CSCI1311: Discrete Structures I

Solutions Lab 1

January 27, 2025

Problem 1: Show that for any two integers m and n, $m^2 + n^2$ has the same parity as m + n.

Solution:

Consider the following two cases which cover all possibilities.

Case 1: m + n is odd

We can write m + n = 2k + 1, for some k $\in$ Z.

Then, we have,

$m^2 + n^2 = (m + n)^2 - 2mn$

$= (2k + 1)^2 - 2mn$

$= 4k^2 + 4k + 1 - 2mn$

$= 2(2k^2 + 2k - mn) + 1$ which is odd, as $2k^2 + 2k - mn \in$ Z.

Case 2: m + n is even

We can write m + n = 2k, for some k $\in$ Z.

Similarly, we have,

$m^2 + n^2 = (m + n)^2 - 2mn$

$= (2k)^2 - 2mn$

$= 4k^2 - 2mn$

$= 2(2k^2 - mn)$which is even, as $2k^2 - mn \in$ Z.

We have shown that m + n has the same parity as $m^2 + n^2$ and we are done.

Problem 2: Prove that $\sqrt{6}$ is irrational.

Solution:

We will use a proof by contradiction.

Assume for the sake of contradiction that $\sqrt{6}$ is rational.

By the definition of a rational number, write $\sqrt{6}$ as $\frac{a}{b}$ where a and b have no common divisors other than 1 (we call them relatively prime natural numbers) and b≠0.

This means that 6 = $\frac{a^2}{b^2}$

$6b^2 = a^2$, If 6 | a^2, then 2 | a^2 which implies that a must be even (recall the lemma from slide 11 from Lecture 3.5).

Because a is even, let a = 2c for some integer c.

$6b^2 = a^2$

$2.3.b^2 = (2c)^2$

$2.3.b^2 = 2.2.c^2$

$3.b^2 = 2.c^2$ If 2 | ($3.b^2$), then 2 | b^2 which implies that b must be even (see Lemma above). So, clearly, a and b are both even.

However, this presents a contradiction: a and b must be relatively prime natural numbers, and thus cannot both be divisible by the same factor, 2.

Therefore, we can say that $\sqrt{6}$ is irrational.

Alternate Solution:

We will use a proof by contradiction. Assume for the sake of contradiction that $\sqrt{6}$ is rational. By the definition of a rational number, we write $\sqrt{6}$ as $\frac{a}{b}$ where a and b have no common divisors other than 1 (we call them relatively prime natural numbers) and b≠0.

This means that 6 = $\frac{a^2}{b^2}$

$6b^2 = a^2$

From here, we case the possible parities of a and b. Note that a and b cannot both be even or else they will share a common factor of 2.

Case 1: a and b are both odd, let a = 2k + 1 and let b = 2l + 1

$6b^2 = a^2$

$6(2l + 1)^2 = (2k + 1)^2$

$6(4l^2 + 4l + 1) = 4k^2 + 4k + 1$

$2(12l^2 + 12l + 3) = 2(2k^2 + 2k) + 1$

The LHS is even, and the RHS is odd; thus, the two sides cannot be equal. This is a contradiction.

Case 2: a is even and b is odd, let a = 2k and let b = 2l + 1

$6b^2 = a^2$

$6(2l + 1)^2 = (2k)^2$

$6(4l^2 + 4l + 1) = 4k^2$

$3(4l^2 + 4l + 1) = 2k^2$

$12l^2 + 12l + 2 + 1 = 2k^2$

$2(6l^2 + 6l + 1) + 1 = 2(k^2)$

The LHS is odd, and the RHS is even; thus, the two sides cannot be equal. This is a contradiction.

Case 3: a is odd and b is even, let a = 2k + 1 and let b = 2l

$6b^2 = a^2$

$6(2l)^2 = (2k + 1)^2$

$6(4l^2) = 4k^2 + 4k + 1$

$2(12l^2) = 2(2k^2 + 2k) + 1$

The LHS is even, and the RHS is odd; thus, the two sides cannot be equal. This
is a contradiction. These cases cover all possibilities, and we reach a contradiction in every case. Therefore, we can say that $\sqrt{6}$ is irrational.