NORMAL LIBOR IN ARREARS

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ABSTRACT. We derive the in arrears covexity adjustment in a normal black model.

1. Introduction

Consider a Libor rate F with fixing time t_f, index start date t_s and index end date t_e which is paid at t_p. We are interested in the valuation of a coupon involving such a Libor rate fixed in arrears, i.e. $t_f = t_p - \Delta$ with Δ denoting the settlement lag (e.g. 2 business days). For the following derivations we assume $\Delta = 0$. We also assume $t_s = t_p =: t, t_e = t_s + \tau$ with $\tau = \tau_{\text{accr}}$ the accrual period of the coupon to be valued. These are typically all minor approximations to the reality.

2. The model

We assume a normal black model for F, i.e.

$$dF = \nu dW \tag{2.1}$$

with ν denoting the normal volatility.

3. The adjustment

The Libor coupon can then be valued (working in the $t + \tau$ forward measure) as

$$C = P(0, t + \tau) E^{t+\tau} \left(\frac{F(t)\tau}{P(t, t + \tau)} \right) \tag{3.1}$$

with $P(t, T)$ denoting the price at time t of a zero bond with maturity T and the expectation being taken in the $t + \tau$-forward measure $\mathbb{Q}^{t+\tau}$. We note here that our derivation will be done in a single curve setup for the moment, i.e. with identical forwading and discounting curves.

We write

$$F(t) = \frac{F(t)}{1 + F(t)\tau}(1 + F(t)\tau) \tag{3.2}$$

and get

$$C = P(0, t + \tau) E^{t+\tau} (\tau F(t) + \tau^2 F^2(t)) \tag{3.3}$$

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Now $F(\cdot)$ is a $\mathbb{Q}^{t+\tau}$ - martingale. Furhtermore by Ito

$$dF^2 = 2FdF + \nu dFdF = 2F\nu dW + \nu^2 dt \tag{3.4}$$

hence

$$E^{t+\tau}(F^2(t)) = F^2(0) + \nu^2 t \tag{3.5}$$

and therefore

$$C = P(0,t+\tau)(\tau F(0) + \tau^2 F^2(0) + \tau^2\nu^2 t) \tag{3.6}$$

which can be written as

$$P(0,t)\tau\left(F(0) + \frac{\tau\nu^2 t}{1+\tau F(0)}\right) \tag{3.7}$$

since $P(0,t+\tau) = P(0,t)(1+\tau F(0))$. It is interesting to note that this adjustment is the same as the in arrears limit case in (3.14) of [1] which is derived in a lognormal model if one relates the normal volatility ν and the lognormal volatility σ by $\nu := F(0)\sigma$, i.e. the drift freezing *approximation* in [1] effectively means to compute the convexity adjustment *exactly* in a normal black model when connecting the parameters of the two models appropriately.

4. Caplets

Now we consider an in arrears caplet. Just as above it can be valued

$$C = P(0,t+\tau)E^{t+\tau}\left(\frac{(F(t)-K)^+\tau}{P(t,t+\tau)}\right) \tag{4.1}$$

We write

$$(F(t)-K)^+ = \frac{(F(t)-K)^+}{1+F(t)\tau}(1+F(t)\tau) \tag{4.2}$$

to arrive at

$$C = P(0,t+\tau)E^{t+\tau}(\tau(F(t)-K)^+ + \tau^2 F(t)(F(t)-K)^+) \tag{4.3}$$

We know from above that $F(t)$ is normally distributed with mean $F(0)$ and standard deviation $\nu\sqrt{t}$. Introducing a new variable

$$X = \frac{F(t)-F(0)}{\nu\sqrt{t}} \tag{4.4}$$

which is obviously standard normal we can evaluate the components from the expectation above as

$$A := \frac{1}{\sqrt{2\pi}}\int_{(K-F(0))/\nu\sqrt{t}}^{\infty}(X\nu\sqrt{t} + F(0) - K)e^{-X^2/2}dX \tag{4.5}$$

and

$$B := \frac{1}{\sqrt{2\pi}} \int_{(K-F(0))/\nu\sqrt{t}}^{\infty} (X\nu\sqrt{t} + F(0))(X\nu\sqrt{t} + F(0) - K)e^{-X^2/2} dX \tag{4.6}$$

These integrals can be computed using the general identity

$$\frac{1}{\sqrt{2\pi}} \int (aX^2 + bX + c)e^{-X^2/2} dX = \frac{a+c}{2} \operatorname{erf}\left(\frac{X}{\sqrt{2}}\right) - \frac{1}{\sqrt{2\pi}} e^{-X^2/2}(aX + b) \tag{4.7}$$

and putting all together we retrieve the caplet price as

$$C = P(0, t + \tau)(\tau A + \tau^2 B) \tag{4.8}$$

References

[1] Caspers Peter, *Libor Timing Adjustments*, Electronic copy available at: http://ssrn.com/abstract=2170721

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