$\beta - \eta$ MODEL IMPLEMENTATION

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INCOMPLETE DRAFT

ABSTRACT. We describe the implementation of the $\beta - \eta$ model ([1], [2] 11.3.2.6) in QuantLib [3].

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1. Model

The driving process is given by

$$dx(t) = \alpha(t)(1 + \beta x(t))^\eta dW(t) \tag{1.1}$$

with $x(0) = 0$, $\alpha(\cdot) > 0$, $\beta > 0$ and $0 \leq \eta \leq 1$. The dynamics is expressed in the measure $\mathbb{Q}^N$ associated to the numeraire

$$N(t, x(t)) = \frac{1}{P(0, t)} e^{\lambda(t)x(t) + M(0,0;T)} \tag{1.2}$$

with a function $\lambda : \mathbb{R} \to \mathbb{R}$ subject to constraints $\lambda(0) = 0$ and $\lambda'(0) = 1$. We define

$$M(t, x; T) = \log E\left(e^{-\lambda(T)(X(T)-x)} \middle| X(t) = x\right) \tag{1.3}$$

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which is central to the computation of numeraire and zerobond prices in the model. The transition density is given in [1], (4.6a), (4.6b) and (4.8). For $\eta < 0.5$ a reflecting barrier at $x = -1/\beta$ can be specified in 1.1, the corresponding amendments in the density are given in [1], p. 241, last paragraph. In the following we assume that the reflecting barrier condition is put in 1.1. The reason is that the calculations can be implemented in a more compact way under this assumption, while the barrier should not impact the pricing too much anyway (see the comments in [1] on this).

For the special case of $\eta = 0$ a closed form representation for the density ([1], (C.3)) and for $M(t, x; T)$ (C.5) is given, but only for the case of no barrier. For $\eta = 0.5$ a closed form expression for $M(t, x; T)$ is given (4.15). For $\eta = 1$ we have (4.10b) for the density (however there is no closed form expression for $M(t, x; T)$ in this case).

We follow [2] and describe λ by a constant mean reversion $\kappa \neq 0$ via

$$\lambda = \frac{1 - e^{-\kappa t}}{\kappa} \tag{1.4}$$

The normalization constraints $\lambda(0) = 0, \lambda'(0) = 1$ are immediately verified. Furthermore

$$\kappa = -\frac{\lambda''(t)}{\lambda'(t)} \tag{1.5}$$

Note that $\lambda > 0$ by construction.

2. Numerical Issues

2.1. Computation of $p(t, x; \bar{t}, \overline{x})$. The formulas for the density ([1], (4.6a), (4.6b) and (4.8)) contain expressions

$$I_\nu(y\overline{y}/(\overline{\tau} - \tau))e^{-(\overline{y}^2+y^2)/(2(\overline{\tau}-\tau))} \tag{2.1}$$

with I_ν denoting the modified Bessel function of the first kind (likewise $I_{-\nu}$ and K_ν appear at other places). We can rewrite 2.1 as

$$I_\nu(y\overline{y}/(\overline{\tau} - \tau))e^{-y\overline{y}/(\overline{\tau}-\tau)}e^{-(\overline{y}+y)^2/(2(\overline{\tau}-\tau))} \tag{2.2}$$

so that we can use the exponentially weighted implementation of the modified Bessel function to cover the first two factors. This is necessary for a numerically stable result.

For values η close to, but not equal to 1 the closed form representation from the paper gets numerically unstable. Therefore we interpolate the density between some threshold value η_M and $\eta = 1$ linearly. The threshold value is set to the largest tabulated value for η (see below, this is typically near 0.99), since this guarantees to deliver stable values.

2.2. **Tabulation of $M(t,x,T)$.** For $\eta = 0.5$ we can use a closed form expression for M, see above. For $\eta = 1$ we can compute

$$M(t,x;T) = \int_{-\infty}^{\infty} e^{-\lambda(T)\beta^{-1}(\exp(\beta\overline{y})-\exp(\beta y))} e^{-z^2} dz \tag{2.3}$$

using a Gauss Hermite scheme (we use 8 points which already seem to give enough accuracy). Here we write $\overline{y} = z\sqrt{2v} + y - \beta v/2$, $v = \overline{\tau} - \tau$ and use the transformation from [1], (4.4)

$$y = \begin{cases} \frac{|1+\beta x|^{1-\eta}}{\beta(1-\eta)} & \eta \neq 1 \\ \log(1+\beta x)/\beta & \eta = 1 \end{cases} \tag{2.4}$$

and

$$\tau = \int_0^t \alpha(s) ds \tag{2.5}$$

Overlined variables like $\overline{y}$ and $\overline{\tau}$ denote the quantities w.r.t. $\overline{t}$ (or T) instead of t. In the following we look at the remaining case $\eta < 1$.

