# MULTITENOR VOLATILITIES

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Abstract. We propose a possible approach for pricing interest rate options on underlyings that are not directly quoted in the market due to their non standard tenor. To do so we translate the density given by the smile of the quoted underlying appropriately.

## 1. General approach

To fix a setting we assume a generic model for an underlying forward rate $f$ of the form

$$df(t) = \sigma(t, f(t))v(t)dW(t) \tag{1.1}$$

$$dv(t) = \epsilon(t, v(t))dV(t) \tag{1.2}$$

$$dW(t)dV(t) = \rho(t)dt \tag{1.3}$$

under a suitable measure associated to a numeraire $N$. We assume to have a quoted market price $\nu$ for an european option which fixes on $t$ and pays on $T$ the amount $(f(t)-k)^+$. We can assume the relation

$$\nu = N(0)E((f(t)-k)^+) \tag{1.4}$$

$f(0)$ is todays forward projection for $f$. Now assume another rate $g$ exhibiting a basis relative to $f$ which should mean

$$g(0) = f(0) + \alpha \tag{1.5}$$

with $\alpha$ denoting the basis spread of $g$ relative to $f$.
From 1.1 we can imply a dynamics for $g$ by setting

$$g(t) = f(t) + \alpha \tag{1.6}$$

Obviously $g$ is a martingale with

$$g(0) = f(0) + \alpha = E(g(t)) \tag{1.7}$$

and

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$$dg(t) = \eta(t, g(t))v(t)dW(t) \tag{1.8}$$
$$dv(t) = \epsilon(t, v(t))dV(t) \tag{1.9}$$
$$dW(t)dV(t) = \rho(t)dt \tag{1.10}$$

under the same measure as above and with a displaced local volatility

$$\eta(t, g) := \sigma(t, g - \alpha) \tag{1.11}$$

We have thus deduced a price for an option on $g$

$$\mu = M(0)E((g(t) - l)^+) \tag{1.12}$$

with numeraire $M$.

Note that it may well be $M = N$ e.g. for collaterized interest rate options, then $M$ and $N$ both being the zerobond numeraire on the OIS curve.

We also note that $f$ and $g$ share the same density only translated by the basis spread $\alpha$.

Now assume we have a pricing formula for the original option on $f$

$$E((f(t) - k)^+) = \Pi(f(0), k, \sigma(\cdot, \cdot), \epsilon(\cdot, \cdot), \rho(\cdot)) \tag{1.13}$$

Since

$$E((g(t) - l)^+) = E((f(t) + \alpha - l)^+) \tag{1.14}$$

we get

$$E((g(t) - l)^+) = \Pi(g(0) - \alpha, l - \alpha, \sigma(\cdot, \cdot), \epsilon(\cdot, \cdot), \rho(\cdot)) \tag{1.15}$$

i.e. we can use a displaced diffusion variant of the original pricing formula shifting the forward and the strike down by the basis spread $\alpha$.

In the next sections we give special cases of the formulas developed here.

## 2. Lognormal model

Assume we use a lognormal Black pricing. Then we have

$$E((f(t) - k)^+) = B_{\ln}(f(0), k, \sigma_{\ln}(k), \tau) \tag{2.1}$$

for an option on $f$ with forward rate $f(0)$, strike $k$, lognormal implied volatility $\sigma_{ln}(k)$ and expiry time $\tau$. Here $B_{\ln}$ denotes the lognormal Black formula with discount factor equal to unity.

For the option on $g$ we get

$$E((g(t) - l)^+) = B_{\ln}(g(0) - \alpha, l - \alpha, \sigma_{\ln}(l - \alpha), \tau) \tag{2.2}$$

which is the displaced diffusion variant of the Black formula with $\sigma$ taken from the original smile for a displaced strike $l - \alpha$.

## 3. Normal model

Assume now we use a normal Black pricing. Then we have

$$E((f(t)-k)^+) = B_{\mathrm{norm}}(f(0), k, \sigma_{\mathrm{norm}}(k), \tau) \tag{3.1}$$

with notation as before, but a normal implied volatility $\sigma_{norm}(k)$ and the normal Black formula $B_{\mathrm{norm}}$, again with discounting 1. Since the normal formula is translation invariant we can write in this case

$$E((g(t)-l)^+) = B_{\mathrm{norm}}(g(0), l, \sigma_{\mathrm{norm}}(l-\alpha), \tau) \tag{3.2}$$

This can be interpreted as translating the normal volatlility surface by the basis $\alpha$ and keeping the pricing formula itself as before. This reflects the intuition of "keeping the normal volatlility constant" when pricing non quoted tenor options.

