Some ad hoc materials

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Abstract

This document contains some ad hoc dissertations made when I was taking course "Paradox and Infinity".

- Proof that cardinality of set of N to N bijections is equal to power of N.

Introduction

This document was prepared as my private notes from the course "MITx: 24.118x: Paradox and Infinity": `https://courses.edx.org/courses/course-v1:MITx+24.118x+2T2019/course/`.

1 Related to Lecture 1

This is an extension to exercise 2. Original text of problem:

> Let F be the set of *functions* from natural numbers to natural numbers.
>
> Is it the case that $|\mathbb{N}| < |F|$?

I wonder: if we take little sharper assumption if the result would still hold. I.e.

Theorem. *Let F^* be the set of* bijections *from natural numbers to natural numbers.*

Then $|\mathbb{N}| < |F^*|$.

Proof. At first let's introduce following name.

Definition. *Given any set A, we call* perfect-mix *a bijection $f : A \to A$ that additionally fulfill condition:*

$$\forall_{a \in A} f(a) \neq a$$

It occurs:

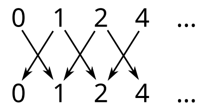
Lema. *For every $A \subset \mathbb{N}$ such as $|A| > 1$ perfect-mix $f : A \to A$ does exist.*

Proof. For finite set: $\{0, 1, 2, \ldots, N\}$:

$$f(n) = \begin{cases} n+1 & : n < N \\ 0 & : n = N \end{cases}$$

For infinite set: $\{0, 1, 2, \ldots\}$:

$$f(n) = \begin{cases} n+1 & : n = \{0, 2, 4, \ldots\} \\ n-1 & : n = \{1, 3, 5, \ldots\} \end{cases}$$



For every subset of $\mathbb{N}$ with at least 2 elements there is bijection b to one of above sets and our perfect-mix function is $b \circ f \circ b^{-1}$. $\square$

Now let's return to proof of main theorem. We will use notation: $\mathscr{P}(A)$ for power set of A (i.e. set of all subsets), it is also being marked 2^A in other papers. Let $\mathscr{P}_{>1}(\mathbb{N})$ stands for power set of natural numbers but such ones that have at least 2 elements. It contains all subsets except empty set and one-element sets. This reminder is obviously countable, so using *Cantor's Theorem* and *No Countable Difference Principle* it occurs $|\mathbb{N}| < |\mathscr{P}(\mathbb{N})| = |\mathscr{P}_{>1}(\mathbb{N})|$.

We will now show that $|\mathscr{P}_{>1}(\mathbb{N})| \leq |F^*|$. This will be sufficient to prove our theorem because of above observation. To proof this inequality we will construct an injective function from our limited power set of naturals to set of bijections $\mathbb{N} \to \mathbb{N}$.

Let us consider special case first:

- assign $\mathbb{N}$ to perfect-mix function over entire $\mathbb{N}$ (which exists thanks to our lemma).

Now let $A \subsetneq \mathbb{N}$ be subset such that $|A| > 1$ and $A \neq \mathbb{N}$. We map our subset to bijection f constructed in following way:

- for $n \in A$ let $f(n) = m(n)$ where m is perfect-mix function which exists on set A,
- for $n \in \mathbb{N} \setminus A$ let $f(n) = n$ (i.e. identity on set $\mathbb{N} \setminus A$).

Obviously f is well defined bijection on natural numbers. It also occurs that this bijection is different than one assigned to $\mathbb{N}$: $n \in \mathbb{N} \setminus A$ is not empty, so for at least one element $f(n) = n$, so f is not perfect-mix.

Now let's take different subset $A' \subsetneq \mathbb{N}$, $A' \neq A$, $|A'| > 1$. Like before we can map it to respective bijection f'. We want to show that $f' \neq f$.

Because $A' \neq A$, one of below occurs:

i) $\exists a' \in A'$ and $a' \notin A$,

ii) $\exists a \in A$ and $a \notin A'$.

From construction of our bijections f, f' we have respectively:

i) $f'(a') \neq a'$ and $f(a') = a'$ therefore $f'(a') \neq f(a')$,

ii) $f(a) \neq a$ and $f'(a) = a$ therefore $f(a) \neq f'(a)$.

So f and f' differ on at least one number - what ends the proof. $\square$