# Chapter 6: Dynamic Response Characteristics of More Complicated Processes

Dr. Mukta Bansal

School of Chemistry, Chemical Engineering & Biotechnology (CCEB)

Office: N1.2 – B2 – 28

Email: mbansal@ntu.edu.sg

# Chapter Overview

This chapter consists of the following topics:

1. Definition of Stability and System Stability
2. Poles and Zeros and Their Effects
   - Central Idea
   - Real Pole
   - Special Case: Pole at Zero
   - Complex Poles
   - Special Case: Pure Imaginary Poles
   - Repeated Real Poles
3. Numerator Dynamics
   - Gain/ Time Constant Form
   - Effect of Zeros
   - Process with Inverse Response

# Chapter Overview

This chapter consists of the following topics:

4. Processes with Time Delays
   - Effect of Time Delays
   - Approximation of Dead Time
   - Taylor Series Approximation
   - Pade Approximation
5. Approximation of Higher Order
   - Approximation of Higher Order Transfer Functions
   - Skogestad's "half rule" Approximation
6. Interacting and Non-interacting Processes

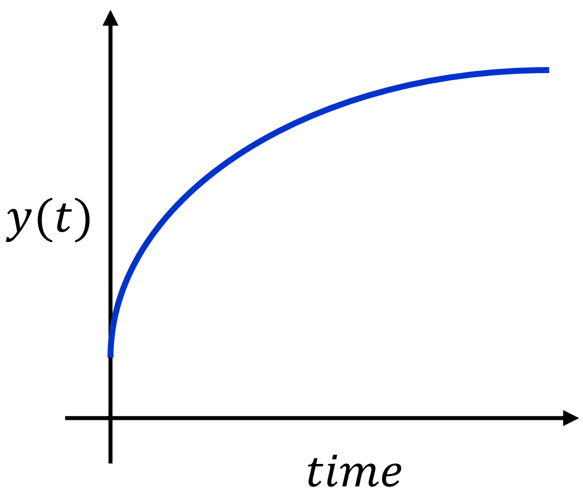
# Learning Objectives

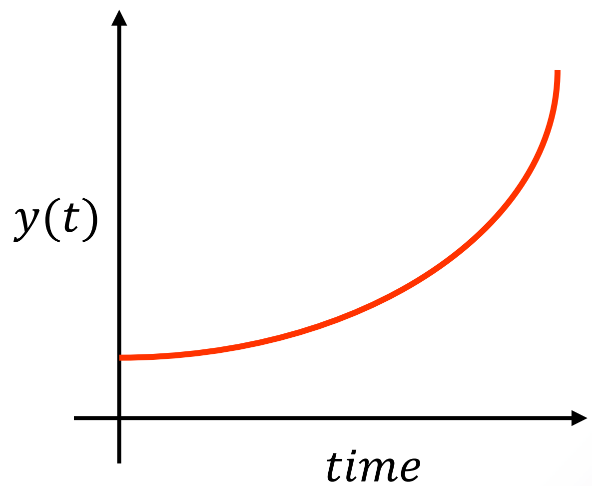
At the end of this chapter, you will be able to:

- Explain the notion of stability of linear systems
- Explain the relationship between location of the poles of the transfer functions (TF) and stability
- Plot the poles and zeros of a TF in a complex plane
- Apply more complex TFs in process response
- Apply TFs in process with time delays
- Approximate complicated TFs by simpler-low order models

# What is Stability?

- An intuitive definition of stability is that the process output must *remain bounded* in response to *all bounded inputs.*



# System Stability

- If the system is stable, the response must be bounded for *all* bounded inputs.
- Checking response for different bounded inputs can be cumbersome (and impossible).
- Non-trivial fact: The response of a linear system is bounded for all bounded inputs,
  - If the response is bounded for unit impulse change in input.
  - The response approaches zero asymptotically.

# Example 1

Consider the following process

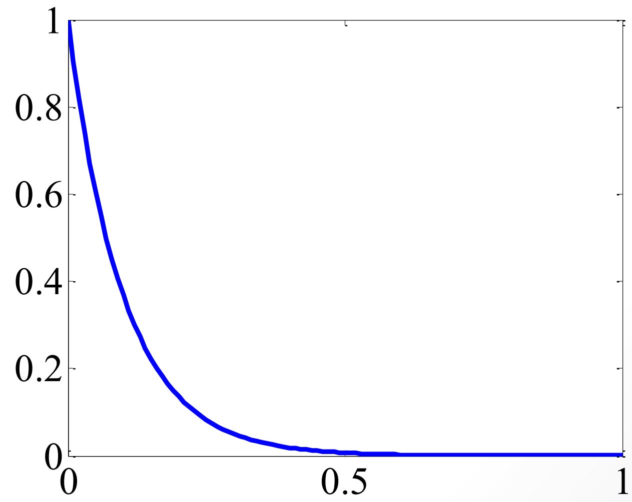
$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+10}$$

For $U(s) = 1$ (unit impulse change)

$$Y(s) = \frac{1}{s+10}$$

$$\Rightarrow \quad y(t) = L^{-1}[Y(s)] = e^{-10t}$$

$G(s)$ is stable!



# Remark

- Being able to access stability of linear systems by simulating the response for unit impulse changes in input makes stability analysis tractable.
- Can we do better?
- It turns out that the stability of linear systems depends only on the location of the poles.

# Poles and Zeros and Their Effects

$$G(s)=\frac{b_0 s^m+\ldots+b_m}{a_0 s^n+\ldots.+a_n}=\frac{Z(s)}{P(s)}$$

- Poles
  - The roots of the polynomial $P(s)$
  - $P(s)$ is also called the characteristic polynomial
- Zeros
  - The roots of the polynomial $Z(s)$
  - Numerator dynamics

- The dynamic behavior of a transfer function model can be characterized by the numerical value of its poles and zeros.

# Central Idea

- For unit impulse, $U(s) = 1$ and $Y(s) = G(s)$.
- Consider the partial fraction expansion of $Y(s) = G(s)$:

real pole

complex poles

$$Y(s) = \frac{a}{s - \alpha} + \frac{bs + c}{(s - \beta - jy)(s - \beta + jy)}$$

$$+ \frac{d_1}{(s-\eta)} + \frac{d_2}{(s-\eta)^2} + \ldots + \frac{d_r}{(s-\eta)^r}$$

repeated real poles

# Central Idea (Cont'd)

- System is stable if $y(t)$ remains bounded:

$$y\,(t) = L^{-1}[Y(s)]$$

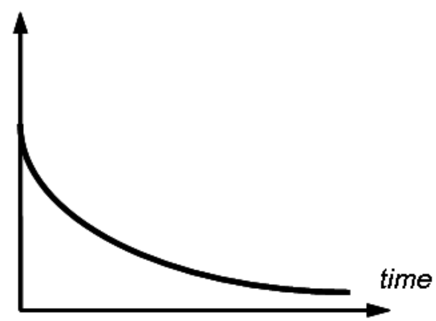
$$= L^{-1}\left[\frac{a}{s-\alpha}\right] + L^{-1}\left[\frac{bs+c}{(s-\beta-jy)(s-\beta+jy)}\right]$$

$$+ L^{-1}\left[\frac{d_1}{(s-\eta)} + \frac{d_2}{(s-\eta)^2} + \dots + \frac{d_r}{(s-\eta)^r}\right]$$

- $y(t)$ is bounded, if each term is finite.
- Analyze the contribution by each term.

