BAYESIAN INFERENCE IN GRAVITATIONAL-WAVE ASTRONOMY

PyCBC Inference online workshop 2020

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OUTLINE

I. OVERVIEW

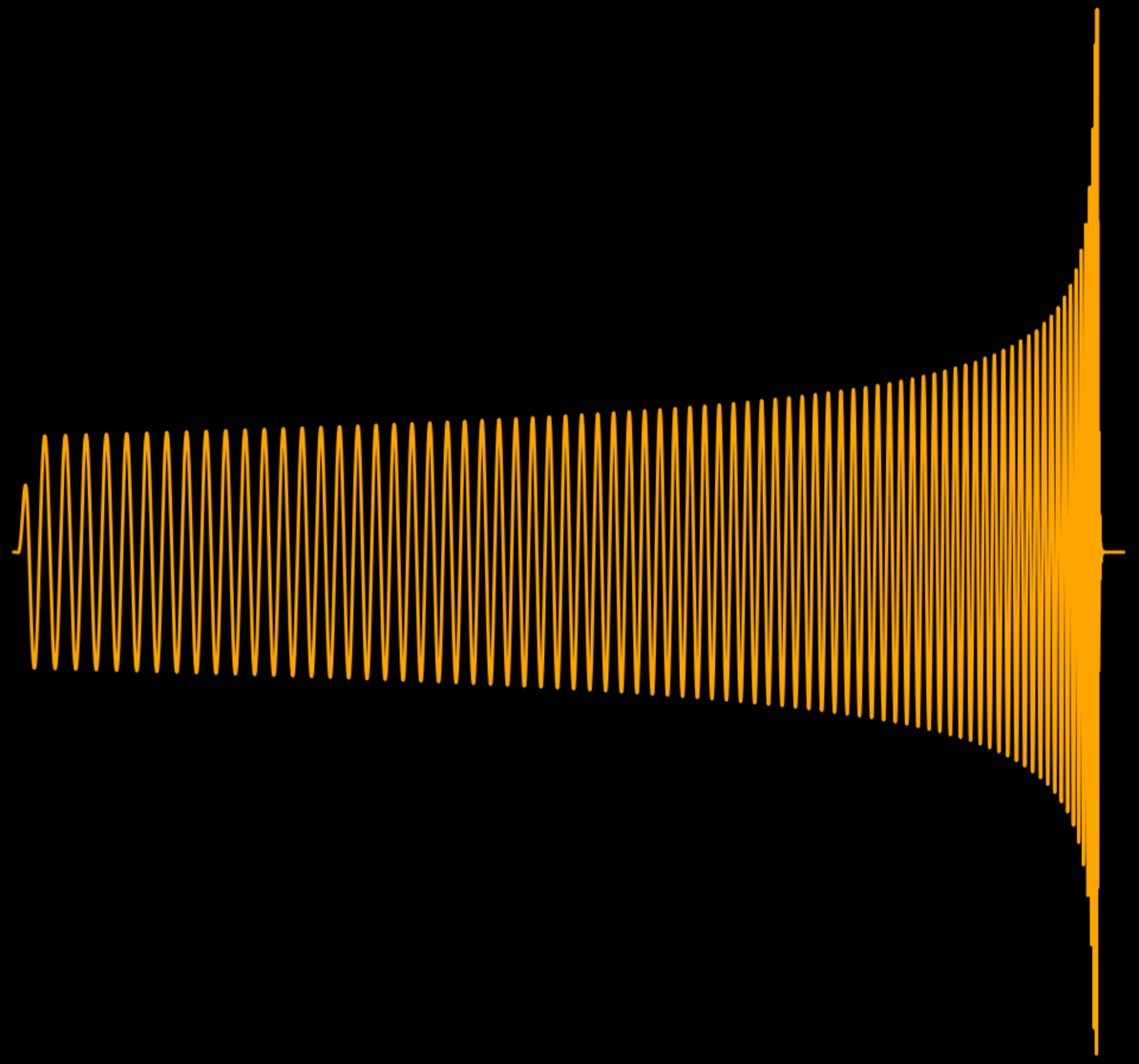
II. COMPACT BINARY PARAMETERS

III. BAYESIAN INFERENCE AND ITS APPLICATION TO GRAVITATIONAL WAVES

IV. MODELS AND SAMPLERS IN PYCBC INFERENCE

STATEMENT OF PROBLEM

- To do gravitational wave astronomy we need to detect signals in noise (search) and determine the parameters of their source (parameter estimation). But…
- Gravitational waves are weak!
- Even “loud” signals detected by Advanced LIGO & Virgo are buried in noise.
- Need tools that produce probabilistic statements about signals’ existence and parameters.



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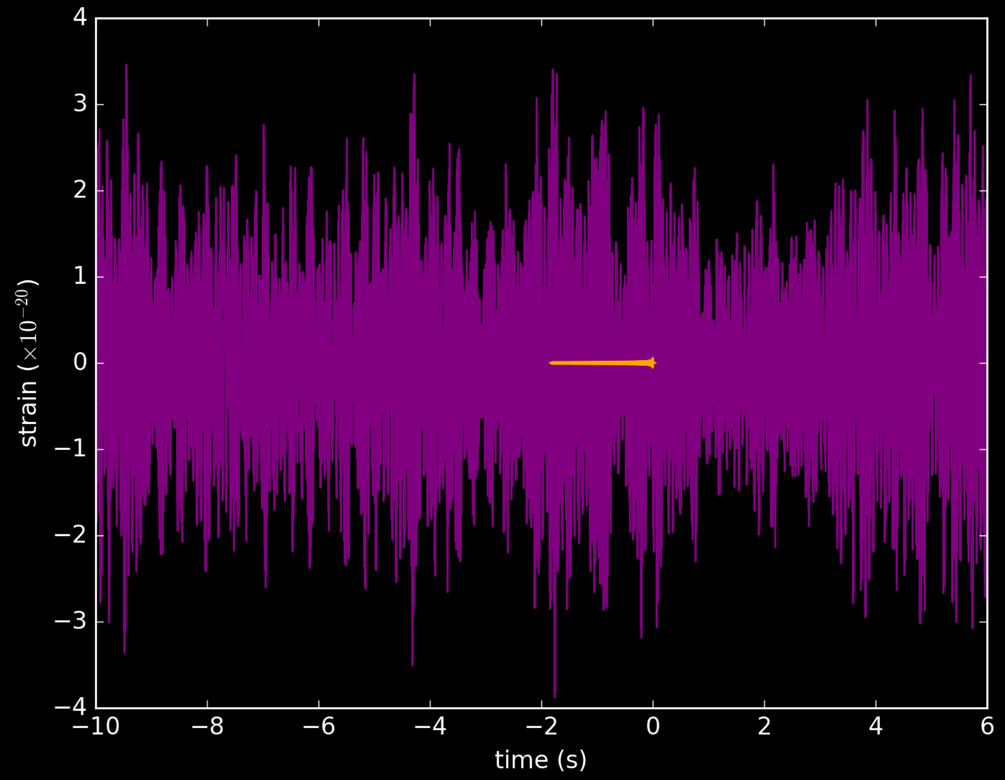


SEARCHES VS PARAMETER ESTIMATION

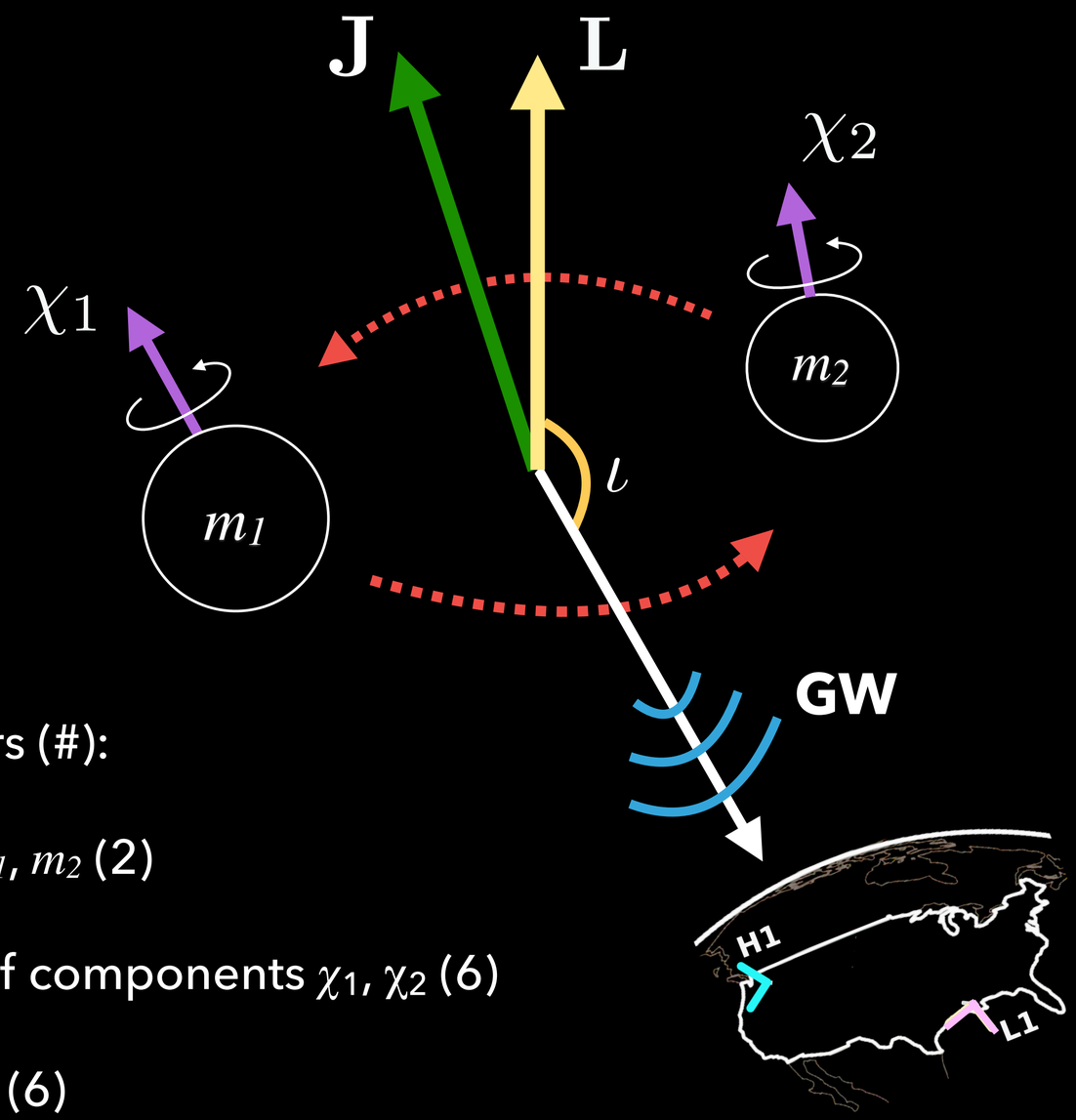
- **Searches**: find times when a measureable gravitational-wave signal (“event”) exists in the detectors & measure statistical significance of candidate events
  - Use match filtering, discrete “template banks”
  - Trade parameter accuracy for computational efficiency
- **Parameter estimation**: followup candidate events provided by searches
  - Assume signal exists in the data, use Bayesian inference to accurately measure source parameters
  - Trade computational efficiency for parameter accuracy
  - Also used for model selection, testing alternative theories

PYCBC

- Python-based open source software suite for gravitational-wave astronomy
- pycbc.org
- Inititally developed to search for gravitational-waves from compact binary mergers
- Has evolved to include parameter estimation
- **PyCBC Inference**: module in PyCBC that does Bayesian inference on gravitational-waves using external stochastic sampling packages

COMPACT BINARY PARAMETERS

BINARY BLACK HOLE PARAMETERS



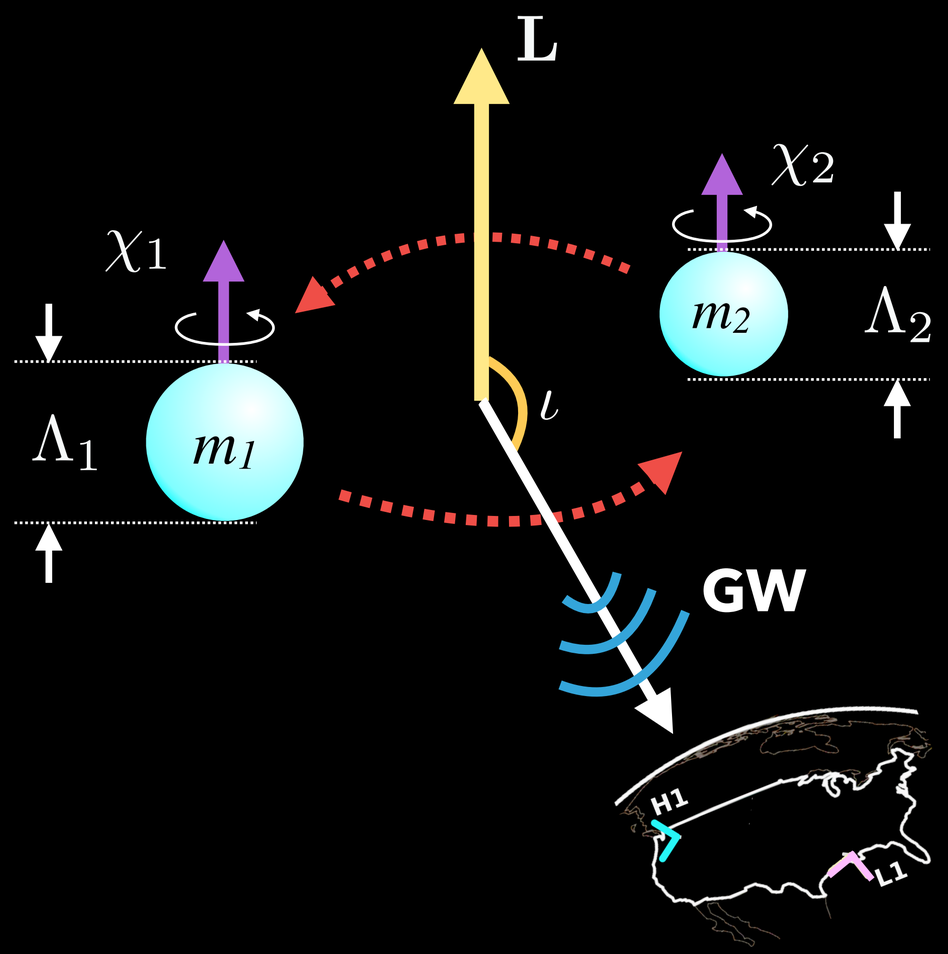
- Possible BBH parameters (#):
  - component masses m_1, m_2 (2)
  - dimensionless spins of components χ_1, χ_2 (6)
  - location & orientation (6)

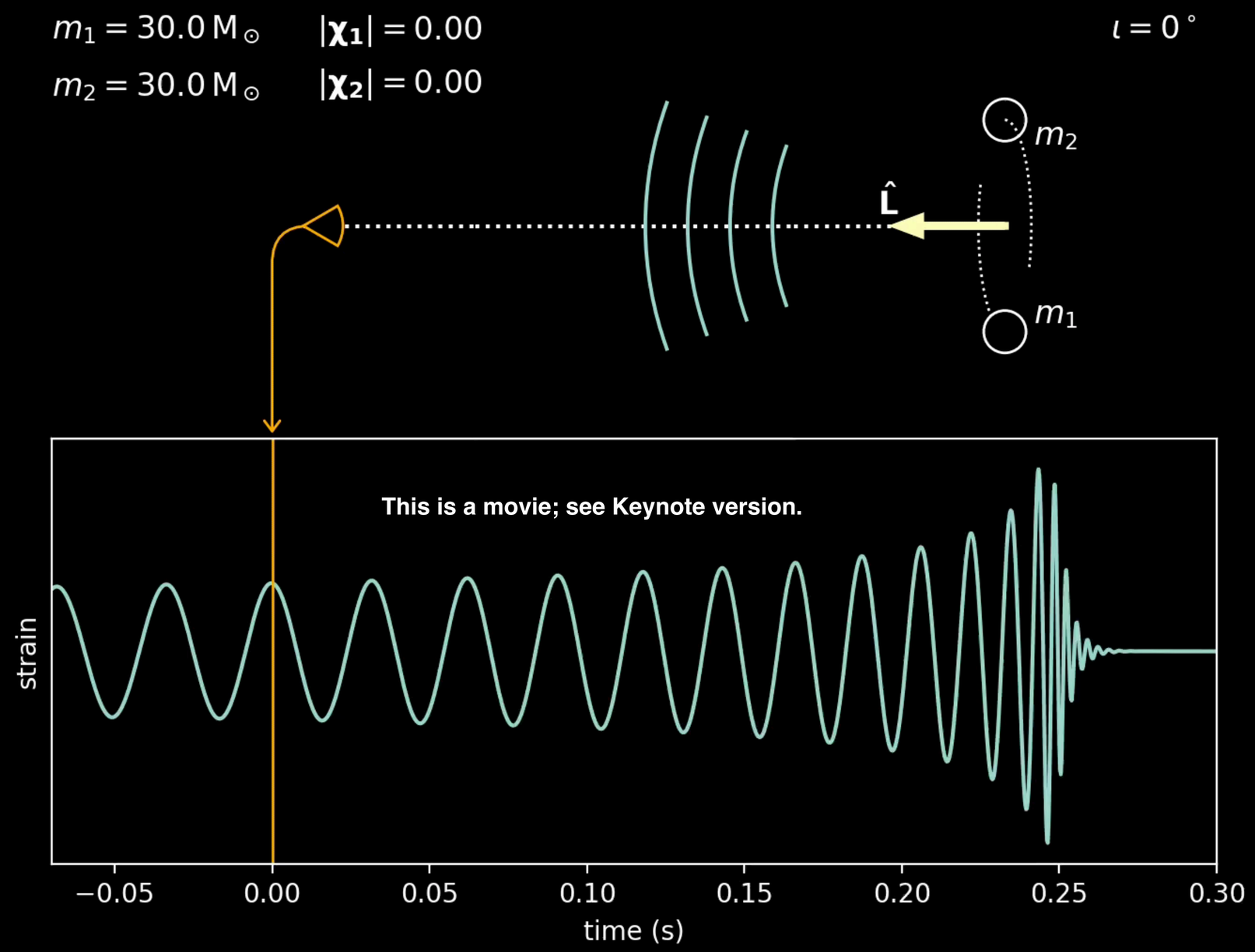
BINARY NEUTRON STAR PARAMETERS

- Tidal forces cause neutron stars to deform

$$\Lambda_i \equiv \frac{2}{3} k_2 \left(\frac{R_i c^2}{G m_i} \right)^5$$

- Tidal deformability depends on the equation of state of the matter in the core
- Stars' deformability is encoded in gravitational wave

