1. 1.3.4 Corollary. In a graph G the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.

Solution:

□

2. 1.3.5 Corollary. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.

Solution:
We proceed by proof by contradiction. Suppose G has an odd number of vertices of odd degree. We can split the sum of the degrees of the vertices of G into even and odd. Let $j, k, m, n \in \mathbb{N}$, where n is odd. Observe:

$$\sum_{v \in V(G)} d(v) = (2k_1 + 1) + (2k_2 + 1) + ... + (2k_n + 1) + 2j_1 + 2j_2 + ... + 2j_m$$
$$= 2(k_1 + k_2 + ... + k_n + \frac{n-1}{2} + j_1 + j_2 + ... + j_m) + 1$$

By the Degree-Sum Formula, $\sum_{v \in V(G)} d(v)$ is even, so this is a contradiction, and the result follows. □

3. 1.3.6 Corollary. A k-regular graph with n vertices has $nk/2$ edges.

Solution:
Let G be a k-regular graph with n vertices. Then $\sum_{v \in V(G)} d(v) = nk$. By the Degree-Sum Formula, $\sum_{v \in V(G)} d(v) = 2e(G)$. Setting the two equal and solving for $e(g)$ yields:

$$e(G) = \frac{nk}{2}$$

as desired. □

4. Exercise 1.3.1. Prove or disprove: If u and v are the only vertices of odd degree in a graph G, then contains a u, v-path.

Solution:
True. We proceed using proof by contradiction. Let u and v be the only vertices of odd degree in a graph G. Suppose there is no u, v-path. Then u and v must be in different components of G. Each of these components is a graph with an odd number of odd vertices. This contradicts Corollary 1.3.5. Thus, G contains a u, v-path. □

5. Exercise 1.3.3: Let u and v be adjacent in a simple graph G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

Solution:
Let u and v be adjacent in a simple graph G. We proceed using proof by cases. We

discuss the three possible cases for adding vertices to G. In the first case, suppose we add a vertex to G that is adjacent to both u and v, creating G', u', and v'. Then $u'v'$ in G' belongs to one more triangle than uv in G Notice that $d(u') = d(u) + 1$, $d(v') = d(v) + 1$, and $n(G') = n(G) + 1$. Observe:

$$\begin{aligned} d(u') + d(v') - n(G') &= d(u) + 1 + d(v) + 1 - (n(g) + 1) \\ &= (d(u) + d(v) - n(G)) + 1 \end{aligned}$$

as desired. In the second case, suppose we create G with u'' and v'' by adding a vertex to G that is adjacent to either u or v, but not both. Then $u''v''$ belongs to the same amount of triangles as uv. In this case, $d(u'') + d(v'') = d(u) + d(v) + 1$, while $n(G'') = n(G) + 1$. Thus,

$$\begin{aligned} d(u'') + d(v'') - n(G'') &= d(u) + d(v) + 1 - (n(G) + 1) \\ &= d(u) + d(v) - n(G) \end{aligned}$$

as desired. Thus, we have shown that whenever we add a vertex adjacent to either u or v or both, the formula $d(u)+d(v)-n(G)$ either increases or stays the same to appropriately represent the least number of triangles uv belongs to. In the third and final case, suppose we add a vertex to G that is not adjacent to either u or v, creating G''' with u''' and v'''. This case will show that the formula in question is only a lower bound and does not track the number of triangles uv belongs to exactly. In this case $d(u) = d(u''')$ and $d(v) = d(v''')$. Therefore, uv belongs to the same amount of triangles as $u'''v'''$. Since $n(G''') = n(G) + 1$, we have:

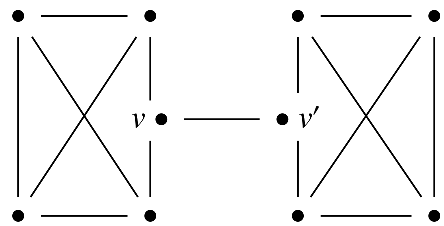
$$\begin{aligned} d(u''') + d(v''') - n(G''') &= d(u) + d(v) - (n(G) + 1) \\ &= d(u) + d(v) - n(G) - 1. \end{aligned}$$

Thus, even though uv and $u'''v'''$ belong to the same amount of triangles, whenever they belong to one triangle or more, respectively, our formula has a smaller lower bound for the number of triangles that $u'''v'''$ belongs to. Therefore, since all cases have been considered, the result follows. □

6. 1.3.12: Prove that an even graph has no cut-edge. For each $k \geq 1$ construct a $2k + 1$-regular simple graph having a cut edge.

