# Chapter 3: Laplace Transforms

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# Chapter Overview

This chapter consists of the following topics:

1. Laplace Transform
   - Definition
   - How Laplace Transform Works
2. Laplace Transform of Typical Functions
   - Constant Function
   - Step Function
   - Exponential Functions
   - Derivatives
   - Time Delay Functions

# Chapter Overview

This chapter consists of the following topics:

3. Laplace Transform Properties
   - Property 1: Linearity
   - Property 2: Multiplication of f(t) by t
   - Property 3: "*s* shifting"
   - Other Properties
4. Solutions of Differential Equations by Laplace Transform Techniques
   - Inverse Laplace Transform
   - Partial Fraction Expansion (PFE): Real Distinct Roots, Complex Roots and Repeated Roots

# Learning Objectives

At the end of this chapter, you will be able to:

- Explain the concept of Laplace transform
- Derive the Laplace transform for various functions
- Explain the different properties of Laplace transform
- Compute the Laplace transform
- Determine the solutions of differential equations by Laplace transform techniques

# Recap

In the previous chapter, we have learnt:

- Process and process variables
- Important elements of the control loop
- Feedback control strategy
- Feedforward control strategy
- Virtually every process needs a control system to operate in presence of disturbances
- A model is necessary for systematic design of controller
- Develop dynamic models represented as ordinary differential equations (ODE) using conservation laws

# Which Equation is Easy to Solve?

First order chemical reaction:

$$\frac{dc}{dt} = -kc$$

$$c\Big|_{t=0} = c_0$$

1

The following algebraic equation:

$$sc - c_0 + kc = 0 \quad \text{solve for } c$$

2

# Which Equation is Easy to Solve?

First order chemical reaction:

$$\frac{dc}{dt} = -kc$$

$$c\Big|_{t=0} = c_0$$

1

$$c(t) = c_0 e^{-kt}$$

4

The following algebraic equation:

$$sc - c_0 + kc = 0 \quad \text{solve for } c$$

2

$$C = \frac{c_0}{s+k}$$

3

If there exist a mapping that transforms Eqs. 1 to Eq. 2 and an inverse mapping that transforms the solution of Eq. 2 (i.e. Eq. 3) to the solution of Eqs. 1 (i.e. Eq. 4). That would be great.

# Laplace Transform

- Laplace Transform is computed as:

$$\boxed{L[f(t)] \equiv F(s) = \int_0^\infty f(t)\, e^{-st} dt}$$

$s$ is a complex variable:

$$s = a + jb;\ j = \sqrt{-1}$$

- It is named after a French mathematician Pierre-Simon Laplace.
- Advantages:
  - The s-domain function $F(s)$ contains same information as time domain function $f(t)$.
  - It is commonly used to produce an easily solvable algebraic equation from an ordinary differential equation.

# Laplace Transform (Cont'd)

Transform produces several changes in the equation.

