Dombi Power Partitioned Heronian Mean Operators of q-Rung Orthopair Fuzzy Numbers for Multiple Attribute Group Decision Making

In this paper, a set of Dombi power partitioned Heronian mean operators of q-rung orthopair fuzzy numbers (qROFNs) are presented and a multiple attribute group decision making (MAGDM) method based on them is proposed. Firstly, the operational rules of qROFNs based on the Dombi t-conorm and t-norm are discussed. According to these rules, a q-rung orthopair fuzzy Dombi partitioned Heronian Mean operator and its weighted form are then established. To reduce the negative effect of unreasonable attribute values on aggregation result, a q-rung orthopair fuzzy Dombi power partitioned Heronian mean operator and its weighted form are constructed through combining the partitioned Heronian mean operator with the power average operator of qROFNs. On the basis of the constructed operators, a method to solve the MAGDM problems based qROFNs is designed. Finally, a practical example, a set of experiments, and comparisons are reported to demonstrate the feasibility and effectiveness of the proposed method.

*Keywords*: q-rung orthopair fuzzy set; partitioned Heronian mean; Dombi t-norm and t-conorm; power average operator; multi-attribute group decision making.

# Introduction

Multi-attribute group decision making (MAGDM) is a process of choosing the best alternative in complex scenarios via evaluating the values of multiple attributes of all alternatives synthetically by a group of decision makers. In this process, the primary task is to accurately express the attribute values, and fuzzy sets have been regarded as effective tools for such expression. So far, over twenty different types of fuzzy sets have been presented within the academia.1 Among them, Zadeh’s fuzzy set (FS)2 is a well-known type of fuzzy sets that uses a degree of membership to quantify the degree of satisfaction. However, it cannot express non-membership and hesitancy degrees. Atanassov3,4 proposed the intuitionistic FS (IFS) to overcome this shortcoming by adding a non-membership function, and thus the hesitancy function can be expressed as one minus the sum of membership and non-membership functions. Because IFSs can describe more complex fuzzy information than FSs, many researchers focus on the research topics of IFSs, such as the operational rules of intuitionistic fuzzy numbers (IFNs)5-7, aggregation operators of IFNs8-12, intuitionistic preference relations13-15, intuitionistic fuzzy calculus16-18, and MAGDM methods based on IFSs19-21. Although IFSs have showed great potential in MAGDM, their application range is limited by their capability to express fuzzy information, i.e. the sum of membership and non-membership degrees should be within the range of 0 to 1. To address this issue, Yager22 proposed the theory of Pythagorean fuzzy set (PFS), in which the condition is extended to the sum of the squares of membership and non-membership degrees falls within the range of 0 to 1. Due to the stronger expressiveness, PFSs have received much researchers’ attention. For example, Yager and Abbasov23 investigated the relationships between Pythagorean fuzzy numbers (PFNs); Zhang and Xu24 presented an extension of TOPSIS to Multi-attribute decision making MADM with PFSs; Ren et al.25 presented an extension of TODIM to MCDM with PFSs; Peng and Yang26 proposed division and subtraction operations on PFSs; Garg27 proposed a new generalized Einstein operator to aggregate PFNs; Dick et al.28 developed interpretations of complex-valued Pythagorean membership grades; Liang et al.29 proposed a new model of three-way decision based on PFSs.

Recently, to further improve the expressiveness of PFS, Yager30 presented the theory of generalized orthopair fuzzy set, i.e. the q-rung orthopair fuzzy set (q-ROFS), in which the membership and non-membership degrees satisfy the condition that the sum of their q-th power within the range of 0 to 1. Obviously, FS and IFS and PFS are special cases of qROFS at q=1 and q=2. This feature makes the expressiveness of q-ROFS more powerful than IFS and PFS via assigning an appropriate value to q. For example, suppose a decision maker gives attribute value whose membership is 0.8 and non-membership is 0.8. Then, this value is not valid for IFS and PFS because 0.82 + 0.82 > 1, while the value can be expressed using qROFS. Thus, qROFSs have also get extensive attention in recent years. Various research topics regarding qROFSs are gaining importance within the academia. For example, Peng31 defined new exponential operational laws of q-ROFNs in which the bases are positive real numbers and the exponents are q-ROFNs and proposed a new score function for comparing two q-ROFNs; Du 32 defined some Minkowski-type distance measures for q-ROFS and investigated the application of the distance measure in decision making; Li et al.33 combined the q-ROFS with a picture fuzzy set and proposed a q-rung picture linguistic set; Liu and Wang34 proposed a family of simple weighted averaging and geometric operators for solving the MAGDM problems; Liu and Liu35 and Wei et al.36 respectively proposed some q-rung orthopair fuzzy Bonferroni mean operators and some q-rung orthopair fuzzy Heronian mean operators, which can consider the interrelationships between any two q-ROFNs; Liu and Wang37 proposed some q-rung orthopair fuzzy Archimedean Bonferroni mean (q-ROFABM) operators, which apply Bonferroni mean in the q-ROFS based on Archimedean T-norm and T-conorm for solving the q-rung orthopair fuzzy MCDM problems.

To solve MAGDM problems, there are generally two groups of methods. One is conventional methods, such as TOPSIS, VIKOR, ELECTRE, and so on. The other is methods based on aggregation operators. Aggregation operators can more effectively solve MAGDM problems than traditional approaches. Because they can provide comprehensive values and then give the ranking results, while conventional methods can only generate rankings. Aggregation operators are usually considered in terms of operational rules and functions: (1) For operational rules, some aggregation operators are the special and significant members in the T-norm (TN) and T-conorm (TC) family and the Archimedean T-norm and T-conorm (ATT) are the generalization of a lot of TNs and TCs. To date, many operational rules have been developed according to the special type of ATT, such as Einstein operational rules38, Hamacher operational rules39, Frank operational rules40 and so on. (2) For functions, according to the type of relationship between attributes, Yager 41 proposed the Power average (PA) operator which is a new tool to aggregate input arguments by considering the relationship between the attribute values. It allows attribute values to support and reinforce each other. To consider the relationships between the aggregated arguments, more than ten kinds of aggregation operators for qROFSs have been presented, such as Maclaurin symmetric mean (MSM) operator42, partitioned MSM operator43, power MSM operator44, Bonferroni mean (BM) operator45, partitioned BM operator46, geometric Heronian mean (HM) operator47 and partitioned Heronian mean operators48. In existing literature, Yu et al.49 explained the advantages of the HM operator over the BM operator in details and analyzed how HM operator has meliority over BM operator. Although the two aggregation operators can consider the inter- relationships between the aggregated parameters, they cannot consider the interactions between membership function and non-membership function and they cannot deal with these decision making problems with interrelationships of the attributes only in the same partition and with no interrelationships in different partitions. Liu at al.50 proposed the partitioned Heronian mean operators to overcome this shortcoming by dividing the attribute values into different sorts.

The recently proposed Dombi t-conorm and t-norm (DTT)51 are powerful in information aggregation and, recently, have been applied to the aggregation of IFS52, Hesitant fuzzy set53, and single-valued neutrosophic information54. However, they have not yet been applied to the aggregation of qROFS. In addition, there is not an aggregation operator that combines PA operator and partitioned Heronian mean operator to reflect the relationships between the input arguments. To this end, a q-rung orthopair fuzzy Dombi power partitioned Heronian mean operator and its weighted form are presented and a MAGDM method based on them are proposed in this paper.

The remainder of this paper is organized as follows. Section 2 briefly recalls some basic concepts of q-rung orthopair fuzzy sets, DTT, PA operator, PHM operator and operational rules of qROFNs based on DTT. Section 3 presents a set of operators for qROFNs. Section 4 proposes a novel MAGDM method based on the presented operators. Section 5 provides a practical example, a set of experiments, quantitative comparisons and further comparative analysis. The last section summarizes the paper.

