22.8.'23

Module I

LINEAR PROGRAMMING PROBLEM (LPP)

Q: A company produces 2 types of articles 1, 2. Each of the 1st type required twice as much labor time as the 2nd type. If all the products are the 2nd type only the company can produce 500 articles. The market limits daily sales of 1st and 2nd type to 150 and 250 no. of products. Assuming that the profits from each of 1, 2 ~~3~~ type are respectively ₹80 and ₹50. Formulate the problem into an LPP in order to determine the no. of products type 1, 2 to be produce so as to max. the profit.

A: Let x & y be the no. of limits of type 1 & 2 products to be produced.

profit : $z = 80x + 50y$

Let 't' be the time taken to produce 1 unit of type 2, then $2t$ will be the time taken to produce 1 unit of type 1.

$\therefore$ total time $= 2tx + ty$

If they produce type 2 products only then they can produce 500 units

$\therefore$ Available time $= 500t$

$2tx + ty \leq 500t$

$\cancel{x + y \leq 500}$

$x \leq 150$

$y \leq 250$

$2x + y \leq 500$

∴ The mathematical model of the given problem is

max; $z = 80x + 50y$ → objective function

subject to, x, y → decision variables

$$\left.\begin{aligned} 2x + y &\leq 500 \\ x &\leq 150 \\ y &\leq 250 \end{aligned}\right\} \text{constraints}$$

with $x \geq 0$, $y \geq 0$ → non-negative restrictions

Note that,

the above problem has 3 parts

- a function to be maximized (or minimized) called Objective function, subject to
- a set of conditions resulted from the limitations of available resources such as money, space, time, man-power, machine power etc..,
- subject with non-negative restrictions.

We can generalize this model into the following form

The mathematical form of an LPP having 'n' no. of decision variables, $x_1, x_2, \ldots x_n$ connected by 'm' no. of constraints, may be expressed in the form

Optimize

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_n$$

with

$$x_1, x_2, \ldots, x_n \geq 0$$

This can be expressed in the matrix form

Optimize

$$z = CX$$

st

$$AX \leq B$$

with

$$X \geq 0$$

$$C = (c_1, c_2, \ldots, c_n)$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We may be able to express the given LPP with all the constraints of the less or equal type constraints then the LPP is said to be in canonical form.

The LPP in which all the constraints are = type, is said to be in standard form.

→ Solution of LPP by graphical method :-

Q: A revenue produces 2 types of dolls. A type & type B. Each dolls of type B takes twice as long to produce as one type A. Company have time to make max. of 2000 per day, if it produce only type B dolls. The supply of plastics is sufficient to produce 1500 ~~daily~~ dolls / day. (Both A & B combined). ~~Type~~ Type B doll requires a fancy ~~doll~~ dress of which there are 600/day available. If the company makes profit ₹30 & ₹50 per doll of type A & B respectively. How many of each should be produced / day so as to maximize profit.

A→

	A	B	Available
time	t	2t	2000t
plastic			1500
fancy dress	0		600
profit	30	50	

Let x and y be the no. of dolls to be produced of type A and B.

$t \rightarrow$ time taken to produce 1 doll of type A

Time required to produce X no. of type A and y no. of type B is,

$$tx + 2ty$$

Available time = 2000t

$$tx + 2ty \leq 2000t$$

$$x + 2y \leq 2000$$

$$x + y \leq 1500$$

$$y \leq 600$$

~~300~~

The mathematical model of the given problem is

max. , $z = 30x + 50y$

st , $y \leq 600$

$x + y \leq 1500$

with $x, y \geq 0$

⇒ (Graphical Representation)

$x + 2y = 2000$

x	0	2000
y	1000	0

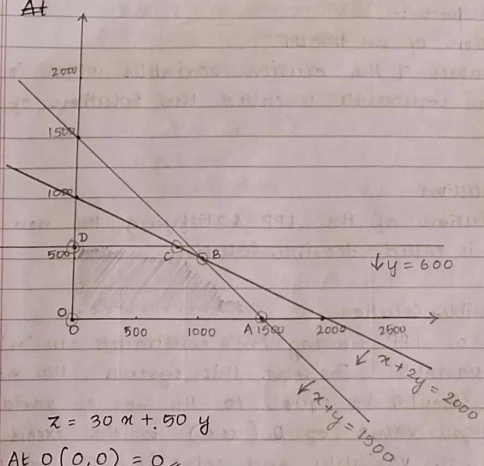
$y = 600$

x	0	50
y	600	600

$x + y = 1500$

x	0	1500
y	1500	0

~~At~~



$z = 30x + 50y$

- At O (0,0) = 0
- At A (1500, 0) ⇒ $z = 30 \times 1500 = 45000$
- At B (1000, 500) ⇒

