CSCI2312: Discrete Structures II

Solutions Lab 7

October 21, 23, 25

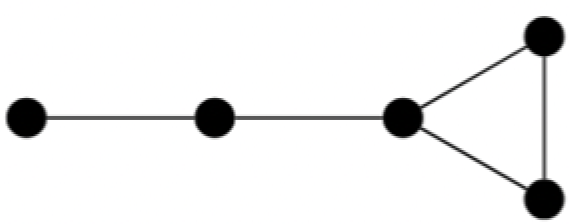
Q1: Consider the graph whose edges are the seams of a standard soccer ball and whose vertices are the places where these seams meet. Each vertex in this graph lies at the corner of one of the 12 black pentagons and has degree 3. How many edges are in this graph?

Solution:
There are 5 vertices on each of the 12 pentagons, and each one has degree 3, so the sum of the degrees of all vertices is 5.12.3 = 180.

By the Handshaking Lemma, we know $\sum_{v \in V} \deg(v) = 2m$.

Therefore, there are 90 edges.

Q2: We say that a connected graph G with at least two vertices is a three-graph if every vertex is of degree at most 3 and it has a vertex of degree exactly 3. We say that G is a strict three-graph if it is a three-graph and every vertex has degree 3.
(a) Prove or disprove that there exists a three-graph G with an odd number of vertices.
(b) Prove or disprove that there exists a strict three-graph G with an odd number of vertices.
Solution:
(a)



(b)
We disprove the claim. Every vertex in a strict three-graph must have an odd degree. Thus, in a three-graph with an odd number of vertices, there must be an odd number of odd degree vertices. We know that every graph must have an even number of odd degree vertices (proved in lecture), so this is impossible. Thus, the claim must be false.

Q3: Prove that for a graph G = (V, E) or its complement $\overline{G}$, denoted by $\overline{G}$ = (V, E′), at least one of the two graphs must be connected. The complement graph $\overline{G}$ is defined such that V′ = V and an edge {u, v} $\in$ E′ if and only if {u, v} $\notin$ E.

Solution:

We are tasked with proving that either G or $\overline{G}$ is connected.

There are two cases to consider:

Case 1: G is Connected If G is connected, then by definition there is a path between any two arbitrary vertices u and v in G. In this case, we are done, as the claim holds.

Case 2: G is Not Connected If G is not connected, then it must consist of multiple connected components. We aim to show that in this case, $\overline{G}$ must be connected.

Let u and v be two arbitrary vertices in G, and consider two subcases:

Subcase 1: u and v do not share an edge in G Since u and v do not share an edge in G, by the definition of the complement, they must share an edge in $\overline{G}$. Therefore, there is a direct path from u to v in $\overline{G}$.

Subcase 2: u and v share an edge in G This means that u and v belong to the same connected component of G.

Now, consider an arbitrary vertex x that is part of a different connected component in G. Since x is in a different component, neither u nor v can be connected to x in G, meaning both u and v must share edges with x in $\overline{G}$. Thus, in $\overline{G}$, there is a path u − x − v, connecting the vertices u and v through x.

Conclusion: In both subcases, we have shown that there is always a path between any two arbitrary vertices in $\overline{G}$. Therefore, if G is not connected, $\overline{G}$ must be connected. Hence, we have proven that either G or $\overline{G}$ is connected.