# Real Pole

- Consider $G_1(s) = \dfrac{a}{s-\alpha} \quad \alpha \neq 0$
- Inverse Laplace Transform: $L^{-1}[G_1(s)] = g_1(t) = ae^{\alpha t}$
- As time increases, $g(t)$ becomes:

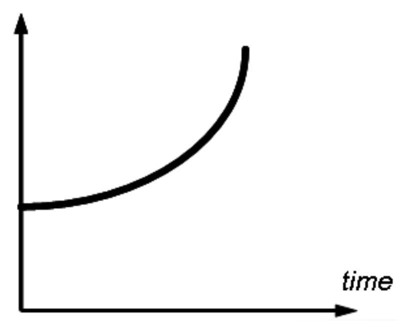



$\alpha < 0$

bounded

$\alpha > 0$

unbounded



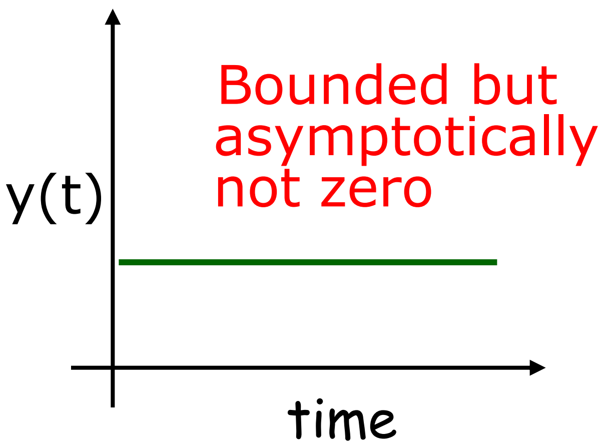

- **The system is unstable if it has positive poles.**

# Special Case: Pole at Zero

- When $\alpha = 0$, $Y(s) = G(s)$ has a pole at zero.
- In this case, partial fraction expansion gives:

$$Y(s) = \frac{a}{s} + \text{Other terms}$$

$$y(t) = L^{-1}[Y(s)]$$
$$= a + \text{Other terms}$$




- Systems with pole at zero lie on the boundary of stability and are called marginally or critically stable systems.

# Complex Poles

- Consider $G_2(s) = \dfrac{bs + c}{(s - \beta - j\gamma)(s - \beta + j\gamma)}$

- Inverse Laplace Transform

$$L^{-1}[G_2(s)] = g_2(t) = e^{\beta t}\left(b\cos(\gamma t) + \frac{c + \beta b}{\gamma}\sin(\gamma t)\right)$$

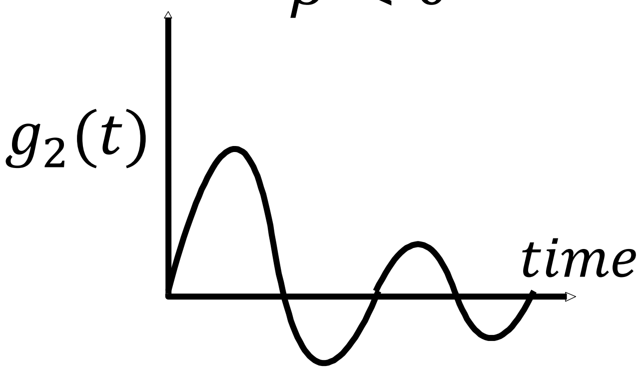
- The sine and cosine terms cause oscillatory behavior but their contribution is always finite.

- Stability depends on the exponential term.

# Complex Poles (Cont'd)

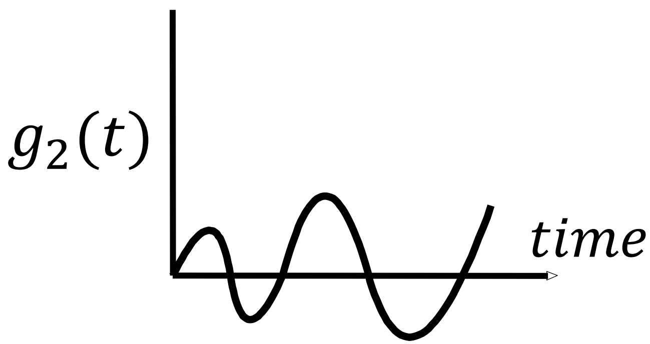
- Consider $g_2(t) = e^{\beta t}\left(b\cos(\gamma t) + \dfrac{c+\beta b}{\gamma}\sin(\gamma t)\right)$

$\beta < 0$




Decaying oscillations

$\beta > 0$



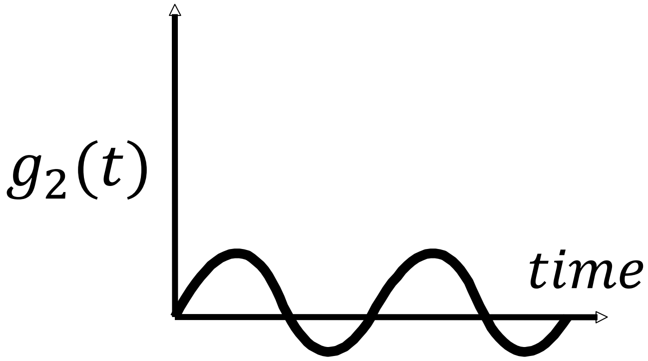

Growing oscillations

- System is unstable, if it has complex poles with positive real parts.

# Special Case: Pure Imaginary Poles

- When $\beta = 0$, the system has pure imaginary poles.
- In this case, $g_2(t)$ becomes:

$$g_2(t) = b\cos(\gamma t) + \frac{c}{\gamma}\sin(\gamma t)$$




- $g_2(t)$ does not grow with time, but shows sustained oscillations.
- Systems with pure imaginary poles lie on the boundary of stability and are called marginally or critically stable systems.

