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Fréchet space

In functional analysis and related areas of mathematics, **Fréchet spaces**, named after Maurice Fréchet, are special topological vector spaces. They are generalizations of Banach spaces (normed vector spaces that are complete with respect to the metric induced by the norm). All Banach and Hilbert spaces are Fréchet spaces. Spaces of infinitely differentiable functions are typical examples of Fréchet spaces, many of which are typically *not* Banach spaces.

A Fréchet space X is defined to be a locally convex metrizable topological vector space (TVS) that is complete as a TVS,[1] meaning that every Cauchy sequence in X converges to some point in X (see footnote for more details).[note 1]

> **Important note**: Not all authors require that a Fréchet space be locally convex (discussed below).

The topology of every Fréchet space is induced by some translation-invariant complete metric. Conversely, if the topology of a locally convex space X is induced by a translation-invariant complete metric then X is a Fréchet space.

Fréchet was the first to use the term "Banach space" and Banach in turn then coined the term "Fréchet space" to mean a complete metrizable topological vector space, without the local convexity requirement (such a space is today often called an "F-space").[1] The local convexity requirement was added later by Nicolas Bourbaki.[1] It is important to note that a sizable number of authors (e.g. Schaefer) use "F-space" to mean a (locally convex) Fréchet space while others do not require that a "Fréchet space" be locally convex. Moreover, some authors even use "*F*-space" and "Fréchet space" interchangeably. When reading mathematical literature, it is recommended that a reader always check whether the book's or article's definition of "*F*-space" and "Fréchet space" requires local convexity.[1]

Definitions

Fréchet spaces can be defined in two equivalent ways: the first employs a translation-invariant metric, the second a countable family of seminorms.

Invariant metric definition

A topological vector space X is a **Fréchet space** if and only if it satisfies the following three properties:

1. It is locally convex.[note 2]
2. Its topology ***can*** be induced by a translation-invariant metric, that is, a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$. This means that a subset U of X is open if and only if for every $u \in U$ there exists an $r > 0$ such that $\{v : d(v, u) < r\}$ is a subset of U.
3. Some (or equivalently, every) translation-invariant metric on X inducing the topology of X is complete.

- Assuming that the other two conditions are satisfied, this condition is equivalent to X being a complete topological vector space, meaning that X is a complete uniform space when it is endowed with its canonical uniformity (this canonical uniformity is independent of any metric on X and is defined entirely in terms of vector subtraction and X's neighborhoods of the origin; moreover, the uniformity induced by any (topology-defining) translation invariant metric on X is identical to this canonical uniformity).

Note there is no natural notion of distance between two points of a Fréchet space: many different translation-invariant metrics may induce the same topology.

Countable family of seminorms definition

The alternative and somewhat more practical definition is the following: a topological vector space X is a **Fréchet space** if and only if it satisfies the following three properties:

1. It is a Hausdorff space.
2. Its topology may be induced by a countable family of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}_0}$. This means that a subset $U \subseteq X$ is open if and only if for every $u \in U$ there exist $K \geq 0$ and $r > 0$ such that $\{v \in X : \|v-u\|_k < r \text{ for all } k \leq K\}$ is a subset of U.
3. It is complete with respect to the family of seminorms.

A family $\mathcal{P}$ of seminorms on X yields a Hausdorff topology if and only if[2]

$$\bigcap_{\|\cdot\|\in\mathcal{P}} \{x \in X : \|x\| = 0\} = \{0\}.$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x in the Fréchet space defined by a family of seminorms if and only if it converges to x with respect to each of the given seminorms.

As webbed Baire spaces

Theorem[3] (de Wilde 1978)—A topological vector space X is a Fréchet space if and only if it is both a webbed space and a Baire space.

Comparison to Banach spaces

In contrast to Banach spaces, the complete translation-invariant metric need not arise from a norm. The topology of a Fréchet space does, however, arise from both a total paranorm and an *F*-norm (the *F* stands for Fréchet).

Even though the topological structure of Fréchet spaces is more complicated than that of Banach spaces due to the potential lack of a norm, many important results in functional analysis, like the open mapping theorem, the closed graph theorem, and the Banach–Steinhaus theorem, still hold.

