CSCI2312: Discrete Structures II

Solutions Lab 7

November 4, 6, 8

Problem #1: Given a graph with n ≥ 2 vertices, prove that there are at least two vertices with the same degree.

Solution:
We consider the following cases
Case 1: There exists an isolated vertex. Thus, there can be no vertex which is connected to all other vertices, in other words, there are no vertices with degree n-1. Thus, the possible degrees for each vertex are 0, 1, 2, …., n-2. We let the n-1 possible degrees to be holes and let the n vertices be the pigeons. By PHP, we know there are at least $\lceil \frac{n}{n-1} \rceil = 2$ vertices with the same degree.

Case 2: There does not exist an isolated vertex. Thus, there can be no vertex with degree 0. Thus, the possible degrees of each vertex are 1, 2, ……, n-1. We let the n-1 possible degrees be the holes and let the n vertices be pigeons. By PHP, we know there are at least $\lceil \frac{n}{n-1} \rceil = 2$ vertices with the same degree.
Thus, the graph must have at least two vertices with the same degree.

Problem #2: Prove that if G is a DAG (has zero cycles) with at least two distinct sinks such that there is a path from every source to every sink, then G must have at least one node of outdegree ≥ 2.

Solution:
Since there is a path from every source to every sink, we know from the lecture that every DAG has at least one source, let's consider the paths from this specific source, u to two distinct sinks, v and w.
Since both these paths start from the same source, u, and end at different vertices, there must exist some vertex k such that the paths diverge. In other words, pick k such that k is the final similar vertex between paths u · · · k · · · v and u · · · k · · · w.
Hence, k must have an outdegree of at least 2.

Problem#3: We say that a tree, T is a 1-3 tree if each of the vertices in the tree have either a degree of 3 or a degree of 1.
a) Prove that if a 1-3 tree has l leaves, then it has l − 2 vertices of degree 3. (HW Question)
b) Let T be a 1-3 tree with at least 4 leaves. Prove that in T there exists an internal vertex that is adjacent to exactly two leaves.

Solution:
a.
We know that there is a 1-3 tree, T, that has l leaves, and we must prove that such a 1-3 tree, T has $l - 2$ vertices of degree 3.
Let's denote the total number of vertices this 1-3 tree has as n. Every leaf in this tree has a degree of 1, which means that besides the l leaves, the rest of the vertices are of degree 3.
Thus, the number of internal vertices are given by $n - l$, where n ia the total number of vertices. We know that a tree has $n - 1$ edges and by the handshake lemma, the sum of the degrees of all nodes in a graph is twice the number of edges.
Thus, twice the number of edges is given by the expression: $2(n - 1)$, and the sum of the degrees of all vertices in a graph is given by: $1(l) + 3(n - l)$, as each leaf has a degree of 1, and the rest of the vertices have a degree of 3. Since we are trying to find the number of vertices of degree 3, we need to solve for $n - l$.
$2(n - 1) = l + 3(n - l)$
$2n - 2 = l + 3n - 3l$
$-n = -2l + 2$
$n = 2l - 2$
$n - l = l - 2$
Therefore, we have proven that if a 1-3 tree has l leaves, then it has $l - 2$ vertices of degree 3.

b.
Option 1:
Let T be a 1-3 tree with at least 4 leaves. Notice that every leaf must be adjacent to exactly one node of degree three. This is because a leaf must be adjacent to exactly one other node, and this node cannot be another leaf or these two leaves would be disconnected from the rest of the graph. Thus, we can imagine mapping each leaf to its respective node of degree three. From part a, we know there are more leaves than there are nodes of degree three, so by the pigeonhole principle, we know two or more leaves must map to the same node of degree three.

We have shown that there exists a node of degree three that is adjacent to at least two leaves. Notice that this node cannot be adjacent to three leaves, as each node would have its maximum number of neighbors and there would be no way to incorporate the forth leaf into this graph. Therefore, this node must be adjacent to exactly two leaves.

Option 2:
Maximal Length Path Proof:
Let T be a 1-3 tree with at least 4 leaves. We will prove that this tree has some internal vertex that is adjacent to two leaves. Let's look at a maximal length path in this tree, T. The last vertex in this maximal length path is a vertex labeled, a. We know that a is a leaf since it is at the end of the maximal length path and has no neighbors other than its inly neighbor on the path. We will call a's only neighbor that is also on the maximal length path, b. We know that b is not a leaf as if it were then there would only be a total of 2 leaves in the graph, violating the condition that T has at least four leaves. Thus, b is a vertex of degree 3, and is therefore connected to two other vertices, one of which must be on the maximal length path (since otherwise the path would not be maximal length). There are three cases for what the internal vertex b, could be connected to.

Case 1: b is connected to two leaves. This cannot be true, as if b is connected to two leaves then there can only be a maximum/total of 3 leaves in the graph, which violates the condition that there are greater than or equal to 4 leaves

Case 2: b is connected to two internal vertices. This cannot be the case as then this would no longer be a maximal length path as one could go from b to the internal vertex that it is connected to, to another vertex that is connected to that internal vertex, which is longer than going straight from b to a.

Case 3: Therefore the case we are left with is that b is connected to one internal vertex and a leas, which is the only case that is possible, as the other two cases have been shown to be impossible with the given conditions.

Thus, we have shown that the internal vertex, b, is adjacent to two leaves, a, and the other leaf that is connected to in case 3.