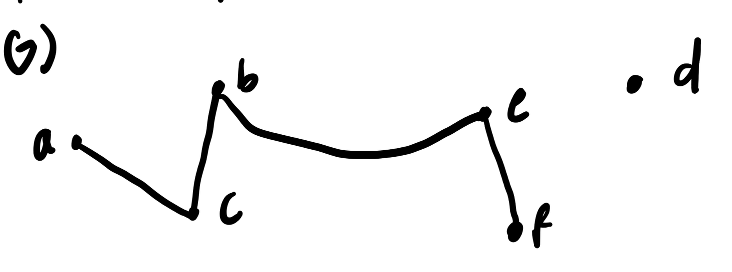
Lecture #15: Trees
Sunday, October 13, 2024 5:57 PM

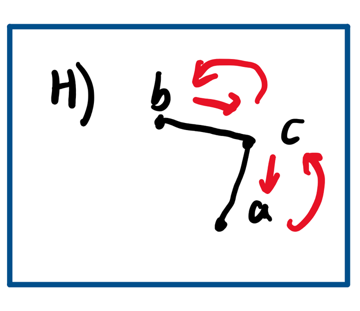
Announcements:

1. Homework #6 due Tuesday
2. Feedback #1 due Thursday
3. Exam #2: November 21st (Thursday)

Review of Graph Definitions

G)





0. $deg(h) = 2$

1. Give me a walk in H that is NOT a path
   b - c - a - c - b   b, {b,c}, c, {c,a}, a, ...

2. Give me a path in H
   b - c - a

3. Is there a cycle in G? If so, what is the cycle?
   Yes, g-h-i-g

4. Is G connected? Is H connected?
   N. Yes

5. Is H a subgraph of G?
   Yes

6. How many connected components does G have? H?
   3 1

Proved claims:

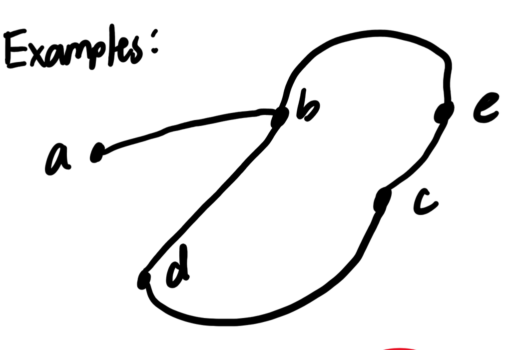
1. Handshaking Lemma: $\sum_{v \in V} deg(v) = 2 \cdot |E|$ ↓ # of edges

2. $|CC| \geq |V| - |E|$

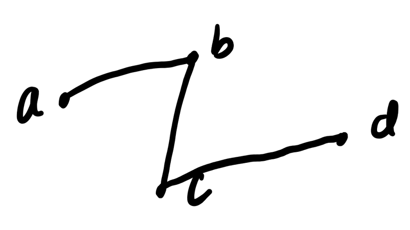
Definitions:

- Graphs with no cycles are called acyclic graphs
- A tree is an acyclic graph that is connected
- In a tree, a vertex with degree 1 is a leaf, all other vertices in a tree are called internal vertices
- A forest is an acyclic graph

Examples:



Tree? Forest? Neither? both?



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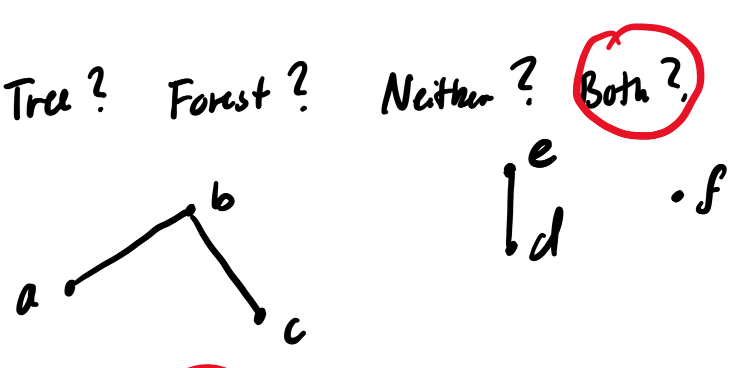