The transition density is originally expressed in variables $y, \overline{y}$ and $\tau, \overline{\tau}$ instead of $x, \overline{x}$ and $t, \overline{t}$. It is obvious from [1], (4.6a), (4.6b), (4.7) and (4.8) that p can be written as a function of $v = \overline{\tau} - \tau$. Furthermore for any $a > 0$ we have the homogenity relation

$$p\left(a^2(\overline{\tau} - \tau), ay, a\overline{y}\right) = a^{1/(\eta-1)} p\left(\overline{\tau} - \tau, y, \overline{y}\right) \tag{2.6}$$

We have to compute

$$M(t,x;\overline{t}) = \log \int_{-\infty}^{\infty} p(t,x;\overline{t},\overline{x}) e^{-\lambda(\overline{t})(\overline{x}-x)} d\overline{x} \tag{2.7}$$

The integral can be rewritten using a variable transformation and 2.6 as

$$\int_0^{\infty} p^*(\lambda^{2-2\eta}\beta^{2\eta}(1-\eta)^{2\eta} v, u^{1-\eta}(1-\eta)^{\eta-1}, \overline{u}^{1-\eta}(1-\eta)^{\eta-1}) e^{-(\overline{u}-u)} d\overline{u} \tag{2.8}$$

where $p^* = p \cdot [(1-\eta)\beta]^{-\eta/(\eta-1)}$ with $u = \lambda(\overline{t})\beta^{-1}|1+\beta x|$. Introducing $S := \beta^2 v/(1+\beta x)^{2-2\eta}$ this reads

$$\int_0^{\infty} p^*(S(1-\eta)^{2\eta} u^{2-2\eta}, u^{1-\eta}(1-\eta)^{\eta-1}, \overline{u}^{1-\eta}(1-\eta)^{\eta-1}) e^{-(\overline{u}-u)} d\overline{u} \tag{2.9}$$

which can be tabulated. Actually we tabulate M in (S^*, u) instead of (S, u) with

$$S^* = S u^{2-\eta/2} \tag{2.10}$$

since this allows to cover a larger area of "useful" values by a rectangular set for (S^*, u).

We tabulate (S^*, u) over $[10^{-4}, 20] \times [10^{-4}, 1000]$ with 100 grid points in each direction with a density of 10^{-4} and concentrating point at $(10^{-4}, 10^{-4})$ (in the sense of QuantLib's concentrating mesher). In η direction we use 100 grid points as well (with a concentrating point at 0.5 and density 1.0).

The rationale for using non-uniform grids is a pure visual inspection of the generated surfaces $M(S^*, u)$ for each fixed η showing more non-linear behaviour near the concentrating points.

The interpolation is done linearly. A bicubic spline interpolation would actually give better accuracy (or allow for a coarser tabulation), but is too slow for pricing purposes. Since for $\eta = 1$ we do not have a tabulation we interpolation M between the largest tabulated point for η and $\eta = 1$, for the latter we use 2.3.

2.3. **Tabulation of $P(y = 0)$ and cutoff of small values.** For $\eta \geq 0.5$ there is a closed form expression for the probability of y being zero given in [1], (4.8). The paper does not give such an expression for $\eta < 0.5$ when a reflecting barrier condition at $y = 0$ is chosen. In any case we need a tabulation of these values because the evaluation of the incomplete upper gamma function $\Gamma(\cdot, \cdot)$ slows down the pricing considerably if implemented explicitly.

The tabulation is straightforward. For η we use the same grid as for the tabulation of M. For y and v we tabulate 50×50 points covering the rectangle $[10^{-4}, 10] \times [10^{-4}, 1]$ with a density of 10^{-4} and concentrating point $(10^{-4}, 10^{-4})$.

Since when computing M very huge terms are multiplied with very small values of $P(y = 0)$ we choose to set this probability to zero in case it is below a threshold (we use 10^{-6}).

3. Examples

3.1. **Introductory comment: yield-state anomaly.** For (simultaneously) larger values of β and α we observe an anomaly in the functional dependency of the zero yield on the model's state variable. The usual expectation would be that the zero yield is monotone ascending in the state variable. For small β this can indeed be verified, see figure 1 for an example with $\beta = 0.1$ (the reversion κ is 1%, the yield term structure flat at 3% in this example).

For higher values of β however, the dependency of the zero yield on the state flattens and even reverses, so that for high volatilities α the zero yield is monotone *descending* in the state. Obviously this can produce unintuitive results like swaption prices that are not monotone ascending w.r.t. the model volatility. Figure 2 shows an example in this direction.

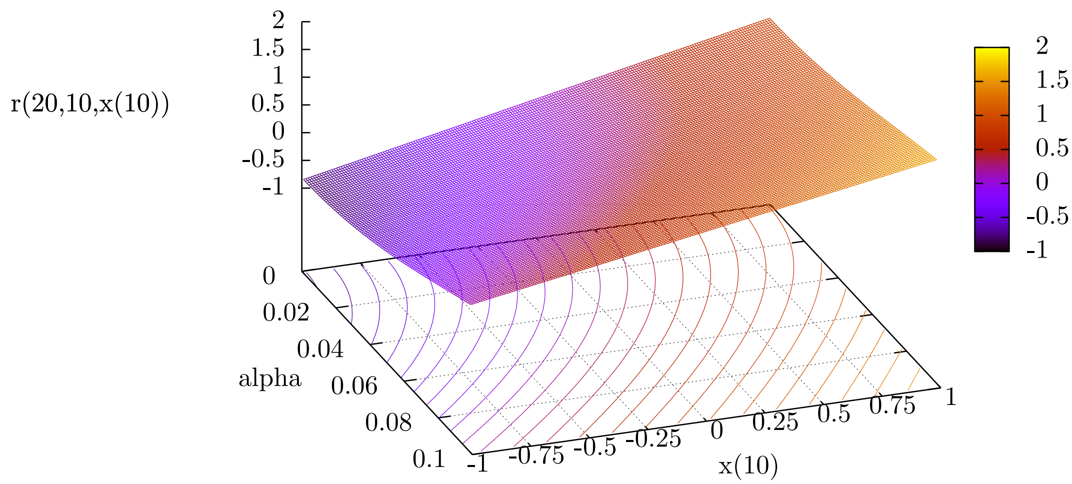
3.2. **Hull White case.** For $\eta = 0, \kappa = 0$ according to [1], C1 we can expect to replicate the Hull White model. Table 1 shows the pricing of 10y into 10y swaptions in the GSR and $\beta - \eta$ model for this case. We use the respective integral engines with 64 integration points covering 7 standard deviations of the state variable. The yield term structure is flat at 3% and we price swaptions with strikes

