Sphere Drag and Settling Velocity Revisited

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Abstract: Sphere drag data from throughout the twentieth century are available in tabular form. However, much of the data arose from experiments in small diameter cylindrical vessels, where the results might have been influenced by the wall effect. Wall effect corrections developed by others were applied to 178 of the 480 data points collected. This corrected data set is believed to be free of the influence of wall effects. Existing drag and settling velocity correlations were compared to this data set. In addition, new correlations of the same forms were developed using the corrected data. Two new correlations of sphere terminal velocity are proposed, one applicable for all Reynolds numbers less than 2×10^5, and the other designed to predict settling velocities with exceptional accuracy for terminal Reynolds numbers less than 4,000, a region that contains almost all applications of interest in environmental engineering. The trial and error solution for settling velocity using the Fair and Geyer equation for drag should be retired in favor of the direct calculation available from these new correlations.

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Introduction

The fluid dynamic drag on a sphere and the terminal settling velocity of a single spherical particle in an infinite fluid are of interest in numerous fields, including chemical and mechanical engineering as well as environmental engineering. As such, they have been the subjects of many experimental and theoretical investigations; despite such interest, existing knowledge is less than perfect, and improvements in predictions could be useful in mathematical modeling of particle behavior. In environmental engineering, such predictions are used for modeling all particle processes in water (flocculation, sedimentation, flotation, and filtration, including backwashing of filter media) and particle capture and deposition in air. The objective of the research reported herein was to improve the correlations for drag and terminal settling velocity that have been in use, based on the collected data from these many previous investigations.

The terminal settling velocity occurs when the net gravitational force (gravity minus buoyancy) equals the drag force, i.e.,

$$F_g - F_b = F_d \tag{1}$$

As is well known, the net gravitational force on a sphere is given as

$$F_{g-b} = \pi(\rho_p - \rho_f) g \frac{d^3}{6} \tag{2}$$

and the drag force is described as follows:

$$F_d = \frac{1}{8} C_D \pi d^2 \rho_f \nu^2 \tag{3}$$

Hence, combining Eqs. (1)–(3), the drag coefficient is calculated from measurements of terminal velocity, assuming the fluid and particle properties are known, as follows:

$$C_D = \frac{4}{3} \frac{\rho_p - \rho_f}{\rho_f} g \frac{d}{\nu^2} \tag{4}$$

For creeping flow conditions where inertial effects are negligible, Stokes (1880) derived the following equation for the drag force:

$$F_D = 3\pi\mu d\nu \tag{5}$$

or, equivalently

$$C_D = \frac{24}{\mathsf{R}} \tag{6}$$

where the Reynolds number, R, is defined as follows:

$$\mathsf{R} = \frac{\rho_f \nu d}{\mu} \tag{7}$$

For these creeping flow conditions, Stokes used Eqs. (1), (2), and (5) to derive the famous expression for the settling velocity of a sphere, now known as Stokes' settling velocity

$$\nu_s = \frac{g}{18\mu}(\rho_p - \rho_f) d_p^2 \tag{8}$$

When the inertial effects cannot be neglected, the drag coefficient cannot be predicted theoretically, and so the approach has been to correlate experimental data for C_D and ν to develop predictive tools. Improving these correlations was our interest.

Historical Data

A partial list of the past experimental efforts includes Allen (1900), Arnold (1911), Shakespear (1914), Wieselsberger (1922,

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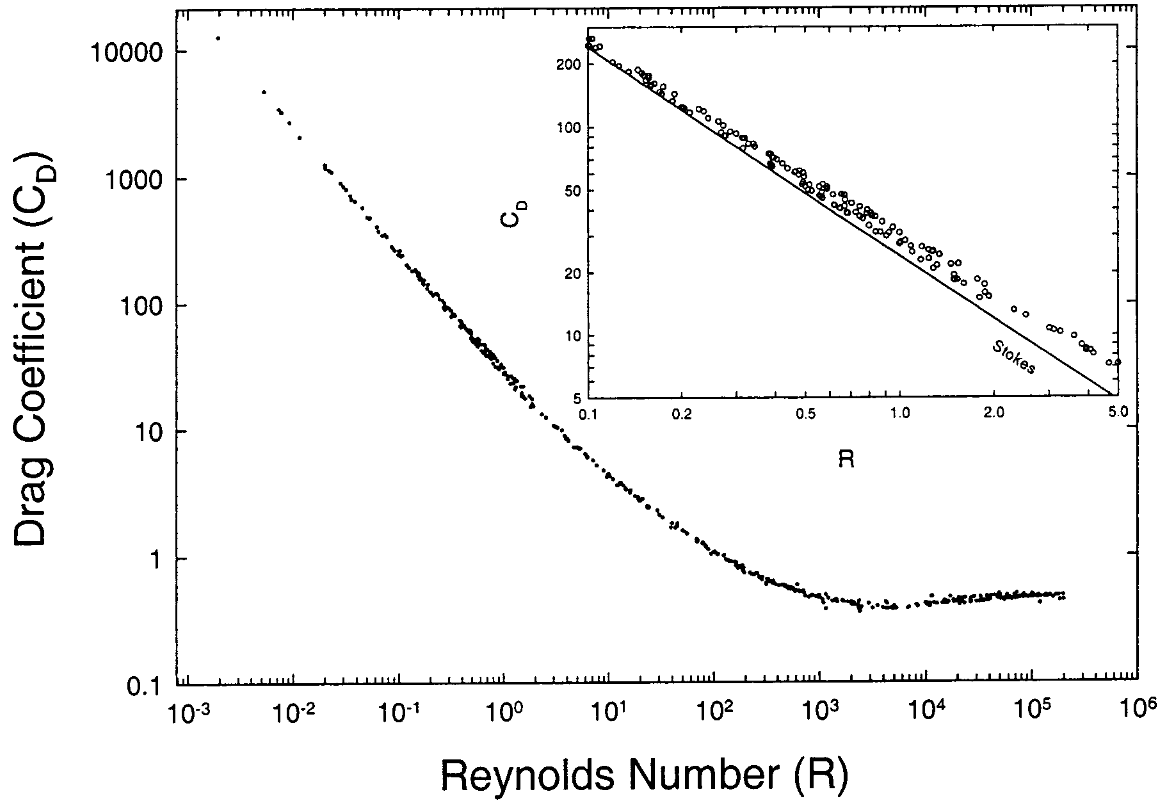

Fig. 1. Raw sphere drag data for $R < 2 \times 10^5$ (480 points)

1923), Liebster (1927), Lunnon (1928), Schmiedel (1928), Moller (1938), Davies (1945), Pettyjohn and Christiansen (1948), Gunn and Kinzer (1949), Maxworthy (1965), Goin and Lawrence (1968), Pruppacher and Steinberger (1968), Beard and Pruppacher (1969), Rimon and Cheng (1969), Bailey and Hiatt (1971, 1972), Dennis and Walker (1971), Roos and Willmarth (1971), Achenbach (1972), Vlajinac and Covert (1972), and Hartman et al. (1994). Of these works, Wieselsberger (1922), Maxworthy (1965), Goin and Lawrence (1968), Pruppacher and Steinberger (1968), Beard and Pruppacher (1969), Rimon and Cheng (1969), Achenbach (1972), and Bailey and Hiatt (1972) presented their data only in graphical form, so it is not possible to obtain their data with precision.

