MODULE 5

QUEUING THEORY

Queue

A Group of item waiting for service in a servicing center is called queue or waiting line.

Queue can be finit or infinit.

Time spend by a customer in a system

= Time spend starting from the moment the customer enter the system until he leave the system after getting the service.

Time spend = waiting time + time spend in service

Waiting time

The time upto which a customer has wait before entering the service.

Queue Length

No. of customers waiting in the queue. This include those who wait for service and those who are in service.

Average length of queue

The no. of customers in the queue per unit time.

Average rate of arrival

~~The rate number~~

The average no. of customers arriving in the system per unit time.

Idle period

It is the time during which the server remains free.

$\lambda \rightarrow$ average rate of arrival

$\mu \rightarrow$ average rate of departure.

$\lambda_n \rightarrow$ average rate of arrival when there are n customers in the system.

$\mu_n \rightarrow$ average rate of departure when there are n customers in the system.

$P_n(t) \rightarrow$ probability that there are n customers in the system during the time interval $[0-t]$

$\phi_n(t) \rightarrow$ probability that n customers leave the system during the time interval $[0-t]$

$\rho = \frac{\lambda}{\mu}$ ⇒ Traffic Intensity / utilization factor of the system.

Depending on the rate of arrival and rate of departure, a queing system can be classified into 3 types.

1) Steady state system
2) Transient state system
3) Explosive state system

when the rate of arrival > rate of departure, the queue length goes on increasing indefinitly then the system said to be in Explosive state system.

A system is said to be in Transient state, if the operating measures like queue length, waiting time ... flactuated with time.

~~ie, these functions are~~

If the average rate of arrival < average rate of departure, and both are constant, then the system will eventually set down into a state called steady state. ~~In~~ In this system, the ~~machinets~~ measures of characteristics of the queue system will be independent of the initial ~~time~~ state.

In this state, the probability of finding a fixed no. of customers will be independent of the time ~~elapst~~ elapse since the beginning of the system.

Poisson Process

An infinit sequence of independent events occurring at an instant of time is said to form a poisson process, if the following conditions are satisfied

1) Total no. of events occurring in any time ~~tuval~~ interval of length x does not depend on the event which has occured before the beginning of that period.

2) The probability of occurence of an event in a small time interval of length Δt

$$\Delta t = \lambda \Delta t + O(\Delta t)^2$$

Δ is a constant. $O(\Delta t)^2$ means it is sufficiently small. so it is negligible.

$$\frac{O(\Delta t)^2}{\Delta t} \to 0 \qquad \lim_{\Delta t \to 0} \frac{O(\Delta t)^2}{\Delta t} \to 0$$

3) probability occurence of 2 or more events during the time interval t to $t+\Delta t$ is negligible.

(5) Ques) Show that if the arrival pattern of customer to a system follows poisson process.

Note: The probability of finding n customers in time interval t is given by,

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,1,2,3,\ldots$$

$\lambda \to$ rate of arrival

If the departure pattern follow a poisson process, the probability that n customers depart from the system during the time interval t is given by,

$$\phi_n(t) = \frac{e^{-\mu t}(\mu t)^n}{n!}$$

Inter arrival time

Let $t_1, t_2, t_3 \ldots$ be the inter-arrival time gap between two successive arrivals of customers to the queuing system.

This random variable follows an exponential distribution if the arrival pattern follows poisson distribution.

It is given by,

$$f(t) = \lambda e^{-\lambda t}$$

if λ is the rate of arrival. then $\frac{1}{\lambda}$ is the average time gap b/w two successive arrivals.

Queuing Models

we will consider the following 4 queuing models.

1) single channel single phase
2) Single channel multi phase
3) multi channel single phase
4) multi channel multi phase.

Qn) write the behaviour of customers in the queuing system.

Ans) There are 4 types of behaviour of customer who enter in the queuing system.

i) Balking : A customer's behaviour is said to be balking if he leave the system due to Lack of time or space without entering the queuing line.

ii) Reneging : A customer's behaviour of leaving the queue due to impatient after joining the queuing Line.

iii) Jockeying : customer's behaviour of jumping from one queue to another.

iv) Collusion : customer's collaboration of asking one person to joint the queue in place of a group of more than one customer.