(1) 0%, atm-200bp, atm-150bp, atm-100bp, atm-50bp, atm-25bp (Receiver)
(2) atm+25bp, atm+50bp, atm+100bp, atm+150bp, atm+200bp (Payer)

Table 1 shows the results. GSR and $\beta - \eta$ pricings are consistent. Also the usage of tabulated values does not impact the pricing accuracy. The parity error is defined as

$$\frac{\pi - \rho}{A} - (f - k) \tag{3.1}$$

FIGURE 1. 10y zero yield at $t = 10$ conditional on $x(10)$ for different levels of α, β is fixed at 0.1

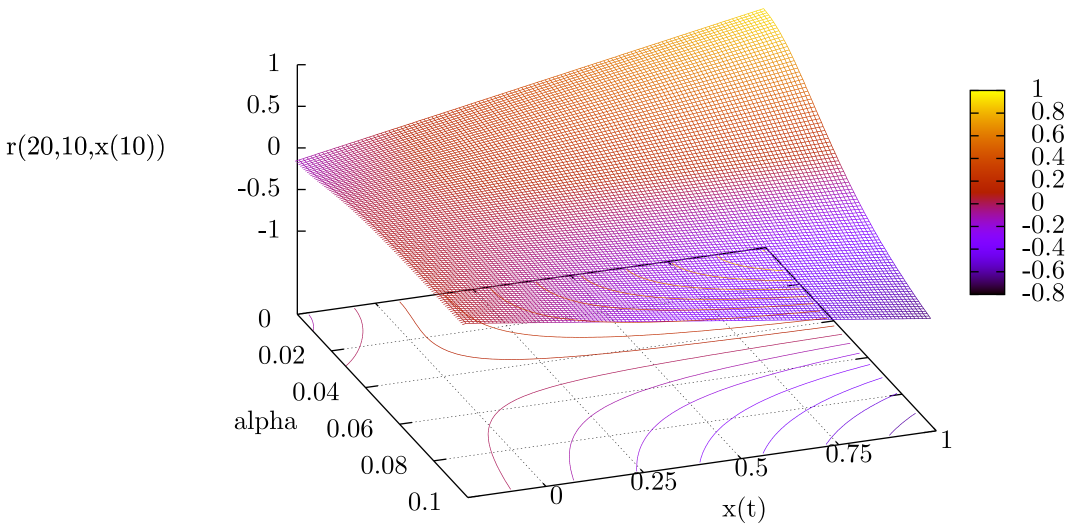


with π and ρ the payer resp. receiver swaption price, A the annuity, f the forward swap rate and k the strike, where f is computed on the inital rate curve outside the model as the reference value. In general we find the parity error to be a good measure for both the numerical error and differences arising from approximations through the tabulation.

TABLE 1. Swaption pricing check against GSR for $\eta = 0$, $\beta = 0.1$, $\alpha = 0.0075$, $\kappa = 0$ using tabulated values and full integration

strike	npv GSR	npv $\beta\eta$ tab	npv $\beta\eta$ full	parity error tab	parity error full
0	0.0073954	0.0073943	0.0073943	6.3182e-08	6.2763e-08
0.010472	0.017449	0.017447	0.017447	7.5887e-08	7.5468e-08
0.015472	0.024959	0.024957	0.024957	-5.7966e-08	-5.8384e-08
0.020472	0.034625	0.034623	0.034623	-1.5272e-07	-1.5313e-07
0.025472	0.04672	0.046729	0.046729	2.1575e-08	2.1157e-08
0.027972	0.053726	0.053734	0.053734	1.2001e-07	1.196e-07
0.030472	0.061376	0.06138	0.06138	-1.9706e-07	-1.9665e-07
0.032972	0.053905	0.053909	0.053909	-2.2308e-07	-2.2267e-07
0.035472	0.04707	0.047073	0.047073	-2.4932e-07	-2.4891e-07
0.040472	0.035247	0.035248	0.035248	-2.9785e-07	-2.9743e-07
0.045472	0.025739	0.025737	0.025737	-3.3749e-07	-3.3707e-07
0.050472	0.018306	0.018303	0.018303	-3.6517e-07	-3.6475e-07

FIGURE 2. 10y zero yield at $t = 10$ conditional on $x(10)$ for different levels of α, β is fixed at 5



3.3. **Influence of β, η and κ on the smile.** We continue to examine the influence of the model parameters β, η and κ on the implied smile. We look at reversion levels $\kappa \in \{-0.05, -0.01, 0, 0.01, 0.05\}$. The exponent η is chosen as 0, 0.2, 0.5, 0.8 and 1 respectively.