Constructing Fréchet spaces

Recall that a seminorm $\|\cdot\|$ is a function from a vector space X to the real numbers satisfying three properties. For all $x, y \in X$ and all scalars c,

$$\|x\| \geq 0$$

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\|c \cdot x\| = |c|\|x\|$$

If $\|x\| = 0 \iff x = 0$, then $\|\cdot\|$ is in fact a norm. However, seminorms are useful in that they enable us to construct Fréchet spaces, as follows:

To construct a Fréchet space, one typically starts with a vector space X and defines a countable family of seminorms $\|\cdot\|_k$ on X with the following two properties:

- if $x \in X$ and $\|x\|_k = 0$ for all $k \geq 0$, then $x = 0$;
- if $x_\bullet = (x_n)_{n=1}^\infty$ is a sequence in X which is Cauchy with respect to each seminorm $\|\cdot\|_k$, then there exists $x \in X$ such that $x_\bullet = (x_n)_{n=1}^\infty$ converges to x with respect to each seminorm $\|\cdot\|_k$.

Then the topology induced by these seminorms (as explained above) turns X into a Fréchet space; the first property ensures that it is Hausdorff, and the second property ensures that it is complete. A translation-invariant complete metric inducing the same topology on X can then be defined by

$$d(x,y) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|x-y\|_k}{1+\|x-y\|_k} \qquad x, y \in X.$$

The function $u \mapsto \frac{u}{1+u}$ maps $[0,\infty)$ monotonically to $[0,1)$, and so the above definition ensures that $d(x,y)$ is "small" if and only if there exists K "large" such that $\|x-y\|_k$ is "small" for $k = 0, \ldots, K$.

Examples

From pure functional analysis

- Every Banach space is a Fréchet space, as the norm induces a translation-invariant metric and the space is complete with respect to this metric.
- The space $\mathbb{R}^\omega$ of all real valued sequences (also denoted $\mathbb{R}^{\mathbb{N}}$) becomes a Fréchet space if we define the k-th seminorm of a sequence to be the absolute value of the k-th element of the sequence. Convergence in this Fréchet space is equivalent to element-wise convergence.

From smooth manifolds

- The vector space $C^\infty([0,1])$ of all infinitely differentiable functions $f:[0,1]\to\mathbb{R}$ becomes a Fréchet space with the seminorms

  $$\|f\|_k = \sup\{|f^{(k)}(x)| : x\in[0,1]\}$$

  for every non-negative integer k. Here, $f^{(k)}$ denotes the k-th derivative of f, and $f^{(0)}=f$. In this Fréchet space, a sequence $(f_n)\to f$ of functions converges towards the element $f\in C^\infty([0,1])$ if and only if for every non-negative integer $k\geq 0$, the sequence $\left(f_n^{(k)}\right)\to f^{(k)}$ converges uniformly.
- The vector space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions $f:\mathbb{R}\to\mathbb{R}$ becomes a Fréchet space with the seminorms

  $$\|f\|_{k,n} = \sup\{|f^{(k)}(x)| : x\in[-n,n]\}$$

  for all integers $k,n\geq 0$. Then, a sequence of functions $(f_n)\to f$ converges if and only if for every $k,n\geq 0$, the sequences $\left(f_n^{(k)}\right)\to f^{(k)}$ converge compactly.
- The vector space $C^m(\mathbb{R})$ of all m-times continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ becomes a Fréchet space with the seminorms

  $$\|f\|_{k,n} = \sup\{|f^{(k)}(x)| : x\in[-n,n]\}$$

  for all integers $n\geq 0$ and $k=0,\ldots,m$.
- If M is a compact C^∞-manifold and B is a Banach space, then the set $C^\infty(M,B)$ of all infinitely-often differentiable functions $f:M\to B$ can be turned into a Fréchet space by using as seminorms the suprema of the norms of all partial derivatives. If M is a (not necessarily compact) C^∞-manifold which admits a countable sequence K^n of compact subsets, so that every compact subset of M is contained in at least one K^n, then the spaces $C^m(M,B)$ and $C^\infty(M,B)$ are also Fréchet space in a natural manner. As a special case, every smooth finite-dimensional *complete manifold* M can be made into such a nested union of compact subsets: equip it with a Riemannian metric g which induces a metric $d(x,y)$, choose $x\in M$, and let

  $$K_n = \{y\in M : d(x,y)\leq n\}\ .$$

  Let X be a compact C^∞-manifold and$V$ a vector bundle over X. Let $C^\infty(X,V)$ denote the space of smooth sections of V over X. Choose Riemannian metrics g and connections D, which are guaranteed to exist, on the bundles TX and V. If s is a section, denote its j^{th} covariant derivative by $D^j s$. Then

  $$\|s\|_n = \sum_{j=0}^{n} \sup_{x\in M} |D^j s|_g$$

  (where $|\cdot|_g$ is the norm induced by the Riemannian metric g) is a family of seminorms making $C^\infty(M,V)$ into a Fréchet space.