$$x + 2y = 2000$$
$$x + y = 1500 \quad (-)$$
$$y = 500$$
$$x = 1000$$

⇒ $z = 30 \times 1000 + 50 \times 500 = 30000 + 25000$

$= 55000$

- At D (0, 600) ⇒ $z = 50 \times 600 = 30000$

B(1000, 500)

∴ Solution is $x = 1000$, $y = 500$

i.e., 1000 type A dolls can be made & 500 type B dolls can be m

- Simplex Method :-

~~Some~~ solution of an ~~SPP~~ LPP :-

The values of the decision variables which ~~is~~ will satisfy the constraints is called the solution of an LPP.

- Feasible Solution :-

A solution of the LPP satisfying the non-negative restriction is called Feasible Solution.

- Basic Feasible Solution :-

Consider a LPP having 'm' constraints in 'n' decision variables. To solve this system, the no. of equations should be equal to the no. of variables. So we assign value ~~of~~ 0 (zero) to the excess 'n-m' no. of variables and solve it. The resulting solution is called Basic ~~Feasible~~ Solution.

The 'n-m' variables whose values are 0 are called Non-Basic Variables & the other 'm' variables are called Basic Variables.

- Basic solution which is also (non-ve) feasible is known as Basic Feasible Solution.

- Optimal Solution :-

The basic feasible solution which optimize the objective function is called the Optimal Solution.

Q: Find the ~~optimal~~ basic feasible solution of the s/m

$x + y + z = 100$

$2x - y + z = 70$

A: CASE I :

$z = 0$

There are 2 constraints & 3 variables, so we ~~chose~~ choose 1 variable & assign value 0.

$x + y = 100 \longrightarrow (1)$

$2x - y = 70 \longrightarrow (2)$

(1)+(2) $-3x = 170$

$x = -\frac{170}{3}$ ~~$= 56.2$~~

$x + y = 100$ ($x = -30$)

$-\frac{170}{3} + y = 100$

$y = 100 - \frac{170}{3}$

$= \frac{300 - 170}{3} = \frac{130}{3}$

$(170/3, 130/3, 0)$

CASE II :-

put $y = 0$,

$x + z = 100$

$2x + z = 70$

$x = -30$

~~$y = 100$~~

$z = 100 + 30$

$= 130$

$\Rightarrow (-30, 0, 130)$

CASE III :

put $x = 0$,

$y + z = 100$

$-y + z = 70$

$2z = 170$

$z = \frac{170}{2} = 85$

$y = 100 - 85$

$= 15$

$\Rightarrow (0, 15, 85)$

26.9.23

$\leq$ + $\geq$ -

→ Slack and Surplus Variables :-

To convert a $\leq$ type inequality constraint to an equality constraint, we add a +ve variable S_1 to left side of the constraint, such a variable is called a Slack variable.

$2x + 4y + 7z \leq 19 \Rightarrow 2x + 4y + 7z + S_1 = 19$

To convert a greater than type inequality constraint we subtract a +ve variable S_2 from ~~to~~ the left side of the side, ~~so~~ to convert it into equality constraint, such a variable is called Surplus variable.

$x - 4y + z \geq 100 \Rightarrow x - 4y + z - S_2 = 100$

- Degenerate solutions :-

- A basic solution in which ~~the~~ one or more of the basic variable take zero value is called Degenerate Solution.

If all the basic variables take non-zero values is called Non-Degenerate Solution.

- Solution of an LPP using simplex method :-

Q. Solve the following LPP using simplex method :

Maximise $z = 3x + 4y$

st $2x + 5y \le 120$

$4x + 2y \le 80$

with $x, y \geq 0$

A) Introducing to slack variable s_1, s_2, we get the std form of the ~~LLP~~ LPP

max $z = 3x + 4y + 0s_1 + 0s_2$

st $2x + 5y + s_1 = 120$

$4x + 2y + s_2 = 80$

with $x, y, s_1, s_2 \geq 0$

To find (IBFS) Initial Basic Feasible Solution, put $x = y = 0 \Rightarrow$

$s_1 = 120$,

$s_2 = 80$

TABLE - 1 c_j (3 4 0 0)

