Review Session: Midterm #2

Sunday, November 17, 2024 11:11 PM

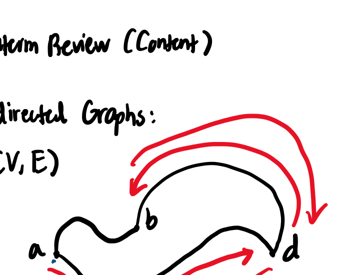
Announcements

- Midterm #2 on 11/21 (Thursday) in Phil B152
  - Graphs, Trees, Spanning Trees, Hamiltonian Cycles, Eulerian Circuits, Directed Graphs, Graph Coloring
- My office hours are after class (midterm review)
- TA application for Discrete Math I (to be released soon...)

Midterm Review (Content)

Undirected Graphs:

$G(V, E)$

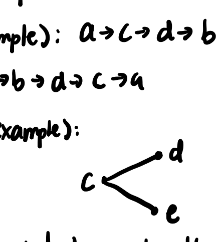


$V = \{a, b, c, d, e\}$

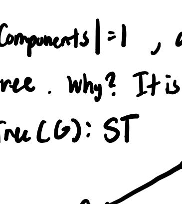
$E: [\{a, c\}, \{c, e\}, \{b, d\}, \{c, d\}, \{a, b\}]$

- $N(c) = \{a, d, e\}$
- $\delta(G) = 1$
- $\Delta(G) = 3$
- Walk(G) (example): $a \to c \to d \to b \to d \to c \to e$
- Path(G) (example): $a \to c \to d \to b$
- Cycle(G): $a \to b \to d \to c \to a$
- Subgraph(G) (example):

- |Connected Components| = 1, aka the graph is connected
- G is a NOT tree. Why? It is has a cycle
- Spanning Tree(G): ST



- Rooted Tree of ST, with c as the root
  RT



level(b) = 2

height(RT) = 3

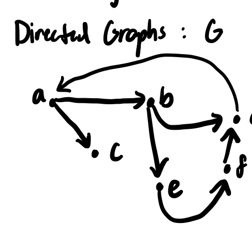
ancestors(d): $\{b, a, c\}$

- Handshaking Lemma: Sum of the degree of all the vertices in a graph is = $2|E|$

$$\sum_{v \in V} \deg(v) = 2|E|$$

- Every graph with n vertices and m edges has at least n-m connected components
- Properties of a tree:
  - acyclic and connected
  - connected and n-1 edges
  - minimally connected
  - acyclic, but adding a new edge creates a cycle
  - Every pair of vertices are linked by a unique path
- Every connected graph $G = (V, E)$ contains a spanning tree
- A hamiltonian cycle in G, is a cycle in which every vertex of G appears exactly once
- An eulerian circuit is a closed walk in which every edge appears exactly once
- A connected graph is Eulerian (has an Eulerian circuit) iff every vertex has even degree

Directed Graphs: G



outdegree(b) = 2

indegree(d) = 2

total degree(b) = 3 = $\overset{2}{\text{outdegree}(b)} + \overset{1}{\text{indegree}(b)}$

sources(G) = [ ]

sinks(G) = [c]

- SCCs(G): $\underset{[a,b,d,e,f]}{SCC_1} \to \underset{[c]}{SCC_2}$
- G_{scc} is a DAG
- $\sigma(G_{scc}) = [SCC_1, SCC_2]$
- Sum of outdegree of all nodes = Sum of indegree of all nodes
- First vertex in a topo-sort is a source, last is a sink

Graph Coloring:

- A graph is k-colorable if each vertex can be colored using one of the k colors so that adjacent vertices are colored using different colors
- The chromatic number of a graph, G, $\chi(G)$, is the smallest value of k for which G is k-colorable.
- A bipartite graph, is a graph that is 2-colorable

Practice Questions:

1. Let $G = (V, E)$ be a connected graph such that $|E| = |V|$. Prove that G has exactly one cycle.

1. Assume f.p.o.c that G doesn't have a cycle and is connected, therefore G is a tree and thus G has n-1 edges

   Contradiction! G has n edges!

