FARMER'S CMS SPREAD OPTION FORMULA FOR NEGATIVE RATES

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ABSTRACT. We extend the cms spread option formula in [1], 13.16.2 to the case of shifted lognormal or normal swap rate dynamics and describe the implementation in [2].

CONTENTS

1. Original lognormal model

The original model in [1], 13.16.2 for cms spread option pricing is as follows. If $S_i, i = 1, 2$ are the underlying swap rates, t is the fixing time, T the payment time of the coupon with payoff ($a, b, K \in \mathbb{R}, \phi \in \{-1, +1\}$)

$$\Pi(t) = \max(\phi(aS_1(t) + bS_2(t) - K), 0) \tag{1.1}$$

then under the T-forward measure we assume (with $\mu_1, \mu_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0, \rho \in [-1, 1]$)

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dZ_1 \tag{1.2}$$
$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dZ_2 \tag{1.3}$$
$$dZ_1 dZ_2 = \rho dt \tag{1.4}$$

[1] gives a formula for the price $P(0, T)E^T(\Pi(t))$ in formula (13.34). This involves a one dimensional integral, which can be efficiently calculated with a Gauss-Hermite scheme for example. Here, the drifts μ_i are implied from exogeneously given convexity adjustments for S_i, e.g. computed in a replication model on the respective underlying smiles.

Note that the formulation in [1] is a little more restrictive, because $t = T$ is assumed. However, provided that the swap rate adjustments implying $\mu_i, i = 1, 2$

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are computed w.r.t. the same fixing time t and payment time $T > t$ as for the spread option, one can just replace the pricing expectation E^t by E^T without changing anything in the derivation (inside the expectation a factor $P(t,T)/P(t,T) = 1$ occurs, that's all).

2. Shifted lognormal extension

We look at the following straightforward extension of the model, introducing shifts $d_1, d_2 > 0$ for the underlying rates

$$dS_1 = \mu_1(S_1 + d_1)dt + \sigma_1(S_1 + d_1)dZ_1 \tag{2.1}$$
$$dS_2 = \mu_2(S_2 + d_2)dt + \sigma_2(S_2 + d_2)dZ_2 \tag{2.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{2.3}$$

Writing

$$X_1 = (S_1 + d_1) \tag{2.4}$$
$$X_2 = (S_2 + d_2) \tag{2.5}$$
$$L = K + ad_1 + bd_2 \tag{2.6}$$

we see that we can apply the original solution for 1.2 with underlyings X_1, X_2 and the payoff written as $(aX_1 + bX_2 - L)^+$. Note that the computation of the drifts μ_i change to

$$\mu_i = \frac{\log((S_i(0) + d_i + c_i)/(S_i(0) + d_i))}{t} = \frac{\log((X_i(0) + c_i)/(X_i(0)))}{t} \tag{2.7}$$

for $i = 1, 2$ accordingly with $c_i = E^T(S_i(t)) - S_i(0)$ denoting the convexity adjustment applicable to the rate S_i (note that the expectaion here is taken in the external model for the single swap rate adjustments respectively).

3. Normal extension

The normal flavour of the original model reads

$$dS_1 = \mu_1 dt + \sigma_1 dZ_1 \tag{3.1}$$
$$dS_2 = \mu_2 dt + \sigma_2 dZ_2 \tag{3.2}$$
$$dZ_1 dZ_2 = \rho dt \tag{3.3}$$

with drifts now given by

$$\mu_i = c_i/t \tag{3.4}$$

with c_i again denoting the exogeneously given convexity adjustment for rate S_i with $i = 1, 2$, see above.

The option price ν is given by

$$\nu = P(0,T)E^T(\Pi(t)) \tag{3.5}$$

The expectation can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\Pi(t)p(s_1,s_2)ds_1ds_2 \tag{3.6}$$

with p denoting a bivariate normal distribution

$$p \sim \mathcal{N}\left(\begin{pmatrix}S_1(0)+\mu_1 t\\ S_2(0)+\mu_2 t\end{pmatrix}, \begin{pmatrix}\sigma_1^2 t & \rho\sigma_1\sigma_2 t\\ \rho\sigma_1\sigma_2 t & \sigma_2^2 t\end{pmatrix}\right) \tag{3.7}$$

It is well known that the distribution of $S_1(t)$ conditional on $S_2(t) = s_2$ is given by

(3.8)

$$S_1(t)|\{S_2(t)=s_2\} \sim \mathcal{N}\left(S_1(0)+\mu_1 t+\frac{\rho\sigma_1}{\sigma_2}(s_2-\mu_2 t-S_2(0)), \sigma_1^2 t(1-\rho^2)\right)$$

We denote the density of this distribution by p_{s_2}. We continue to compute the inner integral in 3.6 for a fixed $S_2(t) = s_2$, i.e.

$$\int_{-\infty}^{\infty}\Pi(t)p_{s_2}(s_1)ds_1 \tag{3.9}$$

We have the following

Lemma 1. *For $\alpha, \beta \in \mathbb{R}$ we have*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\alpha x+\beta)^+e^{-x^2/2}dx = \psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\phi(1-N(-\psi\beta/\alpha)) \tag{3.10}$$

where $\psi = \operatorname{sign}(\alpha) \in \{+1,-1\}$ and N denotes the cumulative normal distribution function. For $\alpha = 0$ the right hand should be read simply as β^+.