The range for β is set to 0.1, 1, 2, 3, 4, 5 here. Note that too large values can lead to difficult conditions in the model as shown in 3.1. Actually in one of the following examples the calibration of the model to an atm swaption with 16% market volatility fails for $\beta = 5$ (namely in the case $\eta = 1$, $\kappa = 0.05$) due to the described issue. Hagan himself does not give a hint which values for β are reasonable either.

Figures 3, 4, 5, 6 and 7 show the dependency of the smile of the different values of β for fixed η and κ. In general for higher η and higher κ the dependency of the skew on β gets stronger, where higher values for β decrease the skew. For all these example cases we recalibrate the model volatility for each (η, β, κ) such that an atm swaption with 16% implied lognormal volatility is matched.

Table 2 lists all example cases (except the one where the calibration failes, see above) together with the implied volatilities σ for atm, atm-200bp, atm+200bp and the skew and curvature of the smile w.r.t. to the atm±200bp points, defined as

$$\text{skew} := \sigma_{\text{atm+200bp}} - \sigma_{\text{atm}-200\text{bp}} \tag{3.2}$$

$$\text{curvature} := \sigma_{\text{atm+200bp}} - 2\sigma_{\text{atm}} + \sigma_{\text{atm}-200\text{bp}} \tag{3.3}$$

serving as summarizing characteristics for the smile shape.

FIGURE 3. implied lognormal volatility smiles, $\eta = 0.2$, $\kappa = 0.0$

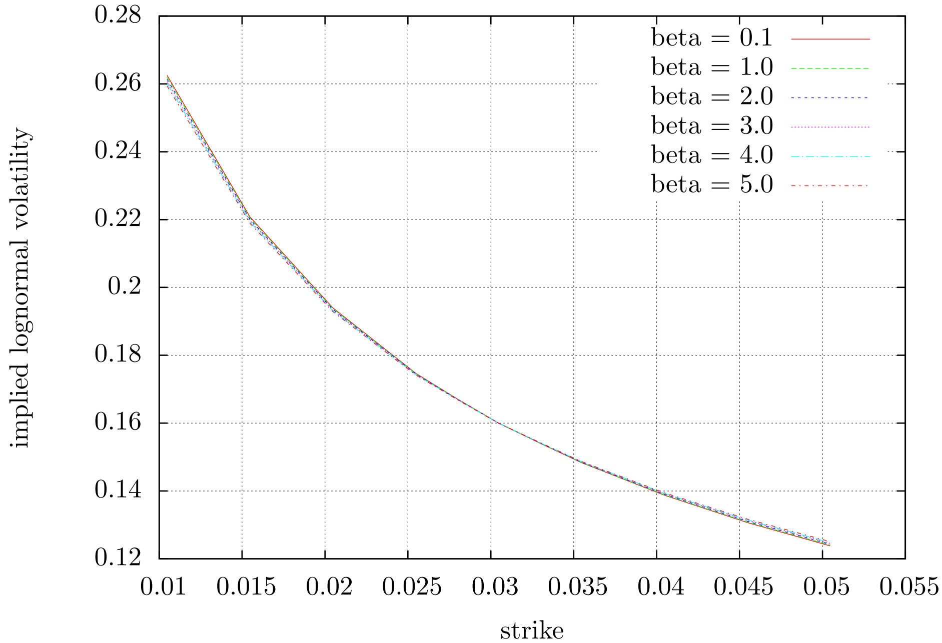


Table 2: Generated skew and curvature for 10y into 10y swaptions

η	κ	β	-200	atm	$+200$	skew	curvature
0	-0.05	·	0.2648	0.1601	0.1227	-0.1421	0.06737
0	-0.01	·	0.2628	0.16	0.1234	-0.1394	0.06615
0	0	·	0.2626	0.16	0.1238	-0.1388	0.06631
0	0.01	·	0.2621	0.16	0.1239	-0.1381	0.066
0	0.05	·	0.2601	0.1599	0.1246	-0.1356	0.06488
0.2	-0.05	0.1	0.2648	0.1601	0.1228	-0.142	0.06736
0.2	-0.05	1	0.2646	0.1601	0.1229	-0.1417	0.06723
0.2	-0.05	2	0.2643	0.1601	0.123	-0.1413	0.06708
0.2	-0.05	3	0.264	0.1601	0.1231	-0.1409	0.06692
0.2	-0.05	4	0.2637	0.1601	0.1232	-0.1405	0.06677
0.2	-0.05	5	0.2635	0.1601	0.1233	-0.1401	0.0666
0.2	-0.01	0.1	0.2627	0.16	0.1234	-0.1393	0.06612
0.2	-0.01	1	0.2623	0.16	0.1236	-0.1387	0.06588
0.2	-0.01	2	0.2618	0.16	0.1239	-0.1379	0.06561
0.2	-0.01	3	0.2613	0.16	0.1241	-0.1372	0.06534
0.2	-0.01	4	0.2608	0.16	0.1243	-0.1365	0.06505
0.2	-0.01	5	0.2602	0.16	0.1245	-0.1357	0.06475

