Lecture 17: Hamiltonian Cycles and Eulerian Graphs

Wednesday, October 23, 2024 10:17 PM

Announcements

1. Homework #7 due Tuesday
2. Feedback #1 due Today
3. Exam #2: 11/21
4. My office hours are after class

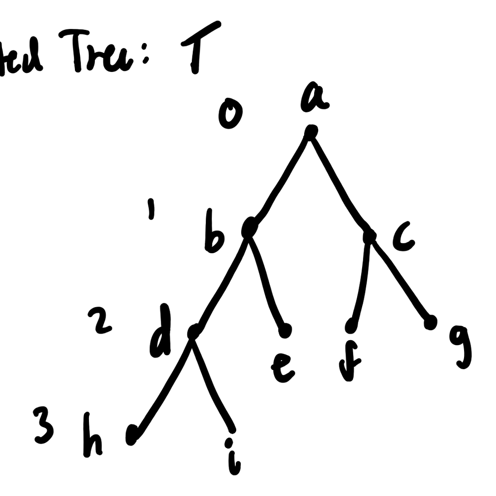
Review:

- What is a spanning tree?

  A spanning tree, T, of G has V(G) and is connected and acyclic. T must be a subgraph of G.

- Lemma: Every connected graph contains a spanning tree

- Rooted Tree: T



1. root (T) = a
2. level (d) = 2
3. height (T) = 3
4. children (b) = {d, e}
5. parent (f) = c
6. ancestors (e) = {b, a}
7. descendants (b) = {d, e, h, i}
8. leaf (T) = {h, i, e, f, g}
9. Is T a binary tree? Yes
10. Is T a full binary tree? Yes

- Lemma: A full binary tree with k internal vertices has 2k+1 total vertices and k+1 leaves

- Lemma: Any binary tree of height at most h has at most 2^h leaves

- The Pigeonhole Principle

  If k+1 or more objects are distributed among k bins, then there is at least one bin with 2 or more objects

  Example: If I have 13 people, at least 2 were born in the same month

  bins → months (12)
  objects → people (13)

- Generalized Pigeonhole Principle:

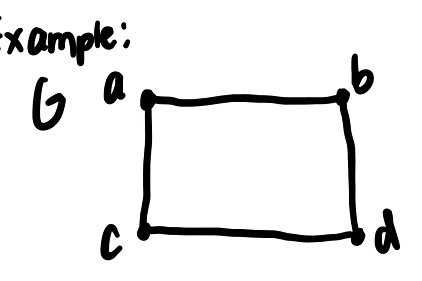
  If n objects are placed into k boxes, then there is at least 1 box that has $\lceil n/k \rceil$ objects

  Example: If I have a 100 people, at least 9 were born in the same month.

  $\frac{100}{12} = 8 R 4 \to 9$

- A hamiltonian cycle in a graph G is a cycle in which every vertex of G appears exactly once. A graph is hamiltonian if it contains a hamiltonian cycle.

  Example:



  a → b → d → c → a

- For any integer $n \geq 3$, let G be a simple graph on n vertices, and assume that all vertices in G are of degree at least n/2. Prove that G has a hamiltonian cycle

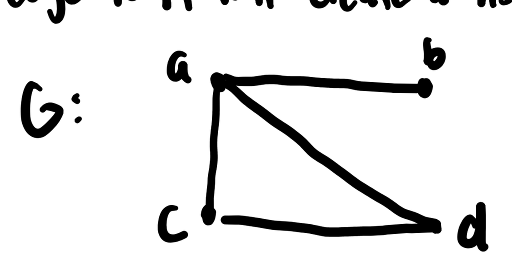
  Basically, $n \geq 3$, $\delta(G) \geq n/2$, prove that G has a hamiltonian cycle:

  Proof by construction:

  Assume for the purpose of contradiction that G does not have a hamiltonian cycle

  Add edges to G until adding anymore edges leads to a hamiltonian cycle

  Let H be the graph, where $\delta(H) \geq n/2$, H does not have a hamiltonian cycle, but adding an edge to H will create a hamiltonian cycle.



