } d_{ij} = C_{ij} - (u_i + v_j)$$

$$\bullet \left( \frac{6}{7} \right) \Rightarrow 4 \Rightarrow -6 \quad [C_{ij} - (u_i + v_j) = 4 - (0 + 10)] = -6$$

highest ~~neg~~ -ve in  $d_{ij}$ , give max. allocation

- construct polygon starting from highest -ve  $d_{ij}$  whose corners are on the basic/occupied cell.

- alternate cell's allocation along the polygon & skip other allocations.

$$\text{find min. of } \{6, 11\} = 6$$

allocate that min. of to the cell that have the highest negative  $d_{ij}$  value.

$$V_1=2, V_2=10, V_3=1, V_4=21, V_5=-16$$

$u_1=0$	20		(-6)	(5)	(-13)	(-14)
	2	4	6	8	9	
$u_2=0$	20	(6)	4		(-16)	(-19)
	2	10	1	9	8	
$u_3=19$	7	(-18)	4	11	40	(-37)
	7	11	20	40	3	
$u_4=-7$	(7)	(2)	9	(15)	7	6
	2	1	9	14	6	16

$$11 - 6 = 5$$

$$6 - 6 = 0$$

$$7 + 6 = 13$$

in polygon  
- skip = add min. value  
- alternate cell  
- sub. min. value

$$\min \{4, 6\} = 4$$

$$V_1=2, V_2=10, V_3=1, V_4=39, V_5=2$$

$u_1=0$	20		(-6)	(5)	(-31)	(-7)
	2	4	6	8	9	
$u_2=0$	20	(2)	(8)		(-34)	(-6)
	2	10	1	9	8	
$u_3=1$	7	(5)	4	(18)	5	(6)
	7	11	20	40	3	
$u_4=-25$	(15)	(2)	(14)	9	(-13)	(-16)
	2	1	9	14	6	16

$$\min \{5, 2\} = 2$$

corners only

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classmate

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Solve the following transportation problem with the following cost matrix

	$w_1$	$w_2$	$w_3$	$w_4$	availability
$P_1$	190	300	500	100	70
$P_2$	700	300	400	600	90
$P_3$	400	100	600	200	180
demand	50	80	70	140	

Using Vogel's method to find initial feasible soln

→

	$w_1$	$w_2$	$w_3$	$w_4$	
$P_1$	190	300	500	100	70 (90) 20(200) 20(400)
$P_2$	700	300	400	600	90 (100) 90 (100) 90 (300) 90 (200) 90 (200) 70 (400)
$P_3$	400	100	600	200	180 (100) 180 (100) 100 (400) 100 (400) -
	50 (210)	80 (200)	70 (100)	140 (100)	

= highest diff. -  
 = lowest cost  
 = (availability & demand) min  
 = strike

$Cost = 90 \times 190 + 2000 + 28000 + 12000 + 8000 + 20000$   
 $= 95000$   
 $= 49,500$

Now, we use ~~modern~~ modified distribution method to find the optimum solution.

$V_1 = 190, V_2 = 0, V_3 = 100, V_4 = 100$   
 $U_1 = 0, U_2 = 300, U_3 = 100$

	$w_1$	$w_2$	$w_3$	$w_4$	
$P_1$	190	300	500	100	70
$P_2$	700	300	400	600	90
$P_3$	400	100	600	200	180
	50	80	70	140	

⇒ Unoccupied cells  
 $d_{12} = 300 - 0 = 300$

Some  $d_{ij} < 0$ , so the current soln is not optimum.  
 $\min(90, 20) = 20$

Allocate a quantity which is equal to the min. of those allocations in the alternate occupied cell connected by the polygon.

$V_1 = 190, V_2 = 0, V_3 = 100, V_4 = 100$   
 $U_1 = 0, U_2 = 300, U_3 = 100$

	$w_1$	$w_2$	$w_3$	$w_4$	
$P_1$	190	300	500	100	70
$P_2$	700	300	400	600	90
$P_3$	400	100	600	200	180
	50	80	70	140	

⑤ This is optimum.

$d_{ij} = c_{ij} - (u_i + v_j) \geq 0$   
since  $d_{ij}$  is greater than 0, it gives the optimum solution.

$$\begin{aligned} \text{The min. cost of transportation} &= \\ &= 50 \times 100 + 20 \times 100 + 20 \times 300 + 40 \times 400 + \\ &60 \times 100 + 120 \times 200 \\ &= 77500 \end{aligned}$$

Q: Solve the following TP to max. profit availability

15	51	42	33	25
80	42	26	81	45
90	40	66	60	35

demand 23 31 16 30 105

The following table giving profit when 1 unit of commodity is transported from origin to destination.

A) Here there is excess resource of 5 units, so we introduce a dummy shop label  $\begin{cases} \geq \text{availability} = 105 \\ \geq \text{demand} = 100 \end{cases}$



while adding profit  $\rightarrow$  min cost  $\rightarrow 0$   
column or row.