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η	κ	β	-200	atm	$+200$	skew	curvature
0.2	0	0.1	0.2625	0.16	0.1238	-0.1387	0.06627
0.2	0	1	0.262	0.16	0.124	-0.1379	0.06598
0.2	0	2	0.2614	0.16	0.1243	-0.1371	0.06563
0.2	0	3	0.2607	0.16	0.1245	-0.1362	0.06527
0.2	0	4	0.2601	0.16	0.1248	-0.1353	0.06488
0.2	0	5	0.2594	0.16	0.1251	-0.1343	0.06448
0.2	0.01	0.1	0.262	0.16	0.124	-0.138	0.06595
0.2	0.01	1	0.2614	0.16	0.1242	-0.1372	0.06561
0.2	0.01	2	0.2607	0.16	0.1245	-0.1362	0.0652
0.2	0.01	3	0.2599	0.16	0.1248	-0.1351	0.06477
0.2	0.01	4	0.2592	0.16	0.1251	-0.134	0.06431
0.2	0.01	5	0.2584	0.16	0.1254	-0.1329	0.06383
0.2	0.05	0.1	0.2598	0.1599	0.1246	-0.1352	0.06456
0.2	0.05	1	0.2587	0.1599	0.1251	-0.1336	0.06395
0.2	0.05	2	0.2573	0.1599	0.1257	-0.1316	0.06318
0.2	0.05	3	0.2557	0.1599	0.1263	-0.1294	0.06226
0.2	0.05	4	0.254	0.1599	0.127	-0.1271	0.06126
0.2	0.05	5	0.252	0.1599	0.1276	-0.1244	0.0598
0.5	-0.05	0.1	0.2648	0.1601	0.1228	-0.142	0.06734
0.5	-0.05	1	0.2642	0.1601	0.123	-0.1411	0.067
0.5	-0.05	2	0.2635	0.1601	0.1233	-0.1402	0.06663
0.5	-0.05	3	0.2628	0.1601	0.1236	-0.1392	0.06624
0.5	-0.05	4	0.2621	0.1601	0.1239	-0.1382	0.06585
0.5	-0.05	5	0.2615	0.1601	0.1242	-0.1373	0.06545
0.5	-0.01	0.1	0.2627	0.16	0.1235	-0.1392	0.06609
0.5	-0.01	1	0.2616	0.16	0.124	-0.1376	0.06551
0.5	-0.01	2	0.2604	0.16	0.1245	-0.1358	0.06486
0.5	-0.01	3	0.2591	0.16	0.1251	-0.134	0.06418
0.5	-0.01	4	0.2579	0.16	0.1257	-0.1322	0.06348
0.5	-0.01	5	0.2566	0.16	0.1262	-0.1304	0.06275
0.5	0	0.1	0.2624	0.16	0.1238	-0.1386	0.06623
0.5	0	1	0.2611	0.16	0.1244	-0.1367	0.06548
0.5	0	2	0.2596	0.16	0.125	-0.1346	0.06462
0.5	0	3	0.2581	0.16	0.1257	-0.1324	0.06372
0.5	0	4	0.2565	0.16	0.1263	-0.1301	0.06278
0.5	0	5	0.2549	0.16	0.127	-0.1279	0.06185
0.5	0.01	0.1	0.2619	0.16	0.124	-0.1379	0.0659
0.5	0.01	1	0.2604	0.16	0.1247	-0.1357	0.06505
0.5	0.01	2	0.2586	0.16	0.1254	-0.1332	0.06404
0.5	0.01	3	0.2568	0.16	0.1261	-0.1307	0.06298
0.5	0.01	4	0.255	0.16	0.1269	-0.1281	0.06186
0.5	0.01	5	0.253	0.16	0.1276	-0.1254	0.06068
0.5	0.05	0.1	0.2596	0.1599	0.1247	-0.135	0.06447
0.5	0.05	1	0.2568	0.1599	0.1259	-0.1309	0.06295
0.5	0.05	2	0.2536	0.1599	0.1274	-0.1262	0.06121
0.5	0.05	3	0.2502	0.1599	0.1289	-0.1213	0.05926
0.5	0.05	4	0.2462	0.1599	0.1304	-0.1158	0.05684

