CS 1311 : Lecture #9 : Induction

Wednesday, September 4, 2024 9:42 PM

Announcements:

1. Homework 4 and Midterm #1 (Lab) Grades posted by Tuesday
2. Homework #6 Released Tuesday
3. Office Hours after class (in this room)

New Proof Technique: Induction

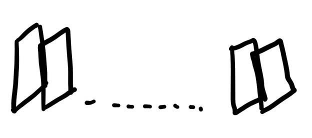
Here is the logic:

I have some statement $P(n)$, and I know that the statement is true for $n=1$, and I know that whenever $P(k)$ is true for $k\geq 1$, that $P(k+1)$ is true, then the statement must be true for all positive integers

Let's build some intuition with an example:

Let's take a ladder :

Let's take dominos :



Formulaic : Base Case, Induction Hypothesis, Induction Step

Base Case $\downarrow$ Prove that $P(x)$ is true where x is the smallest value in domain (usually 0 or 1)

Induction Hypothesis $\downarrow$ Assume that $P(k)$ is true for some $k \in D$

Induction Step $\downarrow$ Prove that $P(k+1)$ is true

Note: Induction is generally only used to prove properties about integers

Question #1) What is $1+2+3+\dots+n$?

Let's prove it: $\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$

① Base Case: Prove that this is true for $n=1$

L.H.S = R.H.S

$1+2+3\dots+n$ $\qquad$ $\frac{n(n+1)}{2}$

1 ✓ $\qquad \frac{1(1+1)}{2} = \frac{2}{2} = 1$

② Induction Hypothesis: Assume that $1+2+3\dots+k = \frac{k(k+1)}{2}$ ✓

③ Induction Step: What do I need to prove?

$1+2+3+\dots+k+1 = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$

L.H.S:

$\underbrace{1+2+3\dots+k}_{=\frac{k(k+1)}{2}} + k+1$

$\frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$

$= \frac{(k+1)(k+2)}{2}$ ✓

Question #2: What is $1+3+5+\dots(2n-1) = n^2$

Want to prove that $\sum_{i=0}^{n-1} 2i-1 = n^2$

① BC: $1 = 1^2$ ✓

② IH: Assume that $1+3+5+\dots(2k-1) = k^2$

$\sum_{i=0}^{k-1} 2i-1 = k^2$

③ IS: W.T.S: $1+3+5+\dots \underbrace{2(k+1)-1}_{2k+1} = (k+1)^2$

L.H.S

$\underbrace{1+3+5+7\dots+(2k-1)}_{=k^2} + (2(k+1)-1)$

By I.H

$k^2 + 2(k+1)-1$

$k^2+2k+2-1$

k^2+2k+1

$= (k+1)^2$ ✓

Question #3: What is $ax^0 + ax^1 + ax^2 + ax^3 \dots + ax^n$ where $x \neq 1$?

$= \frac{a(x^{n+1}-1)}{x-1}$

$\sum_{i=0}^{n} ax^i = \frac{a(x^{n+1}-1)}{x-1}$ where $x \in \mathbb{N}$

BC: $n=0$

L.H.S

ax^0

$= a$ ✓

R.H.S

$\frac{a(x^{0+1}-1)}{x-1} = \frac{a(x-1)}{x-1}$

$= a$ ✓

I.H: Assume that $ax^0 + ax^1 + ax^2 + \dots ax^k = \frac{a(x^{k+1}-1)}{x-1}$

IS: W.T.S $ax^0 + ax^1 \dots + ax^k + ax^{k+1} = \frac{a(x^{k+1+1}-1)}{x-1}$

L.H.S

$\underbrace{ax^0 + ax^1 + \dots + ax^k}_{=\frac{a(x^{k+1}-1)}{x-1}} + ax^{k+1}$

By I.H $\frac{a(x^{k+1}-1)}{x-1} + ax^{k+1}$

$= \frac{ax^{k+1}-a}{x-1} + ax^{k+1}$

$\frac{ax^{k+1}-a}{x-1} + \frac{ax^{k+1}(x-1)}{x-1}$

$= \frac{a}{x-1}\left(x^{k+1} - a + ax^{k+1}(x-1)\right)$

$= \frac{a}{x-1}\left(x^{k+1}(1+x-1)-1\right)$

$= \frac{a}{x-1}\left(x^{k+2}-1\right)$

$= \frac{a(x^{k+2}-1)}{x-1}$

Question #4: What is $2^0 + 2^1 + 2^2 + \dots 2^n$?

$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$a=1$ and $x=2$: $1(2^0) + 1(2^1) + 1(2^2) + \dots 1(2^n)$

$\frac{a(x^{n+1}-1)}{x-1}$

$\frac{2^{n+1}-1}{2-1} = 2^{n+1}-1$

Let's prove it: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

BC:

L.H.S

2^0

1 ✓

R.H.S

$2^{n+1}-1$

2^1-1

1 ✓

IH: $2^0 + 2^1 + 2^2 + \dots + 2^k = 2^{k+1} - 1$

IS: W.T.S $2^0 + 2^1 + 2^2 + 2^3 + \dots 2^k + 2^{k+1} = 2^{k+1+1} - 1$

L.H.S

$\underbrace{2^0 + 2^1 + \dots + 2^k}_{2^{k+1}-1} + 2^{k+1}$

By I.H

$2^{k+1}-1 \quad +2^{k+1}$

$2^{k+1} + 2^{k+1} - 1$

$= 2(2^{k+1}) - 1$

$= 2^{k+1+1} - 1$

$2^{k+2} - 1$

Question #5: Prove or Disprove: $3 \mid 2^{2n}-1 \quad \forall n \in \mathbb{N}$

BC: $n=0$ Does 3 divide into $2^{2(0)}-1$

$= 2^0 - 1$

$= 1-1$

$= 0$ ✓

IH: Assume that $3 \mid 2^{2k}-1$ where $k \in \mathbb{N}$

Assume that $2^{2k}-1 = 3m$ where $m \in \mathbb{N}$

IS: W.T.S $3 \mid 2^{2(k+1)}-1$

W.T.S that $2^{2(k+1)}-1 = 3j$ where $j \in \mathbb{N}$

L.H.S

$2^{2(k+1)}-1$

$2^{2k+2}-1$

$2^2 \cdot 2^{2k} - 1$

$4 \cdot 2^{2k} - 1 = 4(3m+1)-1$

$12m+4-1$

$12m+3$

$3(4m+1)$ ✓

By I.H $2^{2k}-1 = 3m$

$2^{2k} = 3m+1$

Question #6: $\forall n \in \mathbb{N}$, $n > 1$, $n! < n^n$