# Preliminaries

## Q-ROFSs

**Definition 1**.30 A *q*ROFS *Q* in a finite universe of discourseX is:

 (1)

where *μQ*: X → [0, 1] denotes the degree of membership of the element *x* ∈ X to the set *Q* and *νQ*: X → [0, 1] denotes the degree of non-membership of the element *x* ∈ X to the set *Q*, with the condition that 0 ≤ (*μQ*(*x*)*q* + *νQ*(*x*)*q*) ≤ 1 (*q*=1, 2, 3, …). The degree of hesitancy (indeterminacy) of element *x* ∈ X to the set *Q* is:

 (2)

For convenience, a pair <*μQ*(*x*), *νQ*(*x*)> is called as a *q*ROFN30, denoted by Θ = (*μ, v*). To compare two *q*ROFNs, their scores and accuracies are needed to calculate. The followings are the definitions of the score of a *q*ROFN and the accuracy of a *q*ROFN, inspired by the idea of comparing PFNs22.

**Definition 2**.34 Let Θ = (*μ, v*) be a *q*ROFN. Then the score of Θ is:

 (3)

and -1 ≤ *S*(Θ) ≤1.

**Definition 3**.34 Let Θ = (*μ, v*) be a *q*ROFN. Then the accuracy of Θ is:

 (4)

and 0 ≤ A(Θ) ≤1.

Based on *S*(Θ) and the A(Θ), a comparison method of *q*ROFN is presented in Ref. 34. The following is the definition of the method.

**Definition 4**.34 Let Θ1 = (*μ*1*, v*1) and Θ2 = (*μ*2*, v*2)be two arbitrary *q*ROFNs, *S*(Θ1) and *S*(Θ2) be respectively the scores of Θ1 and Θ2, and A(Θ1) and A(Θ2) be respectively the accuracies of Θ1 and Θ2. Then,

1. If *S*(Θ1) > *S*(Θ2), then, Θ1 > Θ2;
2. If *S*(Θ1) = *S*(Θ2), then,
3. If A(Θ1) > A(Θ2), then, Θ1 > Θ2;
4. If A(Θ1) = A(Θ2), then, Θ1 = Θ2.

**Definition 5**.32 Let Θ1 = (*μ*1*, v*1) and Θ2 = (*μ*2*, v*2) be two arbitrary *q*ROFNs, then the Minkowski-type distance between Θ1 and Θ2 is given by:

 (5)

## Dombi t-norm and conorm

In the following, a new operational rule of *q*ROFNs is introduced on the basis of DTT Ref. 51 to put forward a generator to produce t-norm and t-conorm:

 (6)

1. If function *f* (*x*) is a monotonically increasing and meets the conditions that:

:; :; ; .

Then, the TN *T* can be generated as: *T* (*x, y) = f* -1 (*f* (*x*) *+ f* (*y*)).

1. If function *g* (*x*) is a monotonically decreasing and meets the conditions that:

:; :; ; .

Then, the TC *S* can be generated as: *S* (*x, y*) *= g*-1 (g(*x*) *+ g*(*y*)). According to Ref. 58, the relationship of *f*(*x*) and *g*(*x*) *is: f* (*x*) *= g* (*1-x*)*.*

**Definition 6**.51 Let *λ* be a positive real number and *x, y* ∈[0, 1], the DTT and their additive generators are defined as follows:

 (7)

 (8)

, (9)

Then, it can be obtained that:

 (10)

## Operational rules of qROFNs based on DTT

On the basis of DTT, a set of operational rules of *q*ROFNs can be established as follows:

**Definition 7**. Let Θ = (*μ, v*), Θ1 = (*μ*1*, v*1) and Θ2 = (*μ*2*, v*2) re three arbitrary *q*ROFNs, and *δ* and *τ* are two arbitrary positive real numbers. Then the sum, product, multiplication and power operations between *q*ROFNs based on *TD, λ* (*x, y) = f* -1 (*f* (*x*) *+ f* (*y*)) and *SD, λ* (*x, y*) *= g*-1 (g(*x*) *+ g*(*y*)) can be respectively defined as follows:

 (11)

 (12)

 (13)

 (14)

According to Equations (11)-(14), it is easy to prove the following rules:

 (15)

 (16)

 (17)

 (18)

 (19)

 (20)

## HM operator and PA operator

**Definition 8**.57 Let *xi*(*i* = 1, 2, …, *n*) be a series of crisp numbers. If

 (21)

Where *a, b* ≥ 0, then HM*a,b* is called the HM operator.

**Definition 9**.50 LetX*hi*(*i* = 1, 2, …, *n*) be a collection of arguments, which is partitioned into *d* distinct sorts *P1 ,P2 ,…,Pd*, where *Ph* = {X*h1*, X*h2*,…, X*h|Ph|*} (*h*=1,2,…,*d*), denotes the cardinality of *Ph*. For any *a, b* ≥ *0*, the aggregation function:

 (22)

is called the PHM operator.

**Definition 10** [41]. Let *ai*(*i* = 1, 2, …, *n*) be a collection of non-negative real numbers, then

 (23)

is called the PA operator, in this operator,

 (24)

and *Sup* (*a*, *b*) is denoted the support degree for *a* from *b*, which satisfies following properties:

1. 
2. 
3. 

# Dombi Power Partitioned Heronian Mean Operators of QROFNs

## q-Rung Orthopair Fuzzy Dombi Partitioned Heronian Mean Operators

In this section, PHM is extended to the *q*-rung orthopair fuzzy environment and some novel *q*-rung orthopair fuzzy Dombi partitioned Heronian mean *(q*ROFDHM) operators are presented. Their properties are explored.

**Definition 11**. Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = ( *μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} (*h* = 1, 2, … , *d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*. For any *a, b* ≥ *0* but not at same time *a* = *b* = 0, *q*ROFDPHM operator is defined as:

 (25)

Based on Equations (11)-(14) and (25), the following theorem is obtained:

**Theorem 1**. Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} ( *h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*, *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0 and *λ* be a positive real number. The aggregated value by *q*ROFDHM is still a *q*ROFN and

 (26)

where

.

**Proof**. According to Definition 6, it is obtained:





Let ,,, and ,

then



















Since 

Thus, the proof of Theorem 1 is completed. 

**Theorem 2** **(Idempotency)**. Let {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0, then, if Θ*i* = Θ = (*μ*, *v*) for all *i* = 1, 2, …, *n*, then

 (27)

**Proof.** Let , it will be proved that



Since, it is obtained:



where

where



Similarly, it can also be proved that, then,

.

Thus, the proof of Theorem 2 is completed. 

**Theorem 3 (Monotonicity)**. Let {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) and {Θ1’, Θ2’, …, Θ*n’*} (where Θ*i’* = < *μi’*, *vi’* > (*i* = 1, 2, … , *n*) be two collections of *q*ROFNs (*q* = 1, 2, ...) , *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0. If *μi* ≤*μi’*and *vi* ≤ *vi’* for all *i* = 1, 2, …, *n*, then

 (28)

**Proof.** Let



Sinceand,are monotonic decreasing,are monotonic increasing, then it is obtained:



Thereafter,





and





Then



Thus, , similarly, it can be proved that.

Thus,

.

Thus, the proof of Theorem 3 is completed. 

**Theorem 4 (Boundedness)**. {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0, and if Θ*S* = < max(*μi*), min(*vi*) > and Θ*I* = < min(*μi*), max(*vi*) >. Then,

 (29)

**Proof.** From Theorem 2, it is obtained:



From Theorem 3, it is obtained:



Therefore, it can be obtained .

Thus, the proof of Theorem 4 is completed. 

