Some ad hoc materials

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July 6, 2019

Abstract

This document contains some ad hoc dissertations made when I was taking course “Paradox and Infinity”.

- Proof that cardinality of set of N to N bijections is equal to power of N.

Introduction

This document was prepared as my private notes from the course “MITx: 24.118x: Paradox and Infinity”: `https://courses.edx.org/courses/course-v1:MITx+24.118x+2T2019/course/`.

1 Related to Lecture 1

This is an extension to exercise 2. Original text of problem:

> Let F be the set of *functions* from natural numbers to natural numbers.
>
> Is it the case that $|\mathbb{N}| < |F|$?

I wonder: if we take little sharper assumption if the result would still hold. I.e.

Theorem. *Let F^* be the set of* bijections *from natural numbers to natural numbers.*

Then $|\mathbb{N}| < |F^*|$.

Proof. At first let's introduce following name.

Definition. *Given any set A, we call* perfect-mix *a bijection $f : A \to A$ that additionally fulfill condition:*

$$\forall_{a \in A} f(a) \neq a$$

It occurs:

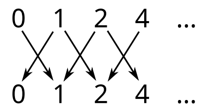
Lema. *For every $A \subset \mathbb{N}$ such as $|A| > 1$ perfect-mix $f : A \to A$ does exist.*

Proof. For finite set: $\{0, 1, 2, \ldots, N\}$:

$$f(n) = \begin{cases} n+1 & : n < N \\ 0 & : n = N \end{cases}$$

For infinite set: $\{0, 1, 2, \ldots\}$:

$$f(n) = \begin{cases} n+1 & : n = \{0, 2, 4, \ldots\} \\ n-1 & : n = \{1, 3, 5, \ldots\} \end{cases}$$



For every subset of $\mathbb{N}$ with at least 2 elements there is bijection b to one of above sets and our perfect-mix function is $b \circ f \circ b^{-1}$. $\square$

Before we return to proof of main theorem, let us define one more symbol: let $2^{\mathbb{N}}_{>1}$ stands for power set (set of all subsets) of natural numbers but such ones that have at least 2 elements. It contains all subsets except empty set and one-element sets. This reminder is obviously countable, so following *Cantor's Theorem* and *No Countable Difference Principle* it occurs $|\mathbb{N}| < |2^{\mathbb{N}}| = |2^{\mathbb{N}}_{>1}|$.

Now let's return to proof of main theorem. We will show that $|2^{\mathbb{N}}_{>1}| \leq |F^*|$. This will be sufficient because of above observation. To proof this inequality we will construct an injective function from our limited power set of naturals to set of bijections $\mathbb{N} \to \mathbb{N}$.

Let us consider special case first:

- assign $\mathbb{N}$ to perfect-mix function over entire $\mathbb{N}$ (which exists thanks to our lemma).

Now let $A \subsetneq \mathbb{N}$ be subset such that $|A| > 1$ and $A \neq \mathbb{N}$. We map our subset to bijection f constructed in following way:

- for $n \in A$ let $f(n) = m(n)$ where m is perfect-mix function which exists on set A,
- for $n \in \mathbb{N} \setminus A$ let $f(n) = n$ (i.e. identity on set $\mathbb{N} \setminus A$).

Obviously f is well defined bijection on natural numbers. It also occurs that this bijection is different than one assigned to $\mathbb{N}$: $n \in \mathbb{N} \setminus A$ is not empty, so for at least one element $f(n) = n$, so f is not perfect-mix.

Now let's take different subset $A' \subsetneq \mathbb{N}$, $A' \neq A$, $|A'| > 1$. Like before we can map it to respective bijection f'. We want to show that $f' \neq f$.

Because $A' \neq A$, one of below occurs:

i) $\exists a' \in A'$ and $a' \notin A$,

ii) $\exists a \in A$ and $a \notin A'$.

From construction of our bijections f, f' we have respectively:

i) $f'(a') \neq a'$ and $f(a') = a'$ therefore $f'(a') \neq f(a')$,

ii) $f(a) \neq a$ and $f'(a) = a$ therefore $f(a) \neq f'(a)$.

So f and f' differ on at least one number - what ends the proof. $\square$