Quantum Winter Hackathon Solutions

Team Oracle

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Goal 1:

Using the Finite Difference Approach, discretize the 1-D wave equation with 2^{nd} order spatial accuracy and 1^{st} order time frequency with the implicit time integration scheme

Solution:

$$\frac{\partial u}{\partial t} = -C.\frac{\partial u}{\partial x}$$

where C is the wave speed.In Implicit Methods there will be more than one unknown.
Consider a grid with 3 points i.e. u_{i-1}, u_i, u_{i+1} at points $i-1, i, i+1$ respectively. Let Δx be the distance between these points and the entire operation runs from time period $n\, to\, n+1$ which is Δt.

Euler's Backward Time and Centered Space(BTCS) method:

Time derivative $\longrightarrow$ Backward Difference Approximation
Space derivative $\longrightarrow$ Central Difference Approximation

$$\frac{\partial u}{\partial t} = -C.\frac{\partial u}{\partial x}$$

will become or is equal to :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -C.\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}$$

On rearranging and defining Courant number $\lambda = \frac{C.\Delta t}{\Delta x}$

$$u_i^{n+1} - u_i^n = \frac{\lambda}{2}.(u_{i+1}^{n+1} - u_{i-1}^{n+1})$$

$$\frac{\lambda}{2}.u_{i-1}^{n+1} - u_i^{n+1} - \frac{\lambda}{2}.u_{i+1}^{n+1} = -u_i^n$$

On rearranging we will get,

$$u_i^{n+1} = u_i^n + \frac{\lambda}{2}.(u_{i+1}^{n+1} - u_{i-1}^{n+1})$$

So,for a discrete point i we have discretaized the 1-D Wave Equation and we got a linear algebraic equation.
Order of Accuracy $\Longrightarrow O[(\Delta t), (\Delta x)^2]$
On doing stability analysis using Von-Neumann's Analysis we will get that this equation is unconditionally stable.

On the L.H.S, there are 3 unknowns i.e. u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1} at each grid point if we discretize this equation we will get a system of linear equations. And if we write in matrix format, we get a tridiagonal matrix which can be solved by using Thomas algorithm.