CMS SPREAD OPTION PRICING WITH NORMAL AND SHIFTED LOGNORMAL DYNAMICS

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Abstract. We extend the cms spread option formula in [1], 13.16.2 to the case of shifted lognormal or normal swap rate dynamics.

Contents

1. Original lognormal model

The original model in [1], 13.16.2 for cms spread option pricing is as follows. If $S_i, i = 1, 2$ are the underlying swap rates, t is the fixing time, T the payment time of the coupon with payoff $(a, b, K \in \mathbb{R}, \phi \in \{-1, +1\})$

$$\Pi(t) = \max(\phi(aS_1(t) + bS_2(t) - K), 0) \tag{1.1}$$

then under the T-forward measure we assume (with $\mu_1, \mu_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0, \rho \in [-1, 1]$)

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dZ_1 \tag{1.2}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dZ_2 \tag{1.3}$$

$$dZ_1 dZ_2 = \rho dt \tag{1.4}$$

[1] gives a formula for the price $P(0, T)E^T(\Pi(t))$ in formula (13.34). This involves a one dimensional integral, which can be efficiently calculated with a Gauss-Hermite scheme for example. Here, the drifts μ_i are implied from exogeneously given convexity adjustments for S_i, e.g. computed in a replication model on the respective underlying smiles.

Note that the formulation is [1] is a little more restrictive, because $t = T$ is assumed. However, provided that the swap rate adjustments implying $\mu_i, i = 1, 2$ are computed w.r.T. the same fixing time t and payment time $T > t$ as for the spread option, one can just replace the pricing expectation E^t by E^T without changing

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anything in the derivation (inside the expectation a factor $P(t,T)/P(t,T) = 1$ occurs, that's all).

2. Shifted lognormal extension

We look at the following straightforward extension of the model, introducing shifts $d_1, d_2 > 0$ for the underlying rates

$$dS_1 = \mu_1(S_1 + d_1)dt + \sigma_1(S_1 + d_1)dZ_1 \tag{2.1}$$
$$dS_2 = \mu_2(S_2 + d_2)dt + \sigma_2(S_2 + d_2)dZ_2 \tag{2.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{2.3}$$

Writing

$$X_1 = (S_1 + d_1) \tag{2.4}$$
$$X_2 = (S_2 + d_2) \tag{2.5}$$
$$L = K + ad_1 + bd_2 \tag{2.6}$$

we see that we can apply the original solution for 1.2 with underlyings X_1, X_2 and the payoff written as $(aX_1 + bX_2 - L)^+$. Note that the computation of the drifts μ_i change to

$$\mu_i = \frac{\log((S_i(0) + d_i + c_i)/(S_i(0) + d_i))}{t} = \frac{\log((X_i(0) + c_i)/(X_i(0)))}{t} \tag{2.7}$$

for $i = 1, 2$ accordingly with $c_i = E^T(S_i(t)) - S_i(0)$ denoting the convexity adjustment applicable to the rate S_i (note that the expectaion here is taken in the external model for the single swap rate adjustments respectively).

3. Normal extension

The normal flavour of the original model reads

$$dS_1 = \mu_1 dt + \sigma_1 dZ_1 \tag{3.1}$$
$$dS_2 = \mu_2 dt + \sigma_2 dZ_2 \tag{3.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{3.3}$$

with drifts now given by

$$\mu_i = c_i/t \tag{3.4}$$

with c_i again denoting the exogeneously given convexity adjustment for rate S_i with $i = 1, 2$, see above.

The option price ν is given by

$$\nu = P(0,T)E^T\left((\phi(aS_1(t) + bS_2(t) - K))^+\right) \tag{3.5}$$

The expectation can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\phi(as_1+bs_2-K))^+p(s_1,s_2)ds_1ds_2 \tag{3.6}$$

with p denoting a bivariate normal distribution

$$p \sim \mathcal{N}\left(\begin{pmatrix} S_1(0)+\mu_1 t \\ S_2(0)+\mu_2 t\end{pmatrix}, \begin{pmatrix}\sigma_1^2 t & \rho\sigma_1\sigma_2 t \\ \rho\sigma_1\sigma_2 t & \sigma_2^2 t\end{pmatrix}\right) \tag{3.7}$$

