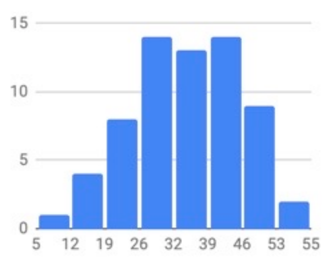
Lecture #14: Graph Theory

Sunday, October 13, 2024 11:30 AM

Announcements

1. Midterm #1 Grades Released (get them back in lab)

Median: 35/55 Max: 54.5/55



2. Homework #6: Released Today (due next Tuesday)
3. My office hours are after class today

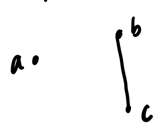
Graphs

A graph is defined as $G(V,E)$

V is a non-empty set of vertices/nodes
"Vertices"

E is a possibly-empty set of edges consisting of 2-element subsets of V
"Edges"

Example:



$V = \{a, b, c\}$

$E = \{(b,c)\}$

In this example, $(b,c) \in E$, connects vertices b and c

Other definitions:

In this example, b and c are adjacent

Two nodes u and v are adjacent if $(u,v) \in E$

Nodes adjacent to u are called neighbors of u: $N(u)$

The number of neighbors of a vertex, u, is called the degree of u: $deg(u)$

$\delta(G)$: the minimum degree of G = $\min_{v \in V} deg(v)$

$\Delta(G)$ = the maximum degree of G = $\max_{v \in V} deg(v)$

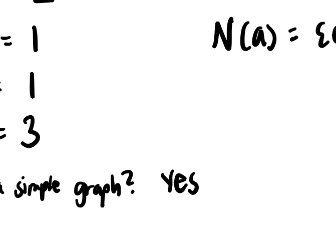
An edge that connects a node to itself is a self loop

Multiple edges between the same vertices are parallel edges

Graphs without loops are called simple graphs

In this class, we will use simple graphs, unless otherwise noted

Example: $G(V,E)$



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$\sum_v deg(v) = deg(a) = 1 + deg(b) = 3 + deg(c) = 1 + deg(d) + 1 + deg(e) + 2 = 8$

$V = \{a, b, c, d, e\}$

$E = \{(a,e), (e,b), (b,c), (b,d)\}$

$N(e) = \{a, b\}$

$deg(e) = 2$

$deg(a) = 1$

$N(a) = \{e\}$

$2 \cdot |E| = 2 \cdot 4 = 8$

$\delta(G) = 1$

$\Delta(G) = 3$

Is this a simple graph? Yes

1. Prove that the sum of degrees of all nodes is equal to 2 times the # of edges.

In other terms prove: $\sum_{v \in V} deg(v) = 2|E|$

Let take an arbitrary edge: e_i

e_i connects a and b

Thus it is adding 1 to $deg(a)$ and 1 to $deg(B)$, so each edge adds 2 to $\sum_v deg(v)$

2. In any graph, there are an even number of vertices with odd degree

$\sum_{v \in V} deg(v)$ is an even number

$$\underbrace{\sum_{v \in V} deg(v)}_{\text{even}} = \underbrace{\sum_{v \in V_{odd}} deg(v)}_{\text{even}} + \underbrace{\sum_{v \in V_{even}} deg(v)}_{\text{even}}$$

$\sum_{v \in V_{odd}} deg(v)$ is even

Thus, there must be an even amount of vertices with odd degree ($|V_{odd}|$ is even)

More definitions:

A walk in G is a non-empty sequence of the form: $v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ $\forall i < k$

The length of a walk: # edges in the walk

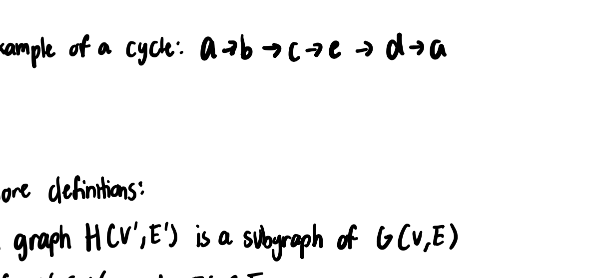
= # vertices in the walk − 1

If all the vertices in a walk are distinct, that walk is called a path, written in the following form:

$v_0, v_1, v_2, \dots v_k$ $\forall i$, $0 \le i < n$, $(v_i, v_{i+1}) \in E$

If $v_0 = v_k$, then the path is closed and that path is referred to as a cycle.

Example:



Example of a walk: $a \to b \to c \to e \to f \to e \to c \to b$

Example of a path: $a \to b \to c \to e \to f$

Example of a cycle: $a \to b \to c \to e \to d \to a$

More definitions:

A graph $H(V',E')$ is a subgraph of $G(V,E)$ if $V' \subseteq V$ and $E' \subseteq E$

A graph G is connected if for all pairs of vertices there is a path between those vertices

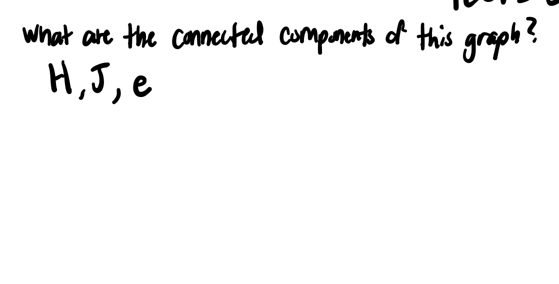
A graph $H(V',E')$ is a connected component ("island") of G if

a) H is a subgraph of G AND

b) H is connected AND

c) H is maximal, if is not contained in any other connected component of G

Example:



Is this graph connected?

No!

7 vertices

4 edges

$|CC| \ge 7-4$

$|CC| \ge 3$

What are the connected components of this graph?

H, J, e

3. Prove that every graph with n vertices and m edges has at least $n-m$ connected components.

In the other words $|CC| \ge |V| - |E|$

Induction:

Induction on m (the # of edges in the graph)

BC: $m=0$ $|CC| \ge |V|$ ← Show this

$|CC| = |V|$ ✓

IH: When the graph has k edges: $|CC| \ge n - k$ ($n = |V|$)

IS: W.T.S that $|CC| \ge n-(k+1)$ when the graph has $k+1$ edges

W.T.S that $|CC| \ge n-k-1$

G is a graph with $k+1$ edges and n vertices

Let's remove an arbitrary edge, $e = \{a, b\}$

G' is G without e, G' is a subgraph of G

In G', $|CC| \ge n-k$

Let's add $e = \{a,b\}$ back to G' to get G

Case 1: a and b are in the same connected component in G'

$|CC| \ge n-k \ge n-k-1$

Case 2: a and b were in different connected components in G'

the number of CC in G' goes down by 1

$|CC| \ge n-k$ $|CC| \ge n-k-1$

4. Prove that every connected graph with n vertices has at least $n-1$ edges.

$|CC| \ge |V| - |E|$

$|E| \ge |V| - |CC|$

$|E| \ge n - 1$