Markov Functional Model Implementation

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Axiomatic Hull White

Any gaussian one factor HJM model which satisfies *separability*, i.e.

$$\sigma_f(t,T) = g(t)h(T) \tag{1}$$

for the instantaneous forward rate volatility with deterministic $g, h > 0$, necessarily fulfills

$$dr(t) = (\theta(t) - a(t)r(t))dt + \sigma(t)dW(t) \tag{2}$$

for the short rate r, which means, it is a Hull White one factor model.

The T-forward numeraire

Set $x(t) := r(t) - f(0,t)$ and fix a horizon T, then in the T-forward measure the numeraire can be written

$$N(t) = P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-x(t)A(t,T)+B(t,T)} \tag{3}$$

with A, B dependent on the model parameters. The Hull White model is called an *affine* model.

Smile in the Hull White Model

- The distribution of $N(t)$ is lognormal. The shape of the distribution can not be controlled by any of the model parameters.
- For fixed t you can calibrate the model to one market quoted interest rate optoin (typically a caplet or swaption).
- You can choose the strike of the option, but the rest of the smile is implied by the model.

Callable vanilla swaps

Pricing of callable fix versus Libor swaps may be done in a Hull White model which is calibrated as follows:

- For each call date find a market quoted swaption which is equivalent to the call right (in some sense, e.g. by matching the npv and its first and second derivative of the underlying at $E(x(t))$).
- Calibrate the volatility function $\sigma(t)$ to match the basket of these swaptions.
- Choose the mean reversion of the model to control serial correlations.

Intertemporal correlations

To understand the role of the reversion parameter assume σ and a constant for a moment. Then it is easy to see

$$\text{corr}(x(T_1), x(T_2)) = \sqrt{\frac{e^{2aT_2} - 1}{e^{2aT_1} - 1}} = e^{-a(T_2 - T_1)} \sqrt{\frac{1 - e^{-2aT_1}}{1 - e^{-2aT_2}}} \tag{4}$$

which shows that for $a = 0$ the correlation is $\sqrt{T_1/T_2}$ and goes to zero if $a \to \infty$ and to one if $a \to -\infty$.

Callable cms swaps

The calll rights in a callable cms swap are options on a swap exchanging cms coupons against fix or Libor rates. Such underlying swaps are drastically mispriced in the Hull White model in general.

- cms coupons are replicated using swaptions covering the whole strike continuum $(0, \infty)$
- The swaption smile in the Hull White model is generally not consistent with the market smile and so are the prices of cms coupons

Obviously we need a more flexible model to price such structures

Model requirements

The wishlist for the model is as follows

- We want to be capable of calibrating to a whole smile of (constant maturity) swaptions, not only to one strike, for all fixing dates of the cms coupons. This is to match the coupons of the underlying.
- In addition we would like to calibrate to (possibly strike / maturity adjusted) coterminal swaptions to match the options representing the call rights.
- Finally we need some control over intertemporal correlations, i.e. something operating like the reversion parameter in the Hull White model

The idea to do so is to relax the functional dependency between the state variable x and the numeraire $N(t, x)$.

The driving process

We start with a markov process driving the dynamics of the model as follows:

$$dx = \sigma(t) e^{at} dW(t) \tag{5}$$

and $x(0) = 0$. The intertemporal correlation of the state variable x is the same as for the Hull White model, see (4), i.e. the parameter a can be used to control the correlation just as the reversion parameter in the Hull White model.

The numeraire surface

The model is operated in the T-forward measure, T chosen big enough to cover all cashflows relevant for the actual pricing under consideration. The link between the state $x(t)$ and the numeraire $P(t,T)$ is given by

$$P(t,T,x) = N(t,x) \tag{6}$$

which we allow to be a non parametric surface to have maximum flexibility in calibration.

