Chapter 5: Dynamic Behaviour of First-Order and Second-Order Processes

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Learning Objectives

At the end of this chapter, you will be able to:

- Explain the different types of standard process inputs
- Explain the response of first-order processes
- Explain the response of integrating processes
- Explain the response of second-order processes

Chapter Overview

1. Standard process inputs:
   - Step input
   - Ramp input
   - Rectangular pulse
   - Sinusoidal input
   - Impulse input
2. Response of first-order processes:
   - Step response
   - Ramp response
   - Sinusoidal response

Chapter Overview

3. Response of integrating processes
4. Response of second-order processes
    - Example of non-interacting tanks
    - Three cases of second-order process
        - Overdamped
        - Critically damped
        - Underdamped

Response of Processes

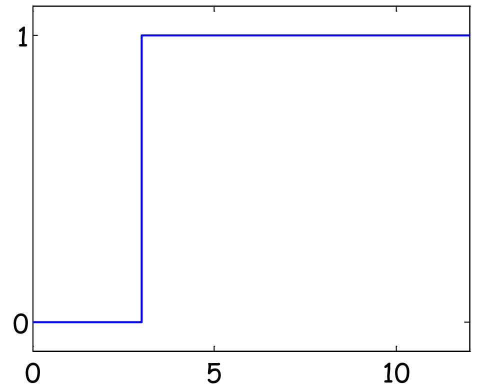
- It is important to know how the process responds to changes in the process inputs.
- A number of standard types of input changes are widely used for two reasons:
  - They are representative of the types of changes that occur in plants.
  - They are easy to analyze mathematically.
- Process input falls into two categories:
  - Input that can be manipulated to control the process and
  - Input that are not manipulated (disturbance).

1. Standard Process Input: Step Input

- A sustained and sudden change in a process variable can be approximated by a step change of magnitude, *M*.

$$u_s = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases} \qquad (4-4)$$

$$U_s(s) = \frac{M}{s} \qquad (4-6)$$



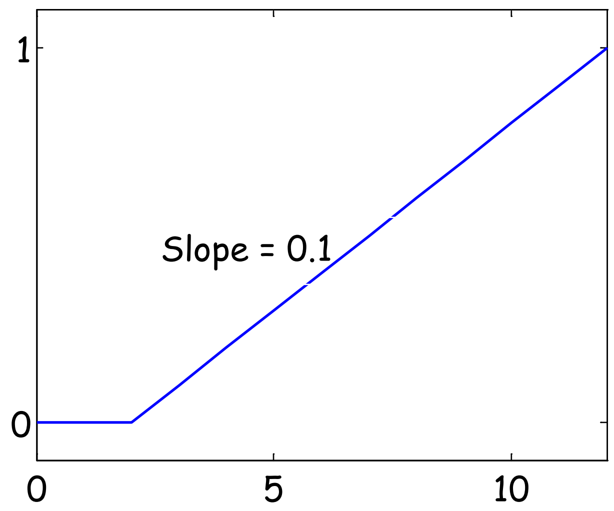
Step Input (Cont'd)

- *Special Case:* If $M = 1$, we have a "unit step change".
- *Example:*
  - A reactor feedstock is suddenly switched from one supply to another, causing sudden changes in feed concentration, flow, etc.
  - The heat input to the stirred-tank heating system is changed from 8000 to 10,000 *kcal/hr* by changing the electrical signal to the heater.

$$q(t) = 8000 + 2000u(t), \qquad u(t) = unit\ step$$
$$q'(t) = q - \bar{q} = 2000u(t), \quad \bar{q} = 8000\ kcal/hr$$

Standard Process Input: Ramp Input

- Industrial processes often experience "drifting disturbances", that is, relatively slow changes up or down for some period of time.
- The rate of change is approximately constant.



Ramp Input (Cont'd)

- Drifting disturbance is approximated by a ramp function:

$$u_R(t) = \begin{cases} 0 & t < 0 \\ at & t \geq 0 \end{cases} \qquad (4-7)$$

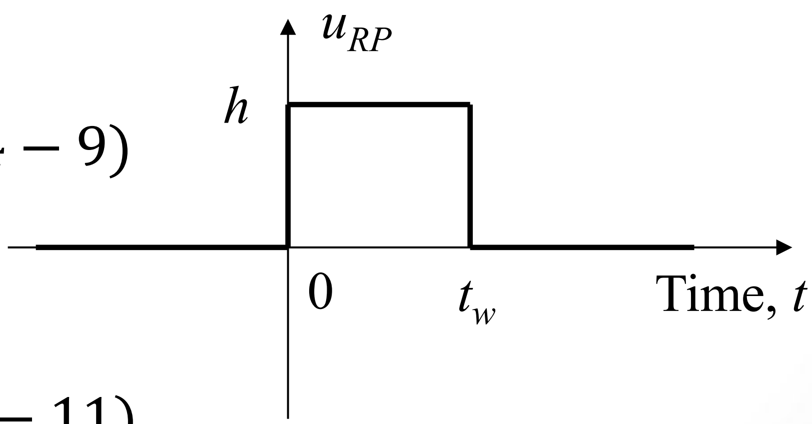
$$U_R(s) = \frac{a}{s^2} \qquad (4-8)$$

- Examples
  - Gradual change in feed composition, heat exchanger fouling, catalyst activity, and ambient temperature.

Standard Process Input: Rectangular Pulse

- Processes are subjected to a sudden step change and then returns to its original value.

$$u_{RP}(t) = \begin{cases} 0 & for \quad t < 0 \\ h & for \quad 0 \le t < t_w \\ 0 & for \quad t \ge t_w \end{cases} \quad (4-9)$$

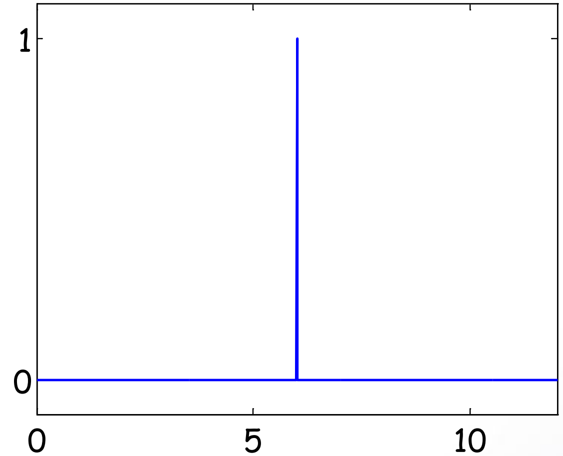


$$U_{RP}(s) = \frac{h}{s}[1 - e^{-t_w s}] \qquad (4-11)$$

- Examples:
  - Reactor feed is shut off for one hour.
  - The fuel gas supply to a furnace is briefly interrupted.

