Lecture 18: Directed Graphs

Monday, October 28, 2024 9:17 PM

Announcements:

1. Homework #7 due Today
2. Homework #8 Released Soon, due 11/7
3. 10/31: Halloween Class
4. My office hours after class

Review

- The Pigeonhole Principle

  If $k+1$ or more objects are distributed among k bins, then there is at least one bin with 2 or more objects

- Generalized Pigeonhole Principle:

  If n objects are placed into k boxes, then there is at least 1 box that has $\lceil n/k \rceil$ objects

1. There are n pairs of socks. How many socks must you pick without looking to ensure that you have at least one matching pair.

- A hamiltonian cycle in a graph G is a cycle in which every vertex of G appears exactly once

  Lemma: When $n \geq 3$ and $\delta(G) \geq n/2$, G has a hamiltonian cycle.

  Lemma: $\delta(G) \geq 2 \Rightarrow G$ has a cycle

- An Eulerian Circuit is a closed walk in which each edge appears exactly once

- Lemma: A graph is Eulerian iff every vertex in G has even degree

$\Rightarrow$ If G is Eulerian then every vertex in G has even degree

Let C denote the Eulerian circuit in G. Each passage of C through a vertex uses two incident edges and the first edge is paired with the last at the first vertex. Hence every vertex has even degree

$\Leftarrow$ If every vertex in G has even degree then G is Eulerian

Induction on m, the # of edges

BC: $m=0$, $\bullet v$, The graph is Eulerian

$1 \leq j < k$

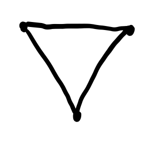
IH: Assume for some k, that a graph with every vertex having even degree, the graph is Eulerian

IS: W.T.S that a graph, G, with $k+1$ edges, and every vertex has even degree, is Eulerian

$\delta(G) \geq 2 \Rightarrow G$ has a cycle

G has a cycle, C

Let's remove the edges in the cycle, C

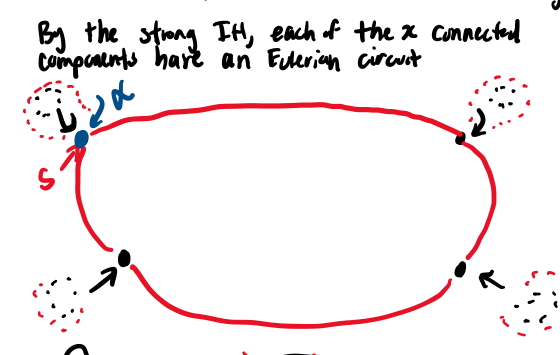


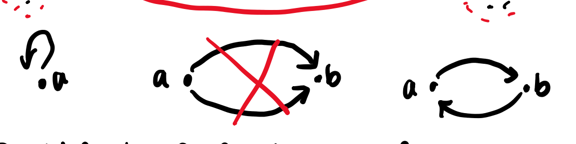
When I remove the edges in the cycle, the degree of each vertex in the cycle goes down by 2 and the degree of all other vertices goes down by 0

My new graph G', could be disconnected into x connected components.

Each of these x connected components have $< k$ edges and each of the vertices in them have even degree

By the strong IH, each of the x connected components have an Eulerian circuit





Directed Graphs: $G=(V,E)$ consists of a non-empty set V of vertices and a set $E \subseteq V \times V$ of directed edges

Self-loops and anti-parallel edges are ok, parallel edges are not

The outdegree of a vertex v is the number of outgoing edges from v, denoted $out(v)$

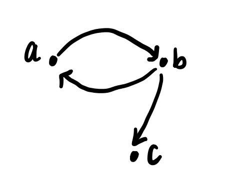
The indegree of a vertex v is the number of incoming edges from v, denoted $in(v)$

The total degree of a vertex is $out(v) + in(v)$

Vertices with indegree 0 is a source

Vertices with outdegree 0 is a sink

Example:



$out(b) = 2$
$in(a) = 1$
Any Sources? No
Any sinks? c

Prove the following:

The sum of the outdegrees of all vertices

=

The sum of the indegrees of all vertices

Each edge $(a \to b)$, adds 1 to $in(b)$ and 1 to $out(a)$, adding 1 to the outdegree of all nodes and 1 to indegree of all nodes

A directed walk is a non-empty sequence:

$\underbrace{v_1, v_2, v_3, \dots v_k}_{\text{length } k}$ such that $v_1 \to v_2 \to v_3, \dots \to v_k$

A directed path is a walk with no repeated vertices

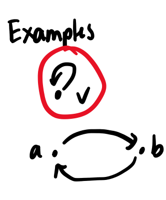
Examples

1) $\bullet v$ Both

2) $\circlearrowleft_{v}$ Walk $v \to v$

- A directed cycle is a closed walk: $\underbrace{v_0 \to \dots \to v_k \to v_0}_{\text{length} = k+1}$ with $v_0, v_1 \dots v_k$ all distinct

  Examples

  - $\circlearrowleft_{v}$ $v \to v$
  -  $a \to b \to a$

- A vertex v is reachable from a vertex u when there is a walk/path from u to v

  $u \overset{+}{\Rightarrow} v$

Prove that reachability is transitive,

$a \overset{+}{\Rightarrow} b$, $b \overset{+}{\Rightarrow} c$, $\Rightarrow a \overset{+}{\Rightarrow} c$ ✓

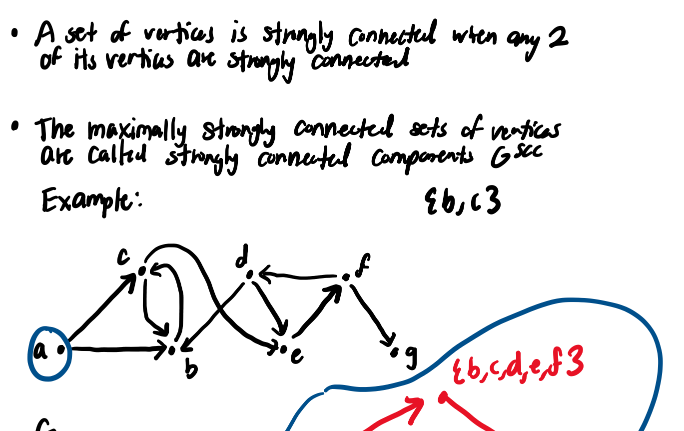
- Two vertices of a directed graph, a and b are strongly connected if $a \overset{+}{\Rightarrow} b$ and $b \overset{+}{\Rightarrow} a$.

  $a \overset{+}{\Leftrightarrow} b$

- A set of vertices is strongly connected when any 2 of its vertices are strongly connected

- The maximally strongly connected sets of vertices are called strongly connected components G^{SCC}

  Example: $\{b, c\}$



Gscc:

1. $\bullet a$ $\in^{SCC}$

2. $\bullet b$ What vertices are reachable from b? c, e, f, d, g

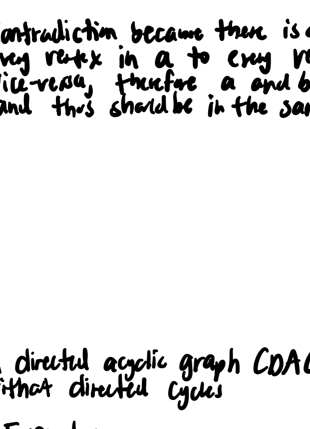
- c What vertices are reachable from c? b, e, f, d, g
- d b, c, e, f, g
- e b, c, d, f, g
- f b, c, d, e, g
- g [ ]

- Prove that G^{SCC} has no directed cycles

  Assume f.p.o.c that G^{SCC} does have a directed cycle

  Let's examine 2 of the vertices in that cycle:

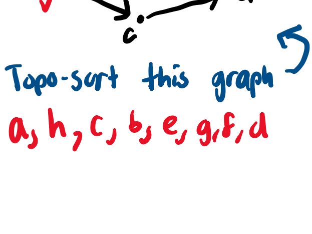
  a and b



Contradiction because there is a path from every vertex in a to every vertex in b and vice-versa, therefore a and b are not maximal and thus should be in the same SCC.

- A directed acyclic graph (DAG) is a directed graph without directed cycles

  Example:



Topo-sort this graph

a, h, c, b, e, g, f, d

- A topological sort of a DAG, is a sequence σ, in which every vertex appears once, such that for any edge $a \to b$ in the graph, the vertex a appears before b in the sequence

  Prove the following:

  1. The first vertex in a topo-sort is a source
  2. The last vertex in a topo-sort is a sink
  3. Only DAGs can have topo-sorts

1)