Analysis of United Boundary-Domain Integro-Differential and Integral Equations to the Mixed BVP for a Compressible Stokes System with Variable Viscosity

T.G. Ayele and G.W. Hagos

Abstract. The mixed (Dirichlet-Neumann) boundary-value problem for a compressible Stokes system with variable viscosity of partial differential equations which is considered in a three-dimensional bounded domain is reduced to *united* boundary-domain integro-differential or integral equations (BDIDEs or BDIEs) based on a specially constructed parametrix. The BDIDPs/BDIDEs/BDIEs contain integral operators defined on the domain under consideration as well as potential-type operators defined on open sub-manifolds of the boundary and acting on the trace and/or traction of the unknown solution or on an auxiliary function. Some of the considered BDIDEs are to be supplemented by the original boundary conditions, thus constituting boundary-domain integro-differential problems (BDIDPs). Solvability, solution uniqueness, and we also prove equivalence of the BDIDPs/BDIDEs/BDIEs to the original BVP in appropriate Sobolev spaces.

Keywords: Mixed BVP, United, Parametrix, BDIEs, BDIDPs, BDIDEs and Equivalence.

1 Introduction

Boundary integral equations and the hydrodynamic potential theory for the Stokes PDE system with constant viscosity have been extensively studied in numerous publications, cf., e.g.,[**?**, LiMa73, **?**, **?**]. The reduction of different boundary value problems for the Stokes system to boundary integral equations in the case of constant viscosity was possible since the fundamental solution for both, velocity and pressure, are readily available in an explicit form. Such reduction was used not only to analyze the properties of the Stokes system and BVP solutions, but also to solve BVPs by solving numerically the corresponding boundary integral equations.

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In this paper, We consider the stationary Stokes PDE system with variable viscosity and compressibility, in a bounded domain that models the motion a laminar compressible viscous fluid, e.g., through a variable temperature field that makes both, viscosity and compressibility depending on coordinates. Reduction of the BVPs for the Stokes system with arbitrarily variable viscosity to explicit boundary integral equations is usually not possible. Since the fundamental solution needed for such reduction is generally not available in an analytical form (Except for some special dependence of the viscosity on coordinates). Using a *Parametrix (Levi function)* as a substitute of a fundamental solution, in the spirit of it is possible however to reduce such a BVPs to some systems of Boundary-Domain Integral Equations, BDIEs, (cf. e.g. [**?**, sec. 18], [Pom98] where the Dirichlet, Neumann and Robin problems for some PDEs were reduced to indirect BDIEs).

We will extend here the approach developed in [**?**] for a mixed boundary value problem with a compressible Stokes system of PDE reducing in to BDIEs of segregate approach, and will reduce the mixed boundary value problem for a compressible Stokes system of the partial differential to different systems of united direct BDIDP(E)s expressed in terms of surface and volume parametrix-based potential type operators. A parametrix for a given PDE (or PDE system) is not unique and special care will be taken to choose a parametrix that leads to the BDIDP(E)s systems simple analyzed. The mapping properties of the parametrix- based hydrodynamic surface and volume potentials will be obtained and equivalence theorems for the associated BDIDP(E) systems will be proved.

Note that the paper is mainly aimed not at the mixed BVP for the Stokes system, which properties are well-known nowadays, but rather analysis of the BDIDP(E) system per se. The analysis is interesting not only in its rights but it also to pave the way for studying the corresponding localized BDIDP(E)s and analyzing convergence and stability of BDIDP(E)-based numerical methods for PDEs, cf.,e.g., [Mik02, GMR13].

2 Formulation of the Mixed BVP

Let $\Omega = \Omega^+ \subset \mathbb{R}^3$ be a bounded and simply connected domain and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. We will assume that the boundary $\partial\Omega$ is simply connected, closed, infinitely smooth surface of dimension 2. Furthermore, $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ where $\partial\Omega_D$ and $\partial\Omega_N$ an open, non-empty, non-intersecting, simply connected manifold of $\partial\Omega$ with infinitely smooth boundary curve, $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \in C^\infty$ i.e. the interface between these two sub-manifolds is also infinitely differentiable.

Let $\boldsymbol{v}$ be the velocity vector field; p the pressure scalar field and $\mu \in C^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(x) > c > 0$. For an arbitrary couple $(p, \boldsymbol{v})$ the stress tensor operator σ_{ij} and the Stokes operator, $\mathcal{A}_j(p, \boldsymbol{v})$, for a compressible fluid is defined as

$$\sigma_{ij}(p, \boldsymbol{v})(x) := -\delta_i^j p(x) + \mu(x)\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} - \frac{2}{3}\delta_i^j div\, \boldsymbol{v}(x)\right),$$

$$\begin{aligned}\mathcal{A}_j(p, \boldsymbol{v})(x) &:= \frac{\partial}{\partial x_i}\sigma_{ij}(p, \boldsymbol{v})(x)\\ &= \frac{\partial}{\partial x_i}\left(\mu(x)\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} - \frac{2}{3}\delta_i^j div\, \boldsymbol{v}(x)\right)\right) - \frac{\partial p(x)}{\partial x_j}, \quad j, i \in \{1, 2, 3\}\end{aligned}$$

where δ_i^j is a Kronecker symbol.

Here and hereafter we assume the Einstein summation in repeated indices from 1 to 3. We also denote the Stokes operator as $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^3$ and $\mathring{\boldsymbol{\mathcal{A}}} := \boldsymbol{\mathcal{A}}|_{\mu=1}$.
We will also use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ with $j = 1, 2, 3$; $\nabla := (\partial_1, \partial_2, \partial_3)$.

In what follows $H^s(\Omega) = H_2^s(\Omega)$ and $H^s(\partial\Omega)$ are the Bessel potential spaces, where s is a real number see, e.g., [LiMa73, ?]. We recall that H^s coincide with the Sobolev-Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$, $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^3), \text{supp g} \subset \overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1) = \{g : g \in H^s(\partial\Omega), \text{supp g} \subset \overline{S_1}\}$ is the Sobolev space of functions having support in $S_1 \subset \partial\Omega$. We will also use the notations $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$, $\mathbf{L}_2(\Omega) = [L_2(\Omega)]^3$, $\boldsymbol{\mathcal{D}}(\Omega) = [\mathscr{D}(\Omega)]^3$ for 3-dimensional vector space.
We will also make use of the following spaces, see, e.g. [?, ?, ?].

$$\mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \boldsymbol{v}) \in H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) : \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \in \mathbf{L}_2(\Omega)\}$$

endowed with the norm

$$\|(p, \boldsymbol{v})\|_{\mathbf{H}^{s,0}(\Omega;\mathcal{A})} := \left(\|v\|^2_{\mathbf{H}^s(\Omega)} + \|p\|^2_{H^{s-1}(\Omega)} + \|\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})\|^2_{\mathbf{L}_2(\Omega)}\right)^{1/2}.$$

Remark 2.1. *Note that $\boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}) = \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ if $s \geq 1$.*
Indeed, $\mathcal{A}_j(p, \boldsymbol{v}) = \mu\mathring{\mathcal{A}}_j(p, \boldsymbol{v}) + B_j(p, \boldsymbol{v})$, where

$$B_j(p, \boldsymbol{v}) := \frac{\partial\mu(x)}{\partial x_i}\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} - \frac{2}{3}\delta_i^j div\, \boldsymbol{v}(x)\right) \in \boldsymbol{L}_2(\Omega)$$

if $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$ and $s \geq 1$.

To prove the following theorem one can follow similar fashion as in [?, Theorem 3.12]

Theorem 2.2. . *The space $\mathscr{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$, $s \in \mathbb{R}$.*

For sufficiently smooth functions $(p, \boldsymbol{v}) \in H^{s-1}(\Omega^\pm) \times \mathbf{H}^s(\Omega^\pm)$ with $s > 3/2$, we can state the classical traction operators, $\mathbf{T}^{c\pm} = \{T_j^{c\pm}\}_{j=1}^3$on the boundary $\partial\Omega$ as

$$\begin{aligned} T_j^{c\pm}(p, \boldsymbol{v})(x) &:= [\gamma^\pm\sigma_{ij}(p, \boldsymbol{v})](x)n_i(x) \\ &= -n_j(x)\gamma^\pm p(x) + n_i(x)\mu(x)\gamma^\pm\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} - \frac{2}{3}\delta_i^j div\, \boldsymbol{v}(x)\right), x \in \partial\Omega \end{aligned}$$

where $n_i(x)$ denote components of the unit outward normal vector $\mathbf{n}(x)$ to the boundary $\partial\Omega$ of the domain and $\gamma^\pm$ is the trace operator from inside and outside Ω. We will sometimes write γu if $\gamma^+ u = \gamma^- u$, and similarly for $\mathbf{T}^c$, etc.

