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Question:

I came across a very interesting problem in a blog by Datascope Analytics[1]. Given a fair six sided dice, the probability of rolling a 1 or a 6 is $\frac{1}{3}$. But what happens to this probability if the length and width of the dice remain the same, but the height changes to some number h?

Proposed solution:

$$P(1,6) = 1 - \frac{4}{\pi}\tan^{-1}\frac{h}{\sqrt{h^2+2}}$$

Equation 1: Proposed solution for the probability of rolling a 1 or 6 in a dice with height h

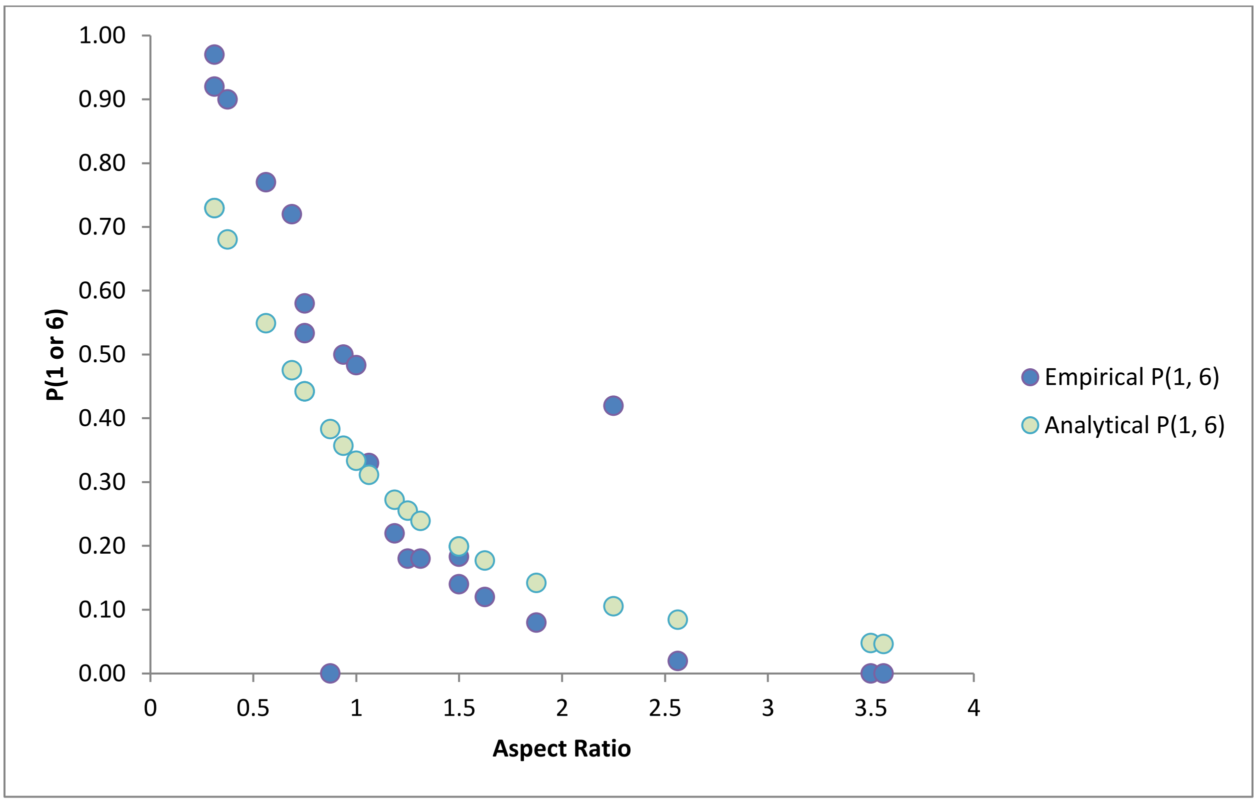


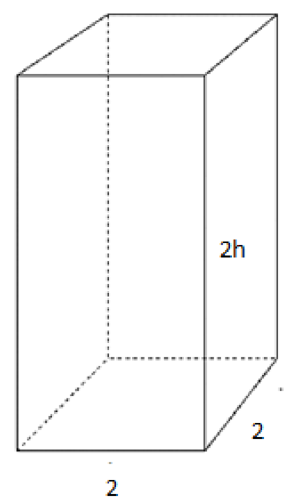
Figure 1: Distribution of the probability of rolling a 1 or a 6, plotted against the aspect ratio (height over length) Empirical results obtained from blog site; Analytical result corresponds to Equation 1. Note that Analytical result corresponding to an aspect ratio of 1 is equal to 1/3, which is consistent with the default case of a regular dice.