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$m_2 = 30.0\,\mathrm{M}_\odot$
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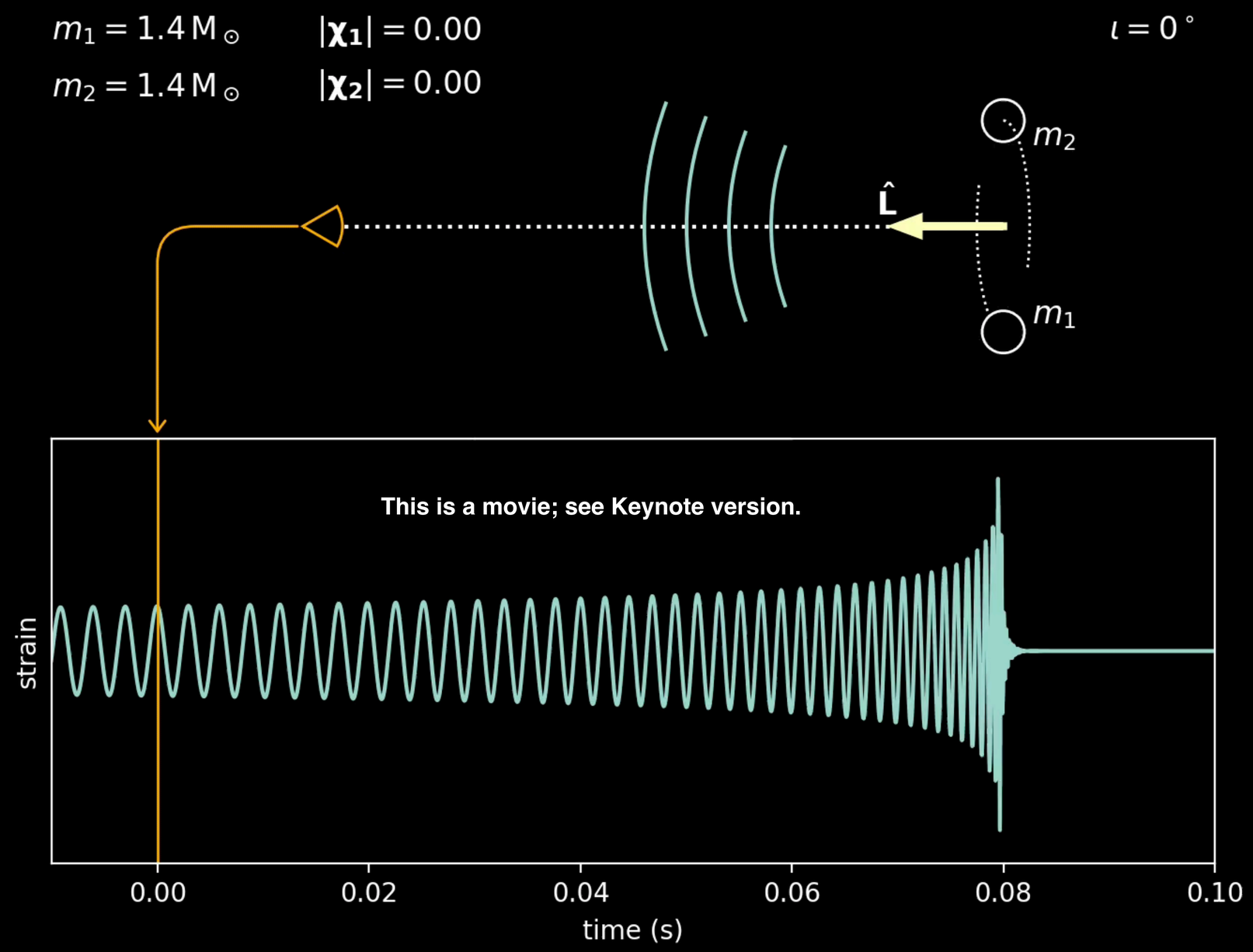
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BAYESIAN INFERENCE PRIMER

BAYES' THEOREM

- GW's source parameters are estimated using Bayesian inference.
- Assume a signal h exists in some data d.
- Probability that the signal has parameters $\vartheta = \{m_1, m_2, \ldots\}$ is:

$$p(\vec{\vartheta}|d,h) = \frac{p(d|\vec{\vartheta},h)\,p(\vec{\vartheta}|h)}{p(d|h)}$$

- **Prior**: Represents our state of knowledge about the true parameters *before* measuring the data
- **Likelihood**: The probability of observing the data assuming that a particular set of parameters is true.
- **Posterior**: Represents our state of knowledge about the parameters *after* measuring the data.

Recommend: D.S. Sivia & J. Skilling, "Data Analysis: A Bayesian Tutorial"

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Posterior = $\frac{\textbf{Likelihood} \times \textbf{Prior}}{\textbf{Evidence}}$

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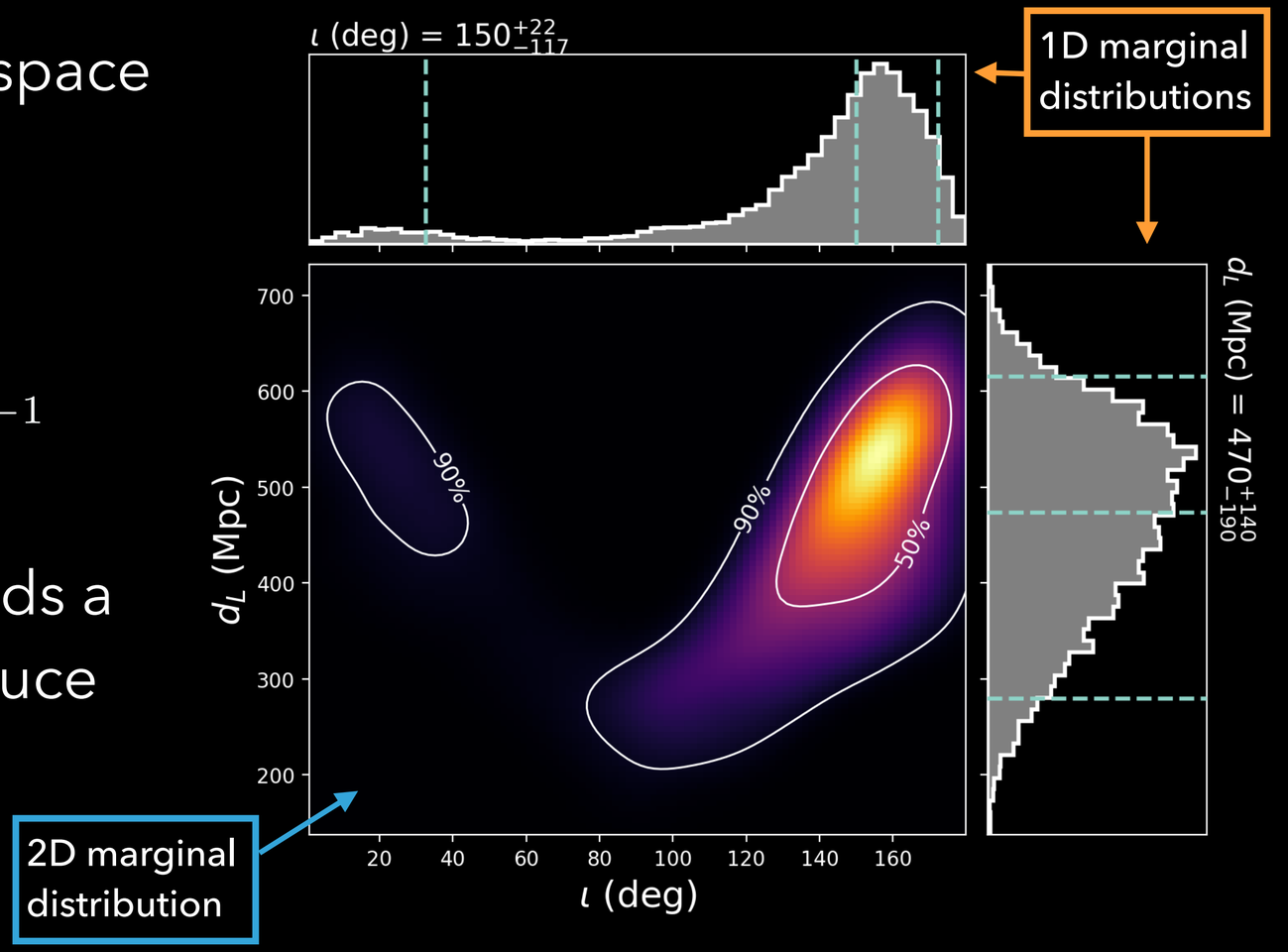
MARGINALIZATION

- Integrating the posterior over the prior space of a subset of parameters yields the **marginal** probability:

$$p(\vartheta_n|d,h) \propto \int p(d|\vec{\vartheta},h)p(\vec{\vartheta}|h)\mathrm{d}\vartheta_1 \cdots \mathrm{d}\vartheta_{n-1}$$

- Marginalizing all but one parameter yields a 1D distribution from which we can produce credible intervals.



Created using `pycbc_inference_plot_posterior` using data from gwcatalog.org,

EVIDENCE

$$p(\vec{\vartheta}|d,h) = \frac{p(d|\vec{\vartheta},h)\,p(\vec{\vartheta}|h)}{p(d|h)}$$

- Marginalizing the likelihood over all parameters yields the **evidence**:

$$p(d|h) = \int p(d|\vec{\vartheta},h)p(\vec{\vartheta}|h)\mathrm{d}\vec{\vartheta}$$

The evidence can be used for model selection. Taking the ratio of evidences for two models yields the **Bayes factor**:

$$\mathcal{B}(A,B|d) = \frac{p(d|B)}{p(d|A)}$$

- A Bayes factor > 1 indicates model B is favored over model A
- Generally only necessary to estimate evidence for model selection.
- If you are only doing parameter estimation for a single model, it is sufficient to measure the posterior up to a proportionality constant.

LIKELIHOOD

$$p(\vec{\vartheta}|d,h) = \frac{p(d|\vec{\vartheta},h)p(\vec{\vartheta}|h)}{p(d|h)}$$

- The **likelihood** function requires both a signal model and a noise model.
- In gravitational-wave analyses it is common to assume wide-sense stationary (WSS) Gaussian noise. In that case:

$$\log p(d|\vec{\vartheta},h) \propto -\frac{1}{2}\sum_{i=1}^{N_d}\left\langle d_i - h_i(\vec{\vartheta}),\, d_i - h_i(\vec{\vartheta})\right\rangle$$

where:

$$\langle a,\, b\rangle = 4\Re\int_0^\infty \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_n(f)}\mathrm{d}f$$

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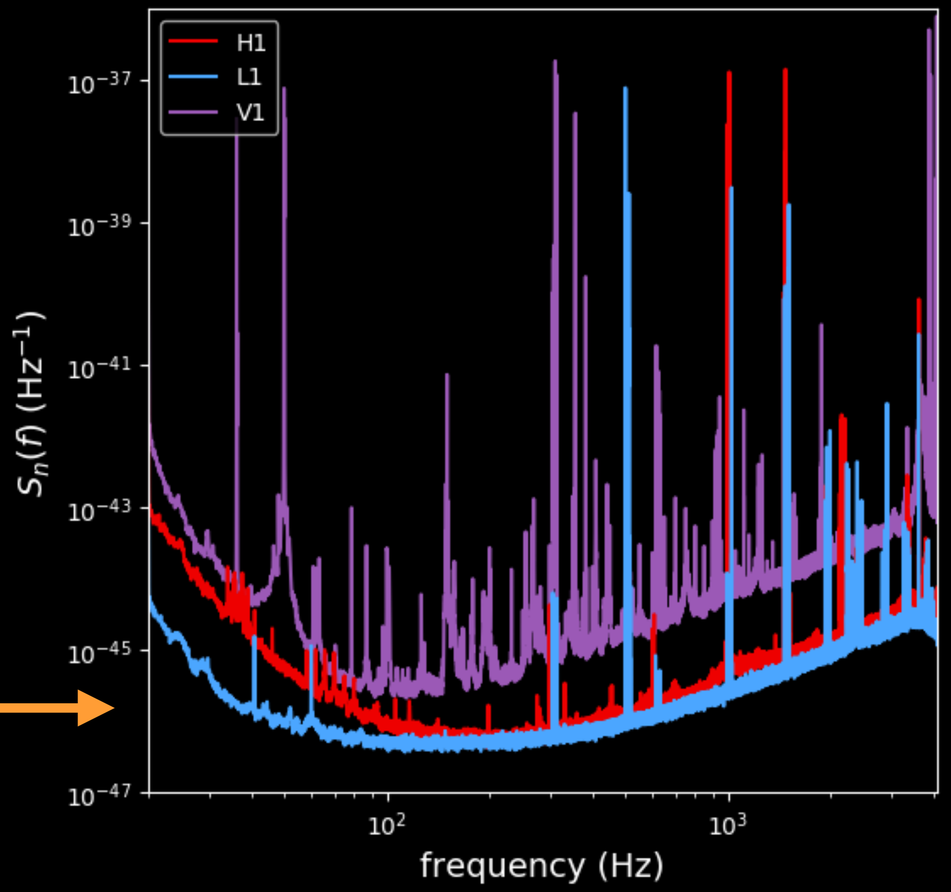
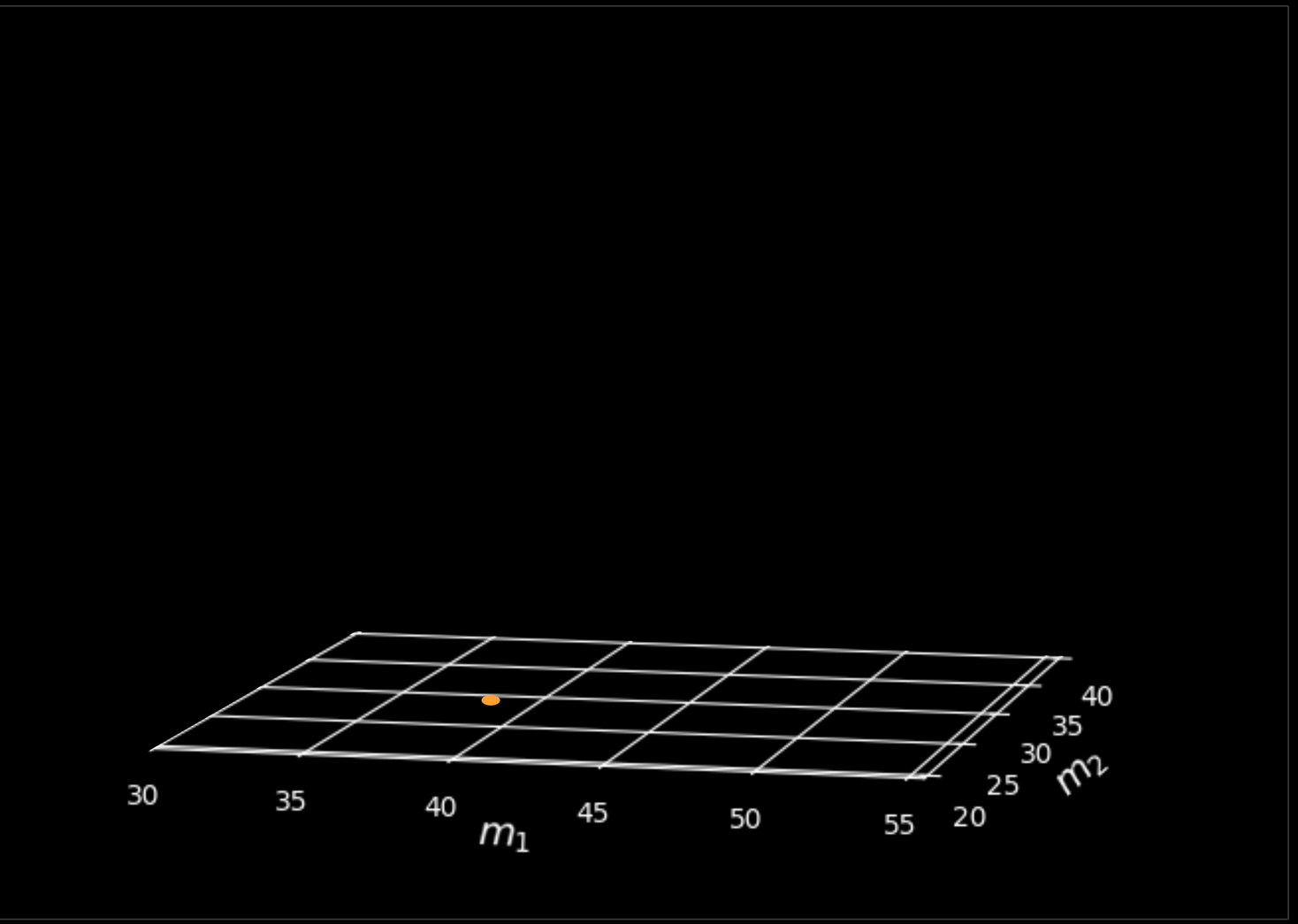
Power Spectral Density (PSD)

