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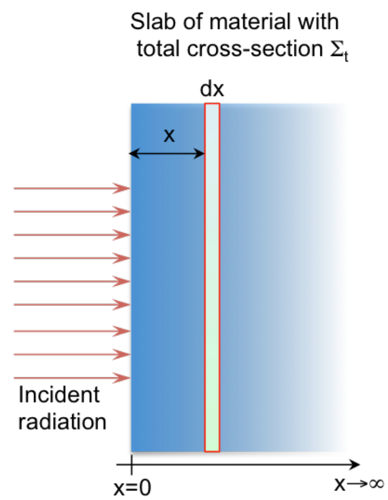
RADI.6060 Monte Carlo Simulation of Radiation Transport
Homework 2, Due March 21, 2024

1. Consider the geometry shown in the figure. A plane-parallel beam of 100 keV x-rays is normally incident on the left side of a semi-infinite water-equivalent parallelepiped. The dimension of the slab is infinite in the y and z directions, and semi-infinite in the x direction. The total interaction cross-section of water for these x-rays is $\Sigma_t = 0.1707$ cm^{-1}.

   We have shown in class that the first-collisional PDF is written as $f(x) = \Sigma_t e^{-\Sigma_t x}$. Here, $f(x)dx$ represents the probability that the incident particle will have its first interaction with the material (first collision) within *dx* about *x*. In class, we also derived the analytical expectation value of this PDF, which is the true mean distance the incident particles travel before their first collision: $E(x) = \int_0^\infty x f(x) dx = \int_0^\infty x \Sigma_t e^{-\Sigma_t x} dx = \frac{1}{\Sigma_t}$. This is also known as the mean free path (mfp). In addition, we also derived analytically the true variance, $var(x) = \frac{1}{\Sigma_t^2}$.



   A. Write a Monte Carlo code to sample the first collisional pdf and estimate the mean free path. Compare the result with the true mean free path for 100, 1,000, and 10,000 histories.
   B. Determine the unbiased variance and standard deviation for the same number of histories, and compare them to the true variance.
   C. Assume that the slab is a pure absorber and it is 10 cm thick, that is, $x \in [0,10]$. Estimate the uncollided leakage flux using MC sampling of 10 000 histories and compare it to the theoretical value, which may be obtained using the exponential attenuation formula.

   In all cases, do not forget to report the MC estimate as $\hat{x} \pm \hat{\sigma}$.

2. Consider a uniform homogeneous sphere made of water-equivalent material with radius R=10 cm. There is a radioactive source, emitting monoenergetic x-rays at 100 keV, which is uniformly distributed within this sphere. The total cross-section of water for these x-rays is $\Sigma_t = 0.1707$ cm^{-1}. We derived in class that the PDF for determining the radial position of a source particle for a spherical source is $f(r) = \frac{3r^2}{R^3}, r \in [0, R]$. We found that the expectation value, which is the mean radius at which particles are emitted, is $E(x) = \int_0^\infty x f(x) dx = \frac{3}{R^3} \int_0^R r^3 dr = \frac{3}{4} R$, and the true variance is $var(x) = \sigma^2 = \frac{3}{80} R^2$.

   A. Write a Monte Carlo code to sample the source pdf and estimate the mean radial position of a particle emission. Compare the result with the true mean for 100, 1,000, and 10,000 histories.
   B. Determine the unbiased variance and standard deviation for the same number of histories, and compare them to the true variance.
   C. Repeat A and B but at this time estimate the median radius of source particle emission.

Question 1 — Analysis

$$\int_0^\infty f(x)dx = \int_0^\infty \Sigma_t e^{-\Sigma_t x} dx = -e^{-\Sigma_t x}\Big|_0^\infty = 1$$ normalized ✓

$$CDF = F(x) = \Sigma_t \int_0^x e^{-\Sigma_t x'} dx' = -e^{-\Sigma_t x'}\Big|_0^x = 1 - e^{-\Sigma_t x}$$

$$\xi_i = F(x_i) \Rightarrow x_i = F^{-1}(\xi_i)$$

$$e^{-\Sigma_t x_i} = 1 - \xi_i \Rightarrow -\Sigma_t x_i = \ln(1-\xi_i) = \ln \xi_i \Rightarrow x_i = -\frac{\ln \xi_i}{\Sigma_t}$$

leakage flux

theoretical:

$$\bar{x} = e^{-\Sigma_t \cdot 10\,cm} = 0.18141$$

$$var = \bar{x} \Rightarrow \sigma = \sqrt{\bar{x}} = 0.42592$$

experimental:

$$R = \text{leakage},\quad S = 10^4 \longrightarrow \bar{x} = \frac{R}{S}$$

$$var = \sum_0^1 (x_i - \bar{x})^2 f_i$$

Question 2 — Analysis

$$\int_0^R f(r)dr = 3\int_0^R \frac{r^2}{R^3}dr = \frac{3}{3}\frac{r^3}{R^3}\Big|_0^R = 1$$ normalized ✓

$$\text{CDF} = F(r) = \frac{3}{R^3}\int_0^r r'^2 dr' = \frac{r^3}{R^3}$$

mean $= \frac{3}{4}R$

var $= \frac{3}{80}R^2$

$$\xi_i = F(r_i) \Rightarrow r_i = F^{-1}(\xi_i)$$

$$\xi_i = \frac{r_i^3}{R^3} \Rightarrow r_i = R\sqrt[3]{\xi_i}$$

<u>median:</u> $P(r \le \eta \cdot R) = \frac{1}{2}$

$$P(r \le \eta \cdot R) = F(\eta R) = \frac{1}{2} = \frac{3}{R^3}\int_0^{\eta R} r'^2 dr' = \frac{r'^3}{R^3}\Big|_0^{\eta R} = \eta^3$$

$$\eta = \sqrt[3]{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}} = 0.7937$$

$$\Rightarrow \text{median}(R) = 0.7937R = 7.937\,\text{cm}$$

<u>median variance:</u>

MC: $\text{medvar}(R) = \frac{1}{N}\sum_{i=1}^{N}(r_i - 0.7937R)^2$

analytical:

$$\text{medvar}(R) = \frac{3}{R^3}\int_0^R (r' - \text{median}(R))^2 r'^2 dr'$$

$$= \frac{3}{R^3}\int_0^R (r'^2 - 2\eta R r' + \eta^2 R^2) r'^2 dr'$$

$$= \frac{3}{R^3}\int_0^R (r'^4 - 2\eta R r'^3 + \eta^2 R^2 r'^2) dr'$$

$$= \frac{3}{R^3}\left[\frac{r^5}{5} - \frac{2\eta R r^4}{4} + \frac{\eta^2 R^2 r^3}{3}\right]_0^R$$

$$= \frac{3}{R^3}\left[\frac{R^5}{5} - \frac{\eta R^5}{2} + \frac{\eta^2 R^5}{3}\right] = \left[\frac{3}{5} - \frac{3}{2}\eta + \eta^2\right]R^2$$

$$= 0.03941\,R^2 = 3.941\,\text{cm}^2$$