From holomorphicity

- Let H be the space of entire (everywhere holomorphic) functions on the complex plane. Then the family of seminorms

$$|f|_n = \sup\{|f(z)| : |z| \leq n\}$$

makes H into a Fréchet space.

- Let H be the space of entire (everywhere holomorphic) functions of exponential type τ. Then the family of seminorms

$$|f|_n = \sup_{z\in\mathbb{C}} \exp\left[-\left(\tau + \frac{1}{n}\right)|z|\right]|f(z)|$$

makes H into a Fréchet space.

Not all vector spaces with complete translation-invariant metrics are Fréchet spaces. An example is the space $L^p([0,1])$ with $p < 1$. Although this space fails to be locally convex, it is an F-space.

Properties and further notions

If a Fréchet space admits a continuous norm then all of the seminorms used to define it can be replaced with norms by adding this continuous norm to each of them. A Banach space, $C^\infty([a,b])$, $C^\infty(X,V)$ with X compact, and H all admit norms, while $\mathbb{R}^\omega$ and $C(\mathbb{R})$ do not.

A closed subspace of a Fréchet space is a Fréchet space. A quotient of a Fréchet space by a closed subspace is a Fréchet space. The direct sum of a finite number of Fréchet spaces is a Fréchet space.

A product of countably many Fréchet spaces is always once again a Fréchet space. However, an arbitrary product of Fréchet spaces will be a Fréchet space if and only if all *except* for at most countably many of them are trivial (that is, have dimension 0). Consequently, a product of uncountably many non-trivial Fréchet spaces can not be a Fréchet space (indeed, such a product is not even metrizable because its origin can not have a countable neighborhood basis). So for example, if $I \neq \varnothing$ is any set and X is any non-trivial Fréchet space (such as $X = \mathbb{R}$ for instance), then the product $X^I = \prod_{i\in I} X$ is a Fréchet space if and only if I is a countable set.

Several important tools of functional analysis which are based on the Baire category theorem remain true in Fréchet spaces; examples are the closed graph theorem and the open mapping theorem. The open mapping theorem implies that if $\tau \text{ and } \tau_2$ are topologies on X that make both (X,τ) and (X,τ_2) into complete metrizable TVSs (such as Fréchet spaces) and if one topology is finer or coarser than the other then they must be equal (that is, if $\tau \subseteq \tau_2 \text{ or } \tau_2 \subseteq \tau \text{ then } \tau = \tau_2$).[4]

Every bounded linear operator from a Fréchet space into another topological vector space (TVS) is continuous.[5]

There exists a Fréchet space X having a bounded subset B and also a dense vector subspace M such that B is *not* contained in the closure (in X) of any bounded subset of M.[6]

All Fréchet spaces are stereotype spaces. In the theory of stereotype spaces Fréchet spaces are dual objects to Brauner spaces. All metrizable Montel spaces are separable.[7] A separable Fréchet space is a Montel space if and only if each weak-* convergent sequence in its continuous dual converges is strongly convergent.[7]

The strong dual space X'_b of a Fréchet space (and more generally, of any metrizable locally convex space[8]) X is a DF-space.[9] The strong dual of a DF-space is a Fréchet space.[10] The strong dual of a reflexive Fréchet space is a bornological space[8] and a Ptak space. Every Fréchet space is a Ptak space. The strong bidual (that is, the strong dual space of the strong dual space) of a metrizable locally convex space is a Fréchet space.[11]

Norms and normability

If X is a locally convex space then the topology of X can be a defined by a family of continuous *norms* on X (a **norm** is a positive-definite seminorm) if and only if there exists *at least one* continuous *norm* on X.[12] Even if a Fréchet space has a topology that is defined by a (countable) family of *norms* (all norms are also seminorms), then it may nevertheless still fail to be normable space (meaning that its topology can not be defined by any single norm). The space of all sequences $\mathbb{K}^{\mathbb{N}}$ (with the product topology) is a Fréchet space. There does not exist any Hausdorff locally convex topology on $\mathbb{K}^{\mathbb{N}}$ that is strictly coarser than this product topology.[13] The space $\mathbb{K}^{\mathbb{N}}$ is not normable, which means that its topology can not be defined by any norm.[13] Also, there does not exist *any* continuous norm on $\mathbb{K}^{\mathbb{N}}$. In fact, as the following theorem shows, whenever X is a Fréchet space on which there does not exist any continuous norm, then this is due entirely to the presence of $\mathbb{K}^{\mathbb{N}}$ as a subspace.