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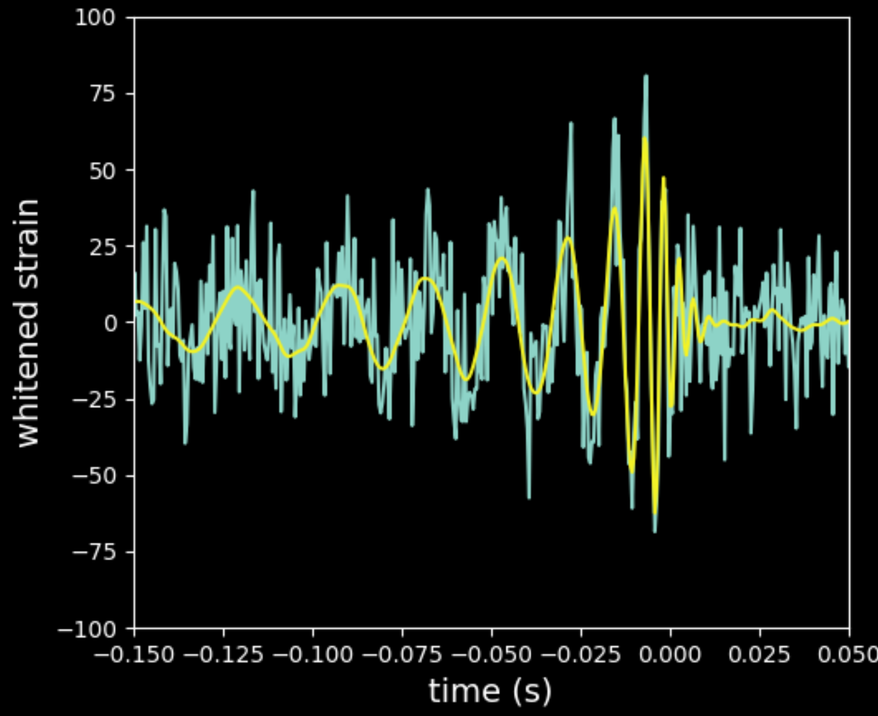
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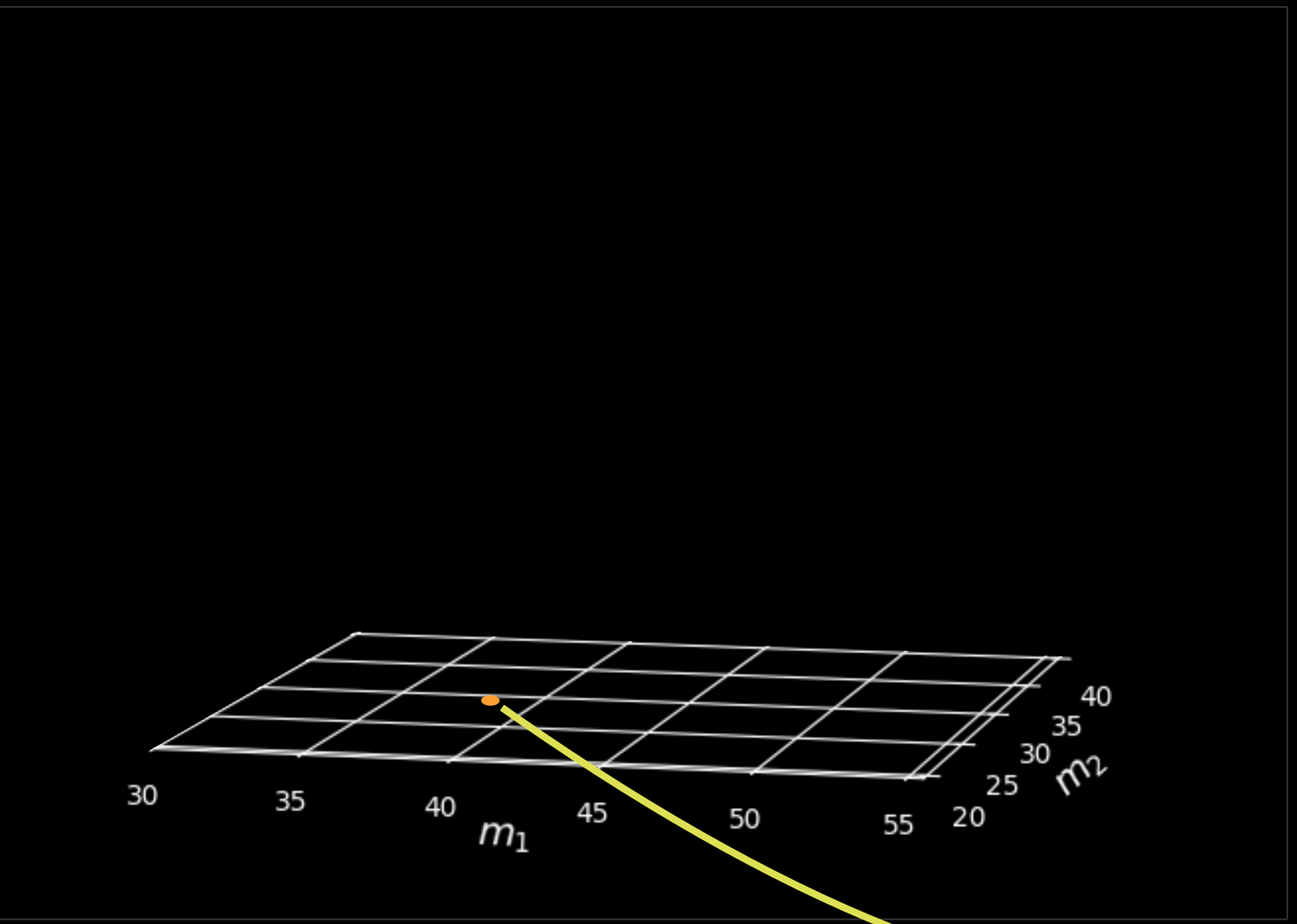
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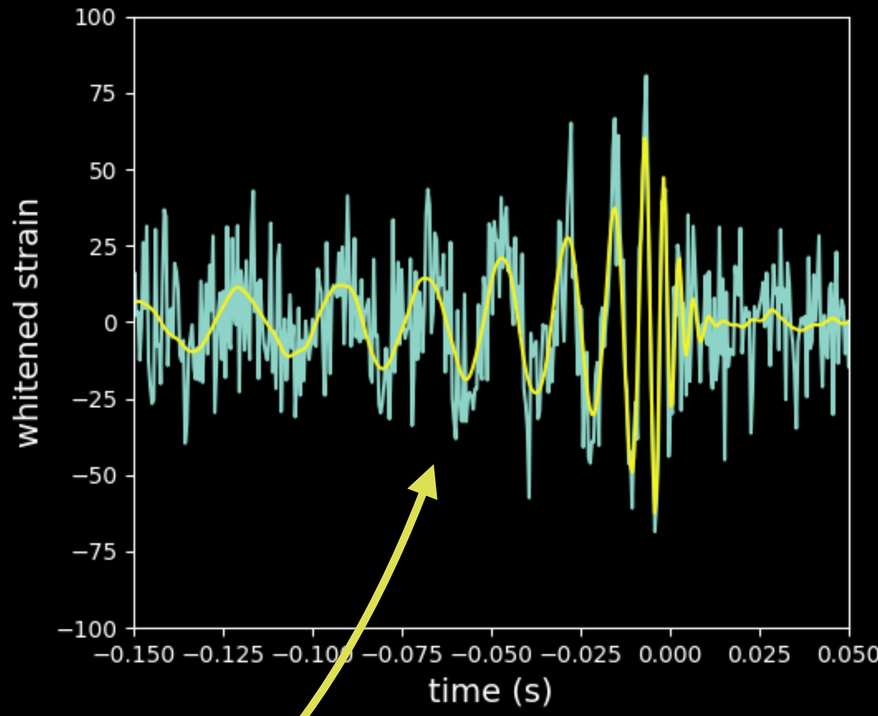
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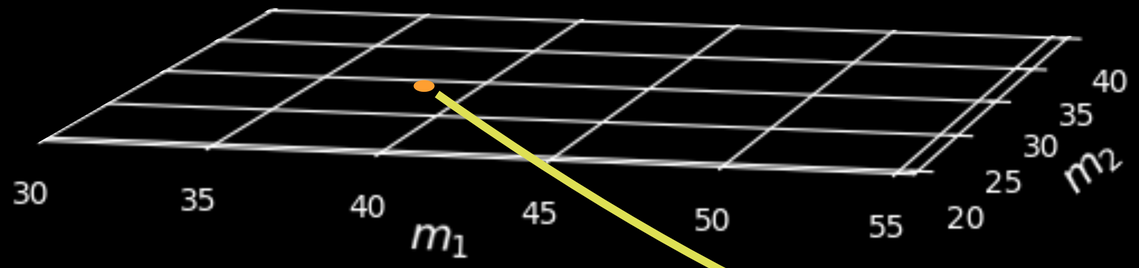

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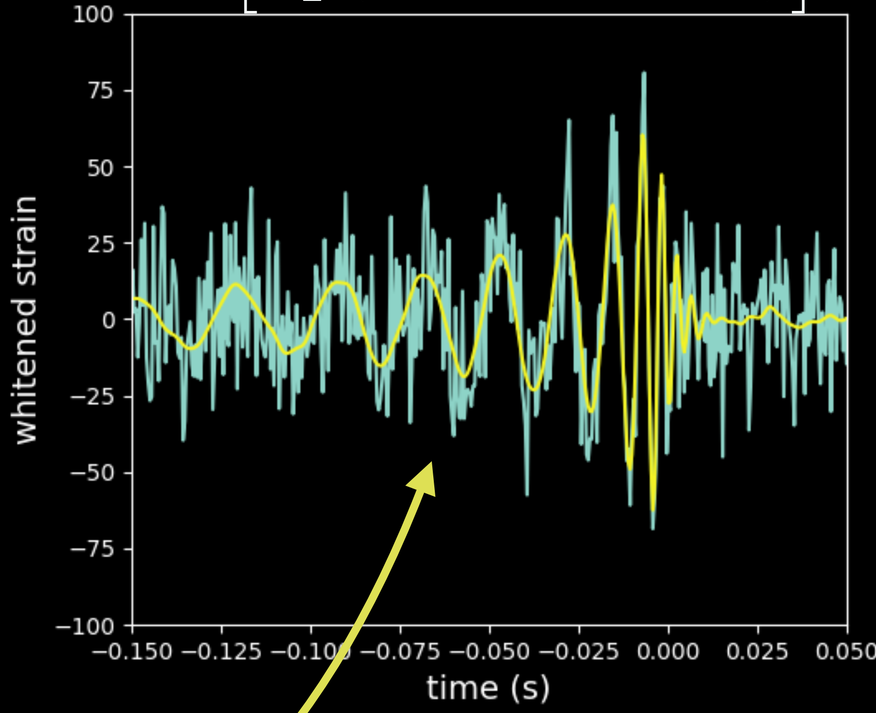

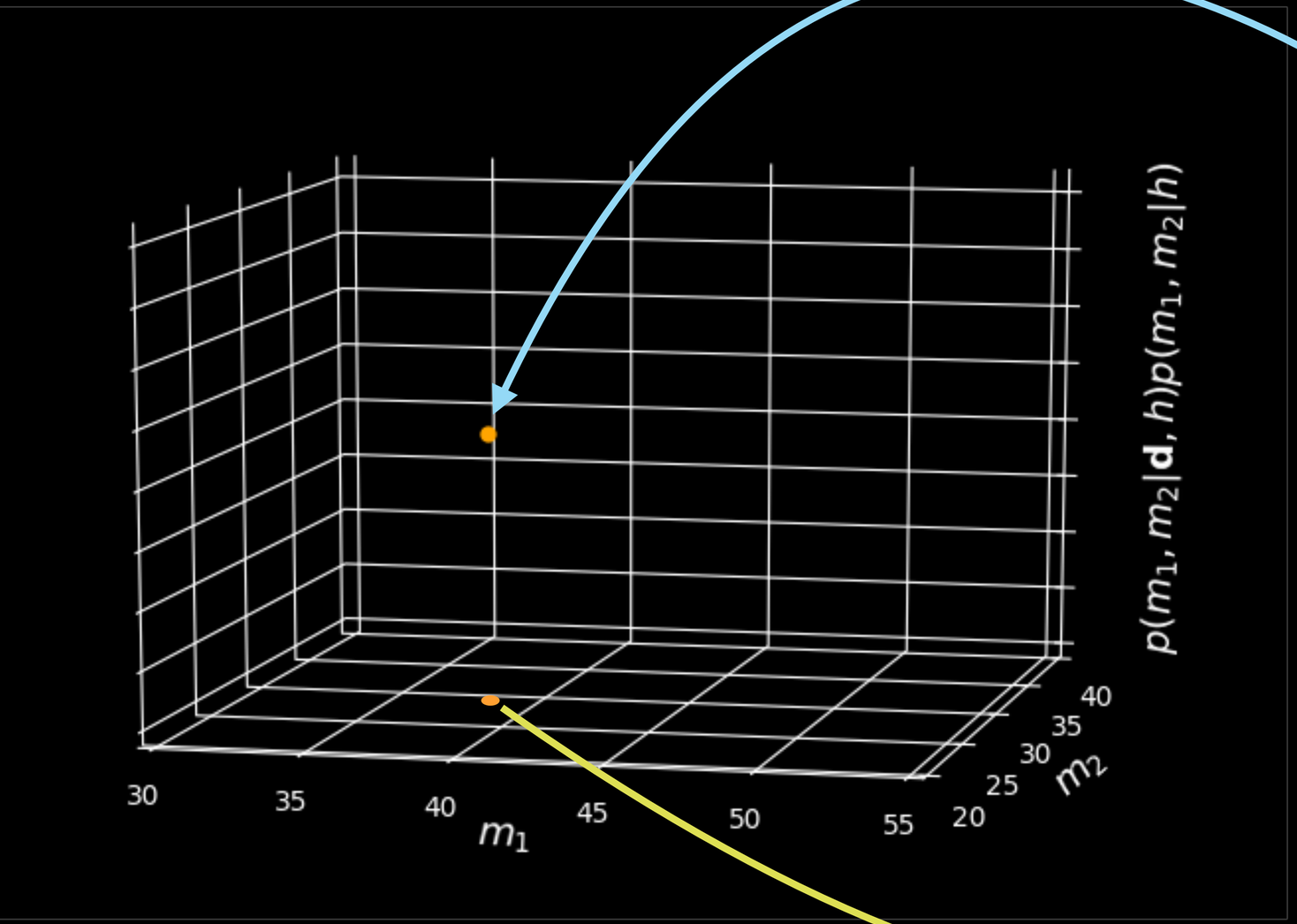

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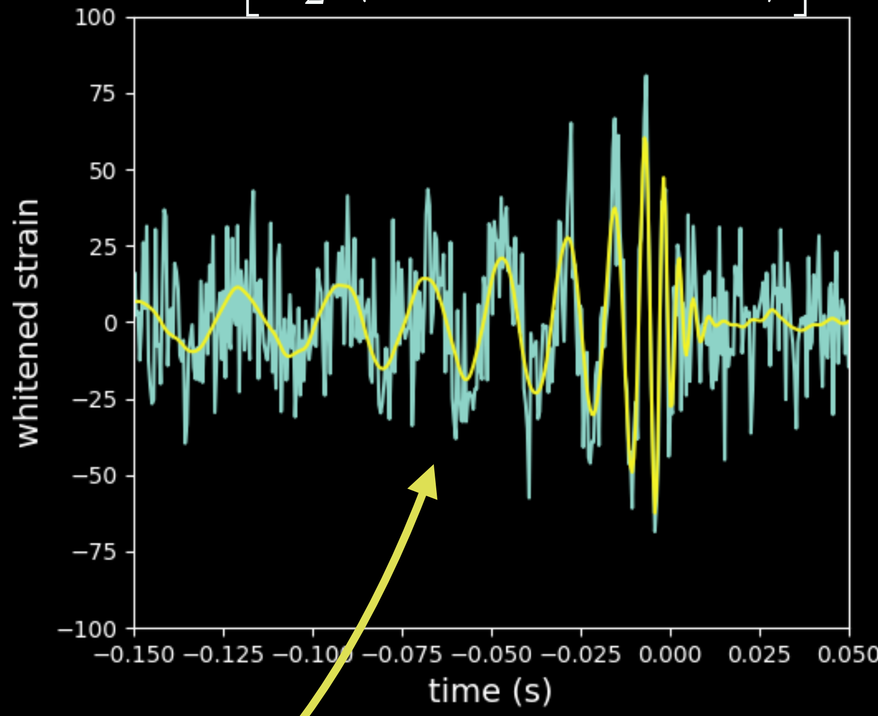

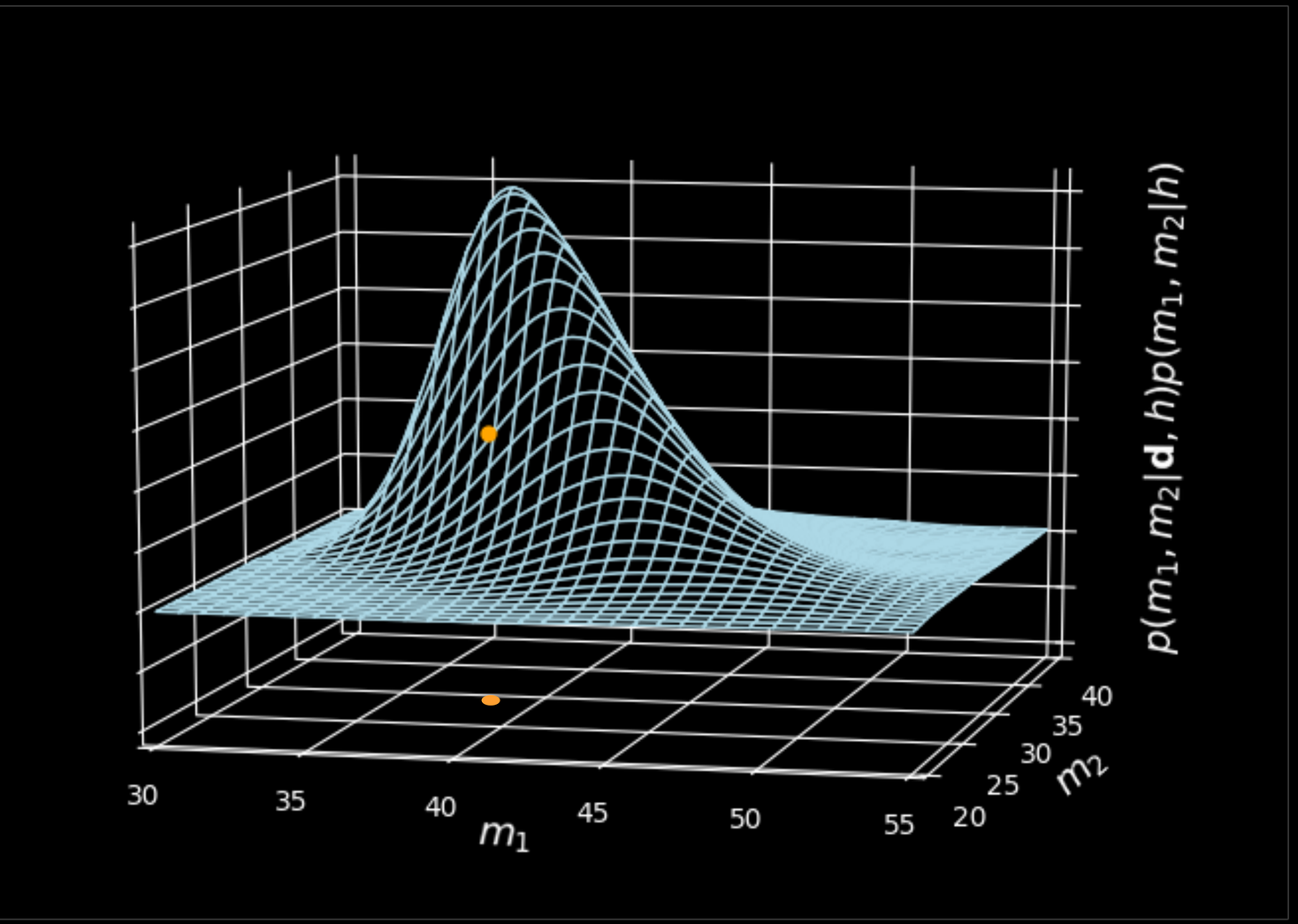
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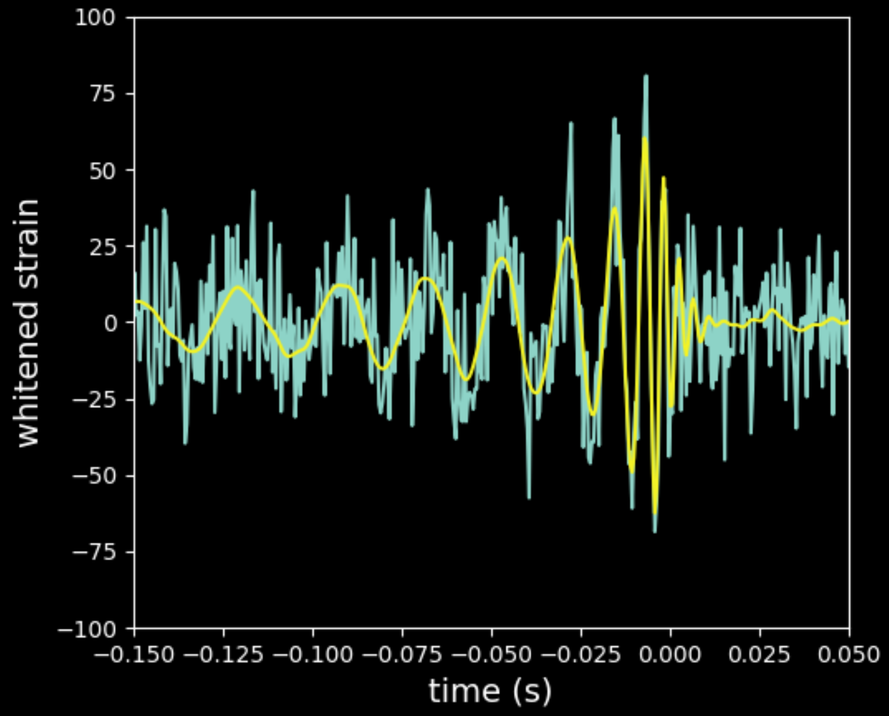
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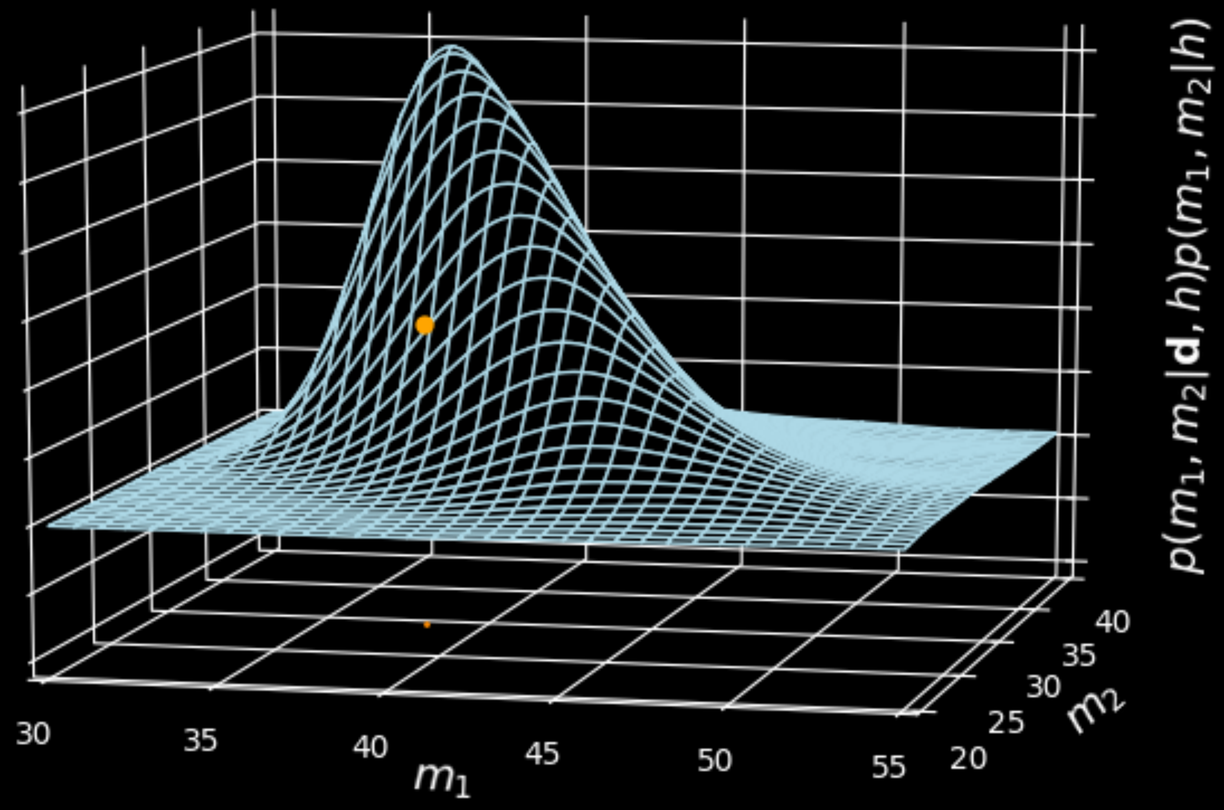
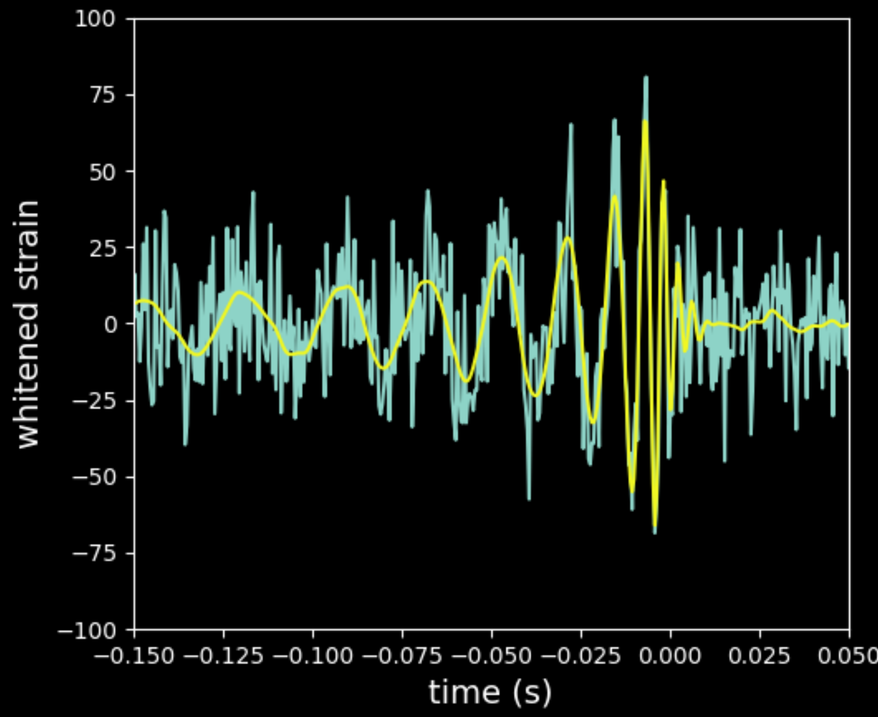
STOCHASTIC SAMPLERS

