CSci 2312
Discrete Structures II: Asymptotic Analysis: “Big-Oh” Notation

Poorvi L. Vora

We attempt to understand how to compare the efficiency of distinct algorithms using asymptotic analysis. For example, we consider the functions n, 2^n, $4n^2$ and n^3.

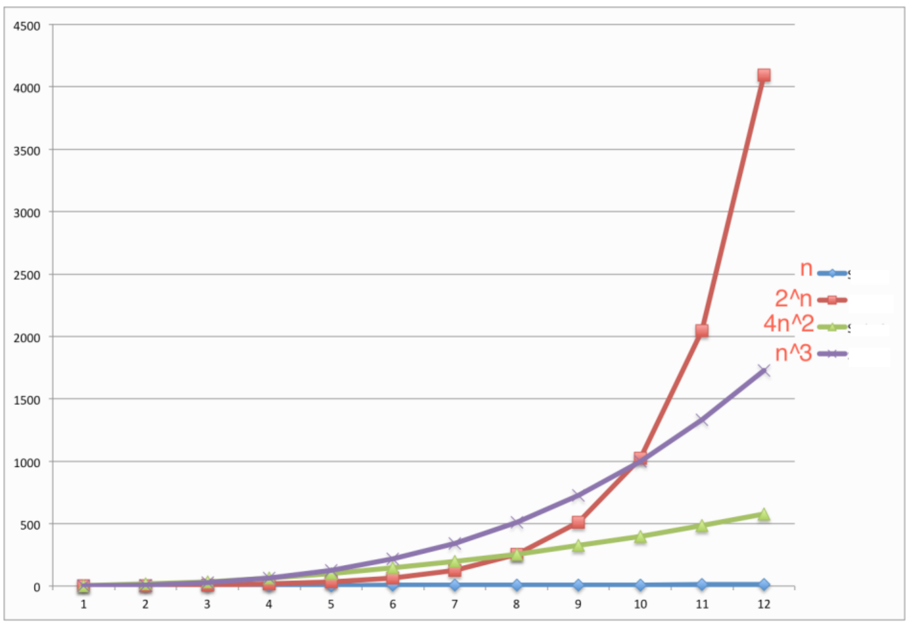

Figure 1: Relative Computational Complexity

We notice that which function represents fewer operations and hence a more efficient algorithm is dependent on the value of n. But we also see that, as n gets really large, 2^n gets much larger than any of the other functions. We also see that n^3 gets larger than $4n^2$ which gets larger than n as n gets large, even though one cannot say this behavior holds for small n. One way of thinking about this is to notice that the derivative of 2^n, which is a multiple of 2^n itself, gets large very quickly. Thus, even if the function 2^n starts off smaller than the others, this large derivative implies it grows faster than the others, and, once it catches up, it still continues to grow faster.

We are most interested in the behavior of the functions for large values of n, which represent, for example, large graphs or large integers or strings with a large number of bits.

We define the notion of “upper bounding” a function for large values of n, using the definition from the text, section 11.2.

Let $f : \mathbb{N} \to \mathbb{R}$. We say that $f \in O(g(n))$ if $\exists B, b$, both positive, such that

$$0 \leq | f(n) | \leq Bg(n) \ \forall \, n \geq b$$

What does it mean for $f \in O(g(n))$? Loosely speaking, it means that f is, in some sense "smaller" (more accurately, "not larger than" because they can be equal) than g, for very large values of n, and ignoring constant multiples. From the perspective of algorithmic complexity, suppose algorithm F takes time f and algorithm G takes time g. Then F is at least as efficient as G (because f is not larger than g, and larger implies the algorithm takes more time)—ignoring constant multiples and for very large values of n.

Let's consider a few examples.

1. Show that $0.1n^2 \in O(n^2)$.

To show that f is $O(g)$ we need to show that "$\exists B, b$, both positive, such that $0 \leq f(n) \leq Bg(n) \ \forall \, n \geq b$".

One can show that such b and B exist by simply producing them and showing they satisfy the inequality.

Step 1: Find such B and b.

Need to find B, b, both positive, such that: $0 \leq 0.1n^2 \leq Bn^2 \ \forall \, n \geq b$.

Because $n^2 \geq 0 \, \forall n$, multiplying both sides by 0.1 gives us $0.1n^2 \geq 0 \, \forall n$. Hence we need only deal with finding b, B such that:

$$0.1n^2 \leq Bn^2 \ \forall \, n \geq b$$

We are not yet proving anything, but are focused on finding the values of b and B. Eyeballing the inequality, we see something that works: $b = B = 1$.