Calibrating the numeriare surface to market smiles

The price of a digital swaption paying out an annuity $A(t)$ on expiry t if the swap rate $S(t) \geq K$ in our model is

$$\mathsf{dig}_{\mathsf{model}} = P(0,T) \int_{y^*}^{\infty} \frac{A(t,y)}{P(t,T)} \phi(y) dy \tag{7}$$

where y^* is the strike in the normalized state variable space (the correspondence between y and $S(t)$ is constructed to be monotonic).

Implying the swap rate

Given the market smile of $S(t)$ we can compute the market price $\mathsf{dig}_{\mathsf{mkt}}(\mathsf{K})$ of digitals for strikes K. For given y^* we can solve the equation

$$\mathsf{dig}_{\mathsf{mkt}}(K) = P(0,T) \int_{y^*}^{\infty} \frac{A(t,y)}{P(t,T)} \phi(y) dy \tag{8}$$

for K to find the swap rate corresponding to the state variable value y^*. For this $\mathrm{dig}_{mkt}(\cdot)$ should be a monotonic function whose image is equal to the possible digital prices $(0, A(0)]$. We will revisit this later.

Computing the deflated annuity

To compute the deflated annuity

$$\frac{A(t)}{P(t,T)} = \sum_{k=1}^{n} \tau_k \frac{P(t,t_k)}{P(t,T)} \tag{9}$$

we observe that

$$\left.\frac{P(t,u)}{P(t,T)}\right|_{y(t)} = E\left(\frac{1}{P(u,T)}\middle| y(t)\right) \tag{10}$$

i.e. we have to integrate the reciprocal of the numeraire at future times. Working backward in time we can assume that we know the numeraire at these times (starting with $N(T) \equiv 1$).

Converting swap rate to numeraire

Having computed the swap rate $S(t)$ we have to convert this value to a numeraire value $N(t)$. Since

$$S(t)A(t) + P(t,t^*) = 1 \tag{11}$$

we get (by division by $N(t)$)

$$N(t) = \frac{1}{S(t)\frac{A(t)}{N(t)} + \frac{P(t,t^*)}{N(t)}} \tag{12}$$

all terms on the right hand side computable via deflated zerobonds as shown above. Note that we use a slightly modified swap rate here, namely one without start delay.

Calibration to a second instrument set

Up to now we have not made use of the volatility $\sigma(t)$ in the driving markov process of the model. This parameter can be used to calibrate the model to a second instrument set, however only a single strike can be matched obviously for each expiry. A typical set up would be

- calibrate the numeraire to an underlying rate smile, e.g. constant maturity swaptions for cms coupon pricing
- calibrate $\sigma(t)$ to (standard atm or possibly adjusted) coterminal swaptions for call right calibration

Note that after changing $\sigma(t)$ the numeraire surface needs to be updated, too.