## q-Rung Orthopair Fuzzy Dombi Weighted Partitioned Heronian Mean Operators

The *q*ROFDPHM operator has advantages in offering more flexibility for describing fuzzy information, generating versatile operational rules for aggregating fuzzy information, and reflecting the interrelationships between different attribute. But it does not consider the relative importance of attributes. Aiming of this issue, weights are introduced and a weighted *q*ROFDPHM operator is presented. The formal definition of this operator is as follow:

**Definition 12**. Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = { Θ*h1*, Θ*h2*,…, Θ*h|Ph|* } ( *h* = 1, 2, … ,*d*) and|*P*1|+|*P*2|+…+|*Pd*|= *n*. The *wi* denotes the weight of Θ*i* with *wi* ∈ [0, 1] and *w*1 + *w*2 + … + *wn* = 1. For any *a, b* ≥ *0* but not at same time *a* = *b* = 0, the *q*-rung orthopair fuzzy Dombi weighted partitioned Heronian mean (*q*ROFDWPHM) operator is defined as follow:

(30)

**Theorem 5.** Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*,, where *Ph* = {Θ*h1*, Θ*h2*, … , Θ*h|Ph|*}, (*h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*, *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0 and *λ* be a positive real number. The *wi* denotes the weight of Θ*i* with *wi* ∈ [0, 1] and *w*1 + *w*2 + … + *wn* = 1. The aggregated value by *q*ROFDWPHM is still a *q*ROFN and

 (31)

where

.

The proof of this theorem is similar to the proof of Theorem 1, please refer to Appendix A. In addition, it is easy to prove that the *q*ROFDWPHM satisfies the Monotonicity and Boundedness properties. Such proofs are omitted.

## q-Rung Orthopair Fuzzy Dombi Power Partitioned Heronian Mean Operators

**Definition 13**. Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} ( *h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*. For any *a, b* ≥ *0* but not at same time *a* = *b* = 0, the q-rung orthopair fuzzy Dombi power partitioned Heronian mean (*q*ROFDPPHM) operator is defined as follow:

 (32)

Where , andis the support degree for Θ*hi* from Θ*hj*, which satisfies following properties:

(1) 

(2) 

(3) , where is the distance of *q*-ROFNs

To simplify Equation (32), let

 (33)

*w*’ = (*w*1’, *w*2’, …, *wn*’), *wi*’ ∈ [0, 1] and . Then Equation (32) can be expressed as:

 (34)

**Theorem 6.** Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} ( *h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*, *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0 and *λ* be a positive real number. The aggregated value by *q*ROFDPPHM is still a *q*ROFN and

 (35)

where



The proof of this theorem is similar to the proof of Theorem 5, so it is omitted here.

**Theorem 7 (Idempotency)**. Let {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0, then, if Θ*i* = Θ = (*μ*, *v*) for all *i* = 1, 2, …, *n*, then

 (36)

**Theorem 8 (Monotonicity).** Let {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) and {Θ1’, Θ2’, …, Θ*n’*} (where Θ*i’* = (*μi’*, *vi’* ) (*i* = 1, 2, … , *n*) be two collections of *q*ROFNs (*q* = 1,2,3,...) , *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0. If *μi* ≤*μi’*and *vi* ≤ *vi’* for all *i* = 1, 2, …, *n*, then

 (38)

**Theorem 9 (Boundedness)**. Let {Θ1, Θ2, …, Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …, *n*), *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0, Θ*S* = ( max(*μi*), min(*vi*)) and Θ*I* = ( min(*μi*), max(*vi*)), then,

 (37)

The proofs of Theorem 7, Theorem 8 and Theorem 9 are similar to the proofs of Theorem 2, Theorem 3 and Theorem 4, respectively. They are omitted here.

## q-Rung Orthopair Fuzzy Dombi Weighted Power Partitioned Heronian Mean Operators

**Definition 14**. Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*-ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} ( *h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*. The *wi* denotes the weight of Θ*i* with *wi* ∈[0, 1] and *w*1 + *w*2 + … + *wn* = 1. For any *a, b* ≥ *0* but not at same time *a* = *b* = 0, the *q*-rung orthopair fuzzy Dombi weighted partitioned Heronian mean (*q*ROFDWPHM) operator is defined as follow:

(39)

Where,and is the support degree for Θ*hi* from Θ*hj*, which satisfies following properties:

(1) 

(2) 

(3) , whereis the distance of *q*-ROFNs.

To simplify Equation (39), let

 (40)

Therefore Equation (39) can be expressed as follows:

 (41)

**Theorem 10.** Let {Θ1, Θ2, … , Θ*n*} (where Θ*i* = (*μi* , *vi* ) (*i* = 1, 2, … , *n*) be a collection of *q*ROFNs (*q* = 1, 2, …), which is partitioned into *d* distinct sorts *P1,P2,…,Pd*, where *Ph* = {Θ*h1*, Θ*h2*,…, Θ*h|Ph|*} ( *h* = 1, 2, … ,*d*) and |*P*1|+|*P*2|+…+|*Pd*|= *n*. The *wi* denotes the weight of Θ*i* with *wi* ∈ [0, 1] and *w*1 + *w*2 + … + *wn* = 1, *a* and *b* be two real numbers and *a, b* ≥ *0* but not at same time *a* = *b* = 0 and *λ* be a positive real number. The aggregated value by *q*ROFDPPHM is still a *q*ROFN and

(42)

where



The proof of the theorem is similar to the proof of Theorem 5, which is omitted here.

# Novel MAGDM Method Based on The Presumed Operators

In this section, a novel MAGDM method is proposed based on the presumed *q*ROFDWPPHM operator.

A MAGDM problem based on *q*ROFNs can be founded through a set of alternatives *A* = {*A*1, *A*2, …, *A*m}, a set of attributes *C* = {*C*1, *C*2, …, *C*n}, a set of weights *w* = {*w*1, *w*2, …, *w*n} (where *w*i ∈ [0,1], *w*1 + *w*2 + … + *wn* = 1 and each element respectively stands for the relative importance of the attribute *C*), and a group of decision makers *D* = {*D*1, *D*2, …, *Dt*} whose weight vector is 𝜔 = {𝜔1, 𝜔2, …, 𝜔*t*} (where 𝜔*i* ∈ [0,1] (*i* = 1, 2, … , *t*) and𝜔1 + 𝜔2+ … + 𝜔*t* = 1). Suppose that these attributes are divided into d different classes *P1, P2, …, Pd* and there are interrelationships among any *kh* attributes in each class *Ph* (*h* = 1, 2, …, *d*) whereas the attributes in different classes are not related. The problem is always coupled with a *q*-rung orthopair fuzzy decision matrix *Mk* =[Θ*k ij*]*m,n* (where *i* = 1, 2, … , *m, j=1, 2,…,n* and Θ*k ij* =(*μk ij*, *vk ij*) is a *q*ROFN that stands for the evaluation value of alternative *Ai* with respect to attribute *Cj*given by decision maker *Dk*, where *j* = 1, 2, …, *n, k=1, 2,…,t*.

On the basis of the components above, the problem can be described as: Making a decision with the help of a ranking of the elements of *A* based on *Mk*, *w* and 𝜔. Using the *q*ROFDWPPHM operators, the problem is solved according to the following steps:

1. Normalize the decision matrix. In real decision making, the attributes in each MAGDM problem are divided into two types of attribute, i.e. cost attributes and benefit attributes. They respectively have positive and negative effects on aggregation results. To eliminate the difference in the attribute types, it is necessary to convert them to the same type. In general, because most attributes are the benefit type, to convert the cost type into the benefit type by the following method and the decision matrix *M* = [*μ ij*, *v ij*]*m,n* is normalized as

 (43)

1. Calculate the evaluation information of decision makers into the collective information. Taking the normalized decision matrix *Mk*’ and the weight set 𝜔 as inputs, the collective information of each alternative can be computed by the *q*ROFDWPPHM operator, shown as follow:

 (44)

1. Calculate the evaluation information of each attribute into the comprehensive evaluation value of each alternative. Taking each of the columns of the collective information decision matrix and the weight set *w* as inputs, the collective information of each alternative can be computed by the proposed *q*ROFDWPPHM operator, shown as follow:

 (45)

1. Calculate the score and accuracy of the comprehensive evaluation value of each alternative. According to Equations (3) and (4), calculate the scores and accuracies of the comprehensive evaluation value of each alternative.
2. Rank all the alternatives and select a proper alternative. According to the comparison rules in Definition 4, a ranking of the alternatives is generated. With the help of the generated ranking, an appropriate alternative can be selected by the decision maker.

