Chapter 4: Transfer Function Models

Dr. Mukta Bansal

School of Chemistry, Chemical Engineering & Biotechnology (CCEB)

Office: N1.2 – B2 – 28

Email: mbansal@ntu.edu.sg

Chapter Overview

This chapter consists of the following topics:

1. Introduction to Transfer Functions
    - Modeling Perspectives
    - Definition
    - Development
    - Transfer Functions for Stirred Tank
    - Important Aspects of Transfer Functions
    - Steps to Derive Transfer Functions

Chapter Overview

This chapter consists of the following topics:

2. Properties of Transfer Functions
   - Steady State Gain
   - Time Constant
   - Order of Transfer Function
   - Physically Realizable
   - Transfer Functions in Series and in Parallel

3. Linearization of Nonlinear Models
   - Principle of Linearization
   - Ordinary Differential Equations (ODEs)

Learning Objectives

At the end of this chapter, you will be able to:

- Explain what is a transfer function
- Describe the development of transfer functions
- Apply the specific steps to derive transfer functions
- Explain the properties of transfer functions
- Explain how to linearize nonlinear processes

Modeling Perspectives

- Modeling activity often leads to a set of nonlinear ODEs.
  - Difficult to work with
  - May not yield analytical solutions
- An alternative model is transfer functions which are based on Laplace transforms.
- Controllers can often be designed based on linearized models that represent process dynamics locally.
- A linear(ized) ODE can be equivalently expressed in the Laplace domain.
- An important concept associated with Laplace transforms is “transfer function model”.

Why Transfer Functions?

- A transfer function(TF) contains the same information as a linearized ODE.
- TFs are still preferred:
  - Independent of the input function.
  - Response of the process to an input change can be generalized for the standard transfer function.
  - Play a key role in the design and analysis of control systems.
  - Characterize the dynamic relationship.
    - ✓ Of two process variables, a dependent (output) & an independent (input) variables.
  - Only applicable to processes that exhibit linear dynamic behavior.
    - ✓ For nonlinear process, TF provides an approximate linear model.
- **Advantages of using TFs will be gradually clear.**

All this fuss over nothing?

Transfer Functions

A transfer function:

- Convenient representation of a linear, dynamic model
- Relates one input and one output

$\underline{u:}$

- Input
- Forcing function
- "Cause"

$\underline{y:}$

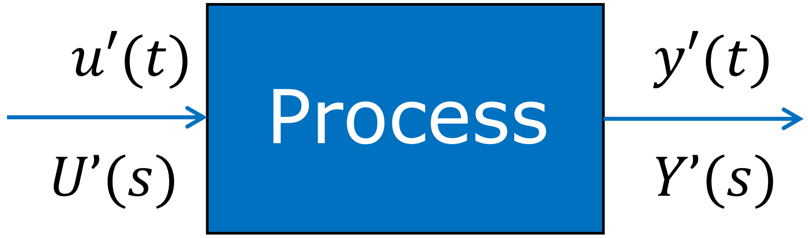
- Output, CV
- Response
- "Effect"

Transfer Functions (Cont’d)

- TF of a linear dynamic system is the ratio of the Laplace transform of the output to the Laplace transform of the input.

$$G(s) = \frac{Y'(s)}{U'(s)}$$

- where



$$Y'(s) = L[y'(t)]$$
$$U'(s) = L[u'(t)]$$
$$y'(t) = y(t) - \bar{y}$$
$$u'(t) = u(t) - \bar{u}$$

Development of Transfer Functions

- Stirred Tank Heating Process:
  - Case 1: Constant holdup

$$V\rho C\frac{dT}{dt} = wC(T_i - T) + Q \qquad (2-36)$$

  - Previous dynamic model, assuming constant liquid holdup and flow rates
- Steady State model:

$$0 = wC(\bar{T}_i - \bar{T}) + Q \qquad (2)$$

- Subtract (2) from (2-36):

$$V\rho C\frac{dT}{dt} = wC[(T_i - \bar{T}_i) - (T - \bar{T})] + (Q - \bar{Q}) \qquad (3)$$