Input smile preconditioning

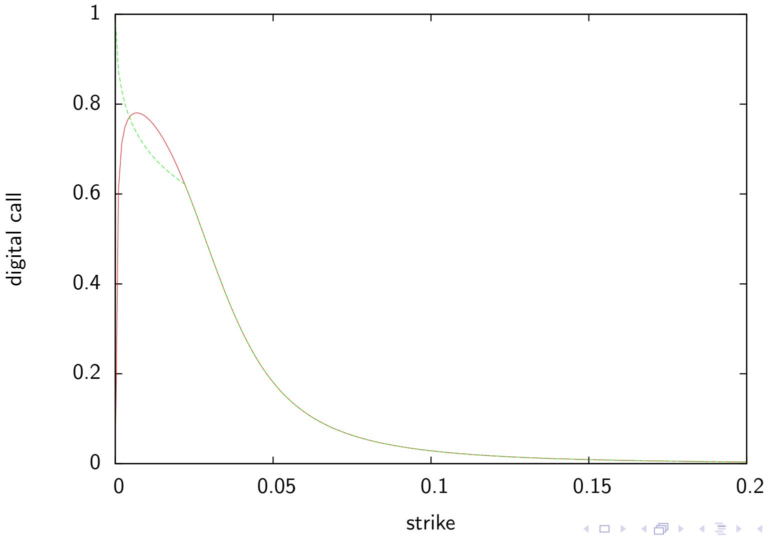
To ensure a bijective mapping

$$\mathsf{dig}_{\mathsf{mkt}} : (0, \infty) \to (0, A(0)) \tag{13}$$

it is sufficient to have an arbitrage free input smile. It is possible to allow for negative rates and generalize the interval $(0, \infty)$ to $(-\kappa, \infty)$ with some suitable $\kappa > 0$, e.g. $\kappa = 1\%$. In general input smiles are not arbitrage free, so some preconditioning is advisable, since arbitrageable smiles will break the numeraire calibration.

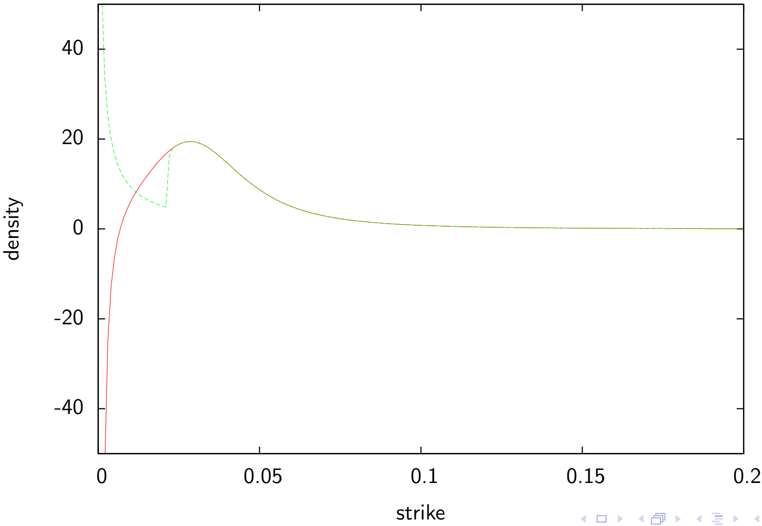
Kahale extrapolation

SABR 14y/1y digital prices as of 14-11-2012, input (solid) and Kahale (dashed)



Kahale extrapolation

SABR 14y/1y density as of 14-11-2012, input (solid) and Kahale (dashed)



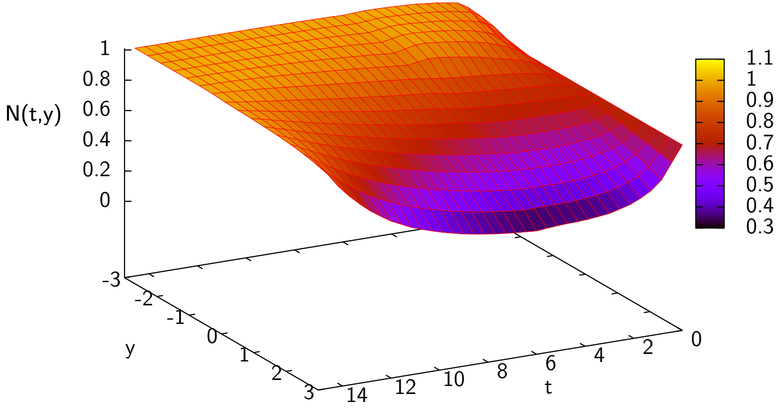
Interpolation of the numeraire

In a numerical implementation of the model we will need to discretize the numeraire surface on a grid (t_i, y_j).

- in y-direction we interpolate $N(t,y)$ (normalized by todays market forward numeraire) with monotonic cubic splines with Lagrange end condition. Outside a range of specified standard deviations (defaulted to 7), we extrapolate flat.
- in t-direction we interpolate the reciprocal of the normalized numeraire linearly. This ensures a perfect match of todays input yield curve even for interpolated times as can easily be seen from (10). After the horizon T we extrapolate flat ($N \equiv 1$ there anyway), only for technical reasons.