# Repeated Real Poles

- Consider $G_3(s) = \dfrac{d_1}{(s-\eta)} + \dfrac{d_2}{(s-\eta)^2} + \ldots + \dfrac{d_r}{(s-\eta)^r}$

- Inverse Laplace Transform:

$$h(t)$$

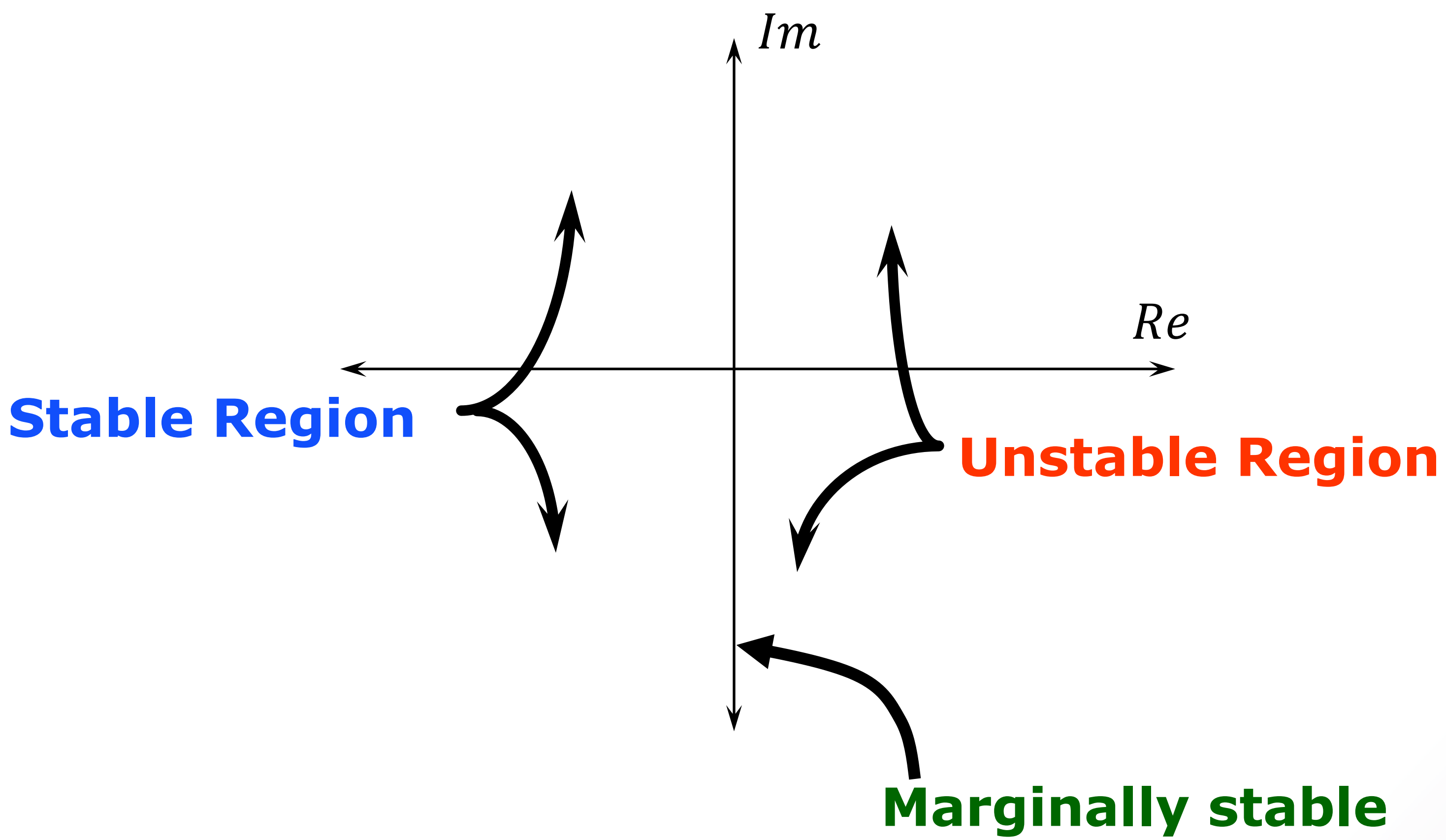
$$L^{-1}[G_3(s)] = g_3(t) = e^{\eta t}\left( d_1 + d_2 t + \frac{d_3 t^2}{2!} + \ldots + \frac{d_r t^{r-1}}{(r-1)!} \right)$$

- $\eta \geq 0$: $g_3(t)$ becomes unbounded, as one or both of $e^{\eta t}$ and h(t) grow with time.
- $\eta < 0$: $g_3(t)$ remains bounded, as $e^{\eta t}$ decays faster than $h(t)$ grows (verify using L'Hospital's rule).
- **Same conclusions as obtained for non-repeated poles.**

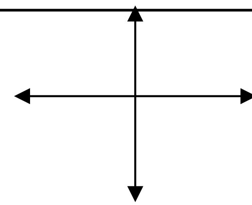
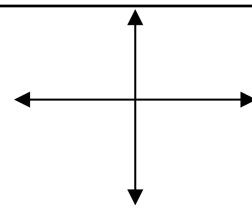
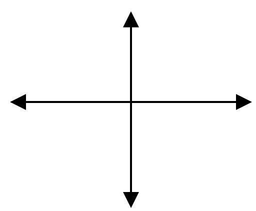
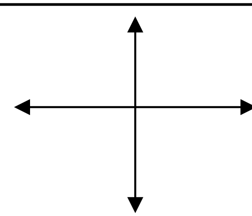
# Summary

- A linear dynamic system is stable if all its poles have negative real parts.
- Even if one pole has positive real part, system is unstable.
- These conclusions still hold, when the system has repeated complex poles.
- Locations of the zeros have no effect on the stability of the process.

# Stability of Linear Systems



# Exercise

| System | Poles | Stable/Unstable? |
|---|---|---|
| $\dfrac{1}{(s-1)}$ |  | |
| $\dfrac{1}{(s^2+s+1)}$ |  | |
| $\dfrac{(s-10)}{(s^2+s+1)}$ |  | |
| $\dfrac{(s-10)}{s(s+1)}$ |  | |

# Numerator Dynamics

- Additional complexity arises if the transfer function contains functions of *s* in the numerator
- Example: Lead-lag element
- Differential equation

$$\tau_1 \frac{dy}{dt} + y = K\left(\tau_a \frac{du}{dt} + \mathrm{u}\right) \qquad (5-3)$$

- Transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K(\tau_a s + 1)}{\tau_1 s + 1}$$

Numerator dynamics

Zero at $s = -1/\tau_a$

# Numerator Dynamics

- Standard transfer function form:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{m} b_i\, s^i}{\sum_{i=0}^{n} a_i\, s^i} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \qquad (3-41)$$

- Quantitative dynamics characteristics can be determined.
  - Analyzing the form of numerator & denominator:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m(s - z_1)(s - z_2)\ldots(s - z_m)}{a_n(s - p_1)(s - p_2)\ldots(s - p_n)} \qquad (5-7)$$

- Poles? Zeros? Characteristic equation?