Transfer Functions for Stirred Tank (Cont'd)

- Deviation Variables:

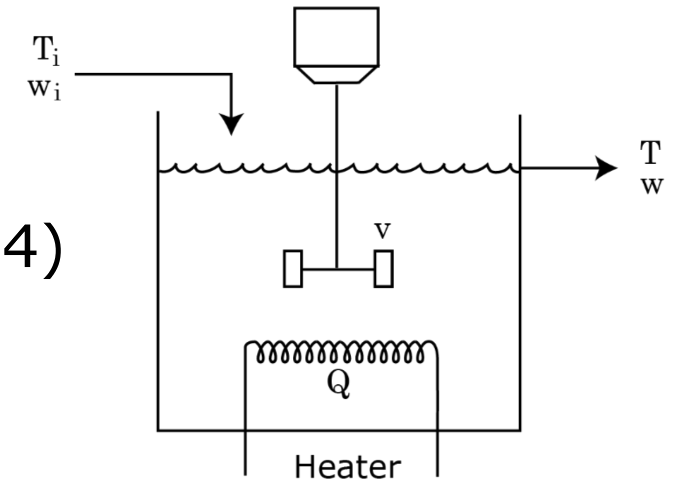
$$T' = T - \bar{T},\ T_i' = T_i - \bar{T}_i,\ Q' = Q - \bar{Q}$$

$$V\rho C \frac{dT'}{dt} = wC(T_i' - T') + Q' \qquad (4)$$

- Take the Laplace of Eq. (4):

$$V\rho C[sT'(s) - T'(0)] = wC[T_i'(s) - T'(s)] + Q'(s)$$

- At the initial steady state, $T'(0) = 0$.
- In process control, we are interested in deviation from steady-state values.



Stirred-tank heating process with constant holdup, V.

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.27). Hoboken, NJ: Wiley.

Transfer Functions for Stirred Tank (Cont’d)

$$V\rho C[sT'(s) - T'(0)] = wC[T_i'(s) - T'(s)] + Q'(s)$$

- Rearranging after substituting T’(0) = 0:

$$T'(s) = \left(\frac{K}{\tau s + 1}\right)Q'(s) + \left(\frac{1}{\tau s + 1}\right)T_i'(s) \qquad (6)$$

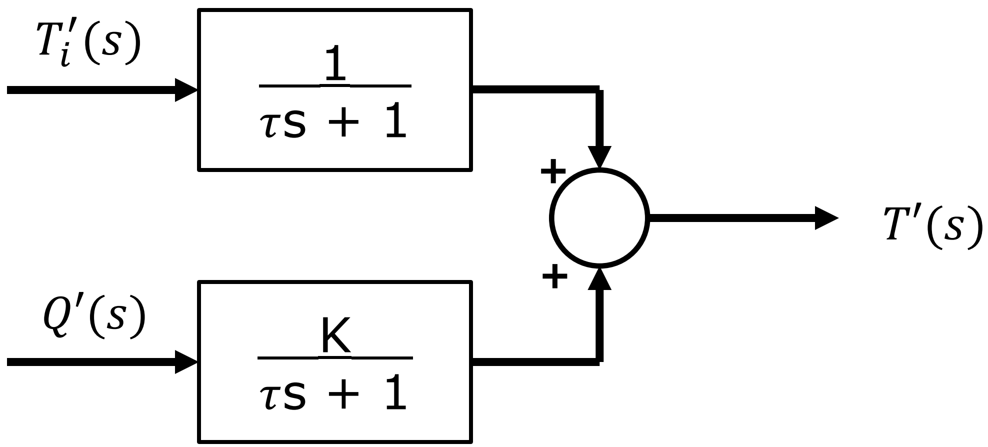
- Where $k = \frac{1}{wC}$ and $\tau = \frac{V_\rho}{w}$

$$T'(s) = G_1(s)Q'(s) + G_2(s)T_i'(s)$$

*G*1, *G*2: transfer functions & independent of the inputs, *Q’* and *T’*
K, τ : depend on operating conditions

- If no change in T_i, what is the TF?