A queuing model is represented by,

(a/b/c) : (e/f)

a → probability distribution describing the pattern of arrival

b → probability distribution describing the pattern of departure

c → no. of servicing centers

e → capacity of the system whether ~~where~~ it can accomodate infinit no. of customers.

f → servicing nature

~~eg) M/M/2 : ∞/FIFO~~

single channel, infinit capacity

1) M/M/1 : ∞/FIFO

finit capacity

2) M/M/1 : N/FIFO

3) M/M/c : ∞/FIFO

4) M/M/C : N/FIFO

multi channel

Consider the queuing system in which the arrival of customers follows poisson process with λ being the average rate of arrival. and the departure follows poisson distribution with μ being the average rate of departure such that the interarrival time distribution follows exponential distribution. If λ_n and M_n are the average rate of arrivals and average rate of departures when there are n customers in the system.

Parkesson's...

The probability that there will be n customers in the system in steady state is denoted by,

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{M_1 M_2 M_2 \cdots M_n} P_0$$

where P_0 stands for probability that system is empty.

Model 1

Kendall's notations

$(a/b/c):(d/f)$

$(M/M/1):(\infty/FIFO)$

arrival pattern follows poisson distribution or markovial process

departure pattern described by poisson process

only one servicing centere (server)

Infinite capacity to allocate any no. of customer

way of servicing of arrivals. (queue discipline)

eg) FIFO
FILO
Random
priority

The generalized queuing model gives the probability that there are n customers in the system is -

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \cdots \mu_n} P_0$$

In this model, $\lambda_n = \lambda$ and $\mu_n = \mu \; \forall n$

because the capacity is ∞

$$P_n = \rho^n P_0 \qquad \rho = \frac{\lambda}{\mu}$$

Ques) Compute the probability that the system is empty in $(M/M/1):(\infty/FIFO)$ Queuing System.

Ans) The probability that there are n customers on the system in steady state is given by,

$$P_n = \rho^n P_0 \quad \text{where } \rho = \frac{\lambda}{\mu} < 1$$

Since P_n is a pdf, $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 + \rho P_0 + \rho^2 P_0 + \cdots = 1$$

$$P_0 (1 + \rho + \rho^2 + \cdots) = 1$$

$$P_0 \times \frac{1}{1-\rho} = 1$$

$$P_0 = 1 - \rho$$

Expected no. of customers in the system (average queue length) expected

$$= E(n) = \sum_{n=0}^{\infty} n P_n$$

$$= 0P_0 + 1P_1 + 2P_2 + 3P_3 + \cdots$$

$$= \rho P_0 + 2 \cdot \rho^2 P_0 + 3 \cdot \rho^3 P_0 + 4 \cdot \rho^4 P_0 + \cdots$$

$$= \rho P_0 (1 + 2\rho + 3\rho^2 + 4\rho^3 + \cdots)$$

$$= \rho P_0 (1-\rho)^2$$

$$= \frac{\rho}{1-\rho}$$

Expected no. of customers waiting for the system, (average length of the waiting line)

$$= E(n-1) = \frac{\rho^2}{1-\rho} = E(L)$$

average waiting time of arrival,

$$E(w) = \frac{\rho}{\mu \times (1-\rho)}$$

average time spend in the system, waiting time + servicing time

$$E(V) = \frac{1}{\mu(1-\rho)}$$

$$\boxed{E(V) = E(W) + \frac{1}{\mu}}$$

$$\boxed{\lambda E(V) = E(n)}$$

$$\boxed{\lambda E(W) = E(L)}$$

Que) Arrival at a telephone booth are following poisson law of distribution with an average time of 10 minutes b/w 2 successive arrivals. Length of a phone call is distributed exponentially with mean 3 minute.

a) what is the probability that a person arriving at a booth will have to wait.

b) what is the average length of queue

c) The Telephone department will install a second booth when.. that an arrival would expect to wait atleast 3 minutes for the phone. By how much must the flow of arrival be increased inorder to justify a ~~second~~ second booth.