# Example, Experiments and Comparisons

In this section, the process of the proposed MAGDM method is firstly illustrated via a practical example. Then a set of experiments are carried out to explore the influence of different parameter values on the aggregation results. Finally, the validity of the method is verified by quantitative comparison with the existing MAGDM methods.

## Example

A MAGDM problem about company location selection Ref. 11 is provided to illustrate the proposed approach. In this example, an investment enterprise wants to invest some money into a company, where there are five possible companies *A*1, *A*2, *A*3, *A*4, *A*5. To make a proper decision, the investment enterprise invited three experts *D*1, *D*2, *D*3 to evaluate the alternatives with respect to four attributes *C*1, *C*2, *C*3, *C*4, where *C*1 denotes the risk analysis, *C*2 denotes the growth analysis, *C*3 denotes the social-political impact analysis, and *C*4 denotes the environmental impact analysis. The relative importance of the four attributes and three decision makers are measured by the weights in 𝜔 = {0.2,0.1,0.3,0.4} and 𝜔 = {0.35,0.40,0.25}, respectively. Assume that the decision makers *D*1, *D*2, *D*3 give the decision matrices of these attribute of the five companies shown in Tables 1-3. In order to make a reasonable decision, the interrelationships between attributes should be considered. Therefore, assume that the attributes are divided into two classes *P*1 = {*C*1, *C*2} and *P*2 = {*C*3, *C*4}, and there is interrelationship between any two attributes in each class. In other words, *k*1 = *k*2 = 2.

Table 1. The q-rung orthopair fuzzy decision matrix M1 given by D1

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *C1* | *C2* | *C3* | *C4* |
| *A*1 | (0.5,0.4) | (0.5,0.4) | (0.2,0.6) | (0.4,0.4) |
| *A*2 | (0.7,0.3) | (0.7,0.3) | (0.6,0.2) | (0.6,0.2) |
| *A*3 | (0.5,0.4) | (0.6,0.4) | (0.6,0.2) | (0.5,0.3) |
| *A*4 | (0.8,0.2) | (0.7,0.2) | (0.4,0.2) | (0.5,0.2) |
| *A*5 | (0.4,0.3) | (0.4,0.2) | (0.4,0.5) | (0.4,0.6) |

Table 2. The q-rung orthopair fuzzy decision matrix M2 given by D2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *C1* | *C2* | *C3* | *C4* |
| *A1* | (0.4,0.5) | (0.6,0.2) | (0.5,0.4) | (0.5,0.3) |
| *A2* | (0.5,0.4) | (0.6,0.2) | (0.6,0.3) | (0.7,0.3) |
| *A3* | (0.4,0.5) | (0.3,0.5) | (0.4,0.4) | (0.2,0.6) |
| *A4* | (0.5,0.4) | (0.7,0.2) | (0.4,0.4) | (0.6,0.2) |
| *A5* | (0.6,0.3) | (0.7,0.2) | (0.4,0.2) | (0.7,0.2) |

Table 3. The q-rung orthopair fuzzy decision matrix M3 given by D3

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *C1* | *C2* | *C3* | *C4* |
| *A1* | (0.4,0.2) | (0.5,0.2) | (0.5,0.3) | (0.5,0.2) |
| *A2* | (0.5,0.3) | (0.5,0.3) | (0.6,0.2) | (0.7,0.2) |
| *A3* | (0.4,0.4) | (0.3,0.4) | (0.4,0.3) | (0.3,0.3) |
| *A4* | (0.5,0.3) | (0.5,0.3) | (0.3,0.5) | (0.5,0.2) |
| *A5* | (0.6,0.2) | (0.6,0.4) | (0.4,0.4) | (0.6,0.3) |

In the following, the proposed method is used to solve the MAGDM problem. The process of the selection consists of the following five steps:

(1) Normalize the decision matrix. Since all attribute are benefit attributes, this step is skipped. That is the normalized decision matrix *Mk*’ is equal to *Mk*, i.e. *Mk*’= *Mk*.

(2) Calculate the evaluation information of decision makers into the collective information. Based on equation (44) and taking the normalized decision matrix *Mk*’ and the weight set 𝜔 as inputs, it can be aggregated the evaluation information of individual decision maker is aggregated into the collective information by the proposed *q*ROFDWPPHM operator (let the parameters *a* =1, *b* =2 and *λ* =1.5, and let the decision makers matrix divided into two classes *P*1 = {*M*1} and *P*2 = {*M*2, *M*3},then, *k*1 = 1 and *k*2 = 2), and the collective decision matrix is presented in Table 4.

Table 4. Collective q-rung orthopair fuzzy decision matrix

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *C1* | *C2* | *C3* | *C4* |
| *A1* | (0.6040,0.5294) | (0.5533,0.6681) | (0.8054,0.3848) | (0.6386,0.5698) |
| *A2* | (0.4503,0.6202) | (0.4212,0.6575) | (0.4673,0.7300) | (0.4400,0.7307) |
| *A3* | (0.6031,0.5133) | (0.6270,0.5159) | (0.5550,0.6818) | (0.7048,0.5429) |
| *A4* | (0.4113,0.6820) | (0.3985,0.7357) | (0.6767,0.6269) | (0.5549,0.7525) |
| *A5* | (0.6239,0.6550) | (0.6199,0.7132) | (0.6635,0.4719) | (0.6147,0.3928) |

(3) Calculate the evaluation information of each attribute into the comprehensive evaluation value of each alternative. Based on Equation (45) and each of the columns of the collective information decision matrix and the weight setas inputs, the evaluation information is aggregated of each attribute into the comprehensive evaluation value by the proposed *q*ROFDWPPHM operator and the comprehensive evaluation value is presented as follows:



(4) Calculate the score and accuracy of the comprehensive evaluation value of each alternative. According to Equations (3) and (4), the score and accuracy of the comprehensive evaluation value of each company is computed. The computed results are shown in Table 5.

Table 5. The calculated scores and accuracies

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Indicator | *A1* | *A2* | *A3* | *A4* | *A5* |
| Score | -0.1265 | 0.2523 | -0.0520 | 0.1816 | -0.0773 |
| Accuracy | 0.8043 | 0.8449 | 0.7868 | 0.7818 | 0.8053 |

(5) Rank all the alternatives and select a proper alternative. On the basis of the calculated results in Table 5, a ranking of the five companies is obtained according to Definition 4:



With the help of the ranking, the company A2 may probably be selected for investment by the investment enterprise.

## Experiments

In the following, the influences of assigning different parameter values on the ranking results of decision making of the example are explored.

