A gentle introduction to IR modeling

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What it is all about

Price a derivative contract !

Constraints

Do that consistently with all observable relevant market prices.

Second Thoughts

What do you mean by "price" after all ?

1. the amount of money that can be realized in the market
2. the amount of money that we would require to receive or pay to close the deal

... we will focus on the first kind of price.

Price Adjustments

1. CVA
2. DVA
3. FVA
4. KVA
5. MVA
6. TVA

... we exclude all of them in the following.

Randomness

We need a source of randomness to model the market which is (or appears to be) random.

Stochastic Processes

To model a random observable we assume $X(0) = x_0$ and

$$dX(t,\omega) = \mu(t,\omega)dt + \sigma(t,\omega)dW \tag{1}$$

which means, knowing $X(t_0)$ we can evolve forward by

$$X(t_0 + \Delta t) \approx X(t_0) + \mu(t)\Delta t + \sigma(t)Z \tag{2}$$

with $Z \sim N(0, \Delta t)$ a normal distributed random variable with zero mean and variance Δt.

Asset Classes

Usually we distinguish

1. Equity
2. FX
3. Interest Rates
4. Credit
5. Inflation
6. Commodity

Today we focus on Interest Rates.

Interest Rate Building Blocks

Quoted, liquidly traded instruments:

1. Cash deposits
2. Forward Rate Agreements
3. Vanilla Swaps
4. Short Futures
5. Bond Futures

and in addition options

1. (Ibor) Caps, Floors
2. Swaptions
3. Futures options

Market Models for Options

Prices for Caps, Floors and Swaptions are quoted using volatilities σ, assuming a black model for the underlying F, either in lognormal style

$$dF = F\sigma dW \tag{3}$$

or shifted lognormal style

$$dF = (F+\alpha)\sigma dW \tag{4}$$

or in normal style

$$dF = \sigma dW \tag{5}$$

all of them having a simple closed form solution (Black formula).

Volatility smile

Options with different strike and maturity are priced using different volatilities. This seems weird at first sight, but simply means that the Black model is more a quoting convention than a global model.

Underlying density

It is simple to derive the underlying density as

$$\phi(K) = \frac{\partial^2 c}{\partial K^2}(K) \tag{6}$$

for undiscounted call prices. The density is in the natural pricing measure (T-forward measure for caplets, Annuity measure for swaptions). This means you can get (in principle) the market density for an expiry time from a continuum of quoted option prices in strike direction.

IR Smile modeling

In interest rates by far the most common single maturity smile model is the (shifted) SABR model

$$dF(t) = (F(t)+\alpha)^{\beta}\sigma(t)dW \quad (7)$$

$$d\sigma(t) = \sigma(t)\nu dV \quad (8)$$

$$dV dW = \rho dt \quad (9)$$

with parameters β (skew), $\alpha = \sigma(0)$ (initial level for σ), ν (volatility of volatility) and ρ (correlation between forward and vol process).

Model independent pricing

Coupons of the following type can be priced only using a no arbitrage argument on today's yield curves:

1. fixed coupons
2. natural floating rate coupons

Natural means an IBOR - styled coupon with payment date = index accrual period end.

Single maturity model pricing

Coupons that depend on the distribution of an underlying variable on a single time instance, like

1. in arrears fixed IBOR coupons
2. capped / floored IBOR coupons
3. CMS coupons
4. CMS Spread Coupons

Usually the effective rate r for coupon estimation is then written

$$r = r_0 + c \tag{10}$$

as the sum of the zero volatility forward (i.e. the forward on today's curve) r_0 plus a convexity adjustment c.

Closed form convexity adjustments

Assuming a black model for the underlying one can derivate closed form approximations for convexity adjustments

1. timing adjustments
2. CMS adjustments

Volatility smile

Usually options with different strikes are prices

Questions / Discussion

French Erlkönig