It is well known that the distribution of $S_1(t)$ conditional on $S_2(t)=s_2$ is given by

$$S_1(t)|\{S_2(t)=s_2\} \sim \mathcal{N}\left(S_1(0)+\mu_1 t+\frac{\rho\sigma_1}{\sigma_2}(s_2-\mu_2 t-S_2(0)), \sigma_1^2 t(1-\rho^2)\right) \tag{3.8}$$

We denote the density of this distribution by p_{s_2}. We continue to compute the inner integral in 3.6 for a fixed $S_2(t)=s_2$, i.e.

$$\int_{-\infty}^{\infty}(\phi(as_1+bs_2-K))^+p_{s_2}(s_1)ds_1 \tag{3.9}$$

We have the following

Lemma 1. *For $\alpha,\beta\in\mathbb{R}$ we have*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\alpha x+\beta)^+e^{-x^2/2}dx = \psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\phi(1-N(-\psi\beta/\alpha)) \tag{3.10}$$

where $\psi=\mathrm{sign}(\alpha)\in\{+1,-1\}$ and N denotes the cumulative normal distribution function. For $\alpha=0$ the right hand should be read simply as β^+.

Proof. Let $\alpha>0$. The proof for $\alpha<0$ is similar and the case $\alpha=0$ is obvious from the following steps as well. Obviously $\alpha x+\beta>0$ iff $x>-\beta/\alpha$, so the integral can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\beta/\alpha}^{\infty}(\alpha x+\beta)e^{-x^2/2}dx \tag{3.11}$$

Furthermore,

$$\int_{-\beta/\alpha}^{\infty}xe^{-x^2/2}dx=\left[-e^{-x^2/2}\right]_{-\beta/\alpha}^{\infty}=e^{-\beta^2/(2\alpha^2)} \tag{3.12}$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\beta/\alpha}^{\infty}e^{-x^2/2}dx=1-N(-\beta/\alpha) \tag{3.13}$$

which proves the identity. □

The integral 3.9 is

$$\frac{1}{v\sqrt{2\pi}}\int_{-\infty}^{\infty}(\phi(as_1+bs_2-K))^+e^{-\frac{(s1-\mu)^2}{2v^2}}ds_1 \tag{3.14}$$

with $\mu=S_1(0)+\mu_1 t+\rho\sigma_1/\sigma_2(s_2-\mu_2 t-S_2(0))$ and $v^2=\sigma_1^2 t(1-\rho^2)$.
Substituting $x=(s_1-\mu)/v$ this becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\phi(a(\mu+vx)+bs_2-K)\right)^{+}e^{-x^2/2}dx \tag{3.15}$$

Setting $\alpha=\phi av$ and $\beta=\phi(a\mu+bs_2-K)$ and $\psi=\mathrm{sign}(\alpha)$ we can write the npv (using lemma 1) ν as

(3.16)

$$P(0,T)\frac{1}{\sigma_2\sqrt{2\pi t}}\int_{-\infty}^{\infty}\psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\psi\beta/\alpha))e^{-\frac{(s_2-\mu_2 t-S_2(0))^2}{2\sigma_2^2 t}}ds_2$$

with

$$\begin{aligned}
\alpha &= \phi a\sigma_1\sqrt{t(1-\rho^2)} && (3.17)\\
\beta &= \phi\left(a(S_1(0)+\mu_1 t+\rho\sigma_1/\sigma_2(s_2-S_2(0)-\mu_2 t))+bs_2-K\right) && (3.18)\\
\psi &= \mathrm{sign}(\alpha) && (3.19)
\end{aligned}$$

Substituting $s=(s_2-\mu_2 t-S_2(0))/(\sigma_2\sqrt{t})$ yields

$$\nu=P(0,T)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\psi\beta/\alpha))e^{-s^2/2}ds \tag{3.20}$$

with

$$\begin{aligned}
\alpha &= \phi a\sigma_1\sqrt{t(1-\rho^2)} && (3.21)\\
\beta &= \phi\left(a(S_1(0)+\mu_1 t)+b(S_2(0)+\mu_2 t)-K+\sqrt{t}(\rho\sigma_1+b\sigma_2)s\right) && (3.22)\\
\psi &= \mathrm{sign}(\alpha) && (3.23)
\end{aligned}$$

which is the final version of our cms spread option formula under bivariate normal dynamics.

References

[1] Brigo, Mercurio: Interst Rate Models - Theory and Practice, 2nd Edition, Springer, 2006

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