The remaining sixteen papers presented a total of 606 data points in tabular form. Some of these data points must be removed from consideration for various reasons. Gunn and Kinzer's data (1949) were for water droplets, so 24 data points with Reynolds numbers greater than 270 do not apply for spheres, since the droplets deformed at higher velocities. Neither Shakespear (1914) nor Allen (1900) provided Reynolds number or drag data, so their 15 data points are of no use in describing the drag characteristics of spheres. Limiting the Reynolds number range of interest to less than 2×10^5 eliminates 26 data points from the measurements of Wieselsberger (1922, 1923) and those of Vlajinac and Covert (1972). Thus, the works referenced above yield a preliminary raw set of 541 data points describing the drag behavior of spheres. However, further selection improves the quality of this data set. In fact, Clift et al. (1978) appear to have used a subset of this data (478 points) for the bulk of their multisegment drag correlation. Also, other previous analyses of sphere drag (Turton and Levenspiel 1986; Turton and Clark 1987; and Haider and Levenspiel 1989) have used a 408-point subset of these data that was apparently derived by excluding the data of Arnold (1911), Bailey and Hiatt (1971), Hartman et al. (1994), and Dennis and Walker (1971).

In selecting among the remaining data points, several factors were considered. Roos and Willmarth (1971) indicated their experimental apparatus experienced much vibration at higher Reynolds numbers, so 44 of their data points for $R > 5{,}500$ were deleted. Vlajinac and Covert (1972) provided estimates of the total error in their drag coefficients, ranging as high as 10%; so the ten data points with total errors greater than 2% were excluded. Bailey (1974) concluded that, at the upper Reynolds number range of Lunnon's work (1928), the vertical fall distance was insufficient to achieve terminal velocity, so the six points with $R > 50{,}000$ were also removed from consideration. Finally, one point in Arnold's data (1911) was excluded because the time of fall appears anomalous. These additional excisions produced a final raw data set of 480 points of very high quality. The raw data set is plotted in Fig. 1. Fig. 1 displays the classic shape of the drag curve for spheres, but it is notable that there is visible scatter in the data points for all Reynolds numbers greater than approximately 0.05. Both the scatter and the gradually increasing departure from the Stokes equation [Eq. (6)] as the Reynolds number increases are obvious in the expanded view included in Fig. 1.

Wall Effects

Some of the scatter seen in Fig. 1 is due simply to the inherent uncertainty or error in experimental work. However, it is our contention that some of the scatter (in the low Reynolds number range) is due to the influence of the wall effect on terminal velocities of spheres in small cylinders. The experiments of Arnold

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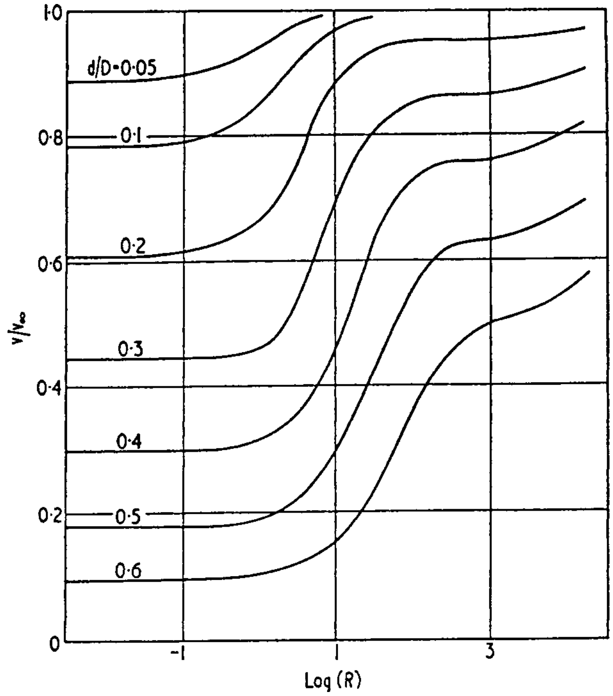

Fig. 2. Wall effect velocity corrections in circular cylinders [reproduced from Fidleris and Whitmore (1961)]

(1911), Schmiedel (1928), and Hartman et al. (1994) were conducted in circular cylinders, while those of Liebster (1927), Pettyjohn and Christiansen (1948), Lunnon (1928), and Moller (1938) were performed in square vessels. Most of these experimenters took pains to minimize the influence of the wall effect on their data [Arnold (1911) was an exception—much of the scatter at Reynolds numbers less than 13 is due to the inclusion of Arnold's data]. Some investigators applied limited corrections to account for it, but the cylinder walls had a discernible effect on the data obtained by most of these researchers.

The wall effect refers to the retardation of the motion of an object in a cylinder due to the displacement and opposing motion of the surrounding fluid. The wall effect for spheres was thoroughly investigated by Fidleris and Whitmore (1961), among others. Interpreting the results of over 3,000 velocity measurements, these investigators found that the wall effect depended on two variables, the ratio of the sphere diameter to the cylinder diameter (d/D) and the sphere Reynolds number. Fidleris and Whitmore (1961) presented wall effect correction factors graphically, with the ratio of the terminal velocity in a cylinder to the terminal velocity in an infinite fluid (V/V_∞) plotted versus the Reynolds number at V_∞ for various diameter ratios. For convenience, Fidleris and Whitmore's graph is reproduced here in Fig. 2. Reference to the graph shows that the wall effect is greatest at low Reynolds numbers and high diameter ratios. For small diameter ratios (less than 0.1), the effect becomes minuscule at Reynolds numbers above 100.

