Lectures 19: Directed Graphs
Wednesday, October 30, 2024 11:29 PM

Announcements:

1. Homework #8 Released Today
2. My office hours after class until 6:45 PM
3. Exam: 11/21, Kasia to teach next 2 classes
4. Halloween Costume Contest

Who is participating?

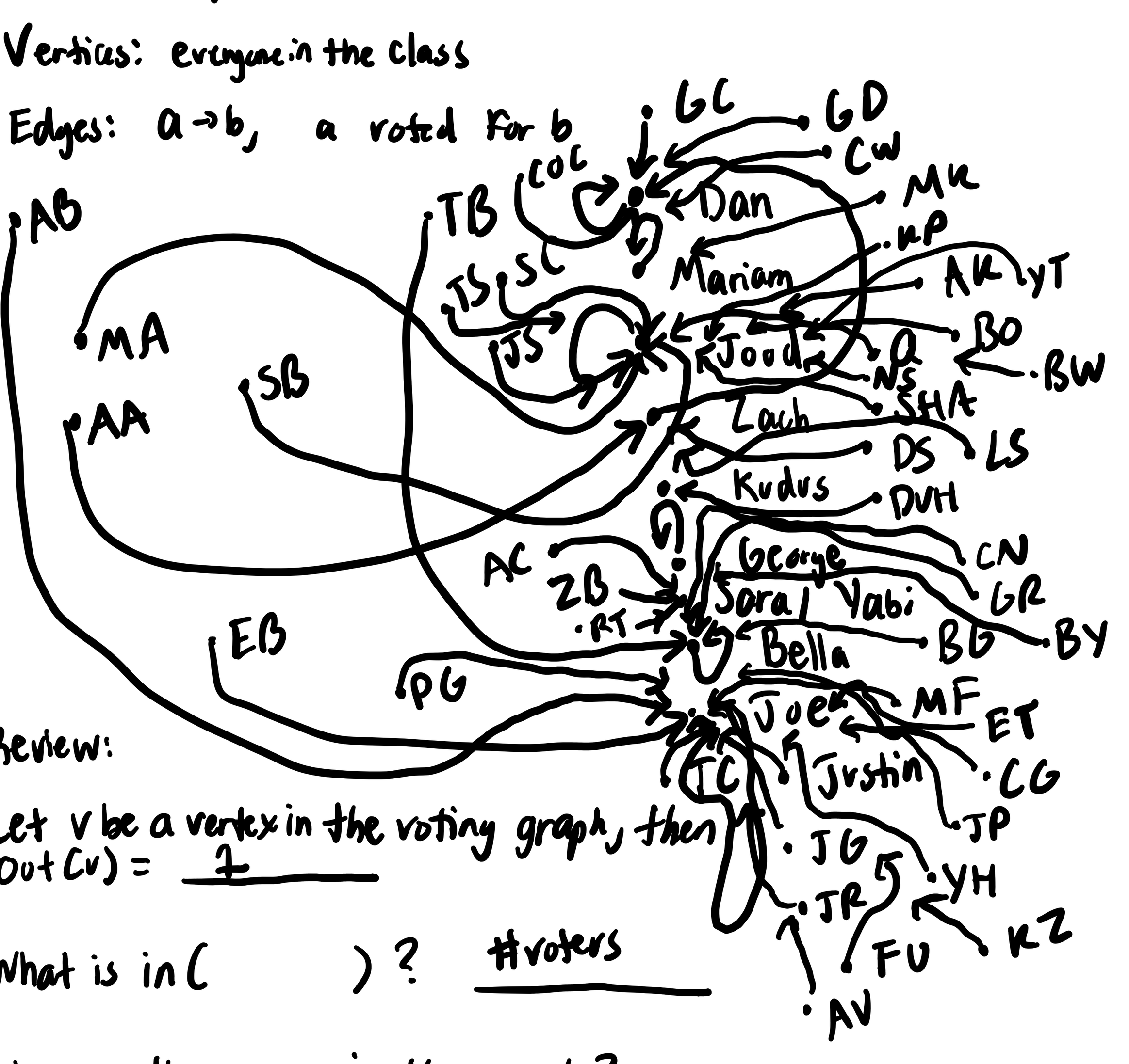
1. Dan
2. Mariam
3. Jood
4. Zach
5. Kudus
6. George
7. Bella
8. Sara / Yabi

Let's construct the voting graph

Directed Graph:

Vertices: everyone in the class

Edges: $a \to b$, a voted for b



Review:

Let v be a vertex in the voting graph, then Out(v) = 1

What is in( )? #voters

Who are the sources in the graph?

No vote or no costume

Who are the sinks in the graph?