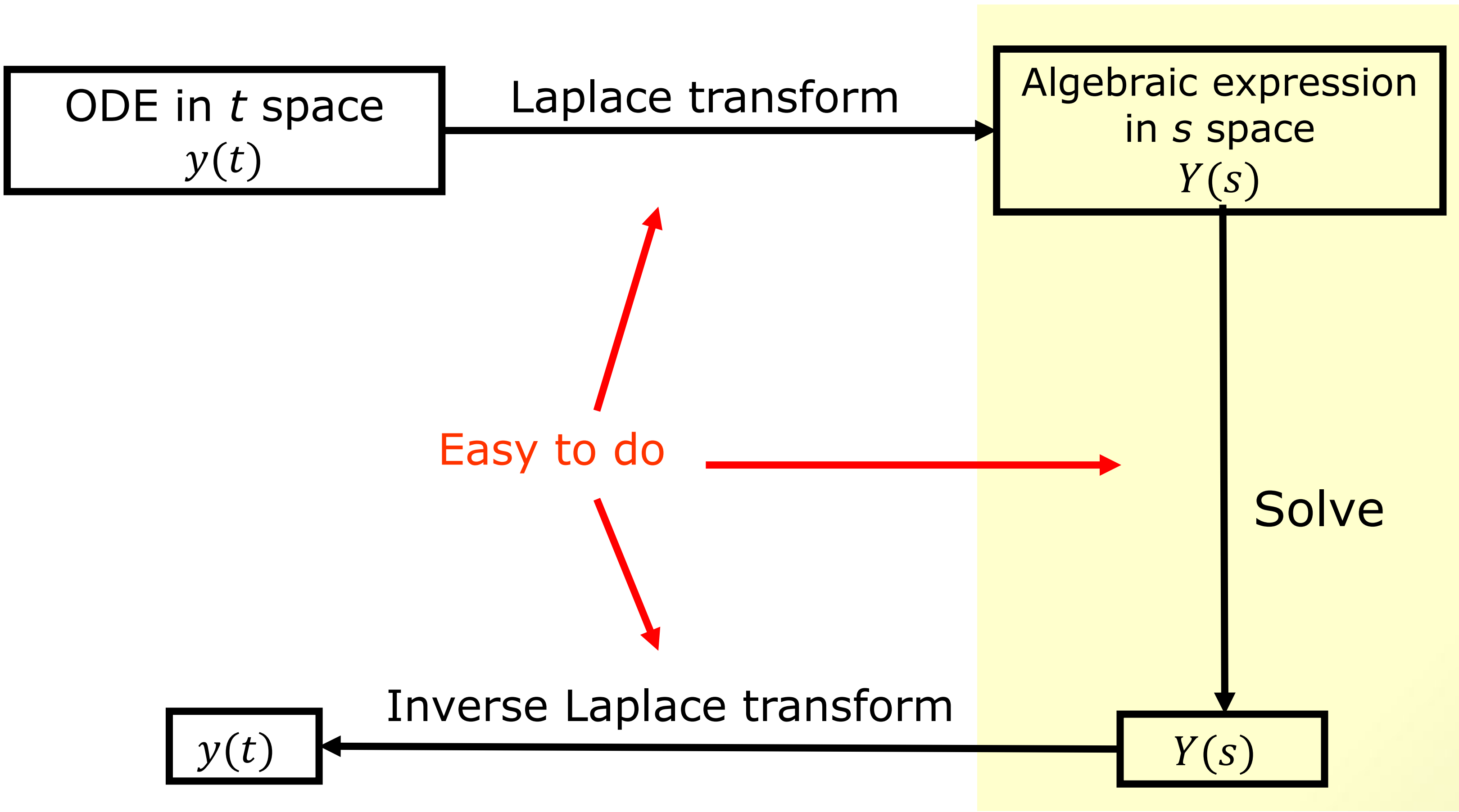
**Variable is time ($t$)**

- $t$ is real number
- Sol$^n$ from time domain
- Differential equation

**Variable is $s$**

- $s$ is complex number
- Sol$^n$ from Laplace domain
- Algebraic equation

# How Laplace Transform Works



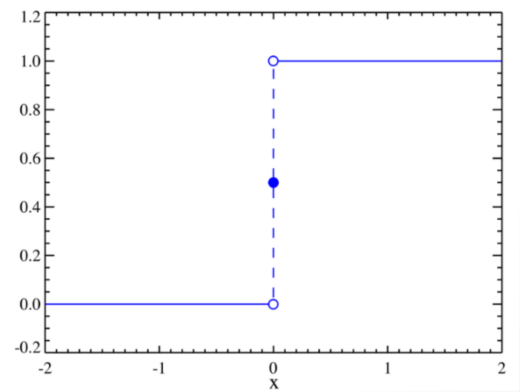
# Laplace Transform of Typical Functions

- Constant Function

$$L[a] = \int_0^\infty a e^{-st} dt = -\frac{a}{s} e^{-st} \Big|_0^\infty = 0 - (-\frac{a}{s}) = \frac{a}{s} \qquad (A-4)$$