Important Aspects of Transfer Functions



- Important aspects of TFs:
  - Effects of different inputs are additive.
  - TF model enables us to determine the output response to any change in an input.
  - Deviation variables help to eliminate initial conditions for TF models.

Steps to Derive the Transfer Function

Step 1: Write dynamic balance equation.

↓

Step 2: Write steady state equation.

↓

Step 3: Subtract steady state from dynamic equation and get it in terms of deviation variables.

↓

Step 4: Take the Laplace transform of both sides.

↓

Step 5: Rearrange the equation to get the output variable in terms of input.

Example 1: Stirred-tank Heating Process

- Investigate time behavior of outlet temperature T to a disturbance in either inlet temperature T_i or Q or both.

$$T'(s) = \left(\frac{K}{\tau s + 1}\right) Q'(s) + \left(\frac{1}{\tau s + 1}\right) T_i'(s) \qquad (6)$$

- Inlet temperature T_i same:

$$T' = \frac{0.05}{2s + 1} Q' \qquad K = 0.05 \qquad \tau = 2.0$$

- Step change in Q from $1500\ cal/\mathrm{sec}$ to $2000\ cal/\mathrm{sec}$:

$$Q' = \frac{500}{s}$$

- What is $T'(t)$?

Example 1: Stirred-tank Heating Process (Cont'd)

$$T' = \frac{0.05}{2s+1}\frac{500}{s} = \frac{25}{s(2s+1)}$$

From line 13, table A.1 page 27 in the textbook:

$$T'(t) = 25\left[1 - e^{-t/\tau}\right] \longleftarrow T(s) = \frac{25}{s(\tau s+1)}$$

$$T'(t) = 25\left[1 - e^{-t/2}\right]$$

Example 1: Stirred-tank Heating Process (Cont'd)

Table A.1 Laplace Transform for Various Time-Domain Functions

$f(t)$		$F(s)$
1. $\delta(t)$	(unit impulse)	1
2. $S(t)$	(unit step)	$\frac{1}{s}$
3. t	(ramp)	$\frac{1}{s^2}$
4. t^{n-1}		$\frac{(n-1)!}{s^n}$
5. e^{-bt}		$\frac{1}{s+b}$
6. $\frac{1}{\tau} e^{-t/\tau}$		$\frac{1}{\tau s+1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$	$(n>0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n(n-1)!} t^{n-1}e^{-t/\tau}$		$\frac{1}{(\tau s+1)^n}$
9. $\frac{1}{b_1-b_2}(e^{-b_2 t}-e^{-b_1 t})$		$\frac{1}{(s+b_1)(s+b_2)}$
10. $\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$		$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11. $\frac{b_3-b_1}{b_2-b_1}e^{-b_1 t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2 t}$		$\frac{s+b_3}{(s+b_1)(s+b_2)}$

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.27). Hoboken, NJ: Wiley.

Example 1: Stirred-tank Heating Process (Cont'd)

Table A.1 Laplace Transform for Various Time-Domain Functions

12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$		$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13. $1-e^{-t/\tau}$		$\frac{1}{s(\tau s+1)}$
14. $\sin \omega t$		$\frac{\omega}{s^2+\omega^2}$
15. $\cos \omega t$		$\frac{s}{s^2+\omega^2}$
16. $\sin(\omega t+\phi)$		$\frac{\omega\cos\phi+s\sin\phi}{s^2+\omega^2}$
17. $e^{-bt}\sin \omega t$	b, ω real	$\frac{\omega}{(s+b)^2+\omega^2}$
18. $e^{-bt}\cos \omega t$		$\frac{s+b}{(s+b)^2+\omega^2}$

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.27). Hoboken, NJ: Wiley.

