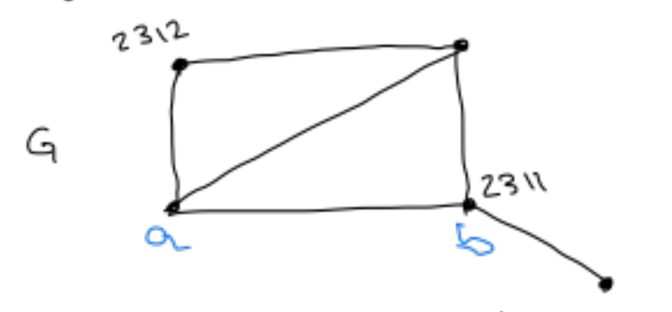
Announcements:
- HW9 will be posted soon, due Thurs 11/14
- Exam 2 is on Nov 21st (can bring double-sided handwritten sheet)
  - covers graphs through 11/14
- Lab next week is midterm review
- No lab week of midterm (Nov 18/20/22)

## Graph Coloring

Example: scheduling exams
vertices = classes
edges = represent overlap in students

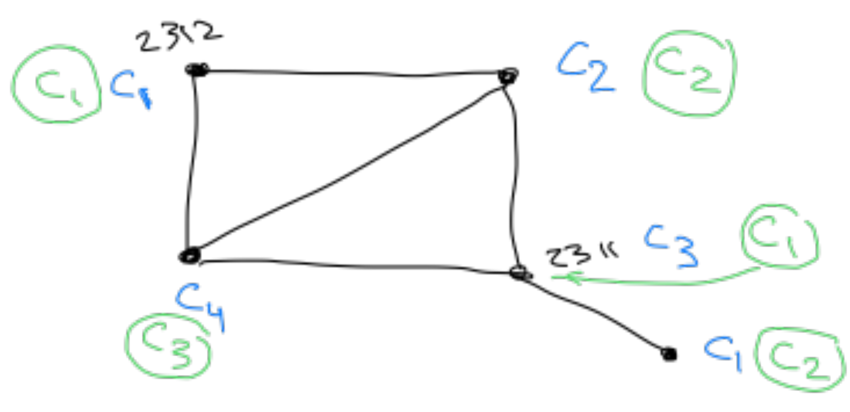



| Slots | = Colors |
|---|---|
| Wed 5-7pm | $c_1$ |
| 7-9pm | $c_2$ |
| 9-11pm | $c_3$ |
| 11-1am | $c_4$ |
| 1-3am | $c_5$ |

Goal: Assign slots to classes & try to use as few slots as possible.