- Step Function

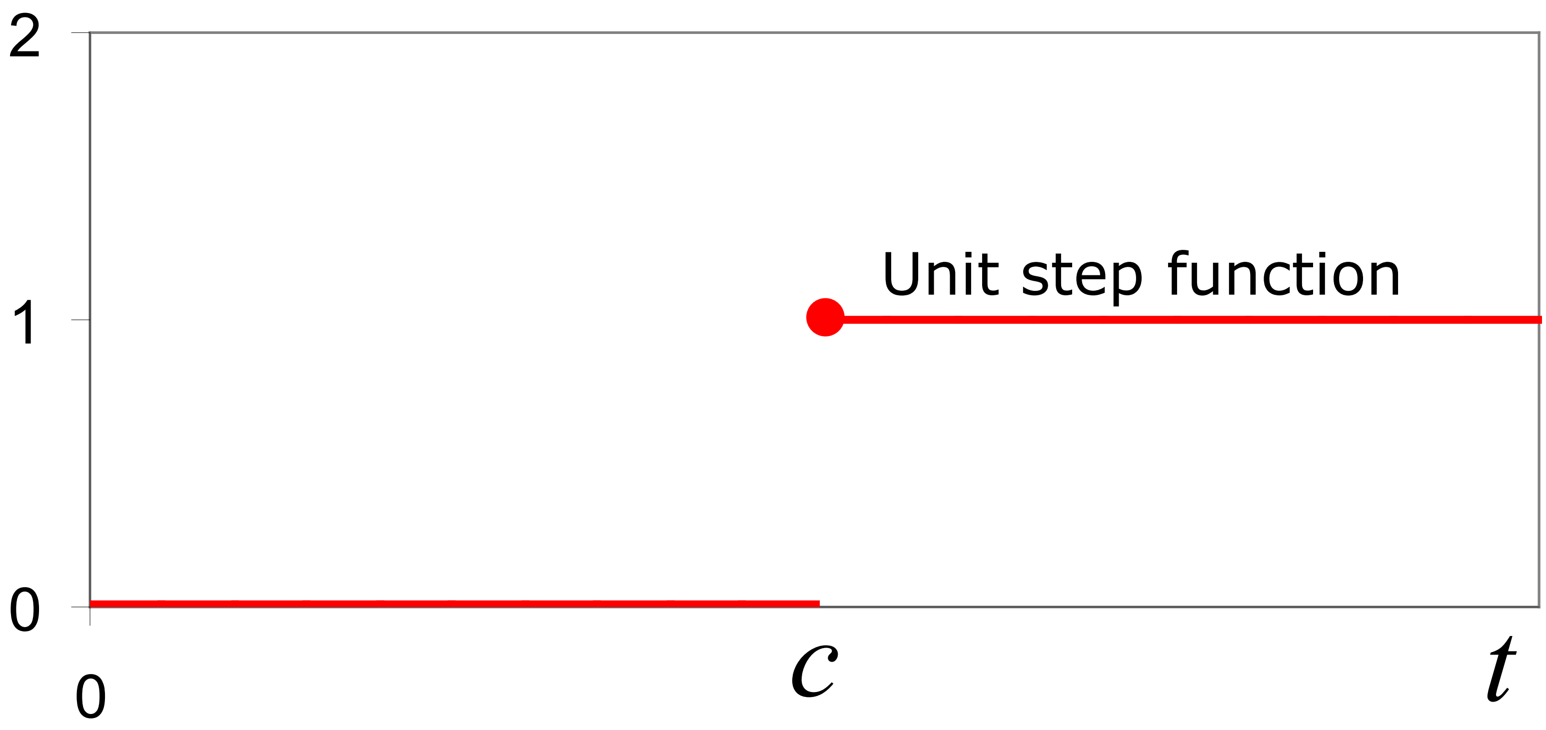
$$f(t) = \begin{cases} 0, & t < 0 \\ A & t > 0 \end{cases}$$




$$L[f(t)] = L[A] = \int_0^\infty A e^{-st} dt = 0 - (-\frac{A}{s}) = \frac{A}{s}$$