Standard Process Input: Impulse Input

- Here, $u_I(t) = \delta(t)$ and $U_I(s) = 1$.
- It represents a short and transient disturbance.
- It is the limit of a rectangular pulse for $t_w \to 0$ and $h = 1/t_w$.
- Examples:
  - Electrical noise spike in a thermo-couple reading and
  - Injection of a tracer dye.
- For this case, $Y(s) = G(s)$.



Standard Process Input: Sinusoidal Input

- Processes are also subject to periodic, or cyclic, or disturbances.
  - Approximated by a sinusoidal function:

$$u_{sin}(t) = \begin{cases} 0 & t < 0 \\ A \sin \omega t & t \geq 0 \end{cases} \qquad (4-14)$$

- A = amplitude, ω = angular frequency, P = period $= 2\pi/\omega$

$$U_{sin}(s) = \frac{A\omega}{s^2+\omega^2} \qquad (4-15)$$

- Examples:
  - 24 hour variations in cooling water temperature
  - 60-Hz electrical noise

2. Response of First-Order Processes

- Standard form in ODE representation

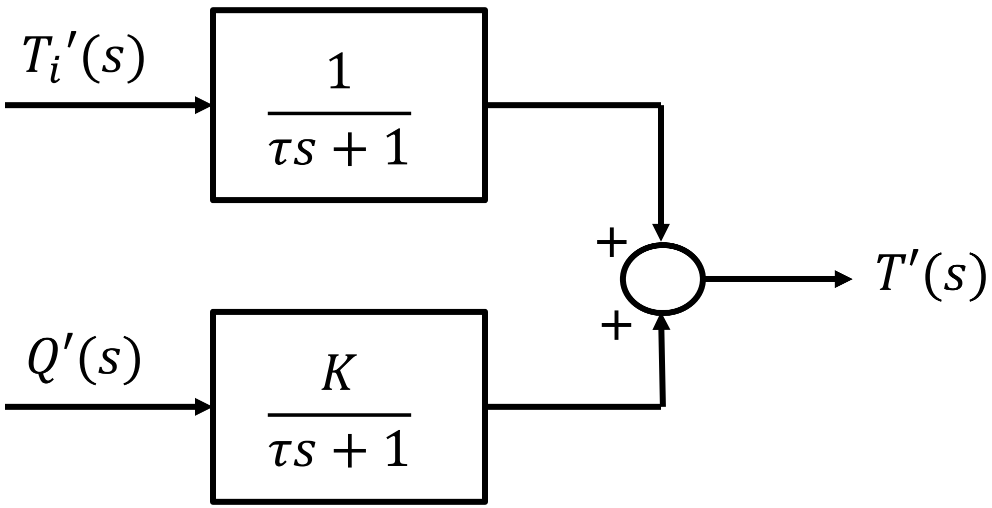
$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

$$\Big\Downarrow \ \text{LT}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

- K – Process Gain
  - Ultimate value of the response (new steady state) for a unit step change in the input.
- τ – Time Constant
  - Measure of the time necessary for the process to adjust to a change in its input.

Example of First-Order Processes



- Stirred tank

$$K = \frac{1}{wC} \text{ and } \tau = \frac{V\rho}{w}$$

Step Response of First-Order Processes

- For a step input of magnitude M, i.e.

$$U(s) = \frac{M}{s}$$

$$\Rightarrow \quad Y(s) = \frac{KM}{(\tau s + 1)s}$$

PFE $\Rightarrow$

$$Y(s) = KM\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right)$$

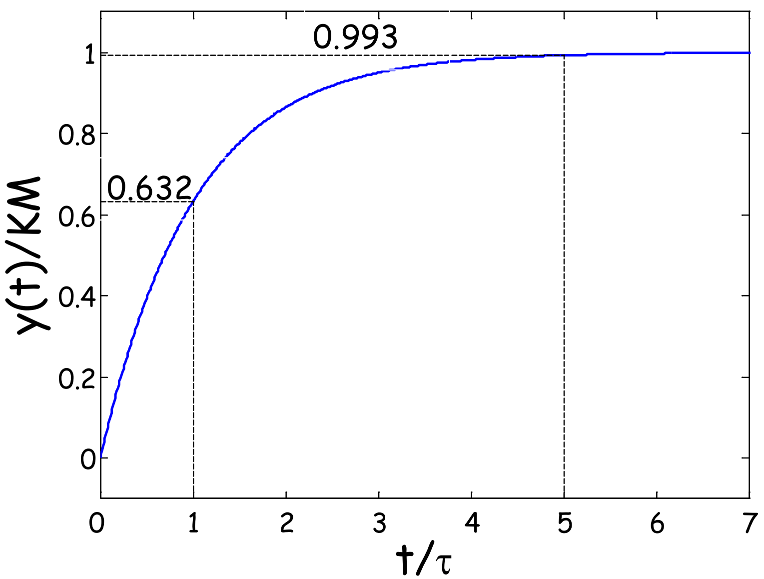
Inverse LT $\Rightarrow$

$$\boxed{y(t) = KM\left(1 - e^{-t/\tau}\right)}$$

Step response of a first-order process

Characteristics of Step Response

$$y(t) = KM\left(1 - e^{-t/\tau}\right)$$



t/τ	y/y_∞
1	0.632
2	0.865
3	0.950
4	0.982
5	0.993

Essentially settled

- First-order process does not respond instantaneously to a sudden change in its input
- $y_\infty = y(t = \infty) = KM$ (new steady state)
- Larger τ implies slower response
- Effect of K and τ on the response

Food for Thought

- Consider two first-order processes:

  $G_1(s) = K_1/(\tau_1 s + 1)$

  $G_2(s) = K_2/(\tau_2 s + 1)$

- The step response of $G_1(s)$ reaches 50% of its ultimate value in 10 min. When will step response of $G_2(s)$ reach 50% of its ultimate value?, if:
  - Case 1: $K_2 = 2K_1$ and $\tau_1 = \tau_2$
  - Case 2: $K_2 = K_1$ and $\tau_1 = 2\tau_2$

Recap of First-Order Processes

In response to a step change in input:

- The output reaches a new steady state, KM.
- The output takes $\sim 5\tau$ to reach new steady-state.
- The larger the τ, the slower is the response.
- The output reaches 63.2% of its final value at time equal to time constant.