Ans) Given $\frac{1}{\lambda} = 10$

$\Rightarrow \lambda = \frac{1}{10}$ /minute

$\frac{1}{\mu} = 3$

$\Rightarrow \mu = \frac{1}{3}$ /minute

$\rho = \frac{\lambda}{\mu} = \frac{1/10}{1/3} = \frac{3}{10} = 0.3$

The problem belongs to model:

$(M/M/1) : (\infty/FIFO)$

with $\lambda = \frac{1}{10}$, $\mu = \frac{1}{3}$ & $\rho = \frac{3}{10}$

a) probability of waiting = p (no. of customers, $n \geq 1$)

$= 1 - p(n < 1)$

$\Rightarrow 1 - P_0 = \rho = 0.3$ $\qquad P_0 = 1 - \rho$

$P_n = \rho^n P_0$

b) $E(n) = \frac{\rho}{1-\rho} = \frac{0.3}{0.7} = \frac{3}{7}$ $\qquad \left[\frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu - \lambda)}\right]$

c) $E(W) = \frac{\rho^2}{1-\rho} = \frac{(0.3)^2}{0.7} = \frac{0.09}{0.7} = \frac{9}{70}$ $\Longrightarrow$

if the average rate of arrival could be made 10/hour, then each customer will have to wait atleast 3 min for getting the phone

ie, the dept will install 2nd booth if

$\lambda = 1/6$ /min.

Let λ' be the new rate of arrival such that,

$$E(w) = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$\Rightarrow E(w) \geq 3$$

$$\Rightarrow \frac{\lambda'}{\mu(\mu-\lambda')} \geq 3$$

$$\Rightarrow \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3}-\lambda'\right)} \geq 3$$

$$\lambda' \geq \cancel{3} \times \frac{1}{\cancel{3}}\left(\frac{1}{3}-\lambda'\right)$$

$$\Rightarrow 2\lambda' \geq 1/3$$

$$\Rightarrow \lambda' \geq 1/6$$

Que) A TV repair man finds that the time spend on his job has exponential distribution with mean 30 min. If he requires sets in the order in which they come in and if the arrival of sets is poisson distribution with an average of 10 per 8 hour day. what is the repair men's expected idle time each day ? How many jobs are ahead of the average set just brought in

Ans) This queuing problem is of the model $(M/M/1):(\infty/FIFO)$

Here, $\frac{1}{\mu} = 30$

$\mu = \frac{1}{30}$ /minute $= 2$/hrs

$\lambda = \frac{10}{8} = \frac{5}{4}$ /hr

$\rho = \frac{\lambda}{\mu} = \frac{5}{8}$

probability that repair man is idle a probability that system is empty.

$\Rightarrow P_0 = 1 - \rho$

$= 1 - \frac{5}{8} = \frac{3}{8}$

$\therefore$ The idle time of the repair man in average everyday =

probability of system empty $\times$ total hour

$= \frac{3}{8} \times 8 = 3$ hrs

$E(n) = \frac{\rho}{1-\rho} = \frac{\frac{5}{8}}{1 - 5/8} = \frac{\frac{5}{8}}{\frac{3}{8}} = \frac{5}{3}$

Model II $(M/M/1):(N/FIFO)$

$$P_n = \rho^n P_0 , \quad n \leq N$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

we know $\sum_{n=0}^{N} P_n = 1$

$$P_0 + P_1 + \cdots + P_n = 1$$

$$P_0 + \rho P_0 + \cdots \rho^N P_0 = 1$$

$$\Rightarrow P_0 (1 + \rho + \rho^2 + \cdots + \rho^n) = 1$$

$$\Rightarrow P_0 \left(\frac{1-\rho^{N+1}}{1-\rho} \right) = 1$$

$$\Rightarrow P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$E(n) = \left(\frac{1 + N\rho^{N+1} - (N+1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \right) \rho$$

or $\frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}$

$$E(L) = \left(\frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \right) \rho^2$$

$$E(V) = \left(\frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \right) \rho^2$$

$E(n) = \lambda E(v)$

$E(L) = \lambda E(w)$

Model III $(M/M/C) : (\infty / FIFO)$

$$P_n = \frac{\lambda^n P_0}{n!\, \mu^n}, \quad 1 \le n \le C-1$$

$$P_0 = \frac{1}{C!} \frac{\lambda^n}{\mu^n} P_0$$

$$\frac{1}{P_0} = \sum_{n=0}^{C-1} \frac{(\rho C)^n}{n!} + \frac{(\rho C)^C}{C!\,(1-\rho)}, \quad \text{where } \rho = \frac{\lambda}{C\mu}$$