If we wished to get into this more deeply (you need not), we see that our requirement needs us to find a B and b such that $0.1 \leq B \, \forall n \geq b$. And if we choose $B \geq 0.1$, the inequality will be satisfied for all non-negative values of n, and so $b \geq 1$ works.

To simplify things and make our upcoming proof simpler, we choose $B = 1$ and $b = 1$.

Step 2: The Proof.

We now need to prove that, *for these values of b and B that we are proposing* the inequality is true. So far, we were just playing with the inequalities to guess good values of B and b, but we haven't proven anything.

Plugging in our chosen values, $B = 1$ and $b = 1$, we need to show that:

$$0.1n^2 \leq n^2 \ \forall \, n \geq 1$$

To do so, we should start with something that is known and end at the inequality we need:

$$0.1 < 1 \Rightarrow 0.1n^2 < n^2 \, \forall \, n \ > \ 1$$

The multiplication by n^2, which is positive for $n \geq 1$, does not change the direction of the inequality.

We begin with an equation or inequality that is known to be true (as above) and then manipulate it to get the equation or inequality that you need to prove. **But in all cases, you need to begin with what is true**. In this case, you could not have done the following:

$$0.1n^2 < 1n^2 \Rightarrow 0.1\cancel{n^2} < 1\cancel{n^2} \Rightarrow 0.1 < 1 \, \forall \, n \ > \ 1$$

because it begins with what you need to show and not what is known to be true.

Example 2: Show that n is $O(n^2)$.

Need to show that "$\exists B, b$, both positive, such that $0 \leq n \leq Bn^2 \;\; \forall\, n \geq b$".

Step 1: Find B, b. Notice that b is positive, hence $b \geq 0$ and $n \geq b \geq 0$ hence $n \geq 0$ and we can focus on: "$\exists B, b$, both positive, such that $n \leq Bn^2 \;\; \forall\, n \geq b$". Here too it looks like $B = 1$ and $b = 1$ would work.

Looking at it more closely (which you don't need to do for this problem) we also see that

$$n \leq Bn^2 \, \forall\, n \geq b \Rightarrow 1 \leq Bn \, \forall\, n \geq b \Rightarrow 1 \leq Bb$$

Notice that here we begin with $n \leq Bn^2 \, \forall n$. Which means that **if** there are values B and b such that this first inequality we began with above is true, then they are such that $1 \leq Bb$. In fact, we can use all sorts of choices for B and b that satisfy this, $b = 100$, $B = 0.001$, etc. This can also be seen by eyeballing the inequality:

$$n \leq Bn^2 \Rightarrow n \leq (Bn)n$$

which means that

$$Bn \geq 1$$

.

Again, to keep it simple, we choose $b = 1$ and $B = 1$.

Step 2: We now need to show that the inequality is true for our choice of b and B. We need to show that: $n \leq n^2 \;\; \forall\, n \geq 1$.

In order to show it, we should begin with a known fact or a clearly-stated assumption. Assume $n \geq b = 1$.

$$1 \leq n \Rightarrow n \leq n^2$$

Because $n \geq 1 \geq 0$, multiplying both sides by n does not change the direction of the inequality.

Example 3. Let's consider a function: $f(n) = 0.1n^2 + 5n + 20$ and show that $f \in O(n^2)$.

We need to find b and B, both positive such that $f(n) \leq Bn^2 \;\; \forall n \geq b$, and then prove it. We see that, in addition to the n^2 terms, f also has the constant term and the n term.

We see that

$$n \geq 1 \Rightarrow 5n \times n \geq 5n \times 1 \Rightarrow 5n^2 \geq 5n$$

This was a proof because we began with a known fact.

Further,

$$n \geq 1 \Rightarrow n^2 \geq 1 \Rightarrow 20n^2 \geq 20$$

Again, this was a proof because we began with a known fact.

Hence

$$f(n) = 0.1n^2 + 5n + 20 \leq 0.1n^2 + 5n^2 + 20n^2 = 25.1n^2 \;\; \forall\, n \geq 1$$

We can choose $B = 25.1$ and $b = 1$.

Thus we see that if we are careful to begin with known facts, we can combine steps 1 and 2.

Now, since f is $O(n^2)$, is it also $O(n^3)$? Why?