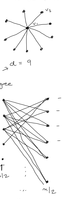
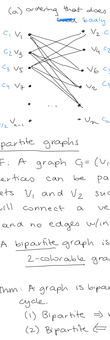
NOVEMBER 14, 2024

Announcements
- Exam 2 in one week — Location TBD (sorry!)
- If taking 0 at 2351, please submit request ASAP

Last time:
- Introduced a basic greedy algorithm for graph coloring
  1. Order the vertices $v_1, v_2, \ldots, v_n$
  2. Order the colors $c_1, c_2, \ldots$
  3. Loop through vertices, assigning lowest legal color
- Proved: a graph with maximum degree d is $(d+1)$-colorable

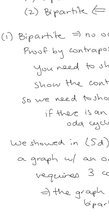




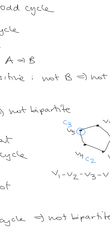
$d = \frac{n}{2} - 1$

at most we need $\frac{n}{2}$ colors

(a) ordering that does CAUSE bad alg:



(b) better ordering



Bipartite graphs

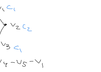
Def: A graph $G = (V, E)$ is bipartite if all of its vertices can be partitioned into two distinct sets V_1 and V_2 such that every edge in G will connect a vertex in V_1 to a vertex in V_2 (and no edges w/in V_1 or V_2).

A bipartite graph is another word for a 2-colorable graph.

Thm: A graph is bipartite iff it contains no odd cycle.
(1) Bipartite $\Rightarrow$ no odd cycle
(2) Bipartite $\Leftarrow$ no odd cycle

(1) Bipartite $\Rightarrow$ no odd cycle
Proof by contrapositive
You need to show $A \Rightarrow B$
Show the contrapositive: not $B \Rightarrow$ not A
So we need to show:
if there is an odd cycle $\Rightarrow$ not bipartite

We showed in (5d) that a graph w/ an odd cycle requires 3 colors
$\Rightarrow$ the graph is not bipartite.



$v_1 - v_2 - v_3 - v_4 - v_5 - v_1$

We've shown: odd cycle $\Rightarrow$ not bipartite
which is equivalent to bipartite $\Rightarrow$ no odd cycle.

(2) no odd cycle $\Rightarrow$ bipartite
We'll show that not bipartite $\Rightarrow \exists$ odd cycle
We'll provide an algorithm to color any graph with 2 colors and show that if it fails, then there is an odd cycle in the graph.

Algorithm: Color the graph as follows:
(1) Begin with any node. Color it c_1.
(2) Color all adjacent nodes c_2
(3) Color all of their adjacent nodes c_1
(4) continue, alternating colors

This algorithm fails when a node is already colored 1 color and you try to color it another.

Note that the color of a node depends on the parity of path length from first node
(even = c_1, odd = c_2)

When a single node is colored with 2 different colors at two different times using this algorithm, it means there's one path from the start to this node of even length and another path from the start to this node of odd length.

$\Rightarrow$ Hence, $\exists$ a cycle of odd length
(odd + even = odd)

We've shown:
not bipartite $\Rightarrow$ this alg will fail $\Rightarrow \exists$ odd cycle
$\Rightarrow$ No odd cycle $\Rightarrow$ bipartite
this is b/c
If a graph is not 2-colorable then any alg (including this one) that tries to color it using only 2 colors will fail

Taken together,
A graph is bipartite iff it contains no odd cycle

- Prove: In an undirected bipartite graph, the total degree of each vertex partition (i.e., V_1 and V_2) is always equal.

total degree = the sum of the degrees of each vertex in the partition (the # of edges that connect to any vertex in the partition)

When an edge connects two vertices, it contributes one to the degree of each vertex. Since every edge in a bipartite graph must connect a vertex in V_1 to a vertex in V_2, every edge adds 1 to the total degree of each partition set. (and no other edges exist)
$\Rightarrow$ total degrees of the partition sets will be equal

(3) If G is bipartite, then so is every subgraph of G.

$\Rightarrow$ you can't construct an odd cycle just by getting rid of nodes/edges
$\Rightarrow$ we know that any subgraph of an n-colorable graph is also n-colorable
$\downarrow$ 2 (bipartite) $\qquad$ $\downarrow$ 2 (bipartite)

There are many ways to prove this

One approach: If G is bipartite, then it doesn't have an odd cycle. Removing edges or vertices from a graph cannot create an odd cycle. If it didn't already exist in G. So, every subgraph of a bipartite graph is also bipartite. (we saw this above)

Second approach: On the homework, we proved that any subgraph of an n-colorable graph is also n-colorable. Because a "bipartite" graph is another word for a "2-colorable" graph, this means that any subgraph of a 2-colorable/bipartite graph is also 2-colorable/bipartite.

Third approach: If a graph G is bipartite then there exists a bipartition V_1 and V_2 of its vertices, such that every edge in G connects a vertex from V_1 to a vertex in V_2. Consider a subgraph $H = (V', E')$ of G. We can partition the vertices V' into 2 sets V_1' and V_2' such that any vertex in H that was in V_1 in G is now in V_1' and any vertex that was in V_2 in G is now in V_2'. Because every edge that is in H was also in G, this means that every edge in H connects a vertex from V_1' to a vertex in V_2'. Therefore, H is bipartite and any subgraph of a bipartite graph is also bipartite.