Example 1: Stirred-tank Heating Process (Cont'd)

Table A.1 Laplace Transform for Various Time-Domain Functions

19. $\dfrac{1}{\tau\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left(\sqrt{1-\zeta^2}\, t/\tau\right)$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20. $1 + \dfrac{1}{\tau_2 - \tau_1}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)$ $(\tau_1 \ne \tau_2)$	$\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left[\sqrt{1-\zeta^2}\, t/\tau + \psi\right]$ $\psi = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}\left[\cos\left(\sqrt{1-\zeta^2}\, t/\tau\right) + \dfrac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\, t/\tau\right)\right]$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.27). Hoboken, NJ: Wiley.

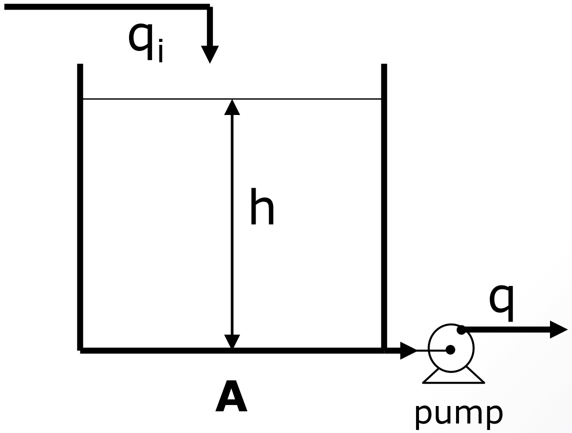
Example 2

- Transfer Function?
  - Steps 1 and 2: Dynamic model & steady state.

$$A\frac{dh}{dt} = q_i - q_0 \qquad \bar{q}_i = \bar{q}_0 \text{ at s.s.}$$

  - Step 3: Subtract steady state from dynamic model to get deviation variables.

$$A\frac{dh'}{dt} = q_i' - q_0'$$



Example 2 (Cont'd)

- Step 4: Taking Laplace transform:

$$AsH'(s) = Q'_i(s)'_i - Q'_0(s)$$

- Step 5: Rearranging (suppose q_0 is constant):

$$AsH'(s) = Q'_i(s), \quad \frac{H'(s)}{Q'_i(s)} = \frac{1}{As}$$

- Pure integrator (ramp) for **step change** in $q_{i.}$

Properties of Transfer Function Models

1. Steady-State Gain (K) from TF

- Ratio of the change in output to the change in input at steady state. Thus, for an input u and the corresponding output y:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1} \qquad (3-38)$$

- For a linear system, K is a constant.
- But for a nonlinear system, K will depend on the operating condition $(\bar{u},\bar{y})$.

Properties of Transfer Function Models

1. Steady-State Gain (K) from TF

- If a TF model has a steady-state gain, then:

$$K = \lim_{s \to 0} G(s) \qquad (14)$$

- This important result is a consequence of the Final Value Theorem.
- *Note*: Some TF models do *not* have a steady-state gain (e.g., integrating process).

Properties of Transfer Function Models

2. Time Constant (τ)

- Indicative of the speed of the system response
- Smaller τ means faster response
- Larger τ means slower response

Properties of Transfer Function Models

3. Order of a TF Model

- Consider a general nth-order differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 \frac{dy}{dt} + a_0 y$$
$$= b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u \qquad (3-39)$$

- Taking Laplace transform & assuming all initial conditions zero

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

Properties of Transfer Function Models

3. Order of a TF Model (Cont'd)

- Order of the TF is defined to be the order of the denominator polynomial

- Steady State Gain?