1. Experiment 1 was carried out to show the influences of assigning different values to the parameters *q* on the ranking results. The results of the experiment are the scores and rankings of the five alternatives, which are shown in Table 6 (suppose *a*=1, *b*=2, *λ* =1.5 and *p*=3). From the Table 6, it can be found that the ranking will change as the value of the parameters *q* changes. When *q*=1, the ranking is *A*2 *> A*4 *> A*3 *> A*5 *> A*1, the *q*ROFNs will become Atanassov’s IFNs. When *q* = 2, 3, the rankings are both *A*2 *> A*4 *> A*5 *> A*3 *> A*1. If *q* = 2, the *q*ROFNs will become Yager’s PFNs. When *q* = 4,5, the rankings are both *A*2 *> A*4 *> A*5 *> A*1 *> A*3. When *q* =6,7,8, the rankings are all *A*2 *> A*4 *> A*1 *> A*5 *> A*3. Although the rankings have changed, the first and second alternatives remain the same. The assignment of a reasonable *q* depends on the values of attributes because it must satisfy the condition that 0 ≤ *vq* + *μq*≤ 1. From the Table 4，the value of a criterion cannot satisfy *v* + *μ*≤ 1 but *v2* + *μ2*≤ 1, thus the value of *q* in the example should be assigned from 2.

Table 6**.** The results of experiment 1

|  |  |  |
| --- | --- | --- |
| *q* | Scores of the five alternatives | Ranking |
| *q*=2 | S1=-0.0632, S2=0.2373, S3=-0.0244,  S4=0.1682, S5=-0.0155 |  |
| *q*=3 | S1==-0.0211, S2=0.1623, S3=-0.0110,  S4=0.1083, S5=-0.0060 |  |
| *q*=4 | S1==-0.0046, S2=0.1008, S3=-0.0047,  S4=0.0627, S5=0.0061 |  |
| *q*=5 | S1=0.0008, S2=0.0604, S3=-0.0015,  S4=0.0344, S5=0.0034 |  |
| *q*=6 | S1=0.0019, S2=0.0355, S3=--0.0012,  S4=0.0188, S5= 0.0013 |  |
| *q*=7 | S1=0.0023, S2=0.0211, S3=-0.0520,  S4=0.1816, S5=0.0014 |  |
| *q*=8 | S1=0.0009, S2=0.0125, S3=0.0004,  S4=0.0051, S5=0.0008 |  |

1. Experiment 2 was carried out to show the influences of assigning different values to the parameters *p* (*p*>1) on the ranking results. The results of the experiment are the scores and rankings of the five alternatives, which are shown in Table 7 (suppose a=1, b=2, *λ* =1.5 and *q*=2). From the Table 7, it can be found that the rankings and the values of the score function almost remain the same with respect to different parameters *p*, which indicates that using different parameters *p* has no obvious influence on the ranking results for the example.

Table 7**.** The results of experiment 2

|  |  |  |
| --- | --- | --- |
| *p* | Scores of the five alternatives | Ranking |
| *p*=1.1 | S1=-0.0625, S2=0.2373, S3=-0.0243,  S4=0.1682, S5=-0.0152 |  |
| *p*=1.5 | S1=-0.0629, S2=0.2374, S3=-0.0244,  S4=0.1682, S5=-0.0154 |  |
| *p*=2 | S1=-0.0629, S2=0.2373, S3=-0.0244,  S4=0.1682, S5=-0.0154 |  |
| *p*=3 | S1=-0.0632, S2=0.2373, S3=-0.0244,  S4=0.1682, S5=-0.0155 |  |
| *p*=5 | S1=-0.0633, S2=0.2372, S3=-0.0245,  S4=0.1682, S5=-0.0154 |  |
| *p*=10 | S1=-0.0632, S2=0.2373, S3=-0.0245,  S4=0.1683, S5=-0.0154 |  |
| *p*=50 | S1=-0.0631, S2=0.2373, S3=-0.0245,  S4=0.1683, S5=-0.0151 |  |
| *p*=100 | S1=-0.0631, S2=0.2373, S3=-0.0245,  S4=0.1682, S5=-0.0151 |  |

1. Experiment 3 was carried out to show the influences of assigning different values to parameters *a* and *b* on the ranking results. The results of the experiment are the scores and rankings of the five alternatives, which are shown in Table 8 (suppose *λ* =1.5, *q*=2, *p*=3). It can be seen from the Table that although the best choice for all parameters *a* and *b* is *A*2, the ranking results are slightly different. As the parameters *a* and *b* increase, the interrelation between attribute values becomes stronger and stronger. When *a*=5, *b*=5, the best choice has changed from *A*2 to *A*4. Thus, the interaction strength significantly affects the ranking results. In practice, the risk degree of decision makers can be expressed by assigning reasonable parameters *a* and *b*. The greater the parameter, the greater the risk.

Table 8. The results of experiment 3

|  |  |  |
| --- | --- | --- |
| *a and b* | Scores of the five alternatives | Ranking |
| *a=0, b=1* | S1=-0.0408, S2=0.2546, S3=-0.0459,  S4=0.1532, S5=-0.0674 |  |
| *a=1, b=0* | S1=-0.1244, S2=0.2112, S3=-0.0351,  S4= 0.1024, S5=-0.0603 |  |
| *a=1, b=1* | S1=-0.0816, S2=0.2356, S3=-0.0295,  S4=0.1428, S5=-0.0300 |  |
| *a=1, b=0.5* | S1=-0.0953, S2=0.2323, S3=-0.0314,  S4= 0.1262, S5=-0.041 |  |
| *a=0.5, b=1* | S1=-0.0823, S2= 0.2417, S3=-0.037,  S4= 0.1345, S5=-0.0442 |  |
| *a=1, b=3* | S1=-0.0504, S2=0.2379, S3=-0.0193,  S4=0.1878, S5=-0.0064 |  |
| *a=3, b=1* | S1=-0.0621, S2=0.2315, S3=-0.0083,  S4=0.1840, S5=-0.0002 |  |
| *a=3, b=3* | S1=-0.0419, S2= 0.2383, S3=-0.005,  S4=0.2167, S5= 0.0096 |  |
| *a=1, b=5* | S1=-0.034, S2=0.2395, S3=-0.0112,  S4=0.2178, S5=0.0045 |  |
| *a=5, b=1* | S1=-0.0452, S2=0.236, S3= 0.0053,  S4=0.2184, S5= 0.0137 |  |
| *a=5, b=3* | S1=-0.0313, S2=0.2431, S3=0.0059,  S4=0.2429, S5= 0.0185 |  |
| *a=3, b=5* | S1=-0.0294, S2=0.2424, S3=-0.0009,  S4=0.2401, S5= 0.0152 |  |
| *a=5, b=5* | S1=-0.0222, S2=0.2470, S3=0.0076,  S4=0.261, S5=0.0214 |  |

(4) Experiment 4 was carried out to show the influences of assigning different values to parameters *λ* on the ranking results. The results of the experiment are the scores and rankings of the five alternatives, which are shown in Table 9 (suppose *a*=1, *b*=2, *q*=2, *p*=3). As can be seen from the table, the rankings remain the same when *λ* ≥5) and the scores of *A*4 firstly gradually increase and then gradually decrease and the scores of *A*1*, A*2*, A*3 and *A*5gradually increase as *λ* increases. This indicates that the parameter *λ* (*λ* ≤5) can be regarded as the “decision maker’s attitude.” The smaller the value of the parameter *λ* is, the more pessimistic the decision makers are and vice versa. Thus, it can be assigned *λ* = 5 for optimism, *λ* = 4 for slightly optimistic, *λ* =3 for neutral, *λ* =2 for slightly pessimistic and *λ* =1 for pessimistic of decision makers. As decision makers’ attitudes change, the ranking keep changing but the best alternatives is always *A*2.