The operator $\boldsymbol{\mathcal{A}}$ acting on $(p, \boldsymbol{v})$ is well defined in the weak sense provided $\mu(x) \in L^\infty(\Omega)$ as

$$\langle\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}), \boldsymbol{u}\rangle_\Omega := -\mathcal{E}((p, \boldsymbol{v}), \boldsymbol{u}) \quad \forall \boldsymbol{u} \in \tilde{\mathbf{H}}^{2-s}(\Omega), 1 \leq s < \frac{3}{2}$$

where the form $\mathcal{E}[H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega)] \times \tilde{\mathbf{H}}^{2-s} \to \mathbb{R}$ is defined as

$$\mathcal{E}((p, \boldsymbol{v}), \boldsymbol{u}) := \int_\Omega E((p, \boldsymbol{v}), \boldsymbol{u})(x)\, dx,$$

and the function $E((p,\boldsymbol{v}),\boldsymbol{u})(x)$ is defined as

$$E((p,\boldsymbol{v}),u)(x) := \frac{1}{2}\mu(x)\left(\frac{\partial u_i(x)}{\partial x_j}+\frac{\partial u_j(x)}{\partial x_i}\right)\left(\frac{\partial v_i(x)}{\partial x_j}+\frac{\partial v_j(x)}{\partial x_i}\right) - \frac{2}{3}\mu(x) div\,\boldsymbol{v}(x) div\,\boldsymbol{u}(x) - p(x) div\,\boldsymbol{u}(x).$$

Furthermore, if $(p,\boldsymbol{v})\in\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u}\in\mathbf{H}^{2-s}(\Omega)$ for $1\le s<\dfrac{3}{2}$ the following first Green identity holds, [?, Eq.(34.2)], cf. also [?, Lemma 3.4(i)], [?, Theorem 3.9]

$$\left\langle \mathbf{T}^+(p,\boldsymbol{v}),\gamma^+\boldsymbol{u}\right\rangle_{\partial\Omega} = \int_\Omega [\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{u}+E((p,\boldsymbol{v})\boldsymbol{u})]\,dx.$$

Applying the first Green identity to the pairs $(p,\boldsymbol{v}),(q,\boldsymbol{u})\in\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, cf. [?, Lemma 3.4(ii)], [?, Eq 4.8],

$$\int_\Omega [\mathcal{A}_j(p,\boldsymbol{v})u_j - \mathcal{A}_j(q,\boldsymbol{u})v_j + q div\,\boldsymbol{v} - p div\,\boldsymbol{u}]\,dx = \left\langle \mathbf{T}^+(p,\boldsymbol{v}),\gamma^+\boldsymbol{u}\right\rangle_{\partial\Omega} - \left\langle \mathbf{T}^+(q,\boldsymbol{u}),\gamma^+\boldsymbol{v}\right\rangle_{\partial\Omega}.$$

The classical traction operators can be continuously extended to the canonical traction operators $\mathbf{T}^\pm:\mathbf{H}^{s,0}(\Omega^\pm;\boldsymbol{\mathcal{A}})\to\mathbf{H}^{s-3/2}(\partial\Omega)$ for $1\le s<\frac{3}{2}$ defined in the weak form [?, section 34.1] similar to [?, Lemma 3.2], [?, Definition 3.8] with the help of first Green identity as

$$\left\langle \mathbf{T}^\pm(p,\boldsymbol{v}),\boldsymbol{w}\right\rangle_{\partial\Omega} := \pm\int_{\Omega^\pm}[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\gamma^{-1}\boldsymbol{w} + E((p,\boldsymbol{v}),\gamma^{-1}\boldsymbol{w})]\,dx \quad \forall (p,\boldsymbol{v})\in\mathbf{H}^{s,0}(\Omega^\pm;\boldsymbol{\mathcal{A}}), \forall\boldsymbol{w}\in\mathbf{H}^{3/2-s}(\partial\Omega).$$

Here the operator $\gamma^{-1}:\mathbf{H}^{s-1/2}(\partial\Omega)\to\mathbf{H}^s(\mathbb{R}^3)$ denotes a continuous right inverse of the trace operator $\gamma:\mathbf{H}^s(\mathbb{R}^3)\to\mathbf{H}^{s-1/2}(\partial\Omega)$.

Now we are ready to define the following mixed BVP for $1\le s<\dfrac{3}{2}$ for which we aim to derive equivalent BDIDP(E)s and investigate the existence and uniqueness of their solutions. For $\boldsymbol{f}\in\mathbf{L}_2(\Omega), g\in H^{s-1}(\Omega), \boldsymbol{\varphi}_0\in\mathbf{H}^{s-\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0\in\mathbf{H}^{s-\frac{3}{2}}(\partial\Omega_N)$, find $(p,\boldsymbol{v})\in\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ such that:

$$\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})(x) = \boldsymbol{f}(x), \quad x\in\Omega, \tag{1a}$$

$$div\,\boldsymbol{v} = g(x), \quad x\in\Omega, \tag{1b}$$

$$\gamma^+\boldsymbol{v}(x) = \boldsymbol{\varphi}_0(x), \quad x\in\partial\Omega_D, \tag{1c}$$

$$\mathbf{T}^+(p,\boldsymbol{v})(x) = \boldsymbol{\psi}_0(x), \quad x\in\partial\Omega_N. \tag{1d}$$

We have the following uniqueness theorem.

Theorem 2.3. *Mixed BVP (1a)–(1d) has a unique solution in the space* $\boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}), 1\le s<\dfrac{3}{2}$.

Proof. The proof for $s=1$ is provided in [?, Theorem 2.3], which evidently implies the theorem claims also for $s>1$. □

Corollary 2.4. *Let* $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega), g \in H^{s-1}(\Omega), \boldsymbol{\varphi}_0 \in \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega_D)$ *and* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega_N)$ *for* $1 \le s < \frac{3}{2}$. *Then, the BVP (1a)–(1d) is uniquely solvable in* $\boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ *and the operator*

$$A^{DN} : \boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) \to \boldsymbol{L}_2(\Omega) \times \in H^{s-1}(\Omega) \times \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega_D) \times \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega_N)$$

is continuously invertible.

Proof. For $s = 1$ the proof is provided in Reference [?, Corollary 7.4], which evidently implies the corollary claims also for $s > 1$. □

3 Parametrix and Parametrix-Based Hydrodynamic Potentials

3.1 Parametrix and Remainder

When $\mu = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes the constant-coefficient Stokes operator $\mathring{\boldsymbol{\mathcal{A}}}$, for which we know an explicit fundamental solution defined by the pair of distributions $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$, where $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution, see e.g. [?, ?, ?, ?, ?, Ste07].

$$\mathring{q}^k(x,y) = \frac{-(x_k - y_k)}{4\pi|x-y|^3} = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi|x-y|}\right), \tag{2}$$

$$\mathring{u}_j^k(x,y) = -\frac{1}{8\pi}\left(\frac{\delta_j^k}{|x-y|} + \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^3}\right), \quad j,k \in \{1,2,3\}. \tag{3}$$

Therefore $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$ satisfies

$$\frac{\partial}{\partial x_k}\mathring{q}^k(x,y) = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{4\pi|x-y|}\right) = -\delta(x-y),$$

$$\mathring{\mathcal{A}}_j(x)(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y)) = \sum_{i=1}^{3}\frac{\partial^2 \mathring{u}_j^k(x,y)}{\partial x_i^2} - \frac{\partial \mathring{q}^k(x,y)}{\partial x_j} = \delta_j^k\delta(x-y), \quad div_x \mathring{\boldsymbol{u}}^k(x,y) = 0.$$

Let us denote $\mathring{\sigma}_{ij}(p,\boldsymbol{v}) := \sigma_{ij}(p,\boldsymbol{v})|_{\mu=1}, \mathring{T}_j^c(p,\boldsymbol{v}) := T_j^c(p,\boldsymbol{v})|_{\mu=1}$. Then in the particular case, for $\mu = 1$ and the fundamental solution $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)_{k=1,2,3}$ of the operator $\mathring{\boldsymbol{\mathcal{A}}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(x,y)$ reads

$$\mathring{\sigma}_{ij}(x)(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y)) = \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^5}.$$

And the classical boundary traction of the fundamental solution becomes,

$$\begin{aligned}\mathring{T}_j^c(x)(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y)) &:= \mathring{\sigma}_{ij}(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y))n_i(x)\\ &= \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^5}n_i(x).\end{aligned}$$

Let us define a pair of functions $(q^k, \boldsymbol{u}^k)_{k=1,2,3}$ as

$$q^k(x,y) = \frac{\mu(x)}{\mu(y)}\mathring{q}^k(x,y) = \frac{\mu(x)}{\mu(y)}\frac{y_k - x_k}{4\pi|x-y|^3}, \quad j,k \in \{1,2,3\}.$$

$$u_j^k(x,y) = \frac{1}{\mu(y)}\mathring{u}_j^k(x,y) = -\frac{1}{8\pi\mu(y)}\left(\frac{\delta_j^k}{|x-y|} + \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^3}\right).$$