# Definition of Heaviside Function

$$u(t-c) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$



# Laplace Transform of Typical Functions (Cont'd)

- Exponential functions, $f(t) = e^{-at}$

$$F(s) = L[e^{-at}] = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{-1}{s+a} [e^{-(s+a)t}]_0^\infty$$

$$= -\frac{1}{s+a}[0-1] = \frac{1}{s+a}$$

Similarly *for* $f(t) = eat$*:*

$$F(s) = L[e^{at}] = \frac{1}{s-a}$$

# Revision 1

Compute the Laplace transform of $f(t) = 1$.

Answer: $$L\{f(t)\} = \lim_{A \to \infty} \int_0^A e^{-st}(1)\,dt = \lim_{A \to \infty} \left[-\frac{1}{s} e^{-st}\right]_0^A$$

$$= \lim_{A \to \infty} -\frac{1}{s}\left[e^{-As} - 1\right]$$

$$= \begin{cases} 1/s & s > 0 \\ \text{undefined} & s = 0 \\ \infty & s < 0 \end{cases}$$

$$\boxed{\therefore L\{1\} = \frac{1}{s}}$$

# Revision 2

In a similar manner, we can obtain the Laplace transform of the following functions:

$$\boxed{\begin{aligned} L\{e^{at}\} &= \frac{1}{s-a} \\ L\{\sin \omega t\} &= \frac{\omega}{s^2+\omega^2} \\ L\{\cos \omega t\} &= \frac{s}{s^2+\omega^2} \end{aligned}}$$

$a \in \Re$

$\omega \in \Re$

# Laplace Transform of Typical Functions (Cont'd)

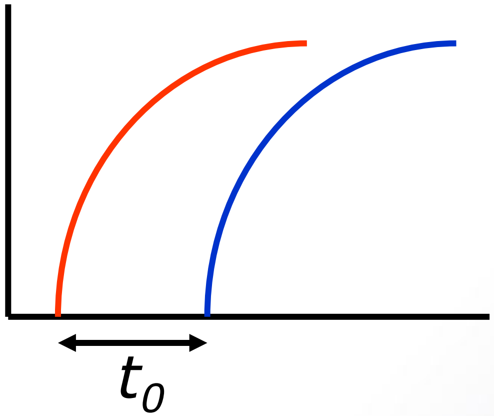
- Derivatives

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + \int_0^\infty s\, e^{-st} f(t) dt$$

$$= [0 - f(0)] + s\int_0^\infty f(t) e^{-st} dt = sF(s) - f(0)$$

- Time Delay Function

$$L[f(t - t_0)] = e^{-st_0} F(s)$$




**In inverting a function that contains $e^{-st}$, first invert $F(s)$ then replace $t$ by $(t - to)$ and multiply the entire function by $S(t - t_0)/u(t - t_0)$.**

# Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$

then $\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$

Proof:

Recall: $L\{f'(t)\} = sF(s) - f(0)$

Note that $f''$ is just the first order derivative of $f'$.

$$\begin{aligned} \therefore L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s[sF(s) - f(0)] - f'(0) \\ &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

# Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$

Then, in general:

$$\boxed{L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)}$$

For example:

$$\begin{aligned} L\{f''''(t)\} &= sL\{f'''(t)\} - f'''(0) \\ &= \cdots \\ &= s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0) \end{aligned}$$

# Summary of Laplace Transform Application

Given below are the summary of Laplace transform equations we have covered:

$$L\{1\} = \frac{1}{s}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

Memorize all of these.