[1] http://datascopeanalytics.com/blog/from-coins-to-rods-properties-of-elongated-dice/

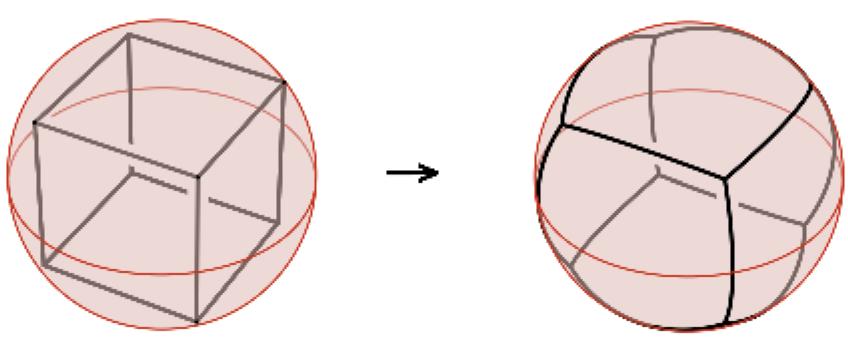
Procedure:

Let's make a few simplifying assumptions:

- The dice is tossed in such a way that the probability of landing at any angle around the center of mass is equally likely. In other words, the dice can assume all possible orientations in the three dimensional space[2].
- Ignore forces such as momentum, torque, air resistance, etc.
- The dice lands in a mud-like surface, such that depending on the angle it lands on relative to the center of mass, it will slowly reorient itself on one of its six sides.
- Without loss of generality, assume the dice is of dimension 2 x 2 x 2h, so that the aspect ratio (height over length) is still h. It can be proven that scaling the dice dimensions by some constant does not impact the probability outcome in question.



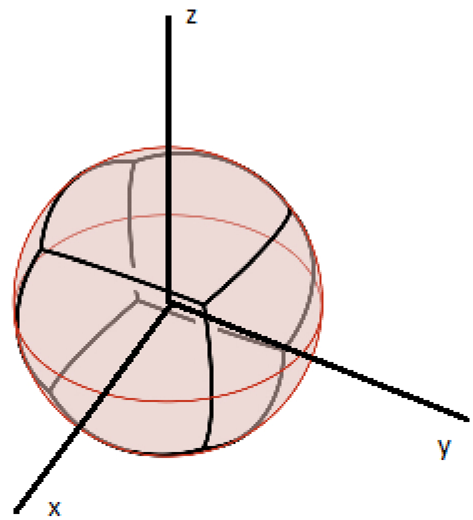
Given the above assumptions, we encircle the dice inside a sphere, such that all 8 corners of the dice are touching the surface of the sphere. Now if we project the center of mass of the dice onto the sphere, we get something similar to the following figure.



[2] Similar to that of John von Neumann's solution to the coin problem of "What height should a coin be to have a 1/3 chance of landing on Sides?", in the book "Fifty Challenging Problems in Probability".

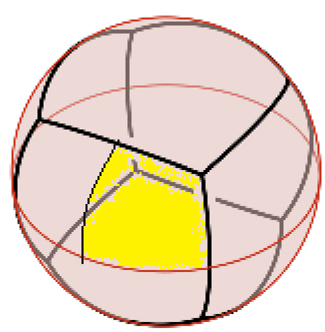
Since we’ve assumed that the dice is equally like to land in any orientation, it’s equivalent to assuming that any point on the sphere is equally likely to be selected as the initial point of impact. Hence, we can calculate the surface area of the projected 1 and 6 face, as a proportion of the whole sphere.

Calculating this turned out to be fun. Assume the following axis-orientation, such that the center of mass is placed at the origin.

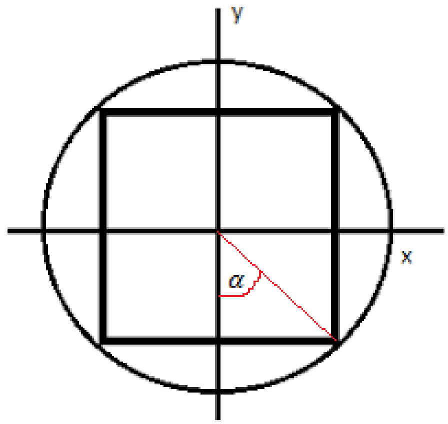


Solving for the proportion of 1 and 6 face is the same as 1 minus the proportion of the 2, 3, 4, and 5 faces (which are equivalent).

