CMS SPREAD OPTION PRICING WITH NORMAL AND SHIFTED LOGNORMAL DYNAMICS

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First Version October 29, 2015 - This Version October 29, 2015

Abstract. We extend the cms spread option formula in [1], 13.16.2 to the case of shifted lognormal or normal swap rate dynamics.

Contents

1. Original lognormal model

The original model in [1], 13.16.2 for cms spread option pricing is as follows. If $S_i, i = 1, 2$ are the underlying swap rates, t is the fixing time, T the payment time of the coupon with payoff ($\phi \in \{-1, +1\}$)

$$\Pi(t) = \max(\phi(aS_1(t) + bS_2(t) - K), 0) \tag{1.1}$$

then under the T-forward measure we assume

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dZ_1 \tag{1.2}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dZ_2 \tag{1.3}$$

$$dZ_1 dZ_2 = \rho dt \tag{1.4}$$

[1] gives a formula for the price $P(0, T)E^T(\Pi(t))$ involving a one dimensional integral as formula (13.34). The drifts μ_i are implied from exogeneously given convexity adjustments for S_i, e.g. computed in a replication model on the respective underlying smiles).

Note that the formulation is [1] is a little more restrictive, because there it is assumed $t = T$. However, provided that the swap rate adjustments implying $\mu_i, i = 1, 2$ are computed w.r.T. the same fixing time t and payment time $T > t$, one can just replace the pricing expectation E^t by E^T without changing anything (inside the expectation a factor $P(t, T)/P(t, T) = 1$ occurs, which doesn't change anything).

Date: October 29, 2015.

2. Shifted lognormal extension

We look at the following straightforward extension of the model, introducing shifts for the underlying rates

$$dS_1 = \mu_1(S_1+d_1)dt + \sigma_1(S_1+d_1)dZ_1 \tag{2.1}$$
$$dS_2 = \mu_2(S_2+d_2)dt + \sigma_2(S_2+d_2)dZ_2 \tag{2.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{2.3}$$

with $d_1, d_2 \geq 0$, possibly different. Writing

$$X_1 = (S_1+d_1) \tag{2.4}$$
$$X_2 = (S_2+d_2) \tag{2.5}$$
$$L = K + ad_1 + bd_2 \tag{2.6}$$

we see that we can apply the original solution for 1.2 with underlyings X_1, X_2 and the payoff written as $(aX_1 + bX_2 - L)^+$. Note that the computation of the drifts μ_i change to

$$\mu_i = \frac{\log((S_i(0)+d_i+c_i)/(S_i(0)+d_i))}{t} = \frac{\log((X_i(0)+c_i)/(X_i(0)))}{t} \tag{2.7}$$

for $i = 1, 2$ accordingly with c_i denoting the convexity adjustment applicable to rate S_i.

3. Normal extension

The normal flavour of the original model reads

$$dS_1 = \mu_1 dt + \sigma_1 dZ_1 \tag{3.1}$$
$$dS_2 = \mu_2 dt + \sigma_2 dZ_2 \tag{3.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{3.3}$$

with drifts now given by

$$\mu_i = c_i/t \tag{3.4}$$

with c_i again denoting the exogeneously given convexity adjustment for rate $S_i, i = 1, 2$.

The option price ν is given by

$$\nu = P(0,T)E^T\left((\phi(aS_1(t) + bS_2(t) - K))^+\right) \tag{3.5}$$

The expectation can, more explicity, written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\phi(as_1 + bs_2 - K))^+ p(s_1, s_2) ds_1 ds_2 \tag{3.6}$$

with p denoting a bivariate normal distribution with expectation μ and covariance matirx Σ

$$\mu = \begin{pmatrix} \mu_1 t \\ \mu_2 t \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 t & \rho\sigma_1\sigma_2 t \\ \rho\sigma_1\sigma_2 t & \sigma_2^2 t \end{pmatrix} \tag{3.7}$$

It is well known that the distribution of $S_1(t)$ conditional on $S_2(t) = s_2$ is given by

$$S_1(t)|\{S_2(t) = s_2\} \sim \mathcal{N}\left(\mu_1 t + \frac{\rho\sigma_1}{\sigma_2}(s_2 - \mu_2 t), \sigma_1^2 t(1-\rho^2)\right) \tag{3.8}$$

