CSci 2312 - Discrete Structures II
In-Class Problems, Asymptotics, Solutions

Definition:

$$f(n) \in O(g(n)) \Leftrightarrow \exists B, b > 0 \; such \; that \; 0 \leq \mid f(n) \mid \leq Bg(n) n \; \geq b$$

is defined for $g(n) \geq 0$.

1. Prove or disprove: $f(n) = O(g(n)) \Rightarrow g(n) = O(f(n))$

   That is, prove:

   $$f(n) = O(g(n)) \Rightarrow g(n) = O(f(n))$$

   or show that

   $$f(n) = O(g(n)) \not\Rightarrow g(n) = O(f(n))$$

   Before we begin this solution, general notes on this type of proof:

   Note that

   $$f(n) = O(g(n)) \Rightarrow g(n) = O(f(n))$$

   is a *conditional* statement. It is a claim that says that if some P is true, then some other statement, Q, is also true. That is, it is of the form

   $$P \Rightarrow Q$$

   where P corresponds to:

   $$f(n) = O(g(n))$$

   and Q to:

   $$g(n) = O(f(n))$$

   Note that the statement claims absolutely nothing about what happens if P is not true.

   - To show that a conditional statement of this form is *true*, you have to show that if P is true, so is Q. Without exception.
   - In order to show that a conditional statement is *false*, you have to show that there is at least one instance when P is true, but Q is not.

   So, for the given statement in our case:

   - To show it is true, you have to show that whenever $f(n) \in O(g(n))$, $g(n) \in O(f(n))$.
   - To show it is false, you need to find just one pair f, g for which $f(n) \in O(g(n))$ is true, but $g \in O(f(n))$ is not.

   Note that, in general, if the statement is false, there will be cases when:

   - $f(n) \in O(g(n))$ but $g(n) \notin O(f(n))$
   - $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$.

   Thus, if you show that the given statement is false, you are *not* showing that

   $$f(n) \in O(g(n)) \Rightarrow g(n) \notin O(f(n))$$

   (in fact, that is not true and there are counterexamples for that statement as well).

Now to the solution.

Solution:

$$f \in O(g) \Rightarrow \exists\, B, b > 0,\ \textit{such that}\ 0 \leq \mid f(n) \mid \leq Bg(n)\ \forall\ n \geq b$$

is the hypothesis. Using it, you need to show that $g \in O(f)$ or:

$$\exists\, B', b' > 0,\ \textit{such that}\ 0 \leq \mid g(n) \mid \leq B'f(n)\ \forall\ n \geq b'$$

You are allowed a multiplier B', and you are allowed to specify a lower limit for the value of n, but, essentially, you have to switch the inequality. So maybe it isn't possible.

In fact, the statement is not true. The way to show that a statement is not true is to find something that satisfies its assumptions and for which the conclusion or claim does not hold, a *counterexample*.

In this case, you can choose a pair of f and g for which the first statement (hypothesis) is true and then show that the second (conclusion) is not true for this particular pair f and g. Note that you are not showing that it would not be true for any pair f and g that satisfied the assumptions. But just that there is some example pair f and g for which it is not true.

This way you will show that the second statement does not always follow from the first one. You are not showing that $f(n) \in O(g(n))$ means that g(n) is always *not* $O(f(n))$. It might be, it might not be. You're just showing that you cannot always say it is.

Thus, because it is a conditional statement, you prove it is false with a counterexample.

You have to choose a counterexample and then show it is has all the characteristics of a counterexample. That is, you need to show that the counterexample satisfies P but not Q. Thus, a counterexample always arises from the set over which P is true.

Consider $f(n) = 1$ a constant, and $g(n) = n$. Then you can see that

$$0 \leq \mid f(n) \mid \leq Bn\, \forall\, n \geq b$$

for $B = 1 > 0$ and $b = c$ for any $c \geq 1 > 0$, we can choose, for example, $c = 1$. Hence we can see that $f(n) \in O(g(n))$ and the hypothesis is satisfied.

