Shape Analysis of Elastic Curves in Euclidean Spaces

Rayane Mouhli, Gabriel Watkinson

MVA - ENS Paris Saclay

January 30, 2023

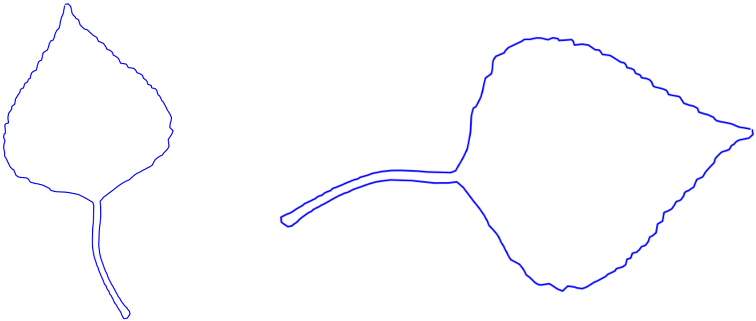
Introduction

- How can we **represent and analyze shapes** in 2D, 3D ?
- Shape spaces have a **Riemannian structure** : **how to build a metric ?**
- **Invariant** to scaling, translation, rotation and re-parametrization.
- We need such metrics for **statistical analysis on shapes :** clustering, classification, hypothesis testing, probability models, ...
- *Shape Analysis of Elastic Curves in Euclidean Spaces* by Srivastava et al. (2010)

Introduction

The goal of shape analysis is to **define a mathematical framework** describing shapes.

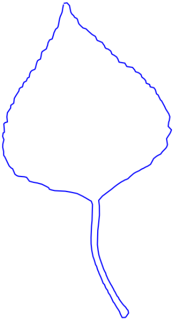
- Transformation invariant :

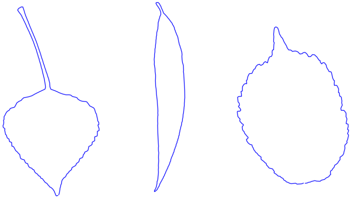


Introduction

The goal of shape analysis is to **define a mathematical framework** describing shapes.

- What is the distance to other shapes :





Tables of contents

Fields of application

- **Biology** : cells, organs, . . .

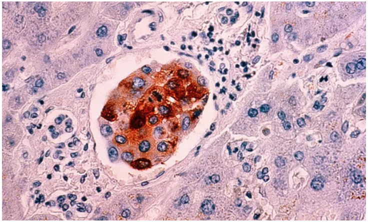


Figure: Microscopic image of a cancerous cell

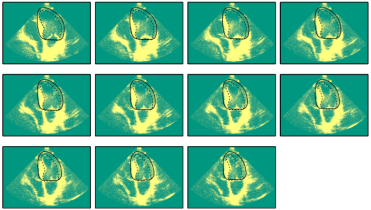
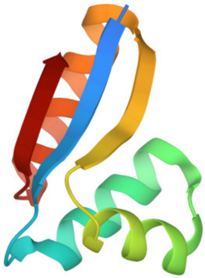


Figure: Stages of a heartbeat in time

Fields of application

- **Medical** : Proteins, RNA structure analysis



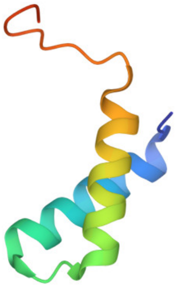


Figure: Examples of proteins structures

Fields of application

- **Agriculture** : Leaves analysis, ...

Figure: Pictures and contouring of leaves

Goals

Shape analysis is a **set of theoretical and computational tools**, that can be used for :

- **Shape space :** With desirable proprieties and tools
- **Shape Metric :** Quantify the difference between two given shapes regardless of rotation or parametrization
- **Geodesics :** Finding the optimal deformation from one shape to another
- **Shape statistics :** Calculate mean shape, covariance, PCA
- **Clustering and classification**

Table of Contents

Mathematical framework for Shape Analysis

- The objective of the paper is to construct a space of shapes with a Riemannian structure
- To do so, there are two main steps :
  1. **Pre-shape space :** Define a representation of the curves with appropriate constraints
  2. **Shape preserving transformation :** remove duplicates w.r.t. translation, scaling, rotation and re-parametrization

Square-Root Velocity

- **Parametrized Euclidean curves :** A curve is a map of the form :

