CS2312: Lecture #5: Induction

Wednesday, September 4, 2024 9:42 PM

Announcements:

1. Homework 1 Solutions Posted
2. Homework 2 Released Today (due next Tuesday)
3. My Office Hours Start Today (Here → 4^{th} Floor)

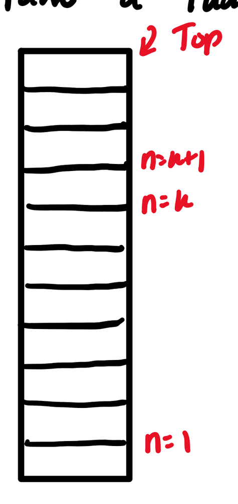
New Proof Technique: Induction

Here is the logic:

I have some statement $P(n)$, and I know that the statement is true for $n=1$, and I know that whenever $P(k)$ is true for $k \geq 1$, that $P(k+1)$ is true, then the statement must be true for all positive integers

Let's build some intuition with an example:

Let's take a ladder:



Let's take dominos:

Formulaic: Base Case, Induction Hypothesis, Induction Step

- Base Case → Prove that $P(x)$ is true where x is the smallest value in domain (usually 0 or 1)
- Induction Hypothesis → Assume that $P(k)$ is true for some $k \in D$
- Induction Step → Prove that $P(k+1)$ is true

Note: Induction is generally only used to prove properties about integers

Question #1) What is $1+2+3+\ldots+n$?

$$\frac{(n)(n+1)}{2}$$

Let's prove it: $\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$

BC: $n=1$ 1 ✓

IH: Assume for some $k \in \mathbb{Z}^+$ that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

IS: W.T.S that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

L.H.S: $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + \sum_{i=k+1}^{k+1} i$

$1+2+3+\ldots+k+(k+1)$

By I.H

$$\frac{k(k+1)}{2} + k+1$$

$$\frac{k(k+1)}{2} + \frac{(k+1)(2)}{2}$$

$$\boxed{\frac{(k+1)(k+2)}{2}} \checkmark$$

Question #2: What is $1+3+5+\ldots(2n-1) = n^2$

Let's prove it: $\sum_{i=0}^{n-1} 2i+1 = n^2$ $\sum_{i=1}^{n} 2i-1 = n^2$

$2(1)-1 = 1$

BC: $n=1$ $n^2 = 1$

IH: Assume that $P(k)$ is true, for some k:

~~$1+3+5+\ldots 2n-1 = n^2$~~

$\boxed{1+3+5+\ldots 2k-1 = k^2}$ → $\sum_{i=1}^{k} 2i-1 = k^2$

IS: W.T.S that $\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$

$$\sum_{i=1}^{k}(2i-1) + 2(k+1)-1$$

By IH

$$k^2 + 2k+2-1$$

$$k^2 + 2k+1$$

$$(k+1)^2 \checkmark$$

Question #3: What is $ax^0 + ax^1 + ax^2 + ax^3 \ldots + ax^n$ where $x \neq 1$?

Let's prove it: $\sum_{i=0}^{n} ax^i = \frac{a(x^{n+1}-1)}{x-1}$ where $x \neq 1$

BC: $n=0$ $ax^0 = a$

$$\frac{a(x^{n+1}-1)}{x-1}$$

$$\frac{a(x-1)}{x-1} = a \checkmark$$

IH: Assume for some k: $\sum_{i=0}^{k} ax^i = \frac{a(x^{k+1}-1)}{x-1}$

IS: W.T.S that $\sum_{i=0}^{k+1} ax^i = \frac{a(x^{k+2}-1)}{x-1}$

L.H.S $\boxed{\sum_{i=0}^{k} ax^i} + ax^{k+1}$

→ By I.H: $\frac{a(x^{k+1}-1)}{x-1}$

$$\frac{a(x^{k+1}-1)}{x-1} + ax^{k+1} =$$

$$\frac{a(x^{k+1}-1)}{(x-1)} + \frac{(ax^{k+1})(x-1)}{(x-1)}$$

$$\frac{a}{(x-1)}\left(x^{k+1}(1+x-1)-1\right)$$

$$= \frac{a}{(x-1)}\left(x^{k+2}-1\right)$$

$$= \frac{a(x^{k+2}-1)}{x-1}$$

Question #4: What is $2^0 + 2^1 + 2^2 + \ldots 2^n$?

$$\sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

Let's prove it: $\sum_{i=0}^{n} 2^i = 2^{n+1}-1$

$a=1$ and $x=2$ → $\frac{a(x^{n+1}-1)}{(x-1)}$

$$\frac{1(2^{n+1}-1)}{1}$$

$$= 2^{n+1}-1$$

Question #5: Prove or Disprove: $3 \mid 2^{2n}-1 \quad \forall n \in \mathbb{N}$

BC: $n=0$ $3 \mid 0$ ✓

IH: Let's assume for some k: $3 \mid 2^{2k}-1 \quad k \in \mathbb{N}$

IS: W.T.S $3 \mid 2^{2(k+1)}-1$

$$3 \mid 2^{2k+2}-1$$

R.H.S: $2^{2k+2}-1$

$$2^{2k} \cdot 2^2 - 1$$

$$4(2^{2k})-1$$

$$(3+1)(2^{2k})-1$$

$$3 \cdot 2^{2k} + 2^{2k}-1$$ = $3q$ for some $q \in \mathbb{N}$

$$3 \cdot 2^{2k} + 3q$$

$$3(2^{2k}+q) \in \mathbb{N} \checkmark$$

Question #6: $\forall n \in \mathbb{N}$, $n > 1$, $n! < n^n$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots (2) \cdot (1)$

$n^n = n \cdot n \cdot n \cdot \ldots (n) \cdot n$ ✓