Ramp Response of First-Order Processes

- Time domain $u(t) = at, t \geq 0$
- Laplace $U(s) = a/s^2$ $\qquad G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s+1}$

$$Y(s) = \frac{K}{\tau s+1} U(s) = \frac{Ka}{s^2(\tau s+1)}$$

- Partial fraction

$$Y(s) = \frac{Ka}{s^2(\tau s+1)} = \frac{Ka\tau^2}{\tau s+1} - \frac{Ka\tau}{s} + \frac{Ka}{s^2} \qquad (4-20)$$

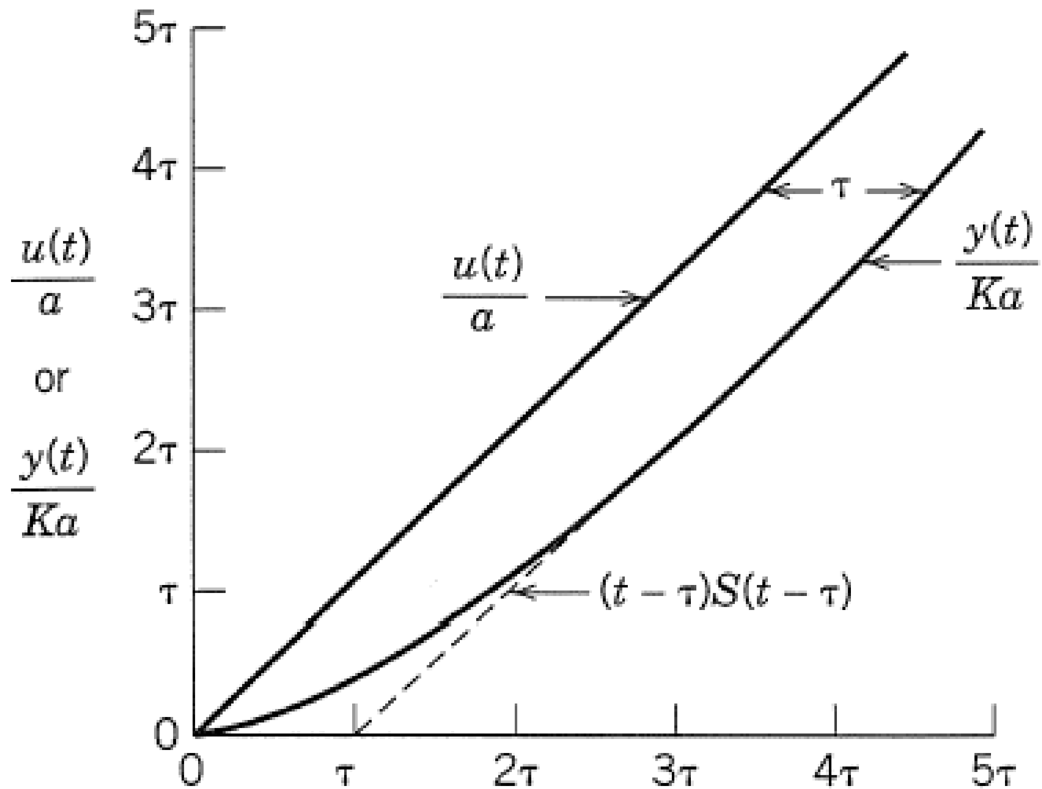
- Response

$$y(t) = Ka\tau\left(e^{-t/\tau} - 1\right) + Kat \qquad (4-21)$$

- Large value of time $(t \gg \tau)$

$$y(t) = Ka(t - \tau) \qquad (4-22)$$

Ramp Response of First-Order Processes (Cont'd)



Ramp response of a first-order process
(Comparison of input and output)

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.111). Hoboken, NJ: Wiley

Sinusoidal Response of First-Order Processes

- For a sine input $U(s) = \frac{\omega}{s^2+\omega^2}$

- Output is

$$Y(s) = \frac{K_p}{\tau s + 1} \cdot \frac{\omega}{s^2 + \omega^2} = \frac{\alpha_0}{\tau s + 1} + \frac{\alpha_1 s}{s^2 + \omega^2} + \frac{\alpha_2}{s^2 + \omega^2}$$

- Partial fraction

$$\alpha_0 = \frac{\omega K_p \tau^2}{\omega^2 \tau^2 + 1}$$

$$\alpha_1 = \frac{-\omega K_p \tau}{\omega^2 \tau^2 + 1}$$

$$\alpha_2 = \frac{\omega K_p}{\omega^2 \tau^2 + 1}$$

Sinusoidal Response of First-Order Processes (Cont'd)

- Inverting:

This term dies out for large 't'

$$y(t) = \frac{K_p \omega \tau}{\omega^2 \tau^2 + 1} e^{-t/\tau} + \frac{K_p}{\sqrt{\omega^2 \tau^2 + 1}} \sin(\omega t + \phi)$$

$$\phi = -arc\tan(\omega \tau)$$

- Note that "y" is not a function of t but of t and ω.
- For large t, $y(t)$ is also sinusoidal, output sine is attenuated by:

$$\frac{Kp}{\sqrt{\omega^2 \tau^2 + 1}}$$ (fast vs. slow ω)

3. Response of Integrating Processes

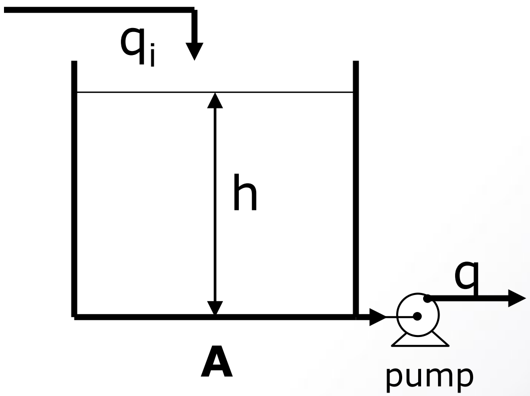
- Recall the tank with pump attached to the outflow line.