# Laplace Transform Properties

- Property 1: Linearity
  - Principle of superposition holds

$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)]$$

*but,*

$$L(f_1(t) f_2(t)) \neq F_1(s) F_2(s)$$

# Laplace Transform Properties

- Property 1: Linearity

  Given below are some examples of linearity:

$$\cosh a\,x = \frac{e^{ax} + e^{-ax}}{2}$$

$$L\{\cosh a\,x\} = L\left\{\frac{e^{ax} + e^{-ax}}{2}\right\} = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2 - a^2}$$

$$\sinh a\,x = \frac{e^{ax} - e^{-ax}}{2}$$

$$L\{\sinh a\,x\} = L\left\{\frac{e^{ax} - e^{-ax}}{2}\right\} = \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{a}{s^2 - a^2}$$

# Laplace Transform Properties

- Property 2: Multiplication of $f(t)$ by $t$

$$L\{tf(t)\} = -\frac{d}{ds}L\{f(t)\}$$

$$L\{tf(t)\} = -\frac{d}{ds}F(s)$$

- If we know the Laplace transform $F(s)$ of $f(t)$, we can just apply Property 2 to determine the Laplace of $tf(t)$.

# Laplace Transform Properties

- Example 1: Compute the Laplace transform of *tsin(3t).*

Answer:

From Property 2:

$$L\{t\sin(3t)\} = -\frac{d}{ds}L\{\sin 3\,t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$L\{t\sin(3t)\} = -(-1)(2s)\frac{3}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

# Laplace Transform Properties

- Property 3: "s shifting"

If $$L\{f(t)\} = \lim_{A\to\infty}\int_0^A e^{-st} f(t)dt\ (s)$$

then $$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

Proof: $$L\{e^{at} f(t)\} = \lim_{A\to\infty}\int_0^A e^{-st} e^{at} f(t)dt$$

$$= \lim_{A\to\infty}\int_0^A e^{-(s-a)t} f(t)dt\ (s-a)$$

If we know the Laplace transform of $f(t)$ to $F(s)$, to determine the Laplace of $e^{at} f(t)$, we just replace every $s$ term in $F(s)$ by $(s - a)$.

# Laplace Transform Properties

- Example 2: Compute the Laplace transform of $e^{3t}\sin t$.

Answer: $$L\{\sin t\} = \frac{1}{s^2+1}$$

By property 3: $$L\{e^{3t}\sin t\} = \frac{1}{(s-3)^2+1}$$

# Laplace Transform Properties

- Example 3: What is the inverse Laplace transform of

$$\frac{s-7}{81+(s-7)^2} \quad ?$$

Answer:

Observe that:

By property 3: $\dfrac{s-7}{81+(s-7)^2} = L(e^{7t}\cos 9\,t)$

# Other Laplace Properties

- Initial and Final Value theorem
  - Useful in evaluating the initial and final value of a function

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s)$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

- Laplace of an Integral

# Summary of Laplace Properties

Given below is the Laplace Transform of Various Time-Domain Functions.

| $f(t)$ | | $F(s)$ |
|---|---|---|
| 1. $\delta(t)$ | (unit impulse) | $1$ |
| 2. $S(t)$ | (unit step) | $\frac{1}{s}$ |
| 3. $t$ | (ramp) | $\frac{1}{s^2}$ |
| 4. $t^{n-1}$ | | $\frac{(n-1)!}{s^n}$ |
| 5. $e^{-bt}$ | | $\frac{1}{s+b}$ |
| 6. $\frac{1}{\tau} e^{-t/\tau}$ | | $\frac{1}{\tau s+1}$ |
| 7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$ | $(n > 0)$ | $\frac{1}{(s+b)^n}$ |
| 8. $\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$ | | $\frac{1}{(\tau s+1)^n}$ |
| 9. $\frac{1}{b_1 - b_2}(e^{-b_2 t} - e^{-b_1 t})$ | | $\frac{1}{(s+b_1)(s+b_2)}$ |
| 10. $\frac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$ | | $\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$ |
| 11. $\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$ | | $\frac{s+b_3}{(s+b_1)(s+b_2)}$ |

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.54-55). Hoboken, NJ: Wiley.