Then

$$\sigma_{ij}(x)(q^k(x,y), \boldsymbol{u}^k(x,y)) = \frac{\mu(x)}{\mu(y)}\mathring{\sigma}_{ij}(x)(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y)),$$

$$T_j(x)(q^k(x,y), \boldsymbol{u}^k(x,y)) := \ \sigma_{ij}(x)(q^k(x,y), \boldsymbol{u}^k(x,y))n_i(x) = \frac{\mu(x)}{\mu(y)}\mathring{T}_j(x)(\mathring{q}^k(x,y), \mathring{\boldsymbol{u}}^k(x,y)).$$

We will say that an order pair $(q^k, \boldsymbol{u}^k)$ of two variables $x, y \in \Omega$ is a parametrix for the Stokes operator $\mathcal{A}_j$ if it satisfy;

$$\mathcal{A}_j(x)(q^k(x,y), \boldsymbol{u}^k(x,y)) = \delta_j^k\delta(x-y) + R_{kj}(x,y),$$

where $\delta(x-y)$ is the Dirac distribution and

$$\begin{aligned} R_{kj}(x,y) &= \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{u}^k, \mathring{q}^k)(x-y) \\ &= \frac{3}{4\pi\mu(y)}\frac{\partial\mu(x)}{\partial x_i}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^5} = \mathcal{O}(|x-y|^{-2}) \end{aligned}$$

is the remainder possesses a weak (integrable) singularity at $x = y$, i.e. $R_{kj}(x,y) = O(|x-y|^{\alpha})$ with $\alpha < 3$.

3.2 Volume and Surface Potentials Based on Parametrix

Definition 3.1. *Let ρ and $\boldsymbol{\rho}$ be sufficiently smooth scalar and vector functions on $\overline{\Omega}$, e.g., $\rho \in \mathscr{D}(\overline{\Omega}), \boldsymbol{\rho} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. The parametrix-based Newton type and remainder vector potentials for the velocity,*

$$\boldsymbol{\mathcal{U}}_k\boldsymbol{\rho}(y) = \mathcal{U}_{kj}\rho_j(y) := \int_\Omega u_j^k(x,y)\rho_j(x)\,dx, \tag{5}$$

$$\boldsymbol{\mathcal{R}}_k\boldsymbol{\rho}(y) = \mathcal{R}_{kj}\rho_j(y) := \int_\Omega R_{kj}(x,y)\rho_j(x)\,dx, \tag{6}$$

and the scalar Newton-type and remainder potentials for the pressure,

$$\boldsymbol{\mathcal{Q}}\rho(y) = \mathcal{Q}_j\rho(y) := \int_\Omega q^j(y,x)\rho(x)\,dx = -\int_\Omega q^j(x,y)\rho(x)\,dx, \tag{7}$$

$$\boldsymbol{\mathcal{Q}}\boldsymbol{\rho}(y) = \boldsymbol{\mathcal{Q}}\cdot\boldsymbol{\rho}(y) = \mathcal{Q}_j\rho_j(y) := \int_\Omega q^j(y,x)\rho_j(x)\,dx = -\int_\Omega q^j(x,y)\rho_j(x)\,dx, \tag{8}$$

$$\mathcal{R}^{\bullet}\boldsymbol{\rho}(y) = \mathcal{R}^{\bullet}_j\rho_j(y) := -2\,p.v.\int_{\Omega}\frac{\partial\mathring{q}^j(x,y)}{\partial x_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(x)\,dx - \frac{4}{3}\rho_j\frac{\partial\mu}{\partial y_j}, \tag{9}$$

$$= -2\left\langle\partial_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\Omega} - 2\rho_i(y)\partial_i\mu(y), \tag{10}$$

for $y\in\mathbb{R}^3$.

The integral in (9) is understood as a 3D strongly singular integral in the Cauchy sense(Cauchy principal value sense). The bi-linear form in (10) should be understood in the sense of distribution, and the equality (9) and (10) hold since

$$\begin{aligned}\left\langle\partial_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\Omega} &= -\left\langle\mathring{q}^j(.,y),\partial_i(\rho_i\partial_j\mu)\right\rangle_{\Omega} + \left\langle n_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\partial\Omega}\\ &= -\lim_{\epsilon\to 0}\left\langle\mathring{q}^j(.,y),\partial_i(\rho_i\partial_j\mu)\right\rangle_{\Omega_\epsilon} + \left\langle n_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\partial\Omega}\\ &= \lim_{\epsilon\to 0}\left\langle\partial_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\Omega_\epsilon} - \lim_{\epsilon\to 0}\left\langle n_i\mathring{q}^j(.,y),\rho_i\partial_j\mu\right\rangle_{\partial\Omega_\epsilon\setminus\partial\Omega}\\ &= p.v.\int_{\Omega}\frac{\partial\mathring{q}^j(x,y)}{\partial x_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(x)\,dx - \frac{1}{3}\rho_j\frac{\partial\mu}{\partial y_j}\end{aligned}$$

where $\Omega_\epsilon = \Omega\setminus\overline{B}_\epsilon(y)$ and $B_\epsilon(y)$ is the ball of radius ϵ and centered in y which implies that

$$\begin{aligned}-2\left\langle\partial_i\mathring{q}_j(.,y),\rho\partial_j\mu\right\rangle_{\Omega} - 2v_i(y)\partial_i\mu(y) &= -2v.p\int_{\Omega}\frac{\partial\mathring{q}^j(x,y)}{\partial x_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(x)\,dx - \frac{4}{3}\rho_j\frac{\partial\mu}{\partial y_j}\\ &= \mathcal{R}^{\bullet}\boldsymbol{\rho}(y).\end{aligned}$$

Let us now define the parametrix-based velocity single layer potential, double layer potential and their respective direct values on the boundary, as follows:

Definition 3.2. *For the velocity, the parametrix-based single layer and double layer potentials are defined for* $y\notin\partial\Omega$,

$$\begin{aligned}\mathbf{V}_k\boldsymbol{\rho}(y) = V_{kj}\rho_j(y) &:= -\int_{\partial\Omega}u^k_j(x,y)\rho_j(x)\,dS_x,\\ \mathbf{W}_k\boldsymbol{\rho}(y) = W_{kj}\rho_j(y) &:= -\int_{\partial\Omega}T^c_j(x;q^k,u^k)(x,y)\rho_j(x)\,dS_x.\end{aligned}$$

For pressure in the variable coefficient Stokes system, the single layer and double layer potentials are defined for $y\notin\partial\Omega$,

$$\begin{aligned}\Pi^s\boldsymbol{\rho}(y) = \Pi^s_j\rho_j(y) &:= \int_{\partial\Omega}\mathring{q}^j(x,y)\rho_j(x)\,dS_x,\\ \Pi^d\boldsymbol{\rho}(y) = \Pi^d_j\rho_j(y) &:= 2\int_{\partial\Omega}\frac{\partial\mathring{q}^j(x,y)}{\partial n(x)}\mu(x)\rho_j(x)dS_x.\end{aligned}$$

It is easy to observe that the parametrix-based integral operators, with the variable coefficient μ, can be expressed in terms of the corresponding integral operators for the constant coefficient case $\mu = 1$, marked by

$$\boldsymbol{\mathcal{U}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \tag{11}$$

$$\boldsymbol{\mathcal{R}}\boldsymbol{\rho} = -\frac{1}{\mu}\left[\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu) + \frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu) + \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)\right], \tag{12}$$

$$\boldsymbol{\mathcal{Q}}\rho = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu\rho), \tag{13}$$

$$\mathcal{R}^{\bullet}_j\rho_j = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu) - 2\rho_j\frac{\partial\mu}{\partial y_i}, \tag{14}$$

$$\mathbf{V}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{V}}\boldsymbol{\rho}, \qquad \mathbf{W}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\rho}), \tag{15}$$

$$\Pi^s\boldsymbol{\rho} = \mathring{\Pi}^s\boldsymbol{\rho}, \qquad \Pi^d\boldsymbol{\rho} = \mathring{\Pi}^d(\mu\boldsymbol{\rho}). \tag{16}$$

Note that to show the above relations (11)–(16) hold, we simply used their corresponding definitions.