Solution:

We proceed by proof by contradiction. Suppose G is an even graph with cut edge uv. Let the vertex degrees for G be $d_1, ..., d_n$. Then the vertex degrees for $G - uv$ is $d_1, ..., d_u - 1, ..., d_v - 1, ...d_n$. Since uv is a cut-edge of G, u and v are in different components of $G - uv$. Since G is even, every vertex, including u and v, is even, which means $d_u - 1$ and $d_v - 1$ are odd. Thus the degree sum of the vertices in the components containing u and v, respectively, will be odd. This is a contradiction because by Proposition 1.3.28., the degree sum must be odd. The result follows. Now, for each $k \geq 1$, to construct a $2k + 1$-regular simple graph having a cut edge we simply take two $K_{2(k+1)}$, put a new vertex on an exterior edge of each, v on one and v' on the other, then put in the edge vv'. For $k = 1$, see below:



□

7. 1.3.17: Let G be a graph with at least two vertices. Prove or disprove:

a) Deleting a vertex of degree $\Delta(G)$ cannot increase average degree.

b) Deleting a vertex of degree $\delta(G)$ cannot reduce average degree.

Solution, part a:
True. Let G be an n vertex graph with degree sequence in nonincreasing order $d_\Delta, ..., d_{n-1}, d_\delta$. Since $2e(G)/n(G) \leq \Delta(G) = d_\Delta$, we have:

$$\begin{aligned}
\frac{2e(G)}{n(G)} &\leq \Delta(G) \\
\frac{2(d_\Delta, ..., d_{n-1}, d_\delta)}{n} &\leq d_\Delta \\
2(d_\Delta, ..., d_{n-1}, d_\delta) &\leq d_\Delta n \\
2(d_\Delta, ..., d_{n-1}, d_\delta) &\leq 2d_\Delta n \\
2(d_\Delta, ..., d_{n-1}, d_\delta) - 2d_\Delta n &\leq 0 \\
(2d_2 n + ... + 2d_\delta n) - (2d_2 n + ... + 2d_\delta n) - 2d_\Delta n + 2(d_\Delta, ..., d_{n-1}, d_\delta) &\leq 0 \\
\frac{2n(d_2 + ... + d_\delta) - 2(n-1)(d_\Delta + d_2 + ... + d_\delta)}{(n)(n-1)} &\leq 0 \\
\frac{2(d_2 + ... + d_\delta)}{n-1} &\leq \frac{2(d_\Delta + d_2 + ... + d_\delta)}{n} \\
\frac{e(G - \Delta(G)}{n(G - \Delta(G))} &\leq \frac{2e(G)}{n(G)}
\end{aligned}$$

as desired. □

Solution, part b:
True. We use a similar argument. Let G be an n vertex graph with degree sequence in

nonincreasing order $d_\Delta, ..., d_{n-1}, d_\delta$. Since $d_\delta = \delta(G) \leq 2e(G)/n(G)$, we have:

$$\frac{2e(G)}{n(G)} \geq \delta(G)$$
$$\frac{2(d_\Delta, ..., d_{n-1}, d_\delta)}{n} \geq d_\delta$$
$$2(d_\Delta, ..., d_{n-1}, d_\delta) \geq d_\delta n$$
$$2(d_\Delta, ..., d_{n-1}, d_\delta) \geq 2d_\delta n$$
$$2(d_\Delta, ..., d_{n-1}, d_\delta) - 2d_\delta n \geq 0$$
$$(2d_\Delta n + ... + 2d_{n-1}n) - (2d_\Delta n + ... + 2d_{n-1}n) - 2d_\delta n + 2(d_\Delta, ..., d_{n-1}, d_\delta) \geq 0$$
$$\frac{2n(d_\Delta + ... + d_{n-1}) - 2(n-1)(d_\Delta + d_2 + ... + d_\delta)}{(n)(n-1)} \geq 0$$
$$\frac{2(d_\Delta n + ... + d_{n-1})}{n-1} \geq \frac{2(d_\Delta + ... + d_{n-1} + d_\delta)}{n}$$
$$\frac{e(G - \delta(G)}{n(G - \delta(G))} \geq \frac{e(G}{n(G)}$$

as desired. □

8. 1.3.18. For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

Solution:
Let $k \geq 2$. We proceed by proof by contradiction. Suppose that a, $k-$regular bipartite graph G has a cut edge uv. Without loss of generality, we assume that G is connected, since the components of k-regular graphs are connected. u and v must be in different partite sets.