If we cut the sphere into two hemispheres, horizontally along the xy plane, then we have 8 equal areas, each piece corresponds to one half of a face. If we cut this face again in half, down the xy plane, we receive 16 equivalent regions that sum up to the area of 2, 3, 4, and 5. The region is shown as the highlighted portion of the sphere. Let's call this region F.

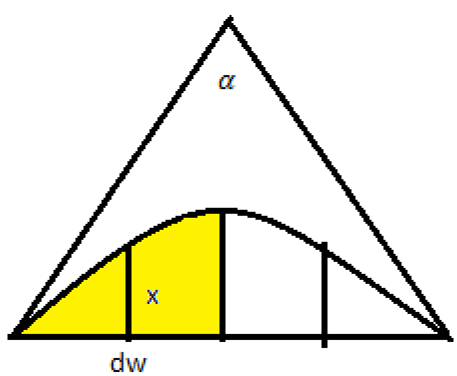


If we look from a top down angle, observing the xy plane, then we can see that the shaded area F corresponds to is bounded within some angle α, where α ranges from 0 to $\frac{\pi}{4}$.



To simplify the calculation of surface area F, we use Archimedes result that surface area is preserved when a sphere is projected onto a cylinder from its z-axis[3]. Thus we are looking for a specific portion of the cylinder that corresponds to the highlighted area above.

Denote the height of the area F (on the cylinder) as x, and the width as w. Note that x is a function of the angle α. Then the area of F is equivalent to the incremental width, dw, multiplied by the corresponding height, x, integrated over the whole range of possible α. The area F we are looking to calculate is highlighted as below.



[3] http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

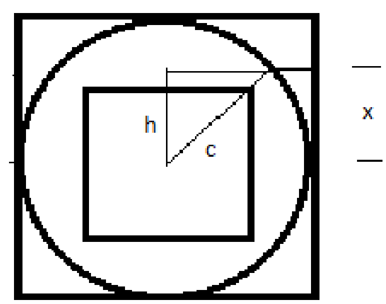
Then the area of F, A(F), is the following:

$$A(F) = \int x\, dw$$

since a section of a circle is calculated as the circumference multiplied by the angle proportion, $dw = 2\pi r * \frac{d\alpha}{2\pi} = r d\alpha$, then above equation becomes

$$A(F) = \int_0^{\frac{\pi}{4}} x\, r\, d\alpha$$

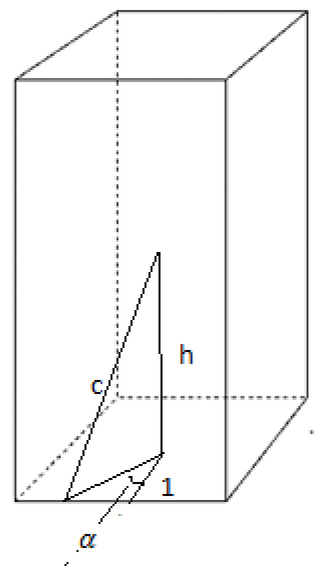
To solve for x, we cut open the sphere down the yz plane, and observe the cross section, which is a square inside a circle inside a larger square (the cylinder).



Using properties of similar triangles, we can solve for x as:

$$\frac{h}{x} = \frac{c}{r}$$

c is a function of α, as seen below.



Hence c is solved as :

$$c = \sqrt{h^2 + \frac{1}{\cos^2(\alpha)}}$$

Cross multiply to solve for x:

$$x = h * \sqrt{\frac{2+h}{h^2 + \frac{1}{\cos^2(\alpha)}}}$$

Plugging x into the original integral:

$$A(F) = \int_0^{\frac{\pi}{4}} hr \sqrt{\frac{2+h}{h^2 + \frac{1}{\cos^2(\alpha)}}} d\alpha$$