$$\frac{d(\rho V)}{dt} = pq_i - pq \qquad (2-53)$$

- Assume constant density and constant A,

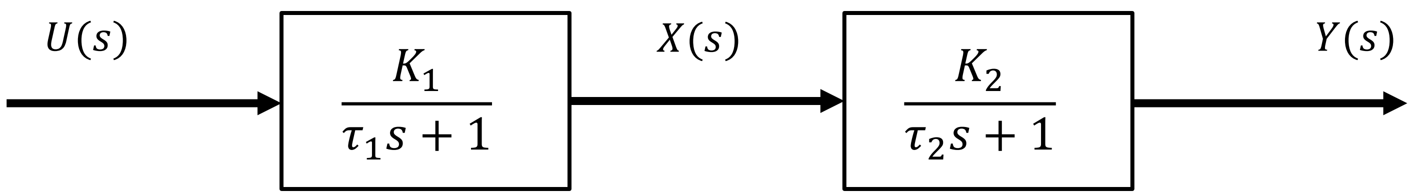
$$AsH'(s) = Q_i'(s), \qquad \frac{H'(s)}{Q_i'(s)} = \frac{1}{As}$$

- Integrate models characterized by the term $1/s$.
- Do not have a steady-state gain.



4. Second-Order Processes

- Second-order TF can arise whenever two first-order processes are connected in series.
  - Example: Two stirred-tank blending processes:



$$G(s) = \frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)}\frac{Y(s)}{X(s)} = \frac{\overbrace{K_1 K_2}^{K}}{(\tau_1 s+1)(\tau_2 s+1)} \qquad (4-39)$$

Second-Order Processes (Cont'd)

- Alternatively, a second-order process transfer function will arise upon transforming a second-order ODE:

$$\tau^2 \frac{d^2 y(t)}{dt^2} + 2\zeta\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

$$\Downarrow \text{ LT}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad (4-40)$$

- Three adjustable parameters:
    - Static gain (K) and time constant (τ): are same as first-order processes.
    - Damping factor (ζ, zeta): determines whether the system has <u>oscillatory behavior</u> or not.

Second-Order Processes (Cont'd)

- Eqn $(4-39)$ and $(4-40)$ differ only in the form of denominator. Equating the denominators:

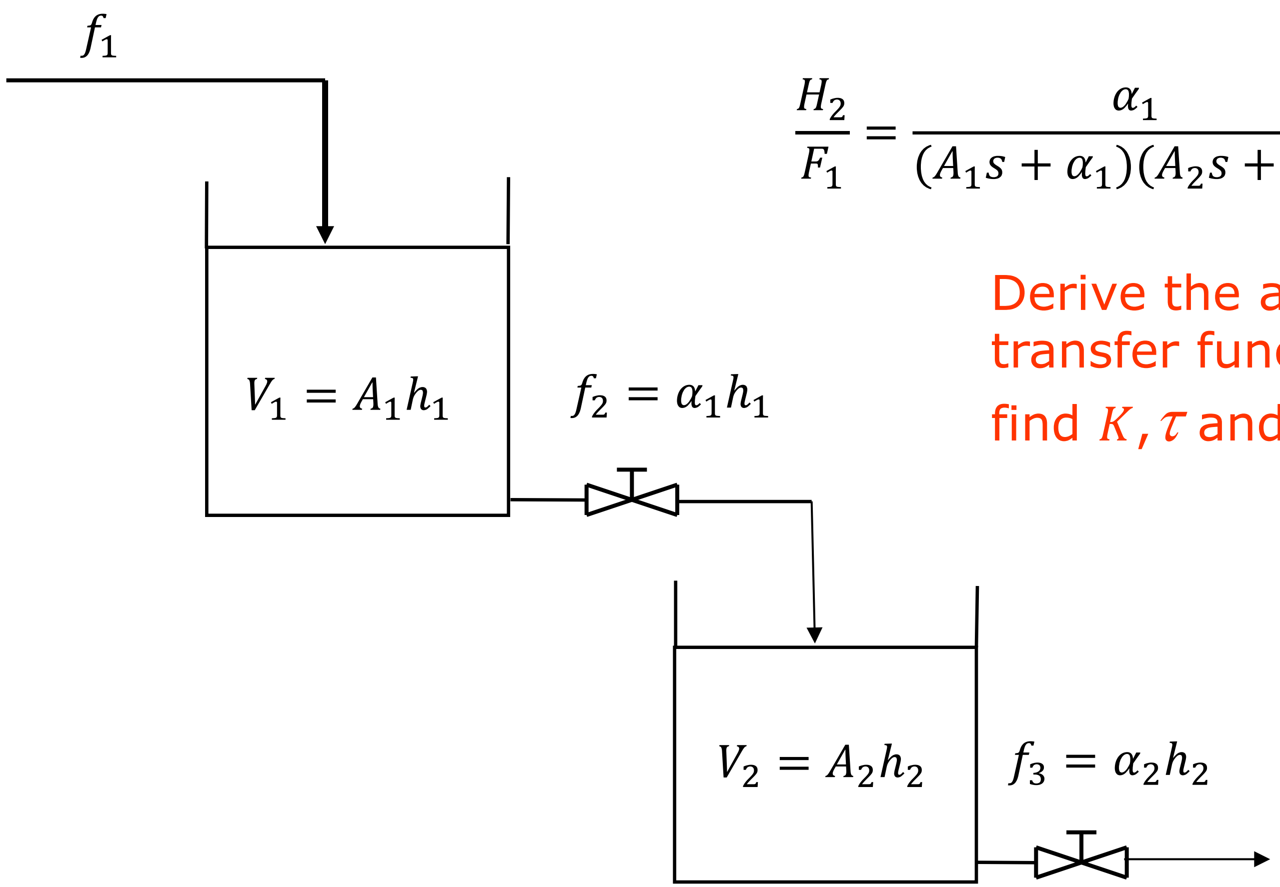
$$\tau^2 s^2 + 2\xi\tau s + 1 = (\tau_1 s + 1)(\tau_2 s + 1) \qquad (4-41)$$

- Equating:

$$\tau = \sqrt{\tau_1 \tau_2} \qquad (4-42)$$

$$\xi = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}} \qquad (4-43)$$

Typical Example: Non-Interacting Tanks



$$\frac{H_2}{F_1} = \frac{\alpha_1}{(A_1 s + \alpha_1)(A_2 s + \alpha_2)}$$

Derive the above transfer function and find K, τ and ζ?

