Q.1

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 4\frac{du}{dt} + 2u$$

Take Laplace Transforms. (Note zero initial conditions)

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = 4sU(s) + 2U(s)$$

Rearranging $\frac{Y(s)}{U(s)} = \frac{4s+2}{s^3+6s^2+11s+6}$

$$U(s) = \frac{1}{s}(unit\ step\ input)$$

$$Y(s) = \frac{4s+2}{s^3+6s^2+11s+6} * \frac{1}{s}$$

Partial Fraction expansion

$$s(s^3 + 6s^2 + 11s + 6) \equiv s(s+1)(s+2)(s+3)$$

$$\frac{4s+2}{s(s^3+6s^2+11s+6)} \equiv \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$= \frac{A(s+1)(s+2)(s+3) + B(s)(s+2)(s+3) + \cdots}{s(s+1)(s+2)(s+3)}$$

For A: s=0 → 2=6A → A=1/3

For B: s=-1 → -2=-2B → B=1

For C: s=-2 → -6=2C → C=-3

For D: s=-3 → -10=-6D → D=5/3

$$Y(s) = \frac{1}{3s} + \frac{1}{s+1} + \frac{-3}{(s+2)} + \frac{5/3}{s+3}$$

Take inverse

$$y(t) = \frac{1}{3} + e^{-t} - 3e^{-2t} + \frac{5}{3}e^{-3t}$$

As t→∞

y(t)=1/3

Applying final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{S\to 0}[sY(s)]$$

$$Y(s) = \frac{4s+2}{s(s+1)(s+2)(s+3)}$$

$$\lim_{S\to 0}[sY(s)] = \frac{2}{6} = \frac{1}{3}$$

Q.2

(a)

$$L^{-1}\left[\frac{11}{(s-1)^3}\right] = \frac{11}{2}e^t t^2$$

Note $L(t^2) = \frac{2}{S^3}$

$$L[t * f(t)] = \frac{-dL[f(t)]}{ds} = \frac{-dF(s)}{ds}$$

$$L[e^{at}f(t)] = F(s-a)$$

Where F(S)=L[f(t)]

(b)

$$\frac{4s-2}{s^2-4s+13} = L^{-1}(\frac{4s-2}{s^2-4s+13})$$

$s^2-4s+13 \rightarrow s_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{4\pm\sqrt{16-4*13}}{2} = 2 \pm \mathrm{j}6$

Imaginary roots

$$s^2-4s+13 \equiv (s-2)^2+9$$

$$L^{-1}\left(\frac{4s-2}{s^2-4s+13}\right) = L^{-1}[\frac{4s-2}{(s-2)^2+9}]$$

We know

$$L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin\omega t$$

$$L^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos\omega t$$

$$L^{-1}\left[\frac{4s-2}{(s-2)^2+9}\right] = L^{-1}\left[\frac{4(s-2)}{(s-2)^2+9}\right] + L^{-1}\left[\frac{6}{(s-2)^2+9}\right] = 4e^{2t}\cos 3t + 2e^{2t}\sin 3t$$

(c)

$$\frac{s+1}{s^2(s^2+4s+5)}$$

First evaluate types of roots for

$$s^2+4s+5 = as^2+bs+c$$

$$b^2 = 16 \quad 4ac = 26$$

$\frac{b^2}{4ac} < 1 \rightarrow$complex roots

$$Y(S) = \frac{s+1}{s^2(s^2+4s+5)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4s+5}$$

$$\frac{s+1}{s^2(s^2+4s+5)} \equiv \frac{As(s^2+4s+5)+B(s^2+4s+5)+s^2(Cs+D)}{s^2(s^2+4s+5)}$$

Equate coefficients of like powers of S

S^3: 0=A+C A=-C

S^2: 0=4A+B+D

S: 1=5A+4B

S^0: 1=5B → b=1/5=0.2 → 5A+4/5=1 → A=1/25=0.04 → A+C=0 → C=-0.04 → 4(0.04)+0.2+D=0 →D=-0. 36

$$y(t) = L^{-1}\left(\frac{0.04}{s}\right) + L^{-1}\left(\frac{0.2}{s^2}\right) + L^{-1}\left(\frac{-0.04s-0.36}{s^2+4s+5}\right)$$

$$\text{Third Term} \equiv \frac{-0.04s-0.36}{(s+2)^2+1} = \frac{-0.04(s+2)}{(s+2)^2+1} - \frac{0.28}{(s+2)^2+1}$$

$$L^{-1}\left(\frac{-0.04s-0.36}{(s+2)^2+1}\right)=L^{-1}\left(\frac{0.04(s+2)}{(s+2)^2+1}\right)+L^{-1}\left(\frac{-0.28}{(s+2)^2+1}\right)$$
$$=0.04e^{-2t}\cos t-0.28e^{-2t}\sin t$$
$$y(t)=0.04+0.2t-0.04e^{-2t}\cos t-0.28e^{-2t}\sin t$$

(d)

$$\frac{1+e^{-2s}}{(4s+1)(3s+1)}\equiv\frac{1}{(4s+1)(3s+1)}+\frac{e^{-2s}}{(4s+1)(3s+1)}$$

$Y(s)=Y_1(s)+Y_2(s)$

$Y_1(s)$ can be found directly from Table A 1 [Number 10]

$$y_1(t)=e^{-t/4}-e^{-t/3}$$

$Y_2(s)=e^{-2s}Y_1(s)$ Inverse transformer can be written immediately by replacing t by (t-2) in $y_1(t)$

$$y_2(t)=[e^{-\frac{t-2}{4}}-e^{-\frac{t-2}{3}}]u(t-2)$$

$$y(t)=e^{-t/4}-e^{-\frac{t}{3}}+[e^{-\frac{t-2}{4}}-e^{-\frac{t-2}{3}}]u(t-2)$$