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⑤

Since all  $d_{ij}$  are non-negative, this given table

min cost  $\rightarrow$  0 in dummy cell/row

max  $\rightarrow$  max. profit

to min to  $w_3$ , then

$$\begin{aligned} \text{The min cost of transportation} &= 50 \times 100 + 20 \times 100 + \\ &20 \times 300 + 40 \times 400 + \\ &40 \times 100 + 120 \times 200 \\ &= 75500 \end{aligned}$$

15	51	42	33	90	25
80	42	26	81	90	45
90	40	66	60	90	35
23	31	16	30	5	

max. profit

Remove Replace each cell by highest profit  $\rightarrow$  minus each cell value.

15	51	42	33	90	25
80	42	26	81	90	45
90	40	66	60	90	35
23	31	16	30	5	



70	39	48	57	0
10	48	64	9	0
0	50	24	30	0

When we are given an unbalanced TP, to make it balanced, we will be giving a dummy row when demand is higher than availability. ~~For~~ or a dummy col. (demand is lesser than availability) with 0 cost in each cell if the given problem is minimising the cost.

If the given TP is of max profit, we will give the max. profit to ~~the~~ <sup>each cell of the</sup> adding dummy row or col.

MODI method is used to minimize the total cost, so when we are given a TP in which our aim is to maximize the profit, we will convert the given TP into 1 in which our aim is to min. the cost, for this we will replace the each cell entry by the highest profit <sup>minus</sup> the corresponding cell entry.

To solve the following problem, Vogan's method used to find the initial solution.



20(9)	16(29)	45	39	48	57	0	25(39)	20(9)	20(9)	20(9)
15(16)	15(16)	10	48	64	9	0	45(9)	45(1)	45(39)	15(16)
-	-	23	31	16	30	5	35(0)	35(24)	12(6)	12(26)
		0	50	24	30	0				
		23(10)	31(9)	16(24)	30(21)	5(0)				
		23(10)	31(9)	16(24)	30(21)	-				
		-	31(9)	16(24)	30(21)	-				
		-	31(9)	16(24)	-	-				
		-	31(9)	4(24)	-	-				
		-	31(9)	-	-	-				



	16	4		5	25
15	51	42	33	90	
	15		30		45
80	42	26	81	90	
23		12			35
90	40	66	60	90	
23	31	16	30	5	



### - Assignment problem :-

It is a special case of TP in which the objective is to assign a no. of origins (person, machines etc) to an equal no. of jobs/tasks in such a way that the total cost of assignment is min (total profit max).

eg: Suppose there are 4 persons in a dept available for doing 4 different jobs,

Now our objective is to assign 4 jobs to 4 persons in such a way that total cost/time is min. Let  $c_{ij}$  be the cost if the  $i^{\text{th}}$  person is assigned to  $j^{\text{th}}$  job. Then the cost matrix can be represented as

	$J_1$	$J_2$	...	
$P_1$	$c_{11}$	$c_{12}$	...	$\vdots$
$P_2$	$c_{21}$	$c_{22}$	...	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$c_{ij}$	$\vdots$
	1	1	1 ... 1	

### Mathematical formulation of assignment problem :-

$$\min z =$$

Let  $x_{ij}$  represents the no. of persons assigned from  $i^{\text{th}}$  origin to  $j^{\text{th}}$  destination. Then the mathematical formulation of given problem is

$$\min z = \sum_{j=1}^n \sum_{i=1}^n c_{ij} x_{ij}$$

$$\text{st } \sum_{j=1}^n x_{ij} = 1, \quad i=1, 2, 3, \dots, n$$

$$\sum_{i=1}^n x_{ij} = 1, \quad j=1, 2, 3, \dots, n$$

$$\text{with } x_{ij} = 0 \text{ or } 1$$

→ Method for solving AP :- Hungarian method

Q: Solve the following minimal AP

	J <sub>1</sub>	J <sub>2</sub>	J <sub>3</sub>	J <sub>4</sub>
P <sub>1</sub>	12	30	21	15
P <sub>2</sub>	18	33	9	31
P <sub>3</sub>	44	25	21	21
P <sub>4</sub>	14	30	28	14

-A→

- Find the min. of each row & subtract it with ~~from the~~ cell value in each row.

0	18	9	3
9	24	0	22
23	4	0	0
0	16	14	0

Find the min. of each col & subtract this quantity min. from the cell of the respective cells cols.

0	14	9	3
9	20	0	22
23	0	X	X
X	12	14	0

Identify the row ~~having~~ having exactly one 0. Then mark a square ~~(assign)~~ to that cell. Next, each col & strike the 0's of that column.

- Subtract row min with each cell value.

\* The optimum assignment is ~~Assign~~

P<sub>1</sub> → J<sub>1</sub>, P<sub>2</sub> → J<sub>2</sub>, P<sub>3</sub> → J<sub>3</sub>, P<sub>4</sub> → J<sub>4</sub>

Minimum cost of assignment = 12 + 9 + 25 + 14 = 60

Q: Solve the following AP involving 5 jobs & 5 persons

	J <sub>1</sub>	J <sub>2</sub>	J <sub>3</sub>	J <sub>4</sub>	J <sub>5</sub>
P <sub>1</sub>	1	3	2	3	6
P <sub>2</sub>	2	4	3	1	5
P <sub>3</sub>	5	6	3	4	6
P <sub>4</sub>	3	1	4	2	2
P <sub>5</sub>	1	5	6	5	4

Form a table, by replacing each

Find the min. of each row & subtract with this min. ~~with~~ from the cells of respective rows.