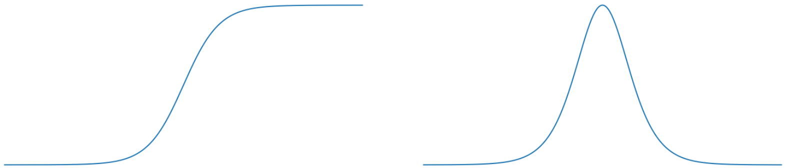
$$\gamma \colon D \to \mathbb{R}^n$$

- **SRV :** One of the main contribution of this paper is the introduction of **Square-Root Velocity** to represent the curves
- We define the SRV of a curve $\gamma \colon D \to \mathbb{R}^n$ as

$$q \colon \begin{cases} D & \to \mathbb{R}^n \\ t & \mapsto F(\dot{\gamma}(t)) = \dfrac{\dot{\gamma}}{\sqrt{\|\dot{\gamma}\|}} \end{cases}$$

Example of curves

- Example of an open curve

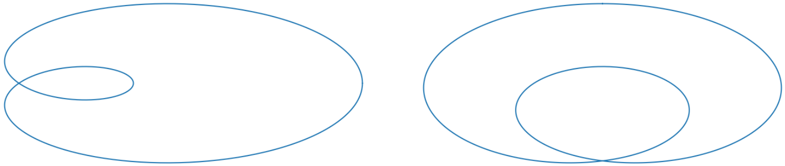


(a) The original curve (b) The associated SRV

Figure: An open curve in $\mathbb{R}^2$: $\gamma_1(t) = \frac{1}{1+e^{-t}}$

Example of curves

- Example of a closed curve



(a) The original curve (b) The associated SRV

Figure: A closed curve in $\mathbb{R}^2$: $\gamma_2(t) = \begin{pmatrix} \cos(t)(0.5+\cos(t)) \\ \sin(t)(0.5+\cos(t)) \end{pmatrix}$

Pre-shape spaces

- Before building the shape space, we introduce **pre-shape spaces :**
- **Open curves :**

$$\mathcal{C}^o = \left\{ q \in \mathbb{L}^2(D, \mathbb{R}^n) \mid \int_{[0,1]} \|q(t)\|^2 \,\mathrm{d}t = 1 \right\}$$

- **Closed curves :**

$$\mathcal{C}^c = \left\{ q \in \mathbb{L}^2(D, \mathbb{R}^n) \mid \int_{\mathbb{S}^1} \|q(t)\|^2 \,\mathrm{d}t = 1, \int_{\mathbb{S}^1} q(t)\|q(t)\| \,\mathrm{d}t = 0 \right\}$$

- We define **Riemannian structures** on those spaces
- On those spaces, the **geodesic between** q_0 **and** q_1 is :

$$d_c(q_0, q_1) = \inf_{\substack{\alpha:[0,1]\to\mathcal{C} \\ \alpha(0)=q_0,\, \alpha(1)=q_1}} \int_0^1 \sqrt{\langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle} \mathrm{d}t$$

Construction of the shape space

- **To build a shape space** $\mathcal{S}$, we have to **remove shape preserving transformations** from its pre-shape space $\mathcal{C}$
- The **shape space** $\mathcal{S}$ **is the quotient** $\mathcal{C}/(\Gamma \times \mathcal{SO}(n))$
- The **geodesic distance between two points of** $\mathcal{S}$ is :

$$d_s([q_0],[q_1]) = \inf_{(\gamma,O)\in\Gamma\times\mathcal{SO}(n)} d_c(q_0, O(q_1 \circ \gamma)\sqrt{\dot{\gamma}})$$

Computation of geodesics

- **For open curves :** Since $\mathcal{C}^o$ is a sphere, we have an explicit expression for the geodesics

$$\alpha : \begin{cases} [0,1] & \to \mathcal{C}^o \\ t & \mapsto \dfrac{1}{\sin(\theta)} \left(\sin(\theta(1-t))\, q_0 + \sin(\theta t)\, q_1 \right) \end{cases}$$

with $\theta = \cos^{-1}(\langle q_0, q_1 \rangle)$

Computation of geodesics

- Srivastava et al. (2010) proposed a **gradient based method** to numerically compute the **geodesics for closed curves** in $\mathbb{R}^n$
- **Path-Straightening Method for closed curves**
- **Energy of a path** $\alpha \in \mathcal{H}_0$ (connecting two points q_0 and q_1 of $\mathcal{C}^c$) :

$$E(\alpha) = \frac{1}{2} \int_0^1 \langle \frac{d\alpha}{dt}(t), \frac{d\alpha}{dt}(t) \rangle \, \mathrm{d}t$$