Table 9**.** The results of experiment 4

|  |  |  |
| --- | --- | --- |
| *λ* | Scores of the five alternatives | Ranking |
| *λ*=0.5 | S1=-0.2228, S2=0.0859, S3=-0.1331,  S4= 0.0348, S5=-0.1869 |  |
| *λ*=1 | S1=-0.1068, S2= 0.2021, S3=-0.0419,  S4= 0.135, S5=-0.0607 |  |
| *λ*=2 | S1=-0.0361, S2= 0.2518, S3=-0.0201,  S4=0.185, S5=0.007 |  |
| *λ*=3 | S1=-0.0016, S2=0.2616, S3=-0.0152,  S4=0.2033, S5=0.025 |  |
| *λ*=4 | S1=0.0212, S2= 0.2635, S3=-0.0115],  S4=0.2118, S5=0.0312 |  |
| *λ*=5 | S1= 0.0383, S2= 0.2636, S3=-0.009,  S4=0.2144, S5=0.035 |  |
| *λ*=10 | S1=0.0865, S2=0.2646, S3=-0.0043,  S4= 0.2136, S5=0.0501 |  |
| *λ*=20 | S1=0.1221, S2=0.2673, S3=-0.0023,  S4=0.2121, S5=0.0605 |  |
| *λ*=50 | S1=0.1449, S2=0.269, S3=-0.0011,  S4= 0.2107, S5=0.0662 |  |
| *λ*=100 | S1=0.1523, S2=0.2693, S3=-0.0004,  S4=0.2103, S5=0.0681 |  |

## Comparisons

In the following, to verify the effectiveness of the proposed method and to explore its advantages, four representative methods are selected to address the example and to compare the proposed method. The comparison results are shown in Table 10 (suppose *λ* = 1.5, *q*=2, *p*=3, *a=*1 and *b=*2).

1. Compared with Liu et al.’s method from Ref. 11 based on intuitionistic fuzzy weight Archimedean Heronian aggregation (IFWAHA) operator: Form Table 10, it can be seen that the proposed method gets the same first three alternatives as Liu et al.’s method, even though the rankings are slightly different. This shows the effectiveness and validity of the proposed method. In the following, the characteristics of the proposed method and Liu et al.’s method are compared and the comparable characteristics are the expressiveness of fuzzy information, whether takes into account the interrelationships between different attribute, and whether considers the attitudes of the decision makers.
2. Expressiveness: The proposed method was presented based on *q*ROFNs, whereas Liu et al.’s method based on IFNs which is special *q*ROFNs (*q*=1). The expressiveness of fuzzy information of Liu et al.’s method is limited to IFNs, whereas the proposed method can express fuzzy information more widely via assigning different *q*. Thus, the proposed method is more flexible for MAGDM methods.
3. Interrelationships: The proposed method were presented based on PHM operator, whereas Liu et al.’s method based on the HM operator. Both the HM and PHM operators have the capability to describe the interrelationships between different attributes, but the PHM operator inherit the features of HM and consideres the interrelationships between the attribute and partitions them into some different parts. In addition to this, the proposed method also uses the PA operator, which can reduce the influence of unreasonable data. Thus, the proposed method can obtain more reliable aggregation results via considering the interrelationship and partitioned attributes.
4. Attitudes: The attitudes of a decision maker usually have important influence on the results of decision making from Ref. 56. In the Liu et al.’s method, the attitudes are reflected by a parameter (*λ*) in the Hamacher operator. As the value of *λ* increases, the attitude will shift from pessimistic to optimistic. Although *λ* can stands for the attitudes of a decision maker, how to set a desirable value to is *λ* also not provided. In the proposed method, *λ* =5 for optimism, *λ* =4 for slightly optimistic, *λ* =3 for neutral, *λ* =2 for slightly pessimistic and *λ* =1 for pessimistic of decision makers. Thus, the proposed method can be used different values of *λ* to set different levels of attitudes of decision makers.
5. Compared with Zhang et al.’s method from Ref. 40 based on intuitionistic fuzzy Frank power aggregation (IFFPA) operator: Form the Table 10, it can be seen that the proposed method gets the same first three alternatives as Zhang et al.’s method, even though the ranking of *A*1 and *A*3 are opposite. In the following, the characteristics of the proposed method and Zhang et al.’s method are compared and the comparable characteristics are the expressiveness of fuzzy information and whether takes into account the interrelationships between different attributes.
6. Expressiveness: Like comparison (1), the expressiveness of fuzzy information of Zhang et al.’s method is also limited to IFNs, whereas the proposed method can express fuzzy information more widely via assigning different *q*. Thus, the proposed method is more flexible for MAGDM methods.
7. Interrelationships: The proposed method were presented based on the PHM operator which have the capability to describe the interrelationships between different attributes and the PA operator which can reduce the influence of unreasonable data and consider the relationships between the input values of attributes, whereas Zhang et al.’s method only based on the PA operator. Thus, the proposed method can not only consider the relationships between the input arguments but also obtain more reliable aggregation results via considering the interrelationships and partitioned these attributes.
8. Compared with Wei et al.’s method from Ref. 47 based on *q*-rung orthopair fuzzy weight geometric Heronian mean (*q*ROFWGHM) operator: As can be seen from the Table, the proposed method gets the same first three alternatives as the Wei et al.’s method, even though the ranking of *A*1 and *A*3 are opposite. However, the GHM operator assumes that each attribute is related to all the other attribues and that there are some decision cases in reality that do not satisfy these preconditions. Like the above investment selection example, the attributes *C*1 (the risk analysis) and *C*2 (the growth analysis) have no relationship with the attributes *C*3 (the social-political impact analysis) and *C*4 (the environmental impact analysis). Thus, the proposed method is more suiTable than Wei et al.’s method for dealing with MAGDM problems.
9. Compared with Liu et al.’s method from Ref. 48 based on *q*-rung orthopair fuzzy weight partitioned Heronian mean (*q*ROFWPHM) operator: Form Table 10, it is easy to find that the proposed method gets different results from Liu et al.’s method in the first two rankings, while the other rankings are the same. There are two reasons for this: operational rules and operators. Firstly, Liu et al.’s method is based on simple operational rules of qROFNs, whereas the operational rules of the proposed method are based on DTT, which makes the information aggregation flexible as there is parameter *λ*. Secondly, the proposed qROFDWPPHM operator is combined with the PA operator on the basis of the PHM operator, whereas the Liu et al.’s method is only based on the PHM operator. Thus, the proposed method is more powerful than Liu et al.’s method for considering interrelationships of the aggregated arguments.

Table 10. The results of comparisons

|  |  |  |
| --- | --- | --- |
| Operator | Scores of the five alternatives | Ranking |
| IFWAHA11 | S1=0.1800, S2=0.4040, S3=0.0880,  S4=0.3030, S5=0.2800 |  |
| IFFPA41 | S1=0.5570, S2=0.6860, S3=0.5180,  S4=0.6390, S5=0.6120 |  |
| *q*ROFWGHM47 | S1=-0.2212, S2=0.4559, S3=0.1290,  S4=0.3248, S5=-0.2866 |  |
| *q*ROFWPHM48 | S1=-0.8272, S2=0.1352, S3=-0.8167,  S4=0.1490, S5=-0.8064 |  |
| *q*ROFDWPPHM | S1=-0.0632, S2=0.2373, S3=-0.0244,  S4=0.1682, S5=-0.0155 |  |

## Further comparative analysis considering the interrelationships between attributes

In the previous subsection, the proposed method is compared with some current methods, and the advantages of the proposed method are analyzed. However, it will be almost the same ranking result, and it does not reflect the advantages of the proposed method. In this section, Liu et al.’s method from Ref. 11 was selected. By modifying specific inputs to compare with the actual ranking in the example. It can be known the change of ranking result by reducing other evaluation values of alternative *A*2. For example, the value *μ1 21* and *μ1 22* are modified from 0.7 to 0.01 and the value *v1 21* and *v1 22* from 0.3 to 0.99. That is the evaluation values of *A*2 with respect to the attributes *C*1 and *C*2 are reduced from (0.7,0.3) to (0.01,0.99). In order to clearly display the effectiveness of the proposed method, Liu et al.’s method from 11 and Liu’s method from Ref. 39 (where two of them have also been compared in Ref. 11) are adopted for this experiment, and Liu et al.’s method and the proposed method are based on the HM operators and Liu’s method is based on the Hamacher operators. The former considers interrelationships between attributes, while the latter does not. The score and the ranking results based on these three methods are shown in Tables 11 and Tables 12.