Is $g(n) \in O(f(n))$? In this case, to show that $g(n) \in O(f(n))$, we need to show $\exists B', b' > 0$ such that

$$0 \leq \mid g(n) \mid \leq B'f(n) \quad \forall \quad n \ \geq \ b'$$

Suppose the statement is true, and $\exists B', b' > 0$ such that $g(n) \leq B'f(n) \quad \forall \quad n \ \geq \ b'$. That is, $\exists B', b' > 0$ such that $n \leq B'\ \forall\, n \geq b'$. This seems strange **whatever** the value of B' and b'. So, it seems the statement is false. In order to show that a statement of the kind

$$\exists x, y\ \textit{such that}\ R(x, y)\ \textit{is true}$$

is false, we need to show that $\forall\ x,\ y\ R$ is false. So that not even a single set of x, y exist where R is true.

How will we show it? To show it, I have to show that there is a problem for **any/all** values of $B' > 0$ and $b' > 0$. That is, I have to show that, $\forall B', b' > 0$ the statement: $g(n) \leq B'f(n) \quad \forall \quad n \ \geq \ b'$ is false.

So my proof has to work for any $B', b' > 0$. If I don't constrain my B', b' any more, then whatever I prove for a single $B', b' > 0$ will be true for any/all $B', b' > 0$. So, given some $B', b' > 0$, how will I show the statement is false?

Note that this statement in turn is a conditional statement:

$$n \geq b' \Rightarrow g(n) \leq B' f(n)$$

To show that a conditional statement is false, one draws from values that satisfy the hypothesis. We want to show this for any $B', b' > 0$. Further, we have chosen $f(n) = 1$ and $g(n) = n$. We choose a counterexample from the set of n that satisfy the hypothesis, $n \geq b'$.

Choose $n = \lceil B' + b' + 1 \rceil$ (thanks to a student in a previous year for this suggestion). Then $n \geq b'$. Also, $n > B'$ (because $B', b' > 0$). Thus $n \leq B' \; \forall \, n \geq b'$ is not true, because here we have a single value of n, $n = b' + B' + 1$ for which it is not true. Hence we have a counterexample, for all possible values of B' and b' and thus the conditional statement

$$n \geq b' \Rightarrow g(n) \leq B' f(n)$$

is false for all $B', b' > 0$. Hence

$$\nexists \, B', b' > 0, \; such \; that \; 0 \leq \mid g(n) \mid \leq B' f(n) \; \forall \, n \geq b'$$

And the original statement, $g(n) \in O(f(n))$ is shown to be not true.

Thus, we have found a counterexample, $f(n) = 1$ and $g(n) = n$ such that $f(n) \in O(g(n))$ and $g(n) \notin O(f(n))$. Hence the statement $f(n) = O(g(n)) \Rightarrow g(n) = O(f(n))$ is not true.

Summary: The original statement was of the form $P \Rightarrow Q$. To show it was false, we chose a counterexample. We showed that the counterexample satisfied P. We wanted to show that Q was not true for this counterexample. The statement Q was of the form

$$\exists x, y \; such \; that \; R \; is \; true$$

In order to show that it was false, we needed to show that R was false for all the x, y possible. In our case the x, y corresponded to B', b'. Finally, R itself was a conditional statement, so we could show it was false through the use of a counterexample, taking care to let f, g, B' and b' remained fixed, as they are given. We hence provided a counterexample for $R(x, y)$ for all values of x, y and hence showed

$$\nexists x, y \; such \; that \; R(x, y) \; is \; true$$

which is exactly the statement that Q is false.

2. Is $2^{n+1} \in O(2^n)$? What about 2^{2n}?

Solution: To show that $f(n) \in O(g(n))$, we need to find $B, b > 0$ such that $f(n) \leq Bg(n) \quad \forall \, n \geq b$. Let's try to do that.

$$2^{n+1} = 2(2^n) \; \forall \, n$$

Hence, if $b = 1$ and $B = 2$,

$$2^{n+1} \leq B2^n \; \forall \, n \; \geq \; 1$$

Hence we have shown that $2^{n+1} \in O(2^n)$.

On the other hand, 2^{2n} is not $O(2^n)$, that is, $2^{2n} \in O(2^n)$ is false.