- Necessary to use stochastic samplers to map out the posterior
- PyCBC Inference has support for two general classes of samplers:
  - Markov chain Monte Carlo (MCMC): emcee, emcee_pt, epsie
  - Nested samplers: dynesty, PyMultiNest, cpnest, ultranest
- Hamiltonian Monte Carlo (e.g., Stan) is another method, but difficult to apply to gravitational waves*

* Bouffanis & Porter, PRD 100 104023 (2019), arXiv:1810.07443

MARKOV CHAIN MONTE CARLO (MCMC)





Metropolis-Hastings algorithm:

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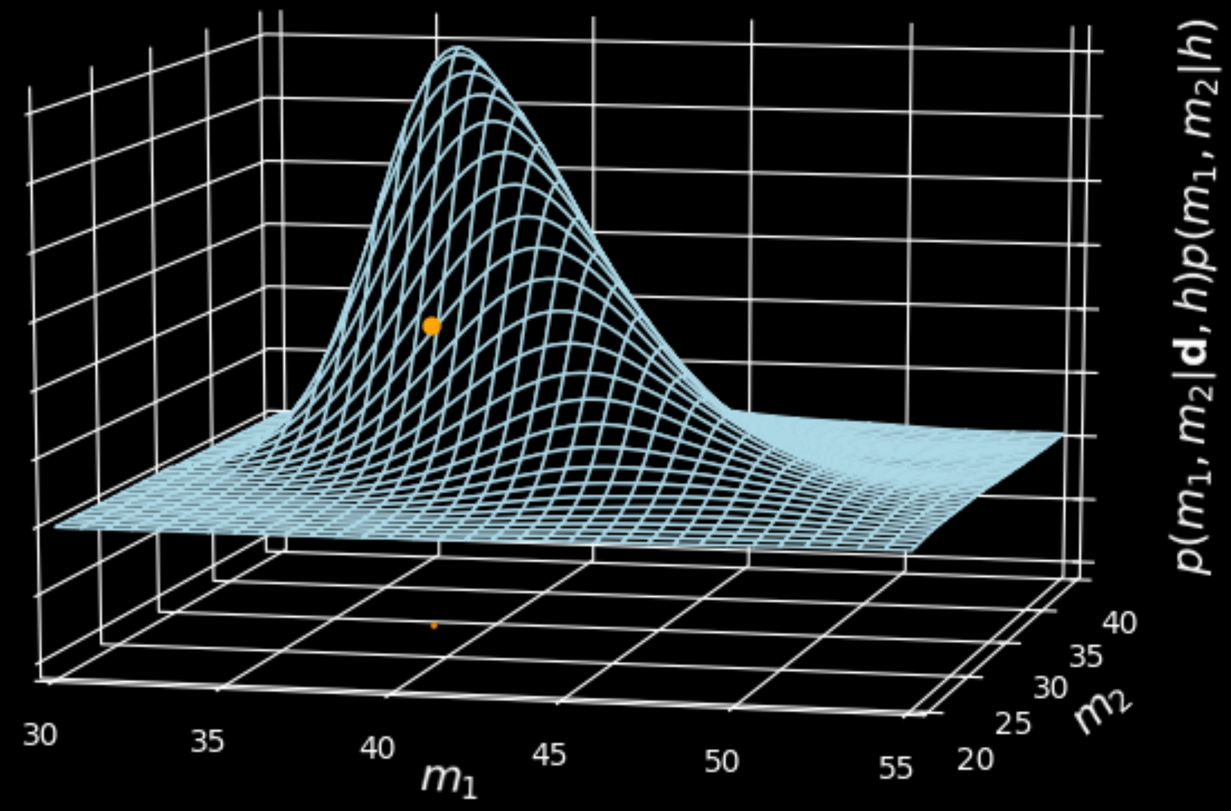

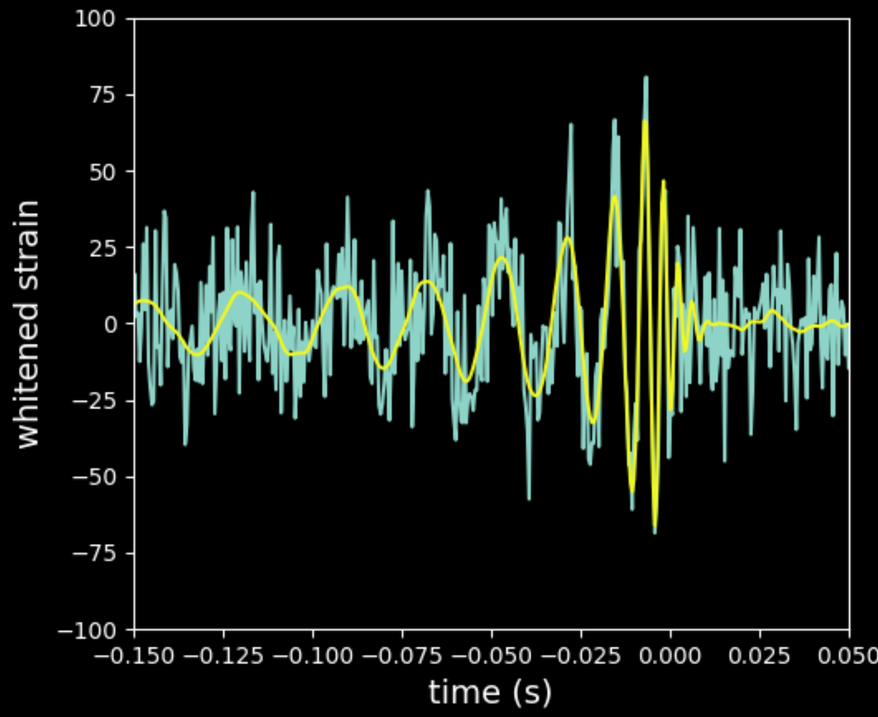

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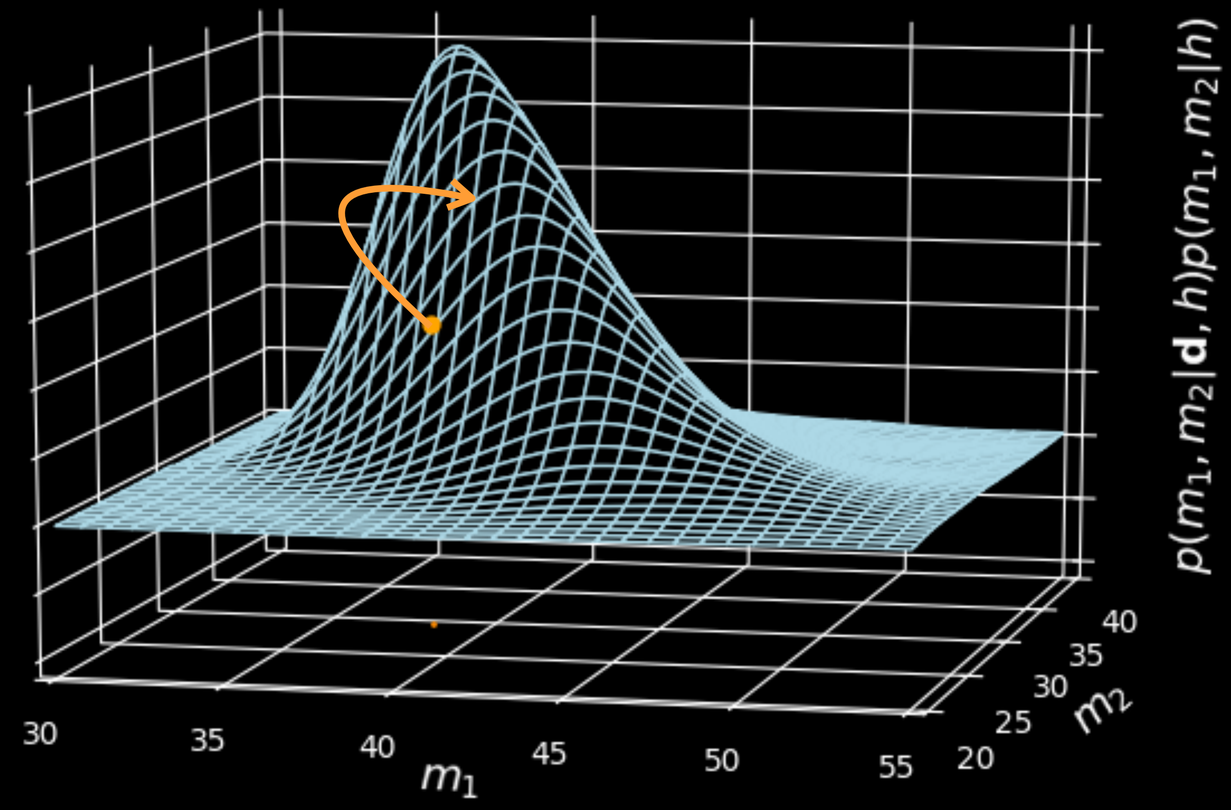

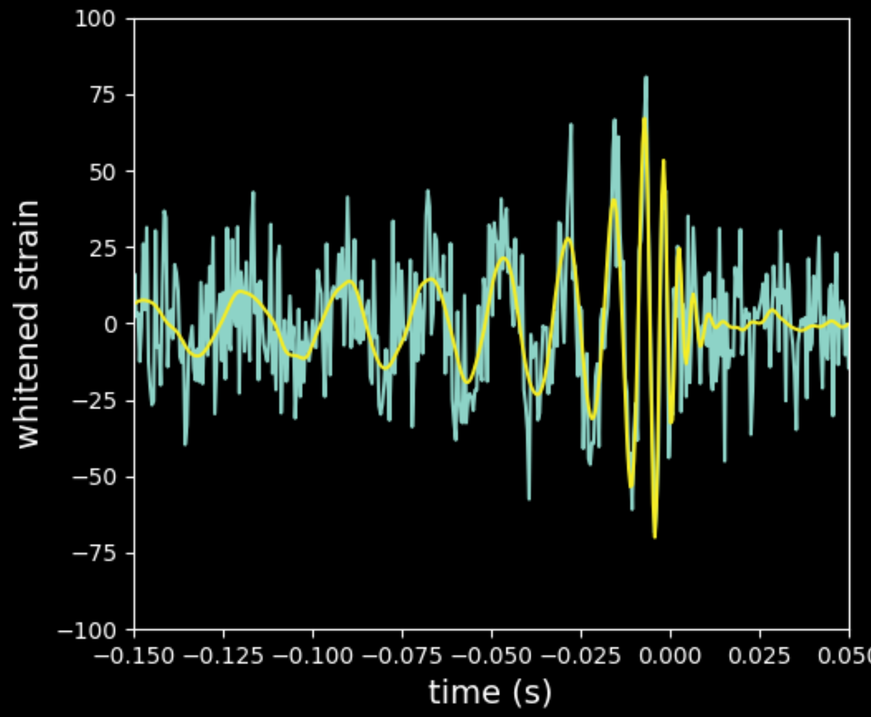

Metropolis-Hastings algorithm:

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2. Draw a new point ϑ_1 from arbitrary distribution $Q(\vartheta_1 \mid \vartheta_0)$

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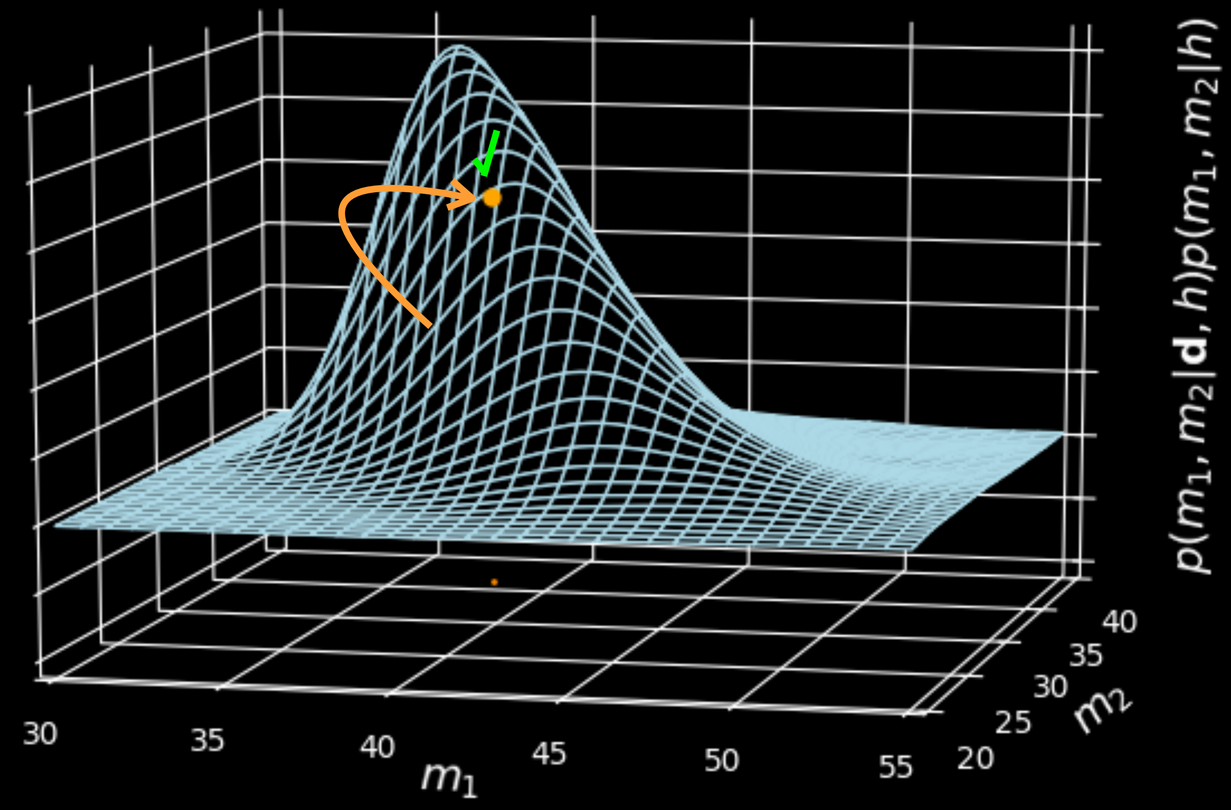

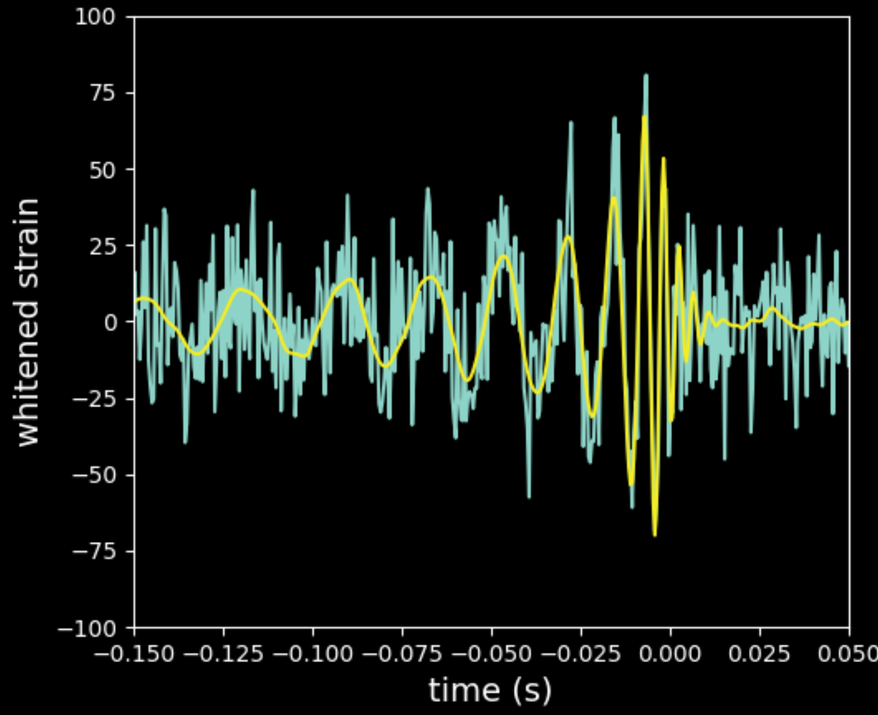

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3. Accept with probability

$$A = \min\left[1,\ \frac{p(\vartheta_1|\mathbf{d},h)}{p(\vartheta_0|\mathbf{d},h)}\frac{Q(\vartheta_0|\vartheta_1)}{Q(\vartheta_1|\vartheta_0)}\right]$$