  H

  We now know that H has a hamiltonian path

  This path has 2 endpoints: x and y

  → $x, v_2, v_3, \ldots, v_{n-1}, y$ ← n vertices

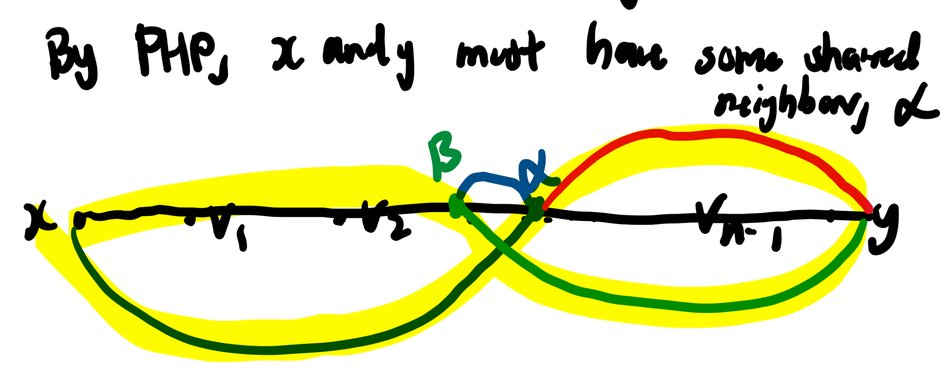
  We know that $\deg(x) \geq \underline{n/2}$

  $\deg(y) \geq \underline{n/2}$

  x and y have a combined n neighbors

  n-2 vertices besides x and y

  By PHP, x and y must have some shared neighbor, α



  Consider y and α

  $\deg(y) \geq n/2$

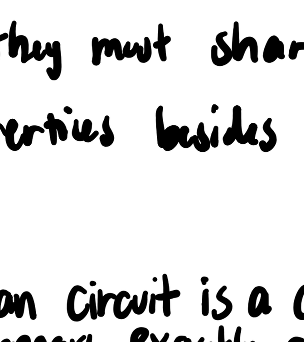
  $\deg(\alpha) \geq n/2$

  y and α have combined n neighbors

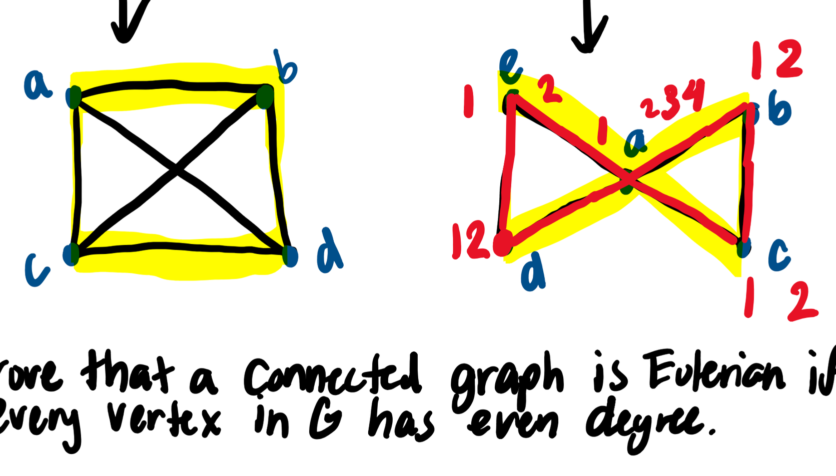
  By PHP → Thus, they must share some neighbor, β

  n-2 vertices besides y and α

- An Eulerian circuit is a closed walk in which each edge appears exactly once. A connected graph is Eulerian if it contains an Eulerian circuit.



Hamiltonian Cycle vs. Eulerian Circuit



Prove that a connected graph is Eulerian iff every vertex in G has even degree.

=> If G is Eulerian, then every vertex in G has even degree

If G is Eulerian, then G has a Eulerian circuit, C.

Every passage of C through a vertex uses 2 incident edges and the first edge is paired with the last at the first vertex. Thus, every vertex has even degree.

<= If every vertex in G has even degree, then G has an eulerian circuit

Induction: on the number of edges, m

BC: m = 0 • ✓

IH: Assume that if every vertex in G has even degree, then G has an Eulerian circuit, when G has K edges

IS: Let G be a graph with k+1 edges, where every vertex has even degree

W.T.S that G is Eulerian.

$\delta(G) \geq 2$, that means G has a cycle