Three Important Cases of Second-Order Processes

$$\tau^2 s^2 + 2\zeta\tau s + 1 = 0$$

$$s_1 = -\frac{\zeta}{\tau} + \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad \text{and} \quad s_2 = -\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2 - 1}}{\tau}$$

	Roots	Response
$\zeta > 1$	Distinct real roots	Overdamped or non-oscillatory
$\zeta = 1$	Real and equal roots	Critically damped
$\zeta < 1$	Complex roots	Underdamped or oscillatory

Step Response of Second-Order Processes

- Consider system response to a step change of magnitude M.

$$U(s) = \frac{M}{s}$$

$$\Rightarrow \quad Y(s) = \frac{KM}{(\tau^2 s^2 + 2\zeta\tau s + 1)s}$$

- After inverting to the time domain, the response can be categorized into three classes (depending on the value of zeta).

Step Response of Second-Order Processes (Cont'd)

- Overdamped (zeta > 1)

$$(\tau^2 s^2 + 2\zeta\tau s + 1) = (\tau_1 s + 1)(\tau_2 s + 1)$$

where $\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}}$ $\quad \tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}}$

$$y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \qquad (4-48)$$

$$y(t) = KM\left[1 - e^{-\zeta t/\tau}\left(\cos h\,(\frac{\sqrt{\zeta^2-1}}{\tau}t) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sin h\,(\frac{\sqrt{\zeta^2-1}}{\tau}t)\right)\right] \qquad (4-49)$$

Step Response of Second-Order Processes (Cont'd)

- Critically damped (Zeta = 1)

$$y(t) = KM\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right] \qquad (4-50)$$

Ordinary Differential Equations

- Underdamped (0 ≤ Zeta < 1)

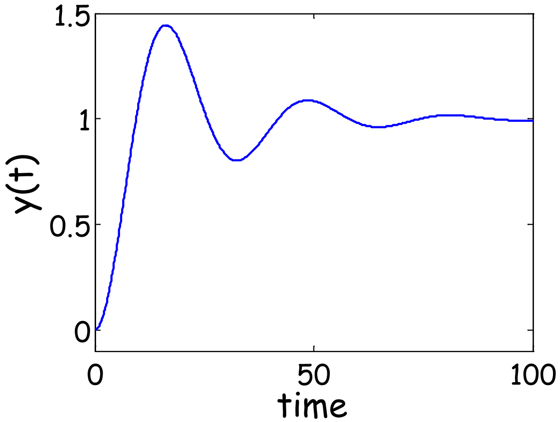
$$y(t) = KM\left[1 - e^{-\zeta t/\tau}\left(\cos\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)\right)\right]$$