C_B	X_B	x	y	s_1	s_2	Solution	ratio = $\frac{\text{Solution}}{y>0}$
0	s_1	2	5	1	0	120	$120/5 = 24$ → Leaving variable
0	s_2	4	2	0	1	80	$80/2 = 40$
	z_j	0	0	0	0	0	

$(0\times2)+(0\times4)$

KEY ELEMENT / ~~PIVOT~~ PIVOTAL LEADING ELEMENT

$\Delta_j = z_j - c_j$: $-3 \quad -4 \quad 0 \quad 0$

To check the optimality of the problem, compute $\Delta_j = z_j - c_j$ for maximization problem (for minimization problem $\Delta_j = c_j - z_j$)

If all Δ_j's are non-negative, the current sol is optimum.

$x = 0, y = 0$
$s_1 = 120$
$s_2 = 80$

Since, some Δ_j's are negative, the current sol is not optimum.

The non-basic variable corresponding to highest -ve Δ_j value is called <u>Entering Variable</u>.

TABLE - 2 c_j (3 4 0 0)

C_B	X_B	x	y	s_1	s_2	Solution	Ratio
4	y	$2/5$	$5/5$ 1	$1/5$	$0/5$ 0	$120/5$ 24	$24/2\times5 = 60$
0	s_2	$\frac{4\times5-2\times2}{5}$ $16/5$	0	$-2/5$	$5/5$	$\frac{80\times5-120\times2}{5}$ $160/5$	$160/5\times5/16 = 10$

$z_j = \quad 8/5 \quad 4 \quad 4/5 \quad 0 \quad 96$

$\Delta_j = z_j - c_j$

$\Delta_j \quad -7/5 \quad 0 \quad 4/5 \quad 0$

$x = 0, y = 24$, $s_1 = 0$, $s_2 = \frac{160}{5}$

~~$400-240$~~
~~$\frac{160}{5}=32$~~
~~$\frac{40\times5-24\times2}{5}$~~
~~$\frac{200-48}{5}=\frac{152}{5}$~~
$\frac{3}{5}$

c_j 3 4 0 0

C_B	X_B	x	y	s_1	s_2	Sol	Ratio
4	y	0	1	$1/4$	$-1/8$	20	
3	x	1	0	$-1/8$	$5/16$	10	

$x = 10$, $y = 20$, $s_1 = 0$, $s_2 = 0$ } s_1 & s_2 not in table (X_B)

$z_j \quad 3 \quad 4 \quad 5/8 \quad 7/16 \quad 110$

$\Delta_j \quad 0 \quad 0 \quad 5/8 \quad 7/16$

All the Δ_j's are non-negative, so the current sol is optimum.

The optimum solution is $x = 10$, $y = 20$,

The optimum value of objective fn, $z = 3x + 4y$
$= 3\times10 + 4\times20$
$= 110$

Q: Solve the following LPP using simplex method:

minimize $z = x_1 - 3x_2 + 2x_3$

st $3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

with $x_1, x_2, x_3 \geq 0$

A: Introducing 3 slack variables s_1, s_2, s_3, we get the std form of LPP as

min $z = x_1 - 3x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3$

st $3x_1 - x_2 + 3x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

with $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

To find IBFS,

put $x_1 = x_2 = x_3 = 0 \Rightarrow s_1 = 7, s_2 = 12, s_3 = 10$

table - 1 $\quad c_j$ (1 -3 2 0 0 0)

C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	sol	ratio	
0	s_1	3	-1	3	1	0	0	7	-7	-ve don't take
0	s_2	-2	4	0	0	1	0	12	$12/4 = 3$	
0	s_3	-4	3	8	0	0	1	10	$10/3$	$3 + 1/3$
	z_j	0	0	0	0	0	0	0		
	Δ_j	1	-3	2	0	0	0			

