TUESDAY SEPTEMBER 17 2024

Announcements:
- HW4 will be posted tonight, due next week
- Exam 1 in 2 weeks
- You can bring one HANDWRITTEN, SINGLE-SIDED sheet of notes that you'll turn in w/ your exam

Big-O

We have two functions $f(n)$ & $g(n)$.

$$f(n) \in O(g(n)) \rightarrow \text{"belongs to"}$$
"is"
"is in"

means

there are some constants
$b > 0$ and $B > 0$
such that
$$0 \leq |f(n)| \leq B \cdot g(n)$$
for all $n \geq b$

$$f(n) \in O(g(n)) \iff \exists b, B > 0$$
such that
$$0 \leq |f(n)| \leq B\, g(n)$$
$$\forall n \geq b$$

To remember:
- $|f(n)|$ → make sure $f(n) \geq 0$ before dropping absolute value sign
- $|f(n)| \leq B \cdot g(n)$ → B can be really large and g can be small
- $n \geq b$ → we don't care what happens for n smaller than b

$\boxed{\text{Ex 1}}$ Show that $0.1n^2 \in O(n^2)$

We need to show
$$\exists b, B > 0 \text{ s.t. } 0 \leq |0.1n^2| \leq B \cdot n^2 \quad \forall n \geq b$$
$$0 \leq 0.1n^2 \leq Bn^2 \quad \forall n \geq b$$

$b = 1$; $B = 1$ $\qquad 0.1n^2 \leq 1 \cdot n^2 \quad \forall n \geq 1$

We know that $0.1 \leq 1$
$\iff 0.1n^2 \leq 1 \cdot n^2$ for all $b \geq 1$

This means that $\exists b, B$ $\; b = 1; B = 1$ s.t.
$$0 \leq |0.1n^2| \leq B \cdot n^2$$
By the definition of $O()$, $0.1n^2 \in O(n^2)$

$\boxed{\text{Ex2}}$ Show that $f(n) = n^2 + 2n + 1$ is $O(n^2)$.

$\exists b, B > 0$ s.t. $0 \leq |f(n)| \leq B \cdot g(n) \quad \forall n \geq b$

$\exists b, B > 0$ s.t. $0 \leq |n^2 + 2n + 1| \leq B \cdot n^2 \quad \forall n \geq b$
$$0 \leq n^2 + 2n + 1 \leq B \cdot n^2 \quad \forall n \geq b$$

1. We know that $n^2 \leq n^2 \quad \forall n \geq 1$
$2n \leq 2n^2 \quad \forall n \geq 1$
$1 \leq n^2 \quad \forall n \geq 1$

$\Rightarrow n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 \quad \forall n \geq 1$
$\Rightarrow n^2 + 2n + 1 \leq 4n^2 \quad \forall n \geq 1$

$b = 1$; $B = 4$

2. $n \geq 1 \Rightarrow f(n) \leq B \cdot g(n) \quad \forall n \geq b$
$\Rightarrow n^2 \geq n$
$\Rightarrow 2n^2 \geq 2n$
$\Rightarrow n^2 + 2n^2 \geq n^2 + 2n$
$\Rightarrow n^2 + 2n^2 + n^2 \geq n^2 + 2n + 1$
$\Rightarrow 4n^2 \geq n^2 + 2n + 1$

Some more:

Show: $7x^2 \in O(x^3)$
n is $O(n^2)$
$\sqrt{n}$ is $O(n)$

$\boxed{\text{Ex3}}$ Show that $n^2 \notin O(n)$

PROOF BY CONTRADICTION

Suppose there are constants $b, B > 0$ s.t.
$$n^2 \leq B \cdot n \quad \text{whenever } n \geq b$$

We can rewrite this as
$$n \leq B \quad \text{whenever } n \geq b$$

Let's consider $n' = \max(b+1, B+1)$

We know that $n' = \max(b+1, B+1) > b$
$n' = \max(b+1, B+1) > B$

$n' > b$, so the inequality $n^2 \leq B \cdot n$ should hold. However, $n^2 \leq B \cdot n$ is only the case when $n \leq B$. But we know that $n' > B$, so we have a contradiction.

$\boxed{\text{Ex 4}}$ Show that x^3 is not $O(7x^2)$.

Assume $x^3 \in O(7x^2)$ and that
$\exists b, B > 0$ s.t. $0 \leq x^3 \leq B \cdot (7x^2) \quad \forall x \geq b$

$$\Rightarrow x \leq 7B \quad \forall x \geq b$$

$(b \leq x \leq 7B)$

Consider $x' = \max(7B+1, b+1)$