The statement $2^{2n} \in O(2^n)$ is defined as the statement:

$$\exists b, B' > 0 \; such \; that \; 2^{2n} \leq B' 2^{2n} \; \forall n \geq b'$$

Again, this is a statement of the form $\exists B', b' > 0$ such that statement R is true. To show it is false, that $\nexists B', b' > 0$ such that R is true, we have to show that, $\forall B', b' > 0$, R is false.

Here the statement R is: $2^{2n} \leq B'(2^n) \; \forall \; n \; \geq b'$.

It may be expressed as a conditional statement:

$$n \geq b' \Rightarrow 2^{2n} \leq B'(2^n)$$

To show it is false, we need a counterexample n which satisfies $n \geq b'$ but not $2^{2n} \leq B'(2^n)$.

Given any B', $2^n > B' \;\; \forall \; n > log_2 B'$, and hence $n > max(log_2 B', b)$ provides a counterexample. It satisfies the hypothesis, as $n \geq b'$. But $n > log_2 B' \Rightarrow 2^n > B'$ which shows the conclusion is false. Thus we have shown that:

$$\forall b, B' > 0, \quad n \geq b' \not\Rightarrow \; 2^{2n} \leq B' 2^{2n}$$

and hence

$$\nexists b, B' > 0 \; such \; that \; 2^{2n} \leq B' 2^{2n} \; \forall n \geq b'$$

and hence $2^{2n} \notin O(2^n)$.

3. Is $2^{2n^2} \in O(2^{n^2})$?

   Solution: Let $f(n) = 2^{2n^2}$. We will show that $f(n) \notin O(2^{n^2})$. Again, the statement $f(n) \in O(2^{n^2})$ is of the form

   $\exists B, b > 0$ such that

$$0 \leq 2^{2n^2} \leq B 2^{n^2} \; \forall \; n \geq b$$

   It is of the form: $\exists B, b > 0$ such that R is true, where R is the above, which can be rewritten as a conditional statement:

$$n \geq b \Rightarrow 0 \leq 2^{2n^2} \leq B 2^{n^2}$$

   To show it is false, we need to provide a counterexample given B, b, f and g:

$$n \geq b \Rightarrow 0 \leq 2^{2n^2} \leq B 2^{n^2} \Rightarrow 0 \leq 2^{n^2} \leq B \; \Rightarrow 0 \leq n^2 \leq log_2 B$$

   But a counterexample is $n > max(b, \sqrt{log_2 B})$, because it satisfies the hypothesis ($n \geq b$) but $n^2 > log_2 B$ and the conclusion is false. Hence the statement is false.

   Hence $f(n) \notin O(2^{n^2})$.

4. Is the following statement true? $f(n) \in O(g(n)) \Rightarrow 2^{f(n)} \in O(2^{g(n)})$ What about the statement: $f(n) \in O(g(n)) \not\Rightarrow 2^{f(n)} \in O(2^{g(n)})$?

   Solution:

   First: There is an easy counterexample if it occurs to you, based on Problem 2, which says: $2^{2n} \notin O(2^n)$.

   Remember, we are not asserting a property that universally holds for f or g. We are simply saying that a particular conditional statement, for some types of f and g that satisfy some property P (in this case, $f \in O(g)$), is not always true (we are not saying it is always false). So we provide a counterexample: an example where the assumptions (left hand side of the statement, $f(n) \in O(g(n))$) are true but the conclusion (right hand side of the statement, $2^{f(n)} \in O(2^{g(n)})$) is not. That is, the counterexample shows that the first half does not imply the second. We choose f and g for this purpose from Problem 2.

   Let $f(n) = 2g(n)$ and $g(n) = n$. Then $\exists B = 2$ and $b = 1$ such that

   $$0 \leq f(n) \leq Bg(n)n \ \geq b$$

   and this shows that $f(n) \in O(g(n))$.

   We showed in problem 2 above that $2^{2n} \notin O(2^n)$, hence this is an example of $f(n) \in O(g(n)) \not\Rightarrow 2^{f(n)} \in O(2^{g(n)})$. And the given claim is not true.