$$(4-51)$$

Three Distinct Cases

$$K = 1, M = 1, \tau = 1$$

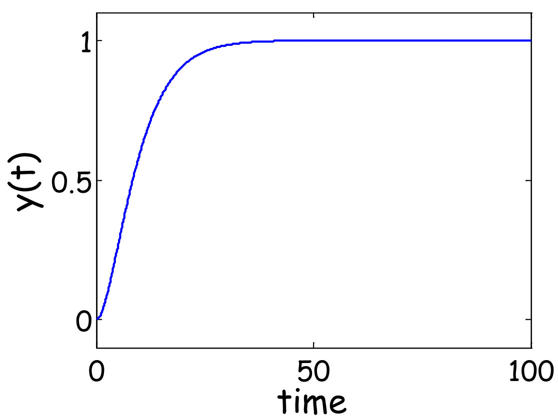
$\zeta = 0.25$



Underdamped

Oscillatory

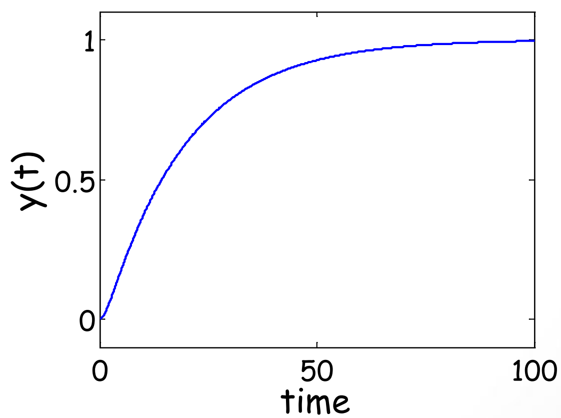
$\zeta = 1$



Critically damped

Smooth

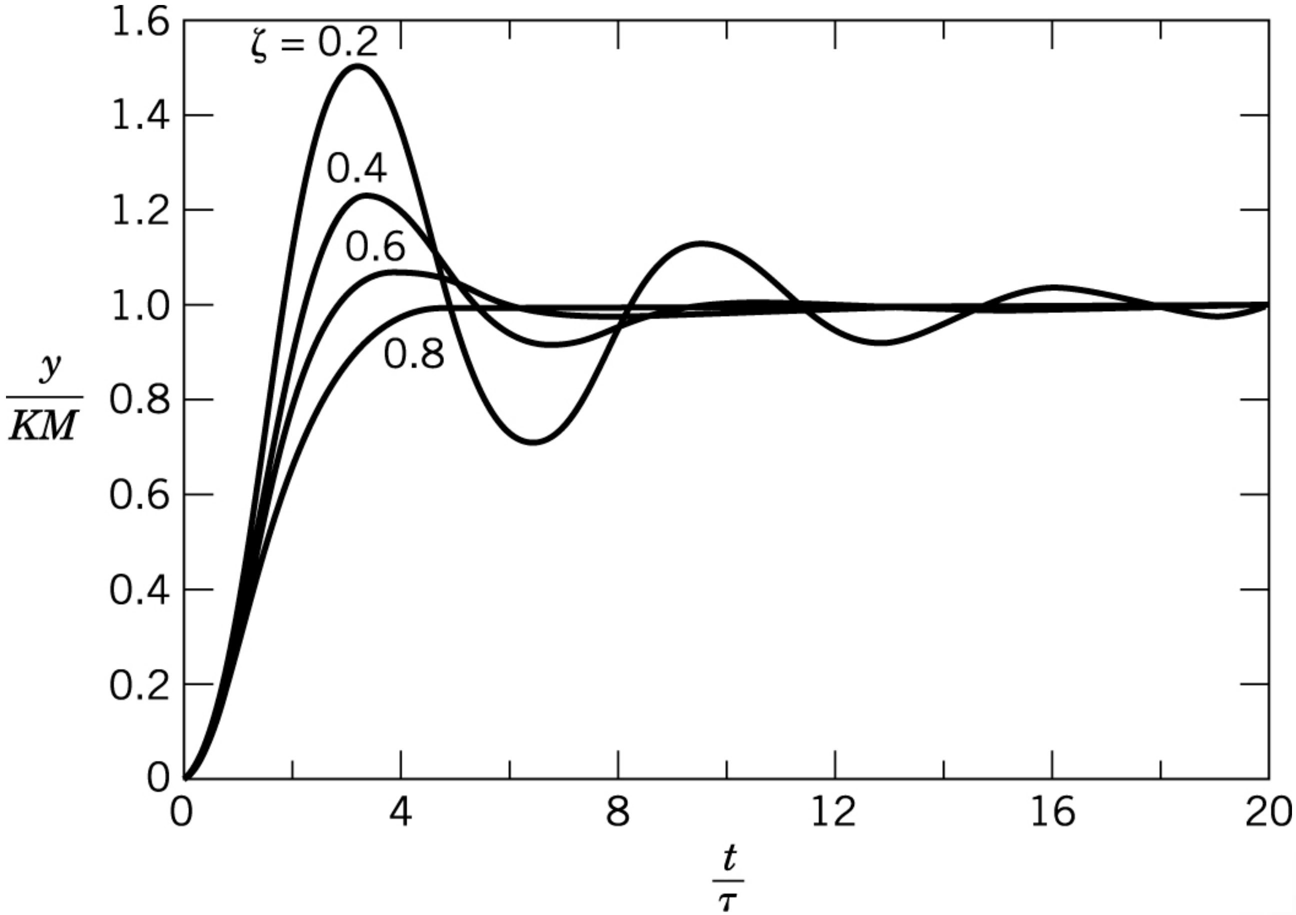
$\zeta = 2$



Overdamped

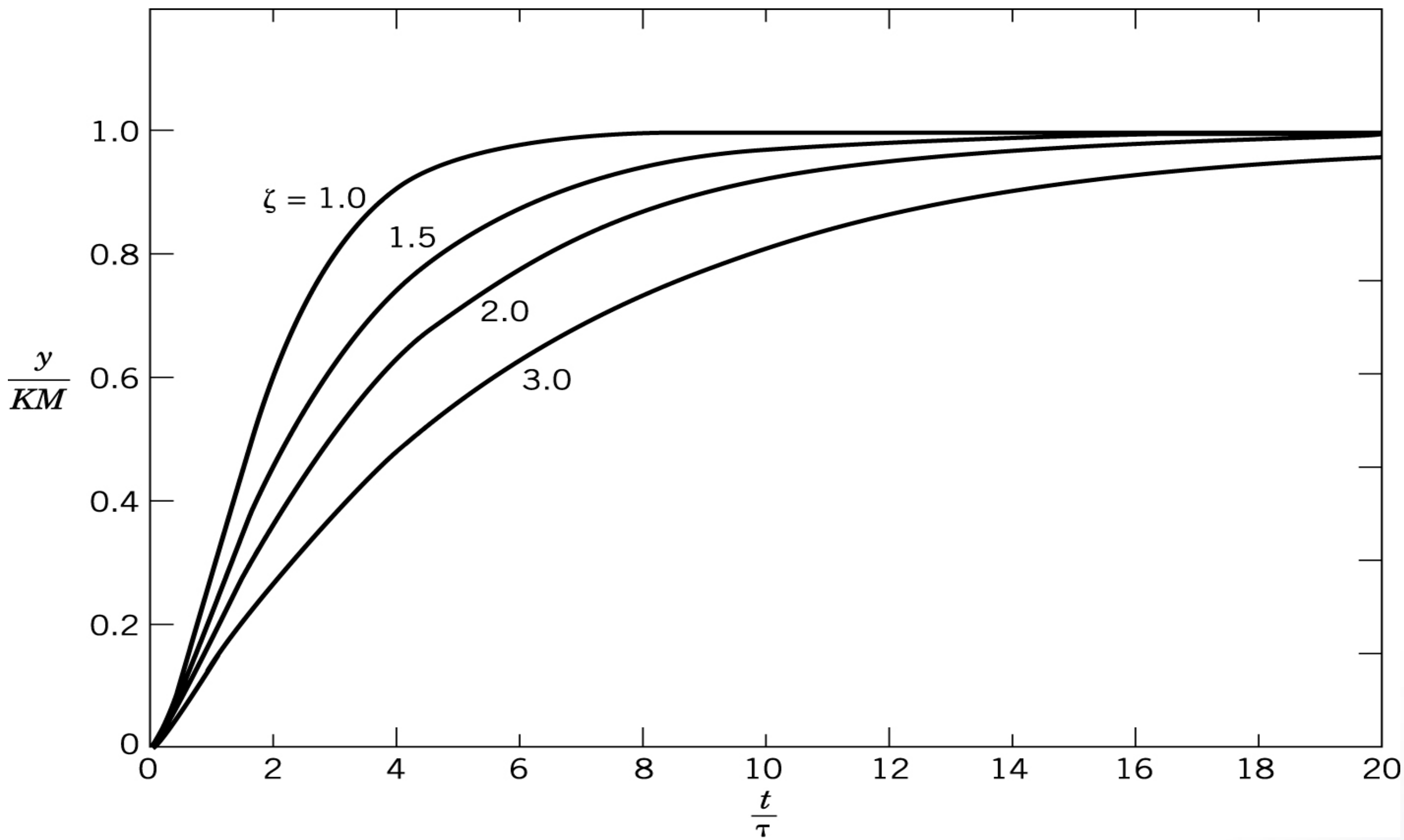
Sluggish

Step Response of Underdamped Second-Order Processes



Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.117). Hoboken, NJ: Wiley

Step Response of Critically Damped & Overdamped Second-Order Processes



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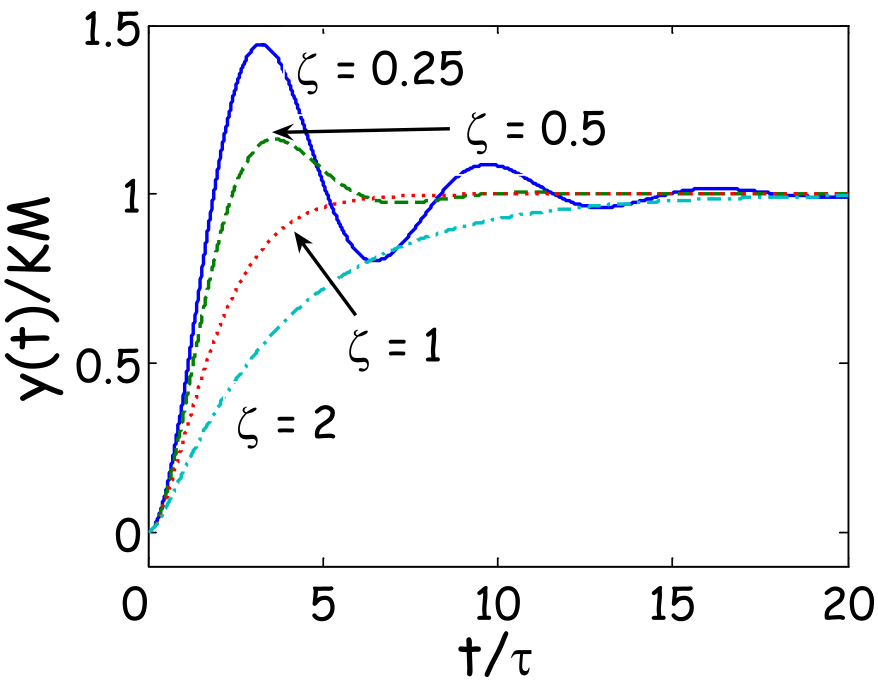
Second-Order Processes: Effect of ξ



- $y_\infty = y(t = \infty) = KM$ (new steady state).
