Lecture 16: Advanced Graph Theory

Monday, October 21, 2024 9:18 PM

Announcements:

1. Homework #6 due Today
2. Homework #7 Released Today
3. Feedback Survey due Thursday
4. My office hours today after class
5. Exam #2: November 21st

Review Trees:

What is a tree?

An acyclic and connected graph

Lemma: Every tree with at least 2 vertices has at least 2 leaves and deleting a leaf from an n-vertex tree produces a tree with n-1 vertices

Properties of a tree:

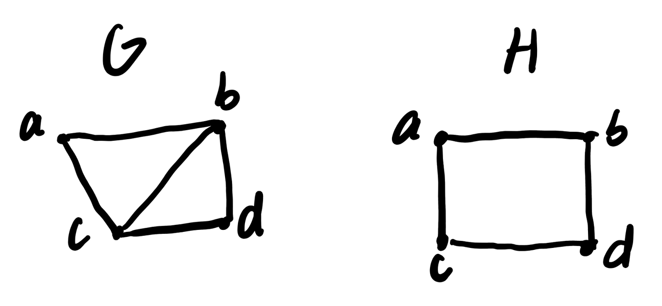
1. G is a tree
2. G is connected and has exactly n-1 edges
3. G is minimally connected
4. G contains no cycles but $G + \{x, y\}$ does for any 2 non-adjacent vertices, $x, y \in V$
5. Any two vertices of G are linked by a unique path in G

What is a subgraph?

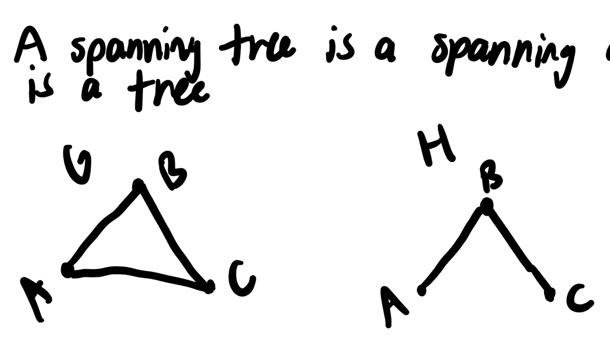
If $H \subseteq G$, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

- A spanning subgraph of G is a subgraph with vertex set: $V(G)$

Example:



- A spanning tree is a spanning subgraph that is a tree



Prove: Every connected graph $G=(V,E)$ contains a spanning tree

Proof by construction:

We will show this is true by constructing a graph, J, as follows:

Start with G

Remove edges, e, from G, such that the removal of e does not disconnect G

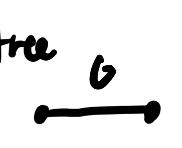
The new graph after this process is called J

J is connected, acyclic $\rightarrow$ J is a tree

J also has the same vertices as G

J is a spanning tree

BC: 1 edge:



IH: Assume that a graph with $1 \le j \le k$ edges a spanning tree

IS: W.T.S that graph with $k+1$ edges has spanning tree.

Remove an arbitrary edge:

Case 1: Did not disconnect graph: 1 CC

By IH, J has a spanning tree

Add the removed edge back to J that edge is not required in the spanning tree and G has the same spanning tree as J

Case 2: Disconnected graph into 2 CC: X and Y

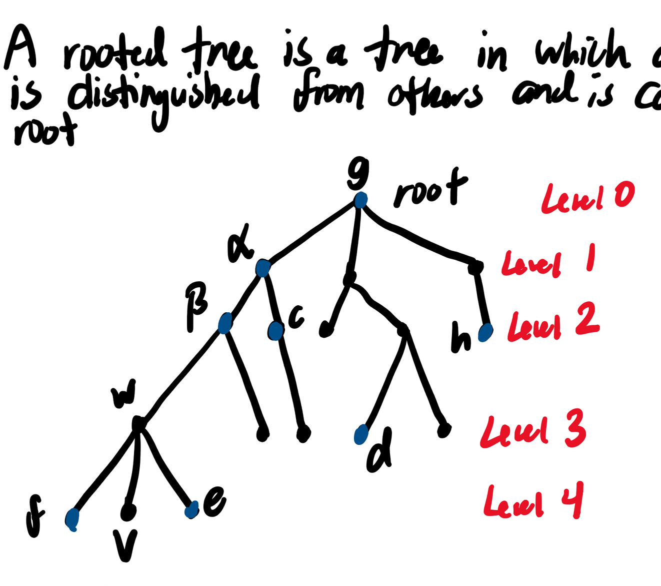
Call IH on X, X has a spanning tree $- T_1$