# Gain/ Time Constant Form

- If all the roots are real then the equation $(3-41)$ can be written:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0(\tau_a s + 1)(\tau_b s + 1) \dots (\tau_m s + 1)}{a_0(\tau_1 s + 1)(\tau_2 s + 1) \dots (\tau_n s + 1)} \qquad (5-8)$$

$$z_1 = -1/\tau_a, z_2 = -1/\tau_b, \dots \qquad (5-9)$$

$$p_1 = -1/\tau_1, p_2 = -1/\tau_2, \dots \qquad (5-10)$$

$$K = \frac{b_0}{a_0} = G(s=0)$$

- $K$: steady state gain.

# Effect of Zeros

- The presence or absence of zeros has no effect on the number and location of the poles and their associated response modes.
  - Unless there is an exact cancellation of a pole by a zero with the same numerical value.

- The zeros exert a profound effect on the coefficients of the response modes (coefficients are found by partial fraction expansion).

# Example 2

- For the case of a single zero in an overdamped second order transfer function, calculate the response to a step input of magnitude M and plot the results for $\tau_1 = 4$ and $\tau_2 = 1$ and several values of $\tau_a$.

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

- For a step input, the response is:

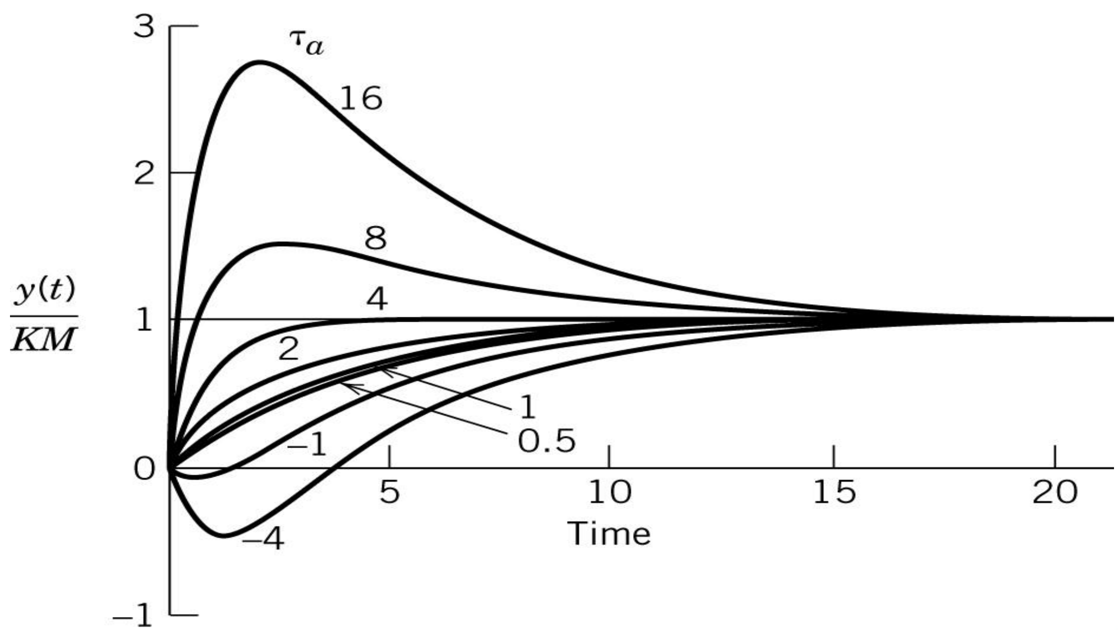
$$y(t) = KM\left(1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}\right)$$

# Important Points to Note

- Zeros do not change the ultimate value, $y_\infty = KM$.
- Zeros do not change the number or location of poles.
- Zeros do not change the number of response modes but they do effect the coefficient of response modes.
- Three distinct cases arise:
  - Case 1: $\tau_a > \tau_1 > \tau_2$
  - Case 2: $0 < \tau_a \leq \tau_1$
  - Case 3: $\tau_a < 0$

# Solution to Example 2

- Effect of $\tau_a$ $(\tau_1 = 4, \tau_2 = 1)$
  - Case 1: $\tau_a > \tau_1 > \tau_2$ (Overshoot)
  - Case 2: $0 < \tau_a \leq \tau_1$ (First-order process response)
  - Case 3: $\tau_a < 0$ (Inverse response)




Source: Seborg, D. E., Edgar, T. F., & Mellichamp, D. A., (2011). *Process dynamics and control* (2nd ed.)(pp.134). Hoboken, NJ: Wiley.

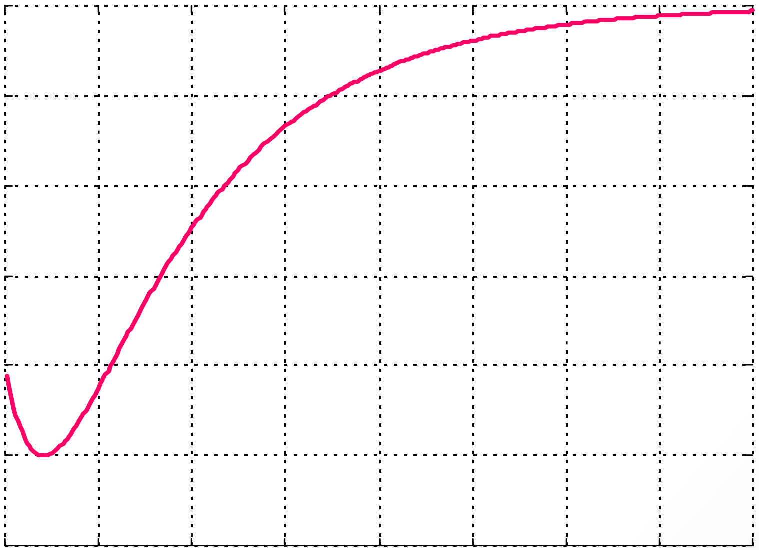
# Message from Example 2

- Zeros affect the process response.
  - Stable (or left half plane) zeros can cause overshoot in response for even overdamped processes.
  - The case of unstable (or right half plane) zeros is of particular interest. They give rise to “inverse” response.

- This will not occur for an overdamped second-order transfer function containing two poles but no zero.