> **Theorem**[13]—Let X be a Fréchet space over the field $\mathbb{K}$. Then the following are equivalent:
>
> 1. X does *not* admit a continuous norm (that is, any continuous seminorm on X can *not* be a norm).
> 2. X contains a vector subspace that is TVS-isomorphic to $\mathbb{K}^{\mathbb{N}}$.
> 3. X contains a complemented vector subspace that is TVS-isomorphic to $\mathbb{K}^{\mathbb{N}}$.

If X is a non-normable Fréchet space on which there exists a continuous norm, then X contains a closed vector subspace that has no topological complement.[14]

A metrizable locally convex space is normable if and only if its strong dual space is a Fréchet–Urysohn locally convex space.[9] In particular, if a locally convex metrizable space X (such as a Fréchet space) is *not* normable (which can only happen if X is infinite dimensional) then its strong dual space X'_b is not a Fréchet–Urysohn space and consequently, this complete Hausdorff locally convex space X'_b is also neither metrizable nor normable.

The strong dual space of a Fréchet space (and more generally, of bornological spaces such as metrizable TVSs) is always a complete TVS and so like any complete TVS, it is normable if and only if its topology can be induced by a complete norm (that is, if and only if it can be made into a Banach space that has the same topology). If X is a Fréchet space then X is normable if (and only if) there exists a complete norm on its continuous dual space X' such that the norm induced topology on X' is finer than the weak-* topology.[15] Consequently, if a Fréchet space is *not* normable (which can only happen if it is infinite dimensional) then neither is its strong dual space.

Anderson–Kadec theorem

> **Anderson–Kadec theorem**—Every infinite-dimensional, separable real Fréchet space is homeomorphic to $\mathbb{R}^{\mathbb{N}}$, the Cartesian product of countably many copies of the real line $\mathbb{R}$.

Note that the homeomorphism described in the Anderson–Kadec theorem is *not* necessarily linear.

> **Eidelheit theorem**—A Fréchet space is either isomorphic to a Banach space, or has a quotient space isomorphic to $\mathbb{R}^{\mathbb{N}}$.

Differentiation of functions

If X and Y are Fréchet spaces, then the space $L(X,Y)$ consisting of all continuous linear maps from X to Y is *not* a Fréchet space in any natural manner. This is a major difference between the theory of Banach spaces and that of Fréchet spaces and necessitates a different definition for continuous differentiability of functions defined on Fréchet spaces, the Gateaux derivative:

Suppose U is an open subset of a Fréchet space X, $P : U \to Y$ is a function valued in a Fréchet space Y, $x \in U$ and $h \in X$. The map P is **differentiable at x in the direction h** if the limit

$$D(P)(x)(h) = \lim_{t\to 0} \frac{1}{t} \left(P(x+th) - P(x)\right)$$

exists. The map P is said to be **continuously differentiable** in U if the map

$$D(P) : U \times X \to Y$$

is continuous. Since the product of Fréchet spaces is again a Fréchet space, we can then try to differentiate $D(P)$ and define the higher derivatives of P in this fashion.

The derivative operator $P : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $P(f) = f'$ is itself infinitely differentiable. The first derivative is given by

$$D(P)(f)(h) = h'$$

for any two elements $f, h \in C^\infty([0,1])$. This is a major advantage of the Fréchet space $C^\infty([0,1])$ over the Banach space $C^k([0,1])$ for finite k.

If $P : U \to Y$ is a continuously differentiable function, then the differential equation

$$x'(t) = P(x(t)), \quad x(0) = x_0 \in U$$

need not have any solutions, and even if does, the solutions need not be unique. This is in stark contrast to the situation in Banach spaces.

In general, the inverse function theorem is not true in Fréchet spaces, although a partial substitute is the Nash–Moser theorem.

