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Natural Transformation

Key sources: `nLab:natural transformation`, *functor*

Summary. A natural transformation compares two functors by providing componentwise morphisms that commute with every arrow in the source category, so one functor transforms into the other uniformly.

Idea

Natural transformations are the morphisms between functors: they assign component maps that commute with every arrow in the source category, making the transformation uniform and functorial.

Prerequisites: *functor category*

Definition 1 (Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ assigns to each object $A \in \mathcal{C}$ a morphism $\eta_A : F(A) \to G(A)$ such that for every $f : A \to B$ in $\mathcal{C}$:

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. The morphisms η_A are called the *components* of η.

§2. Examples

Example 3 (Double dual)**.** For finite-dimensional vector spaces, there is a natural isomorphism $V \to V^{**}$ given by $v \mapsto (\varphi \mapsto \varphi(v))$.

Example 4 (Determinant)**.** $\det : \mathrm{GL}_n \Rightarrow (-)^\times$ is a natural transformation of functors $\mathsf{Ring} \to \mathsf{Grp}$.

See also:
- *functor*
- *category*

References

- `nLab:natural transformation`, *functor* | `nLab:natural transformation` |