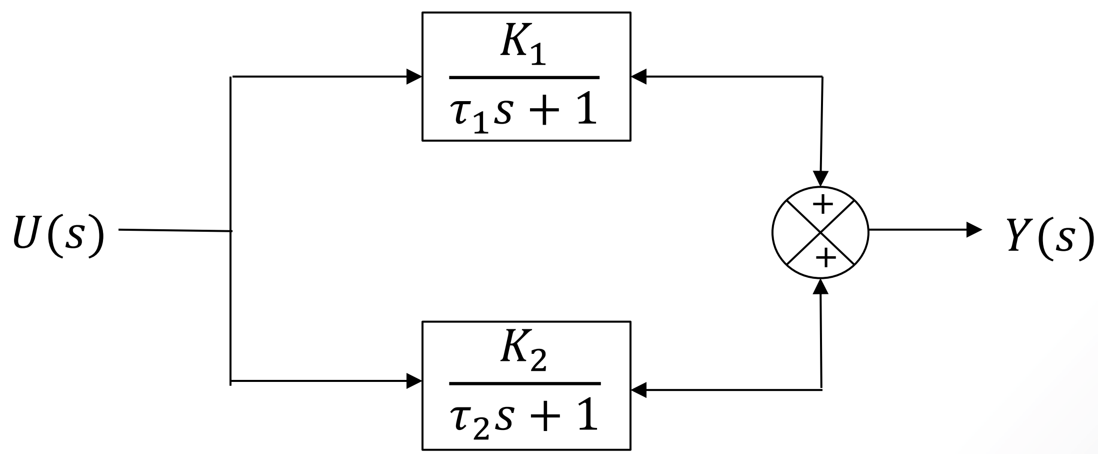
# Processes with Inverse Response

- Initially, the response of the system is in the opposite direction to where it eventually ends up.
- Unstable zeros are bad for control, as the controller may be "fooled" (output moves in opposite direction than intended).



# Inverse Response

Inverse response is the result of two physical effects that act on the process output variable in opposite ways and with different time scales!

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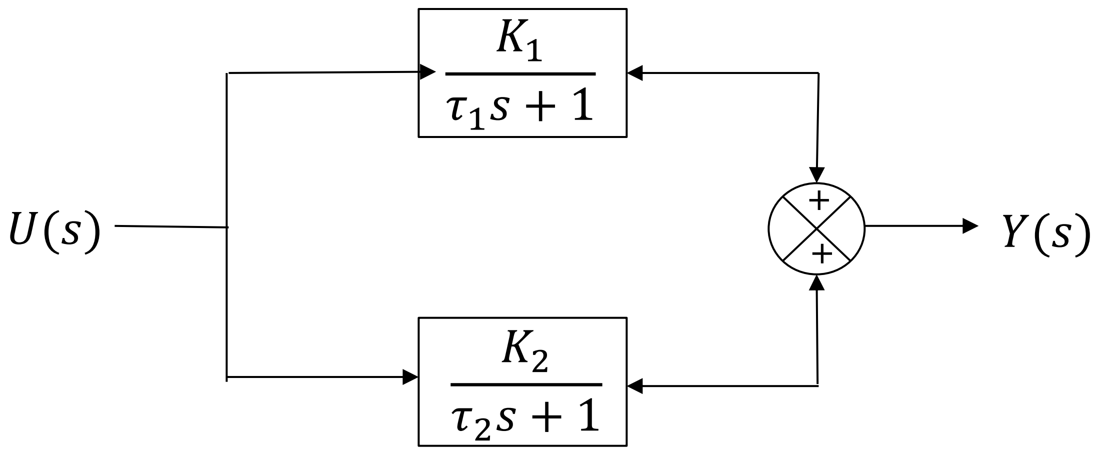
# Origin of Inverse Response




- Inverse response behavior is seen if:

$$\frac{K_1\tau_2 + K_2\tau_1}{K_1 + K_2} < 0$$

Source: Seborg, D. E., Edgar, T. F., & Mellichamp, D. A., (2011). *Process dynamics and control* (2nd ed.)(pp.135). Hoboken, NJ: Wiley.

# Conclusions

- Inverse response is obtained when:
  - $K_1$ and $K_2$ have opposite signs.
  - Sign of the overall transfer function gain $(K)$ is the same as that of the slower process.

- Example
  - Distillation column when the steam pressure to the reboiler is suddenly changed.

# Processes with Time Delays

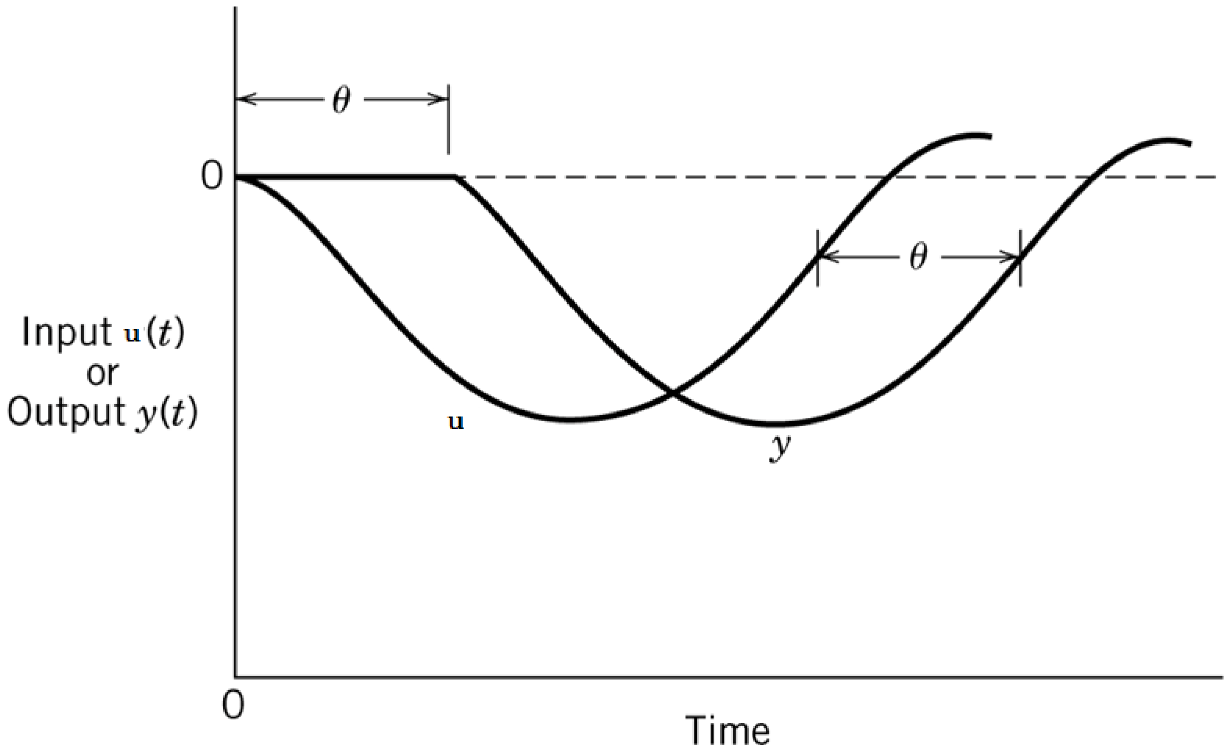
- Time delays occur due to:
  - Fluid flow in a pipe
  - Transport of solid material (e.g., conveyor belt)