In the current work, the results of Fidleris and Whitmore (1961) were used to correct the sphere drag data in the data set described herein. Information in the original papers was used to determine the diameter ratio for each data point. For results obtained in square vessels, the diameter of an equal-area circular cylinder was used in calculating this ratio. The appropriate correction factor (V/V_∞) was then obtained via interpolation of Fidleris and Whitmore's results (1961). Once the velocity correction factor was determined, the originally reported values of R and C_D were corrected. Table 1 summarizes the range of correction factors that were applied to the data of various researchers.

Table 1. Range of Correction Factors Applied to Sphere Drag Data

Researcher	Range of velocity correction factors applied
Liebster (1927)	0.953–1.000
Arnold (1911)	0.853–0.976
Pettyjohn and Christiansen (1948)	0.964–1.000
Hartman et al. (1994)	0.999–1.000
Lunnon (1928)	1.000 (no corrections needed)
Moller (1938)	0.983–0.996
Schmiedel (1928)	0.897–0.979

In total, 178 data points in the 480-point raw data set were corrected. All of the corrected data points have Reynolds numbers less than 80, and the vast majority has Reynolds numbers less than 10. These corrections yielded a data set in which, we believe, the influence of the wall effect has been eliminated to the greatest extent possible. The corrected data set is plotted in Fig. 3. A comparison with Fig. 1 shows the scatter in the low Reynolds number data has been dramatically reduced, and Arnold's data (1911) now coincide with those of other researchers.

Correlations for Drag

The purpose of most of the cited sphere drag experiments was to develop correlations describing either the drag coefficient or terminal velocity of a sphere, as noted herein. Numerous sphere drag correlations have been proposed. Clift et al. (1978) and Khan and Richardson (1987) list many of them. In the environmental (water) field, the relation proposed by Fair and Geyer (1954) has been the most widely used; this correlation is the one that is cited in most reference and text books, including Metcalf and Eddy (1991), Montgomery (1985), American Water Works Association (1999), Droste (1997), and Reynolds and Richards (1996). Later correlations include those given by Clift et al. (1978), Flemmer and Banks (1986), Turton and Levenspiel (1986), Khan and Richardson (1987), and Haider and Levenspiel (1989). These correlations are given in Table 2, as Eqs. (9) through (14), respectively.

The correlations in Table 2 were fitted by the various authors to data sets contaminated (to some extent) by wall effects. We sought to improve upon these correlations by fitting them to the 480-point data set described earlier. Equations of the general forms given in the four most recent correlations were sought, with the parameters determined by a local minimization of the sum of the squared errors, Q, defined in Eq. (15)

$$Q = \sum_{1}^{n} (\log C_{D_{\text{exp}}} - \log C_{D_{\text{calc}}})^2 \tag{15}$$

The local minimization for the parameters in each correlation was obtained by systematically attempting all combinations of the parameters (in which all parameters have three significant digits) near the set of parameters proposed by the original authors. Hence, the parameter combinations given here are truly only local minimizations, and may not be the best global minimizations of the errors. The correlations developed in this manner are given in Eqs. (16) through (19)

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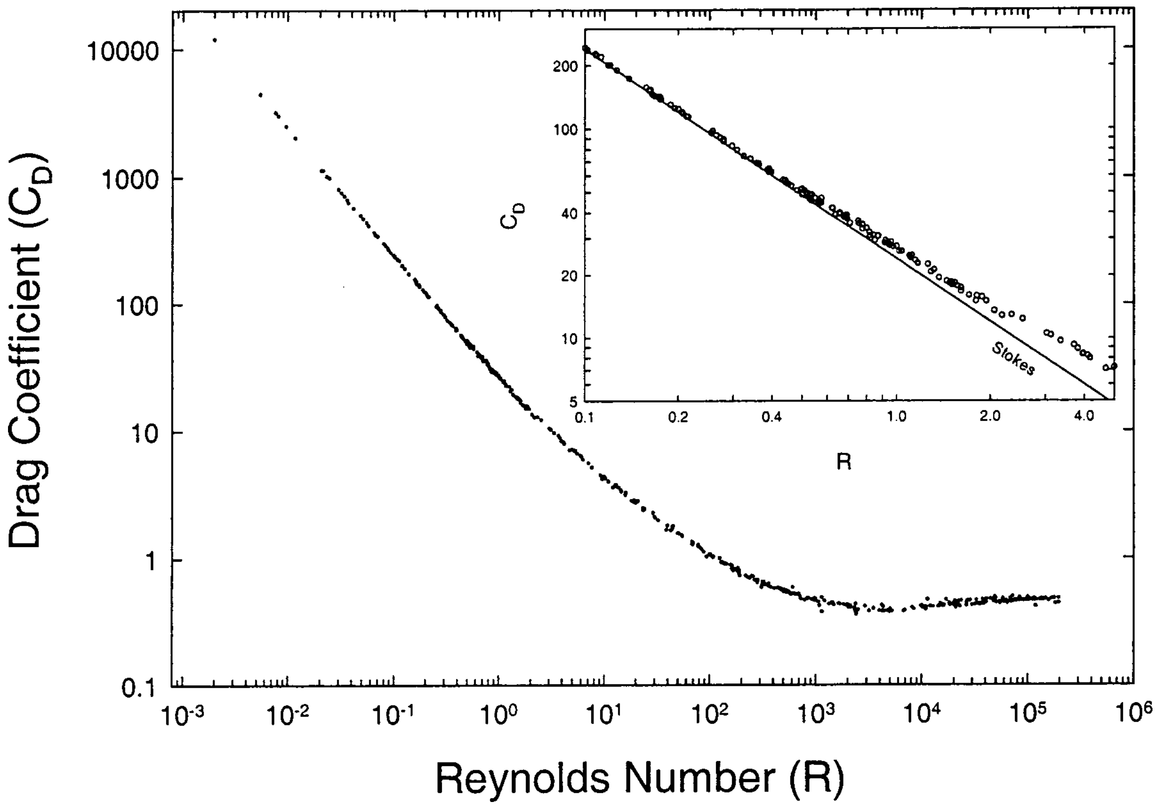

Fig. 3. Subcritical sphere drag data corrected for wall effect (480 points with $\mathsf{R}<2\times10^5$)

$$C_D=\frac{24}{\mathsf{R}}10^E,\quad \text{where}\quad E=0.383\mathsf{R}^{0.356}-0.207\mathsf{R}^{0.396}-\frac{0.143}{1+(\log\mathsf{R})^2} \tag{16}$$

$$C_D=\frac{24}{\mathsf{R}}(1+0.152\mathsf{R}^{0.677})+\frac{0.417}{1+5{,}070\mathsf{R}^{-0.940}} \tag{17}$$

$$C_D=(2.49\mathsf{R}^{-0.328}+0.340\mathsf{R}^{0.0670})^{3.18} \tag{18}$$

$$C_D=\frac{24}{\mathsf{R}}(1+0.150\mathsf{R}^{0.681})+\frac{0.407}{1+\dfrac{8{,}710}{\mathsf{R}}} \tag{19}$$