# Summary of Laplace Properties

Given below is the Laplace Transform of Various Time-Domain Functions.

| | | |
|---|---|---|
| 12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$ | | $\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$ |
| 13. $1-e^{-t/\tau}$ | | $\frac{1}{s(\tau s+1)}$ |
| 14. $\sin \omega t$ | | $\frac{\omega}{s^2+\omega^2}$ |
| 15. $\cos \omega t$ | | $\frac{s}{s^2+\omega^2}$ |
| 16. $\sin(\omega t+\phi)$ | | $\frac{\omega \cos\phi + s\sin\phi}{s^2+\omega^2}$ |
| 17. $e^{-bt}\sin \omega t$ | $b, \omega$ real | $\frac{\omega}{(s+b)^2+\omega^2}$ |
| 18. $e^{-bt}\cos \omega t$ | $b, \omega$ real | $\frac{s+b}{(s+b)^2+\omega^2}$ |

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.54-55). Hoboken, NJ: Wiley.

# Summary of Laplace Properties

Given below is the Laplace Transform of Various Time-Domain Functions.

| | |
|---|---|
| 19. $\dfrac{1}{\tau\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin\left(\sqrt{1-\zeta^2}\,t/\tau\right)$ <br> $(0 \le \lvert\zeta\rvert < 1)$ | $\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$ |
| 20. $1 + \dfrac{1}{\tau_2-\tau_1}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)$ <br> $(\tau_1 \neq \tau_2)$ | $\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$ |
| 21. $1 - \dfrac{1}{\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin\left[\sqrt{1-\zeta^2}\,t/\tau + \psi\right]$ <br> $\psi = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le \lvert\zeta\rvert < 1)$ | $\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$ |
| 22. $1 - e^{-\zeta t/\tau}\left[\cos\left(\sqrt{1-\zeta^2}\,t/\tau\right) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,t/\tau\right)\right]$ <br> $(0 \le \lvert\zeta\rvert < 1)$ | $\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$ |

Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.54-55). Hoboken, NJ: Wiley.

# Summary of Laplace Properties

Given below is the Laplace Transform of Various Time-Domain Functions.

| $f(t)$ | $F(s)$ |
|---|---|
| 23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ <br> $(\tau_1 \neq \tau_2)$ | $\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$ |
| 24. $\frac{df}{dt}$ | $sF(s) - f(0)$ |
| 25. $\frac{d^n f}{dt^n}$ | $s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$ |
| 26. $f(t - t_0) S(t - t_0)$ | $e^{-t_0 s} F(s)$ |