0	2	1	2	5
1	3	2	0	4
2	3	0	1	3
2	0	3	1	1
0	4	5	4	3

↓

0	2	1	2	4
1	3	2	0	3
2	3	0	1	3
2	0	3	1	X
X	4	5	4	2

Using minimal no. of horizontal & vertical lines, cover all 0's

no. of lines = no. of ~~rows~~ rows  $\Rightarrow$  optimum soln.

Find the min. of those entries ~~of~~ which no line passes. Subtract this min from these entries & add ~~this min~~ <sup>it to the</sup> to those cells where 2 lines ~~to~~ crosses & leave the remaining ~~entries~~ entries as such

Find the min. of those entries over which no lines passes. Subtract this min from these entries. & add it to the entries which have cross intersection of lines & leave the remaining entries as such.

→

min = 1

X	1	X	1	3
2	3	2	0	3
3	3	0	1	2
3	0	3	1	X
0	3	5	4	2

Using min. no. of horizontal & vertical lines, cover all zeroes. If the no. of lines = no. of ~~tot~~ rows, then soln is optimum. Hence, the above table is not optimum.

- \* find the min. of those entries over which no line passes.
- \* subtract this min. from those entries.
- \* add this min. to those cells where 2 lines crosses and leave the remaining entries as such.



no line  $\rightarrow$  sub(min)  
 line passing cells  $\rightarrow$  no change  
 intersection  $\rightarrow$  add(min)

1	1	1	1	3	✓
2	3	2	0	3	0
3	3	0	1	2	✓
3	0	3	1	1	0
0	3	1	3	1	✓

using minimal  
no. of lines,  
cover all zeros

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Rule to find the a min. no. of lines

- 1) mark all rows for which there is no assignment
- 2) mark cols which has <sup>zeros</sup> assignments in marked rows
- 3) mark the rows which has assignment in marked ~~rows~~ cols.
- 4) Draw lines through unmarked rows and marked cols.

randomly chosen, not making to choose

0	1	1	1	2
3	3	3	0	3
3	2	0	1	1
4	0	4	1	1
1	2	4	2	0

5 zeros = 5 rows  $\Rightarrow$  optimum

min. cost of assignment =  $1 + 1 + 3 + 1 + 4 = 9$

Q: The following table gives profits for different jobs done by diff. machines. Find the max. profit of assignment.

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	M <sub>5</sub>
J <sub>1</sub>	30	37	40	28	40
J <sub>2</sub>	40	24	27	21	36
J <sub>3</sub>	40	32	33	30	35
J <sub>4</sub>	25	38	40	36	36
J <sub>5</sub>	29	62	41	34	39

The given problem is to max. the profit. we replace each cell ~~entry~~ by  $62 - \text{the respective cell entry}$  (62 is the max. of all entries).

32	25	22	34	22
22	38	35	41	26
22	30	29	32	27
37	24	22	26	26
33	0	21	28	23

for each row  
Find the row minima. - subtract this min. from the elements of respective rows form the new table.

10	3	0	12	0
0	16	13	19	4
0	8	7	10	5
5	2	0	4	4
33	0	21	28	23

Find col minima.

Subtract this minimum from the elements of respective col. form the new table.

10	3	0	8	X
0	16	13	15	4
X	8	7	6	5
5	2	X	0	4
33	0	21	24	23

- subtract min. of uncovered elements to this element.

(2)

14	3	0	8	X
X	12	9	11	0
0	4	3	2	1
19	2	X	0	4
37	0	21	24	23

cover all zeros by min. no. of horizontal & vertical lines.

mark rows without assignment.

Draw lines through unmarked rows & marked cols.

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Q: A company has 4 machines to do 3 jobs. Each job can be assigned to one & only one machine. The following table gives the cost of each job on each machine. Determine an optimal assignment which will minimise the cost.

	1	2	3	4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

→ This AP is unbalanced. we introduce 1 dummy job machine D with 0 cost in the respective cells.

	1	2	3	4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22
D	0	0	0	0

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## MODULE 4

## Network Analysis

A project is composed of a no. of jobs and activities or tasks that are related to each other. And all of these should be completed in order to finish the project. There are 2 prominent methods to do this

- 1) Critical path method (CPM)
- 2) PERT (Project Evaluation Review Technique)

Q: Assume that a statue is to be erected in a village square on a stone base which is to be built in on a cement concrete foundation. The statue is to be made on another place, we move to the base and install. The various operations of the entire project are below in a random order.

- A - make statue,
- B - lift the statue into place
- C - construct concrete foundation,
- D - compact & level the site
- E - move statue to village square,
- F - construct stone base.

Construct network diagram of the project

A→

