Homework 2

1.

(a) Let $f(n) = 3^{n+1}$ and $g(n) = 3^n$. Examine

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \\
&= \lim_{n\to\infty} \frac{3 \cdot 3^n}{3^n} \\
&= \lim_{n\to\infty} 3 \\
&= 3
\end{aligned}
$$

Since 3 is a constant, then $f(n) \in \Theta(g(n))$

(b) Let $f(n) = 3^{3n}$ and $g(n) = 3^n$. Examine

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{3^{3n}}{3^n} \\
&= \lim_{n\to\infty} \frac{(3^n)^3}{3^n} \\
&= \lim_{n\to\infty} 3^{2n} \\
&= \infty
\end{aligned}
$$

So $f(n) \in \Omega(g(n))$.

2.

The list of functions ranked from smallest order of growth to largest is below.

$$
1000, \ln(\ln n), \sqrt{\ln(n)}, \{\log_5 n, \lg n\}, (\lg n)^2, \left(\sqrt{2}\right)^{lgn},
$$

$$
\{n, 1000n + 3, 2^{\lg n}\}, \{n \cdot \lg n, \ln(n!)\}, \{n^2, 4^{\lg n}\},
$$

$$
n^3, \left(\frac{3}{2}\right)^n, 2^n, n2^n, e^n, n!, (n+1)!
$$

3.

Let $T(n) = T(n/4) + B$, with initial condition $T(1) = A$, and let $n = 4^k$. We proceed with backwards substitution.

$$
\begin{aligned}
T(n) = T(4^k) \\
= T(4^{k-1}) + B \\
= T(4^{k-2}) + B + B \\
\vdots \\
= T(1) + \underbrace{B + B + \cdots + B}_{k \text{ times}}
\end{aligned}
$$

Thus, $T(n) = T(1) + Bk = A + Bk$. Since $n = 4^k$, then $k = \log_4 n$ so $T(n) = A + B \log_4 n$.

4.

Let $x(n) = \frac{1}{2}(x(n-1) + x(n-2))$. Then,

$$
x(n) - \frac{1}{2}x(n-1) - \frac{1}{2}x(n-2) = 0
$$

So we have the characteristic equation:

$$
\begin{aligned}
r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \\
2r^2 - r - 1 = 0 \\
(2r+1)(r-1) = 0
\end{aligned}
$$

So $r_1 = -\frac{1}{2}$ and $r_2 = 1$. So the general solution is

$$
\begin{aligned}
x(n) = \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2 (1)^n \\
= \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2
\end{aligned}
$$

5.

(a) Let $T(n) = 2T(\frac{n}{4}) + 1000$. So

$$a = 2, b = 4, d = 0$$

Since

$$a = 2 > 1 = b^d$$

then

$$T(n) \in \Theta\left(n^{\log_4(2)}\right)$$

(b) Let $T(n) = 2T(\frac{n}{4}) + 1000n$. So

$$a = 2, b = 4, d = 1$$

Since

$$a = 2 < 4 = b^d$$

then

$$T(n) \in \Theta(n)$$

(c) Let $T(n) = 2T(\frac{n}{4}) + 1000\sqrt{n}$. So

$$a = 2, b = 4, d = \frac{1}{2}$$

Since

$$a = 2 = 2 = b^d$$

then

$$T(n) \in \Theta\left(\sqrt{n}\log(n)\right)$$

(d) Let $T(n) = 2T(\frac{n}{4}) + 1000n^2$. So

$$a = 2, b = 4, d = 2$$

Since

$$a = 2 < 16 = b^d$$

then

$$T(n) \in \Theta\left(n^2\right)$$

6. In both parts, we denote E and E' to differentiate the two identical characters.

(a) Sec 3.1 #8
   We proceed with selection sort.

E	X	A	M	P	L	E'
A |	X	E	M	P	L	E'
A	E |	X	M	P	L	E'
A	E	E' |	M	P	L	X
A	E	E'	L |	P	M	X
A	E	E'	L	M |	P	X
A	E	E'	L	M	P |	X

(b) Sec 3.1 #11 We proceed with bubble sort.

E	X	A	M	P	L	E'
E	A	M	P	L	E' |	X
A	E	M	L	E' |	P	X
A	E	L	E' |	M	P	X
A	E	E' |	L	M	P	X
A	E |	E'	L	M	P	X
A |	E	E'	L	M	P	X