- Mathematical description:
  - Time delay, θ, between an input u and an output y results in the following expression

$$y(t) = \begin{cases} 0 & \text{for } t < \theta \\ u(t-\theta) & \text{for } t \geq \theta \end{cases} \qquad (5-27)$$

# Processes with Time Delays (Cont'd)

$$G(s) = e^{-\theta s} = \frac{Y(s)}{U(s)}$$

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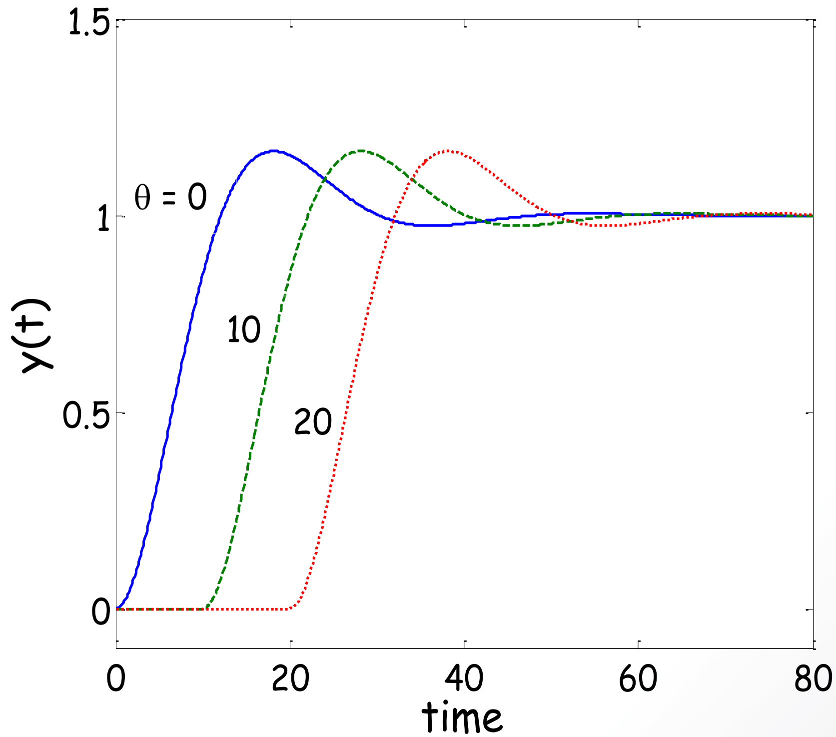
# Effect of Time Delays

- $G_1(s)$ – delay free part.
- If $y_1(t)$ is the step response of $G_1(s)$, then step response of $G(s)$ is $y_1(t-\theta)$.
- Response simply gets shifted by $\theta$ units of time.
- Time delays are bad for control as information gets delayed.

$$G(s) = G_1(s)e^{-\theta s}$$



# Effect of Time Delays (Cont'd)

- Time delay is inconvenient from the point of view of:
  - Analysis/ stability
  - Mathematics $(e^{-\theta s})$
  - Controller design

- Time delays are bad for control as information gets delayed.

# Approximation of Dead Time

- Dead time transfer function
  - Not a rational transfer function (ratio of two polynomials in $s$)
  - Difficult to analyze as cannot be factored into poles and zeros
  - Approximation is necessary
- Taylor series approximation for $e^{-\theta s}$
- Padé approximation

# Taylor Series Approximation

- Approximation for dead time

$$e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} - \frac{\theta^5 s^5}{5!} + \ \ldots$$

- Neglecting, second and higher order terms

$$e^{-\theta s} = 1 - \theta s$$

- Example

$$\frac{2}{25s^2 + 5s + 1} e^{-10\,s} \approx \frac{2(1 - 10s)}{25s^2 + 5s + 1}$$

# Padé Approximation

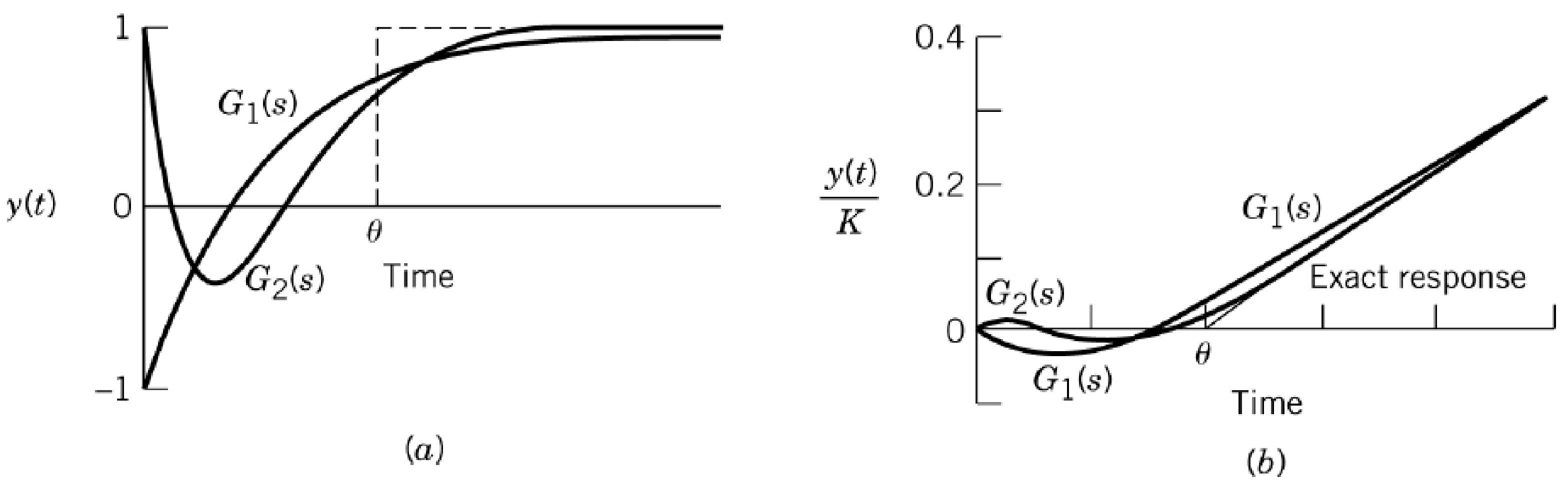
- Padé 1/1 approximation $e^{-\theta s} = \dfrac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s}$

- Second order (2/2) Padé approximations

$$e^{-\theta s} = \frac{e^{-\theta s/2}}{e^{\theta s/2}} \approx \frac{1 - \theta s/\,2 + \theta^2 s^2/\,12}{1 + \theta s/\,2 + \theta^2 s^2/\,12} \qquad \text{(2}^{\text{nd}}\text{ order Padé)}$$

# Comparison of Methods

- Both figures illustrate the unit step input response to 1/1 and 2/2 Padé approximation:
  - Time delay
  - First-order with time delay

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# Summary about Time Delay

- Time delays shift the response, i.e. the output only starts changing after the time delay is elapsed.
- Time delay does not have any other effect on response, e.g. gain remains unchanged.
- To represent time delay as a rational function, first-order Padé approximation is often used. It approximates time delay better than the use of Taylor series expansion.
- Time delay is bad for control.