   Second solution approach:

   Suppose that the counterexample did not occur to you. What should you do? Well, first, you don't know whether the conditional statement is true or not. So, in order to understand it, you see how you might try to prove it is true. Then at some stage, hopefully it will be clear to you whether it is true or false.

   Begin with what is given. Use old results or definitions to progress.

   Given: $f(n) \in O(g(n))$

   $$f(n) \in O(g(n)) \Rightarrow \exists B, b > 0 \ such\ that\ f(n) \leq Bg(n) \quad \forall\, n \geq b$$

   Because B is positive and $f(n), g(n)$ are non-negative, we can raise 2 to the powers above and the inequality is preserved:

   $$\Rightarrow 2^{f(n)} \leq 2^{Bg(n)} \,\forall\, n \ \geq b$$

   Note that, in order to show that $2^{f(n)} \in O(2^{g(n)})$, we need to show $\exists\ B', b'$ such that:

   $$2^{f(n)} \leq B' 2^{g(n)} \,\forall\, n \ \geq b'$$

   Note that B, b need not be the same as B', b' respectively, so we cannot use the same symbols.

   Also note that $2^{Bg(n)} = (2^{g(n)})^B$ which looks quite different from $B'2^{g(n)}$. So we are now clearly stuck.

   This is now the time to explore the possibility that maybe the statement is not true. We should explore it using the simplest possible examples. For example, let's try different values for B:

   Suppose $B \leq 1$. Then $f(n) \leq Bg(n) \leq g(n)$ (because $B \leq 1$). And we get:

   $$f(n) \leq g(n) \,\forall\, n \ \geq b \Rightarrow 2^{f(n)} \leq 2^{g(n)} \,\forall\, n \ \geq b$$

$B' = 1$ and $b' = b$ and we see that $2^{f(n)} \in O(2^{g(n)})$. So we could prove the required statement for $B \leq 1$. But the value of B is not in our hands. In fact, we have seen in previous problems that B could well be larger than 1.

Suppose $B > 1$, let it be $B = 1 + \delta$ where $\delta > 0$: Then our equation becomes:

$$\Rightarrow 2^{f(n)} \leq 2^{Bg(n)} \; \forall \; n \; \geq b \Rightarrow 2^{f(n)} \leq 2^{g(n)} \times 2^{\delta g(n)} \; \forall \; n \; \geq b$$

and we see that the multiplying factor for $2^{g(n)}$ (which we would like to be our B') is not a constant, but a function of n. Perhaps this means our claim is not true, and perhaps this will give us a counterexample. So let's take the simplest values for B, f and g such that $f(n) \leq Bg(n)$ and $B > 1$. For example, what if $B = 2$, $f(n) = 2n$, $g(n) = n$? Then, for $B = 2$ and $b = 1$,

$$f(n) \leq Bg(n) \quad \forall \, n \geq b$$

and $f(n) \in O(g(n))$.

Now we wish to show that $2^{f(n)} \notin O(2^{g(n)})$. We wish to show that a statement of the following form is false: $\exists B', b'$ such that R is true. We need to show it is false $\forall B', b'$.

$$\exists B', b' > 0 \; such \; that \; 2^{2n} \leq B' 2^n \quad \forall \, n \geq b'$$

is of the form $\exists B', b'$ such that:

$$n \geq b' \Rightarrow 2^{2n} \leq B' 2^n \Rightarrow 2^n \leq B'$$

But the above is a conditional statement with a counterexample is $n = max(ceil(log(B'), b') + 1$. It satisfies the hypothesis, $n \geq b$, and the conclusion is false, because $2^n > B'$. Thus we have shown that $f(n) \in O(g(n)) \Rightarrow 2^{f(n)} \in O(2^{g(n)})$ is not true.