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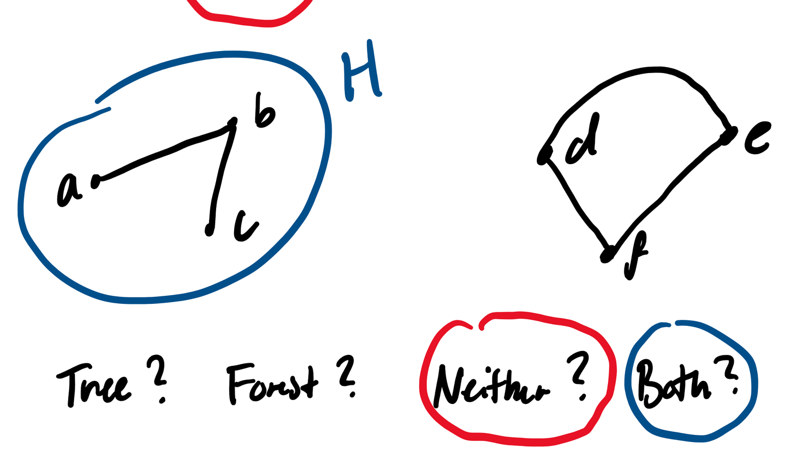
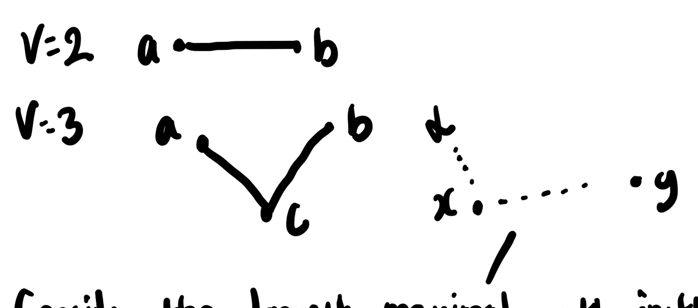


Tree? Forest? Neither? Both?

Example Problems:

1. Prove that every tree, T, with at least 2 vertices ($|V| \geq 2$), has

   a) At least 2 leaves



Consider the longest maximal path in the graph, the two vertices that are the endpoint of that path are leaves.

Why? The graph cannot have a cycle

   b) deleting a leaf from a n-vertex tree results in a tree with n-1 vertices

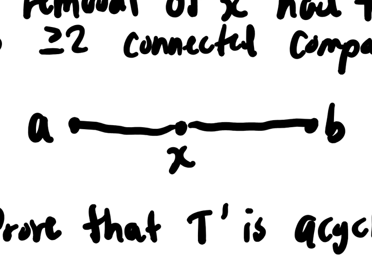
T has n-vertices and if $n \geq 2$, T has ≥ 2 leaves

Let T' be $T - \{x\}$ where x is a leaf along with it's one adjacent edge

Clearly T' has n-1 vertices

   a) Let's prove T' is connected

   Let's assume fpoc that T' is not connected
   In that case, the following had to be true
   The removal of x had to split the graph into ≥ 2 connected components.



Contradiction!

   b) Prove that T' is acyclic ✓

Properties about trees that have n-vertices (T)

1) T is a tree (acyclic and connected graph)
2) T is connected and has exactly n-1 edges
3) T is minimally connected (T is connected, but $T - \{e\}$ for any $e \in E$ is disconnected
4) T is acyclic, but $T + \{x, y\}$ for any pair (x, y) when $x \in V$, and $y \in V$ has a cycle
5) Any two vertices in T are linked by a unique path in T.

a) $1 \rightarrow 2$

If G is a tree with n vertices (acyclic and connected) then, G is connected and has exactly n-1 edges

I know that G is a tree and is connected ✓

$|CC| \geq |V| - |E|$

$1 \geq n - |E|$

$|E| \geq n - 1$

Let's prove using induction on n that T has exactly n-1 edges

BC: $n=1$ •a ✓

$n-1 = 0$, 0 edges

IH: Assume that a tree with k vertices has k-1 edges

IS: Prove that T with k+1 vertices has k edges

Let $T' = T - \{x\}$, where x is a leaf

T' is a tree with k vertices

By IH, I know that T' has k-1 edges

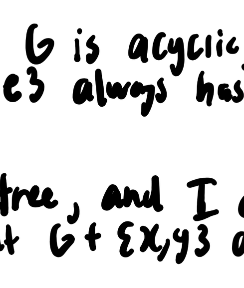
I add back x to the graph along with it's adjacent edge, now

$T = T' + \{x\}$, and T has k edges

b) => G is minimally connected, G is connected, but $G - \{e\}$ for any edge, $e \in E$, disconnects G

Let's assume f.p.o.c that removing an edge, e, does not disconnect the graph.

Consider removing the edge: $\{x, y\}$



If deleting $\{x, y\}$ does not disconnect the graph, there must be another path from $x \rightarrow y$ that does not include the edge $\{x, y\}$

Contradiction, G is a tree, and thus cannot have cycles.

=> While G is acyclic, if I add an edge $G + \{e\}$ always has a cycle

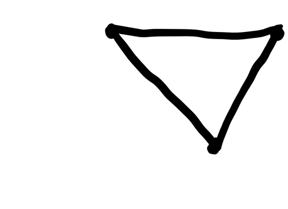
G is a tree, and I add $e = \{x, y\}$, I need to prove that $G + \{x, y\}$ always has a cycle.

Every pair of vertices, i.e x and y have a path in T by the definition of connectivity, and thus adding an edge between those vertices, creates a cycle:

=> Every pair of vertices in T has a unique path

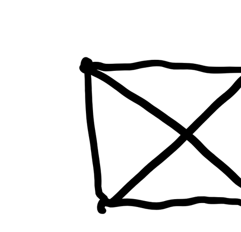
Q4) Complete-Graph

n=2



1

n=3



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n=4

6