$x_1 = 4$,

$x_2 = 5$, $x_3 = 0$

-11

27.9.23

Q: Maximise $z = 30x_1 + 40x_2 + 20x_3$

st $100x_1 + 120x_2 + 70x_3 \le 100000$

$7x_1 + 10x_2 + 8x_3 \le 8000$

$x_1 + x_2 + x_3 \le 1000$

with $x_1, x_2, x_3 \geq 0$

A→

$10x_1 + 12x_2 + 7x_3 \le 10000$

$7x_1 + 10x_2 + 8x_3 \le 8000$

$x_1 + x_2 + x_3 \le 1000$

introducing 3 slack ~~for~~ variables S_1, S_2, S_3, we can convert the given LPP into it's std form

max, $z = 30x_1 + 40x_2 + 20x_3 + 0S_1 + 0S_2 + 0S_3$

st $10x_1 + 12x_2 + 7x_3 + S_1 \le 10000$

$7x_1 + 10x_2 + 8x_3 + S_2 = 8000$

$x_1 + x_2 + x_3 + S_3 = 1000$

with $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

To find IBFS,

put $x_1 = x_2 = x_3 = 0 \Rightarrow S_1 = 10000$

$S_2 = 8000$

$S_3 = 1000$

$\boxed{\Delta_j = Z_j - C_j}$

table 1 C_j (30 40 20 0 0 0)

C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	sol	ratio
0	S_1	10	12	7	1	0	0	10000	$\frac{10000}{12}$ = 833.3
0	S_2	7	**10**	8	0	1	0	8000	$\frac{8000}{10}$ = 800
0	S_3	1	1	1	0	0	1	1000	$\frac{1000}{1}$ = 1000
	Z_j	0	0	0	0	0	0		
	Δ_j	-30	-40	-20	0	0	0		

C_j 30 40 20 0 0 0

C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	sol	ratio
0	S_1	16/10	0	-26/10	1	-12/10	0	400	$\frac{400}{16} \times 10 = 250$
40	x_2	7/10	1	8/10	0	1/10	0	$\frac{8000}{10}$	$\frac{800}{7} \times 10 = 1142.8$
0	S_3	3/10	0	2/10	0	-1/10	1	200	$\frac{200}{3} \times 10 = 666.6$
	Z_j	28	40	32	0	4	0		
	Δ_j	-2	0	12	0	4	0		

$\frac{100-84}{10} = \frac{16}{10}$ $\quad \frac{310}{16}$ $\quad \frac{\frac{16}{10}\times\frac{1}{10} + \frac{12}{10}\times\frac{7}{10}}{16}$ $\quad \frac{800\times16 + 400\times7}{16} = \frac{12800 + 2800}{16}$

$\frac{128 + 182}{16} = \frac{310}{16}$ $\quad \frac{16+84}{16} = \frac{100}{16}$ $\quad \frac{46}{16} = \frac{23}{8}$ $\quad = 15600$ $\quad \frac{-16 - 12\times3}{16}$ $\quad 975$

$70 - 96 = -\frac{26}{10}$ $\quad \frac{200\times16 - 400\times3}{16}$ $\quad \frac{-16-12\times3}{16}$

$\frac{10000\times10 - 8000\times12}{10}$ $\quad \frac{16\times2 - 26\times3}{16}$ $\quad \frac{400\times10}{16}$ $\quad -16 - 36 = -52$

$\frac{100000 - 96000}{10} = \frac{4000}{16} = 400$ $\quad \frac{3200-1200}{16} = \frac{2000}{16} = 125$

C_j (30 40 20 0 0 0)

C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	sol	ratio
30	x_1	1	0	-13/6	10/16	-3/4	0	250	
40	x_2	0	1	$\frac{310}{16}$	-7/16	100/16	0	975	
0	S_3	0	0	23/8	-3/16	-13/4	1	125/2	

while finding ~~initial~~ IBFS corresponding to an LPP having greater than or equal to inequality constraint, we may not be able to get an initial Feasible Sol. because the value of the surplus var in the initial sol becomes -ve.

For eg : consider the constraint

$$x + 2y + 10z \geq 120$$

introducing surplus variable S_1

$$x + 2y + 10z - S_1 = 120$$

when we put $x = y = z = 0$, $S_1 = -120$

which is not feasible

~~To~~ initial soln is not feasible

To overcome this situation ; we introduce a variable called Artificial ~~value~~ variable to the left side of the equation

$$x + 2y + 10z - S_1 + A_1 = 120$$

then putting,

$$x = y = z = S_1 = 0, \quad A_1 = 120$$

But introducing this var. A_1 is illegal ~~operati~~

To compensate this, we will add $-MA_1$ in the objective function if it is to be maximised (we will add ~~in~~ $+MA_1$ in the objective fn if it is to be minimised)

As and when the artificial variable leave the basis ~~be~~ during the iteration process, ~~for the data~~ we will remove all the data of that var. from the table.