Also, note that although the constant coefficient velocity potentials $\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \mathring{\boldsymbol{V}}\boldsymbol{\rho}$ and $\mathring{\boldsymbol{W}}\boldsymbol{\rho}$ are divergence-free in $\Omega^{\pm}$, the corresponding potentials $\boldsymbol{\mathcal{U}}\boldsymbol{\rho}, \boldsymbol{V}\boldsymbol{\rho}$ and $\boldsymbol{W}\boldsymbol{\rho}$ are not divergence-free for the variable coefficient $\mu(y)$. Note also that by (2) and (7),

$$\mathring{\mathcal{Q}}_j\rho = -\partial_j\mathcal{P}_{\Delta}\rho, \tag{17}$$

where

$$\mathcal{P}_{\Delta}\rho(y) = -\frac{1}{4\pi}\int_{\Omega}\frac{1}{|x-y|}\rho(x)\,dx$$

is the harmonic Newtonian potential. Hence

$$div\,\mathring{\boldsymbol{\mathcal{Q}}}\rho = \partial_j\mathring{\mathcal{Q}}_j\rho = -\Delta\mathcal{P}_{\Delta}\rho = -\rho. \tag{18}$$

Moreover, for the constant coefficient potentials we have the following well known relations,

$$\begin{aligned}&\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^s\boldsymbol{\rho}, \mathring{\mathbf{V}}\boldsymbol{\rho}) = \mathbf{0}, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^d\boldsymbol{\rho}, \mathring{\mathbf{W}}\boldsymbol{\rho}) = \mathbf{0},\\ &\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\mathcal{Q}}\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}) = \boldsymbol{\rho} \ \text{ in } \Omega^{\pm}.\end{aligned}$$

In addition, by (17) and (18),

$$\begin{aligned}\mathring{\mathcal{A}}_j(\frac{4}{3}\rho, -\mathring{\boldsymbol{\mathcal{Q}}}\rho) &= -\partial_i\left(\partial_i\mathring{\mathcal{Q}}_j\rho + \partial_j\mathring{\mathcal{Q}}_i\rho - \frac{2}{3}\delta_i^j div\,\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - \frac{4}{3}\partial_j\rho\\ &= -\left(\Delta\mathring{\mathcal{Q}}_j\rho + \partial_j div\,\mathring{\boldsymbol{\mathcal{Q}}}\rho - \frac{2}{3}\partial_j div\,\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - \frac{4}{3}\partial_j\rho = 0.\end{aligned}$$

The following assertions of this section are well-known for the constant coefficient case, see e.g. [**?**, **?**]. Then, by relations (11)–(16) we obtain their counterparts for the variable coefficient case.

The following theorem is proved in [**?**, Theorem 4.1]

Theorem 3.3. *The following operators are continuous ,*

$$\boldsymbol{\mathcal{U}} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{19}$$

$$\boldsymbol{\mathcal{U}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{20}$$

$$\boldsymbol{\mathcal{Q}} : \widetilde{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{21}$$

$$\boldsymbol{\mathcal{Q}} : H^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{22}$$

$$\mathcal{Q} : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{23}$$

$$\mathcal{Q} : \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{24}$$

$$\boldsymbol{\mathcal{R}} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{25}$$

$$\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{26}$$

$$\mathcal{R}^\bullet : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}, \tag{27}$$

$$\mathcal{R}^\bullet : \boldsymbol{H}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{28}$$

$$(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}) : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 0, \tag{29}$$

$$(\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}) : H^{s-1}(\Omega) \to \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 1, \tag{30}$$

$$(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 1. \tag{31}$$

The following result is proved in [**?**, Theorem 4.2]

Proposition 3.4. *Let $s > 1/2$ The following operators are compact,*

$$\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^s(\Omega), \qquad \mathcal{R}^\bullet : \boldsymbol{H}^s(\Omega) \to H^{s-1}(\Omega),$$
$$\gamma^+\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega), \qquad T^\pm(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{H}(\Omega; \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega).$$

The following assertion can be proved similar to [**?**, Theorem 4.3] besides for (34), which evidently implies the claim also for $s > 1$.

Theorem 3.5. *The following operators are continuous.*

$$\boldsymbol{V} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+3/2}(\Omega), \quad \boldsymbol{W} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1/2}(\Omega), \quad s \in \mathbb{R}, \tag{32}$$

$$\Pi^s : \boldsymbol{H}^{s-3/2}(\partial\Omega) \to H^{s-1}(\Omega), \quad \Pi^d : \boldsymbol{H}^{s-1/2}(\partial\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}, \tag{33}$$

$$(\Pi^s, \boldsymbol{V}) : \boldsymbol{H}^{s-3/2}(\partial\Omega) \to \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), \; (\Pi^d, \boldsymbol{W}) : \boldsymbol{H}^{s-1/2}(\partial\Omega) \to \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), 1 \leq s. \tag{34}$$

Let us now define direct values on the boundary of the parametrix-based velocity single layer and double layer potentials and introduce the notations for the conormal derivative of the latter,

$$\boldsymbol{\mathcal{V}}_k\boldsymbol{\rho}(y) = \mathcal{V}_{kj}\rho_j(y) := -\int_{\partial\Omega} u^k_j(x,y)\rho_j(x)dS_x,$$
$$\boldsymbol{\mathcal{W}}_k\boldsymbol{\rho}(y) = \mathcal{W}_{kj}\rho_j(y) := -\int_{\partial\Omega} T^+_j(x; q^k, u^k)(x,y)\rho_j(x)dS_x,$$
$$\boldsymbol{\mathcal{W}}'_k\rho(y) = \mathcal{W}'_{kj}\rho_j(y) := -\int_{\partial\Omega} T^+_j(y; q^k, u^k)(x,y)\rho_j(x)dS_x,$$
$$\boldsymbol{\mathcal{L}}^\pm\boldsymbol{\rho}(y) := \mathbf{T}^\pm(\Pi^d\boldsymbol{\rho}, \mathbf{W}\boldsymbol{\rho})(y),$$

where $y \in \partial\Omega$ see, e.g, [?]. Here $\mathbf{T}^{\pm}$ are the canonical derivative (traction) operators for the compressible fluid that are well defined due to continuity of the second operator in (34).

Similar to the potentials in the domain, we can also express the boundary operators in terms of their counterparts with the constant coefficient $\mu = 1$,

$$\boldsymbol{\mathcal{V}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}, \quad \boldsymbol{\mathcal{W}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{W}}}(\mu\boldsymbol{\rho}), \tag{35}$$

$$\boldsymbol{\mathcal{W}}'_k\boldsymbol{\rho} = \mathring{\boldsymbol{\mathcal{W}}}'_k\boldsymbol{\rho} - \left(\frac{\partial_i\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_k\boldsymbol{\rho} + \frac{\partial_k\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_i\boldsymbol{\rho} - \frac{2}{3}\delta_i^k\frac{\partial_j\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_j\boldsymbol{\rho}\right)n_i. \tag{36}$$

The following result is proved in [?, Theorem 4.4]

Theorem 3.6. *Let $s \in \mathbb{R}$. Let S_1 and S_2 be two non empty manifolds on $\partial\Omega$ with smooth boundaries ∂S_1 and ∂S_2, respectively. Then the following operators are continuous,*

$$\begin{aligned}
&\boldsymbol{\mathcal{V}} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \quad \boldsymbol{\mathcal{W}} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega),\\
&r_{s_2}\boldsymbol{\mathcal{V}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2), \quad r_{s_2}\boldsymbol{\mathcal{W}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2),\\
&\boldsymbol{\mathcal{L}}^{\pm} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s-1}(\partial\Omega), \quad \boldsymbol{\mathcal{W}}' : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega).
\end{aligned}$$

Moreover, the following operators are compact,

$$\begin{aligned}
r_{s_2}\boldsymbol{\mathcal{V}} &: \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2),\\
r_{s_2}\boldsymbol{\mathcal{W}} &: \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2),\\
r_{s_2}\boldsymbol{\mathcal{W}}' &: \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2),
\end{aligned}$$

Theorem 3.7. *Let $\boldsymbol{\tau} \in \ \boldsymbol{H}^{s-1/2}(\partial\Omega)$ and $\boldsymbol{\rho} \in \boldsymbol{H}^{s-3/2}(\partial\Omega), 1 \leq s$. Then the following jump relations hold on $\partial\Omega$:*

$$\begin{aligned}
&\gamma^{\pm}\,\boldsymbol{V}\boldsymbol{\rho} = \boldsymbol{\mathcal{V}}\boldsymbol{\rho}, \qquad \gamma^{\pm}\,\boldsymbol{W}\boldsymbol{\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}}\boldsymbol{\tau},\\
&\boldsymbol{T}^{\pm}(\Pi^s\boldsymbol{\rho},\, \boldsymbol{V}\boldsymbol{\rho}) = \pm\frac{1}{2}\boldsymbol{\rho} + \boldsymbol{\mathcal{W}}'\boldsymbol{\rho}.
\end{aligned}$$

Proof. For s = 1, c.f. [?], the proof of the theorem directly follows from relations (15), (35)–(36) and the analogous jump properties for the counterparts of the operators for the constant coefficient case of $\mu = 1$, see [?, Lemma 5.6.5], which evidently implies the case $s > 1$. $\square$

Let denote

$$\mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(y) = \mathring{\boldsymbol{\mathcal{L}}}^{\pm}\boldsymbol{\tau}(y) := \mathring{\mathbf{T}}(\mathring{\Pi}^d\boldsymbol{\tau}, \mathring{\mathbf{W}}\boldsymbol{\tau})(y), \quad \hat{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(y) := \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\tau})(y), \quad y \in \partial\Omega,$$

where the first equality is implied by Lyapnove-Tauber theorem for the constant coefficient Stokes potentials.