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η	κ	β	-200	atm	$+200$	skew	curvature
0.5	0.05	5	0.2416	0.1599	0.1319	-0.1097	0.05373
0.8	-0.05	0.1	0.2647	0.1601	0.1228	-0.1419	0.06733
0.8	-0.05	1	0.2638	0.1601	0.1232	-0.1406	0.0668
0.8	-0.05	2	0.2627	0.1601	0.1237	-0.139	0.0662
0.8	-0.05	3	0.2616	0.1601	0.1241	-0.1375	0.06559
0.8	-0.05	4	0.2605	0.1601	0.1246	-0.136	0.06498
0.8	-0.05	5	0.2595	0.1601	0.125	-0.1344	0.06434
0.8	-0.01	0.1	0.2626	0.16	0.1235	-0.1391	0.06604
0.8	-0.01	1	0.2609	0.16	0.1243	-0.1366	0.06513
0.8	-0.01	2	0.259	0.16	0.1252	-0.1338	0.0641
0.8	-0.01	3	0.257	0.16	0.1261	-0.1309	0.06306
0.8	-0.01	4	0.255	0.16	0.127	-0.1281	0.06198
0.8	-0.01	5	0.253	0.16	0.1278	-0.1252	0.06088
0.8	0	0.1	0.2623	0.16	0.1239	-0.1385	0.06619
0.8	0	1	0.2602	0.16	0.1248	-0.1354	0.065
0.8	0	2	0.2579	0.16	0.1258	-0.132	0.06362
0.8	0	3	0.2555	0.16	0.1268	-0.1287	0.06227
0.8	0	4	0.2531	0.16	0.1278	-0.1253	0.06092
0.8	0	5	0.2507	0.16	0.1288	-0.1219	0.05957
0.8	0.01	0.1	0.2618	0.16	0.124	-0.1377	0.06584
0.8	0.01	1	0.2594	0.16	0.1251	-0.1343	0.06448
0.8	0.01	2	0.2566	0.16	0.1263	-0.1304	0.06288
0.8	0.01	3	0.2538	0.16	0.1274	-0.1264	0.06124
0.8	0.01	4	0.2509	0.16	0.1286	-0.1224	0.05948
0.8	0.01	5	0.2479	0.16	0.1297	-0.1182	0.05765
0.8	0.05	0.1	0.2594	0.1599	0.1247	-0.1347	0.06439
0.8	0.05	1	0.2551	0.1599	0.1267	-0.1284	0.0621
0.8	0.05	2	0.2503	0.1598	0.129	-0.1213	0.05957
0.8	0.05	3	0.2448	0.1598	0.1311	-0.1138	0.05623
0.8	0.05	4	0.2387	0.1597	0.1332	-0.1055	0.05244
0.8	0.05	5	0.2317	0.1597	0.1358	-0.09582	0.04799
1	-0.05	0.1	0.2647	0.1601	0.1228	-0.1419	0.0673
1	-0.05	1	0.2635	0.1601	0.1233	-0.1402	0.06664
1	-0.05	2	0.2622	0.1601	0.1239	-0.1383	0.0659
1	-0.05	3	0.2609	0.1601	0.1245	-0.1364	0.06515
1	-0.05	4	0.2595	0.1601	0.125	-0.1345	0.06439
1	-0.05	5	0.2582	0.1601	0.1256	-0.1326	0.06362
1	-0.01	0.1	0.2625	0.16	0.1235	-0.139	0.06602
1	-0.01	1	0.2604	0.16	0.1245	-0.1359	0.0649
1	-0.01	2	0.2581	0.16	0.1256	-0.1324	0.06364
1	-0.01	3	0.2557	0.16	0.1267	-0.129	0.06236
1	-0.01	4	0.2533	0.16	0.1278	-0.1255	0.06104
1	-0.01	5	0.2508	0.16	0.1289	-0.1219	0.05971
1	0	0.1	0.2623	0.16	0.1239	-0.1384	0.06615
1	0	1	0.2597	0.16	0.125	-0.1346	0.06468
1	0	2	0.2567	0.16	0.1263	-0.1305	0.063

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η	κ	β	-200	atm	$+200$	skew	curvature
1	0	3	0.2539	0.16	0.1275	-0.1264	0.0614
1	0	4	0.251	0.16	0.1287	-0.1223	0.05978
1	0	5	0.2481	0.16	0.1299	-0.1182	0.05811
1	0.01	0.1	0.2617	0.16	0.1241	-0.1376	0.06581
1	0.01	1	0.2587	0.16	0.1254	-0.1334	0.06412
1	0.01	2	0.2553	0.16	0.1268	-0.1285	0.06217
1	0.01	3	0.2519	0.16	0.1282	-0.1237	0.06013
1	0.01	4	0.2483	0.16	0.1296	-0.1187	0.05799
1	0.01	5	0.2449	0.16	0.131	-0.1139	0.05591
1	0.05	0.1	0.2593	0.1599	0.1248	-0.1345	0.06431
1	0.05	1	0.2541	0.1599	0.1273	-0.1268	0.06154
1	0.05	2	0.2481	0.1599	0.1299	-0.1182	0.05828
1	0.05	3	0.2416	0.1599	0.1323	-0.1093	0.05422
1	0.05	4	0.2344	0.1598	0.135	-0.0994	0.0498

3.4. **Zero strike floor pricing.** The possibility of negative rates and - if allowed at all - their distributionn is a particular point of interest of a model. We list the premiums for zero strike floors (receiver swaptions) in table 3 for different combinations of η, β and κ and a model volatility α that is calibrated to an atm lognormal implied volatility level of 16%.

Table 3: 0% 10y into 10y receiver swaption premiums, α is recalibrated to an atm lognormal implied volatility of 16%

η	β	κ	npv $\beta\eta$
0	·	-0.05	0.000811375
0	·	-0.01	0.000765611
0	·	0	0.000756554
0	·	0.01	0.000739278
0	·	0.05	0.000689536
0.2	0.1	-0.05	0.000810655
0.2	1	-0.05	0.000804051
0.2	2	-0.05	0.000796619
0.2	3	-0.05	0.000789083
0.2	4	-0.05	0.000781505
0.2	5	-0.05	0.000773891
0.2	0.1	-0.01	0.000764355
0.2	1	-0.01	0.000752335
0.2	2	-0.01	0.000738753
0.2	3	-0.01	0.000724805
0.2	4	-0.01	0.000710552
0.2	5	-0.01	0.000696026
0.2	0.1	0	0.000755057
0.2	1	0	0.000741221
0.2	2	0	0.000725557
0.2	3	0	0.000709303
0.2	4	0	0.000692642