## 4. SABR model

As a final example consider pricing on a SABR smile

$$E((f(t)-k)^+) = S(f(0), k, \alpha, \beta, \nu, \rho) \tag{4.1}$$

with $S$ denoting a (again non discounting) SABR pricing formula. From the discussion above the option on $g$ is priced

$$E((g(t)-l)^+) = S(g(0)-\alpha, l-\alpha, \alpha, \beta, \nu, \rho) \tag{4.2}$$

## 5. Examples

We give examples in abstract form, i.e. we assume generic forward rates $f$ and $g$ and options on these rates. Furthermore we assume the discount factor (or the annuity) to be one. Typically $f$ and $g$ wil be Libor or Swap rates and the options on them caplets / floorlets respectively payer / receiver swaptions.

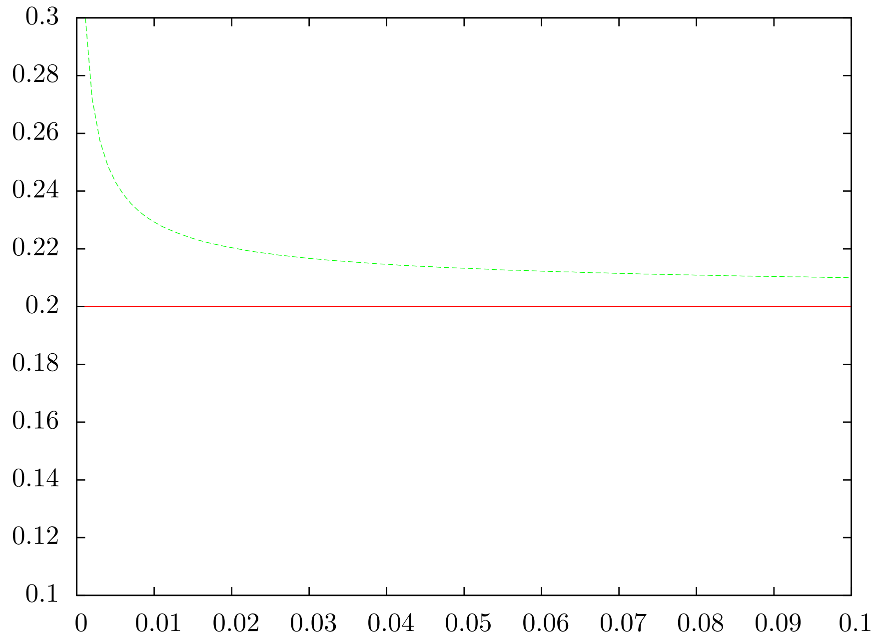
5.1. **Flat smile.** Let $f$ be 2% and $g$ be 2.2%, i.e. the basis spread $\alpha$ is 20 basis points. We consider a flat implied (lognormal) volatility smile at 20% quoted for options on $g$ with expiry time $\tau = 5$.

The premium for an atm call on $g$ is then 38.93 basis points. The premium for an atm call on $f$ is the same by construction of the model. It can be computed with a lognormal black formula using a displacement of 20 basis points. The zero displacement implied volatility for the call on $f$ can be computed with numerical inversion and is 22.0390%.

The smile for $f$ is shown in figure 1. It is not flat though $g$ has a flat smile. This example can be thought of as converting e.g. a 6m caplet volatility to a 3m caplet volatility.

Now we assume that $g$ has a float volatility smile at 20% and imply the smile for $f$. The result is shown in figure 2. This example corresponds e.g. to the conversion of a 3m caplet volatility to a 6m caplet volatility.

FIGURE 1. Flat Smile for $g$ "6m" (solid), Implied Smile for $f$ "3m" (dashed)




5.2. **SABR smile.** With $f$ and $g$ as above we now consider a SABR smile for $g$ with parameters $\alpha = 0.07, \beta = 0.8, \nu = 0.6, \rho = -0.3$ and imply the smile for $f$. The result is shown in figure 3.