The following assertion can be proved similar to [?, Theorem 4.6]

Theorem 3.8. *Let $\boldsymbol{\tau} \in \boldsymbol{H}^{s-1/2}(\partial\Omega)$. Then the following jump relation holds:*

$$(\mathcal{L}_k^{\pm} - \hat{\mathcal{L}}_k)\boldsymbol{\tau} =$$
$$\gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i.$$

Corollary 3.9. *Let S_1 be nonempty submanifold of $\partial\Omega$ with smooth boundary, $0 < s < 1$. Then, the operators*

$$\widehat{\mathcal{L}} : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega), \quad (\mathcal{L} - \widehat{\mathcal{L}}) : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega),$$

are continuous and the operator

$$(\mathcal{L} - \widehat{\mathcal{L}}) : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega),$$

is compact

Proof. The proof for $s = \dfrac{1}{2}$ follows from [?, Corollary 4.7] and then implies for $0 < s < 1$. $\square$

4 The Third Green Identities and Integral Relations

Theorem 4.1. *For any $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), 1 \le s < \dfrac{3}{2}$, the following third Green identities hold.*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^{s}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \Pi^{d}\gamma^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) + \frac{4\mu}{3} div\, \boldsymbol{v} \ \text{ in } \Omega, \tag{37}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{W}\gamma^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{\mathcal{Q}} div\, \boldsymbol{v} \ \text{ in } \Omega. \tag{38}$$

Proof. For $s = 1$ the proof is provided in [?, Theorem 5.1], which evidently implies the lemma claims also for $s > 1$. $\square$

Finally if $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ then by Theorem 2.2, and the mapping properties of the operators involved in (37) and (38) extend these relations to any $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$. If the couple $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDEs (1a)–(1b) with variable viscosity coefficient, then the third Green identities (37) and (38) reduce to

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^{s}\mathbf{T}^{+}(p, \boldsymbol{v}) + \Pi^{d}\gamma^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3} g \ \text{ in } \Omega, \tag{39}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \mathbf{V}\mathbf{T}^{+}(p, \boldsymbol{v}) + \mathbf{W}\gamma^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}} g \ \text{ in } \Omega. \tag{40}$$

We will also need the trace and traction of the third Green identities for $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ on $\partial\Omega$,

$$\frac{1}{2}\gamma^{+}\boldsymbol{v} + \gamma^{+}\boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\mathbf{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{\mathcal{W}}\gamma^{+}\boldsymbol{v} = \gamma^{+}\boldsymbol{\mathcal{U}}\boldsymbol{f} - \gamma^{+}\boldsymbol{\mathcal{Q}} g, \tag{41}$$

$$\frac{1}{2}\mathbf{T}^{+}(p, \boldsymbol{v}) + \mathbf{T}^{+}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\mathbf{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{\mathcal{L}}^{+}\gamma^{+}\boldsymbol{v} = \mathbf{T}^{+}(\mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3} g, \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}} g). \tag{42}$$

Note that the traction operators in (42) are well defined by virtue of the continuity of operators (29)–(31) in Theorem 3.3 and operators (34) in Theorem 3.5. Let us now prove the following three assertions that are instrumental for proving the equivalence of the BDIDP/Es systems to the mixed BVP. One can prove the following two assertions that are instrumental for proof of equivalence of the BDIDP/Es and the mixed PDE.

Lemma 4.2. *Let* $1 \leq s < \dfrac{3}{2}$. *Suppose some functions* $p \in H^{s-1}(\Omega), \boldsymbol{v} \in \boldsymbol{H}^s(\Omega), g \in H^{s-1}(\Omega), \boldsymbol{f} \in \boldsymbol{L}_2(\Omega), \boldsymbol{\Psi} \in \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega), \boldsymbol{\Phi} \in \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega)$ *satisfy equations;*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^s\boldsymbol{\Psi} + \Pi^d\boldsymbol{\Phi} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in } \Omega,$$
$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\Psi} + \boldsymbol{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \quad \text{in } \Omega.$$

Then $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ *and solves the equation*

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) = \boldsymbol{f}, \quad div\, \boldsymbol{v} = g.$$

Moreover, the following relations hold true:

$$\Pi^s(\boldsymbol{\Psi} - \boldsymbol{T}^+(p, \boldsymbol{v})) - \Pi^d(\boldsymbol{\Phi} - \gamma^+\boldsymbol{v}) = 0 \quad \text{in } \Omega,$$
$$\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{T}^+(p, \boldsymbol{v})) - \boldsymbol{W}(\boldsymbol{\Phi} - \gamma^+\boldsymbol{v}) = \boldsymbol{0} \quad \text{in } \Omega.$$

Proof. The proof follows from Theorem 3.3 ((28)–(31)) and operators in (34) in Theorem 3.5 similar to [?, Theorem 5.2]. $\square$

Lemma 4.3. *Let* $\partial\Omega = \overline{S_1} \cup \overline{S_2}$, *where* S_1 *and* S_2 *an open non-empty non-intersecting simply connected sub-manifolds of* $\partial\Omega$ *with infinitely smooth boundaries. Let* $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{s-3/2}(S_1)$, $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{s-1/2}(S_2)$ *for* $1 \leq s$. *If*

$$\Pi^s\boldsymbol{\Psi}^* - \Pi^d\boldsymbol{\Phi}^* = 0, \quad \boldsymbol{V}\boldsymbol{\Psi}^* - \boldsymbol{W}\boldsymbol{\Phi}^* = \boldsymbol{0}, \quad \text{in} \quad \Omega,$$

then $\boldsymbol{\Psi}^* = \boldsymbol{0}$ *and* $\boldsymbol{\Phi}^* = \boldsymbol{0}$ *on* $\partial\Omega$.

Proof. For $s = 1$ the proof is provided in [?, Lemma 5.3], which evidently implies the lemma claims also for $s > 1$. $\square$

5 Boundary-Domain Integral and Integro-Differential Equations and Problems

5.1 United Boundary-Domain Integro-Differential Problem (GDN)

Theorem 5.1. *Let* $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$. *A pair functions* $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), 1 \leq s < \frac{3}{2}$, *is a solution PDE* (1a)–(1b) *in* Ω *if and only if it is a solution of* (39)–(40).

Proof. If $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), 1 \leq s$, solves PDE (1a)–(1b), then, as follows from (37) and (38), it satisfies (39)–(40). On the other hand, if $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), 1 \leq s < \frac{3}{2}$, solves (39)–(40), then using Lemma 4.2 for $\boldsymbol{\Psi} = \mathbf{T}^+(p, \boldsymbol{v}), \boldsymbol{\Phi} = \gamma^+\boldsymbol{v}$ completes the proof. $\square$

The proved equivalence of the above theorem now allows to supplement (39)–(40) with the original mixed boundary conditions (1c)–(1d) and arrive at the following BDIDP (GDN):

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^s\mathbf{T}^+(p, \boldsymbol{v}) + \Pi^d\gamma^+\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in } \Omega,$$
$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \mathbf{V}\mathbf{T}^+(p, \boldsymbol{v}) + \mathbf{W}\gamma^+\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \quad \text{in } \Omega,$$
$$\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0 \quad \text{on } \partial\Omega_D,$$
$$\mathbf{T}^+(p, \boldsymbol{v}) = \boldsymbol{\psi}_0 \quad \text{on } \partial\Omega_N.$$

We can rewrite the system in matrix form:

$$\mathscr{A}^{GDN}\mathcal{X} = \mathscr{F}^{GDN}, \tag{43}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system;

$$\mathcal{X}^T = (p, \boldsymbol{v}, \mathbf{T}^+(p,\boldsymbol{v}), \gamma^+\boldsymbol{v}) \in H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) \times \mathbf{H}^{s-3/2}(\partial\Omega_N) \times \mathbf{H}^{s-1/2}(\partial\Omega_D)$$

and

$$\mathscr{A}^{GDN} := \begin{bmatrix} I & \mathcal{R}^\bullet & -\Pi^s & \Pi^d \\ 0 & I+\boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ 0 & 0 & 0 & r_{\partial\Omega_D} \\ 0 & 0 & r_{\partial\Omega_N} & 0 \end{bmatrix}, \qquad \mathscr{F}^{GDN} := \begin{bmatrix} \mathring{\mathcal{Q}}\boldsymbol{f} + \dfrac{4}{3}\mu g \\ \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \\ \boldsymbol{\varphi}_0 \\ \boldsymbol{\psi}_0 \end{bmatrix}.$$

The BDIDP system is equivalent to the mixed BVP (1a)–(1d) in Ω, in the following sense.

Theorem 5.2. *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega), \boldsymbol{\varphi}_0 \in \boldsymbol{H}^{s-1/2}(\partial\Omega_D), \boldsymbol{\psi}_0 \in \boldsymbol{H}^{s-3/2}(\partial\Omega_N), 1 \le s < \frac{3}{2}$.*

(i) If a couple $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ solves the mixed BVP (1a)–(1d), then the set $(p, \boldsymbol{v}, \boldsymbol{T}^+(p,\boldsymbol{v}), \gamma^+\boldsymbol{v})$ belongs to $\boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \boldsymbol{H}^{s-3/2}(\partial\Omega_N) \times \boldsymbol{H}^{s-1/2}(\partial\Omega_D)$ solves the BDIDP system (43).