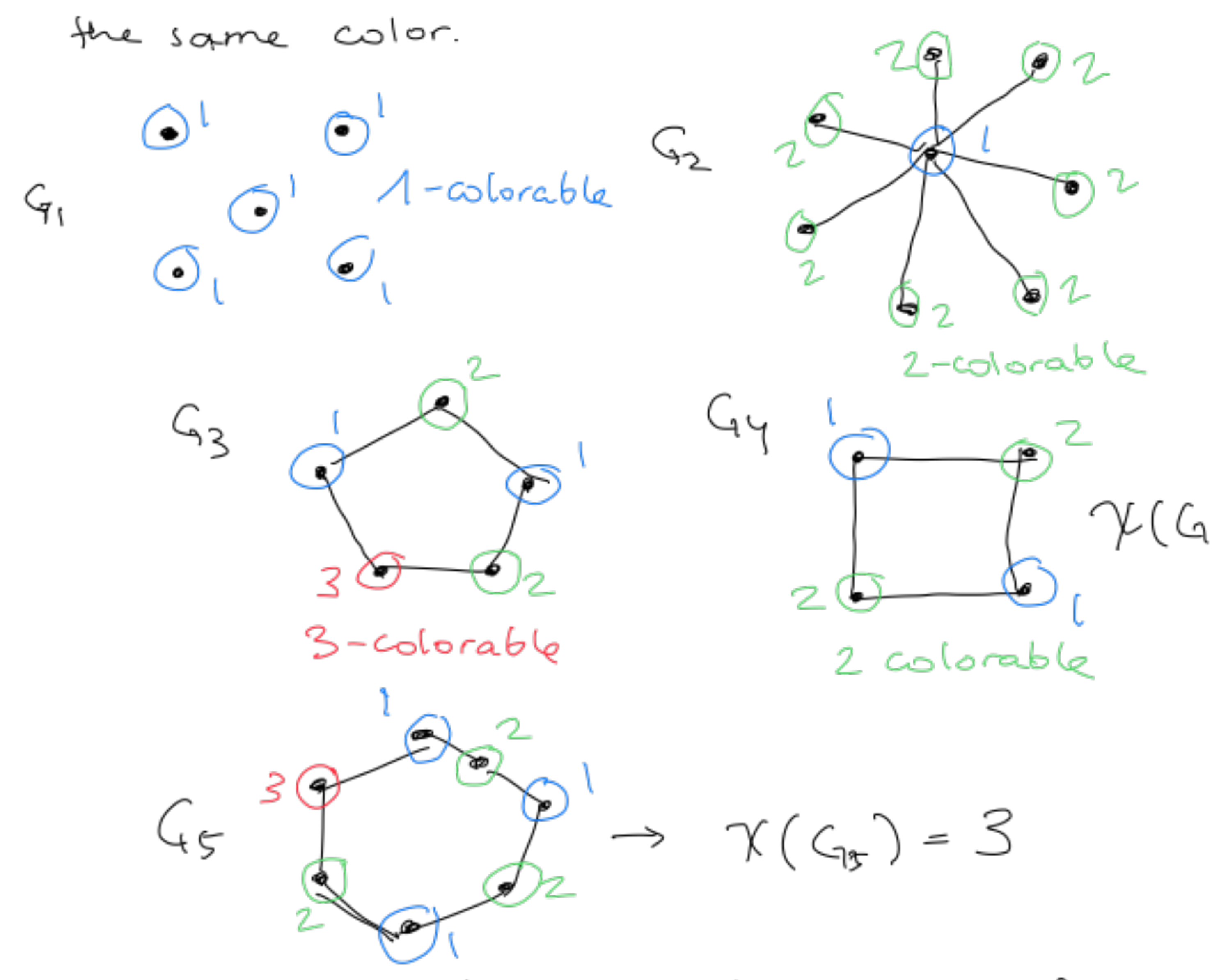
Graph coloring problem: Given a graph $G=(V,E)$ and $K$ colors $c_1, c_2, \ldots c_k$ assign a color to each node so that adjacent nodes have different colors.

k-coloring is a map $f: V \to \{1,2,\ldots,k\}$ s.t. $(a,b) \in E \rightarrow f(a) \neq f(b)$




→ 3 colorable
→ not 2 colorable
(also 4-colorable, 5-colorable, ...)

Definition: A graph $G=(V,E)$ is k-colorable if each node can be assigned one of k colors and no two adjacent nodes have the same color.



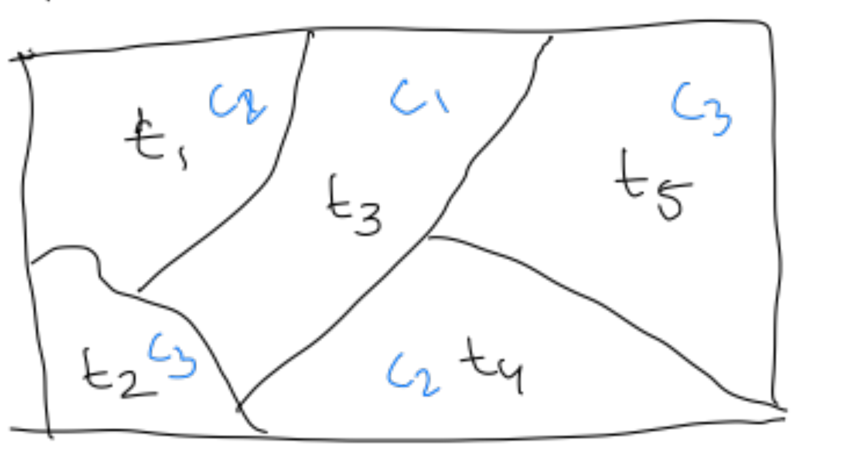

$\rightarrow \chi(G_5) = 3$

Definition: The chromatic number of G is the min # of colors needed so that no adjacent nodes have the same color