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

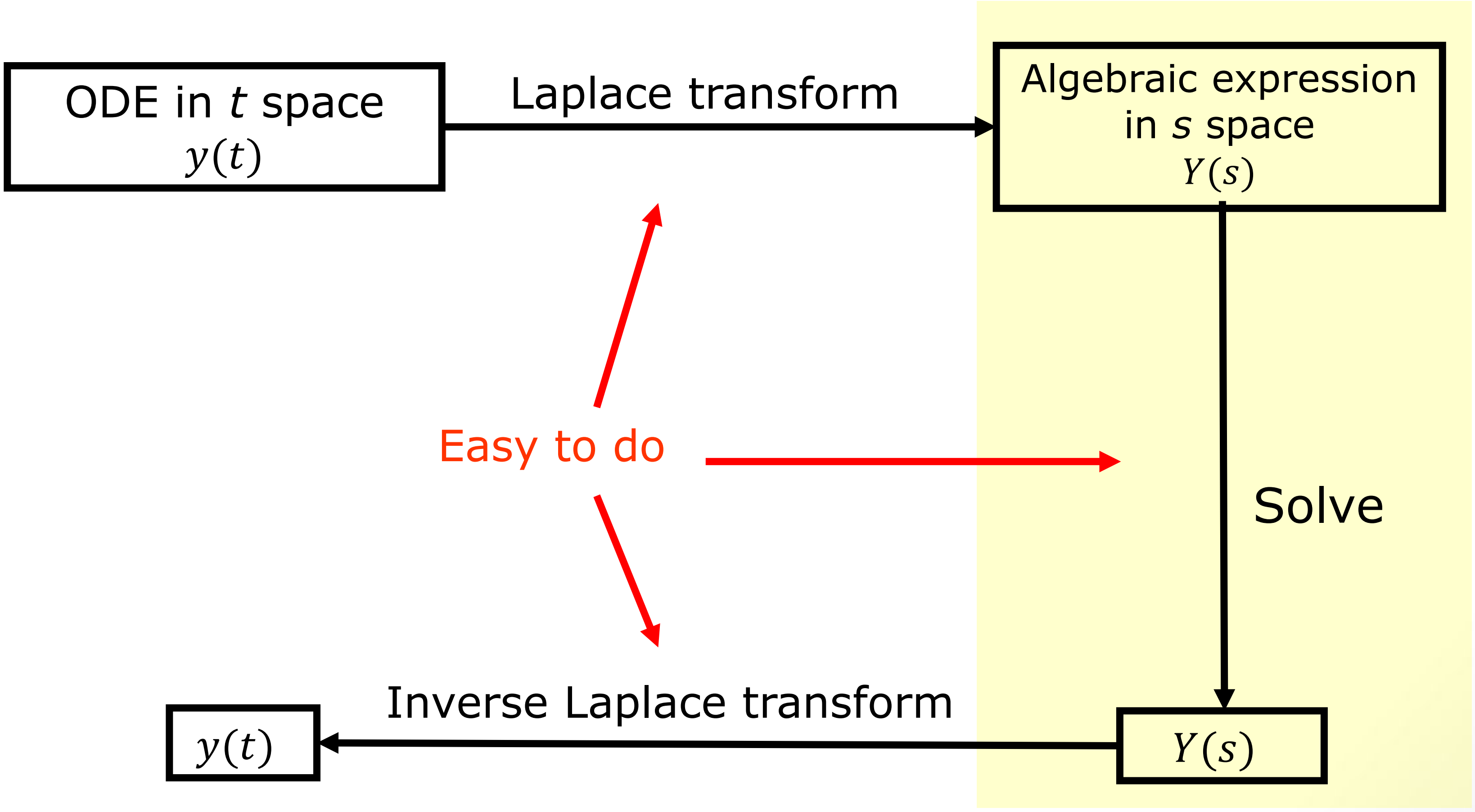
Source: Seborg, D. E., Edgar, T. F., Mellichamp, D. A., & Doyle, F. J., (2011). *Process dynamics and control* (3rd ed.)(pp.54-55). Hoboken, NJ: Wiley.

# Solutions of Differential Equations by Laplace

## Procedure:

- Step 1
  - Laplace transform both sides of the differential equation.
    - Substitute values for initial conditions in the transforms.
- Step 2
  - Rearrange the algebraic equation and solve for dependent variable.
- Step 3
  - Finally, find the inverse of the transformed output variable using partial fraction expansion.

# How Laplace Transforms Work



# Inverse Laplace Transform

- To get the solution in time-domain, we need inverse Laplace transform defined as:

$$L^{-1}[F(s)] \equiv f(t)$$

- It is obtained from Laplace transform table.

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

- What if the LT table does not contain a particular $F(s)$?

# Inverse Laplace Transform (Cont'd)

- $F(s)$ is decomposed into components with known LT, e.g. using partial fraction expansion (PFE).

$$F(s) = F_{1(s)} + F_{2(s)} + \cdots + F_n(s)$$

$$\Downarrow$$

$$f(t) = L^{-1}[F_{1(s)}] + L^{-1}[F_{2(s)}] + \cdots + L^{-1}[F_n(s)]$$

- Each term can be inverted independently as inverse LT is also linear.

