CS1311: Proofs #3

Monday, January 20, 2025 1:43 PM

Announcements:

1) Homework #1 was due before class, Solutions released today, grades sometime this week.
2) Homework #2 released after class today (more proofs!)
3) My office hours: after-class in this room (5PM → no more questions)

Review

- An integer, x, is even if $\exists$ some k where $x = 2k$ and $k \in \mathbb{Z}$
- An integer, x, is odd if $\exists$ some k where $x = 2k+1$ and $k \in \mathbb{Z}$
- An integer, x, is prime if $\forall (r,s)$ where $x = r \cdot s$, $r=1$ or $s=1$
  otherwise x is composite (if $\exists (r,s)$ where $x = r \cdot s$ and both $r \neq 1$ and $s \neq 1$)
- A real number, $r \in \mathbb{R}$, is rational if $\exists$ some pair (x,y) where $\frac{x}{y} = r$ and $y \neq 0$
- $\lfloor x \rfloor$ is the floor of x. If $\lfloor x \rfloor = n$, then $n \leq x < n+1$
- $\lceil x \rceil$ is the ceiling of x. If $\lceil x \rceil = n$, then $n-1 < x \leq n$
- Thus far we have covered direct proofs and proof by contrapositive

Proof the following:

- $\forall x \in \mathbb{R}$ and $\forall m \in \mathbb{Z}$: $\lfloor x+m \rfloor = \lfloor x \rfloor + m$

$\sqrt{2} = \frac{2}{\sqrt{2}}$

Prove the following:

- $\sqrt{2}$ is irrational : p $\quad\quad$ $\sim p$: $\sqrt{2}$ is rational

Remider: a rational number is any real number that can be written as $\frac{a}{b}$ where $b \neq 0$ and $a, b \in \mathbb{Z}$
and a and b have no common factors

New proof technique: Proof by contradiction:

Assume for the purpose of contradiction (f.p.o.c) that the $\sqrt{2}$ is rational

$\sqrt{2} = \frac{a}{b}$ where a and $b \in \mathbb{Z}$ and $b \neq 0$

$2 = \frac{a^2}{b^2}$

1 $a^2 = 2b^2$

a^2 is even become $a^2 = 2 \cdot q$ where $q = b^2 \in \mathbb{Z}$

Since a^2 is even, a must also be even

2 $a = 2k$ where $k \in \mathbb{Z}$

1 and 2 $a^2 = 2b^2$ $\quad$ $(2k)^2 = 2b^2$

$4k^2 = 2b^2$

$b^2 = 2k^2$

b^2 is even

b is even

Since a and b are both even, they have a common factor → contradiction!

- The unique factorization theorem: Fundamental Theorem of Arithmetic

$\forall x \in \mathbb{Z}$, x can be expressed as a product of prime numbers

$x = p_1^{e^1} \cdot p_2^{e^2} \cdots\cdot p_n^{e^n}$

Assume that the $\sqrt{2}$ is rational, f.p.o.c

$\sqrt{2} = \frac{a}{b}$ $\quad\quad$ $S(a^2)$ is even

$\quad\quad$ $S(b^2)$ is even

$2 = \frac{a^2}{b^2}$

$a^2 = 2b^2$ ← a^2 has exactly 1 more prime factor than b^2

Let $S(m)$ be the sum of the number of times each prime factor occurs in the prime factorization of m

$S(3) = 1$

$S(21) =$ $\quad\quad$ $S(9) = 2$

$21 = 7^1 \cdot 3^1$ $\quad$ $1+1 = \boxed{2}$ $\quad$ $9 = 3^2$

$S(25) = 5^2 = \boxed{2}$

$S(a^2)$ is an even number

$S(a^2) = 2 \cdot S(a)$

$S(a) = x^n \cdot y^m$

$S(a^2) = (x^n \cdot y^m)^2 = (x^{2n} \cdot y^{2m})$

$S(a) = m+n$

$S(a^2) = 2m+2n = 2(m+n)$

$S(b^2) = 2 \cdot S(b)$ and is also even

Prime factorization $(b^2) = x_1^{p_1} \cdot x_2^{p_2} \cdots x_n^{p_n}$ $\quad$ $S(b^2) = p_1 + p_2 + \cdots p_n$

Prime factorization $(a^2) = 2^1 \cdot [x_1^{p_1} \cdot x_2^{p_2} \cdots x_n^{p_n}]$ $\quad$ $S(a^2) = 1 + p_1 + p_2 \cdots + p_n$

Contradiction!

$S(b^2) = S(a^2) - 1$

Prove or disprove the following:

- The sum of 2 irrational numbers is irrational
- If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and x and y are irrational then $x+y$ is irrational
- If $x+y$ is rational then x and y are rational

$\sqrt{2} + -(\sqrt{2}) = 0$

Prove or Disprove the following:

- If x and y are irrational numbers, x^y is irrational
- If x^y is rational then x and y are rational numbers

Disprove:

$x = \sqrt{2}$
$y = \sqrt{2}$

$x^y = \sqrt{2}^{\sqrt{2}}$

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational ✓

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational

$x = \sqrt{2}^{\sqrt{2}}$

$y = \sqrt{2}$

$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$

Contradiction!

Prove the following:

- n is even $\Leftrightarrow$ $7n+4$ is even

Prove the following:

- There are an infinite number of prime numbers

Assume for the purpose of contradiction that there are a finite number of primes

$2, 3, 5, 7, 11, \ldots \cdot p$

$n = \underbrace{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots \cdot p)}_{} + 1$

$n > p$

n cannot be prime

Prime factorization of n =

$\frac{n}{\text{any prime}} = \_\_\_ \; R \; 1$

$\frac{100}{2}$ = $\quad$ $2 \cdot 2 \cdot 5 \cdot 5$ $\quad\quad$ $\frac{n}{3} = 3$

$\quad\quad$ $2^2 \cdot 5^2$ $\quad\quad$ 11

$\frac{50}{2}$

$\frac{25}{5}$

$1001 = (5 \cdot 20 \cdot 10) + 1$