Again we now assume a SABR smile with the same parameters as above, but for $f$ and imply the smile for $g$. The result is shown in figure 4.

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FIGURE 2. Flat Smile for $f$ "3m" (solid), Implied Smile for $g$ "6m"(dashed)

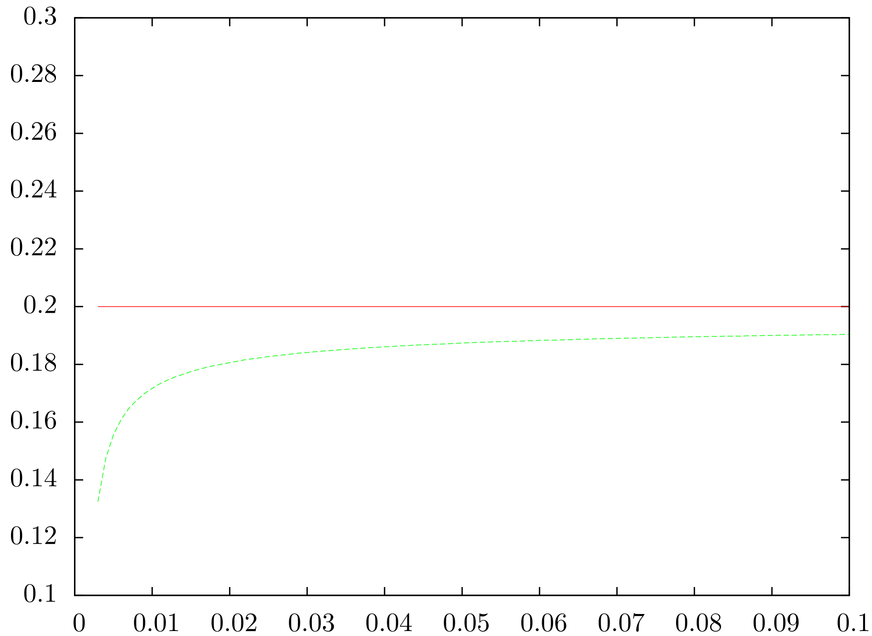

Figure 3. SABR Smile for $g$ "6m" (solid), Implied Smile for $f$ "3m" (dashed)

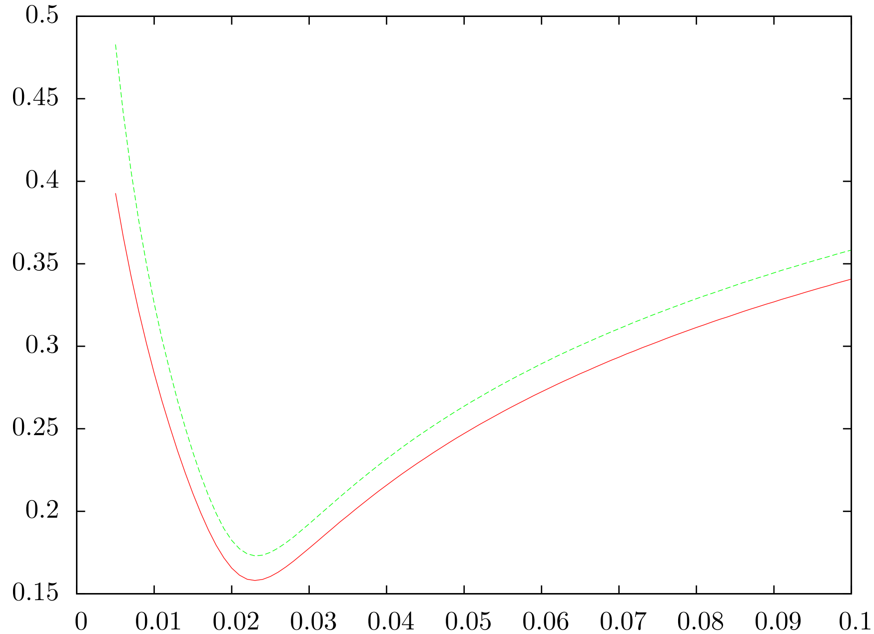

FIGURE 4. SABR Smile for $f$ (solid) "3m", Implied Smile for $g$ (dashed) "6m"