Geometric Transform & Rotation

1. Rotation

There are four methods to represent a rotation: Euler angle, Quaternion Rotation vector/matrix,.

(1). Euler angle

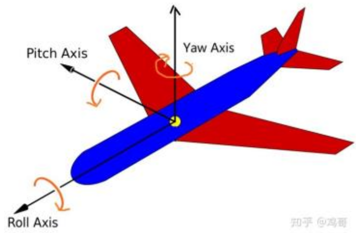


Fig 1. Z-Roll，X-Pitch，Y-Yaw

The rotations around Cartesian coordinate axis *X, Y, Z* are usually called Pitch, Yaw and Roll, which are note as Euler angle of α, β, γ in equations bellow. There are 12 types of Euler angle rotational sequence, including 6 types of Tait-Bryan Angle (*XYZ, XZY, YXZ, YZX, ZXY, ZYX*), and 6 types of Proper Euler Angle (*XYX, YXY, XZX, ZXZ, YZY, ZYZ*). In the following, we will mainly discuss the rotations of Euler angle sequenced *XYZ*.

(2). Quaternion

Irish mathematician Sir William Rowan Hamilton (1805-1865) discovered the algebra of quaternions, and found that imaginary elements of *i, j, k* can be used to express the unit vectors *x, y, z* of Cartesian coordinate systems.

$$\begin{aligned}&\boldsymbol{q} = w + xi + yj + zk, \text{note as:}[w, x, y, z]^T\\&\|\boldsymbol{q}\| = w^2 + x^2 + y^2 + z^2 = 1\end{aligned}$$

The imaginary unit *i, j, k* conforms to the following equation.

$$\begin{aligned}&i^2 = j^2 = k^2 = ijk = -1\\&i^{-1} = -i, j^{-1} == -j, k^{-1} = -k\\&ij = k \rightarrow ji = -k\\&jk = i \rightarrow kj = -i\\&ki = j \rightarrow ik = -j\end{aligned}$$

Which is similar with the operation of cross product of Cartesian coordinate unit vector ***x, y, z***.

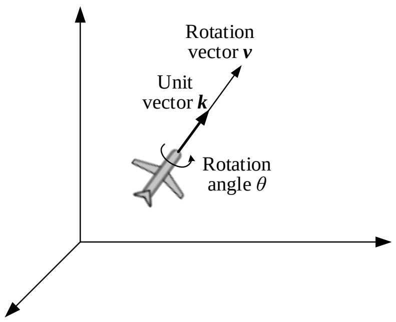
$$\begin{matrix}\boldsymbol{x} \times \boldsymbol{y} = \boldsymbol{z} \rightarrow \boldsymbol{y} \times \boldsymbol{x} = -\boldsymbol{z}\\\boldsymbol{y} \times \boldsymbol{z} = \boldsymbol{x} \rightarrow \boldsymbol{z} \times \boldsymbol{y} = -\boldsymbol{x}\\\boldsymbol{z} \times \boldsymbol{x} = \boldsymbol{y} \rightarrow \boldsymbol{x} \times \boldsymbol{z} = -\boldsymbol{y}\end{matrix}\quad \text{, Unit vector:} \begin{cases}\boldsymbol{x} = [1, 0, 0]^T\\\boldsymbol{y} = [0, 1, 0]^T\\\boldsymbol{z} = [0, 0, 1]^T\end{cases}$$

The add and product operation between quaternions as below.

$$\begin{aligned}&\boldsymbol{q}_1 + \boldsymbol{q}_2 = [w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]^T\\&\boldsymbol{q}_1\boldsymbol{q}_2 = (w_1 + x_1 i + y_1 j + z_1 k)(w_2 + x_2 i + y_2 j + z_2 k)\\&\qquad = \begin{bmatrix}w_1 w_2 - x_1 x_2 - y_1 y_2 - z1_z 2\\w_1 x_2 + x_1 w_2 + y_1 z_2 - z1_y 2\\w_1 y_2 - x_1 z_2 + y_1 w_2 + z1_x 2\\w_1 z_2 + x_1 y_2 - y_1 x_2 + z1_w 2\end{bmatrix}\end{aligned}$$

(3). Rotation vector

Rotation vector is used to indicate the direction of rotation axis, and vector length equal to the rotation angle.



(4). Rotation matrix

Rotation matrix is the most widely used method to implement a rotation by using a matrix multiplication, such as rotations of Euler angle as below.

$$\boldsymbol{R}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos_\alpha & -sin_\alpha & 0 \\ 0 & sin_\alpha & cos_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}(\beta) = \begin{bmatrix} cos_\beta & 0 & sin_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -sin_\beta & 0 & cos_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}(\gamma) = \begin{bmatrix} cos_\gamma & -sin_\gamma & 0 & 0 \\ sin_\gamma & cos_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(5). Convertion