Fréchet manifolds and Lie groups

One may define **Fréchet manifolds** as spaces that "locally look like" Fréchet spaces (just like ordinary manifolds are defined as spaces that locally look like Euclidean space $\mathbb{R}^n$), and one can then extend the concept of Lie group to these manifolds. This is useful because for a given (ordinary) compact C^∞ manifold $M,$ the set of all C^∞ diffeomorphisms $f : M \to M$ forms a generalized Lie group in this sense, and this Lie group captures the symmetries of $M.$ Some of the relations between Lie algebras and Lie groups remain valid in this setting.

Another important example of a Fréchet Lie group is the loop group of a compact Lie group $G,$ the smooth (C^∞) mappings $\gamma : S^1 \to G,$ multiplied pointwise by $(\gamma_1\gamma_2)(t) = \gamma_1(t)\gamma_2(t)..$[16][17]

Generalizations

If we drop the requirement for the space to be locally convex, we obtain F-spaces: vector spaces with complete translation-invariant metrics.

LF-spaces are countable inductive limits of Fréchet spaces.

See also

- Banach space – Normed vector space that is complete
- Brauner space
- Complete metric space – Metric geometry
- Complete topological vector space – Structure in functional analysis
- F-space – Topological vector space with a complete translation-invariant metric
- Fréchet lattice – Topological vector lattice
- Graded Fréchet space – Generalization of the inverse function theorem
- Hilbert space – Type of vector space in math
- Locally convex topological vector space – Vector space with a topology defined by convex open sets
- Metrizable topological vector space – A topological vector space whose topology can be defined by a metric
- Surjection of Fréchet spaces – Characterization of surjectivity
- Tame Fréchet space – Generalization of the inverse function theorem
- Topological vector space – Vector space with a notion of nearness

Notes

1. Here "Cauchy" means Cauchy with respect to the canonical uniformity that every TVS possess. That is, a sequence $x_\bullet = (x_m)_{m=1}^\infty$ in a TVS X is Cauchy if and only if for all neighborhoods U of the origin in $X,$ $x_m - x_n \in U$ whenever m and n are sufficiently large. Note that this definition of a Cauchy sequence does not depend on any particular metric and does not even require that X be metrizable.

2. Some authors do not include local convexity as part of the definition of a Fréchet space.

Citations

1. Narici & Beckenstein 2011, p. 93.
2. Conway 1990, Chapter 4.
3. Narici & Beckenstein 2011, p. 472.
4. Trèves 2006, pp. 166–173.
5. Trèves 2006, p. 142.
6. Wilansky 2013, p. 57.
7. Schaefer & Wolff 1999, pp. 194–195.
8. Schaefer & Wolff 1999, p. 154.
9. Gabriyelyan, S.S. "On topological spaces and topological groups with certain local countable networks (https://arxiv.org/pdf/1412.1497.pdf) (2014)
10. Schaefer & Wolff 1999, p. 196.
11. Schaefer & Wolff 1999, pp. 154–155.
12. Jarchow 1981, p. 130.
13. Jarchow 1981, pp. 129–130.
14. Schaefer & Wolff 1999, pp. 190–202.
15. "The dual of a Fréchet space" (https://math.stackexchange.com/a/112852/239329). 24 February 2012. Retrieved 26 April 2021.
16. Sergeev 2010
17. Pressley & Segal 1986

References

- "Fréchet space" (https://www.encyclopediaofmath.org/index.php?title=Fréchet_space), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Berberian, Sterling K. (1974). *Lectures in Functional Analysis and Operator Theory*. Graduate Texts in Mathematics. Vol. 15. New York: Springer. ISBN 978-0-387-90081-0. OCLC 878109401 (https://search.worldcat.org/oclc/878109401).
- Bourbaki, Nicolas (1987) [1981]. *Topological Vector Spaces: Chapters 1–5*. Éléments de mathématique. Translated by Eggleston, H.G.; Madan, S. Berlin New York: Springer-Verlag. ISBN 3-540-13627-4. OCLC 17499190 (https://search.worldcat.org/oclc/17499190).
- Conway, John (1990). *A course in functional analysis*. Graduate Texts in Mathematics. Vol. 96 (2nd ed.). New York: Springer-Verlag. ISBN 978-0-387-97245-9. OCLC 21195908 (https://search.worldcat.org/oclc/21195908).
- Edwards, Robert E. (1995). *Functional Analysis: Theory and Applications*. New York: Dover Publications. ISBN 978-0-486-68143-6. OCLC 30593138 (https://search.worldcat.org/oclc/30593138).
- Grothendieck, Alexander (1973). *Topological Vector Spaces* (https://archive.org/details/topologicalvecto0000grot). Translated by Chaljub, Orlando. New York: Gordon and Breach Science Publishers. ISBN 978-0-677-30020-7. OCLC 886098 (https://search.worldcat.org/oclc/886098).
- Jarchow, Hans (1981). *Locally convex spaces*. Stuttgart: B.G. Teubner. ISBN 978-3-519-02224-4. OCLC 8210342 (https://search.worldcat.org/oclc/8210342).