Properties of Transfer Function Models

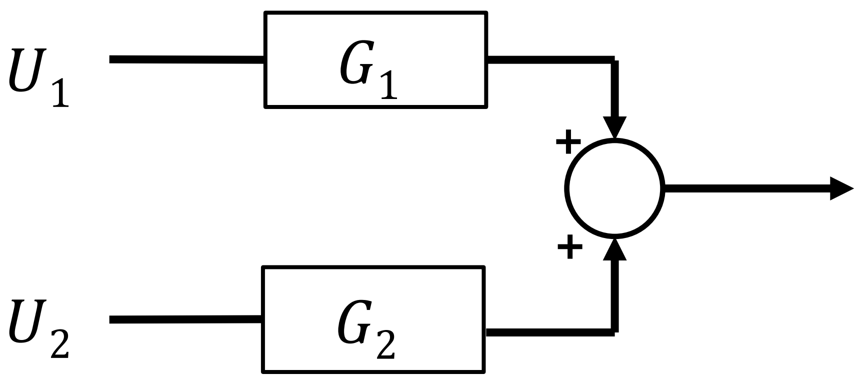
4. Physical Realizable

- For any physical system, $n \geq m$ in (3-39).
- Order of denominator ≥ order of numerator.
- Otherwise, the system response to a step input will be an impulse. This can't happen.
- Example: $$a_0 y = b_1 \frac{du}{dt} + b_0 u$$

Properties of Transfer Function Models

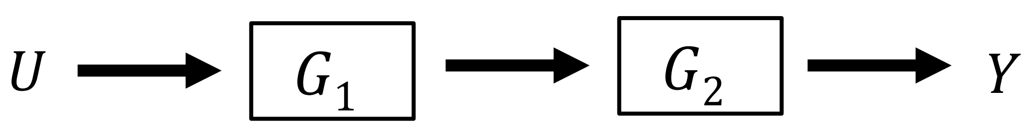
5. TFs of Systems in Series & Parallel

- TFs in Parallel (Additive Rule)



$$Y = G_1U_1 + G_2U_2$$

- TFs in Series (Multiplicative Rule)



$$Y = G_1 \cdot G_2U$$

Linearization Of Nonlinear Models

- Most chemical processes are nonlinear:
  - Model given by nonlinear differential equations.
  - Difficult to solve (require numerical methods).
- To design and analyze control systems for chemical processes, a linear approximation of the model is often used.
- Linear or Nonlinear:

$$y = 10x$$

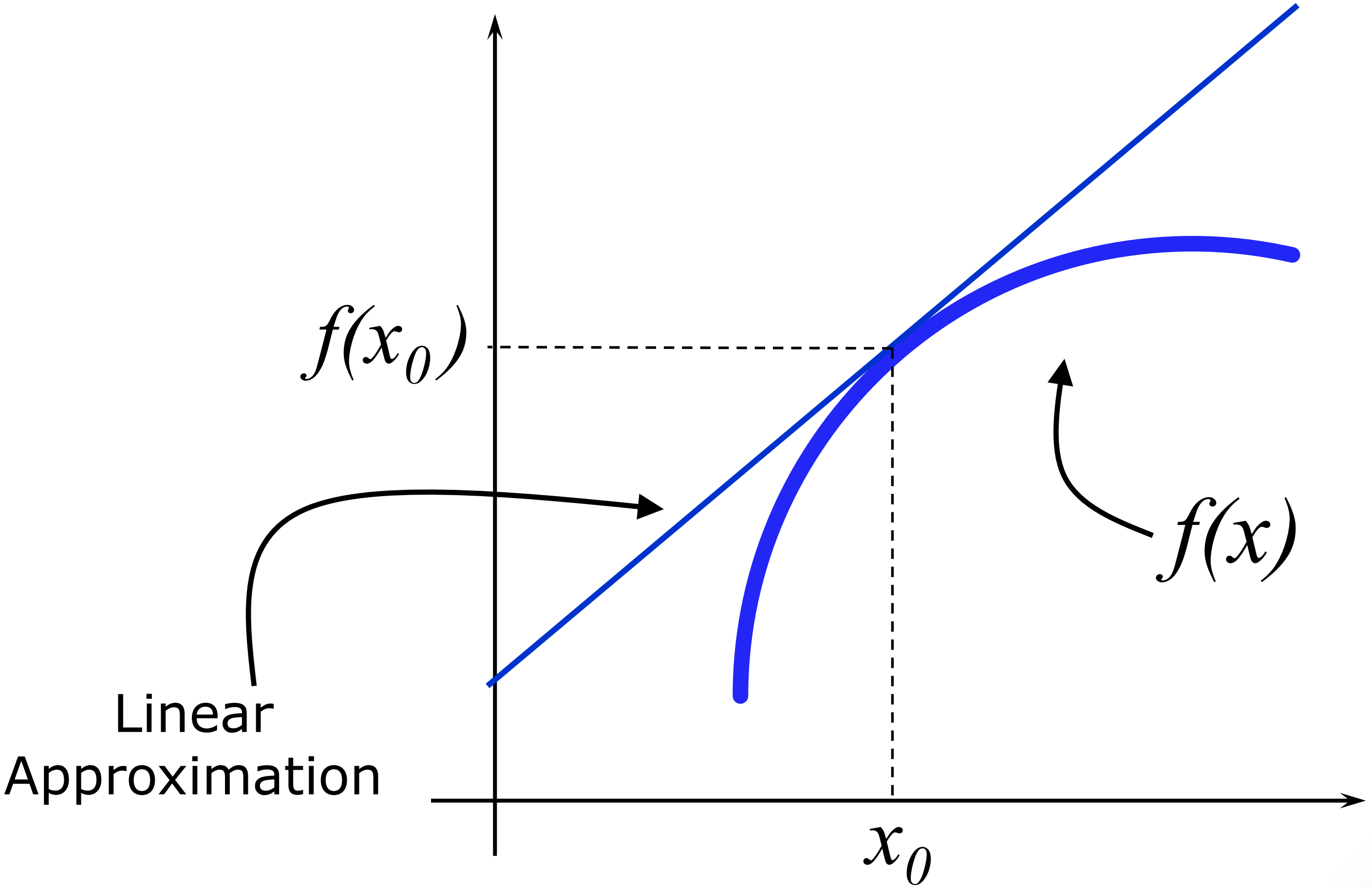
Linearization

- Linear approximation of a nonlinear model is most accurate near the point of linearization:

  - When nonlinear processes remain close to operating point, a linear approximation is reasonably accurate!

  - Moreover, a well-performing controller keeps the process in vicinity of the operating point.

Principle of Linearization



Large changes in operating conditions for a nonlinear process cannot be approximated satisfactorily by linear expressions.

Principle of Linearization

- Use 1st order Taylor series:

$$y = f(x)$$
$$= f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_0} (x - x_0)^2 + \ldots$$

- If the *variation in x is small*, second and higher order terms in $f(x)$ can be neglected to get:

$$y \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

Note that approximation is exact at $x = x_0$

Ordinary Differential Equations

- A nonlinear dynamic model where y is output & u is input:

$$\frac{dy}{dt} = f(y,u) \qquad (3-61)$$

$$f(y,u) \cong f(\bar{y},\bar{u}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{y},\bar{u}}(y-\bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y},\bar{u}}(u-\bar{u}) \qquad (3-62)$$

Transfer Function for Nonlinear Processes

- Steady state condition refers to:

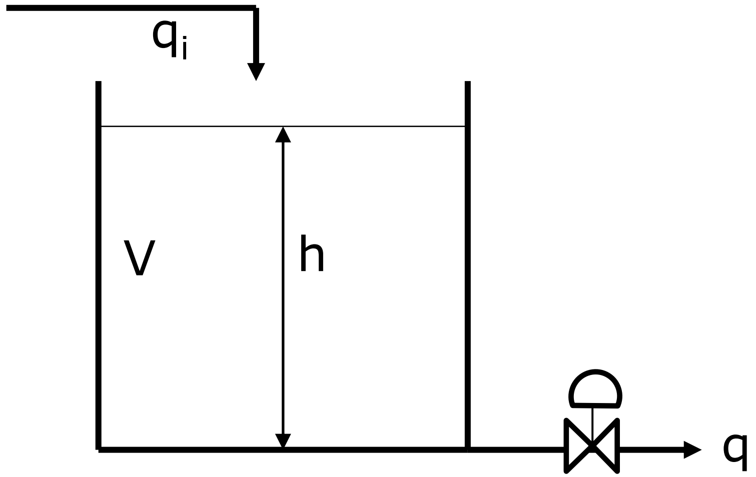
$$f(\bar{y}, \bar{u}) = 0$$

$$\frac{dy'}{dt} = \left.\frac{\partial f}{\partial y}\right|_s y' + \left.\frac{\partial f}{\partial u}\right|_s u' \qquad (3-63)$$

- Note that deviation variables arise naturally in Eq. 3-63.