No sinks

Lemma: Sum of indegree of all nodes = Sum of outdegree of all nodes

What are the strongly connected components of the voting graph? Draw G^{SCC}

There at least num # of sources SCCs in this graph

Lemma: G^{SCC} has no directed cycles.

Is the voting graph a DAG?

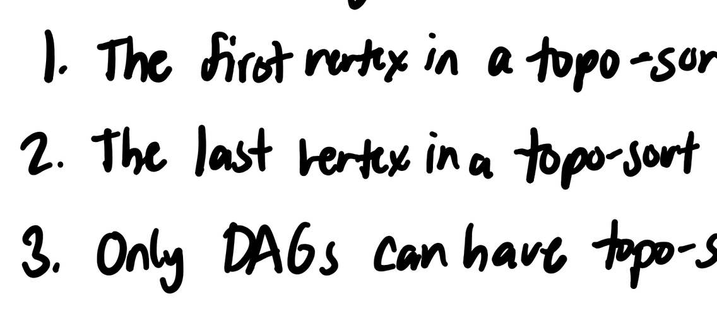
There are cases where the voting graph is NOT a DAG

i.e when 2 people vote for each other

Let's make a topological sort of the voting graph

σ: AB, DVH, BG

AB → JS
BG → BD
DVH → K?



- A topological sort of a DAG, is a sequence σ, in which every vertex appears once, such that for any edge $a \to b$ in the graph, the vertex a appears before b in the sequence

Prove the following:

1. The first vertex in a topo-sort is a source
2. The last vertex in a topo-sort is a sink
3. Only DAGs can have topo-sorts

1) Assume f.p.o.c that first vertex v is Not a source:

This means that there is some a such that

$a \to v$

Contradiction! a cannot come after v in the topo-sort and thus must be the first vertex

2) Assume f.p.o.c that last vertex, b, is not a sink

There must be a vertex, v such that $b \to v$

Thus, v cannot come before b in the topo-sort, and v must be the last vertex, contradiction!

3) Assume f.p.o.c that directed graphs with cycles can have topo-sorts.

Consider the cycle, C,

Consider two vertices in that cycle V_a and V_b

Toposort ---- V_a ---- V_b -----

V_{a-2} V_{a-1} V_{b+1} -- V_{b+2}

V_{a-i} and V_{b+i} will eventually be the same vertex, contradiction

Consider the following graph. What is the "algorithm" for finding a topological sort of a graph?

Step 1: Find source(s) (a, c, e, g) (S, f) (d) and add to ordering

c, e, g, a, f, b, d

Step 2: Delete source(s)

Step 3: Repeat Step 1 until no vertices left

Prove every DAG has at least 1 topo sort

BC: $n=1$ v ✓

IH: Our algorithm works on a DAG with K vertices

It returns a valid topo-sort

IS: We have a DAG with k+1 vertices

Remove a source vertex, v, to create $G' = G / \{v\}$ → k vertices

By IH, G' has a valid topo-sort and our algorithm will find it $= \sigma$

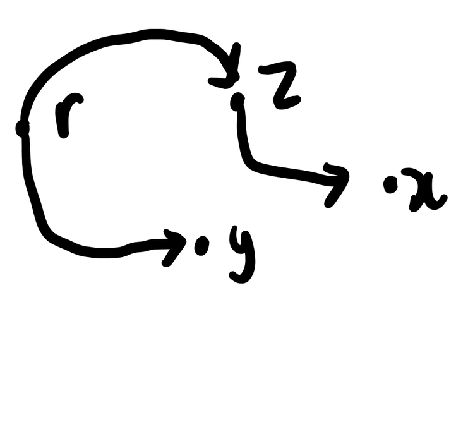
$v + \sigma$ ✓

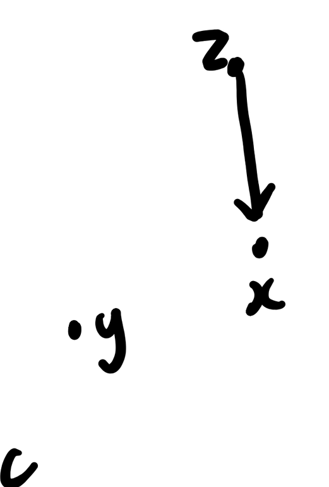
Consider the directed graph (DAG), with nodes:

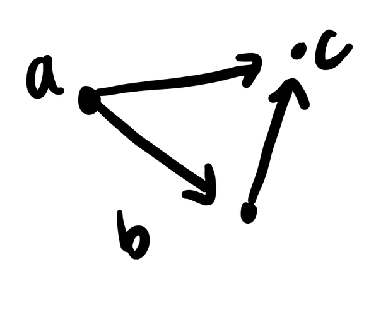
$V = \{r, x, y, z\}$. Here are all the topological sorts of G:

rzxy ryzx rzyx

What are the edges in E?







a, b, c