- Khaleelulla, S. M. (1982). *Counterexamples in Topological Vector Spaces*. Lecture Notes in Mathematics. Vol. 936. Berlin, Heidelberg, New York: Springer-Verlag. ISBN 978-3-540-11565-6. OCLC 8588370 (https://search.worldcat.org/oclc/8588370).
- Köthe, Gottfried (1983) [1969]. *Topological Vector Spaces I*. Grundlehren der mathematischen Wissenschaften. Vol. 159. Translated by Garling, D.J.H. New York: Springer Science & Business Media. ISBN 978-3-642-64988-2. MR 0248498 (https://mathscinet.ams.org/mathscinet-getitem?mr=0248498). OCLC 840293704 (https://search.worldcat.org/oclc/840293704).
- Narici, Lawrence; Beckenstein, Edward (2011). *Topological Vector Spaces*. Pure and applied mathematics (Second ed.). Boca Raton, FL: CRC Press. ISBN 978-1584888666. OCLC 144216834 (https://search.worldcat.org/oclc/144216834).
- Pressley, Andrew; Segal, Graeme (1986). *Loop groups* (https://books.google.com/books?id=MbFBXyuxLKgC). Oxford Mathematical Monographs. Oxford Science Publications. New York: Oxford University Press. ISBN 0-19-853535-X. MR 0900587 (https://mathscinet.ams.org/mathscinet-getitem?mr=0900587).
- Robertson, Alex P.; Robertson, Wendy J. (1980). *Topological Vector Spaces*. Cambridge Tracts in Mathematics. Vol. 53. Cambridge England: Cambridge University Press. ISBN 978-0-521-29882-7. OCLC 589250 (https://search.worldcat.org/oclc/589250).
- Rudin, Walter (1991). *Functional Analysis* (https://archive.org/details/functionalanalys00rudi). International Series in Pure and Applied Mathematics. Vol. 8 (Second ed.). New York, NY: McGraw-Hill Science/Engineering/Math. ISBN 978-0-07-054236-5. OCLC 21163277 (https://search.worldcat.org/oclc/21163277).
- Schaefer, Helmut H.; Wolff, Manfred P. (1999). *Topological Vector Spaces*. GTM. Vol. 8 (Second ed.). New York, NY: Springer New York Imprint Springer. ISBN 978-1-4612-7155-0. OCLC 840278135 (https://search.worldcat.org/oclc/840278135).
- Sergeev, Armen (2010). *Kähler Geometry of Loop Spaces* (http://www.worldscientific.com/worldscibooks/10.1142/e023). Mathematical Society of Japan Memoirs. Vol. 23. World Scientific Publishing. doi:10.1142/e023 (https://doi.org/10.1142%2Fe023). ISBN 978-4-931469-60-0.
- Adasch, Norbert; Ernst, Bruno; Keim, Dieter (1978). *Topological Vector Spaces: The Theory Without Convexity Conditions*. Lecture Notes in Mathematics. Vol. 639. Berlin New York: Springer-Verlag. ISBN 978-3-540-08662-8. OCLC 297140003 (https://search.worldcat.org/oclc/297140003).
- Swartz, Charles (1992). *An introduction to Functional Analysis*. New York: M. Dekker. ISBN 978-0-8247-8643-4. OCLC 24909067 (https://search.worldcat.org/oclc/24909067).
- Trèves, François (2006) [1967]. *Topological Vector Spaces, Distributions and Kernels*. Mineola, N.Y.: Dover Publications. ISBN 978-0-486-45352-1. OCLC 853623322 (https://search.worldcat.org/oclc/853623322).
- Wilansky, Albert (2013). *Modern Methods in Topological Vector Spaces*. Mineola, New York: Dover Publications, Inc. ISBN 978-0-486-49353-4. OCLC 849801114 (https://search.worldcat.org/oclc/849801114).