# Partial Fraction Expansion (PFE)

- Systematic approach for decomposing a complex expression into simpler terms.
- First, we need to write $F(s)$ as a ratio of two polynomials in $s$:

$$F(s) = \frac{p(s)}{q(s)}$$

- Now, three different cases may arise:
  - All roots of $q(s)$ are <u>real and distinct.</u>
  - Some of the roots of $q(s)$ are <u>complex.</u>
  - Some of the roots of $q(s)$ are <u>repeated.</u>

# PFE: Real Distinct Roots

- When $q(s)$ has $n$ distinct real roots, $F(s)$ can be written as:

$$F(s) = \frac{p(s)}{q(s)}$$

$$F(s) = \frac{\alpha_1}{(s-\beta_1)} + \frac{\alpha_2}{(s-\beta_2)} + \ldots .. + \frac{\alpha_n}{(s-\beta_n)}$$

$\alpha_i, \beta_i$ are real numbers

- To find $\alpha_i$, multiply by $(s-\beta_i)$ and evaluate at $s = \beta_i$.

# PFE: Complex Roots

- When $q(s)$ has complex conjugate roots, PFE has the form:

$$F(s) = \frac{p(s)}{q(s)} = \frac{c_1 s + c_0}{s^2 + d_1 s + d_0} \qquad d_1^2 < 4d_0$$

$$F(s) = \frac{\alpha + j\beta}{s + b + j\omega} + \frac{\alpha - j\beta}{s + b - j\omega}$$

- To find $(\alpha + j\beta)$, multiply by $(s + b + j\omega)$ and evaluate at $s = -b - j\omega$, $(\alpha - j\beta)$ is the conjugate.

# PFE: Repeated Roots

- If r roots of $q(s)$ are repeated PFE has the form:

$$Y(s) = \frac{p(s)}{q(s)} = \frac{\alpha_1}{(s+b)} + \frac{\alpha_2}{(s+b)^2} + \ldots + \frac{\alpha_r}{(s+b)^r} + \ldots$$

- Find the coefficient

# Other Examples

Example 4:

- 2nd order process: $$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+y=Ku$$

- Step 1: Laplace Transform:

$$[as^2+bs+1]\cdot Y(s)=KU(s)$$

- Step 2: Rearrange and solve for dependent variable:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K}{as^2+bs+1}$$

2 roots

$$s_{1,2}=\frac{-b\pm\sqrt{b^2-4a}}{2a}$$

$\frac{b^2}{4a}>1$ : real roots

$\frac{b^2}{4a}<1$ : imaginary roots

# Other Examples

Example 4:

- Y(s) = $\frac{2}{3s^2+4s+1}$U(s) $\qquad \frac{b^2}{4a} = \frac{16}{12} = 1.333 > 1$
- Step 3:
  - PFE $3s^2 + 4s + 1 = (3s+1)(s+1) = 3(s + {}^{1}/_{3})(s+1)$
  - *Inverse: transform to* $e^{-t/3}, e^{-t}$ *(real roots)*
- Y(s) modified to $\frac{s+2}{s^2+s+1}$ $\qquad \frac{b^2}{4a} = \frac{1}{4} < 1$ (no oscillation)

$$s^2 + s + 1 = (s + 0.5 + \frac{\sqrt{3}}{2}j)(s + 0.5 - \frac{\sqrt{3}}{2}j)$$

$$y(t) = e^{-0.5t}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}e^{-0.5t}\sin\frac{\sqrt{3}}{2}t$$ (oscillation)

# Other Examples

Example 4:

- From line 17, table A.1 in the textbook:

$$e^{-bt}\sin\omega t \overset{L}{\longleftrightarrow} \frac{\omega}{(s+b)^2+\omega^2}$$

$$\frac{2}{s^2+s+1} = \frac{2}{(s+0.5)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

# Other Examples

Example 5:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 4\frac{du}{dt} + 2u$$

$$y(0) = y'(0) = y''(0) = 0$$

$\dfrac{du}{dt} = u = 0$ At $t = 0$ system at rest (s.s.)

- Transient response for a unit step function at $t > 0$
- Also evaluate the final value and initial value
- Suggested reading: Appendix A of the textbook

# Summary

In this chapter, we have covered:

- Application of Laplace transform techniques to solve linear differential equations
- Properties of Laplace transform

Suggested Reading: Appendix A of SEMD (Seborg, Edgar, Mellichamp and Doyle *Third Edition*)

# Chapter 3: Laplace Transforms

**The End**