Sample numeraire surface

Numeraire surface for market data as of 14-11-2012



Interpolation of payoffs

Payoffs occuring in the numeraire bootstrap (digitals) or later in pricing exotics are also interpolated with Lagrange splines. We leave it as an option to

- restrict to integration of the payoff to a specified number of standard deviations,
- extrapolate the payoff flat
- extrapolate the payoff according to the Lagrange end condition

The results should not depend significantly on this choice, otherwise the numerical parameters should be increased.

Numerical Integration for deflated zero bonds

To compute deflated zerobond prices according to (10) it turns out that it is fast and accurate to

- use the celebrated Gauss Hermite Integration scheme where
- 32 points are more than enough usually to ensure a good accuracy.

This is because the integrand is globally well approximated by polynomials.

Numerical Integration for payoffs

For the numerical integration of

- digitals during numeraire bootstrapping or
- exotic pricing Gauss Hermite

is possible but leading not to satisfactory accuracy. This is due to the non global nature of the integrand in this case. Here we rely on exact integration of the piecewise 3rd order polynomials against the gaussian density, which is possible in closed form only involving the error function erf

How is the yield curve matched after all ?

It is interesting to note that the initial yield curve is matched by calibrating the numeraire to market input smiles. The yield curve is never a direct input though, as it is e.g. for the Hull White model. It is reconstructed by the model via the numeraire density coded in and read off the market smile during numeraire bootstrapping.

Long term constant maturity calibration

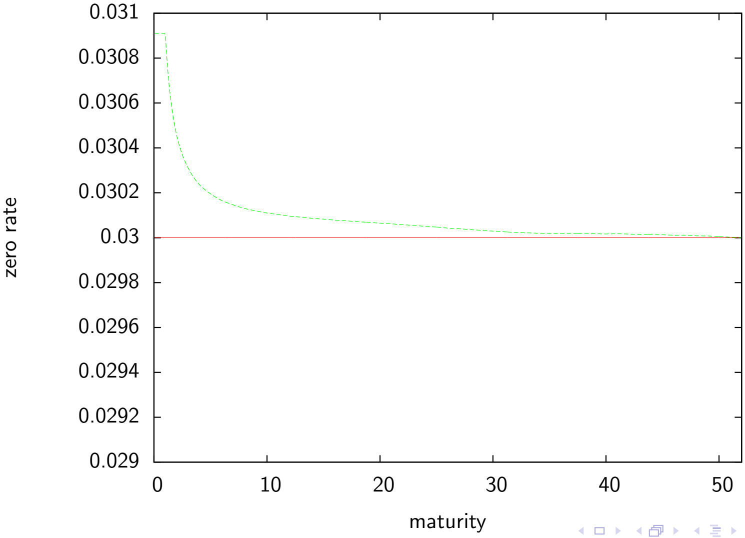
The hardest case of calibration is a long term calibration to constant maturity swaptions (or caplets).

- We start at maturity T and bootstrap the numeraire at some $t_k < T$, introducing some numerical noise in $N(t_k)$.
- We then boostrap $N(t_{k-1})$ for $t_{k-1} < t_k$, relying on the already bootstrapped future numeraire values.
- In case of a coterminal calibration we largely still use $N(T)$ and $N(t_k)$ only to a smaller degree.
- In case of cm calibration however we have to rely on $N(t_k)$ with t_k nearer to our calibration time t.

Therefore in long term cm calibration numerical noise may pile up giving large errors for the shorter term numeraire surface.