The goodness of fit of all these correlations can be evaluated using several measures, as explained next. The sum of the squared errors (i.e., the value of Q) is an obvious one. The root-mean-square deviation gives an indication of the average displacement of measured C_D values from the correlations, as shown in Eq. (20):

$$\mathrm{dev}_{\mathrm{rms}}=\sqrt{\frac{Q}{n}} \tag{20}$$

Khan and Richardson (1987) used two different measures for the goodness of fit, the sum of the squares of the relative errors [Eq. (21)] and a correlation coefficient [Eq. (22)]

$$Q_{\mathrm{rel}}=\sum_1^n\left(\frac{C_{D_{\mathrm{calc}}}-C_{D_{\mathrm{exp}}}}{C_{D_{\mathrm{exp}}}}\right)^2 \tag{21}$$

$$\gamma=\frac{\Sigma C_{D_{\mathrm{exp}}}C_{D_{\mathrm{calc}}}-(\Sigma C_{D_{\mathrm{exp}}}\Sigma C_{D_{\mathrm{calc}}}/n)}{\sqrt{(\Sigma C^2_{D_{\mathrm{exp}}}-((\Sigma C_{D_{\mathrm{exp}}})^2/n))(\Sigma C^2_{D_{\mathrm{calc}}}-((\Sigma C_{D_{\mathrm{calc}}})^2/n))}} \tag{22}$$

Karamanev (1996) used the sum of the relative errors [Eq. (23)]

$$\sigma=\sum_1^n\frac{|C_{D_{\mathrm{calc}}}-C_{D_{\mathrm{exp}}}|}{C_{D_{\mathrm{exp}}}} \tag{23}$$

The degree to which the various correlations match the corrected experimental data, as indicated by a range analysis and the measures just given, is summarized in Table 3. The correlations are listed in order of decreasing sum of squared errors.

As can be seen in Table 3, the correlation of Clift et al. (1978) appears to model sphere drag best. However, it is unwieldy and includes slight discontinuities at the transitions from one Reynolds number range to another. The correlation of Fair and Geyer (1954), widely used in environmental engineering, is the worst, but part of the reason for the poor fit is the inclusion of data at higher Reynolds numbers than its intended region. The equations proposed by Flemmer and Banks (1986) and by Khan and Richardson (1987) are also rather poor; Eqs. (18) and (16) (which are of the same forms) improve the fit considerably. The correlations due to Haider and Levenspiel (1989) and Turton and Levenspiel (1986) are better. But when these last two correlations were further refined in this effort, Eqs. (19) and (17) were found to approach the accuracy of Clift et al.'s multisegment relation (1978). The degree of fit of Eqs. (19) and (17) are essentially the same, with approximately 82% of the data lying within ±5% of the correlations. Thus, Eq. (19) is recommended for use due to its greater simplicity [four fitted parameters versus the five param-

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Table 2. Sphere Drag Correlations by Various Authors

Author	Correlation	Reynolds number range	Equation no.
Fair and Geyer (1954)	$C_D=\frac{24}{\mathsf{R}}+\frac{3}{\sqrt{\mathsf{R}}}+0.34$	$\mathsf{R}<1\times10^4$	(9)
Clift et al. (1978)	$C_D=\frac{24}{\mathsf{R}}+\frac{3}{16}$	$\mathsf{R}<0.01$	(10)
	$C_D=\frac{24}{\mathsf{R}}[1+0.1315\mathsf{R}^{(0.82-0.05w)}]$	$0.01\leqslant\mathsf{R}\leqslant20$	
	$C_D=\frac{24}{\mathsf{R}}[1+0.1935\mathsf{R}^{0.6305}]$	$20\leqslant\mathsf{R}\leqslant260$	
	$\log C_D=1.6435-1.1242w+0.1558w^2$	$260\leqslant\mathsf{R}\leqslant1{,}500$	
	$\log C_D=-2.4571+2.5558w-0.9295w^2+0.1049w^3$	$1{,}500\leqslant\mathsf{R}\leqslant1.2\times10^4$	
	$\log C_D=-1.9181+0.6370w-0.0636w^2$	$1.2\times10^4\leqslant\mathsf{R}\leqslant4.4\times10^4$	
	$\log C_D=-4.3390+1.5809w-0.1546w^2$	$4.4\times10^4\leqslant\mathsf{R}\leqslant3.38\times10^5$	
	where $w=\log\mathsf{R}$		
Flemmer and Banks (1986)	$C_D=\frac{24}{\mathsf{R}}10^E$	$\mathsf{R}<8.6\times10^4$	(11)
	where		
	$E=0.261\mathsf{R}^{0.369}-0.105\mathsf{R}^{0.431}-\frac{0.124}{1+(\log\mathsf{R})^2}$		
Turton and Levenspiel (1986)	$C_D=\frac{24}{\mathsf{R}}(1+0.173\mathsf{R}^{0.657})+\frac{0.413}{1+16{,}300\mathsf{R}^{-1.09}}$	$\mathsf{R}<2.6\times10^5$	(12)
Khan and Richardson (1987)	$C_D=(2.25\mathsf{R}^{-0.31}+0.36\mathsf{R}^{0.06})^{3.45}$	$0.01<\mathsf{R}<3\times10^5$	(13)
Haider and Levenspiel (1989)	$C_D=\frac{24}{\mathsf{R}}(1+0.1806\mathsf{R}^{0.6459})+\frac{0.4251}{1+\frac{6{,}880.95}{\mathsf{R}}}$	$\mathsf{R}<2.6\times10^5$	(14)

eters of Eq. (17)] and is valid for Reynolds numbers less than 2×10^5.

As can be seen in Fig. 4, Eq. (19) fits the data very well throughout the entire range of Reynolds numbers considered. The only discrepancies are a slight underprediction of drag coefficient for Reynolds numbers between 3 and 90 and a minor overprediction of C_D at Reynolds numbers between 100 and 400.

