Lecture #13: Midterm Review Session

Friday, September 27, 2024 9:55 PM

Announcements

Philip's Hall: B152

1. Midterm #1 on Tuesday: 10/8, get here at 4:30 PM
   Two-sided "cheat" sheet that will be turned in with exam
2. Midterm Review Session in Lab (this week)
3. No Lab next week and No Homework next week (Fall Break)
4. My office hours are after class (Midterm Questions!)
5. Happy Red October!

1. Propositions, Proof Notation: Operations ($\wedge, \vee, \rightarrow$, etc)
2. Proving logical equivalence between propositions:
3. Quantified Statements: $\exists$ ("there exists"), $\forall$ ("for all")
4. Key Definitions: What is a prime number? Composite number? Even number? Odd number? Rational number? Irrational number? Natural number? Integer? Floor? Ceiling?
5. Proof Techniques: Direct Proofs, Contrapositive, Contradiction

Prove #1 a) If $a \cdot b$ is irrational, then either a or b or both must be irrational. $p \rightarrow q$ $\neg q \rightarrow \neg p$

Contradiction or Contrapositive

Let's prove instead: If both a and b are rational, then $a \cdot b$ is rational.

$a = p/q$ $p, q, r, s \in \mathbb{Z}$ and
$b = r/s$ p and q do not have a common factor
r and s do not have a common factor
$a \cdot b = \frac{p}{q} \cdot \frac{r}{s}$ q and s cannot be 0
$= pr/qs$

$p \cdot r$ is $\in \mathbb{Z}$
$q \cdot s \in \mathbb{Z}$ and $q \cdot s \neq 0$

Disprove b) If x and y are irrational numbers, then x^y is irrational

Prove c) n is even $\leftrightarrow$ $7n+4$ is even

6. Divisibility: Handout

a) Prove that $\forall a \in \mathbb{Z}$, if a is odd and $a \geq 3$, then $8 \mid a^2-1$

#2 b) Prove that if x, y, and z are 3 consecutive integers, then $3 \mid x+y+z$, and $x^2+y^2+z^2$ is not divisible by 3

1) Prove $3 \mid x+y+z$

$x = x$
$y = x+1$
$z = x+2$

$x + x+1 + x+2 = 3x+3$
$3(x+1)$
$3 \mid 3(x+1)$ ✓

2) Let's assume f.p.o.c that $3 \mid x^2+y^2+z^2$

$x^2 + x^2+2x+1 + x^2+4x+4$

$3 \mid 3x^2+6x+5$

$3 \mid 3(x^2+2x+1) + 2$

7. Proof Techniques: Induction: BC, IH, IS

a) Weak Induction:

#3 Consider the sequence of integers $T(n)$ defined by the following recurrence:

$T(0) = 0$, $T(n+1) = (-1)^n T(n) + 1$
$T(1) = T(0+1) = (-1)^0 T(0) + 1$

Prove that $T(n) = T(n-2)$ $\forall n \geq 2$

BC: 0, $T(0) = 0$
1, $T(1) = (-1)^0 \cdot T(0) + 1 = 1 + 0 = 1$
2, W.T.S that $\boxed{T(2) = T(0)}$ ✓

$T(2) = T(1+1) = (-1)^1 (1) + 1 = -1+1 = 0$ ✓

IH: Assume that $T(k) = T(k-2)$ for some $k \geq 2$

IS: W.T.S that $T(k+1) = T(k-1)$

$T(k+1) = (-1)^k T(k) + 1$
$\hookrightarrow T(k-2)$

$= (-1)^k T(k-2) + 1$

$(-1)^0 = 1$
$(-1)^1 = -1$
$(-1)^2 = 1$
$(-1)^3 = -1$
⋮

$(-1)^k = (-1)^{k-2}$
$\rightarrow (-1)^2 = 1$

$T(k-1) = (-1)^{k-2} T(k-2) + 1$

#4 b) Strong Induction:

Let's play a game:

Some number of marbles are placed in front of you and another player. The two players take turns removing 1, 2, or 3 marbles from the pot. The player to remove the last marble loses.

x_1 x_2 x_3 x_4 x_5

Prove that the first player has a winning strategy iff the number of marbles (n) is not $4k+1$, for any $k \in \mathbb{N}$

BC: $k=0$, $4(0)+1 = 1$
↓
First player has to take marble and loses ✓

$4(0)+2 = 2$ marbles

$4(0)+3 = 3$ marbles

$4(0)+4 = 4$ marbles

IH:

Assume that for all j where $j: 1 \leq j \leq k$, the first player has a winning strategy if the number of marbles in the pot are NOT $4(j)+1$

IS:

W.T.S that for $4(k+1)+1$ marbles, that the first player has a winning strategy when there are NOT $4(k+1)+1$ marbles, and does have a winning strategy otherwise.

$4(k+1)+1$ Marbles

3 cases:

Case 1: First player takes 1 marble

$4(k+1)$

Second player wins

Case 2:

8. Timing Analysis and Asymptotics: Proving Big-Oh, Omega, Theta

#5 a) Prove that $12n\sqrt{n} = O(11/4 n^2 - 6)$

$1 \leq n^6$
$n^2 \leq n^6$

9. Finding Theta Runtime of Code-Snippets

a) Superset / Subset Method

#6

```
for i:=1 to n do  i++
  for j:=1 to i·i do  j++
    for k:=1 to j do  k++
      print ("How are we feeling about the exam?")
```

$n/4$ to n
$n^2/32$ to $n^2/16$
1 to $n^2/32$

Superset:

for $i:=1$ to $\underline{n}$
for $j:=1$ to $\underline{n^2}$
for $k:=1$ to $\underline{n^2}$

for $i:=1$ to n
for $j:=1$ to n n^2

for $i = \underline{n/4}$ to $\underline{3n/4}$ $n/2$
for $j = \underline{n^2/32}$ to $\underline{n^2/16}$ $n^2/16$
for $k = \underline{1}$ to $\underline{n^2/32}$ $n^2/32$

$= \Omega(n^5)$

$n/2 \cdot n^2/16 \cdot n^2/32 = n^5/2 \cdot 16 \cdot 32$

$c = \frac{1}{2 \cdot 16 \cdot 32}$

$= \Omega(n^5)$

b) From Lab #4:

#7

```
for (int i=4; i<n, i=i·i)
  for (int j=2; j<√i; j=j+j)
    print ("                ")
```

```
for (i=1; i≤n, i=i·2)
  for (j=i; j<9i+7; j=j+4)
```

$i = 2^x$

iteration	i
0	1
1	2
2	4
3	8
⋮	
x	2^x

x:

iteration	j
0	i
1	i+4
2	i+8
3	i+12
⋮	
y:	i+4y

$$\sum_{x=0}^{?} \sum_{y=0}^{2i+7/4} 1$$

$2^x \leq n$
$x \leq \lg_2(n)$

$i+4y < 9i+7$
$4y < 8i+7$
$y < 2i+7/4$

$$\sum_{x=0}^{?} 2i+7/4$$

$$\sum_{x=0}^{\lg(n)} 2(2^x) + 7/4$$

$$2 \sum_{x=0}^{\lg(n)} 2^x + 7/4 \lg(n)$$

$$2(2^{\lg(n)+1} - 1) + 7/4 \lg(n)$$

$$2^{\lg(n)+2} - 2 + 7/4 \lg(n)$$

$2^2 \cdot 2^{\lg(n)}$
$= 4 \cdot 2^{\lg(n)}$

$\Theta(n)$

$4n - 2 + 7/4 \lg(n)$