Table 11**.** The score functions based on these therr methods

|  |  |  |  |
| --- | --- | --- | --- |
| () | The Proposed Method | Liu et al.’s Method11 | Liu’s Method39 |
| (0.7,0.3) | S1=-0.0816, S2=0.2356,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.327,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.354,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.6,0.4) | S1=-0.0816, S2= 0.2227,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.306,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.333,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.5,0.5) | S1=-0.0816, S2=0.2110,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.288,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.309  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.4,0.6) | S1=-0.0816, S2=0.2017,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.272,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.281,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.3,0.7) | S1=-0.0816, S2=0.1956,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.258,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.247,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.2,0.8) | S1=-0.0816, S2=0.1923,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.246,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0202,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.1,0.9) | S1=-0.0816, S2=0.1911,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.236,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.130,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.05,0.95) | S1=-0.0816, S2=0.1908,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.232,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=0.063,  S3=-0.020, S4=0.227,  S5=0.153 |
| (0.01,0.99) | S1=-0.0816, S2= 0.1907,  S3=-0.0295, S4=0.1428,  S5=-0.03 | S1=-0.091, S2=0.228,  S3=0.023, S4=0.234,  S5=0.184 | S1=0.065, S2=-0.078,  S3=-0.020, S4=0.227,  S5=0.153 |

It can be seen from Table 11 that as the degree of membership decreases and the degree of non-membership increases, the score *S*2 of alternative *A*2 decreases gradually, but the rate of decrease is different. Among all three methods, the proposed method reduces the rate is slowest, and the reduction is only 0.0449, while Liu et al.'s method has an amplitude of 0.099, and the Liu's method amplitude is 0.432. This can be explained by the interrelationships between attributes. The proposed method is based on the PA operator and the PHM operator, which can well handle the relationships of the input data and classify it considering the correlation between the attributes, while the Liu's method only uses the HM operator. Although the correlation between attributes can also be considered, it is not comprehensive enough, and Liu's method does not consider the correlation. This shows that the proposed method is more reasonable than the other two methods.

Table 12**.** The ranking results based on these three methods

|  |  |  |  |
| --- | --- | --- | --- |
| () | The proposed method | Liu et al.’s method11 | Liu’s method39 |
| (0.7,0.3) |  |  |  |
| (0.6,0.4) |  |  |  |
| (0.5,0.5) |  |  |  |
| (0.4,0.6) |  |  |  |
| (0.3,0.7) |  |  |  |
| (0.2,0.8) |  |  |  |
| (0.1,0.9) |  |  |  |
| (0.05,0.95) |  |  |  |
| (0.01,0.99) |  |  |  |

As can be seen from Table 12, the proposed method maintains the same ranking result in the change of attribute values. Liu et al. 's method starts to change when the attribute value changes to (0.05,0.95), and the ranking of alternative *A*2 is backward. However, Liu's method starts to change when the attribute value changes to (0.2,0.8). It can be inferred intuitively from Table 2 and Table 3 that when the attribute values in the fuzzy matrix of decision maker *D*1 is changed from (0.7,0.3) to (0.01,0.99), the corresponding attribute values of the other two matrices are (0.5,0.4) and (0.5,0.3), which indicates that this set of fuzzy numbers is unreasonable. It is further demonstrated that the latter two methods are greatly affected by unreasonable data, and the proposed method has a strong ability to process unreasonable input data, that is, the correlation between attribute values.

# Conclusions

In this paper, a set of *q*-rung orthopair fuzzy operational rules was discussed based on the Dombi t-conorm and t-norm. Then, the *q*-rung orthopair fuzzy Dombi partitioned Heronian Mean operator and the *q*-rung orthopair fuzzy Dombi weight partitioned Heronian Mean operator were present. In order to reduce the negative impact of unreasonable attribute values on aggregated results, the *q*-rung orthopair fuzzy Dombi power partitioned Heronian mean (*q*ROFDPPHM) operator and the *q*-rung orthopair fuzzy Dombi weight power partitioned Heronian mean (*q*ROFDWPHM) operator are presumed via combining the partitioned Heronian Mean (PHM) with the power average (PA) operator based on *q*-ROFSs. Moreover, a new multiple attribute group decision making (MAGDM) method based on the proposed operators are also proposed. A practical instance and a set of experiments to illustrate the proposed approach and a qualitative comparative analysis is conducted to demonstrate the effectiveness and feasibility of the proposed approach. The results of the experiments and comparisons show that the proposed method is feasible, effective, and flexible. Comparing with the existing MAGDM methods, the proposed MAGDM method has the following advantages:

1. It can fully consider interrelationships of aggregated arguments;
2. It can provide a generalization of most existing operational rules for *q*ROFNs.
3. It can reduce the negative impact of unreasonable attribute values on aggregated results.

In the future studies, some new special cases of DTT will be studied and extended to the power partitioned Heronian aggregation operator based on qROFNs, such as power partitioned geometry Heronian aggregation operator. And we will apply the proposed operators and methods to some practical applications, such as recommendation system, performance evaluation, supplier selection evaluation and pattern recognition system.

# Acknowledgments

This research was funded by the National Natural Science Foundation of China (No. 61562016; No. 51765012), the Guangxi Colleges and Universities Key Laboratory of Intelligent Processing of Computer Images and Graphics (No. GIIP1805), and the Innovation Key Project of Guangxi Province (No. AA18118039-2).

# APPENDIX A. PROOF OF THEOREM 5.

**Proof.** According to Definition 6, it is obtained:





Let ,,, and , then





Thereafter,















Since



Thus, the proof of Theorem 5 is completed. 