# Why Approximation of Higher Order?

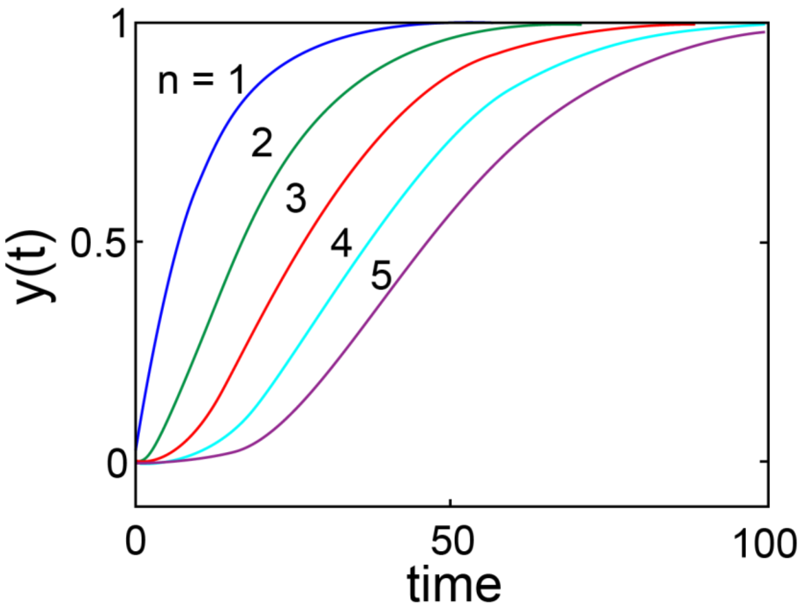
- Most chemical processes have very high-order dynamics due to imperfect mixing, wall effects, flow dynamics, etc.
  - Example: Multiple tanks in series

$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)\ldots.(\tau_n s + 1)}$$

- Unlike first or second order processes, it is difficult to draw any general insights.

# Higher Order Process: Step Response

- With increasing order, the response becomes more sluggish.
- Note that apparent delay increases with increasing order




$$G(s) = \frac{1}{(10s+1)^n}$$

## Approximation of Higher Order Transfer Functions

- Approximating high-order transfer function with low models.
- Low-order models are more convenient for control system design and analysis.
- Transfer function for a time delay can be expressed as a Taylor series expansion:

$$e^{-\theta_0 s} \approx 1 - \theta_0 s \qquad (5-57)$$

## Approximation of Higher Order Transfer Functions (Cont'd)

- An alternative first-order approximation consists of the transfer function,

$$e^{-\theta_0 s} = \frac{1}{e^{\theta_0 s}} \approx \frac{1}{1+\theta_0 s} \qquad (5-58)$$

where the time constant has a value of $\theta$.

- These expressions can be used to approximate the pole or zero term in a transfer function.
- We can turn the same approximation around (right hand side approximated as left hand side) to approximate non-dominant (much smaller than rest) time constant with delay.

# Skogestad's "half rule" Approximation

- Skogestad (2002) has proposed an approximation method for higher-order models that contain multiple time constants.
- He approximates the largest neglected time constant in the following manner.
  - One half of its value is added to the existing time delay (if any) and the other half is added to the smallest retained time constant.
- Time constants that are smaller than the "largest neglected time constant" are approximated as time delays using:

$$e^{-\theta_0 s} \approx 1 - \theta_0 s$$

# Summary of Approximation of Higher Order TFs

- For process with order 3 or higher, it is difficult to draw any general insights.
- Increasing the order makes the response more sluggish.
- Approximate analysis is possible by approximating non-dominant (much smaller than rest) time constants as time delay.

# Example 3

$$G(s) = \frac{K(-0.1s+1)}{(5s+1)(3s+1)(0.5s+1)}$$

Derive an approximate first-order-plus-time-delay model using two methods:

1. Taylor's series expansion
2. Skogestad's half rule

# Example 4

$$G(s) = \frac{K(1-s)e^{-s}}{(12s+1)(3s+1)(0.2s+1)(0.05s+1)}$$

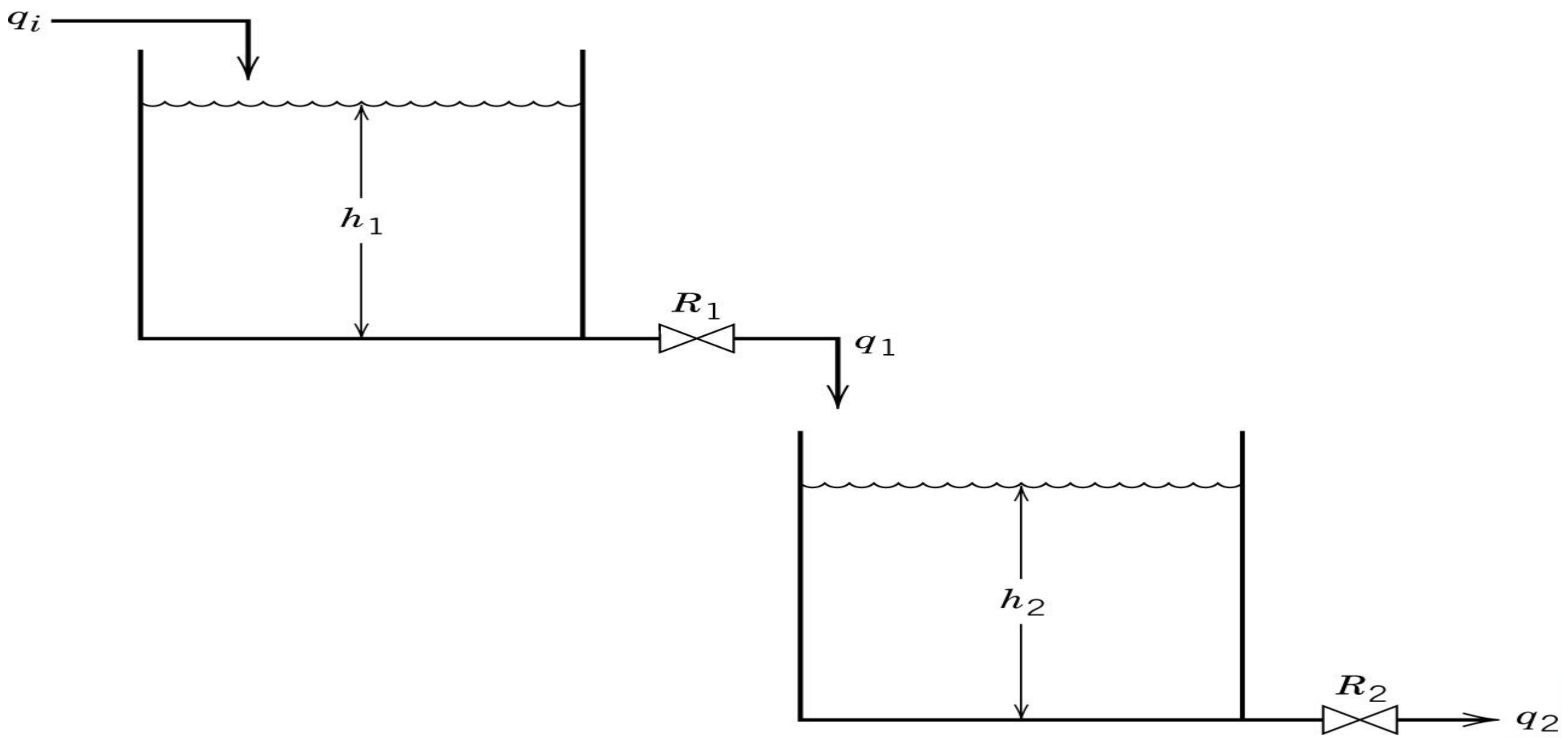
Use Skogestad's method to derive two approximation models:

1. First-order-plus-dead-time model
2. Second-order-plus-dead-time model

# Interacting and Non-interacting Processes

- Process is said to be interacting if:
  - Changes in a downstream unit affect upstream units, and vice versa.
- Otherwise the process is non-interacting.
- Example: The exit stream from the reactor serves as the feed to a distillation column.

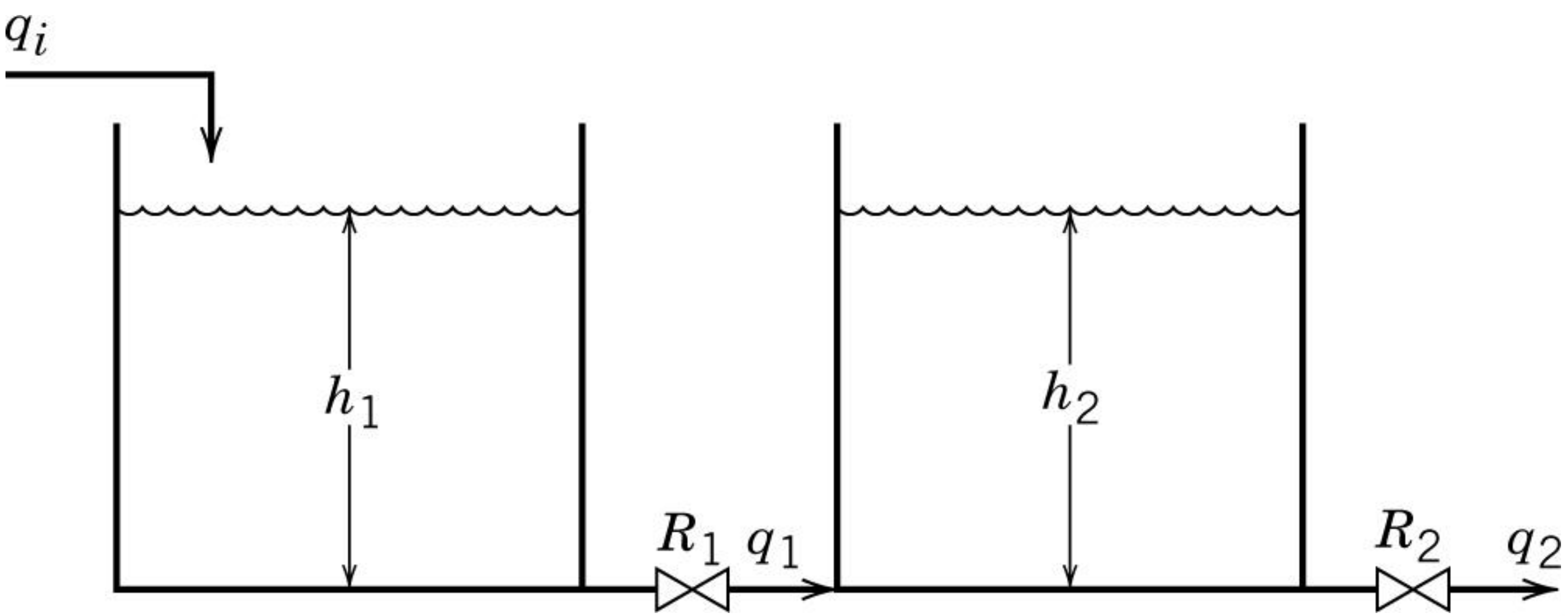
# Non-Interacting Processes




$$\frac{H_2(s)}{Q_i(s)} = \frac{R_2}{(A_1R_1s+1)(A_2R_2s+1)}$$

Source: Seborg, D. E., Edgar, T. F., & Mellichamp, D. A., (2011). *Process dynamics and control* (2nd ed.)(pp.146). Hoboken, NJ: Wiley.

# Interacting Processes




Find $G(s) = \dfrac{H_2(s)}{Q_i(s)}$

Source: Seborg, D. E., Edgar, T. F., & Mellichamp, D. A., (2011). *Process dynamics and control* (2nd ed.)(pp.146). Hoboken, NJ: Wiley.

# To Sum It All Up

| | |
|---|---|
| Gain | Ultimate value, how much? |
| Time Constant | How fast? |
| Underdamped Process | Exhibits overshoot, oscillations |
| Critically Damped Process | Fastest response without overshoot and oscillations |
| Increase in Order | More sluggish response |
| Time Delay | Shifts response |
| Stable Zeros | "May" cause overshoot |
| Unstable Zeros | Inverse response |

# Lecture Summary

- Effect of poles on the process response
- Dynamics of the processes that cannot be described by simple transfer functions
- It may include numerator dynamics, time delay
- An observed time delay is often a manifestation of higher order dynamics
- Approximation of time delay and higher order dynamics
- Interacting and non-interacting processes
- Suggested Reading: Chapter 5 of Seborg

# Review Questions

1. Go to wooclap.com
2. Enter the event code in the top banner

Event code

**CH3101CHAPTER6**

# Chapter 6: Dynamic Response Characteristics of More Complicated Processes

**The End**