Correlations for Terminal Settling Velocity

An analysis similar to the one just presented for sphere drag can likewise be performed for the settling velocity of a sphere in an infinite fluid, using the same 480-point data set. Such correlations are generally cast as a dimensionless settling velocity, u_*, as a function of a dimensionless sphere diameter, d_*. These quantities

Table 3. Fit of Drag Correlations to Corrected Data

						Range Analysis: Percent of Data Points within Specified Range of Correlation			
Correlated	Sum of squared error (Q) Eq. (15)	rms[a] deviation ($\mathrm{dev}_{\mathrm{rms}}$) Eq. (20)	Sum of squared relative error (Q_{rel}) Eq. (21)	Sum of relative error (σ) Eq. (23)	Correlation coefficient (γ) Eq. (22)	±15%	±10%	±7.5%	±5%
Fair and Geyer (1954)	2.131	0.0666	8.988	51.4	0.999 981 4	62.9	53.3	49.0	39.0
Flemmer and Banks (1986)	0.578	0.0347	2.456	22.3	0.999 984 0	95.4	91.0	84.6	67.7
Khan and Richardson (1987)	0.435	0.0301	2.104	23.8	0.999 501 8	94.4	82.1	74.0	62.7
Eq. (18)	0.209	0.0209	1.132	17.4	0.999 983 6	99.0	95.6	89.8	75.2
Eq. (16)	0.198	0.0203	1.044	17.5	0.999 980 7	99.2	96.3	92.3	73.3
Haider and Levenspiel (1989)	0.195	0.0201	1.082	18.4	0.999 981 0	100.0	97.3	91.0	71.9
Turton and Levenspiel (1986)	0.188	0.0198	1.042	18.0	0.999 982 1	100.0	97.7	91.9	72.9
Eq. (19)	0.151	0.0177	0.810	15.5	0.999 984 4	99.8	97.7	94.4	81.3
Eq. (17)	0.150	0.0177	0.807	15.4	0.999 984 3	99.8	97.7	94.2	81.7
Clift et al. (1978)	0.146	0.0174	0.816	14.5	0.999 985 3	99.6	98.1	95.0	81.5

[a]rms indicates root-mean square.

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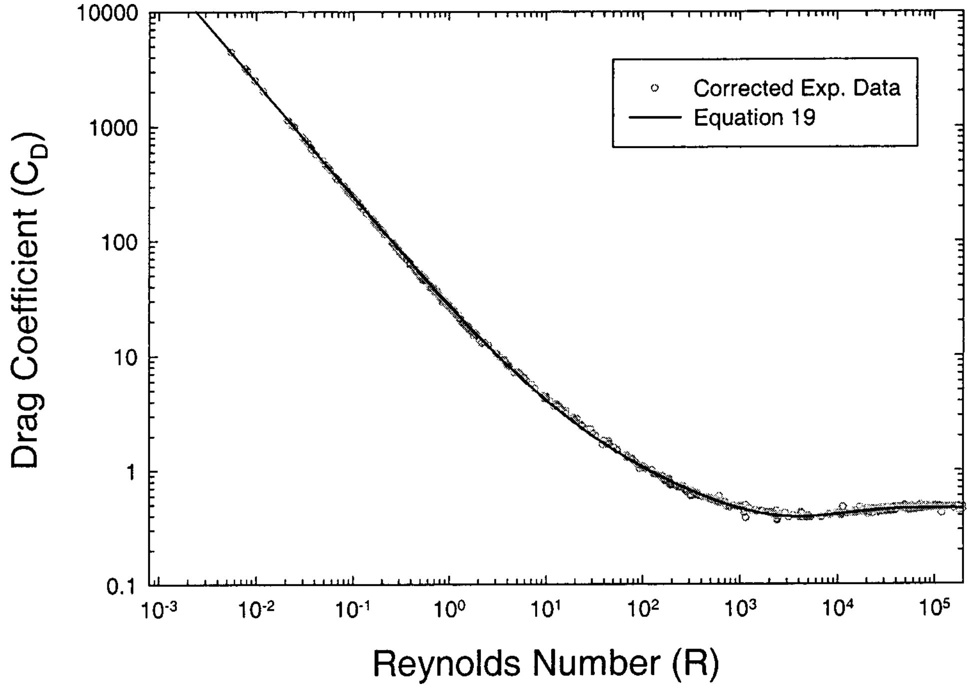

Fig. 4. Comparison of recommended drag correlation [Eq. (19)] to data

account for all of the variables that affect the settling velocity and are directly calculable from particle and fluid properties as follows:

$$u_* = u_t\left[\frac{\rho_f^2}{g\mu(\rho_p-\rho_f)}\right]^{1/3} = \left(\frac{4\mathsf{R}}{3C_D}\right)^{1/3} \tag{24}$$

$$d_* = d_p\left[\frac{g\rho_f(\rho_p-\rho_f)}{\mu^2}\right]^{1/3} = \left(\frac{3}{4}C_D\mathsf{R}^2\right)^{1/3} \tag{25}$$

Settling velocity correlations have been developed by Zigrang and Sylvester (1981), Khan and Richardson (1987), and Turton and Clark (1987). The correlations given by Zigrang and Sylvester (1981) and Turton and Clark (1987) rely on two fitted parameters, while the correlation of Khan and Richardson (1987) uses five fitted parameters. These correlations were designed for use at subcritical Reynolds numbers ($\mathsf{R}<300{,}000$) and are detailed next.

Zigrang and Sylvester (1981) $$u_* = \frac{[(14.51+1.83d_*^{3/2})^{1/2}-3.81]^2}{d_*} \tag{26}$$

Khan and Richardson (1987) $$u_* = \frac{(2.33d_*^{0.054}-1.53d_*^{-0.048})^{13.3}}{d_*} \tag{27}$$

Turton and Clark (1987) $$u_* = \left[\left(\frac{18}{d_*^2}\right)^{0.824} + \left(\frac{0.321}{d_*}\right)^{0.412}\right]^{-1.214} \tag{28}$$

Turton and Clark's correlation (1987) is notable in that it was derived using a weighted combination of the two asymptotic values that u_* assumes at either very low or very high Reynolds numbers. Correlations of the same form as Eqs. (26)–(28) were sought using the same procedure as just described for local minimization of Q, now defined for the dimensionless settling velocity rather than the drag coefficient. In addition, a correlation similar to Turton and Clark's was investigated. This new form of correlation took the general form of

$$u_* = \left[\left(\frac{18}{d_*^2}\right)^{((ad_*+1)/(d_*+1))b} + \left(\frac{c}{d_*}\right)^{b/2}\right]^{-(1/b)} \tag{29}$$

where a, b, and c = constant undetermined parameters. This form defines the exponent on the first term to be a function of d_* equal to $((ad_*+1)/(d_*+1))b$. At low values of d_*, this exponent approaches a value of b, allowing a correlation which is consistent with Stokes' law at low d_*. However, at higher values of d_*, the exponent approaches a value of ab, hopefully allowing a closer fit to the experimental data. This new form of correlation uses three fitted parameters, whereas Turton and Clark's used two, but the correlation collapses to the identical form of Turton and Clark's if the parameter a is set to a value of 1.0.

