Homework 2

1.

(a) Let $f(n) = 3^{n+1}$ and $g(n) = 3^n$. Examine

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \\
&= \lim_{n\to\infty} \frac{3 \cdot 3^n}{3^n} \\
&= \lim_{n\to\infty} 3 \\
&= 3
\end{aligned}$$

Since 3 is a constant, then $f(n) \in \Theta(g(n))$

(b) Let $f(n) = 3^{3n}$ and $g(n) = 3^n$. Examine

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{3^{3n}}{3^n} \\
&= \lim_{n\to\infty} \frac{(3^n)^3}{3^n} \\
&= \lim_{n\to\infty} 3^{2n} \\
&= \infty
\end{aligned}$$

So $f(n) \in \Omega(g(n))$.

2.

The list of functions ranked from smallest order of growth to largest is below.

$$1000, \ln(\ln n), \sqrt{\ln(n)}, \{\log_5 n, \lg n\}, (\lg n)^2, \left(\sqrt{2}\right)^{lgn},$$
$$\{n, 1000n+3, 2^{\lg n}\}, \{n \cdot \lg n, \ln(n!)\}, \{n^2, 4^{\lg n}\},$$
$$n^3, \left(\frac{3}{2}\right)^n, 2^n, n2^n, e^n, n!, (n+1)!$$

3.

Let $T(n) = T(n/4) + B$, with initial condition $T(1) = A$, and let $n = 4^k$. We proceed with backwards substitution.

$$\begin{aligned}
T(n) &= T(4^k) \\
&= T(4^{k-1}) + B \\
&= T(4^{k-2}) + B + B \\
&\;\;\vdots \\
&= T(1) + \underbrace{B + B + \cdots + B}_{k \text{ times}}
\end{aligned}$$

Thus, $T(n) = T(1) + Bk = A + Bk$. Since $n = 4^k$, then $k = \log_4 n$ so $T(n) = A + B\log_4 n$.

4.

Let $x(n) = \frac{1}{2}(x(n-1) + x(n-2))$. Then,

$$x(n) - \frac{1}{2}x(n-1) - \frac{1}{2}x(n-2) = 0$$

So we have the characteristic equation:

$$\begin{aligned}
r^2 - \frac{1}{2}r - \frac{1}{2} &= 0 \\
2r^2 - r - 1 &= 0 \\
(2r+1)(r-1) &= 0
\end{aligned}$$

So $r_1 = -\frac{1}{2}$ and $r_2 = 1$. So the general solution is

$$\begin{aligned}
x(n) &= \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2(1)^n \\
&= \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2
\end{aligned}$$

5.

(a) Let $T(n) = 2T(\frac{n}{4}) + 1000$. So

$$a = 2, b = 4, d = 0$$

Since

$$a = 2 > 1 = b^d$$

then

$$T(n) \in \Theta\left(n^{\log_4(2)}\right)$$

(b) Let $T(n) = 2T(\frac{n}{4}) + 1000n$. So

$$a = 2, b = 4, d = 1$$

Since

$$a = 2 < 4 = b^d$$

then

$$T(n) \in \Theta(n)$$

(c) Let $T(n) = 2T(\frac{n}{4}) + 1000\sqrt{n}$. So

$$a = 2, b = 4, d = \frac{1}{2}$$

Since

$$a = 2 = 2 = b^d$$

then

$$T(n) \in \Theta\left(\sqrt{n}\log(n)\right)$$

(d) Let $T(n) = 2T(\frac{n}{4}) + 1000n^2$. So

$$a = 2, b = 4, d = 2$$

Since

$$a = 2 < 16 = b^d$$

then

$$T(n) \in \Theta\left(n^2\right)$$