# References

1. H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera and B. De Baets, A historical account of types of fuzzy sets and their relationships, *IEEE Transactions on Fuzzy Systems* **24** (2016) 179‒194.
2. L.A. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965) 338–353.
3. K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986) 87–96.
4. K.T. Atanassov, More on intuitionistic fuzzy sets, *Fuzzy Sets Syst.* **33** (1989) 37–45.
5. S.K. De, R. Biswas and A.R. Roy, Some operations on intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **114** (2000) 477‒484.
6. W.Z. Wang and X.F. Liu, Intuitionistic fuzzy information aggregation using Einstein operations, *IEEE Trans. Fuzzy Syst.* **20** (2012) 923–938.
7. E.B. Jamkhaneh and H. Garg, Some new operations over the generalized intuitionistic fuzzy sets and their application to decision-making process. *Granular Computing* **3** (2018) 111‒122.
8. Z.S. Xu and R.R. Yager, Intuitionistic fuzzy Bonferroni means, *IEEE Trans. Syst*. *Man Cybern. B Cybern* **41** (2011) 568–578.
9. M. Xia, Z. Xu and B. Zhu, Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm, *Knowledge-Based Systems* **31** (2012) 78‒88.
10. M.M. Xia, Z.S. Xu and B. Zhu, Geometric Bonferroni means with their application in multi-criteria decision making. *Knowl. -Based Syst.* **40** (2013) 88–100.
11. P. Liu and S.M. Chen, Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers, *IEEE Transactions on Cybernetics* **47** (2017) 2514‒2530.
12. P. Liu, J. Liu, and S.M. Chen, Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making, *Journal of the Operational Research Society* **69** (2018) 1‒24.
13. Z. Xu, Intuitionistic preference relations and their application in group decision making, *Information Sciences* **177 (**2007) 2363‒2379.
14. H. Liao and Z. Xu, Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency, *IEEE Transactions on Fuzzy Systems* **22** (2014) 1669‒1681.
15. Z. Zhang and W. Pedrycz, Models of mathematical programming for intuitionistic multiplicative preference relations, *IEEE Transactions on Fuzzy Systems* **25** (2017) 945‒957.
16. Q. Lei and Z. Xu, Fundamental properties of intuitionistic fuzzy calculus, *Knowledge-Based Systems* **76** (2015) 1‒16.
17. Q. Lei, and Z. Xu, Chain and substitution rules of intuitionistic fuzzy calculus, *IEEE Transactions on Fuzzy Systems* **24** (2016) 519‒529.
18. Z. A and Z. Xu, Multiple Definite Integrals of Intuitionistic Fuzzy Calculus and Isomorphic Mappings, *IEEE Transactions on Fuzzy Systems* **26** (2018) 670‒680.
19. P. Liu, S.M. Chen and J. Liu, Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators, *Inf. Sci.* **411** (2017) 98–121.
20. P. Liu, Multiple attribute decision-making methods based on normal intuitionistic fuzzy interaction aggregation operators, *Symmetry* **9** (2017) 261.
21. P. Liu, T. Mahmood and Q. Khan, Multi-attribute decision-making based on prioritized aggregation operator under hesitant intuitionistic fuzzy linguistic environment, *Symmetry* **9** (2017) 270.
22. R.R. Yager, Pythagorean membership grades in multicriteria decision-making, *IEEE Trans. Fuzzy Syst.* **22** (2014) 958–965
23. R.R. Yager and A.M. Abbasov, Pythagorean membership grades, complex numbers, and decision-making, *Int. J. Intell. Syst.* **28** (2013) 436–452.
24. X. Zhang and Z. Xu, Extension of TOPSIS to Multiple Criteria Decision-Making with Pythagorean Fuzzy Sets, *Int. J. Intell. Syst.* **29 (**2014) 1061–1078.
25. P. Ren, Z.Xu and X. Gou, Pythagorean fuzzy TODIM approach to multi-criteria decision-making, *Appl. Soft Comput.* **42** (2016) 246–259.
26. X. Peng and Y. Yang, Some results for Pythagorean fuzzy sets, *Int. J. Intell. Syst.* **30** (2015) 1133–1160.
27. H. Garg, A new generalized Pythagorean fuzzy information aggregation using einstein operations and its application to decision making, *Int. J. Intell. Syst*. **31** (2016) 886–920.
28. S. Dick, R.R. Yager and O. Yazdanbakhsh, On Pythagorean and complex fuzzy set operations, *IEEE Transactions on Fuzzy Systems* **24** (2016) 1009‒1021.
29. D. Liang, Z. Xu, D.Liu and Y. Wu, Method for three-way decisions using ideal TOPSIS solutions at Pythagorean fuzzy information, *Information Sciences* **435** (2018) 282‒295.
30. R.R. Yager, Generalized orthopair fuzzy sets, *IEEE Trans. Fuzzy Syst.* **25** (2017) 1222–1230.
31. X. Peng, J. Dai and H. Garg, Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function, *Int. J. Intell. Syst* **33** (2018) 2255-2282, doi:10.1002/int.22028.
32. W.S. Du, Minkowski-type distance measures for generalized orthopair fuzzy sets, *Int. J. Intell. Syst*. **33** (2018) 802–817.
33. L. Li, R. Zhang, J. Wang, X. Shang and Bai, K. A novel approach to multi-attribute group decision-making with q-rung picture linguistic information, *Symmetry* **10** 2018 172.
34. P. Liu and P. Wang, Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Intell. Syst.* **33** (2018) 259–280.
35. P. Liu and J. Liu, Some q-rung orthopai fuzzy Bonferroni mean operators and their application to multi-attribute group decision making, *Int. J. Intell. Syst*. **33** (2018) 315–347.
36. G. Wei, H. Gao and Y. Wei, Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making, *Int. J. Intell. Syst*. **33** (2018) 1426–1458.
37. P. Liu and P. Wang, Multiple-attribute decision-making based on Archimedean Bonferroni Operators of q-rung orthopair fuzzy numbers, *IEEE Trans. Fuzzy Syst.* 2018, doi: 10.1109 /TFUZZ. 2018.2826452.
38. P. Liu, Y. Li and Y.Chen, Some generalized Einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making, *Scientia Iranica, Transaction E, Industrial Engineering*  **22** (2015) 2684-2701.
39. P. Liu, Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Transactions on Fuzzy Systems* **22 (**2014) 83-97.
40. X. Zhang, P. Liu and Y. Wang, Multiple attribute group decision making methods based on intuitionistic fuzzy frank power aggregation operators, *Journal of Intelligent & Fuzzy Systems* **29** (2015) 2235-2246.
41. R.R. Yager, The power average operator. *IEEE Trans. Syst. Man Cybern. A*  **31** (2001) 724–731.
42. G. Wei, C. Wei, J. Wang, H. Gao and Wei, Y. Some q‐rung orthopair fuzzy maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization, *International Journal of Intelligent Systems*  **34** (2018) 50-81.
43. K. Bai, X. Zhu, J. Wang, R. Zhang, Some partitioned Maclaurin symmetric mean based on q-rung orthopair fuzzy information for dealing with multi-attribute group decision making, *Symmetry* **10 (**2018) 383.
44. P. Liu, S.M. Chen and P. Wang, Multiple-attribute group decisionmaking based on q-rung orthopair fuzzy power Maclaurin symmetric mean operators, *IEEE Transactions on Systems, Man, and Cybernetics: Systems* (2018) doi: 10.1109/TSMC.2018.2852948.
45. P. Liu and J. Liu, Some q‐rung orthopai fuzzy Bonferroni mean operators and their application to multi‐attribute group decision making, *International Journal of Intelligent Systems*  **33** (2018) 315‒347.
46. W. Yang and Y. Pang, New q‐rung orthopair fuzzy partitioned Bonferroni mean operators and their application in multiple attribute decision making, *International Journal of Intelligent Systems* **34** (2019) 439-476.
47. G. Wei, H. Gao, and Y. Wei, Some q‐rung orthopair fuzzy Heronian mean operators in multiple attribute decision making, *International Journal of Intelligent Systems* **33** (2018) 1426-1458.
48. Z. Liu, S. Wang and P. Liu, Multiple attribute group decision making based on q-rung orthopair fuzzy Heronian mean operators, *International Journal of Intelligent Systems*, **33** (2018) 2341-2363.
49. D.J. Yu, Y.Y. Wu and T. Lu, Interval-valued intuitionistic fuzzy Heronian mean operators and their application in multi-criteria decision making, *Afr. J. Bus. Manag.* **6** (2012) 4158–4168.
50. P. Liu, J. Liu and Merigo, J.M. Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making, *Appl. Soft Comput*. **62** (2018) 395–422.
51. J. Dombi, A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators, *Fuzzy Sets Syst.* **8** (1982) 149–163.
52. P. Liu, J. Liu and S.M. Chen, Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision-making, *J. Oper. Res. Soc.* **69** (2018) 1–24.
53. X.R. He, Typhoon disaster assessment based on Dombi hesitant fuzzy information aggregation operators, *Nat. Hazards* **90** (2018) 1153–1175.
54. J. Chen and J. Ye, Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making, *Symmetry* **9** (2017) 82.
55. X. Zhang, P. Liu, and Y. Wang, Multiple attribute group decision making methods based on intuitionistic fuzzy Frank power aggregation operators, *J. Intell. Fuzzy Syst.* **29** (2015) 2235–2246.
56. T.Y. Chen, Optimistic and pessimistic decision making with dissonance reduction using interval-valued fuzzy sets, *Information Sciences* **181** (2011) 479–502.
57. S. Sykora, Mathematical means and averages: Generalized Heronian means (Stan’s Library, Milan, 2009).
58. E.P. Klement and R. Mesiar, Eds., Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms (Elsevier, Amsterdam, 2005).