The newly fitted correlations corresponding to Eqs. (26)–(29) are given next as Eqs. (30)–(33), respectively

$$u_* = \frac{[(14.82+1.92d_*^{3/2})^{1/2}-3.85]^2}{d_*} \tag{30}$$

$$u_* = \frac{(2.63d_*^{0.0479}-1.82d_*^{-0.0351})^{13.6}}{d_*} \tag{31}$$

$$u_* = \left[\left(\frac{18}{d_*^2}\right)^{0.901} + \left(\frac{0.330}{d_*}\right)^{0.440}\right]^{-1.110} \tag{32}$$

$$u_* = \left[\left(\frac{18}{d_*^2}\right)^{((0.936d_*+1)/(d_*+1))0.898} + \left(\frac{0.317}{d_*}\right)^{0.449}\right]^{-1.114} \tag{33}$$

The adequacies of Eqs. (30)–(33) and the older correlations are summarized in Table 4. Here, the settling velocity correlations

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Table 4. Fit of settling Velocity Correlations to Corrected Data

Correlation	Sum of squared error (Q)	rms[a] deviation ($\mathrm{dev_{rms}}$)	Sum of squared relative error (Q_{rel})	Sum of relative error (σ)	Correlation coefficient (γ)	Range Analysis: Percent of Data Points within Specified Range of Correlation			
						$\pm 15\%$	$\pm 10\%$	$\pm 7.5\%$	$\pm 5\%$
Khan and Richardson (1987)	0.780	0.0403	3.299	25.7	0.997 61	96.5	82.3	76.0	66.5
Zigrang and Sylvester (1981)	0.686	0.0378	3.264	34.4	0.998 18	89.8	70.6	55.6	31.3
Turton and Clark (1987)	0.458	0.0309	2.220	27.6	0.999 11	96.9	78.3	64.8	49.0
Eq. (30)	0.386	0.0284	2.062	25.3	0.998 20	99.2	86.3	67.3	52.9
Eq. (31)	0.363	0.0275	1.633	21.4	0.997 73	98.8	96.5	89.0	62.7
Eq. (32)	0.343	0.0267	1.798	25.1	0.999 20	99.6	90.8	72.5	50.4
Eq. (33)	0.184	0.0196	0.968	18.0	0.999 10	99.8	98.5	92.9	70.8

[a]rms indicates root-mean square.

are gauged against the same measures used in evaluating the drag correlations and are again listed in order of decreasing sum of squared errors.

Inspection of Table 4 shows that the correlation of Khan and Richardson (1987) yielded the highest sum of squared errors. It is worth noting that this correlation, as well as the related correlation of Eq. (31), departs significantly from the experimental data at low d_*. Thus, these two correlations are only applicable for d_* greater than approximately 1.0 (Reynolds number above approximately 0.05). Of the remaining two correlations proposed by other authors, Turton and Clark's (1987) matches the experimental data best. However, the new correlations developed in this work do a significantly better job in modeling settling velocities of spheres. Eq. (33) offers the most notable improvement, with almost 93% of the corrected experimental data lying within $\pm 7.5\%$ of the correlation. Thus, Eq. (33) is recommended for modeling sphere settling velocities at all Reynolds numbers less than 2×10^5.

In spite of the close fit to the data of Eq. (33), it still has significant error associated with it. In fact, all of the aforementioned correlations yield predictions of settling velocities that, in general, are accurate to no more than $\pm 5\%$. In many applications, it would be useful to be able to quickly determine the settling velocity of a sphere with greater accuracy than this. It has proven difficult to develop a single correlation that accurately fits the entire range of settling velocity data, but many engineering applications reside squarely within the low end of the Reynolds number scale. For example, in sedimentation processes, the Reynolds numbers rarely exceed 1,000 or so. Thus, an accurate settling velocity correlation was sought for use in low to moderate Reynolds number ranges.

Settling Velocities at Low and Moderate Reynolds Numbers

In dimensionless terms, the Stokes settling velocity [Eq. (8)] is given as follows:

$$u_* = \frac{d_*^2}{18} \tag{34}$$

As noted, Stokes developed his equation by examining the limiting case of creeping flow, where all inertial effects are deemed negligible. Because inertial effects become more significant with increasing Reynolds numbers, Eqs. (8) or (34) are accurate only for low terminal Reynolds numbers. Specifically, all the corrected experimental data with Reynolds numbers less than 0.1 (30 points) lie within $\pm 2.5\%$ of Stokes velocity. For the corrected data with Reynolds numbers between 0.5 and 0.6 (15 points), Stokes equation overpredicts the settling velocity by an average of 6.4%.

McGauhey (1956) presented a terminal velocity equation derived by Thomas

$$v_t = \frac{v_s}{1+0.17\sqrt{\mathsf{R}_3}} \quad \text{or} \quad u_* = \frac{d_*^2}{18+0.17\sqrt{18}d_*^{3/2}} \tag{35}$$

where R_s = the Reynolds number at Stokes velocity. Eq. (35) provides relatively accurate solutions for settling velocities (to within approximately $\pm 5\%$ of the corrected experimental data) with terminal Reynolds numbers less than 130. Eq. (35) also allows direct calculation of the settling velocity rather than requiring an iterative solution, since it is based on the Reynolds number associated with the Stokes velocity rather than the true terminal velocity. Although the accuracy and simplicity are helpful, Eq. (35) neither offers exceptional accuracy nor does it apply to the larger particles encountered in water, either in the environment or in treatment processes.