CSCI 2312 - Discrete Structures II - Big-Oh Practice Problems Part I Solutions
The George Washington University

Let $f : \mathbb{N} \rightarrow \mathbb{R}$ and $g : \mathbb{N} \rightarrow \mathbb{R}$, such that $g(n) \geq 0 \;\; \forall n$. We say that $f(n) \in O(g(n))$ or $f \in O(g)$ if $\exists B, b > 0$, such that

$$0 \leq | f(n) | \leq Bg(n) \;\; \forall \; n \geq b$$

1. Suppose $f \in O(g)$. Is $kf(n) \in O(g(n))$ for $k \in \mathbb{R}$, $k \neq 0$? If so, prove it. If not, provide a counter-example.

   Note: in the original handout, we did not include the condition that $k \neq 0$. This is included to simplify this solution.

   Solution: Claim: $f \in O(g) \Rightarrow kf(n) \in O(g(n)) \;\; \forall \; k \; \in \; \mathbb{R}$

   Proof: Begin with what is assumed. Progress towards what you need to prove.

$$\begin{aligned} f(n) \in O(g(n)) &\Rightarrow \exists B, b > 0 \textit{ such that } 0 \leq | f(n) | \leq Bg(n) \; \forall \; n \; \geq b \\ &\Rightarrow \exists B, b > 0 \textit{ such that } 0 \leq | k | \times | f(n) | \leq B \mid k \mid g(n) \; \forall \; n \; \geq b \end{aligned}$$

   Because $| k | > 0$, the directions of the inequalities are preserved. Further, the above implies that:

$$\exists B, b > 0 \textit{ such that } 0 \leq | kf(n) | \leq B \mid k \mid g(n) \; \forall \; n \; \geq b$$

   If $B' = | k | B > 0$ and $b' = b > 0$,

$$\Rightarrow 0 \leq | kf(n) | \leq B'g(n) \; \forall \; n \; \geq b'$$

   and hence $kf \in O(g)$.

2. For non-negative functions, is the relation O (a) reflexive? (b) symmetric? (c) transitive?

   Solution:

   (a) The relation O is reflexive.

   $f(n) = 1 \times f(n) \;\; \forall \; n \geq 1$. Hence, for $B = 1$ and $b = 1$:

$$f(n) \leq Bf(n) \;\; \forall \; n \geq b$$

   and $f \in O(f)$.

   You can use any $B \geq 1$ and $b \geq 1$ as long as you show the above.

   (b) The relation O is not symmetric. That is, $f \in O(g) \not\Rightarrow g \in O(f)$. To show this, we provide an f and g such that $f \in O(g)$ but $g \notin O(f)$. You have examples from the two problems on Asymptotics posted online (which we did in class); you may use either of those as long as you cite them correctly. So you may see more counterexamples of this kind, here's another one.

Let $f(n) = 5$ and $g(n) = n$. Then:

$$f(n) \leq 1 \times g(n) \quad \forall n \geq 5$$

and $f \in O(g)$.

There are many other possibilities, as long as you show that $f \in O(g)$.

Notice that $g \notin O(f)$.

$g \in O(f)$ is a statement of the form:

$$\exists B, b > 0 \textit{ such that } 0 \leq g(n) \leq Bf(n) \; \forall \; n \geq b$$

To show it is false we need to show

$$\nexists B, b > 0 \textit{ such that } 0 \leq g(n) \leq Bf(n) \; \forall \; n \geq b$$

or:

$$\forall B, b > 0, \quad n \geq b \not\Rightarrow 0 \leq g(n) \leq Bf(n)$$

Given $B, b > 0$, we need to show that:

$$n \geq b \not\Rightarrow 0 \leq g(n) \leq Bf(n)$$

or

$$n \geq b \not\Rightarrow 0 \leq n \leq 5B$$

Counterexamples are any values of n such that $n \geq b$ and $n > 5B$. Thus, any values:

$$n \geq max(b, 5B) + 1$$

For any of these values of n we see that the hypothesis is true ($n \geq b$) and the conclusion is false, because $n > 5B$. Thus we have shown that

$$f \in O(g) \not\Rightarrow g \in O(f)$$

and O is not symmetric.

(c) The relation O is transitive.

That is, we claim that $f \in O(g)$ and $g \in O(h)$ implies that $f \in O(h)$.

