MULTICURVE REPLICATION OF CMS COUPONS

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Abstract. We summarize formulas for the replication and pricing of Cms coupons in a multicurve setting.

1. Notation

We assume a two curve setting consisting of a discount curve D and a forward curve F. All quantities computed on these curves are denoted with a respective subscript D or F.

A swap rate fixed on t_f may be written as

$$S(t_f) = \frac{\sum \tau_i L_F(t_f, t_i) P_D(t_f, s_i)}{\sum \tau_j^* P_D(t_f, s_j^*)} \tag{1.1}$$

with τ denoting yearfractions, t, s fixing and payment times, $L(u,v)$ the Libor rate fixed at v as seen from u and P a discount factor.

A swaption price p for a swaption withh expiry on t_f is given by the usual Black formula B

$$p = \sum \tau_j^* P_D(0, s_j^*) B(S(0), K, \sigma, t_f) \tag{1.2}$$

2. Curve Scenarios

We use curve scenarios generated from a Hull White model to determine the replication basket. In a single curve setup we can write

$$P(t,t') = \frac{P(0,t')}{P(0,t)} e^{-hG(t,t')} \tag{2.1}$$

with $h = x + \phi$ derived from the normalized short rate $x(t) = r(t) - f(0,t)$ by a constant ϕ which only depends on the Hull White model parameters. Here

$$G(t,t') = \int_t^{t'} e^{-(u-t)\kappa} du = \frac{1 - e^{-(t'-t)\kappa}}{\kappa} \tag{2.2}$$

and for $\kappa = 0$

$$G(t,t') = t' - t \tag{2.3}$$

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For our two curve setup we assume a static instantaneous forward spread between the curves, thus both curves have the same $x(t)$ and therefore equation 2.1 can be used for both curves P_D and P_F.

3. Replication Basket

3.1. **CMS Caplet.** We wish to replicate a CMS caplet with fixing time t_f, payment time t_p and strike K. We assume a discretization $h_i, i = 0, ..., n$ with $h_0 = K$.

The payoff of the caplet in scneario $i = 1$ is $S(t_f, h_1) - K$ which discounts back to t_f by $P_D(t_f, t_p, h_1)$. The npv of a physical settled call swaption with expiry on t_f and strike K is on the other hand $\Delta_1 A(t_f, h_1)(S(t_f, h_1) - K)$ where A denotes the annuity as written out explicitly above. Here Δ_1 denotes the hedge weight. Equating the npv of the cms caplet payoff and the call swaption yields the first hedge weight

$$\Delta_1 = \frac{P_D(t_f, t_p, h_1)}{A(t_f, h_1)} \tag{3.1}$$

We note here that it is possible to use a swaption price w.r.t. cash settlement where the annuity is replaced by

$$\sum_j \frac{\tau}{(1 + \tau S(t_f))^j} \tag{3.2}$$

with a uniform τ corresponding to the frequency on the fixed leg.

For the second scenario we have to revalue the first hedge instrument as $\Delta_1 A(t_f, h_2)(S(t_f, h_2) - K)$, subtract this from the npv of a cms caplet in scenario two which is $P_D(t_f, t_p, h_2)(S(t_f, h_2) - K)$ and solve for the weight Δ_2 of the second hedge instrument which is a call swaption with strike $S(t_f, h_1)$ (therefore contributing nothing in the first scenario):

$$\Delta_2 = \frac{P_D(t_f, t_p, h_2)(S(t_f, h_2) - K) - \Delta_1 A(t_f, h_2)(S(t_f, h_2) - K)}{A(t_f, h_2)(S(t_f, h_2) - S(t_f, h_1))} \tag{3.3}$$

We continue until the last scenario $i = n$.

3.2. **CMS Floorlet.** The case of a floorlet is handled analoguous to the caplet case.

3.3. **CMS Swaplet.** A swaplet is priced via parity.

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