December 5, 2024

Announcements:

- HW 10 due Tuesday
- Final exam on Thursday
  - cumulative
  - double-sided sheet of notes
  - last day to make requests through DSS office
- Course evals are live!

LAST TIME:

Graph isomorphisms: An isomorphism between simple graphs $G = (V, E)$ and $G' = (V', E')$ is a one-to-one and onto function $f: V \to V'$ such that

$(u,v) \in E \iff (f(u), f(v)) \in E'$

PROVE:

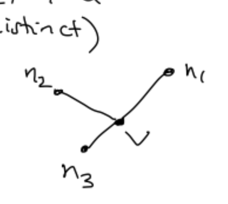
Show that the degree of a node is invariant under isomorphism. That is, if f is an isomorphism from graph G to graph G' and v any node in G, show that the $\deg_G(v) = \deg_{G'}(f(v))$.

Let v have $\deg_G(v) = d$

$\Rightarrow$ v has d neighbors; call them $n_1, n_2, \dots, n_d$

(We know that $v, n_1, n_2, \dots, n_d$ are all distinct)

and $(v, n_i) \in E$ for all $i: 1 \to d$



Because f is an isomorphism,

$\Rightarrow (f(v), f(n_i)) \in E'$ for all $i: 1 \to d$

and because f is a one-to-one function, every input has to map to a distinct output, so $f(v), f(n_1), f(n_2), \dots, f(n_d)$ all must be distinct.

Therefore, $f(v)$ has at least d neighbors in G' and $\deg_{G'} f(v) \geq d$

Could $\deg_{G'} f(v) > d$? That is, could we have added more neighbors to $f(v)$?

No! $\rightarrow$ Suppose there exists $x \in V'$, $x \neq f(n_i)$, $(f(v), x) \in E'$

Because f is onto, $\exists y \in V$ such that

$f(y) = x$

Because f is an isomorphism,

$(f(v), f(y) = x) \in E' \iff (v, y) \in E$

We said $f(y) = x \neq f(n_i)$ for all i, so since f is one-to-one, $y \neq n_i$ for all i

CONTRADICTION! $\rightarrow$ Because we said that $n_1 \dots n_d$ were the only neighbors of v in G.

Therefore, $\deg_{G'} f(v) = d = \deg_G(v)$

$\Rightarrow$ The degree of a node is invariant under isomorphism ✓

PROVE: Show that the number of nodes with given degree d is invariant under isomorphism.

Let $v_1, v_2, \dots, v_k$ be the **k** nodes of degree d in graph G. Their corresponding vertices in G': $f(v_1), f(v_2), \dots, f(v_k)$ all must have degree d (based on the previous proof). So we have at least k vertices in G' that have degree d.

We still need to show that these are the only nodes w/ degree d in G'.

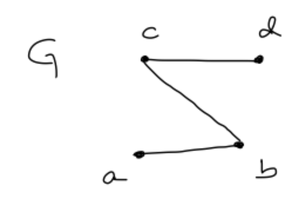
Consider any node x of degree d in G'. Because f is onto, there must exist a y in G such that $f(y) = x$ and the degree of both x and y is d (per the previous proof).

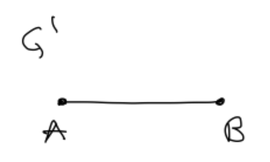
Therefore, $y = v_i$ for some i. So there are no additional nodes beyond $f(v_i)$, so there are exactly k nodes with degree d in G'

HOMOMORPHISMS

Definition: A homomorphism between simple graphs $G = (V, E)$ and $G' = (V', E')$ is a function [not necessarily one-to-one or onto] $h: V \to V'$ such that

$(u,v) \in E \Rightarrow (h(u), h(v)) \in E'$





$h(a) = A$
$h(b) = B$
$h(c) = A$
$h(d) = B$

The edges in G are:

$(a,b), (b,c), (c,d)$

$\downarrow$

$$\left.\begin{array}{l} (h(a), h(b)) = (A,B) \\ (h(b), h(c)) = (B, A) \\ (h(c), h(d)) = (A,B) \end{array}\right\}$$ these are in G' so h is a homomorphism between G and G'

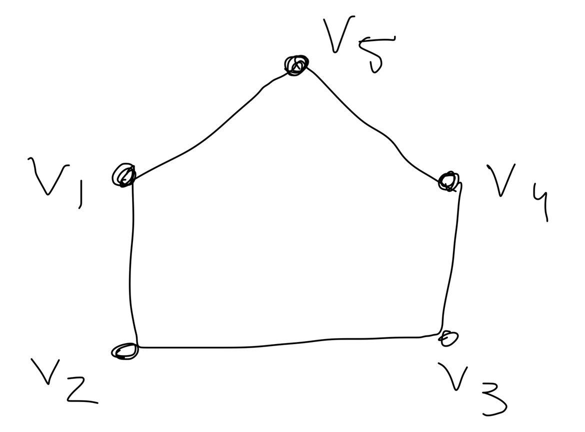
What about the other way $(G' \to G)$?

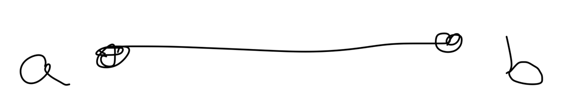
j: $A \to a$
$B \to b$

The edges in G': (A, B)

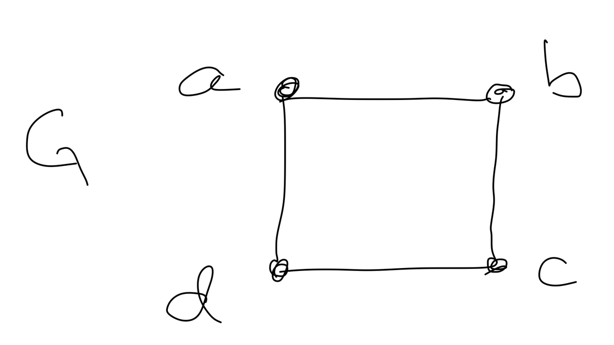
$\downarrow$

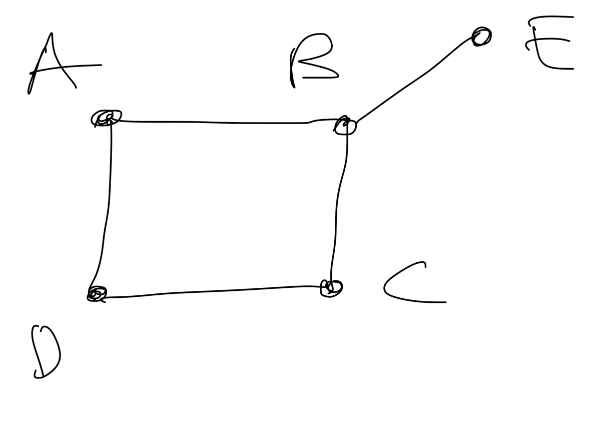
$(j(A), j(B)) = (a, b)$





Can G' have more edges than G?





$a \to A$
$b \to B$
$c \to E$
$d \to B$

PROVE: Show that the degree of a node is not invariant under a homomorphism.

$\Rightarrow$ We can just provide a counterexample 🙂