CSCI2312: Discrete Structures II

Solutions Lab 8

October 28, 30 November 1

Problem 1:
Consider a connected graph G = (V, E) and an arbitrary partition of G's vertex set V into nonempty sets S and V \ S. Prove that if there exists only one edge e between the vertices in S and the vertices in V \ S, then e must be in every spanning tree of G.

Solution:
We are tasked with proving that if there is only one edge e connecting two disjoint vertex sets S and V \ S in a connected graph G, then this edge must be included in every spanning tree of G.

Let us consider any arbitrary spanning tree of G, denoted T . Since T is a tree, it is connected and spans all the vertices in G, which means there is a path between any two vertices in T . Now, take an arbitrary vertex $x \in S$ and another vertex $y \in V \setminus S$. Since T is connected, there must be a path P between x and y in T .

Key observation: For P to connect x and y, it must traverse from a vertex in S to a vertex in V \S. Since the edge e is the only edge that crosses the cut between S and V \ S, the path P must include the edge e. Therefore, e must be part of the spanning tree T .

Conclusion: Since T was an arbitrary spanning tree of G, this argument holds for any spanning tree. Thus, e must be in every spanning tree of G, proving that e is a necessary edge for connectivity in any spanning tree.

Problem 2:
We say a graph is maximally acyclic if adding any edge to the graph creates a cycle. In lecture, we will prove that if T is a tree, then T is maximally acyclic. Prove the converse, that is, if T is maximally acyclic then T is a tree.

Solution:
Since T is maximally acyclic, it is also acyclic, so it suffices to show T is connected.
Assume for contradiction that T is not connected, then that means there exist vertices u,v such that there is not a walk between u and v in T.
Now consider adding the edge u-v to create T'. Since T is maximally acyclic, T' must now have a cycle.

Note that this cycle must contain the new u-v edge, or else the cycle would have been present in the original T.
Consider the portion of the cycle without the edge u-v. This is a path between u and v and it does not contain the edge u-v so it must lie completely inside T. This contradicts the assumption that there is no walk between u and v in T

Problem 3:
Let G be a graph where the minimum degree is d. Prove that if there are no cycles of exactly three vertices, then there must be at least 2d vertices in the graph.
Solution:
We will use direct proof to prove the claim. There are two cases:
The first case is where d=0. In this case we must prove that if there are no cycles of exactly three vertices, there must be at least 2(0) = 0 vertices in the graph which is vacuously true since there needs to be at least 1 vertex in order to have a graph.

The second case is where d>0.
In this scenario since the minimum degree is greater than 0, we know that there exists some vertex in the graph a, which has at least d neighbors, one of which can be denoted as b.
These two vertices (a and b) are adjacent. If there exists another vertex in the graph, x, a and b cannot both be a neighbor of x, because if this is the case then there would be a cycle with three vertices (a contradiction). Thus a and b have disjoint set of neighbors.
Since the minimum degree is at least d, a has at least d neighbors, one of which is b. Thus a must have at least d-1 distinct neighbors from b. By the same reasoning, b also has d-1 distinct neighbors from a.
Now, counting the total number of vertices we have a, b, a's d-1 distinct neighbors, and b's d-1 distinct neighbors, which means that there are at least 1+1+d-1+d-1 = 2d vertices in the graph.