Example 3

- Transfer function where q is manipulated by a flow control valve:

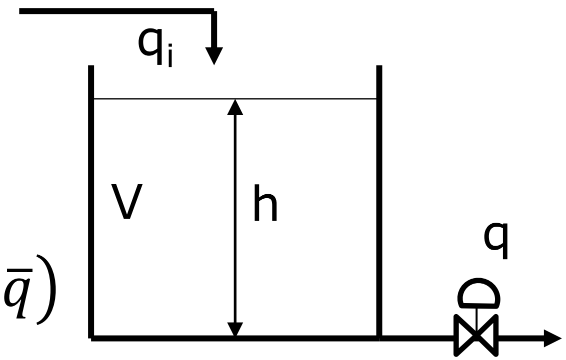


<u>Nonlinear element</u>

$$q = C_v\sqrt{h}$$

Example 3 (Cont'd)

- Linearization
  - Dynamic model $A\dfrac{dh}{dt} = q_i - C_v\sqrt{h}$
  - Linearize about the steady state $(\bar{h}, \bar{q})$



$$A\frac{dh}{dt} = \bar{q}_i - C_v\bar{h}^{0.5} + \frac{\partial f}{\partial q_i}(q_i - \bar{q}_i) + \frac{\partial f}{\partial h}(h - \bar{h})$$

$$A\frac{dh'}{dt} = 0 + 1(q_i - \bar{q}_i) - \frac{1}{2}C_v\bar{h}^{-0.5}(h - \bar{h}) = q_i' - \frac{1}{2}C_v\bar{h}^{-0.5}h'$$