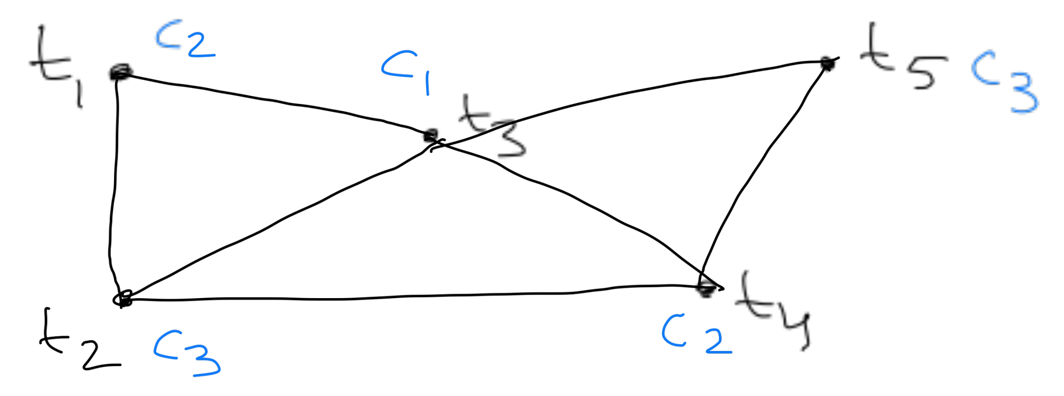
$\chi(G)$

Some other applications

- coloring maps




nodes = territory
edges = border $(t_i, t_j) \in E$ if $t_i$ and $t_j$ share a border




1. Show that a graph that is $n$-colorable is also $n'$-colorable for all $n' \geq n$.

Assume a graph is $n$-colorable (hypothesis)
$\Rightarrow$ there exists a valid coloring (no adjacent nodes share a color)
Now consider a set of $n' > n$ colors.
Use the same assignment of $n$ colors above (leaving $n'-n$ colors unassigned)
$\Rightarrow$ The graph is $n'$-colorable too b/c each node is assigned one of $n'$ colors and no adjacent nodes share a color.

Key: not every color has to be assigned.

3. Show that a graph w/ at least one edge needs at least 2 colors.

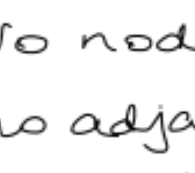
Let the edge be $(x,y) \in E$.
$x$ and $y$ cannot have the same color because they are adjacent, so we need at least 2 colors.

4. Show that any graph with $n$ nodes is $n$-colorable

Assign a different color to each node.
Thus, each node can be assigned one of $n$ colors. No two nodes have the same color so no 2 adjacent nodes have the same color
$\Rightarrow$ The graph is $n$-colorable

5. Find min value of $k$ (w/ proof)

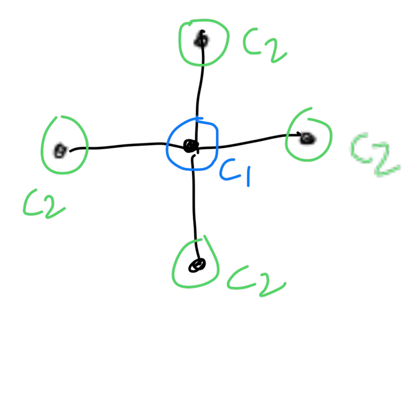
a. Empty graph w/ $n$ nodes



Assign the same color to each node.
No nodes are adjacent to any other, so no adjacent nodes are assigned the same color. $\rightarrow$ 1-colorable

Every graph needs $\geq 1$ node & that node needs a color.

b. Star graph w/ nodes, $n \geq 2$




Assign 1 color to the hub ($c_1$) and color all other nodes w/ $c_2$

All pairs of adjacent nodes are between the hub and an outside vertex, so no adjacent nodes share a color

$\Rightarrow$ 2-colorable

As the graph has at least one edge, we need at least 2 colors from #3

so $k=2$