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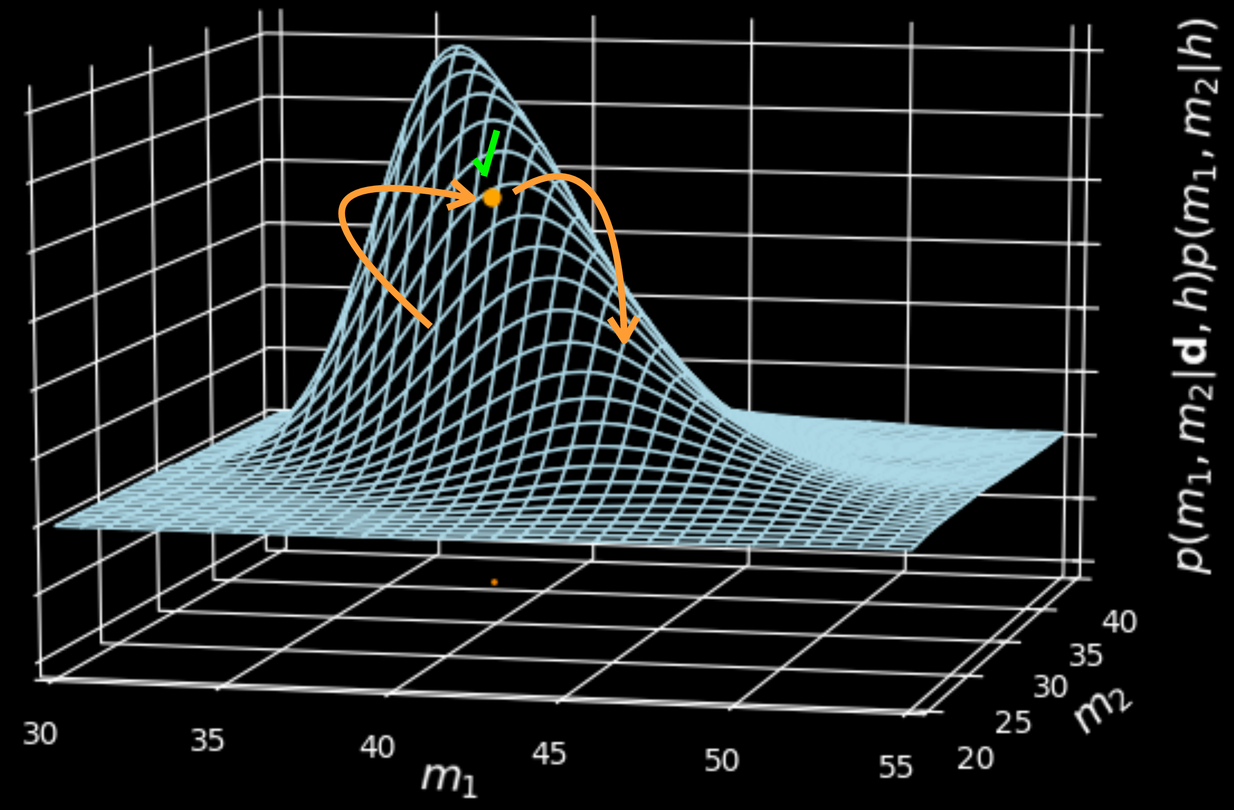

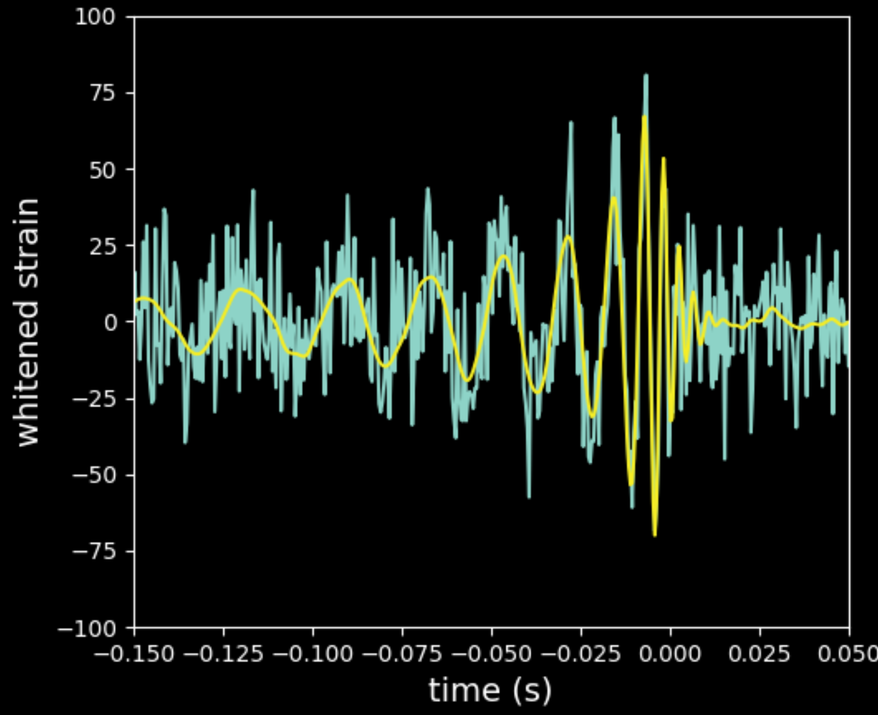

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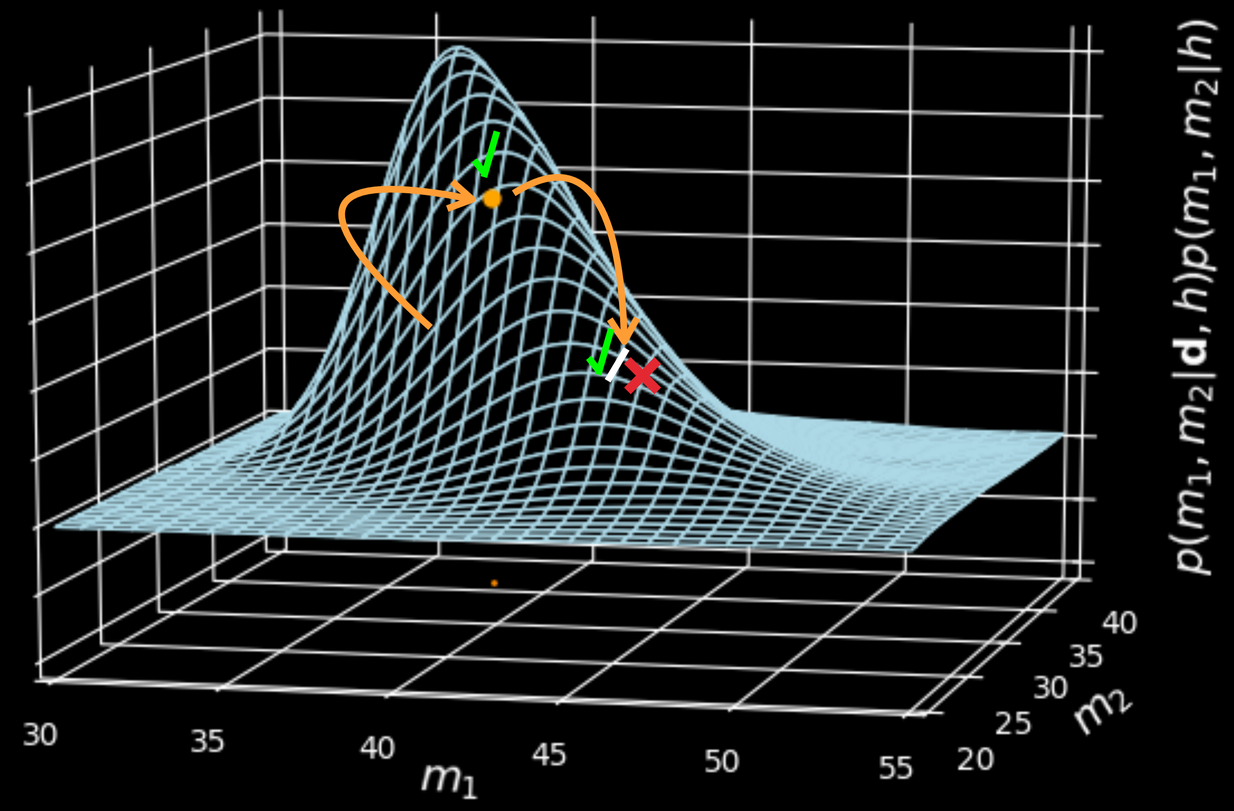
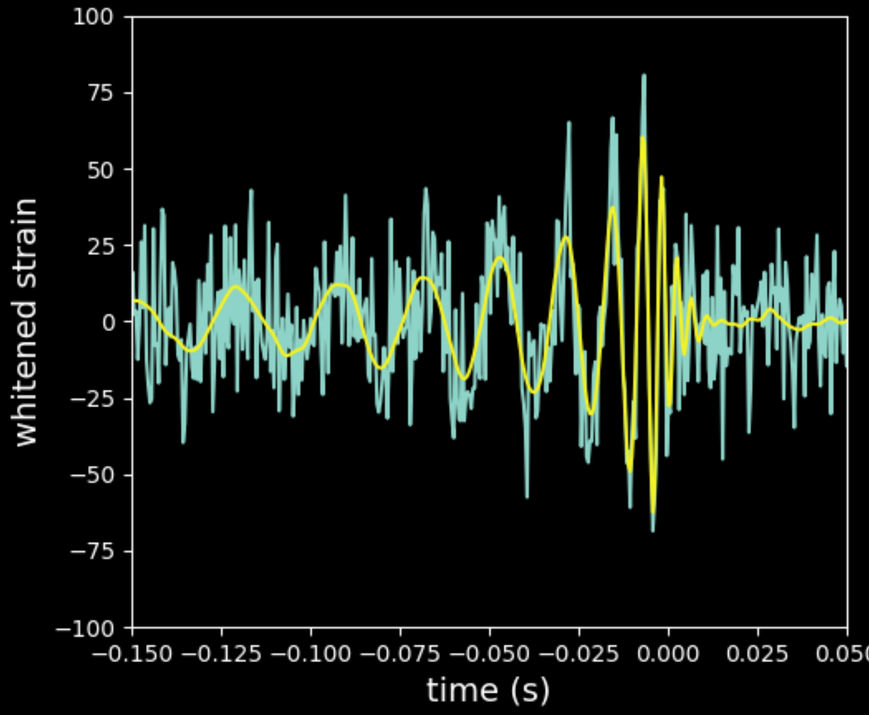
MARKOV CHAIN MONTE CARLO (MCMC)





Metropolis-Hastings algorithm:

1. Start with arbitrary point ϑ_0
2. Draw a new point ϑ_1 from arbitrary distribution $Q(\vartheta_1 \mid \vartheta_0)$
3. Accept with probability

$$A = \min\left[1,\ \frac{p(\vartheta_1|\mathbf{d},h)}{p(\vartheta_0|\mathbf{d},h)}\frac{Q(\vartheta_0|\vartheta_1)}{Q(\vartheta_1|\vartheta_0)}\right]$$

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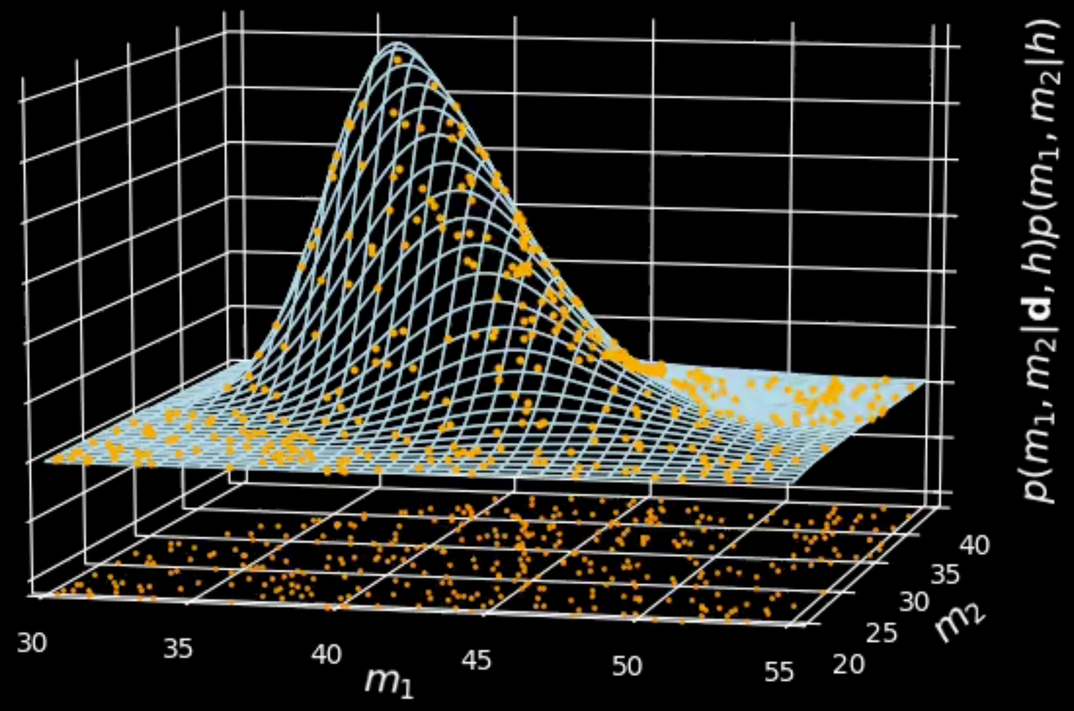
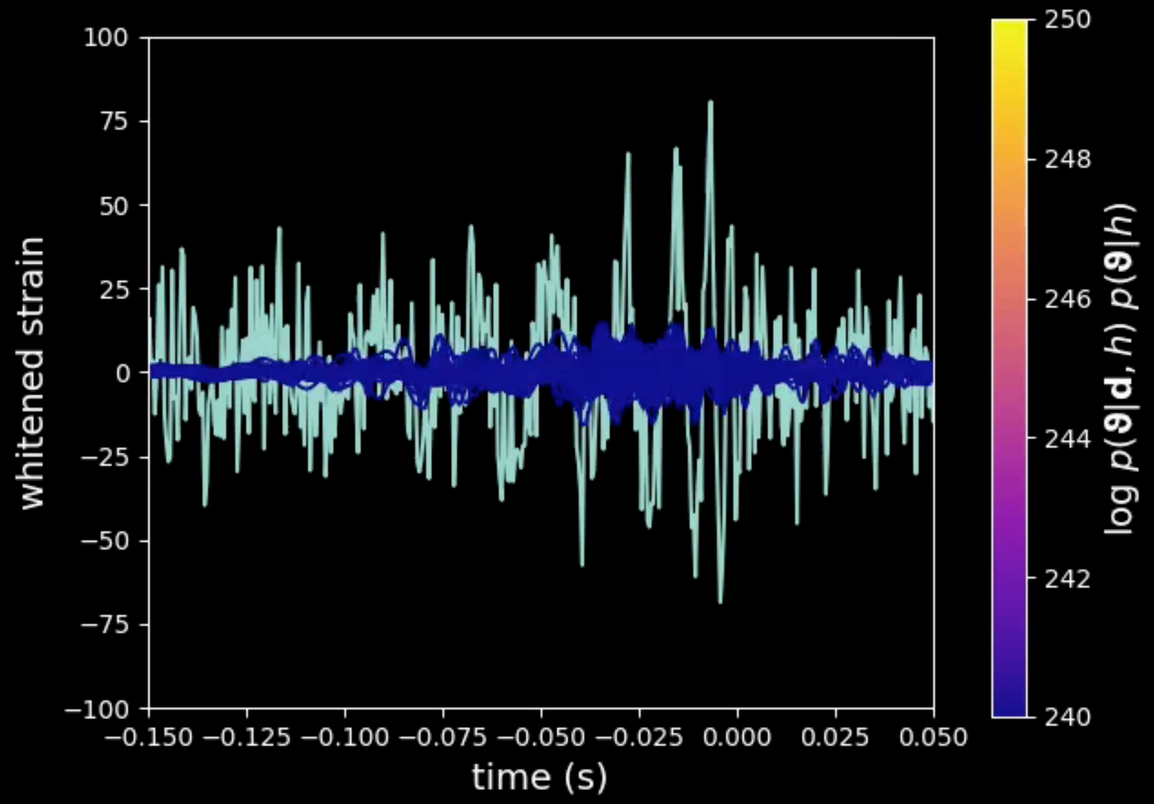
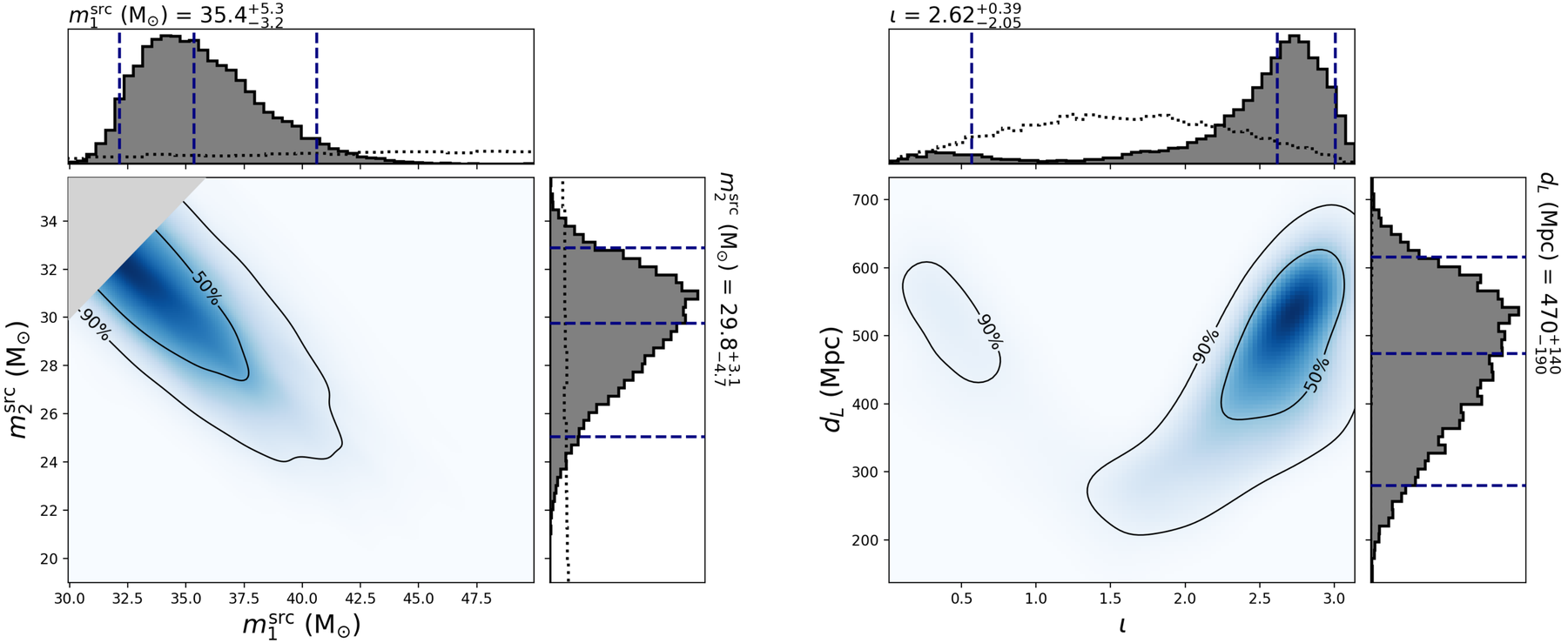
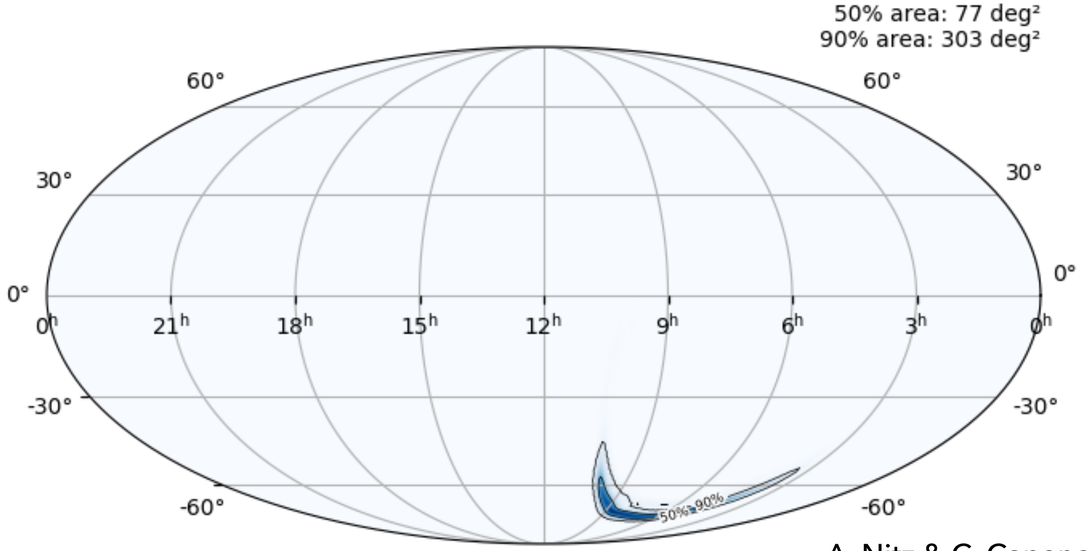