3-10-'23

Q. Solve the following LPP

Minimize $z = 5x_1 - 6x_2$

st $2x_1 + 5x_2 \geq 1500$

$3x_1 + x_2 \geq 1200$

with $x_1, x_2 \geq 0$

A> Introducing 2 surplus variables S_1, S_2, the given LPP can be converted into its standard form,

minimize, $z = 5x_1 - 6x_2 + 0s_1 + 0s_2$

st $2x_1 + 5x_2 - S_1 = 1500$

$3x_1 + x_2 - S_2 = 1200$

with $x_1, x_2, S_1, S_2 \geq 0$

To get an IBFS, we introduce 2 artifical variables A_1 & A_2 with a large +ve panalty MA_1, MA_2 in our objective function. At that time, it takes the form :-

minimise, $z = 5x_1 - 6x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

st $2x_1 + 5x_2 - S_1 + A_1 = 1500$

$3x_1 + x_2 - S_2 + A_2 = 1200$

with $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

$-MA_1 - MA_2$ in case of maximize

To find IBFS,

put $x_1 = x_2 = S_1 = S_2 = 0 \Rightarrow$

$A_1 = 1500$
$A_2 = 1200$

table 1

	C_j	5	-6	0	0	M	M		
C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	sol	ratio = sol / x_2
M	A_1	2	$\boxed{5}$	-1	0	1	0	1500	1500/5 = 300
M	A_2	3	1	0	-1	0	1	1200	1200/1 = 1200
	Z_j	5M	6M	-M	-M	M	M		
	Δ_j	5M - 5	-6-6M	M	M	0	0		

↑ highest -ve Δ_j

Since, so Δ_j are -ve, the ~~con~~ current sol. is not ~~optimum~~ optimum.

x_2 is the entering variable

A_1 is the leaving variable.

	C_j	5	-6	0	0	~~M~~	M		
C_B	X_B	x_1	x_2	S_1	S_2	~~A_1~~	A_2	sol	ratio
-6	x_2	2/5	1	-1/5	0		0	300	$\frac{300 \times 5}{2} = 750$
M	A_2	$\boxed{13/5}$	0	1/5	-1		1	900	$\frac{900 \times 5}{13} = 346.1$
	Z_j	$-12/5 + 13/5 M$	-6	$6/5 + 1/5 M$	-M		M	900M - 1800	
	Δ_j	$\frac{37}{5} - 13/5 M$	0	$-6/5 - 1/5 M$	2M		0		

	C_j	5	-6	0	0		
C_B	X_B	x_1	x_2	S_1	S_2	sol	ratio
-6	x_2	0	1	-3/13	2/13	2100/13	—
5	x_1	1	0	$\boxed{1/13}$	-5/13	4500/13	$4500/13 \times 1/13$
	Z_j	5	-6	$18/13 + 5/13$ = 23/13	$-12/13 - 25/13$ = -37/13		
	Δ_j	0	0	-23/13	37/13		

	C_j	5	-6	0	0		
C_B	X_B	x_1	x_2	S_1	S_2	sol	ratio
-6	x_2	3	1	0	-1	1200	—
0	S_1	13	0	1	-5	4500	
	Z_j	-18	-6	0	6		
	$\Delta_j = C_j - Z_j$	23	0	0	-6		

	C_j	5	-6	0	0		
C_B	X_B	x_1	x_2	S_1	S_2	sol	ratio
-6	x_2	0	1	3/13	-28/13	29100/13	
5	x_1	1	0	1/13	-5/13	4500/13	
	Z_j	5	-6	$-18/13 + 5/13$ $-13/13 = -1$	$16/13 + 25/13$ = 9/13		
	$\Delta_j = C_j - Z_j$	0	0	1	-9/13		

Since, both values are negative, so No solution.

Q: Solve ~~for~~ the following LPP :-

maximize, $z = x_1 + 2x_2 + 3x_3 - x_4$

st, $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

with, $x_1, x_2, x_3, x_4 \geq 0$

$z = 5/2 + 5 + 3 \times \frac{5}{2}$

$\frac{5 + 10 + 15}{2} = \frac{30}{2} = 15$

A→ Introduce 2 artificial variable, the given LPP becomes

maximise, $z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

st, $x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_2 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

with $x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

4 tables

Sol : $x_1 = 5/2$, $x_2 = x_3 = 5/2$

$x_4 = 0$

(In this special problem, we are including x_4 as a basic variable as this variable is not present in the remaining constraint)