- Oscillations and response exceeds final value (overshoot), when $\zeta < 1$ (underdamped).
- Response becomes sluggish as ζ and t increases.
- Fastest response without oscillations and overshoot, when $\zeta = 1$ (critically damped).

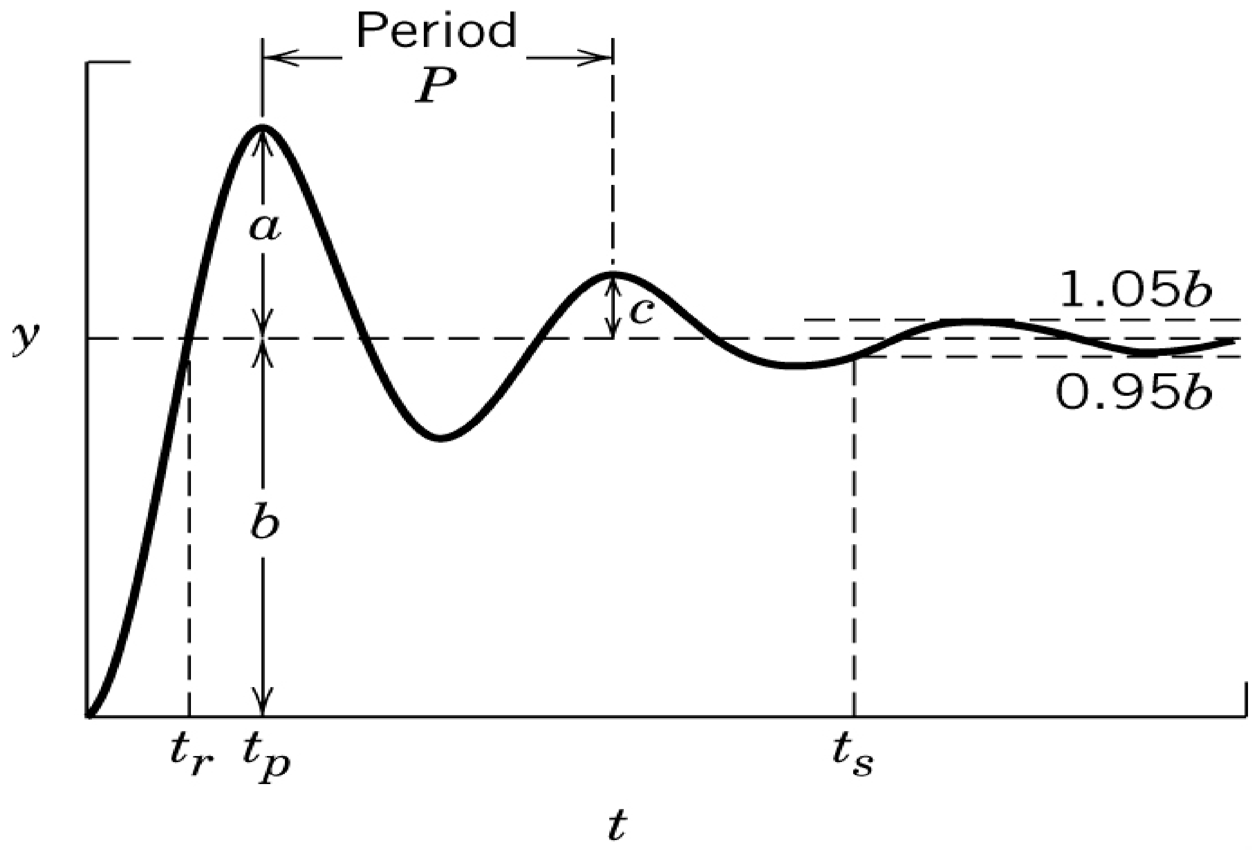
Take Home Message

- For zeta > 1
  - Response is non-oscillatory or overdamped.
  - Response is delayed initially (compared to 1st order) and becomes more sluggish as zeta increases.
  - Approaches its final value (KM) asymptotically as t approaches infinity.
- For zeta = 1
  - Response non-oscillatory; rapid approach to final value without oscillations.
- For zeta < 1
  - Response is oscillatory and oscillations increases as zeta decreases.
  - Response is initially faster.