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η	β	κ	npv $\beta\eta$
0.2	5	0	0.000675345
0.2	0.1	0.01	0.000737633
0.2	1	0.01	0.000721922
0.2	2	0.01	0.000703994
0.2	3	0.01	0.000685254
0.2	4	0.01	0.000665768
0.2	5	0.01	0.000645463
0.2	0.1	0.05	0.00068921
0.2	1	0.05	0.000661121
0.2	2	0.05	0.000627311
0.2	3	0.05	0.000590094
0.2	4	0.05	0.000548571
0.2	5	0.05	0.000498871
0.5	0.1	-0.05	0.000809579
0.5	1	-0.05	0.000793191
0.5	2	-0.05	0.000775007
0.5	3	-0.05	0.000756871
0.5	4	-0.05	0.000738774
0.5	5	-0.05	0.000720711
0.5	0.1	-0.01	0.000762327
0.5	1	-0.01	0.000732723
0.5	2	-0.01	0.000700046
0.5	3	-0.01	0.000667325
0.5	4	-0.01	0.000634662
0.5	5	-0.01	0.000602072
0.5	0.1	0	0.000752778
0.5	1	0	0.000718793
0.5	2	0	0.000681044
0.5	3	0	0.000643325
0.5	4	0	0.000605655
0.5	5	0	0.000568059
0.5	0.1	0.01	0.000735005
0.5	1	0.01	0.000696519
0.5	2	0.01	0.000653697
0.5	3	0.01	0.000610831
0.5	4	0.01	0.000568082
0.5	5	0.01	0.000525403
0.5	0.1	0.05	0.000684826
0.5	1	0.05	0.000617815
0.5	2	0.05	0.000541889
0.5	3	0.05	0.000464455
0.5	4	0.05	0.000384096
0.5	5	0.05	0.000303229
0.8	0.1	-0.05	0.00080834
0.8	1	-0.05	0.000782254
0.8	2	-0.05	0.000753688

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η	β	κ	npv $\beta\eta$
0.8	3	-0.05	0.000725453
0.8	4	-0.05	0.000697698
0.8	5	-0.05	0.000670431
0.8	0.1	-0.01	0.000760358
0.8	1	-0.01	0.000713532
0.8	2	-0.01	0.000663235
0.8	3	-0.01	0.000613854
0.8	4	-0.01	0.000565672
0.8	5	-0.01	0.000518802
0.8	0.1	0	0.000750411
0.8	1	0	0.000696861
0.8	2	0	0.000639136
0.8	3	0	0.000582673
0.8	4	0	0.000528175
0.8	5	0	0.000475088
0.8	0.1	0.01	0.000732469
0.8	1	0.01	0.000671974
0.8	2	0.01	0.000606983
0.8	3	0.01	0.000543971
0.8	4	0.01	0.000483416
0.8	5	0.01	0.000424818
0.8	0.1	0.05	0.000680149
0.8	1	0.05	0.000576421
0.8	2	0.05	0.000466983
0.8	3	0.05	0.000359457
0.8	4	0.05	0.000263279
0.8	5	0.05	0.000173803
1	0.1	-0.05	0.000807795
1	1	-0.05	0.000775516
1	2	-0.05	0.000740422
1	3	-0.05	0.00070609
1	4	-0.05	0.000672507
1	5	-0.05	0.000639869
1	0.1	-0.01	0.000759088
1	1	-0.01	0.000701575
1	2	-0.01	0.000640035
1	3	-0.01	0.000580964
1	4	-0.01	0.000524326
1	5	-0.01	0.000470083
1	0.1	0	0.000749056
1	1	0	0.000683078
1	2	0	0.000612921
1	3	0	0.000546035
1	4	0	0.000482364
1	5	0	0.000421841
1	0.1	0.01	0.000730797

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η	β	κ	npv $\beta\eta$
1	1	0.01	0.000656457
1	2	0.01	0.000578038
1	3	0.01	0.000504151
1	4	0.01	0.000434272
1	5	0.01	0.00036835
1	0.1	0.05	0.0006776
1	1	0.05	0.000552205
1	2	0.05	0.000423188
1	3	0.05	0.000310761
1	4	0.05	0.000213898

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FIGURE 4. implied lognormal volatility smiles, $\eta = 0.5$, $\kappa = 0.0$