Proof: Suppose $f \in O(g)$. Then, $\exists B, b > 0$ such that:

$$f(n) \leq Bg(n) \;\; \forall n \geq b \tag{1}$$

Suppose $g \in O(h)$. Then, $\exists B', b' > 0$ such that:

$$g(n) \leq B'h(n) \;\; \forall n \geq b' \tag{2}$$

From (1) and (2) above, we get:

$$f(n) \leq Bg(n) \leq BB'h(n) \;\; \forall n \geq max(b', b) \tag{3}$$

and $f \in O(h)$ because $BB' = B'' > 0$ because $B, B' > 0$ and $b'' = max(b, b') > 0$ because $b, b' > 0$.

Note here that you have to use $max(b, b')$ or some variation of that, but you cannot ignore the fact that the statement is true only when n is greater than both b and b'.

CSCI 2312 - Discrete Structures II
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Monomials and Big-Oh

Recall the Big-Oh definition:

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}$, such that $g(n) \geq 0$. We say that $f(n) \in O(g(n))$ or $f \in O(g)$ if $\exists B, b$, both positive, such that

$$0 \leq \mid f(n) \mid \leq Bg(n) \ \ \forall\ n \geq b$$

We will show that:

For any $k, d \geq 0$,

$$n^k \in O(n^d) \Leftrightarrow d \geq k$$

Solution:

We show the forward direction first:

$\Rightarrow$

$$n^k \in O(n^d) \Rightarrow \ \exists B, b > 0 \ s.t. \ n^k \leq Bn^d \ \ \forall n \geq b$$

We need to show that $d \geq k$. We show it by contradiction. Suppose $d < k$. The above inequality implies that:

$$n^{k-d} \leq B \ \ \forall n \geq b$$

Note that $n \geq b > 0$ hence we can divide by n, and, also, the inequality retains its direction.

$$n^{k-d} \leq B \Rightarrow n \leq B^{\frac{1}{k-d}} \quad \forall n \geq b$$

The inequality retains direction because $k - d > 0$.

We have a contradiction because $n \leq B^{\frac{1}{k-d}}$ is not true when $n > max(b, B^{\frac{1}{k-d}})$.

We can also show this differently, by using some $n > max(b, B^{\frac{1}{k-d}})$ as a counterexample for the case when $d < k$.

For the reverse direction:

$\Leftarrow$

$$\begin{aligned}
&d \geq k \\
\Rightarrow &d - k \geq 0 \\
\Rightarrow &n^{d-k} \geq n^0 = 1 \ \forall \ n \geq 1 \\
\Rightarrow &n^d \geq n^k \ \forall \ n \geq 1
\end{aligned}$$

Let $B = 1, b = 1$. Then we have found $B, b > 0$ such that:

$$Bn^d \geq n^k \ for \ n \geq b$$

which implies that $n^k \in O(n^d)$ by the definition of Big-Oh.

Let $f(n) = n^2$ and $g(n) = \frac{1}{2}n^2$. Show that:

(a) $2^{f(n)} \notin O(2^{g(n)})$

Solution: To show that $2^{f(n)} \notin O(2^{g(n)})$, that is, $2^{n^2} \notin O(2^{\frac{n^2}{2}})$,we need to show that

$$\nexists B, b > 0 \ s.t.\ 2^{n^2} \leq B2^{\frac{n^2}{2}} \ \forall n \geq b$$

That is, we need to show that,

$$\forall B, b > 0 \quad n \geq b \nRightarrow 2^{n^2} \leq B2^{\frac{n^2}{2}}$$

We cannot choose B, b.

To show the above, given $B, b > 0$, we need a counterexample for:

$$n \geq b \Rightarrow 2^{n^2} \leq B2^{\frac{n^2}{2}} \Rightarrow 2^{\frac{n^2}{2}} \leq B$$
$$\Rightarrow n^2 \leq 2\log_2 B$$
$$\Rightarrow n \leq \sqrt{2\log_2 B}$$

which is not true when $n^2 \geq 2log_2 B$ or $n \geq \sqrt{2log_2 B}$.