Yield curve match with standard numerical parameters

Fit to Yts flat @ 3%, 2y cm swaptions @ 20%, 7 standard deviations, 200 % upper rate bound



Increasing numerical accuracy for long term cm baskets

The first idea in this case is to increase the numerical accuracy by increasing

- the number of covered standard deviations (e.g. from 7 to 12)
- the upper cut off point for rates (e.g. from 200% to 400%)
- (maybe the number of discretization points, though less critical)

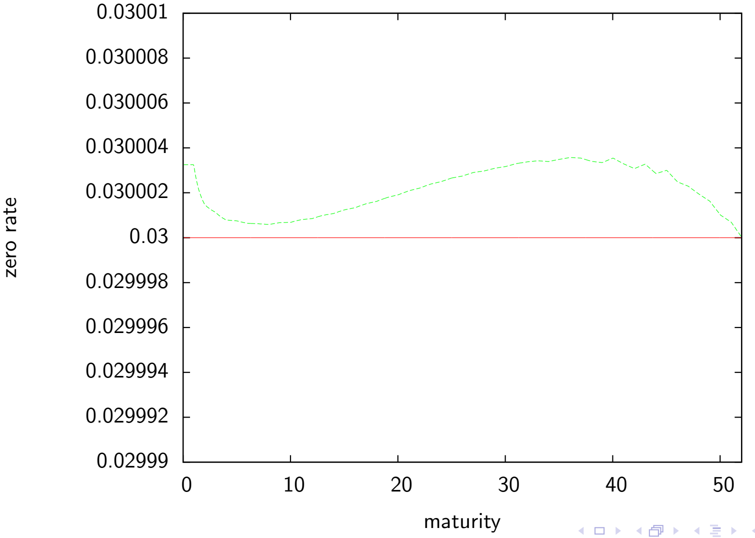
However this breaks the calibration totally, because the standard ("double") 53 bit mantissa numerical precision is not sufficient to do the computations numerically stable any more.

NTL high precision computing

The NTL and boost libraries provide support for arbitrary floating point precision. We incorporated NTL support as an option in the model replacing the standard double precision by an arbitrary mantissa length precision in the critical sections of the computation (which turned out to be the integration of payoffs against the gaussian density, where large integrand values are multiplied by small density values).

Yield curve match using high precision computing

Fit to Yts flat @ 3%, 2y cm swaptions @ 20%, 150bit mantissa, 12 standard deviations, 400 % upper rate bound



A more pragmatic approach: Adjustment factors

Computations with NTL are slow. A more practical way to stabilize the calibration in difficult circumstances are adjustment factors forcing the numeraire to match the market input yield curve. The adjustment factor is introduced by replacing

$$N(t_i, y_j) \rightarrow N(t_i, y_j) \frac{P^{\text{model}}(0, t_i)}{P^{\text{market}}(0, t_i)} \tag{14}$$

This option should be used with some care because it may lower the accuracy of the volatility smile match. In most situations the adjustment factors are moderate though, in the example we had before:

Adjustment factors in the example above

Date	Adjustment Factor
November 14th, 2013	1.00000029079227
November 14th, 2014	0.999999566861981
November 14th, 2015	0.999999720414697
November 14th, 2016	0.999999838009949
November 14th, 2017	0.999999380770489
...	...
November 14th, 2056	0.999804362556495
November 14th, 2057	0.999904959217797
November 14th, 2058	0.99988000473961
November 14th, 2059	0.999816723715493
November 14th, 2060	0.999830021528368
November 14th, 2061	0.999737006370401
November 14th, 2062	0.999761860227793
November 14th, 2063	0.999887598113548

QuantLib Implementation

- http://quantlib.org
- Open Source library for quantitative finance in real life
- The experimental markov functional model will be available in ql/experimental/models
- Including a Bermudan Swaption engine using numerical integration
- Static multi curve suport, more instruments and PDE pricing engines are currently under development

References

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- Shoup, Victor: NTL A Library for doing Number theory, http://www.shoup.net/ntl/
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Thank you

Questions?

../../../Downloads/Beaker2.png