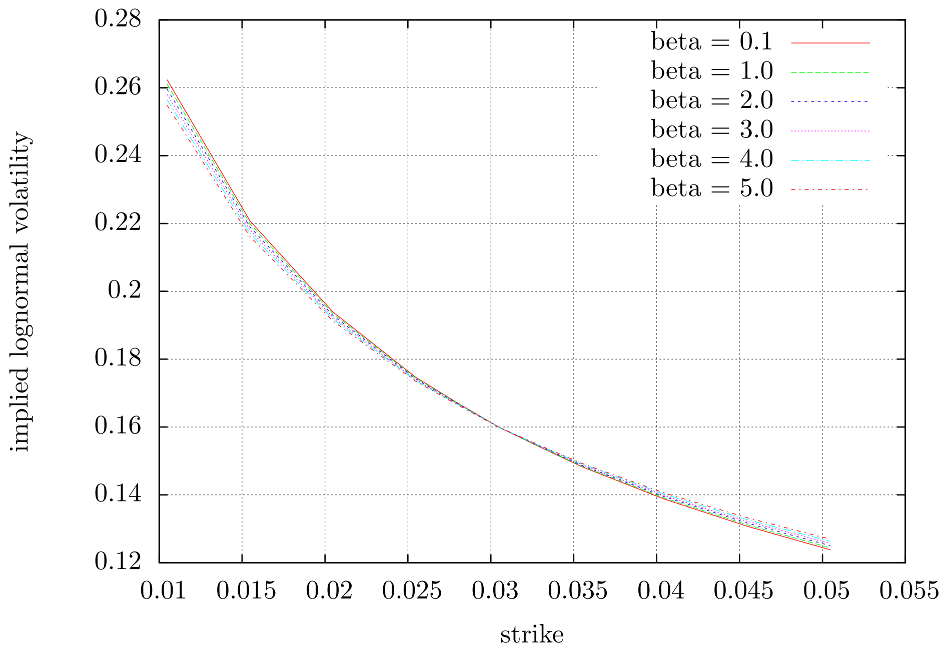

FIGURE 5. implied lognormal volatility smiles, $\eta = 0.8$, $\kappa = -0.01$

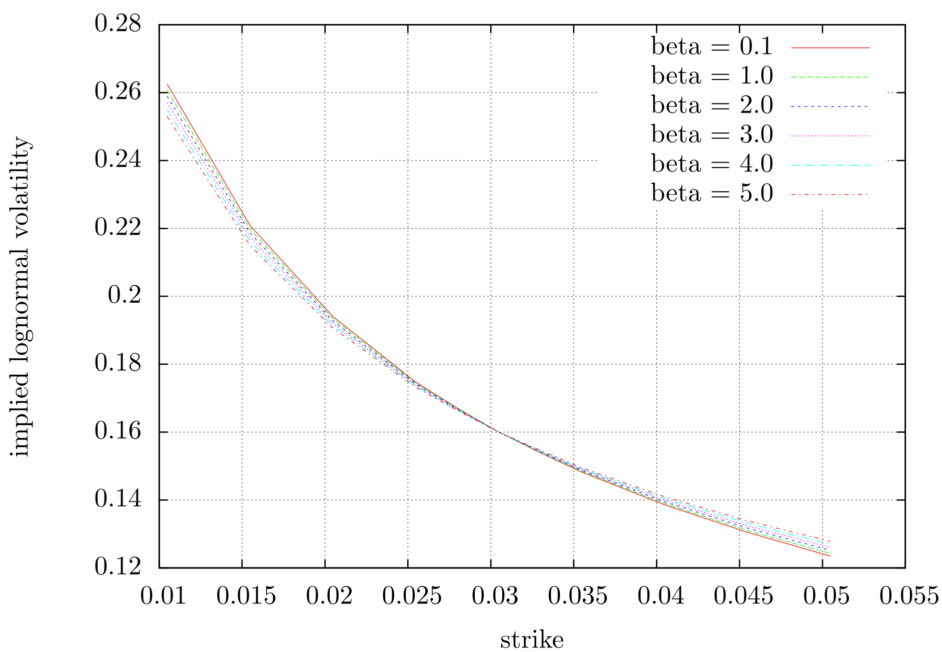

FIGURE 6. implied lognormal volatility smiles, $\eta = 0.8$, $\kappa = 0.01$

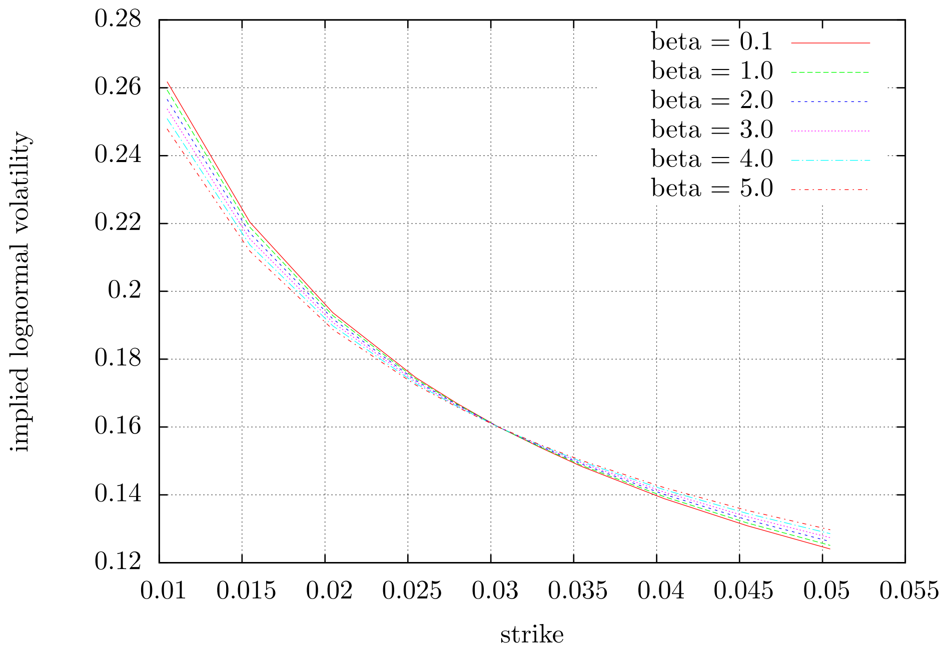

FIGURE 7. implied lognormal volatility smiles, $\eta = 1.0$, $\kappa = 0.05$