Thus $n = max(b, \sqrt{2log_2 B}) + 1$ is a counterexample as it satisfies the hypothesis $n \geq b$, and

$$n \geq \sqrt{2log_2 B} + 1 \Rightarrow n^2 > 2log_2 B \Rightarrow 2^{\frac{n^2}{2}} > B \Rightarrow 2^{n^2} > B2^{\frac{n^2}{2}}$$

and the conclusion is false. Hence we have a counterexample.

Thus, we have shown that

$$\forall B, b > 0 \quad n \geq b \nRightarrow 2^{n^2} \leq B2^{\frac{n^2}{2}}$$

and

$$\nexists B, b > 0 \ s.t.\ 2^{n^2} \leq B2^{\frac{n^2}{2}} \ \forall n \geq b$$

Hence, $2^{f(n)} \notin O(2^{g(n)})$ ✓

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Big-Theta and Big-Omega

Let $f, g : \mathbb{N} \to \mathbb{R}$, such that $g(n) \geq 0$. We say that $f(n) \in O(g(n))$ or $f \in O(g)$ if $\exists B, b$, both positive, such that

$$0 \leq\mid f(n) \mid\leq Bg(n) \;\; \forall \; n \geq b$$

Let $f, g : \mathbb{N} \to \mathbb{R}^+$. We say that $f \in \Omega(g(n))$ or $f \in \Omega(g)$ if $\exists A, a$, both positive, such that

$$0 \leq Ag(n) \leq f(n) \;\; \forall \; n \geq a$$

Let $f, g : \mathbb{N} \to \mathbb{R}^+$. We say that $f \in \Theta(g(n))$ or $f \in \Theta(g)$ if $\exists A, B, k$, all positive, such that

$$Ag(n) \leq f(n) \leq Bg(n) \;\; \forall \; n \geq k$$

Let $f, g : \mathbb{N} \to \mathbb{R}^+$. Show that

1. $f \in O(g) \Leftrightarrow g \in \Omega(f)$ (you showed this last week).

   Solution:

   $\Rightarrow$

   By definition[1] of O,

$$f \in O(g) \Rightarrow \;\; \exists \; B, b > 0 \; such \; that \;\; \mid f(n) \mid\leq Bg(n) \;\; \forall \; n \geq b \tag{1}$$

   Note that, $f(n) \geq 0$ and $\mid f(n) \mid= f(n)$. Similarly, $\mid g(n) \mid= g(n)$.

   To show that $g \in \Omega(f)$, we need[2] to show that $\exists \; A, a > 0 \; such \; that \; Af(n) \leq g(n) \;\; \forall \; n \geq a.$

   Rewriting expression (1):

$$f \in O(g) \Rightarrow \;\; \exists \; B, b > 0 \; such \; that \; g(n) \geq \frac{1}{B} \mid f(n) \mid= \frac{1}{B} f(n) \;\; \forall \; n \geq b \tag{2}$$

   Note that, because $B > 0$, we may divide by B and the inequality retains its direction.

   Let $A = \frac{1}{B}$ and $a = b$. Note that $B > 0 \Rightarrow A > 0$ and $b > 0 \Rightarrow a > 0$. Thus we have shown that

$$\exists A, a > 0 \; s.t. \; Af(n) \leq g(n) \;\; \forall n \geq a$$

   and hence, by definition, $g \in \Omega(f)$.

   $\Rightarrow$

   By definition[3] of Ω,

[1] Beginning with what is given or assumed, and NOT with what you want to prove:

[2] Here you may state your end goal. You haven't shown it yet, so you need to tag it as such, so I know that you know that you haven't shown it yet.

[3] You can pretty much repeat the above procedure for Ω

$$g \in \Omega(f) \Rightarrow \ \exists \ A, a > 0 \ such \ that \ Af(n) \leq g(n) \ \ \forall \ n \geq a \tag{3}$$

Note that, $f(n) \geq 0$ and $\mid f(n) \mid = f(n)$.

To show that $f \in O(g)$, we need to show that $\exists \ B, b > 0 \ such \ that \ \mid f(n) \mid = f(n) \leq Bg(n) \ \ \forall \ n \geq b$.