(ii) If a set $(p, \boldsymbol{v}, \boldsymbol{T}^+(p,\boldsymbol{v}), \gamma^+\boldsymbol{v}) \in H^{s-1}(\Omega) \times \boldsymbol{H}^s(\Omega) \times \boldsymbol{H}^{s-3/2}(\partial\Omega_N) \times \boldsymbol{H}^{s-1/2}(\partial\Omega_D)$ solves the BDIDP system (43), then the pair of functions $(p, \boldsymbol{v})$ belongs to $\boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ solves the mixed BVP (1a)–(1d).

(iii) The BDIDP system (43) is uniquely solvable for $(p, \boldsymbol{v}, \boldsymbol{T}^+(p,\boldsymbol{v}), \gamma^+\boldsymbol{v}) \in H^{s-1}(\Omega) \times \boldsymbol{H}^s(\Omega) \times \boldsymbol{H}^{s-3/2}(\partial\Omega_N) \times \boldsymbol{H}^{s-1/2}(\partial\Omega_D)$.

Proof. A solution of BVP (1a)–(1d) does exist and is unique due to Theorem 2.3 and provides a solution to BDIDP (43) due to Theorem 5.1. On the other hand, any solution of BDIDP (43) satisfies also (1a)–(1b) due to the same Theorem 5.1. Finally, the unique solvability of the BDIDP system (43) in item (iii) follows from the unique solvability of the BVP (1a)–(1d), see Theorem 2.3, and items (i) and (ii). □

Due to the mapping properties of operators $\mathbf{V}, \mathbf{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{U}}, \Pi^s, \Pi^d, \mathcal{R}^\bullet$, and $\mathring{\mathcal{Q}}$ we have $\mathscr{F}^{GDN} \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{s-1/2}(\partial\Omega_D) \times \mathbf{H}^{s-3/2}(\partial\Omega_N)$, and the operator $\mathscr{A}^{GDN} : H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) \times \mathbf{H}^{s-3/2}(\partial\Omega_N) \times \mathbf{H}^{s-1/2}(\partial\Omega_D) \to \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{s-1/2}(\partial\Omega_D) \times \mathbf{H}^{s-3/2}(\partial\Omega_N)$ is continuous. It is also injective due to Theorem 5.2.

5.2 United Boundary-Domain Integro-Differential Problem ($\tilde{G}$N)

Let $1 \le s < \dfrac{3}{2}$. For $S_1 \subseteq \partial\Omega$ we will use the following subspaces of $\mathbf{H}^s(\Omega^\pm)$ and $\mathbf{H}^{s,0}(\Omega^\pm; \boldsymbol{\mathcal{A}})$ respectively;

$$\begin{aligned} \widetilde{\mathbf{H}}^s_{S_1}(\Omega^\pm) &:= \{\boldsymbol{u} \in \mathbf{H}^s(\Omega^\pm) : \ r_{s_1}\gamma^\pm\boldsymbol{u} = 0\} \\ \widetilde{\mathbf{H}}^{s,0}_{S_1}(\Omega^\pm; \boldsymbol{\mathcal{A}}) &:= \{(p, \boldsymbol{u}) \in \mathbf{H}^{s,0}(\Omega^\pm; \boldsymbol{\mathcal{A}}) : \boldsymbol{u} \in \widetilde{\mathbf{H}}^s_{S_1}(\Omega^\pm)\} \end{aligned}$$

with norm $\|\boldsymbol{u}\|_{\widetilde{\mathbf{H}}^s_{S_1}(\Omega^\pm)} = \|\boldsymbol{u}\|_{\mathbf{H}^s(\Omega^\pm)}$ and $\|(p,\boldsymbol{u})\|_{\widetilde{\mathbf{H}}^{s,0}_{S_1}(\Omega^\pm, \boldsymbol{\mathcal{A}})} = \|(p,\boldsymbol{u})\|_{\mathbf{H}^{s,0}(\Omega^\pm, \boldsymbol{\mathcal{A}})}$ respectively. Moreover, when $\boldsymbol{u} \in \widetilde{\mathbf{H}}^s_{S_1}(\Omega^\pm)$, then $\gamma^\pm\boldsymbol{u} \in \widetilde{\mathbf{H}}^{s-1/2}(\partial\Omega \setminus S_1)$.

Now we will formulate another BDIDP departing from the BDIDP (43), which doesn't include the explicit Dirichlet boundary condition. Let $(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ such that $\boldsymbol{v}_0$ be a fixed extension of the given boundary function $\boldsymbol{\varphi}_0 \in \mathbf{H}^{s-1/2}(\partial\Omega_D)$ into the domain. By the uniqueness theorem, Theorem 2.3 such pair functions exist. Denoting $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0)$ and substituting into (39) and (40) and boundary condition (1d) leading to the following BDIDP($\tilde{G}$N) for $(\tilde{p}, \tilde{\boldsymbol{v}}) \in \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega^{\pm}; \mathring{\boldsymbol{\mathcal{A}}})$:

$$\begin{aligned} \tilde{p} + \mathcal{R}^{\bullet}\tilde{\boldsymbol{v}} - \Pi^{s}\mathbf{T}^{+}(\tilde{p}, \tilde{\boldsymbol{v}}) + \Pi^{d}\gamma^{+}\tilde{\boldsymbol{v}} &= \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g - \mathbf{F}_0 \text{ in } \Omega, &(44)\\ \tilde{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}\tilde{\boldsymbol{v}} - \mathbf{V}\mathbf{T}^{+}(\tilde{p}, \tilde{\boldsymbol{v}}) + \mathbf{W}\gamma^{+}\tilde{\boldsymbol{v}} &= \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{F}_1 \text{ in } \Omega, &(45)\\ r_{\partial\Omega_N}\mathbf{T}^{+}(\tilde{p}, \tilde{\boldsymbol{v}}) &= \boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^{+}(p_0, \boldsymbol{v}_0) \text{ on } \partial\Omega_N, &(46) \end{aligned}$$

where

$$\begin{aligned} \mathbf{F}_0 = \quad & p_0 + \mathcal{R}^{\bullet}\boldsymbol{v}_0 - \Pi^{s}\mathbf{T}^{+}(p_0, \boldsymbol{v}_0) + \Pi^{d}\gamma^{+}\boldsymbol{v}_0 = \mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p_0, \boldsymbol{v}_0) + \frac{4\mu}{3}div\, \boldsymbol{v}_0 \\ \mathbf{F}_1 = \quad & \boldsymbol{v}_0 + \boldsymbol{\mathcal{R}}\boldsymbol{v}_0 - \mathbf{V}\mathbf{T}^{+}(p_0, \boldsymbol{v}_0) + \mathbf{W}\gamma^{+}\boldsymbol{v}_0 = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p_0, \boldsymbol{v}_0) - \boldsymbol{\mathcal{Q}}div\, \boldsymbol{v}_0. \end{aligned} \tag{47}$$

BDIDP (44)–(46) is only a reformulation of Theorem 5.2 holds true.

Theorem 5.3. *Let $1 \leq s < \frac{3}{2}$. Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega), \boldsymbol{\varphi}_0 \in \boldsymbol{H}^{s-1/2}(\partial\Omega_D), \boldsymbol{\psi}_0 \in \boldsymbol{H}^{s-3/2}(\partial\Omega_N)$ and $(p_0, \boldsymbol{v}_0) \in \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ such that $\boldsymbol{v}_0$ be a fixed extension of $\boldsymbol{\varphi}_0$ into the domain.*

(i) There exists a unique solution $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ of the mixed BVP (1a)–(1d) in Ω. The pair functions $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ is a solution of (44)–(46).

(ii) There exists a unique solution $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ of BDIDP (44)–(46). The couple $(p, \boldsymbol{v}) = (\tilde{p}, \tilde{\boldsymbol{v}}) + (p_0, \boldsymbol{v}_0) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ is a solution of the mixed BVP (1a)–(1d).

Proof. Let $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$. By Theorem 5.2 it uniquely satisfies (43). Again by letting $(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ such that $\boldsymbol{v}_0$ be a fixed extension of $\boldsymbol{\varphi}_0 \in \mathbf{H}^{s-1/2}(\partial\Omega_D)$ into Ω. Consequently, $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0)$ uniquely solves (44)-(46), which completes the prove of (i). We will follow similar fashion to prove (ii). $\square$

BDIDP (44)-(46) can be written in the form;

$$\mathscr{A}^{\tilde{G}N}\tilde{\mathcal{X}} = \mathscr{F}^{\tilde{G}N} \tag{48}$$

where

$$\mathscr{A}^{\tilde{G}N} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^{s} & \Pi^{d} \\ 0 & I + \boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ 0 & 0 & r_{\partial\Omega_N} & 0 \end{bmatrix}, \quad \mathscr{F}^{\tilde{G}N} = \begin{bmatrix} \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g - \mathbf{F}_0 \\ \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{F}_1 \\ \boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^{+}(p_0, \boldsymbol{v}_0) \end{bmatrix}$$

and $\tilde{\mathcal{X}}^{T} = (\tilde{p}, \tilde{\boldsymbol{v}}, \mathbf{T}^{+}(\tilde{p}, \tilde{\boldsymbol{v}}), \gamma^{+}\tilde{\boldsymbol{v}}) \in \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{s-3/2}(\partial\Omega_N) \times \mathbf{H}^{s-1/2}(\partial\Omega_D)$.
Relation (47) implies that $\mathscr{F}^{\tilde{G}N} \in \mathbf{Y}^{s}_{1}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{s-3/2}(\partial\Omega_N)$.