Proof. Let $\alpha > 0$. The proof for $\alpha < 0$ is similar and the case $\alpha = 0$ is obvious from the following steps as well. Obviously $\alpha x + \beta > 0$ iff $x > -\beta/\alpha$, so the integral can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\beta/\alpha}^{\infty}(\alpha x+\beta)e^{-x^2/2}dx \tag{3.11}$$

Furthermore,

$$\int_{-\beta/\alpha}^{\infty}xe^{-x^2/2}dx = \left[-e^{-x^2/2}\right]_{-\beta/\alpha}^{\infty} = e^{-\beta^2/(2\alpha^2)} \tag{3.12}$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\beta/\alpha}^{\infty}e^{-x^2/2}dx = 1-N(-\beta/\alpha) \tag{3.13}$$

which proves the identity. □

The integral 3.9 is

$$\frac{1}{v\sqrt{2\pi}}\int_{-\infty}^{\infty}(\phi(as_1+bs_2-K))^+e^{-\frac{(s1-\mu)^2}{2v^2}}ds_1 \tag{3.14}$$

with $\mu = S_1(0)+\mu_1 t+\rho\sigma_1/\sigma_2(s_2-\mu_2 t-S_2(0))$ and $v^2 = \sigma_1^2 t(1-\rho^2)$.
Substituting $x = (s_1-\mu)/v$ this becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\phi(a(\mu+vx)+bs_2-K)\right)^+ e^{-x^2/2}dx \tag{3.15}$$

Setting $\alpha = \phi a v$ and $\beta = \phi(a\mu + bs_2 - K)$ and $\psi = \text{sign}(\alpha)$ we can write the npv (using lemma 1) ν as

$$P(0,T)\frac{1}{\sigma_2\sqrt{2\pi t}}\int_{-\infty}^{\infty}\psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\psi\beta/\alpha))e^{-\frac{(s_2-\mu_2 t-S_2(0))^2}{2\sigma_2^2 t}}ds_2 \tag{3.16}$$

with

$$\alpha = \phi a\sigma_1\sqrt{t(1-\rho^2)} \tag{3.17}$$
$$\beta = \phi\left(a(S_1(0)+\mu_1 t+\rho\sigma_1/\sigma_2(s_2-S_2(0)-\mu_2 t))+bs_2-K\right) \tag{3.18}$$
$$\psi = \text{sign}(\alpha) \tag{3.19}$$

Substituting $s = (s_2 - \mu_2 t - S_2(0))/(\sigma_2\sqrt{t})$ yields

$$\nu = P(0,T)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\frac{\alpha}{\sqrt{2\pi}}e^{-\beta^2/(2\alpha^2)}+\beta(1-N(-\psi\beta/\alpha))e^{-s^2/2}ds \tag{3.20}$$

with

$$\alpha = \phi a\sigma_1\sqrt{t(1-\rho^2)} \tag{3.21}$$
$$\beta = \phi\left(a(S_1(0)+\mu_1 t)+b(S_2(0)+\mu_2 t)-K+\sqrt{t}(\rho\sigma_1+b\sigma_2)s\right) \tag{3.22}$$
$$\psi = \text{sign}(\alpha) \tag{3.23}$$

which is the final version of our cms spread option formula under bivariate normal dynamics.

4. Implementation in QuantLib

All three variants of the cms spread option model described above (lognormal, shifted lognormal and normal bivariate dynamics) are implemented in [2] in the class `LognormalCmsSpreadPricer`, which is a coupon pricer applicable to coupons of type `CmsSpreadCoupon` and `CapFlooredCmsSpreadCoupon`. Note that the latter class by default represents capped / floored cms spread coupons. The embedded option can easily be isolated using the class `StrippedCappedFlooredCoupon` though (which is more generally applicable to all cap / floored coupons).

By default the cms spread pricer inherits the volatility type, and in case of shifted lognormal volatilities the respective shifts applicable to the underlyings, from the volatility structure used in the `CmsCouponPricer` that produces the swap rate adjustments from which the drifts μ_i (see above) are implied.

However this can also be overridden with a custom volatility type and shifts. Note that this only applies to the bivariate dynamics, but not to the computation of the single rates' convexity adjustments, which are always relying on the beforehand mentioned volatility structure w.r.t. volatility type and shifts.

References

[1] Brigo, Mercurio: Interst Rate Models - Theory and Practice, 2nd Edition, Springer, 2006
[2] QuantLib A free/open-source library for quantitative finance, http://www.quantlib.org

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