Rewriting expression (3):

$$g \in \Omega(f) \Rightarrow \ \exists \ A, a > 0 \ such \ that \ f(n) \leq \frac{1}{A} g(n) \ \ \forall \ n \geq a \tag{4}$$

Note that, because $A > 0$, we may divide by A and the inequality retains its direction.

Let $B = \frac{1}{A}$ and $b = a$. Note that $A > 0 \Rightarrow B > 0$ and $a > 0 \Rightarrow b > 0$. Thus we have shown that

$$\exists B, b > 0 \ s.t. \ f(n) \leq Bg(n) \ \ \forall n \geq b$$

and hence, by definition, $f \in O(g)$.

Thus we have shown that:

$f \in O(g) \Leftrightarrow g \in \Omega(f)$

2. Show that:

$$f \in \Theta(g) \Leftrightarrow f \in O(g) \ and \ f \in \Omega(g)$$

Solution:

$\Rightarrow$

$$f(n) \in \Theta(n) \Rightarrow \exists A, B, k > 0 \ s.t \ Ag(n) \leq f(n) \leq Bg(n) \ \ \forall \ n \geq k$$

$$\Rightarrow \exists A, k > 0 \ s.t \ Ag(n) \leq f(n) \ \ \forall \ n \geq k \Rightarrow f(n) \in \Omega(g(n))$$

and

$$\Rightarrow \exists B, k > 0 \ s.t \ f(n) \leq Bg(n) \ \ \forall \ n \geq k \Rightarrow f(n) \in O(g(n))$$

$\Leftarrow$

$$f(n) \in O(g(n)) \Rightarrow \exists B, b > 0 \ s.t. \ f(n) \leq Bg(n) \ \ \forall \ n \geq b \tag{5}$$

$$f(n) \in \Omega(g(n)) \Rightarrow \exists A, a > 0 \ s.t. \ Ag(n) \leq f(n) \ \ \forall \ n \geq a \tag{6}$$

Let $k = max(a, b)$. Then, combining (5) and (6), we get:

$$\exists A, B, k > 0 \ s.t \ Ag(n) \leq f(n) \leq Bg(n) \ \ \forall \ n \geq k$$

and, hence, $f(n) \in \Theta(g(n))$.

Thus we have shown that: $f(n) \in \Theta(g(n)) \Leftrightarrow f \in O(g(n))$ and $f \in \Omega(g(n))$.

3. Using Problem (1) and previous results proven in discussion, HW, class and quizzes about properties of O, Ω and Θ, show the following. Do not use A, B, k, a, b, etc.

(a) Ω is reflexive.

Solution: From Discussion section week of 26 September we know that Big-Oh is reflexive. That is, $f \in O(f)$. From Problem (1) we know that $f \in O(g) \Rightarrow g \in \Omega(f)$. Hence,

$$f \in O(f) \Rightarrow f \in \Omega(f)$$

(b) Ω is not symmetric.

Solution: From Discussion section week of 26 September and class on 27 September we know that Big-Oh is not symmetric. That is, we know that there is a counterexample (pair of functions f and g) such that $f \in O(g)$ but $g \notin O(f)$. The same functions will provide a counterexample for the statement: Ω is symmetric.

$f \in O(g) \Rightarrow g \in \Omega(f)$ from Problem (1).

Further, $g \in O(f)$ is false, so, from Problem (1), $f \in \Omega(g)$ is false.

Thus, the same counterexample, show that there are functions f, g such that $g \in \Omega(f)$ but $f \notin \Omega(g)$. Hence Ω is not symmetric.

(c) Ω is transitive.

Solution: Suppose $f \in \Omega(g)$ and $g \in \Omega(h)$.

From Problem (1), we get:

$$f \in \Omega(g) \Rightarrow g \in O(f)$$

$$g \in \Omega(h) \Rightarrow h \in O(g)$$

From transitivity of O, shown in class on September 29, we get:

$$h \in O(g) \;\; and \;\; g \in O(f) \Rightarrow h \in O(f)$$

From Problem (1),

$$h \in O(f) \Rightarrow f \in \Omega(h)$$