After studying the correlations of previous researchers, the following four-parameter equation for settling velocity is proposed

$$v_t = \frac{v_s}{1+\dfrac{a\mathsf{R}_s^{(1/3+n)}+b\mathsf{R}_s^{(2/3+n)}}{(c+\mathsf{R}_s^n)}}$$

or

$$u_* = \frac{d_*^2(18^n c + d_*^{3n})}{18^{1/3} b d_*^{(2+3n)} + 18^{2/3} a d_*^{(1+3n)} + 18 d_*^{3n} + 18^{(1+n)} c} \tag{36}$$

Here, a, b, c, and n = undetermined constant parameters. To select appropriate values for each of these parameters, Eq. (36) was fitted to the 344 corrected experimental data points with $\mathsf{R} < 5{,}000$ $(d_* < 190)$. The same local minimization of the Q procedure was used. The sum of the squared errors was minimized with the parameter values $a = 0.409$, $b = 0.00985$, $c = 3.13$, and $n = 0.682$. Thus, the settling velocity correlation arising from Eq. (36) is

$$u_* = \frac{d_*^2(22.5+d_*^{2.046})}{0.0258d_*^{4.046}+2.81d_*^{3.046}+18d_*^{2.046}+405} \tag{37}$$

The goodness of fit of the various settling velocity correlations to the corrected experimental data for Reynolds numbers less than 5,000 is shown in Table 5. The correlations are again listed in order of decreasing sum of squared errors.

The data in Table 5 show that Eq. (37) offers greatly improved accuracy over the other settling velocity correlations available. In

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Table 5. Fit of Settling Velocity Correlations to Correlated Data in Low-Moderate Reynolds Number Range ($R<5{,}000$)

Correlation	Sum of squared error (Q)	rms[a] deviation (dev_{rms})	Sum of squared relative error (Q_{rel})	Sum of relative error (σ)	Correlation coefficient (γ)	Range Analysis: Percent of Data Points within Specified Range of Correlation			
						±10%	±5%	±2%	±1%
Zigrang and Sylvester (1981)	0.612	0.0422	2.867	28.0	0.999 34	60.8	21.8	7.0	4.1
Khan and Richardson (1987)	0.513	0.0386	1.691	12.4	0.999 12	94.5	84.6	47.7	24.4
Turton and Clark (1987)	0.418	0.0349	2.007	23.0	0.998 96	69.8	36.9	12.5	5.5
Eq. (32)	0.311	0.0301	1.631	21.0	0.998 24	87.2	35.2	11.6	5.2
Eq. (31)	0.288	0.0289	1.217	14.9	0.999 15	96.5	65.4	25.0	12.5
Eq. (30)	0.222	0.0254	1.100	15.4	0.999 33	89.5	60.5	32.9	17.7
Eq. (33)	0.139	0.0201	0.725	13.1	0.998 97	98.0	69.2	28.5	13.1
Eq. (37)	0.034	0.0099	0.177	6.1	0.999 25	100.0	96.5	64.8	36.3

[a]rms indicates root-mean square.

fact, even when the general forms of Eqs. (30)–(33) are re-fitted to only the data for $R<5{,}000$, Eq. (37) provides less than half of the sum of squared errors of its nearest competitor. When used for Reynolds numbers less than 4,000, Eq. (37) predicts terminal settling velocities with an accuracy of approximately ±2.5%. At Reynolds numbers greater than approximately 5,000, however, it systematically underestimates the terminal velocity, and another correlation [e.g., Eq. (33)] should be used. Fig. 5 illustrates how closely the velocity correlations of Eqs. (33) and (37) match experimental data.

An inspection of Fig. 5 demonstrates that Eq. (33) does a good job of modeling sphere settling velocities over the entire range of experimental data. However, this correlation does predict terminal velocities that are mildly higher than the data where d_* lies between 10 and 20 and at d_* greater than approximately 600. The predicted velocities lie below the data for d_* between 40 and 300. No such shortcomings are visible in the plot of Eq. (37) for d_* less than approximately 200. The limitation on the use of Eq. (37) is that it only applies in this low-moderate Reynolds number range and gives erroneous results in the high Reynolds number regime.

Examples

A few examples of the calculations help demonstrate the value of these correlations; two sets of calculations relating to applications of interest in water treatment are indicated in Table 6. In the analysis that follows, the values associated with Eq. (37) are assumed to be the best estimates, given the excellent fit of that equation to the corrected data. Use of the Fair and Geyer equation (1954) (which is not based on d_*) requires an iterative solution to

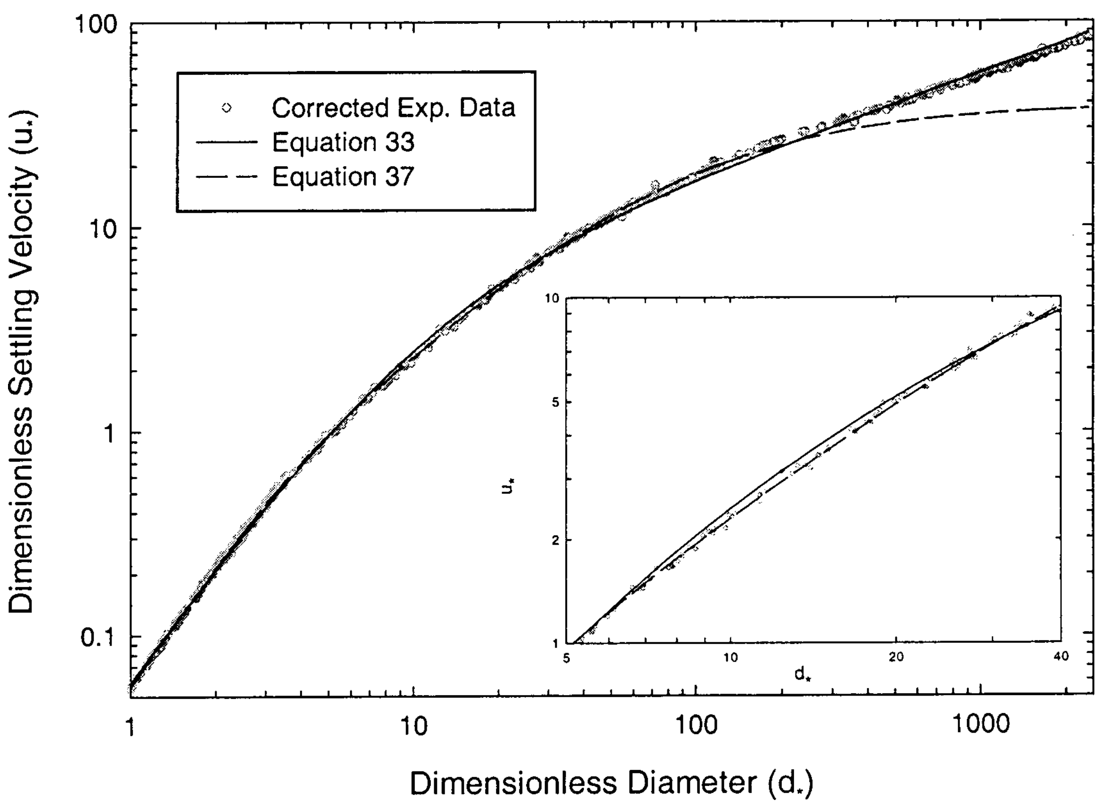

Fig. 5. Comparison of recommended settling velocity correlations to data

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Table 6. Example Calculations for Some Applications