- Rearrange

$$A\frac{dh'}{dt} = q_i' - \frac{1}{R}h'$$

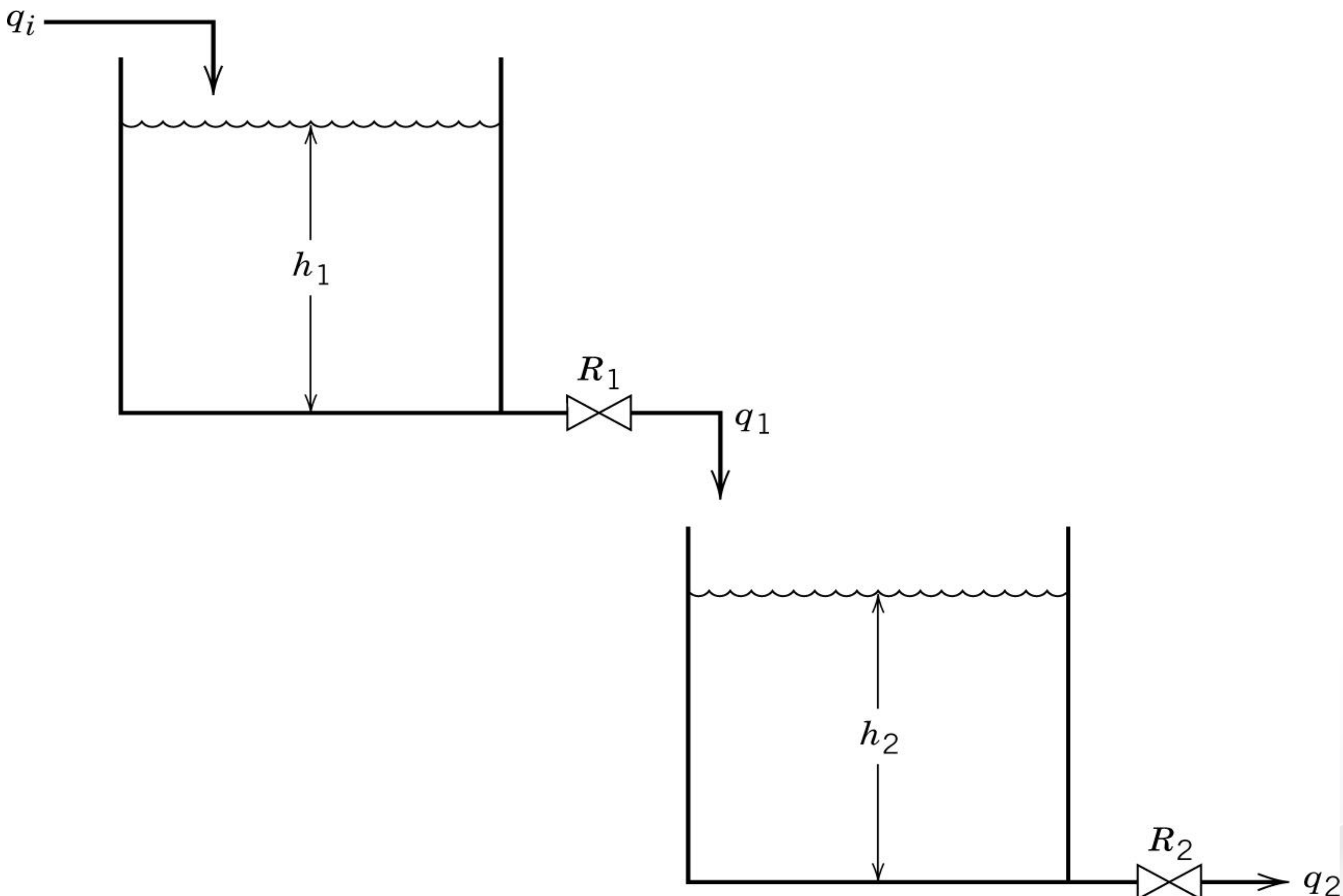
$$R = 2\bar{h}^{0.5}/C_v$$

Summary

In this chapter, we have covered:

- Derive transfer functions
    - Always express in terms if deviation variables
- Properties of transfer functions
    - Steady state gain
    - Time constant
    - Order of TF
    - Physically realizable
    - TFs in series and in parallel
- How to linearize nonlinear processes
- Suggested Reading: Chapter 3 of Seborg (Third Edition)

Example 4



Example 4

- Find the transfer function relating changes in flow rate from second tank, $Q'(s)$, to changes in flow rate into the first tank, $Q_i'(s)$. Assume that the two tanks have different cross-sectional areas A_1 & A_2 and valves resistance R_1 and R_2. Outlet flow rate from each tank is linear to the height of liquid in the tank.

- Show how this transfer function is related to the individual transfer functions, $H_1(s)/Q_i(s)$, $Q_1(s)/H_1(s)$, $H_2(s)/Q_1(s)$, and $Q_2(s)/H_2(s)$.

Review Questions

1 Go to wooclap.com

2 Enter the event code in the top banner

Event code

CH3101CHAPTER4

Review Questions

Question 1. Which of the following is not a step to derive a TF?

A. Write steady state equation
B. Divide steady state equation by dynamic equation and get it in terms of deviation variables
C. Take the Laplace transform of both sides
D. Rearrange the equation to get the output variable in terms of input

Question 2. If f(t) increases to 50 from 25 in an instant and returns to the original value, what is F(s)?

A. 1
B. 25
C. 50
D. 50/s

Review Questions

Question 3. Given the TF $G(s) = 4s + 1$. Given that this TF can reach a steady state when responding to unit step change. What is the steady-state gain?

A. 5
B. 4
C. 1
D. 0

Question 4. Given a physical system with

$$G(S) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

Which of the following is the relationship between m and n?

$A.\ n \geq m$ $\quad$ $B. m \geq n$

$C.\ n > m$ $\quad$ $D. m > n$

Chapter 4: Transfer Function Models

The End.