2. Every connected graph has a spanning tree with n vertices and n-1 edges.

   Add back the missing edge, → creates exactly 1 cycle.

2. Let T be a tree with at least 2 vertices, and no vertex in T has a degree which is larger than 3 ($\Delta(T) \le 3$). Let n_i be the number of vertices of degree exactly i, prove that $n_1 = n_3 + 2$

Every vertex in T — Vertices of degree 1 / Vertices of degree 2 / Vertices of degree 3

Let the total # of vertices in T be n

$n = n_1 + n_2 + n_3$ ①

By the handshaking lemma:

$2(n-1) = n_1(1) + n_2(2) + 3(n_3)$ ②

$\downarrow$

$2(|E|)$

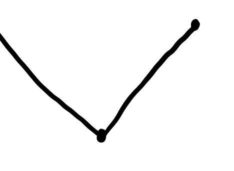
$2(n_1 + n_2 + n_3 - 1) = n_1 + 2n_2 + 3n_3$

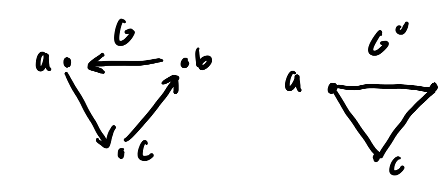
$2n_1 + \cancel{2n_2} + 2n_3 - 2 = n_1 + \cancel{2n_2} + 3n_3$

$n_1 = n_3 + 2$

3. How many edges are required to make a graph with 3 connected components, connected?

2





4. Consider a directed graph, G, with no self-loops and no cycles of length 2 (no parallel edges).

   Define an undirected graph $G_u = (V, E^u)$ where G_u has the same vertices as G, and moreover in G^u, we have an edge (x, y) if $x \to y$ OR $y \to x$ are in G

a) Prove/Disprove: If G is strongly connected, then G^u is connected

Prove: W.T.S that G^u is connected and therefore for any pair of vertices a and b, I can get from a to b

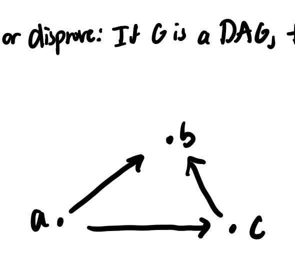
Because G is strongly connected, there is a path from $a \to b$ in G and a path from $b \to a$. Consider this first path: $a \to b$

$a \to x_1 \to \dots \to x_a \to b$

$\downarrow$

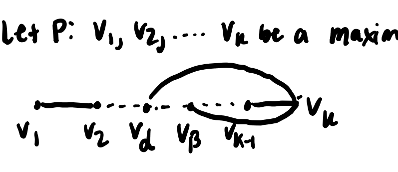
Undirected ✓

b) Prove or disprove: If G is a DAG, then G^u is acyclic



5. Let $G = (V, E)$ be a simple connected graph with at least 3 vertices. Prove that there exists at least 2 vertices a and b, such that a subgraph of G consisting of V without a and b is connected

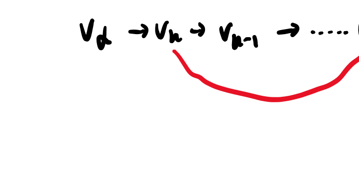
6. An edge $\{a, b\}$ is a chord of the cycle C in an undirected simple graph $G = (V, E)$ if u and v are vertices in the cycle but the edge $\{u, v\} \in E$ is not an edge of the cycle



(a, d)

Prove if $\delta(G) \ge 3$, then G contains a cycle with a chord.

Let P: $v_1, v_2, \dots v_k$ be a maximal path in G



(v_α, v_k) and (v_β, v_k) exist

$v_\alpha \to v_k \to v_{k-1} \to \dots v_\beta \to v_\alpha$

Final Annoucement:

- Very likely, this is my final class this semester! (99%)
- Thanks for being a fantastic 1st class at GW, and don't hesitate to reach out (email / LinkedIn)
- I think I'll be teaching Discrete I next semester, and looking for TAs
  - Official applications out soon... let's chat
- Summer 2025 - Research Endeavor?
- Class social ~ 12/13? Next Semester?
- Good luck on the exam, and the final exam (cumulative)