Call IH on Y, Y has a spanning tree $- T_2$

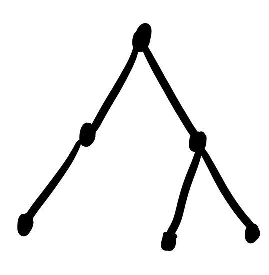
Add the edge, e, back

$T_1 + e + T_2$ = Spanning Tree

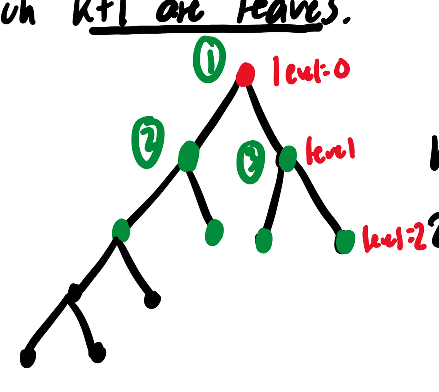
- A rooted tree is a tree in which one vertex is distinguished from others and is called the root



- The level of a vertex, u, is the number of edges on the path from u to the root.
- The height of a rooted tree is the maximum level of any vertex in the tree.
- Given some vertex of a rooted tree, u, Children(u) = neighbors of u (N(u)) that are one level away from the root than u
- If a vertex, v, is in Children(u) then u = parent(v)
- Two vertices that are both children of the same vertex are siblings
- Given vertices v and w, if v lies on the unique path between w and the root, v is an ancestor of w and w is a descendant of v
- leafs in rooted trees have no children
- All internal vertices in a rooted tree have children
- A binary tree is a rooted tree in which each internal vertex has at most 2 children, a left child and a right child
- A full binary tree is a binary tree where each internal vertex has exactly 2 children



- Prove the following: If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of $2k+1$ vertices, of which $k+1$ are leaves.



$k = 3$

$k+1 = 4$ leaves

$2(3)+1 = 7$ total vertices

Start with an internal vertex, for each internal vertex, I count 2 nodes in the graph, the left child and the right children $= 2k + 1$

for root

$2k+1$ total vertices

k internal vertices

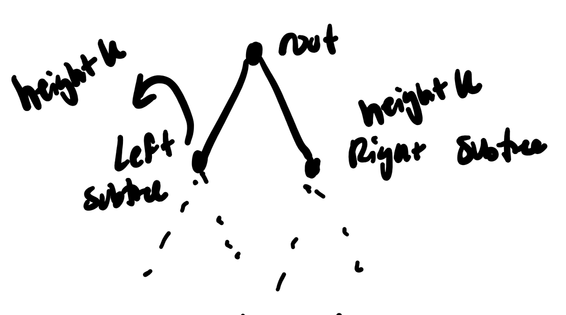
$2k+1-k = k+1$ leaves

- Prove that any binary tree of height at most h has at most 2^h leaves

BC: height = 0, $2^0 = 1$, 1 leaf ✓ root

IH: Assume for some k, that when the height of the tree is k, the tree has at most 2^k leaves

IS: W.T.S when height is $k+1$, the binary tree has at most 2^{k+1} leaves



By IH, left subtree has at most 2^k leaves

Right subtree has at most 2^k leaves

Thus, $2^k + 2^k \le 2^{k+1}$ ✓

- If $\delta(G) \ge 2$, then G has a cycle.

1. We know that the $\sum_{v \in V} \deg(v)$ of $G \ge 2n$
2. #edges $= \sum_{v \in V} \deg(v)$

#edges $\ge n$

Let P be the longest path in the graph G.