MODELLING VOLATILITY SMILES WITH SPLINE DENSITIES

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Abstract. todo...

1. Method

We assume a strike grid $l = k_0 < ... < k_n = u$ and arbitrage free, non discounted call prices $f - l = c_0 < ... < c_n = 0$, where f denotes the underlying forward. We aim to construct a density function ϕ which is smooth and matches the call prices exactly. The idea is to use a cubic spline within $[l, u]$ using the given strike grid and set the density zero outside this interval. As additional conditions we set $\phi'(l) = \phi'(u) = 0$, which ensures that the density is $C^1(\mathcal{R})$ globally on the real line and $C^2(l, u)$.

Denoting the call price function by $c : \mathcal{R} \to [0, f]$ we have the relation

$$\frac{\partial^2 c}{\partial k^2}(t) = \phi(t) \tag{1.1}$$

for the density, as well as

$$\frac{\partial c}{\partial k}(t) = \int_l^u \phi(s)ds - 1 \tag{1.2}$$

for the distribution / negative digital prices and finally

$$c(t) = f - l + \int_l^t \left(\int_l^v \phi(s)ds - 1 \right) dv \tag{1.3}$$

for the call price function itself.

The density ϕ is given by cubic polynomials

$$p_i(s) = a_i(s - k_i)^3 + b_i(s - k_i)^2 + c_i(s - k_i) + d_i \tag{1.4}$$

on each interval $[k_i, k_{i+1}], i = 0, ..., n - 1$. We start with the computation of the inner integral in 1.3. For each i we can compute

$$\int_{k_i}^v p_i(s)ds = \frac{1}{4}a_i(v - k_i)^4 + \frac{1}{3}b_i(v - k_i)^3 + \frac{1}{2}c_i(v - k_i)^2 + d(v - k_i) \tag{1.5}$$

For the integral over a full subinterval we introduce the notation

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$$\int_{k_i}^{k_{i+1}} p_i(s)ds = \eta_i \tag{1.6}$$

We continue with the outer integration in 1.3. We write $c(t) - f$ as

$$\int_l^t \left[\left(\sum_{i=0}^{m(v)-1} \int_{k_i}^{k_{i+1}} \phi(s)ds \right) + \int_{k_{m(t)}}^{v} \phi(s)ds - 1 \right] dv \tag{1.7}$$

with $m(t)$ such that $t \in [k_{m(t)}, k_{m(t)+1})$. The left integral is independent of v and was computed in 1.6. For the right integral we have a closed form expression 1.5. Therefore we can continue to write

$$\int_l^t \left[\left(\sum_{i=0}^{m(v)-1} \eta_i \right) + \frac{1}{4} a_{m(v)} (v - k_{m(v)})^4 + ... - 1 \right] dv \tag{1.8}$$

We decompose the outer integral as well on our strike grid and get

$$\sum_{j=0}^{m(t)} \int_{k_j}^{\min(k_{j+1},t)} \left[\left(\sum_{i=0}^{j-1} \eta_i \right) + \frac{1}{4} a_j (v - k_j)^4 + ... - 1 \right] dv \tag{1.9}$$

which leads to our final formula

$$c(t) = f - l + \sum_{j=0}^{m(t)} \left\{ (\mu_j - k_j)(\lambda_j - 1) + \frac{1}{20} a_j (\mu_j - k_j)^5 + ... + \frac{1}{2} d_j (\mu_j - k_j)^2 \right\} \tag{1.10}$$

with $\mu_j := \min(k_{j+1}, t)$ and $\lambda_j := \sum_{i=0}^{j-1} \eta_i$.

In addition we need to compute the expectation, which is the sum over the expressions

$$\int_{k_i}^{k_{i+1}} sp_i(s)ds = (k_{i+1} - k_i)\eta_i - \left(\frac{1}{20} a_j (\mu_j - k_j)^5 + ... + \frac{1}{2} d_j (\mu_j - k_j)^2 \right) \tag{1.11}$$

for $i = 0, ..., n-1$.

References