5.3 United Boundary-Domain Integro-Differential Equations ($\tilde{G}$)

In this section, we will formulate a system by removing the remaining Neumann boundary condition to deal with integro-differential equation system. Let $1 < s < \dfrac{3}{2}, \boldsymbol{\psi}_0 \in \mathbf{H}^{s-3/2}(\partial\Omega_N), \boldsymbol{\varphi}_0 \in \mathbf{H}^{s-1/2}(\partial\Omega_D)$ and $(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ such that $\boldsymbol{v}_0$ be an extension of $\boldsymbol{\varphi}_0$ into the domain Ω. If $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, then $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \tilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s-3/2}(\partial\Omega_N)$, $r_{\partial\Omega_D} \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) \in \mathbf{H}^{s-3/2}(\partial\Omega_D)$.
Since $\mathbf{H}^{s-3/2}(\partial\Omega_N) = \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_N)$ and $\mathbf{H}^{s-3/2}(\partial\Omega_D) = \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D)$ for $1 < s < \dfrac{3}{2}$ the surface potentials and the corresponding surface pseudo-differential operators for functions from these spaces have the nice mapping properties as we have seen above. Substituting the Neumann boundary condition (46) into (44) and (45) leads to the following BDIE $\tilde{G}$ for $(\tilde{p}, \tilde{\boldsymbol{v}}) \in \tilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$

$$\tilde{p} + \mathcal{R}^{\bullet}\tilde{\boldsymbol{v}} - \Pi^s r_{\partial\Omega_D} \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) + \Pi^d \gamma^+ \tilde{\boldsymbol{v}} = \mathring{\mathcal{Q}} \boldsymbol{f} + \frac{4\mu}{3} g - \mathbf{F}_0 + \Pi^s(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)) \text{ in } \Omega, \tag{49}$$

$$\tilde{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}\tilde{\boldsymbol{v}} - \mathbf{V} r_{\partial\Omega_D} \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) + \mathbf{W}\gamma^+ \tilde{\boldsymbol{v}} = \boldsymbol{\mathcal{U}} \boldsymbol{f} - \boldsymbol{\mathcal{Q}} g - \mathbf{F}_1 + \mathbf{V}(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)) \text{ in } \Omega, \tag{50}$$

and we can rewrite the system for $\tilde{\mathcal{X}}^T = (\tilde{p}, \tilde{\boldsymbol{v}}, \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}), \gamma^+ \tilde{\boldsymbol{v}})$ as

$$\mathscr{A}^{\tilde{G}} \tilde{\mathcal{X}} = \mathscr{F}^{\tilde{G}} \tag{51}$$

where

$$\mathscr{A}^{\tilde{G}} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s r_{\partial\Omega_D} & \Pi^d \\ 0 & I + \boldsymbol{\mathcal{R}} & -\mathbf{V} r_{\partial\Omega_D} & \mathbf{W} \end{bmatrix}, \quad \mathscr{F}^{\tilde{G}} = \begin{bmatrix} \mathring{\mathcal{Q}} \boldsymbol{f} + \frac{4\mu}{3} g - \mathbf{F}_0 + \Pi^s(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)) \\ \boldsymbol{\mathcal{U}} \boldsymbol{f} - \boldsymbol{\mathcal{Q}} g - \mathbf{F}_1 + \mathbf{V}(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)) \end{bmatrix}$$

F_0 and F_1 are given in (47).

Let us prove the equivalence of the BDIDE to the BVP (1a)–(1d).

Theorem 5.4. *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega), \boldsymbol{\varphi}_0 \in \boldsymbol{H}^{s-1/2}(\partial\Omega_D), \boldsymbol{\psi}_0 \in \boldsymbol{H}^{s-3/2}(\partial\Omega_N), 1 < s < \frac{3}{2}$. Let $(p_0, \boldsymbol{v}_0) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ such that $\boldsymbol{v}_0$ be an extension of $\boldsymbol{\varphi}_{\mathbf{0}}$.*

(i) There exists a unique solution $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ of mixed BVP (1a)–(1d) in Ω. The pair of functions $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ is a solution of BDIDE (49) and (50).

(ii) There exists a unique solution $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ of BDIDE (49) and (50). $(p, \boldsymbol{v}) = (\tilde{p}, \tilde{\boldsymbol{v}}) + (p_0, \boldsymbol{v}_0) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ is a solution of mixed BVP (49)–(50) in Ω.

Proof. Any solution of BVP (1a)–(1d) solves BDIDE (49)–(50) due to the third Green identities (37) and (38). On the other hand, if $(\tilde{p}, \tilde{\boldsymbol{v}})$ is a solution of BDIDE (49)–(50), then Lemma 4.2 implies that $(p, \boldsymbol{v}) = (p_0, \boldsymbol{v}_0) + (\tilde{p}, \tilde{\boldsymbol{v}})$ satisfies equation (1a)–(1b) and $\Pi^s(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)) = 0 = \mathbf{V}(\boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0))$ in Ω. Lemma 4.3(i) implies that the Neumann boundary condition (1d) is satisfied for $(p, \boldsymbol{v})$. The Dirichlet boundary condition is satisfied due to the chosen space for $(\tilde{p}, \tilde{\boldsymbol{v}})$ and the extension property of the function $\boldsymbol{v}_0$ of $(p_0, \boldsymbol{v}_0)$. Thus any solution $(\tilde{p}, \tilde{\boldsymbol{v}})$ of BDIDE

(49)–(50) generates a solution $(p_0, \boldsymbol{v}_0) + (\tilde{p}, \tilde{\boldsymbol{v}})$ to the mixed BVP (1a)–(1d). To prove the unique solvability of BDIDE (49)–(50), let us consider its homogeneous counterpart. Since $\mathscr{F}^{\tilde{G}} = 0$ can be associated with $\boldsymbol{f} = 0$, $(p_0, \boldsymbol{v}_0) = 0$, $\boldsymbol{\psi}_0 = 0$, any solution $(\tilde{p}, \tilde{\boldsymbol{v}})$ of homogeneous BDIDE (49)–(50), according to the previous paragraph, is a solution to the homogeneous BVP (1a)–(1d), which is trivial due to uniqueness theorem, Theorem 2.2. $\square$

The mapping properties of operators $\mathbf{V}$, $\mathbf{W}$, Π^s, Π^d, $\boldsymbol{\mathcal{R}}$, $\mathcal{R}^\bullet$, $\boldsymbol{\mathcal{U}}$, $\mathring{\mathcal{Q}}$ and $\boldsymbol{\mathcal{Q}}$ imply the membership $\mathscr{F}^{\tilde{G}} \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ and continuity of the operator $\mathscr{A}^{\tilde{G}} : \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{s-3/2}(\partial\Omega_N) \times \mathbf{H}^{s-1/2}(\partial\Omega_D) \to \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ while Theorem 5.4 implies its injectivity.

5.4 Partly segregated Boundary-Domain Integral equation ($\tilde{G}$D)

In this section, we consider the integral equation only, without the differential term, where the trace of solution is used in the boundary integrals (unlike [?]) but the unknown traction is replaced by an auxiliary boundary function.

Let in this section $1 \leq s < \frac{3}{2}$, $\boldsymbol{\psi}_0 \in \mathbf{H}^{s-3/2}(\partial\Omega_N)$, $\boldsymbol{\varphi}_0 \in \mathbf{H}^{s-1/2}(\partial\Omega_D)$, and $(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ where $\boldsymbol{v}_0$ be an extension of $\boldsymbol{\varphi}_0$ into the domain Ω. If $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ and satisfies the Dirichlet condition (1c), then $(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \in \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, $\boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^+(p_0, \boldsymbol{v}_0) \in \mathbf{H}^{s-3/2}(\partial\Omega_N)$, $r_{\partial\Omega_D}\mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) \in \mathbf{H}^{s-3/2}(\partial\Omega_D)$.

Let $\boldsymbol{\Psi}_0 \in \mathbf{H}^{s-3/2}(\partial\Omega)$ be a fixed extension of the given function $\boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^+(p_0, \boldsymbol{v}_0 \in \mathbf{H}^{s-3/2}(\partial\Omega_N)$ from $\partial\Omega_N$ to the whole of $\partial\Omega$. Note that if $1 < s < \frac{3}{2}$, then $\mathbf{H}^{s-3/2}(\partial\Omega_N) = \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_N)$ and one may simply choose $\boldsymbol{\Psi}_0$ as the canonical extension of $\boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^+(p_0, \boldsymbol{v}_0)$ from $\partial\Omega_N$ to the whole of $\partial\Omega$ by zero.

