Grand Prix of Hailiang, Stage 2

Jury's Solution and Tutorial

一、选择题：每题 2 分，共 15 题，30 分. 在每小题给出的四个选项中，只有一项是符合题目要求的.

题号	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15
答案	B	D	C	D	A	C	D	B	B	A	B	A	C	A	B

二、阅读程序（程序输入不超过数组或字符串定义的范围；判断题正确填 ✓，错误填 ✗；除特殊说明外，判断题每题 1.5 分，选择题每题 4 分，共 40 分）

（一） （12 分）阅读下面一段程序，完成 16 ∼ 21 六道小题。

题号	P16	P17	P18	P19	P20	P21
答案	✓	✓	✓	✗	B	C
分值	1	1	1.5	1.5	3	4

（二） （13 分）阅读下面一段程序，完成 22 ∼ 27 六道小题。

题号	P22	P23	P24	P25	P26	P27
答案	✓	✗	✗	✗	A	D
分值	1.5	1.5	1.5	1.5	3.5	3.5

（三） （15 分）阅读下面一段程序，完成 28 ∼ 33 六道小题。

题号	P28	P29	P30	P31	P32	P33
答案	✓	✓	✗	✓	D	B
分值	1.5	1.5	1.5	1.5	4	5

三、完善程序（单选题，每小题 3 分，共 30 分）

（一） （15 分）阅读下面一段程序，完成 34 ∼ 38 五道小题。

题号	P34	P35	P36	P37	P38
答案	D	D	C	A	D

（二） （15 分）阅读下面一段程序，完成 39 ∼ 43 五道小题。

题号	P39	P40	P41	P42	P43
答案	B	B	A	D	C

四、Solutions to Selected Problems

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Problem 10

Let $\mathcal{D}_n$ be the n-th Derangement number, i.e. $\mathcal{D}_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k}$.

- $4+0$, $\binom{4}{4}\binom{2}{0}\mathcal{D}_4 = 9$.
- $3+1$, $\binom{4}{3}\times\binom{2}{1}\times(\mathcal{D}_3+3\times 3) = 88$
- $2+2$, $\binom{4}{2}\times\binom{2}{2}\times(2!\times\mathcal{D}_2+2\times 4+2!\times 2!) = 84$

Thus, ans $= 9+88+84 = 181$.

Problem 11

Let F be the generating function of $\{f_n\}$, obviously that

$$F(x) = 3xF(x) - 2x^2F(x) + 2 - 3x$$

So $F(x) = \frac{2-3x}{2}\cdot(-2)\left(\frac{1}{1-x}-\frac{1}{1/2-x}\right) = (3x-2)\left(\sum_n (1-2\cdot 2^n)x^n\right)$.
Then $[x^n]F(x) = 3\cdot(1-2^n) - 2\cdot(1-2^{n+1}) = 1+2^{n-1}$ and we got the answer.

Problem 12

Let F be the generating function of a given sequence $p_n(\sum p_n = 1)$ where $\Pr(X=k) = p_k$.
Then we have:

$$\begin{cases} F(1) = \sum_n p_n = 1 & = \mathbb{E}[X^0] \\ F'(1) = \sum_n np_n = \mathbb{E}[X] & = \mathbb{E}[X^1] \\ \dots \\ F^{(d)}(1) = \sum_n n^{\underline{d}}p_n = \mathbb{E}[X] & = \mathbb{E}[X^{\underline{d}}] \end{cases}$$

So we have:

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mathbb{E}[X])^2] = F''(1) + F'(1) - (F'(1))^2$$

and...

$$F(x) = \frac{3}{4}x^2 + \frac{1}{4}x^3$$

then we have $F(1) = 1, F'(1) = \frac{9}{4}, F''(1) = 3$, so $\mathrm{Var}[X] = 3 + \frac{9}{4} - (\frac{9}{4})^2 = \frac{3}{16}$

Problem 13

see Why isn't vector<bool> a STL container?

Problem 14

First note that for $x \le -2$, the LHS is not an integer. When $x=-1$, the LHS is 2 which is not a perfect square. When $x=0$, we have $y=\pm 2$. We can try $x=1,2$ and see that these do not work.

So consider $x \ge 3$. We have $(y-1)(y+1) = 2^x(2^{x+1}+1)$. Now certainly $(y-1)$ and $(y+1)$ are both even, and since they differ by 2 at most one of them can be a multiple of 4.

But the RHS has a factor of 2^x , so what does this mean? It means this power 2^x can only be distributed among $(y-1)$ and $(y+1)$ as 2 and 2^{x-1} or 2^{x-1} and 2 respectively. For $x \ge 3$, any other distribution leaves either one of $(y-1)$,$(y+1)$ to be odd or both of them a multiple of 4.

In other words, it means either $y-1 = 2^{x-1}k$ or $y+1 = 2^{x-1}k$ for some odd k . (Note k is odd because $2x-1$ should already account for all the powers of 2 which divide $y\pm 1$.)

Going back to the original equation, we have $(2^{x-1}k \pm 1)^2 = 2^{2x+1} + 2^x + 1$. Now if $k \geq 5$, we have $y \geq 4{\times}2^{x-1} = 2^{x+1}$ so $y^2 \geq 2^{2x+2}$. This means $2^{2x+1} + 2^x + 1 \geq 2^{2x+2}$, or $2^x + 1 \geq 2^{2x} + 1$, which is evidently impossible.

Thus $k = 1$ or 3 . Let $2x - 1 = t$ for convenience. When $k = 1$, we have $(t \pm 1)2 = 8t^2 + 2t + 1$, or $t \pm 2 = 8t + 2$, which has no possible solutions for t .

When $k = 3$, we have $(3t \pm 1)^2 = 8t^2 + 2t + 1$, or $9t \pm 6 = 8t + 2$. For $9t - 6 = 8t + 2$ this gives $t = 8$ while for $9t + 6 = 8t + 2$ there are no possible $t \in \mathbb{Z}$.

So $t = 8$ which gives $x = 4$ and then $y = \pm 23$.

In conclusion, the only solutions are $(x, y) = (0, \pm 2), (4, \pm 23)$.

Problem 15

When G is not a multigraph, each region would have at least 3 edges, therefore a degree of ≥ 3.

Also note that $2e = |E|$, which is sum of degrees of r regions, where $2e \geq 3r$.

Using Euler's theorem, $2 = v - e + r \leq v - e + (2/3)e = v - (1/3)e$.

Then, we have: $6 \leq 3v - e \Longleftrightarrow e \leq 3v - 6$, that means $|E| = \mathcal{O}(|V|)$