1. Let G be a simple graph such that $\delta(G) = k \geq 2$. Prove that G contains a cycle on at least $k+1$ vertices. (3 pts extra credit: Give an example that demonstrates this result is best possible.)

Solution:
Let G be a simple graph such that $\delta(G) = k \geq 2$. Since every vertex in G has at least two neighbors, G must contain a cycle. Call it C_n. Suppose, for a contradiction, that $n < k+1$. Then when $k = 2$, $n < 3$, but this is a contradiction. The smallest cycle in a simple graph is on 3 vertices. Hence, the opposite must be true. □

2. Let G be a connected graph on n vertices. Prove that G has exactly one cycle if and only if G has exactly n edges.

Solution, part a:
$\rightarrow$ Suppose G has exactly one cycle and G is connected on n vertices. Let C_k be the cycle, $3 \leq k \leq n$. Every $uv \in E(G)$ such that $uv \notin E(C_k)$ forms a forest.

3. 2.1.12. Compute the diameter and radius of the biclique $K_{m,n}$.

Solution:
$\text{diam}(K_{m,n}) = 2$ since no vertex is ever more than two edges away from any other. On the other hand, $\text{rad}(K_{m,n}) = 1$ since every vertex in one partite set is one edge from any vertex in the other partite set. □

4. Exercise 2.1.13. Prove that every graph with diameter d has an independent set with at least $\lceil (1+d)/2 \rceil$ vertices.

Solution:
Let G be a graph with diameter d. Let $u, v \in V(G)$ be such that $d(u, v) = d$. Vertices along this path that are not consecutive are nonadjacent. Then the number of vertices on the path, including u and v, is $d+1$. Then splitting the path in half after $\lceil (1+d)/2 \rceil$ vertices yields two independent sets, one with that size

5. Exercise 2.1.14: Suppose that the processors in a computer are named by binary k-tuples and pairs can communicate directly if and only if their names are adjacent in the k-dimensional cube Q_k. A processor with name u wants to send a message to the processor with name v. How can it find the first step on a shortest path to v?

Solution:
Since pairs must be adjacent to communicate, u must choose from its neighbors. The shortest u, v-path starts along any edge to a neighbor who differs from v in the same spot that it differs from u. This will avoid backtracking, ensuring the communication always gets closer to v. (Viewed solution)

6. 2.1.18. Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. Show that this is best possible by constructing an n-vertex tree with exactly Δ leaves, for each choice of n, Δ with $n > \Delta \geq 2$.

Solution:
We proceed inductively on the number of vertices. First, the base case. For $n \leq 3$, the result is obvious, since a tree with three vertices has two vertices of degree two, a tree with two vertices has two vertices of degree one, and a tree with 0 or 1 vertices has zero degree. For $n > 3$ deleting a leaf u from T produces tree T_1 where $n(T_1) = n - 1$. Thus, we can apply the induction hypothesis to yield that there are at least $k - 1$ leaves in $T - u$. Then, adding u back in adds another vertex of maximum degree, since the maximum degree in T must not be a leaf in $T - u$. (viewed solution) □

7. 2.1.26: For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

Solution:
Let G_i be a graph obtained by deleting one vertex from G. Then G_i has $n - 1$ vertices and is a tree, so G_i has $n - 2$ edges. So, $\sum \lim_{i=1}^{n} e(G_i) = n(n - 2)$. Hence, since each edge has two endpoints, endpoints, each edge appears in $n - 2$ of these graphs, and thus counted $n - 2$ times in the sum. Since G has n vertices and n edges, it must contain a cycle. G_i is just the opposite. Thus G has a spanning cycle because every spanning cycle with n additional edges. And since G has n edges, it has no extra edges. $G = C_k$ (viewed solution) □

8. 2.1.35. Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ have odd order.

Solution, part a:
$\rightarrow$ Assume the vertices of tree T all have odd degree. Then $n(T) = 2k$ for some $k \in \mathbb{N}$, since there must be an even number of vertices of odd degree. Consider the two components in $T - e$, C_1 and C_2 where u, v are the endpoints of e. Let $u' \in C_1$ and $v' \in C_2$ represent u and v, respectively, in $T - e$. Since $n(C_1) + n(C_2) = n(T) = 2k$, $n(C_1)$ and $n(C_2)$ have the same parity. Since u has odd degree, u' has even degree while the rest of the vertices in C_1 have odd degree. Then $n(C_1)$ must be odd, since there must be an even number of vertices of odd degree plus 1 to account for u'. Hence, both components of $T - e$ have odd order.

Solution, part b:
$\leftarrow$ We prove the contrapositive. Let T be a tree. Assume there exists $e \in E(T)$ with endpoints $u, v \in V(T)$such that one of the components of $T - e$ has even order. We show that there exists a vertex in V(T) with even degree. Let C_1 and C_2 be the two components of $T - e$ such that $u' \in C_1$ and $v' \in C_2$. Suppose, without loss of generality, C_1 has even order. Then... not done, stuck