An arbitrary extension $\boldsymbol{\Psi} \in \mathbf{H}^{s-3/2}(\partial\Omega)$ of $\boldsymbol{\psi}_0 - r_{\partial\Omega_N}\mathbf{T}^+(p_0, \boldsymbol{v}_0)$ preserving the function space can be then represented as $\boldsymbol{\Psi} = \boldsymbol{\Psi}_0 + \boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D)$.

To reduce BVP (1a)–(1d) to a pure integral equation, let us consider (39) and (40) in Ω and replace there $(p, \boldsymbol{v})$ with $(p_0, \boldsymbol{v}_0) + (\tilde{p}, \tilde{\boldsymbol{v}})$ in Ω and $\mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}})$ with $\boldsymbol{\Psi}_0 + \boldsymbol{\psi}$ on $\partial\Omega$, where $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D)$ is a new auxiliary function. This leads to the following BDIE ($\tilde{G}$D):

$$\tilde{p} + \mathcal{R}^\bullet\tilde{\boldsymbol{v}} + \Pi^d\gamma^+\tilde{\boldsymbol{v}} - \Pi^s\boldsymbol{\psi} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g - \mathbf{F}_0 + \Pi^s\boldsymbol{\Psi}_0 \quad \text{in } \Omega, \tag{52}$$

$$\tilde{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}\tilde{\boldsymbol{v}} + \mathbf{W}\gamma^+\tilde{\boldsymbol{v}} - \mathbf{V}\boldsymbol{\psi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{F}_1 + \mathbf{V}\boldsymbol{\Psi}_0 \quad \text{in } \Omega, \tag{53}$$

for the triple $(\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi}) \in \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D)$ where $\mathbf{F}_0$ and $\mathbf{F}_1$ are given in (47).

Let us prove the following equivalence statement.

Theorem 5.5. *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{s-1/2}(\partial\Omega_D)$, $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{s-3/2}(\partial\Omega_N)$, $1 \leq s < \frac{3}{2}$. Let $(p_0, \boldsymbol{v}_0) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ such that $\boldsymbol{v}_0$ be an extension of $\boldsymbol{\varphi}_0$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{s-3/2}(\partial\Omega)$ be an extension of $\boldsymbol{\psi}_0 - r_{\partial\Omega_N}\boldsymbol{T}^+(p_0, \boldsymbol{v}_0)$.*

(i) There exists a unique solution $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ of mixed BVP (1a)–(1d) in Ω. The triple $(\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi}) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \widetilde{\boldsymbol{H}}^{s-3/2}(\partial\Omega_D)$, where

$$(\tilde{p}, \tilde{\boldsymbol{v}}) = (p, \boldsymbol{v}) - (p_0, \boldsymbol{v}_0) \quad \textit{in } \Omega, \tag{54}$$

$$\boldsymbol{\psi} = \boldsymbol{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) - \boldsymbol{\Psi}_0 \quad \textit{on } \partial\Omega, \tag{55}$$

is a solution of BDIE system (52)–(53).

(ii) There exists a unique solution $(\tilde{p}, \tilde{\boldsymbol{v}}, \psi) \in \widetilde{\boldsymbol{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \widetilde{\boldsymbol{H}}^{s-3/2}(\partial\Omega_D)$ *of BDIEs (52)–(53). The pair functions* $(p, \boldsymbol{v})$ *defined by (54) is a solution of mixed BVP (1a)–(1d) in* Ω*, and equation (55) holds.*

Proof.

(i) The unique solvability of BVP (1a)–(1d) is implied, e.g. by Theorem 2.3. For any solution $(p, \boldsymbol{v})$ of the BVP, the triple $(\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi})$ defined by (54)–(55) solves BDIE system (52)–(53) due to the third Green identities (37) and (38). Thus, point (i) is proved.

(ii) Existence of a solution to BDIE system (52)–(53) is implied by point (i).

□

Let $(\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi}) \in \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D)$ be a solution of (52)–(53). The function $(p, \boldsymbol{v})$ defined by (54) evidently satisfies Dirichlet boundary condition (1c)
Equations (52)–(53) and Lemma 4.2 for $(\tilde{p}, \tilde{\boldsymbol{v}})$, $\boldsymbol{\Psi} = \boldsymbol{\psi} + \boldsymbol{\Psi}_0$ and $\boldsymbol{\Phi} = \gamma^+ \boldsymbol{v}$ with account of (47) implies that $(p, \boldsymbol{v})$ is a solution of PDE (1a)–(1b) in Ω and

$$(\Pi^s, \mathbf{V})\boldsymbol{\Psi}^* - (\Pi^d, \mathbf{W})\boldsymbol{\Phi}^* = \mathbf{0} \quad \text{in } \Omega,$$

where $\boldsymbol{\Psi}^* = \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}})$ and $\boldsymbol{\Phi^*} = \mathbf{0}$. Eventually, lemma 4.3$(i)$ implies $\boldsymbol{\psi} = \mathbf{T}^+(\tilde{p}, \tilde{\boldsymbol{v}}) - \boldsymbol{\Psi}_0$ on $\partial\Omega$, i.e. condition (55) is satisfied. Its restriction on $\partial\Omega_N$ implies also the Neumann boundary condition (1d) for $(p, \boldsymbol{v})$ if one takes into consideration that $\boldsymbol{\psi} = 0$ and $\boldsymbol{\Psi}_0 = \boldsymbol{\psi}_0 - r_{\partial\Omega_N} \mathbf{T}^+(p_0, \boldsymbol{v}_0)$ on $\partial\Omega_N$.
To prove the unique solvability of BDIEs (52)–(53), let us consider its homogeneous counterpart. Since the right hand side is zero and then can be associated with $\boldsymbol{f} = \mathbf{0}, (p_0, \boldsymbol{v}_0) = \mathbf{0}, \boldsymbol{\psi}_0 = \mathbf{0}$, any solution $(\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi})$ of homogeneous BDIEs (52)–(53), according to (54), (55), gives a solution $(\tilde{p}, \tilde{\boldsymbol{v}})$ to the homogeneous BVP (1a)–(1d), which is trivial due to Theorem 2.2.

BDIEs (52)–(53) can be rewritten in the form

$$\mathscr{A}^{\tilde{G}D}\tilde{\mathcal{U}} = \mathcal{F}^{\tilde{G}D}, \tag{56}$$

where $\tilde{\mathcal{U}}^T := (\tilde{p}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi})$,

$$\mathscr{A}^{\tilde{G}D} = \begin{bmatrix} I & \mathcal{R}^{\bullet} + \Pi^d\gamma^+ & -\Pi^s \\ 0 & I + \boldsymbol{\mathcal{R}} + \mathbf{W}\gamma^+ & -\mathbf{V} \end{bmatrix}, \ \textit{and} \ \ \mathscr{F}^{\tilde{G}D} = \begin{bmatrix} \mathring{\mathcal{Q}}\boldsymbol{f} + \dfrac{4\mu}{3}g - \mathbf{F}_0 + \Pi^s\boldsymbol{\Psi}_0 \\ \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{F}_1 + \mathbf{V}\boldsymbol{\Psi}_0 \end{bmatrix}$$

F_0 and F_1 are given in (47). The mapping properties of the operators involved in, imply $\mathcal{F}^{\tilde{G}D} \in \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, and the operator $\mathscr{A}^{\tilde{G}D} : \widetilde{\mathbf{H}}^{s,0}_{\partial\Omega_D}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \widetilde{\mathbf{H}}^{s-3/2}(\partial\Omega_D) \to \mathbf{H}^{s,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ is continuous, while Theorem 5.5 implies its injectivity.

6 Conclusion

In this paper, we considered the mixed BVP for a compressible Stokes system with variable viscosity with a right-hand side function from $\mathbf{L}_2(\Omega), H^{s-1}(\Omega)$, and with the Dirichlet and the Neumann data from the spaces $\mathbf{H}^{s-1/2}(\partial\Omega_D)$ and $\mathbf{H}^{s-3/2}(\partial\Omega_N)$, respectively, was considered in this paper for $1 \leq s < \dfrac{3}{2}$. It was shown that the BVP can be equivalently reduced to two direct united boundary-domain integro-dierential problems, or to a united BDIDEs or to a partly segregated BDIEs. This implied unique solvability of the BDIDPs/BDIDEs/BDIEs with the right-hand sides generated by the considered BVP. The invertibility of the associated operators in the corresponding Sobolev spaces can also be proved. This study can serve as a starting point for approaching BDIDPs/BDIDEs/BDIEs for mixed problem in exterior domains, incompressible Stokes system as well as the compressible Stokes system, which were analysed for 3D case in [**?**, **?**].

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