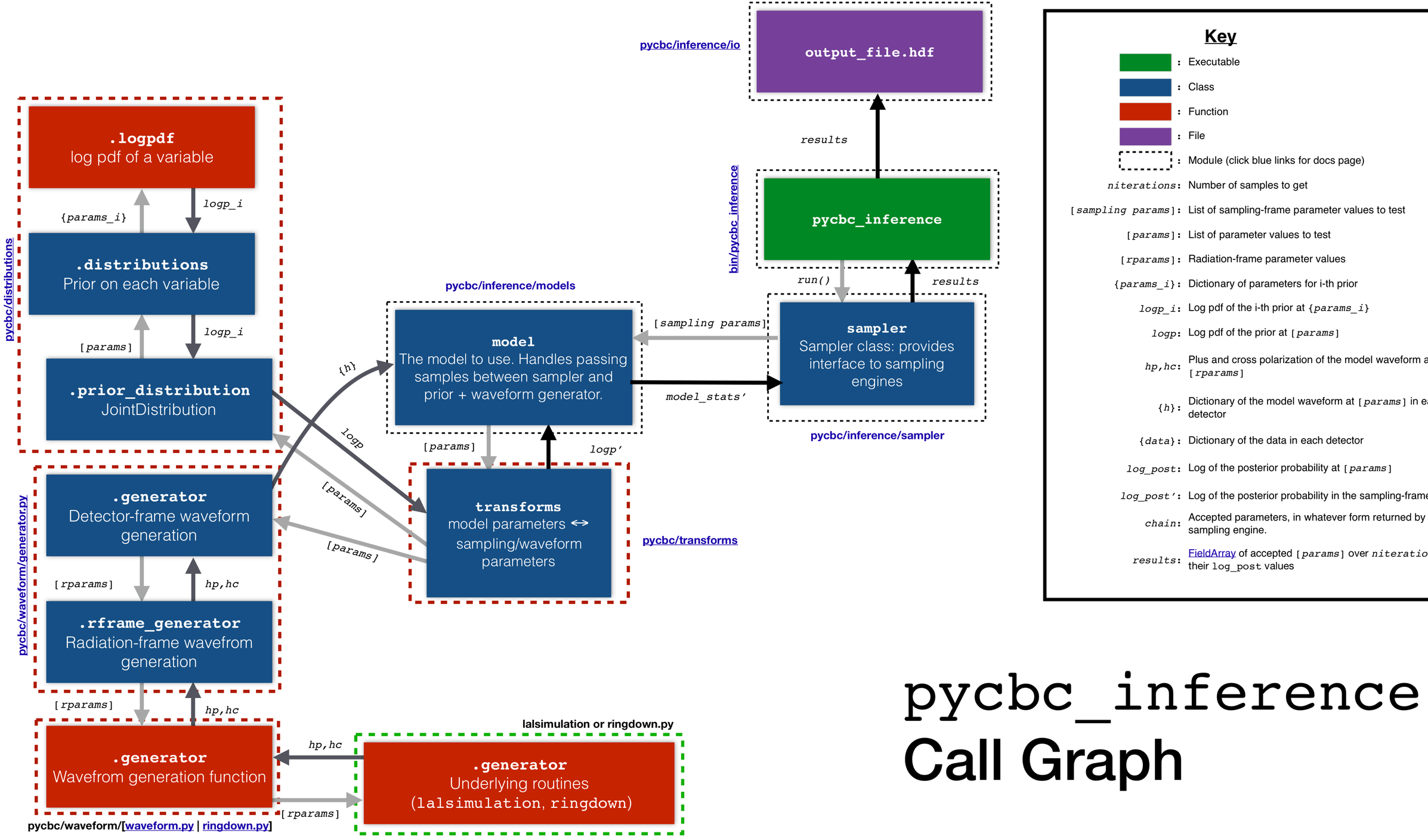
This is a movie; see Keynote version.





A. Nitz & C. Capano, gwcatalog.org/top30_bbh

MODELS & SAMPLERS IN PYCBC INFERENCE



pycbc_inference
Call Graph

MODELS IN PYCBC INFERENCE

- A “model” defines the type of analysis that you are doing
- Defines the likelihood function
- We will be looking at models in **tutorial 1**
- Two general types in PyCBC Inference
- **Analytic models**: define an analytic likelihood, used for testing purposes
  - `test_normal`, `test_eggbox`, `test_rosenbrock`, `test_volcano`
- **Data-based models**: likelihood requires gravitational-wave data and waveform model to calculate inner products
  - `gaussian_noise`, `marginalized_phase`, `relative`, `single_template`

DATA-BASED MODELS

- All of the (current) data-based models in PyCBC assume stationary Gaussian noise
- Differences are various approximations or simplifying assumptions to speed up likelihood evaluation & convergence
- `gaussian_noise`: standard Gaussian likelihood
- `marginalized_phase`: analytically marginalizes over unknown phase; can be used with waveform models that go like $h(f, \phi) \sim h(f, 0) \exp[i\, \phi]$
- `single_template`: fixes intrinsic parameters (masses, spins), just varies extrinsic (sky location, inclination, distance); useful for quickly getting estimate of sky location
- `relative`: uses “relative binning” method from Zackay et al*.; linear interpolation of likeihood around peak to allow for fast calculation

* Zackay et al., arXiv:1806.08792 (2018)

SAMPLERS

- PyCBC Inference uses external packages for samplers
- MCMC: emcee, emcee_pt, epsie
- Nested samplers: dynesty, PyMultiNest, cpnest, ultranest
- Different samplers have strengths and weaknesses

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- PyCBC Inference uses external packages for samplers
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MCMC SAMPLERS

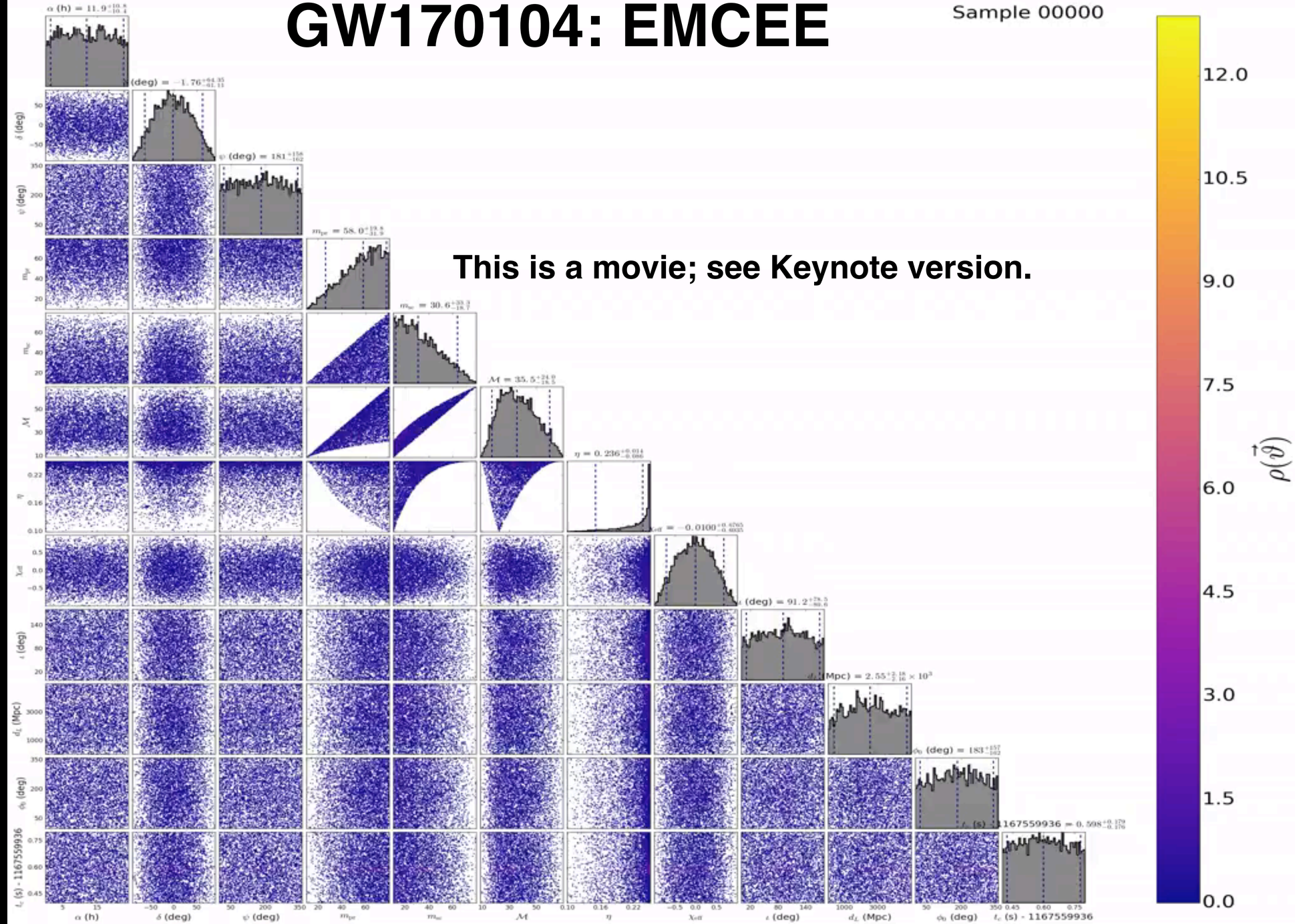
- All MCMC samplers in PyCBC inference use multiple, parallel chains
- emcee, emcee_pt are “ensemble” samplers – the chains share information
- Require some **burn-in** period before the chains resemble the posterior
- Successive samples in a chain are not independent
- Need to calculate the **autocorrelation time** (ACT) of the chains
- ACT gives the number of iterations needed to acquire a new independent sample
- Thinning the chains by the ACT yields independent samples of the posterior

EMCEE

- `emcee`: an ensemble MCMC package
  - Documentation: http://dfm.io/emcee/current/
  - Paper: https://arxiv.org/abs/1202.3665
- Open source, all python; can be installed with pip
- Popular in astronomy

GW170104: EMCEE

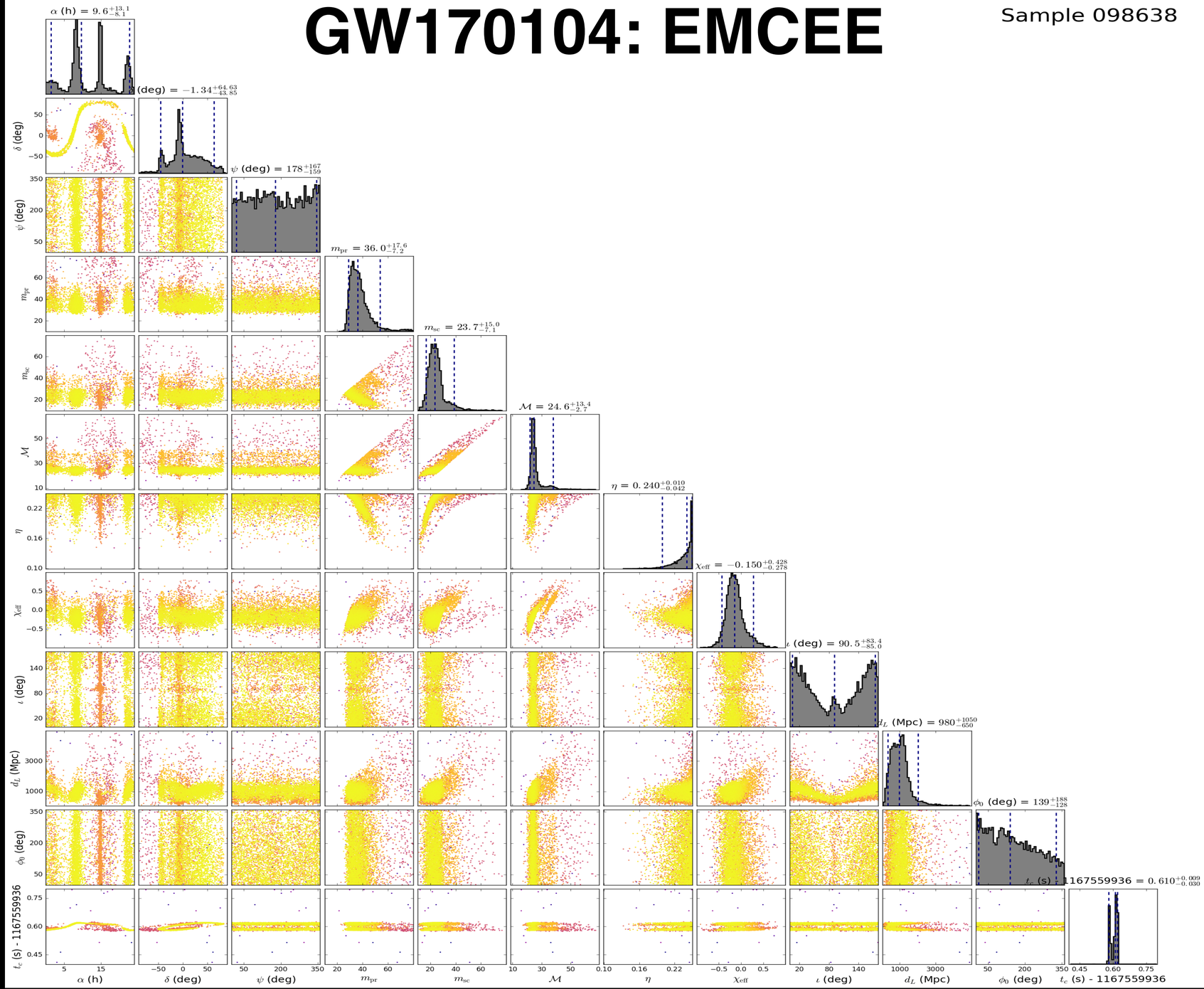
Sample 00000



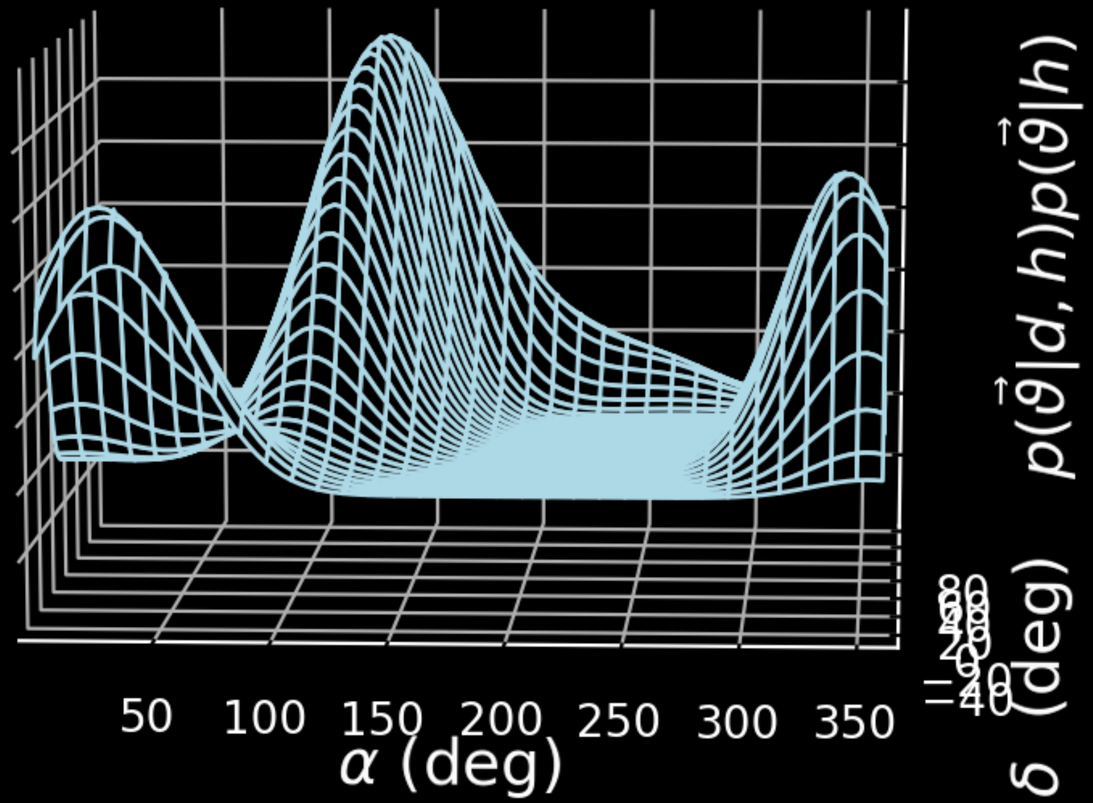
This is a movie; see Keynote version.

GW170104: EMCEE

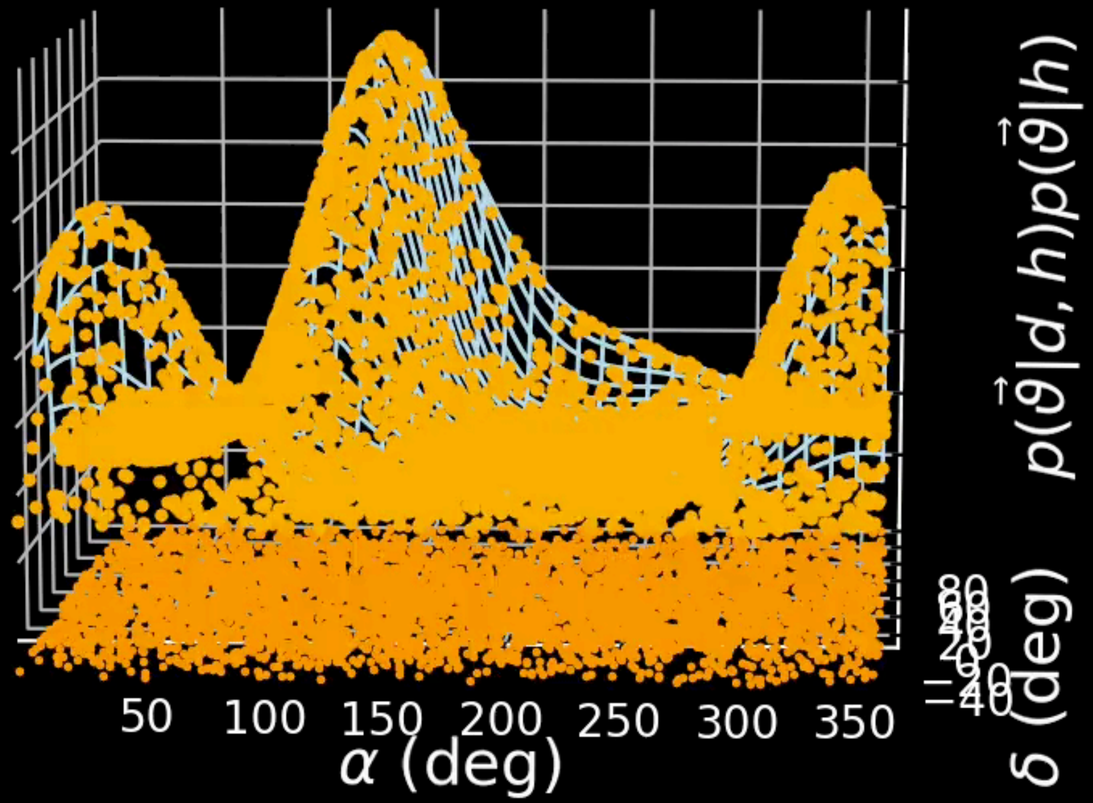
Sample 098638



GW170104: EMCEE



GW170104: EMCEE



This is a movie; see Keynote version.