We denote the density of this distribution by p_{s_2} We continue to compute the inner integral in 3.6 for a fixed $S_2(t) = s_2$, i.e.

$$\int_{-\infty}^{\infty} (\phi(as_1 + bs_2 - K))^+ p_{s_2}(s_1) ds_1 \tag{3.9}$$

We have the following

Lemma 1. *For $\alpha, \beta \in \mathbb{R}$ we have*

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\alpha x + \beta)^+ e^{-x^2/2} dx = \phi \frac{\alpha}{\sqrt{2\pi}} e^{-\beta^2/(2\alpha^2)} + \beta\phi(1 - N(-\beta/\alpha)) \tag{3.10}$$

where $\phi = \operatorname{sign}(\alpha) \in \{+1, -1\}$ and N denotes the cumulative normal distribution function. For $\alpha = 0$ the right hand side simplifies to β^+.

Proof. Let $\alpha > 0$. The proof for $\alpha < 0$ is similar and the case $\alpha = 0$ is obvious from the following steps as well. Obviously $\alpha x + \beta > 0$ iff $x > -\beta/\alpha$, so the integral can be written as

$$\frac{1}{\sqrt{2\pi}} \int_{-\beta/\alpha}^{\infty} (\alpha x + \beta) e^{-x^2/2} dx \tag{3.11}$$

Furthermore,

$$\int_{-\beta/\alpha}^{\infty} x e^{-x^2/2} dx = \left[-e^{-x^2/2}\right]_{-\beta/\alpha}^{\infty} = e^{-\beta^2/(2\alpha^2)} \tag{3.12}$$

and

$$\int_{-\beta/\alpha}^{\infty} e^{-x^2/2} dx = 1 - N(-\beta/\alpha) \tag{3.13}$$

which proves the identity. □

THe integral 3.9 is

$$\frac{1}{\sqrt{2\pi}} v \int_{-\infty}^{\infty} (\phi(as_1 + bs_2 - K))^+ e^{-\frac{(s1-\mu)^2}{2v^2}} ds_1 \tag{3.14}$$

with $\mu = \mu_1 t + \rho\sigma_1/\sigma_2(s_2 - \mu_2 t)$ and $v^2 = \sigma_1^2 t(1-\rho^2)$.
Substituting $x = (s_1 - \mu)/v$ this becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\phi a(\mu + vx) + bs_2 - K)^+ e^{-x^2/2} dx \tag{3.15}$$

Setting $\alpha = \phi a v$ and $\beta = \phi a\mu + bs_2 - K$ and $\psi = \operatorname{sign}(\alpha)$ we can write the npv (using lemma 1) ν as

(3.16)
$$P(0,T)\frac{1}{\sigma_2\sqrt{2\pi t}}\int_{-\infty}^{\infty}\psi\left[\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\beta/\alpha))\right]e^{-\frac{(s_2-\mu_2 t)^2}{2\sigma_2^2 t}}ds_2$$

with

$$\alpha = \phi a\sigma_1\sqrt{t(1-\rho^2)} \tag{3.17}$$
$$\beta = \phi a(\mu_1 t+\rho\sigma_1/\sigma_2(s_2-\mu_2 t))+bs_2-K \tag{3.18}$$
$$\psi = \mathrm{sign}(\alpha) \tag{3.19}$$

A final substitution $s=(s_2-\mu_2 t)/(\sigma_2\sqrt{t})$ yields

$$\nu = P(0,T)\frac{1}{\sigma_2\sqrt{2\pi t}}\int_{-\infty}^{\infty}\psi\left[\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\beta/\alpha))\right]e^{-s^2/2}ds \tag{3.20}$$

with

$$\alpha = \phi a\sigma_1\sqrt{t(1-\rho^2)} \tag{3.21}$$
$$\beta = \phi a(\mu_1 t+\rho\sigma_1/\sigma_2^2 s\sqrt{t})+b(s\sigma_2\sqrt{t}+\mu_2 t)-K \tag{3.22}$$
$$\psi = \mathrm{sign}(\alpha) \tag{3.23}$$

References

[1] Brigo, Mercurio: Interst Rate Models - Theory and Practice, 2nd Edition, Springer, 2006

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