Questions and Conjectures

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For a knot, K, the following are equivalent:

(a) K is tricolorable

(b) K is S_3-colorable

(c) K is S_4-colorable

Note For $n \geq 5$, then S_n-colorable does not imply S_{n-1}-colorable. The proof boils down to the fact that the only nontrivial normal subgroups of S_n are A_n for $n \geq 5$, and $A_n \ncong S_{n-1}$.

Proof $(a) \iff (b)$ and $(c) \implies (b)$ are trivial. Perko proved $(b) \implies (c)$. He used high level homology. **Q:** Is there an easier way?

By G-colorable, we mean that there is a surjective homomorphism from the knot group, K to G. i.e there exists some φ such that

$$\varphi : K \twoheadrightarrow G.$$

Conjecture If T_k is G-colorable, and ~~a, b are the labels of any two strands~~ a is the overstrand and b is an understrand at any crossing, then $G = \langle a, b \rangle$ and a, b are conjugate.

Corollary T_k is not $\binom{n}{2}$-colorable if $n \geq 4$.
(Answer to **Q:** For what n is a twist knot $\binom{n}{2}$-colorable?)

Dr. Rogers provides a short proof of a, b being conjugate.

The sketch of the proof is as follows:
Without loss of generality, we label a, b at the bottom of the knot. We continue up the twist portion labeling each strand on the left a_k and the right b_k. Note that $b_k = a_{k-1}$. By observation we see

$$\begin{array}{ll} k \text{ is odd:} & a_k = (a_{k-1})^{-1} b_{k-1} a_{k-1} \\ k \text{ is even:} & a_k = a_{k-1} b_{k-1} (a_{k-1})^{-1} \end{array}$$

Define $h = a^{-1}b$. Then,

	k odd	k even
Recursive	$a_k = ha_{k-1}$	$a_k = a_{k-1}h^{-1}$
Closed	$a_k = h^{\frac{1+k}{2}} a h^{\frac{1-k}{2}}$	$a_k = h^{\frac{k}{2}} a h^{-\frac{k}{2}}$

At the top of the knot, we have two consistency equations, a left and right as follows:

	n odd	n even
Left	$a = a_n b a_n^{-1}$	$a = a_n^{-1} b a_n$
Right	$a_n = b b_n b^{-1}$	$a_n = b^{-1} b_n b$

It's easy to show that left consistency equations hold if the right consistency equations hold. It's also easy to show that the equations in the above tables hold by induction.

Questions following the Conjecture

1. Can this generalize to other knots? If so, what conditions need to be applied?
    - Prime for sure by paper by Norwood.
2. Is there a prime knot that needs more than two generators?