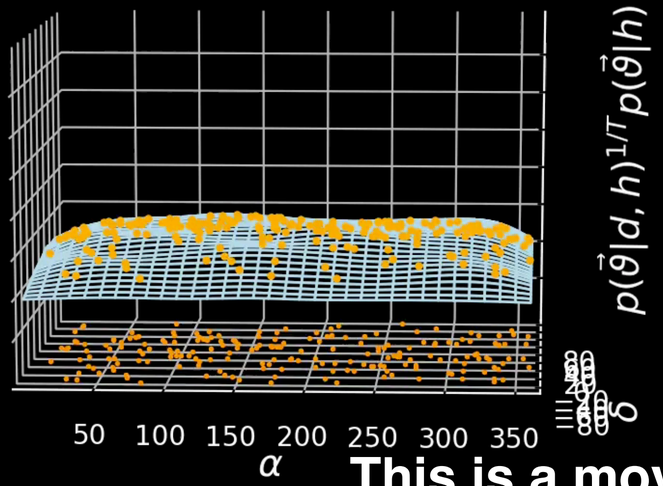
EMCEE PT

- `emcee` struggles to converge for GW problems
- **`emcee_pt`**:* **parallel tempered** version of emcee
- Uses same proposal distribution as emcee, but with parallel tempering

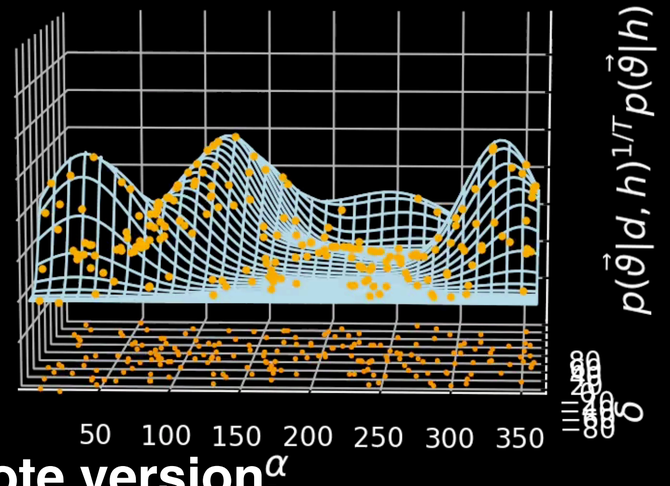
*Note: Current version of emcee_pt is called “ptemcee” & lives in a separate repository. We will probably add support for that in a future update.

PARALLEL TEMPERED MCMC

$T = 10^5$

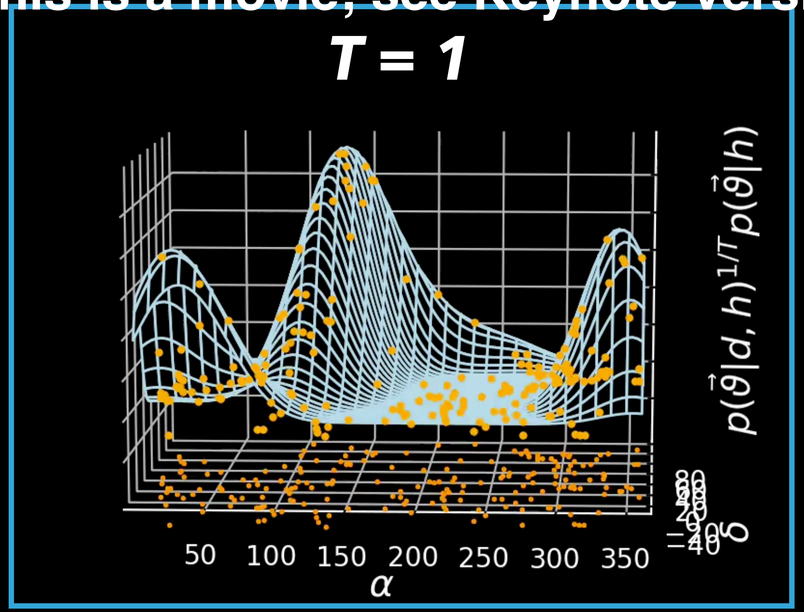


$T = 3.5$



This is a movie; see Keynote version.

$T = 1$



BURN IN TESTS

How do you know when your sampler is converged?

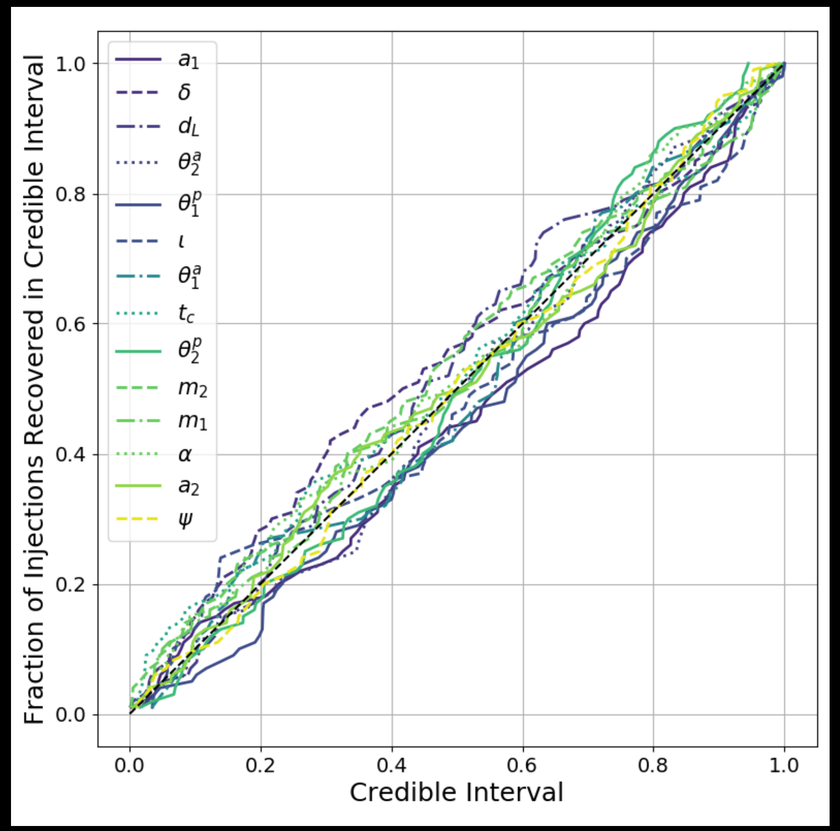
- No single universal test works for all samplers and problems
- Multiple different tests available in PyCBC Inference; see the burn_in module
  - Note: these are for MCMC tests; nested samplers have their own convergence tests
- For emcee_pt, the `nacl` & `max_posterior` tests generally work well
  - `nacl`: checks that the sampler has run for several autocorrelation times
  - `max_posterior`: checks that all chains have sampled a point close to the max posterior

PERCENTILE - PERCENTILE TEST

How do you know the sampler is producing correct results?

- Add signals to realizations of Gaussian noise
- Run PE on each signal, produce marginal posteriors for each parameter
- Test: for each parameter, do X% of the injected values fall within the X% credible interval?

emcee_pt

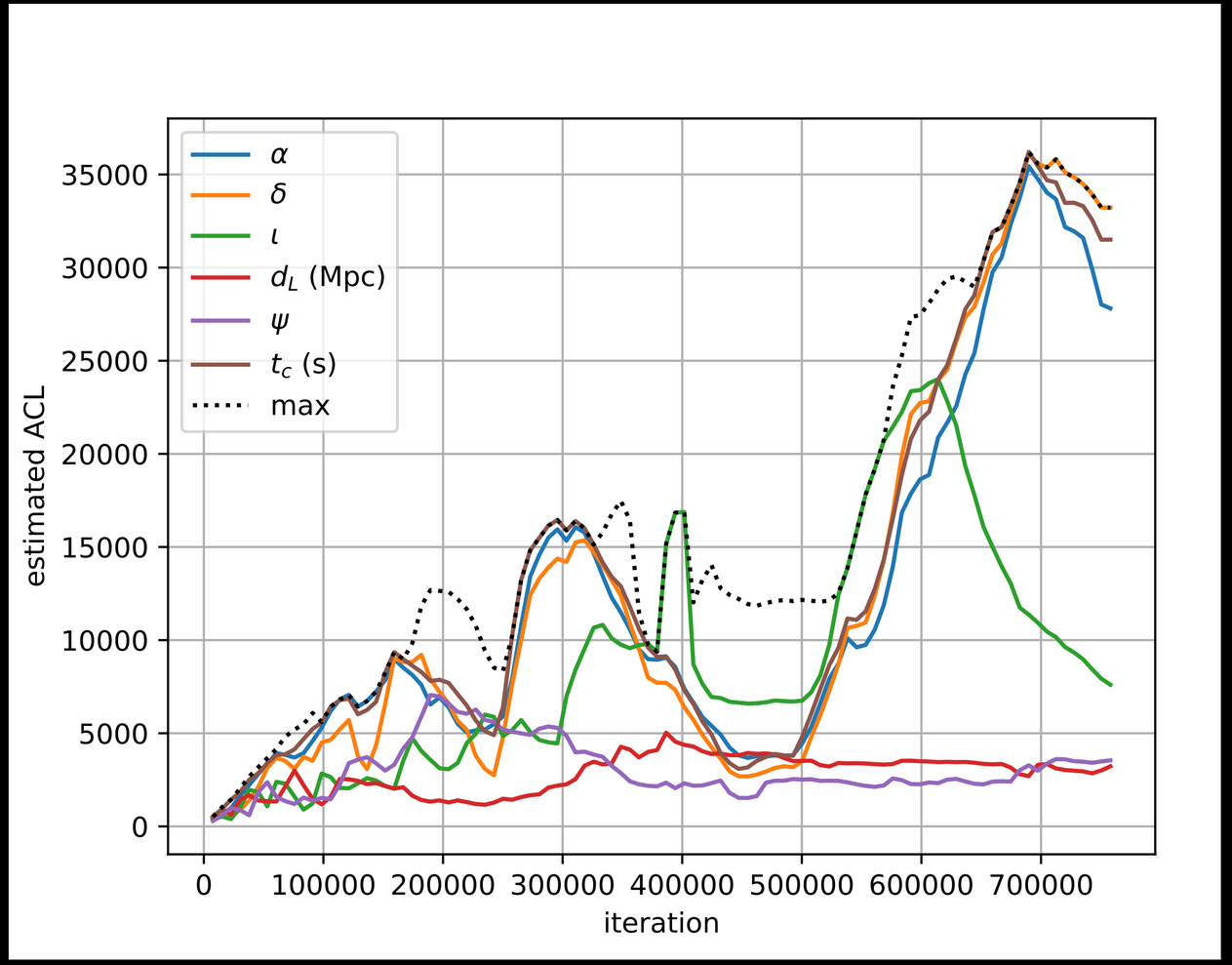


C. Biwer, **C. Capano**, et al., arXiv:1807.10312

Probability of obtaining this graph if emcee_pt provides unbiased parameter estimates: **70% (pass)**

EMCEE PT ISSUES

- Can be slow to converge
- ACT can grow unbounded –> number of effective samples hits a limit (~8 per chain)
- Acquire more samples by running multiple instances, combine results



SAMPLERS

- We offer multiple samplers in PyCBC as each has different strengths & weaknesses.
- MCMC:
  - pros: have been tested extensively; can use any prior with them
  - cons: slow to converge, require parallel tempering
- Nested samplers:
  - pros: provide accurate evidence; fast! (at least dynesty is)
  - cons: only certain priors can be used with them
- **Choose the sampler that best suits your problem!**

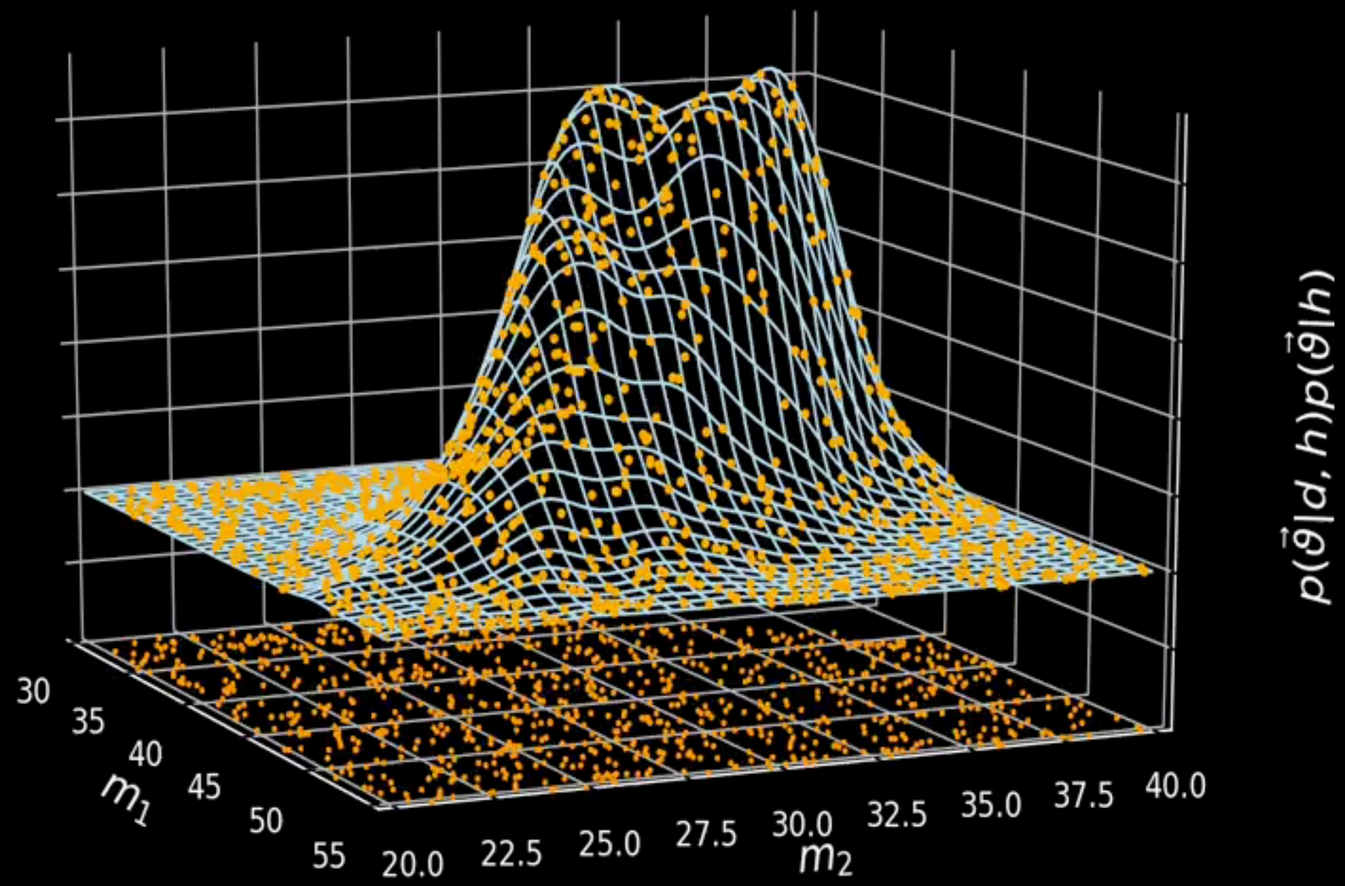
LOOKING AHEAD

- In the tutorials we will go more in-depth into the topics introduced here
  1. Learn about models and do parameter estimation in a notebook
  2. PyCBC Inference code overview
  3. Using PyCBC Inference on an analytic test model
  4. How to estimate the parameters of a binary merger
  5. How to load, plot, and manipulate results
  6. Advanced configuration settings
  7. How to add custom waveform models to PyCBC
- We hope you find this workshop to be engaging and instructive!

THANK YOU!