Particle density (kg/m^3)	Particle diameter (10^{-6} m)	$d*$	R[a]	Predictions of Settling Velocity in Water (m/s): This paper [Eq. (37)]	Stokes (1880) [Eq. (34)]	This paper [Eq. (33)]	Fair and Geyer (1954) [Eqs. (4) and (9)]	Khan and Richardson (1987) [Eq. (27)]	Turton and Clark (1987) [Eq. (28)]
Example no. 1									
2,600	200	5.00	4.80	0.0241	0.0348	0.0239	0.0257	0.0241	0.0227
2,600	400	10.0	23.0	0.0576	0.139	0.0615	0.0684	0.0594	0.0603
2,600	800	20.0	98.2	0.123	0.557	0.130	0.142	0.126	0.129
2,600	1,600	40.0	376	0.236	2.23	0.230	0.247	0.236	0.229
Example no. 2									
1,400	3	0.0472	$5.88E-6$	$1.97E-6$	$1.97E-6$	$2.01E-6$	$1.97E-6$	NA[b]	$1.96E-6$
1,400	10	0.158	0.000218	$2.18E-5$	$2.18E-5$	$2.30E-5$	$2.18E-5$	NA	$2.16E-5$
1,400	30	0.473	0.00587	0.000196	0.000197	0.000211	0.000195	NA	0.000191
1,400	100	1.576	0.212	0.00213	0.00219	0.00215	0.00206	0.00216	0.00194
1,400	300	4.729	4.20	0.0141	0.0197	0.0138	0.0148	0.0140	0.0131

[a]Calculated using the settling velocity predicted by Eq. (37).

[b]NA indicates not applicable.

find a value that simultaneously satisfies Eqs. (4), (7), and (9); the other correlations allow direct calculation of the settling velocity.

The top half of Table 6 (Example No. 1) is based on particles with a density of 2,600 kg/m^3 and diameters ranging from 0.2 to 1.6 mm. This set is intended to represent a range somewhat broader than the sand grains used for deep bed filtration; most filters have sand grains in the range of the middle two sizes shown. When such filter beds are cleaned by backwashing, the settling velocity of the individual particles must be known, as the backwash velocity (and flow rate) are based on that settling velocity. These particles all have dimensionless diameters much greater than one, and Reynolds numbers far higher than the region where Stokes' law applies. Hence, the Stokes velocity is much greater than any other prediction in Table 6 and is only included for reference. Relative to Eq. (37), the Fair and Geyer (1954) estimates are as much as 19% high (for the 0.4 mm particles), a quite substantial error that could lead to poor design or operation of backwashing facilities. The Khan and Richardson (1989) estimates are high for the middle two sizes but agree exactly with Eq. (37) for the two extremes shown. The Turton and Clark (1987) values are low at both extreme sizes and several percent high for the middle sizes chosen; the greatest difference of this correlation from Eq. (37) is just less than 6%.

The bottom half of Table 6 (Example No. 2) reflects particles, flocs, and precipitates that might be obtained in metals removal in drinking water or industrial wastewater treatment. Both the size and density are lower than in the previous example, so that some of the Reynolds numbers are very low. In this range (first three particles with the density of 1,400 kg/m^3), Eq. (37) agrees with the Stokes velocity, as expected. For the larger sizes that might enter a sedimentation facility after growing in flocculation, the corrections to the Stokes velocity are essential. For these particles, the Fair and Geyer (1954) correlation does reasonably well [with a maximum difference from Eq. (37) of 5%], the Khan and Richardson (1987) correlation is very good, and the Turton and Clark (1987) predictions are quite good except at the highest Reynolds number (with a difference of almost 9%). If it is true that Eq. (37) is the best estimate of the true velocity, the errors of the other correlations are sufficient to cause some design, modeling, or operations errors.

Conclusions

The historical experimental data on sphere drag were critically selected to produce a high-quality raw data set of 480 points. The data points in this set that originated from terminal velocity measurements in cylinders were corrected for the wall effect using the results of Fidleris and Whitmore (1961). Thus the final data set is believed to be free of the influence of the wall effect and therefore truly reflects the behavior of spheres in an infinite fluid.

Previously published correlations of sphere drag were compared to the corrected experimental data and the relations proposed by Clift et al. (1978) were found to most closely match the data. However, that correlation consists of seven different equations in the Reynolds number region studied here, making it somewhat inconvenient. Correlations of the forms proposed by four previous research teams were fitted to the new data set. The four-parameter drag correlation of Eq. (19) was found to be very satisfactory and is recommended for use at $\mathsf{R}<2\times10^5$.

A similar analysis of settling velocity correlations was performed. This analysis was conducted both for all data $\mathsf{R}<2\times10^5$ and for the data in the low-moderate Reynolds number range ($\mathsf{R}<5{,}000$). The three-parameter correlation of Eq. (33) was found to most closely match the entire range of corrected experimental results and is recommended for general use. However, for $\mathsf{R}<4{,}000$, Eq. (37) was found to predict settling velocities with an accuracy of approximately $\pm2.5\%$, far better accuracy than any other correlation known to the authors. Eq. (37) should not be used for Reynolds numbers greater than 5,000.

For many years, environmental (water) engineers have used the Fair and Geyer equation (1954) for drag along with a trial and error solution to calculate the terminal settling velocity of a sphere. This approach should be retired in favor of the simpler, and more accurate, direct calculation of the terminal velocity made possible by Eqs. (33) or (37), as appropriate.

Notation

The following symbols are used in this paper:

a,b,c = fitted parameters, dimensionless;

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C_D = drag coefficient, dimensionless;
D = cylinder diameter, m;
d = sphere diameter, m;
F = force, kg m/s^2;
g = gravitational acceleration, m/s^2;
n = fitted parameter, dimensionless, or number of data points;
Q = sum of squared errors;
R = Reynolds number, dimensionless;
u,v = velocity, m/s;
γ = correlation coefficient, dimensionless;
μ = absolute fluid viscosity, kg/m s;
ρ = density, kg/m^3; and
σ = sum of errors.

Subscripts

b = buoyant;
calc = calculated;
d = drag;
exp = experimental;
f = fluid property;
g = gravitational;
p = particle property;
rel = relative;
s = Stokes;
t = terminal;
* = dimensionless; and
∞ = in infinite fluid.

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