Why Study Underdamped Systems?

- Very few processes encountered in chemical industries exhibit natural (when control is not applied) underdamped behavior.
- Feedback controller may cause the closed-loop system to exhibit underdamped dynamics.
- Control engineers often attempt to make the response of the controlled variable to a set point change **approximate the ideal step response of an underdamped second-order system**, that is, make it exhibit a prescribed amount of overshoot and oscillation.

Underdamped Response Characteristics



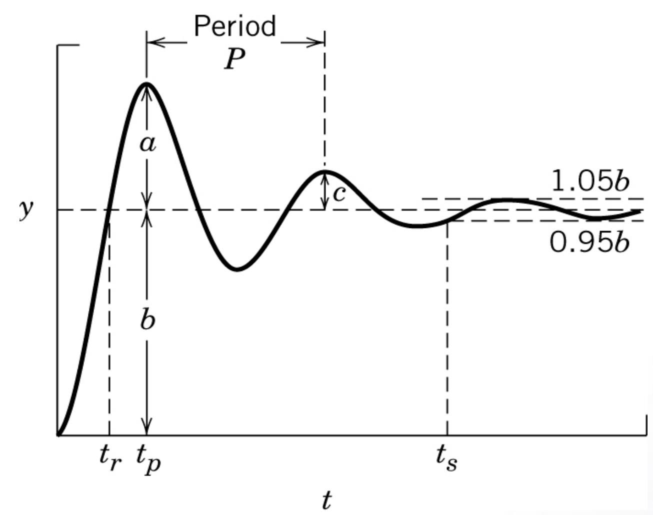
Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.118). Hoboken, NJ: Wiley

Underdamped Response Characteristics

- Rise Time (t_r)
  - Time when output "first" reaches the new steady-state value.
- Time to First Peak (t_p)
  - Time when output reaches its first maximum value.

$$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}} \qquad (4-52)$$

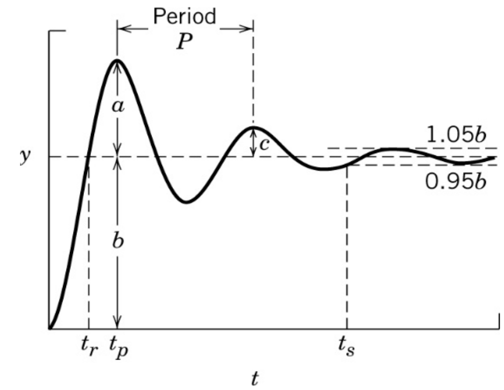
- Settling Time (t_s)
  - Time required for output to reach and remain inside a band with width equal to $\pm 5\%$ of the total change in y.



Underdamped Response Characteristics

- **Overshoot**
  - a/b (% overshoot is $100a/b$)

$$\frac{a}{b} = exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \qquad (4-53)$$



- **Decay Ratio**
  - c/a, where c is height of second peak

$$\frac{c}{a} = exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = \frac{a^2}{b^2} \qquad (4-54)$$

- **Period of Oscillation (P)**
  - Time between two successive peaks or valleys of the response

$$p = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \qquad (4-55)$$

Summary of Second-Order Processes for Step Input

In response to a step change in input:

- The output reaches a new steady state, KM.
- Increasing ζ and τ makes the response more sluggish.
- Oscillations and overshoot are observed, if $\zeta < 1$ (underdamped).
- Fastest response with no oscillations and overshoot, if $\zeta = 1$ (critically damped).

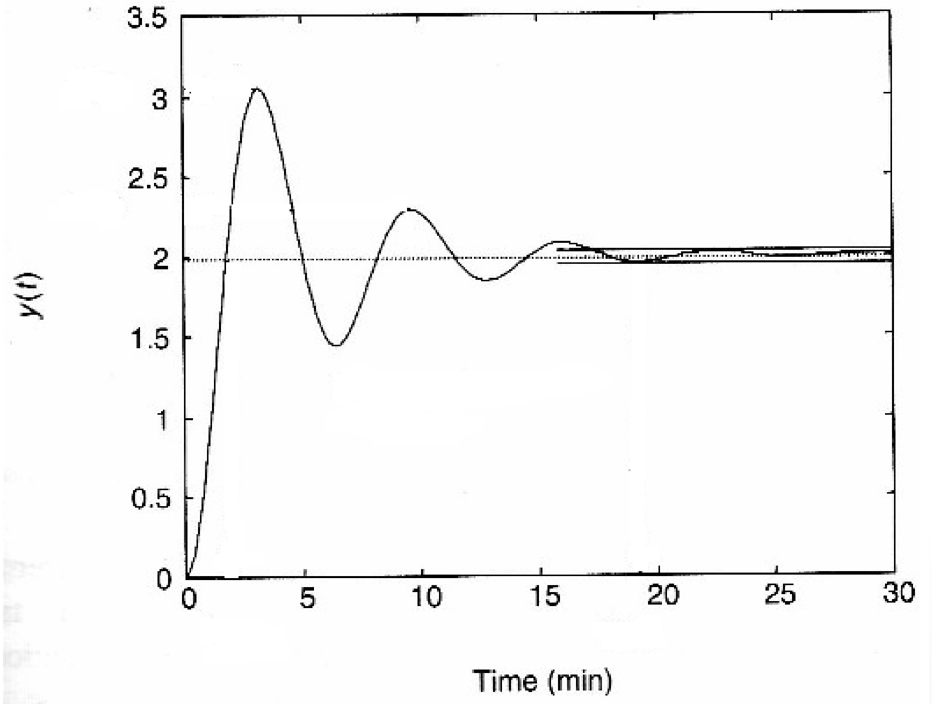
Chapter Summary

- Transfer functions can be used to obtain responses to any type of input change
- Important types of input functions
- Responses of first-order, second-order to these types of input functions
- Integrating processes
- Suggested Reading: Chapter 4 of Seborg

Exercise

- Find overshoot, decay ratio, rise time, settling time and period of oscillation.

$$G(s) = \frac{Y(s)}{U(s)}$$
$$= \frac{2}{s^2+0.4s+1}$$



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Chapter 5: Dynamic Behavior of First-Order and Second-Order Processes

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