Asymptotics In-Class Practice Problems

Recall the definitions we have introduced in class:

Big-Oh

Let $f, g\colon \mathbb{N} \to \mathbb{R}$ such that $g(n) \geq 0$.
We say that $f(n) \in O(g(n))$ or $f \in O(g)$ if $\exists B, b$, both positive, such that

$$0 \leq |f(n)| \leq Bg(n)\ \forall n \geq b$$

Big-Omega

Let $f, g\colon \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) \in \Omega(g(n))$ or $f \in \Omega(g)$ if $\exists A, a$, both positive, such that

$$0 \leq Ag(n) \leq f(n)\ \forall n \geq b$$

Big-Theta

Let $f, g\colon \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) \in \Theta(g(n))$ or $f \in \Theta(g)$ if $\exists A, B, k$, all positive, such that

$$Ag(n) \leq f(n) \leq Bg(n)\ \forall n \geq k$$

Practice Questions:

1. Is $2^{n+1} \in O\left(2^n\right)$? What about 2^{2n}? Provide proof.
2. Suppose $f \in O(g)$. $Is\ kf(n) \in O(g(n))$ for $k \in \mathbb{R}$, $k \neq 0$? If so, prove it. If not, provide a counterexample.
3. For non-negative functions, is the relation big-Oh (provide proof):
   a. Reflexive?
   b. Symmetric? [cf Homework 4]
   c. Transitive?
4. Show that for any $k, d \geq 0$, $n^k \in O\left(n^d\right) \Leftrightarrow d \geq k$
5. Let $f(n) = n^2$ and $g(n) = \frac{1}{2}n^2$. Show that: $2^{f(n)} \notin O\left(2^{g(n)}\right)$
6. Let $f, g\colon \mathbb{N} \to \mathbb{R}^+$. Show that $f \in O(g) \Leftrightarrow g \in \Omega(f)$
7. Let $f, g\colon \mathbb{N} \to \mathbb{R}^+$. Show that $f \in \Theta(g) \Leftrightarrow f \in O(g)\ and\ f \in \Omega(g)$
8. Using any previous results on this practice sheet or from class/HWs/labs, show the following. Do not use *A, B, k, a, b,* etc.:
   a. Ω is reflexive
   b. Ω is not symmetric
   c. Ω is transitive