- A **critical point of** E **is a geodesic** between q_0 and q_1:

$$\alpha^* = \arg\min_{\alpha \in \mathcal{H}_0} E(\alpha)$$

Table of Contents

Application to cell analysis

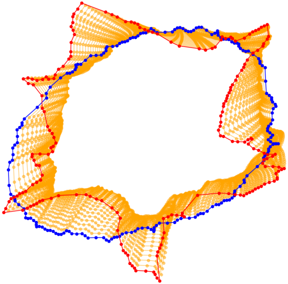


Figure: Geodesic between two types of cells

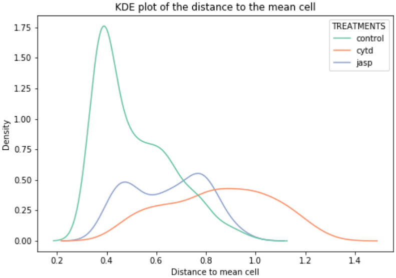


Figure: KDE plot of the distance to the mean cells by treatment

Application to leaf classification

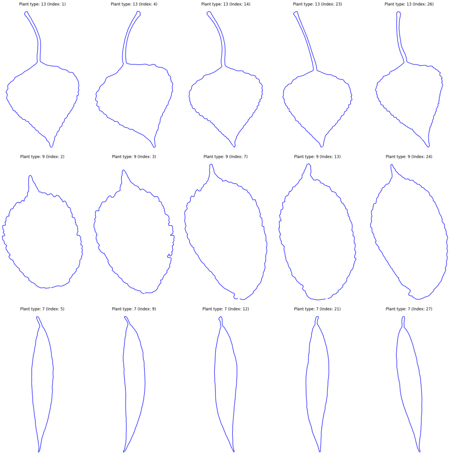


Figure: Subset of leaves to analyze

Results

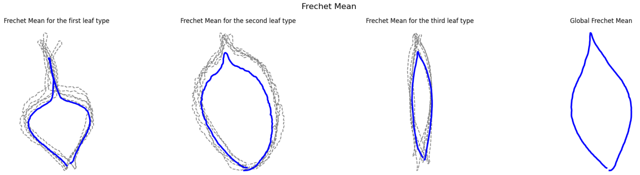


Figure: Frechet mean of each class

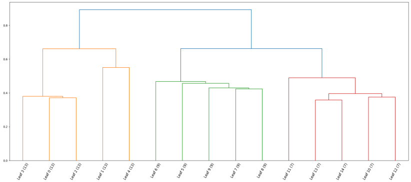


Figure: Agglomerative clustering

Figure: Geodesic between leaves

Table of Contents

Limits and discussion

- **Generalization to surfaces :** $f : [0,1]^2 \to \mathbb{R}^3$
- Normal Vector Fields : For $s = (u, v) \in \mathbb{S}^2$, $\tilde{n}(s) = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$, $n(s) = \frac{\tilde{n}(s)}{\|\tilde{n}(s)\|}$.
- Tensor field : $g(s) = \nabla f(s)^T \nabla f(s)$.
- Area element $r(s) = \|\tilde{n}(s)\| = \sqrt{\det(g(s))}$
- Square root normal field : $q = \sqrt{r} n$
- New Riemannian metric :

$$\begin{aligned}\langle\langle(\delta g_1, \delta n_1), (\delta g_2, \delta n_2)\rangle\rangle_{(g,n)} = a \int_{\mathbb{S}^2} &\mathrm{Tr}((g^{-1}\delta g_1)_0 (g^{-1}\delta g_2)_0) r(s) \mathrm{d}s \\ + b \int_{\mathbb{S}^2} &\mathrm{Tr}(g^{-1}\delta g_1) \mathrm{Tr}(g^{-1}\delta g_2) r(s) \mathrm{d}s \\ + c \int_{\mathbb{S}^2} &\langle \delta n_1, \delta n_2 \rangle r(s) \mathrm{d}s\end{aligned}$$

Limits and discussion

- Hypothesis of **absolute continuity** of β
- **Discontinuous** shapes
- **Unicity** of the geodesic

Table of Contents

Conclusion

- Using **SRV representation** for shapes
- **Build shape spaces** for open and closed curves
- **Numerical method** to find geodesics for closed curves
- Numerical applications with **two examples**

References

Srivastava, A., Klassen, E., Joshi, S. H., and Jermyn, I. H. (2010). Shape analysis of elastic curves in euclidean spaces. *IEEE transactions on pattern analysis and machine intelligence*, 33(7):1415–1428.

Thanks for your attention !