Lecture #6: Induction

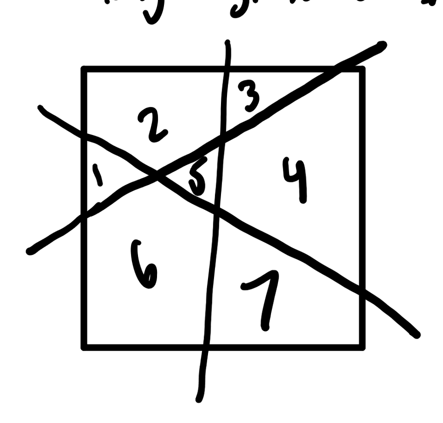
Saturday, September 7, 2024 10:51 PM

Announcements:

1. Homework #2 due Today
2. Homework #3: Induction released today (due next Tuesday): Start early!
3. My office hours are today after Class (here and then outside)
4. Homework #1 Grades Released Today

Geometry?

Let's say I have an infinite plane. Let's say I draw $n \geq 1$ lines such that no 2 lines are parallel, and no three lines pass through a common point, how many regions do I create?



$n=1$ 2 regions

$n=2$ 4 regions

$n=3$ 7 regions

$\frac{n^2+n+2}{2}$ regions

Let's prove it: n lines seperate the plane into $\frac{n^2+n+2}{2}$ regions

BC: $n=1$ $\frac{1+1+2}{2} = \boxed{2}$ ✓

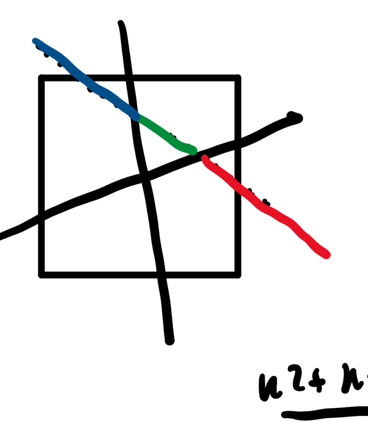
IH: Let's assume that for some K, k lines seperate the plane into $\frac{K^2+K+2}{2}$ regions

IS: W.T.S that $K+1$ lines seperate the plane into $\frac{(k+1)^2+(k+1)+2}{2}$ regions.

I start with a plane with $k+1$ lines

Remove an arbitrary line in the plane

This new plane has $\frac{k^2+k+2}{2}$ regions by the IH



Everytime I draw a new line that line gets split into $\underline{k+1}$ line segments and thus $k+1$ new regions

$$\frac{k^2+k+2}{2} + k+1$$

$$\frac{k^2+k+2}{2} + \frac{2k+2}{2}$$

$$\frac{k^2+3k+4}{2}$$

$$\frac{k^2+2k+1+k+3}{2}$$

Strong Induction

Turns out that $\forall n \in \mathbb{N}$ if $P(0)$ and $P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}\ P(n)$

$=$ if $P(0)$ and $P(1)$ and $P(2)\ldots$ and $P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}\ P(n)$

We can make a stronger assumption then we have before. Instead of assuming that $P(k)$ is true, we can assume that $P(j)$ is true for $1 < j \leq k$, and then as before if we can show that $P(k+1)$ is true using the fact that $P(j)$ is true, we are good.

Prove the unique factorization theorem/ fundamental theorem of arithmetic.

What was that theorem?

For any number $n > 1$, n is either prime or can be expressed as a multiple of prime factors

BC: $n=2$ and n is prime ✓

IH: Assume for some j where $1 < j \leq k$ that j is a prime number or can be expressed as a multiple / product of prime factors

IS: W.T.S that $K+1$ is a prime or can be expressed as a product of primes

Case 1: $k+1$ is prime: ✓

Case 2: $k+1$ is Composite

$k+1 = r \cdot s$ where r and $s \neq 1$

$2 \leq r \leq k$
$2 \leq s \leq k$

r is either prime or a product of primes by IH
s is either prime or a product of primes by IH

Thus, since $K+1 = r \cdot s$, $k+1$ is a product of primes.

✓

Prove using induction that for any positive integer n, if $x_1, x_2, \ldots x_n$ are n distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is $n-1$.

$$x_1 \cdot x_2 \cdot x_3 \ldots \cdot x_n = y$$

BC: $n=1$, x_1, 0 multiplications required

$n-1 = 0$ ✓

IH: Assume for some k, that the number of multiplications required for K numbers is $k-1$

Assume for some j, that the number of multiplications required for j numbers is $j-1$ where $1 \leq j \leq k$

IS:

W.T.S that for $k+1$ numbers, we always need K multiplications

I start with $k+1$ numbers:

$x_1 \cdot x_2 \cdot x_3 \ldots \cdot x_k \cdot x_{k+1}$

$$\underbrace{x_1 \cdot x_2 \ldots \cdot x_i}_{1 \leq \alpha \leq K} \qquad \underbrace{x_{i+1} \cdot x_{i+2} \ldots \cdot x_{k+1}}_{1 \leq \beta \leq K}$$

$\alpha - 1$ multiplications $\qquad$ $\beta - 1$ multiplications

by IH

$$\alpha - 1 + \beta - 1 + 1 = \alpha + \beta - 1$$

$$k+1-1 = \boxed{k}$$

Let's play a game:

Some number of marbles are placed in front of you and another player. The two players take turns, removing 1, 2, or 3 marbles from the pot. The player to remove the last marble loses.

Prove that the first player has a winning strategy iff the number of marbles (n) is not $4k+1$, for any $K \in \mathbb{N}$

Prove that the two forms of induction (weak and strong) are equivalent. Any statement that admits a strong induction proof can be solved with weak induction and vice-versa.