$$\Rightarrow \rho C = \frac{\lambda}{\mu}$$

$$P_n = \frac{1}{C^{n-C}\, C!} \frac{\lambda^n}{\mu^n} P_0$$

[no. of customer is equal to greater then no. of service centers]

$$E(L) = \frac{\rho}{(1-\rho)^2} \cdot P_C$$

$$E(n) = E(L) + C\mu$$

$$E(w) = E(L)/\lambda$$

Model IV $(M/M/C) : (N/FIFO)$

$$\frac{1}{P_0} = \sum_{n=0}^{C-1} \frac{\lambda^n}{n!\, \mu^n} + \sum_{n=C}^{N} \frac{\lambda^n}{C^{n-C}\, C!\, \mu^n}$$

$$P_n = \begin{cases} \frac{1}{n!} \cdot \frac{\lambda^n}{\mu^n} P_0, & 0 \le n < C \\ \frac{1}{C^{n-C}\, C!} \frac{\lambda^n}{\mu^n} P_0, & C \le n \le N \end{cases}$$

$$E(L) = \frac{(\rho C)^C}{C!} \cdot \rho P_0 \left[\frac{1 - \rho^{N-C+1} - (1-\rho)(N-C+1)\rho^{N-C}}{(1-\rho)^2} \right]$$

Single channel Finit capacity

Que) 1 person barber shop has 6 chairs to accomadate for a hair cut. If the service discipline is FIFO and the customers above at the rate of 3/h and spent an average of 15 minutes in the barber shop chair. Then find the following.

a) The probability that a customer can directly go to the barber chair on arrival.

b) expected no. of customers waiting for a haircut.

c) The time a customer can spent in the barber shop.

Ans)

MODEL 2

$(M/M/1) : (N/FIFO)$

$N = 6+1 = 7$

$\lambda = 3/hr$

$= \frac{3}{60} = \frac{1}{20}$ hrs

$\mu = \frac{1}{15} = 4$ hr

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

a) probability = probability (system is empty)

$= P_0$

Que)

Ans)

MODEL 3

~~b) $E(L) = 1 - N\rho^{N+1} + (N+1)^3 \rho^N$~~

b) $$E(L) = \left(\frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \right) \times \rho^2$$

c) $$E(n) = \left(\frac{1 - (N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})} \right) \times \rho$$

$$E(n) = \lambda E(V)$$

Qu) In a supermarket, 2 sales girls ringing up sales at counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in a poisson fashion at the rate of 10/h. Compute the following.

a) what is the probability of having to wait for service?

b) what is the expected % of free time for ~~the~~ each girl.

Ans) $(M/M/2) : (\infty/FIFO)$

MODEL 3

$c = 2$, $\mu = \frac{1}{4}$, $\lambda = \frac{10}{60} = \frac{1}{6}$

$\rho = \frac{\lambda}{c\mu}$,

~~$\frac{1}{P_0} = \sum_{n=0}^{c-1} \frac{(\rho c)^n}{n!}$~~

$$\frac{1}{P_0} = \sum_{n=0}^{c-1} \frac{(\rho c)^n}{n!} + \frac{(c\rho)^c}{c!\,(1-\rho)}$$

Ans: $P_0 = \frac{1}{2}$

a) probability of waiting $= P(n \geq 2)$

$= 1 - [P(n<2)]$

$= 1 - [P(0) + P(1)]$

$$P_n = \frac{\lambda^n}{\mu^n n!} P_0 \quad , \quad 0 < n < c$$

~~$P_1 = \frac{\lambda P_0}{\mu} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$~~

$= 1 - \left\{ \frac{1}{2} + \frac{1}{3} \right\}$

$= \frac{1}{6}$

$P_1 = \frac{\lambda P_0}{\mu} = \frac{4}{6} \times \frac{1}{2}$

$= \frac{4}{12} = \frac{1}{3}$

b) Let x represent no. of free girls ~~in typical~~

$\cancel{X \neq 0, 1, 2}$

$\cancel{p(X = x) =}$

X :	0	1	2
$p(X=x)$:	$\frac{1}{6}$	${}^{2}C_{1} P_{1} = \frac{2}{6} = \frac{1}{3}$	$1/2$

$$E(x) = \sum_{x=0}^{2} x\, p(x)$$

$$= 0 \times P_0 + 1 \times P_1 + 2 \times P_2$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

The total free time of all girls $= 4/3$

so, free time available to each girl

$= 2/3$