The ratio of the total surface area of 2, 3, 4, and 5 (equals to surface area F multiplied by 16) compared to the whole sphere is

$$P(2,3,4,5) = \frac{16 * \int_0^{\frac{\pi}{4}} hr \sqrt{\frac{2+h}{h^2 + \frac{1}{\cos^2(\alpha)}}} d\alpha}{4\pi r^2}$$

$$P(2,3,4,5) = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} hr \sqrt{\frac{2+h}{h^2 + \frac{1}{\cos^2(\alpha)}}} d\alpha$$

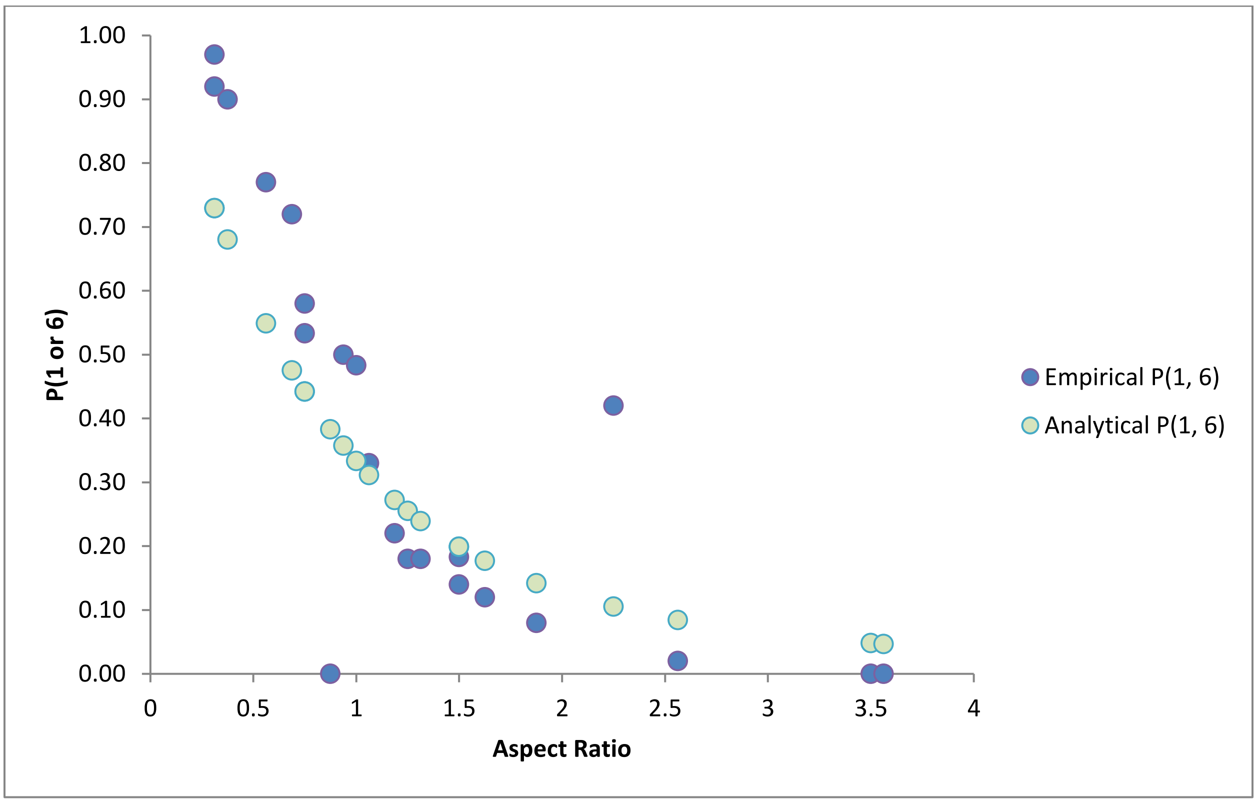
Using wolfram alpha to solve the integral

$$P(2,3,4,5) = \frac{4}{\pi}\tan^{-1}\frac{h}{\sqrt{h^2+2}}$$

Then the probability of rolling a 1 or 6 is

$$P(1,6) = 1 - \frac{4}{\pi}\tan^{-1}\frac{h}{\sqrt{h^2+2}}$$

Plotting the results gives the following comparison with the empirical results.



It appears that the empirical solution has a steeper slope compared to the proposed analytical solution. Given that the conditions which these observed data were collected were perhaps different from the assumptions that were made throughout this problem, i.e. landing surface and tossing technique, the variation in results are not too surprising. Additional empirical work could be performed with varied surfaces and tossing techniques to test the sensitivity and materiality of various attributes.