EXTRAS

emcee uses a
“stretch” proposal



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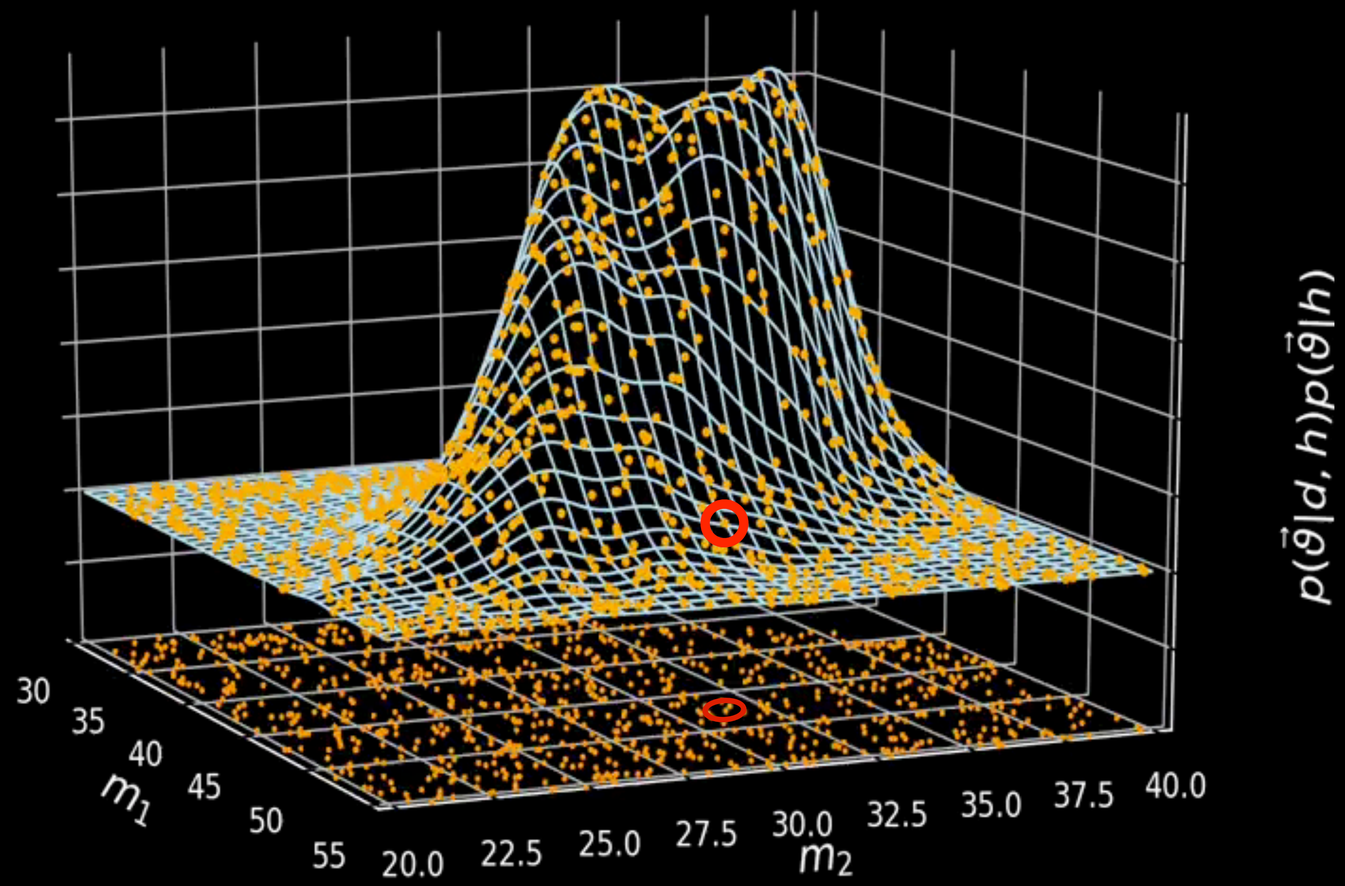

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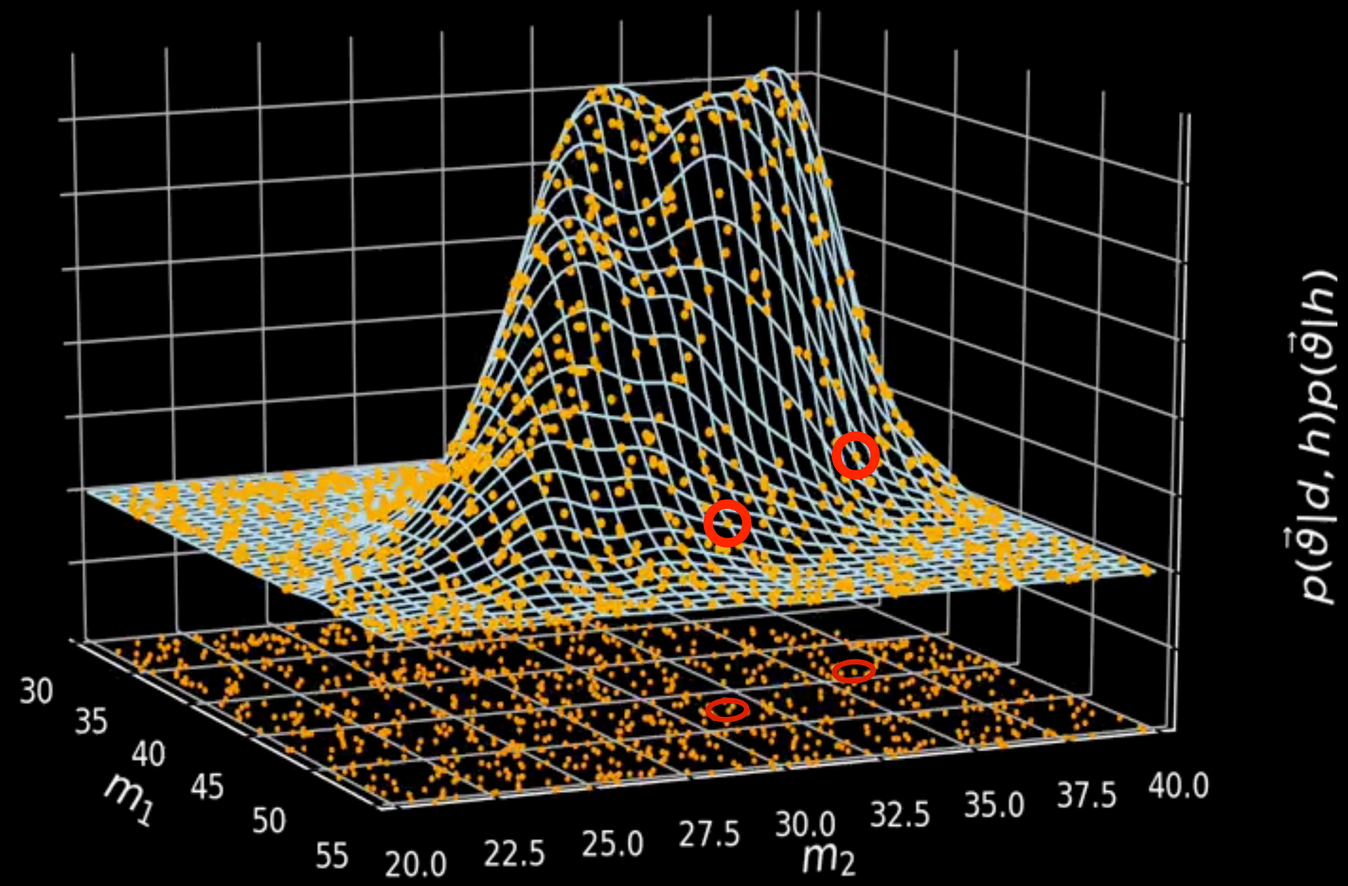

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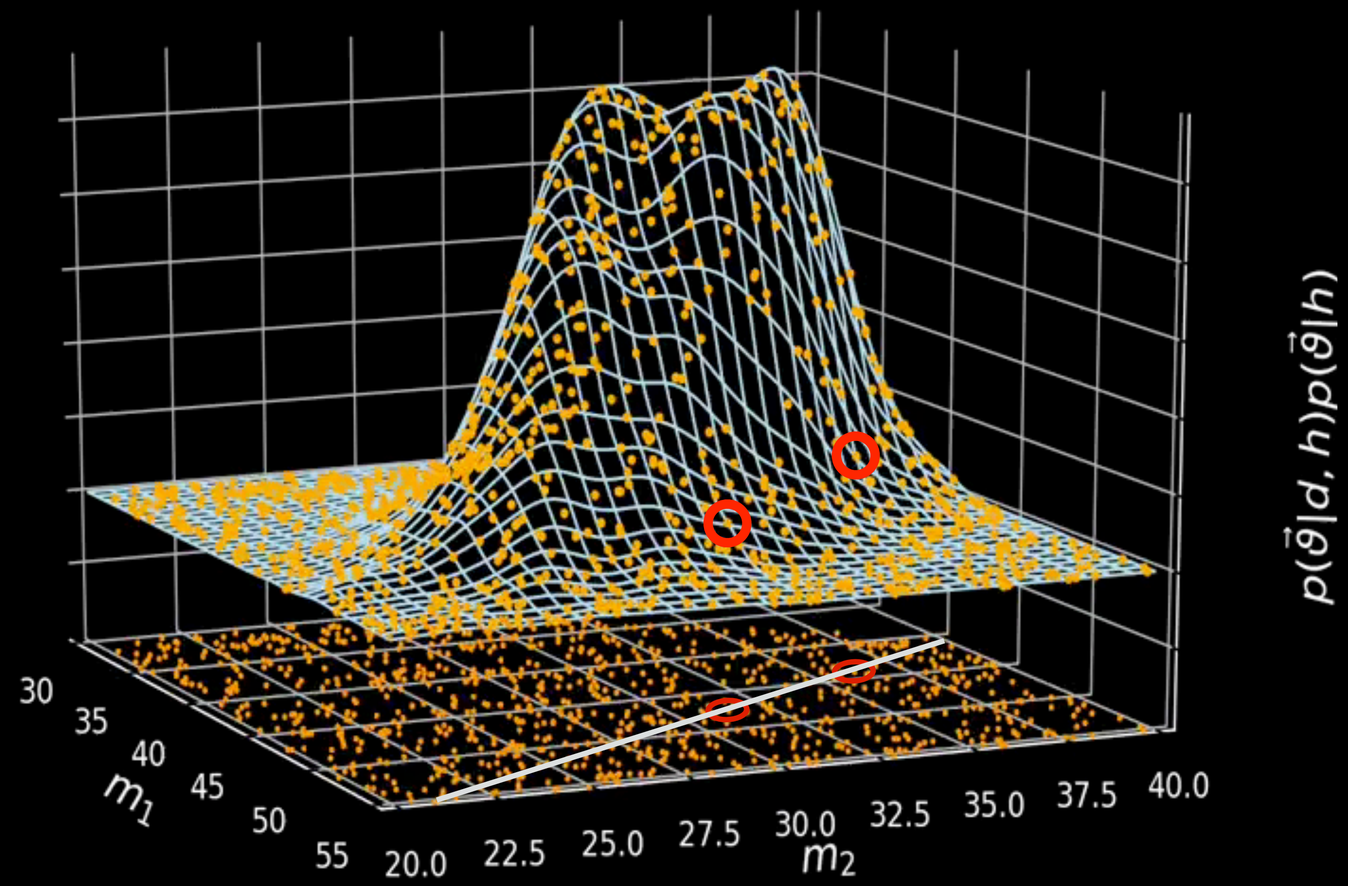

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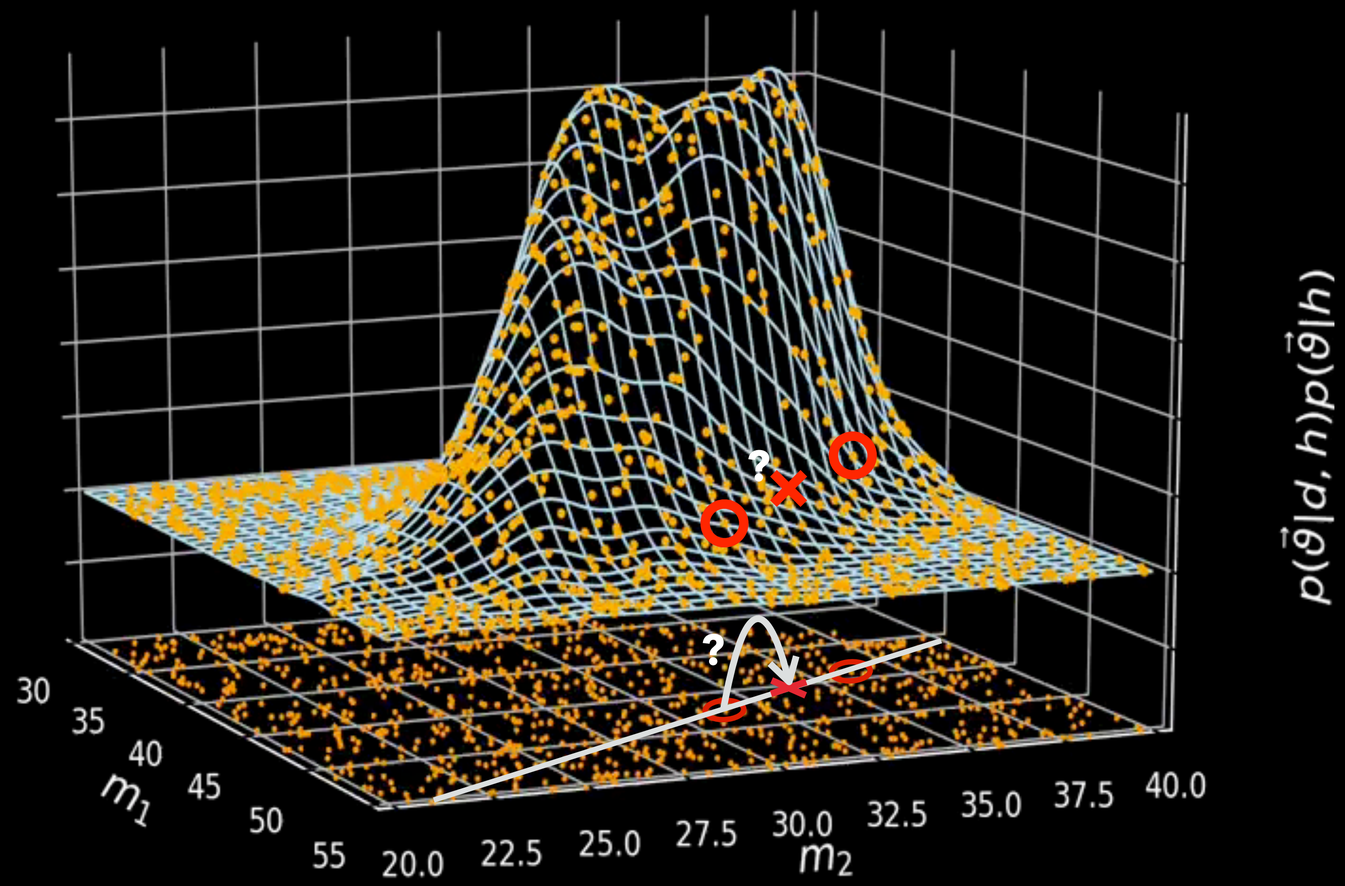

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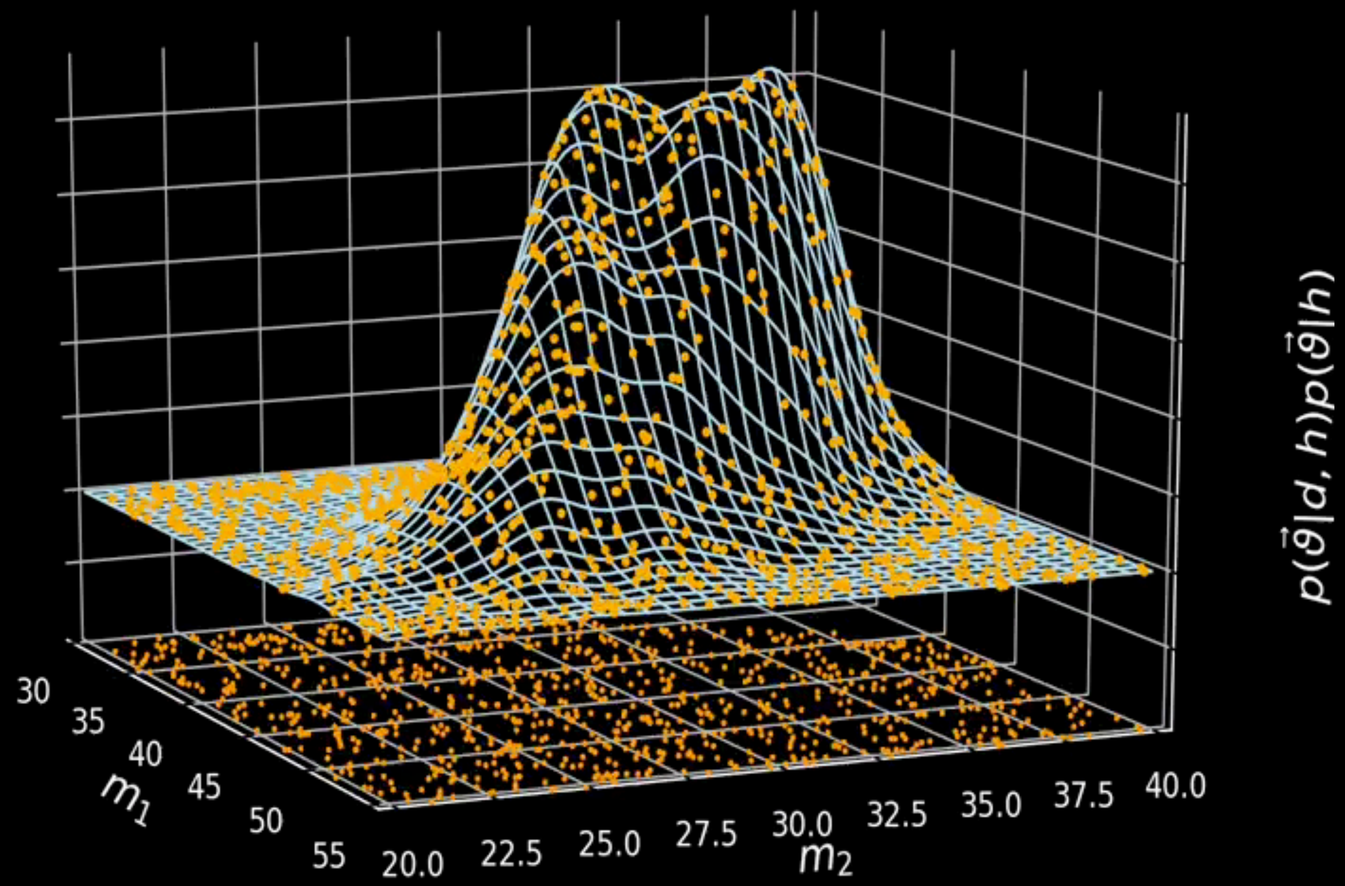

STANDARD LIKELIHOOD DERIVATION

Have some data: $\mathbf{s} = \{s_0, s_1, \cdots, s_N\}$

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Noise hypothesis - assume Gaussian noise:

$$p(\mathbf{s}|n) = \frac{\exp\left[-\frac{1}{2}\mathbf{s}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{s}\right]}{\sqrt{(2\pi)^N \det \boldsymbol{\Sigma}}}$$

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Assume zero-mean, wide-sense stationary (WSS):

$$\boldsymbol{\Sigma} = \frac{1}{2}\begin{bmatrix}
R_{ss}[0] & R_{ss}[1] & \cdots & R_{ss}[\Delta_{\max}] & & & & & \\
R_{ss}[1] & R_{ss}[0] & \ddots & \ddots & \ddots & & & \mathbf{0} & \\
\vdots & R_{ss}[1] & \ddots & \ddots & \ddots & \ddots & & & \\
R_{ss}[\Delta_{\max}] & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & R_{ss}[\Delta_{\max}] \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 & \mathbf{0} & & & \ddots & \ddots & \ddots & \ddots & R_{ss}[1] \\
 & & & & & R_{ss}[\Delta_{\max}] & \cdots & R_{ss}[1] & R_{ss}[0]
\end{bmatrix}$$

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$$\boldsymbol{\Sigma} \approx \frac{1}{2}\begin{bmatrix}
R_{ss}[0] & R_{ss}[1] & \cdots & R_{ss}[\Delta_{\max}] & & & R_{ss}[\Delta_{\max}] & \cdots & R_{ss}[1] \\
R_{ss}[1] & R_{ss}[0] & \ddots & \ddots & \ddots & & & \ddots & \vdots \\
\vdots & R_{ss}[1] & \ddots & \ddots & \ddots & \ddots & \mathbf{0} & & R_{ss}[\Delta_{\max}] \\
R_{ss}[\Delta_{\max}] & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & R_{ss}[\Delta_{\max}] \\
R_{ss}[\Delta_{\max}] & & \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & & & \ddots & \ddots & \ddots & \ddots & R_{ss}[1] \\
R_{ss}[1] & \cdots & R_{ss}[\Delta_{\max}] & & & R_{ss}[\Delta_{\max}] & \cdots & R_{ss}[1] & R_{ss}[0]
\end{bmatrix}$$

STANDARD LIKELIHOOD DERIVATION

Eigenvectors: $$\mathbf{U}_{kp} = \frac{1}{\sqrt{N}} \exp\left[-2\pi i k p / N\right]$$

⇒ eigenvalues: $$\lambda_p = \frac{S_n[p]}{2\Delta t}$$

Eigendecomposition yields:

$$\mathbf{s}^\mathsf{T} \boldsymbol{\Sigma}^{-1} \mathbf{s} = 4\Delta f \sum_{p=1}^{N/2-1} \frac{\tilde{s}^*[p]\tilde{s}[p]}{S_n[p]} \equiv \langle \mathbf{s}, \mathbf{s} \rangle$$

Signal hypothesis:

$$\mathbf{s} = \mathbf{n} + \mathbf{h}(\boldsymbol{\vartheta})$$

$$\Rightarrow p(\mathbf{s}|\boldsymbol{\vartheta}, h) \propto \exp\left[-\frac{1}{2}\langle \mathbf{s} - \mathbf{h}(\boldsymbol{\vartheta}), \mathbf{s} - \mathbf{h}(\boldsymbol{\vartheta}) \rangle\right]$$