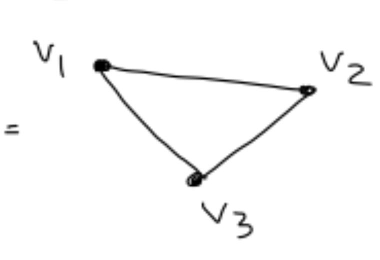
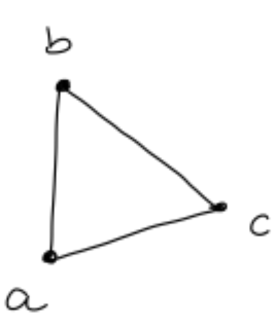
December 3, 2024

Announcements:

- HW10 will be posted today; due Dec 10^{th}
  - can't provide extensions (so that we can release solutions before the exam)
- Final exam : December 12^{th} 5:20-7:20pm in TOMP 208
  - cumulative
  - double-sided [redacted] notes
  - If taking at DSS, submit the request ASAP!
- Course Evals are live!

Today's topic: Graph Isomorphisms

$G =$ 

$G' =$ 

Definition : The simple graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a one-to-one and onto function $f : V \to V'$ such that

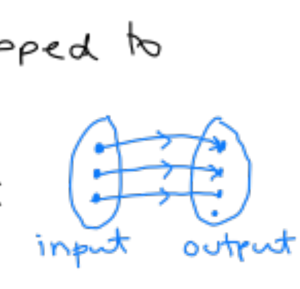
$$(u,v) \in E \iff (f(u), f(v)) \in E'$$

What is a one-to-one function?

$$f(x) = f(y) \Rightarrow x = y$$

* No two different inputs can be mapped to the same output

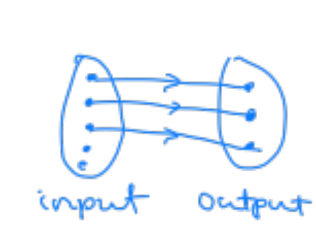
(1) * For every y, there is at most one x



What is an onto function?

For every y, there exists an x s.t. $f(x) = y$

* Every element in the output set ("codomain") is hit

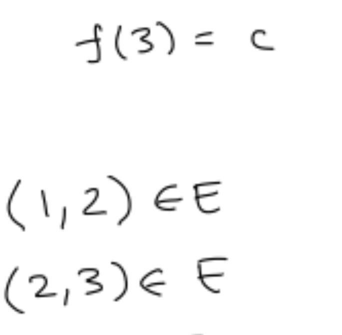


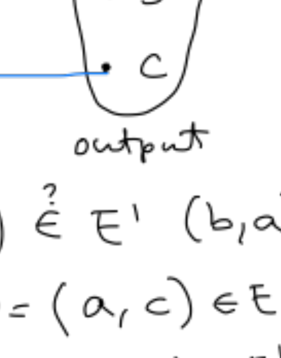
(2) * For every y, there is at least one x

From (1) and (2), if a function is one-to-one and onto, then for every y, there is exactly one x, meaning we have a one-to-one correspondence between the input set and the elements of the output set

Key idea : Isomorphism is a mapping or relabeling between graphs that have the same structure.

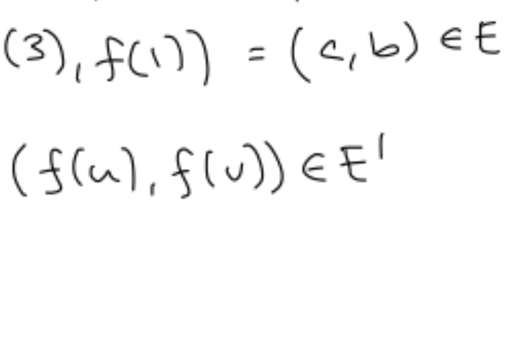
Ex 1

$G = (V, E)$ 

$H = (V', E')$ 

G and H are isomorphic.

f : $f(1) = b$
$f(2) = a$
$f(3) = c$



$(1,2) \in E$ $\quad (f(1), f(2)) \overset{?}{\in} E'$ $(b,a) \in E'$

$(2,3) \in E$ $\quad (f(2), f(3)) = (a,c) \in E'$

$(3,1) \in E$ $\quad (f(3), f(1)) = (c,b) \in E'$

$$\Rightarrow (u,v) \in E \iff (f(u), f(v)) \in E'$$

Ex 2

$G = (V, E)$

$H = (V', E')$

Yes they are!
$1 \to a$
$2 \to b$
$3 \to c$
$4 \to d$

What about? f : $1 \to a$
$2 \to c$
$3 \to b$
$4 \to d$

- f is one to one and onto ✓
- $(u,v) \in E \iff (f(u), f(v)) \in E'$

$E : \{(1,2), (2,4), (4,3), (3,1)\}$
$f \downarrow$
$E' : \{(a,c), (c,d), (d,b), (b,a)\}$

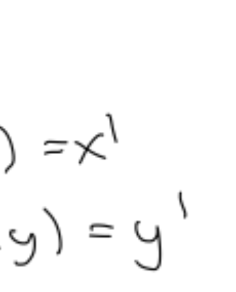
$(u,v) \in E \iff (f(u), f(v)) \in E'$ ✓

Proofs

(1) Show that the inverse of an isomorphism is an isomorphism.

If f is an isomorphism (from G to G'), then f^{-1} is an isomorphism (from G' to G).

- Because f is an isomorphism, it is one-to-one and onto. Therefore, it has an inverse, which is also one-to-one and onto.
  So the first part of the definition is satisfied for $f^{-1} : V' \to V$



- Now we need to show that:

$$(x', y') \in E' \iff (f^{-1}(x'), f^{-1}(y')) \in E$$

Because f is one-to-one onto:

→ there exists a unique x s.t. $f(x) = x'$
→ there exists a unique y s.t. $f(y) = y'$

$$\iff f^{-1}(f(x)) = x = f^{-1}(x')$$
$$\iff f^{-1}(f(y)) = y = f^{-1}(y')$$

Because f is an isomorphism

$$\begin{array}{ccc} (x,y) \in E & \iff & (f(x), f(y)) \in E' \\ \| & & \| \\ (f^{-1}(x'), f^{-1}(y')) \in E & \iff & (x', y') \in E' \end{array}$$ ✓

So the second part of the def holds

$\Rightarrow$ the inverse of an isomorphism $f : V \to V'$ is also an isomorphism.

Definition : A property P is called an invariant for graph isomorphism iff given any simple graphs G and G', if G has property P and G' is isomorphic to G, then G' must also have that property

G H

Some invariants

* # of vertices n
* # of edges m
* degree of a node

(2) Show that the degree of a node is invariant under isomorphism. That is, if f is an isomorphism from G to G', and v is an arbitrary node in G, then

$$\deg_G(v) = \deg_{G'}(f(v))$$