SEPTEMBER 26, 2024

Announcements
- Homework 5 due October 1
- Midterm #1 is October 8
  - please arrive ON TIME
  - if you need any accommodations or need to take a make-up please e-mail us TODAY
  - review session in lab next week
  - Questions?

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① $2^{n+1} \in O(2^n)$??

$2^{n+1} \leq c \cdot 2^n$

$\boxed{2^{n+1} = 2 \cdot 2^n}$

We need to show that

$\exists b, B > 0$ s.t. $0 \leq |f(n)| \leq g(n) \cdot B \quad \forall n \geq b$

$0 \leq 2^{n+1} \leq 2^n \cdot B$

E.g. $b = 1$; $B = 2$

$0 \leq 2 \cdot 2^n \leq 2 \cdot 2^n \quad \forall n \geq 1$

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② Is $2^{2n} \in O(2^n)$?? Assume it is.

No! What would it mean for it to be $O(2^n)$.

$\exists b, B$ s.t. $2^{2n} \leq B \cdot 2^n \quad \forall n \geq b \qquad (*)$

$2^{2n} = 2^n \cdot 2^n \leq B \cdot 2^n$

Divide by 2^n

$\Rightarrow 2^n \leq B \qquad \forall n \geq b$

We can express $(*)$ as a conditional statement

$\exists b, B$ s.t $n \geq b \Rightarrow 2^{2n} \leq B \cdot 2^n$

$\Rightarrow \log_2(2^{2n}) \leq \log(B \cdot 2^n)$

$\Rightarrow 2n \leq \log_2 B + n$

$\Rightarrow n \leq \log_2 B$

Consider $n' > \max(b, \log_2 B)$.

This is : $n' \geq b$ but also $n' \geq \log_2 B$

OCTOBER 1, 2024

(2) from worksheet

$\underbrace{f \in O(g)} \Rightarrow kf(n) \in O(g(n))$ for $k \in \mathbb{R}$, $k \neq 0$

$f \in O(g)$

$\Rightarrow \exists b, B > 0$ st $0 \leq |f(n)| \leq B \cdot g(n) \quad \forall n \geq b$

We want to show:

$\exists b', B' > 0$ s.t $0 \leq |k \cdot f(n)| \leq B' g(n) \quad \forall n \geq b'$

$\exists b, B > 0$ s.t. $0 \leq |k| \cdot |f(n)| \leq |k| \cdot B \cdot g(n) \quad \forall n \geq b$

Because $|k| > 0$, the direction of the inequality is preserved.

Define $B' = |k| \cdot B$

$B' > 0$

So there exists a $B' = |k| \cdot B > 0$ and $b' = b > 0$

s.t $0 \leq |k \cdot f(n)| \leq B' g(n) \quad \forall n \geq b'$

By the def of Big-Oh, $kf \in O(g)$

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(3) Is big-Oh reflexive?

$f(n) \in O(f(n))$

$\exists b, B > 0$ s.t. $0 \leq |f(n)| \leq B \cdot \underbrace{g(n)}_{= f(n)} \quad \forall n \geq b$

$B = 1$; $b = 1$

$f(n) = 1 \times f(n) \quad \forall n \geq 1$

$|f(n)| \leq 1 \times f(n) \quad \forall n \geq 1$

(c) Transitive?

$\underbrace{f(n) \in O(g(n)) \wedge g(n) \in O(h(n))} \Rightarrow f(n) \in O(h(n))$

$\Rightarrow \exists b, B > 0$ s.t. $0 \leq \underbrace{f(n) \leq B \cdot g(n)} \quad \forall n \geq b$

and [Big-Oh def]

$\exists b', B' > 0$ s.t. $0 \leq g(n) \leq B' \cdot h(n) \quad \forall n \geq b'$

We want to show:

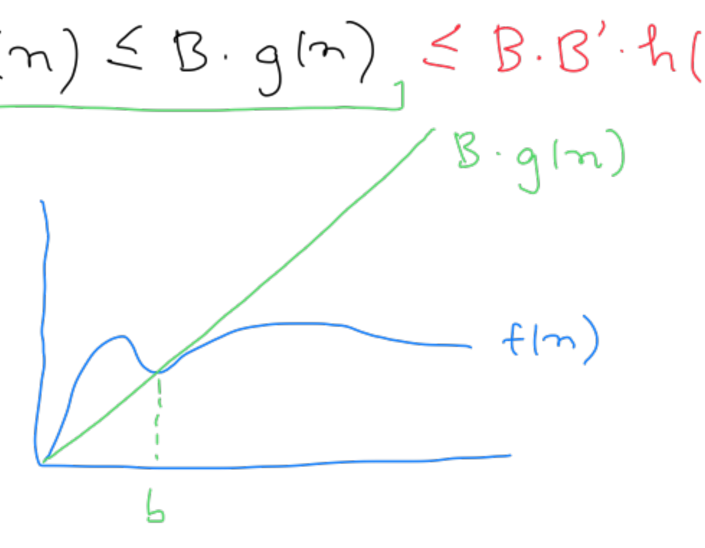
$\exists b'', B'' > 0$ s.t. $0 \leq f(n) \leq B'' \cdot h(n) \quad \forall n \geq b''$

(or that $f \in O(h)$)

$\Rightarrow \exists b', B' > 0$ s.t $0 \leq \underbrace{B \cdot g(n) \leq B \cdot B' \cdot h(n)} \quad \forall n \geq b'$

$\Rightarrow \quad 0 \leq \underbrace{f(n) \leq B \cdot g(n)} \leq B \cdot B' \cdot h(n) \quad \forall n \geq \max(b, b')$

Because this is true when n is bigger than both b and b'



$\Rightarrow B'' = B \cdot B' > 0$ b/c $B > 0$ and $B' > 0$ (by def)

$b'' = \max(b, b') > 0$ b/c $b > 0$ and $b' > 0$ (by def)

$\exists b'', B'' > 0$ s.t. $0 \leq f(n) \leq B'' \cdot h(n) \quad \forall n \geq b''$

$\Rightarrow f \in O(h)$ ✓

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(4) Show that for any $k, d \geq 0$, $n^k \in O(n^d) \Leftrightarrow d \geq k$

(a) First let's show that $n^k \in O(n^d) \Rightarrow d \geq k$ for $k, d \geq 0$

Let's do a proof by contradiction.

Assume $d < k$

$n^k \in O(n^d)$

$\Rightarrow \exists b, B > 0$ s.t. $0 \leq n^k \leq B \cdot n^d \quad \forall n \geq b$

$\Rightarrow \exists b, B > 0$ s.t $0 \leq n^{k-d} \leq B \quad \forall n \geq b$

$\Rightarrow \exists b, B > 0$ s.t. $0 \leq n \leq B^{\frac{1}{k-d}} \quad \forall n \geq b$

(raising everything ^ $\frac{1}{k-d}$)

Consider $n > \max\left(b, B^{\frac{1}{k-d}}\right)$

$n > b$, so the inequality $0 \leq n \leq B^{\frac{1}{k-d}}$ should hold, but $n > B^{\frac{1}{k-d}}$

$\Rightarrow$ contradiction!

(b) Now let's show $\underbrace{d \geq k} \Rightarrow n^k \in O(n^d)$

We'll want to show that:

$\exists b, B > 0$ s.t. $n^k \leq B \cdot n^d \quad \forall n \geq b$

$d \geq k$

$\Rightarrow n^d \geq n^k \quad \forall n \geq 1$

$\Rightarrow n^k \leq 1 \cdot n^d \quad \forall n \geq 1$

so $\exists b, B = 1$ s.t. $n^k \leq B \cdot n^d \quad \forall n \geq b$

$\Rightarrow n^k \in O(n^d)$

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(6) Let $f, g : \mathbb{N} \to \mathbb{R}^+$ Show that $\underbrace{f \in O(g)} \Leftrightarrow \underbrace{g \in \Omega(f)}$

To show that $g \in \Omega(f)$:

$\exists a, A > 0$ s.t. $Af(n) \leq g(n) \quad \forall n \geq a \qquad (1)$

$f \in O(g)$

$\Rightarrow \exists b, B > 0$ s.t $f(n) \leq B \cdot g(n) \quad \forall n \geq b \qquad (2)$

we want to make (2) look like (1)

$\exists b, B > 0$ s.t $\frac{1}{B} f(n) \leq g(n) \quad \forall n \geq b$

$\exists A = \frac{1}{B} > 0$ and $a = b > 0$ s.t.

$Af(n) \leq g(n) \quad \forall n \geq a$