1. 1.3.5. Count the copies of P_3 and C_4 in Q_k.

Solution, part a:
Q_K contains $\binom{k}{2}2^{k-2}$ copies of C_4. To show this is true, we consider Example 1.3.8. for $j = 2$. This example states that we can form a 2-dimensional subcube by keeping any $k-2$ coordinates fixed and letting the values in the remaining 2 coordinates range over all 2^2 possible 2-tuples. The subgraph induced by such a set of vertices is isomorphic to Q_2. Hence, there are $\binom{k}{2}$ ways to pick 2 coordinates to vary and 2^{k-j} ways to specify the values in the fixed coordinates. Thus, we may conclude that there are $\binom{k}{2}2^{k-2}$ such subcubes, and hence, the result follows because Q_2 is isomorphic to C_4.

Solution, part b:
Q_k contains $\binom{k}{2}2^k$ copies of P_3. Since each C_4 has 4 P_3s in it, we simply multiply the previous result by 4, which gives us the result. □

2. 1.3.7. Determine the maximum number of edges in a bipartite subgraph of P_n, of C_n, and of K_n.

Solution, part a:
For $G \subseteq P_n$, $e(G) \leq n-1$. Consider the largest subgraph of P_n without odd-cycles. It has $n-1$ edges. Thus, the result follows.

Solution, part b:
For $G \subseteq C_n$, if n is even, $e(G) \leq n$, otherwise, $e(G) \leq n-2$. If n is even, then let $G = C_n$. Since G has no odd cycle, it is bipartite with n edges. If n is odd, then let $v \in V(C_n)$ and let $G = C_n - v$. Then G contains no odd cycle, and is thus bipartite. Since v has degree 2, $C_n - v$ has $n-2$ vertices, so the result follows.

Solution, part c:
For $G \subseteq K_n$, if n is even, $e(G) \leq n^2/4$. If n is odd, $e(G) \leq (n-1)^2/4$. Consider the largest bipartite subgraph without any odd cycles in K_n. If n is even, this is $K_{\frac{n}{2},\frac{n}{2}}$, with $n^2/4$ edges. If n is odd, this is $K_{\frac{n-1}{2},\frac{n-1}{2}}$, with $(n-1)^2/4$ edges. □

3. 1.3.8. Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

a) $(5, 5, 4, 3, 2, 2, 2, 1)$

b) $(5, 5, 4, 4, 2, 2, 1, 1)$

c) $(5, 5, 5, 3, 2, 2, 1, 1)$

d) $(5, 5, 5, 4, 2, 1, 1, 1)$

Solution, part a:
Graphic. Reversing the Havel Hakimi Theorem, construct $(5, 5, 4, 3, 2, 2, 2, 1)$ as fol-

lows:

$$\begin{aligned}
(1,1,0,0,0)\\
(2,1,1,1,1,0)\\
(4,3,2,2,1,1,1)\\
(5,5,4,3,2,2,2,1).
\end{aligned}$$

Solution, part b:
Graphic. Construct $(5,5,4,4,2,2,1,1)$ as follows:

$$\begin{aligned}
(1,1,0,0,0)\\
(2,2,1,1,0,0)\\
(4,3,3,1,1,1,1)\\
(5,5,4,4,2,2,1,1).
\end{aligned}$$

Solution, part c:
Graphic. Construct $(5,5,5,3,2,2,1,1)$ as follows:

$$\begin{aligned}
(3,1,1,1,0,0)\\
(4,4,2,1,1,1,1)\\
(5,5,5,3,2,2,1,1).
\end{aligned}$$

Solution, part d:
Not graphic. Following the Havel Hakimi Theorem, we have:

$$\begin{aligned}
(5,5,5,4,2,1,1,1)\\
(4,4,3,1,0,1,1)\\
(3,2,1,0,0,0)\\
(1,0,-1,0,0).
\end{aligned}$$

The last degree sequence is not graphic, thus $(5,5,5,4,2,1,1,1)$ is not graphic. □

4. Exercise 1.3.40. Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions. Note: Viewed a solution.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

Solution, part a:
Let $I \subseteq G$ with size a. Then $G = I \cup G \backslash I$. In order to maximize the number of edges in G, we maximize the number of edges in $G \backslash I$ as well as between I and $G \backslash I$. The $\max(e(G \backslash I)) = \binom{n-a}{2}$. The max degree of each vertex in I is $n - a$ because no vertex in I can be connected to a vertex in I. Thus, the max number of edges in G is $\binom{n-a}{2} + (n-a)a$.

Solution, part b:
If G has k components, we maximize the number of edges G has by making one component complete and leaving the others to be isolated vertices. Thus, the only component with edges has $n - k + 1$ vertices. So, G has $\binom{n-k+1}{2}$ edges. Now consider the n-vertex simple graph G' with more edges than G. Suppose that G' has two components with edges. Then there are two components in G' with two or more vertices, call them C and C'. Moving all but one of the vertices from C' to C does not change the number of components of G'. Suppose we do this, then ensure that C is complete. But then this would raise $e(G')$ by at least one, since there will be at least one new adjacency. Thus, G with k components has maximal edges whenever one component is complete and the rest are isolated vertices.

Solution, part c:
If G is diconnected, then $\overline{G}$ is connected. By Proposition 1.3.13, the minimum number of edges in $\overline{G}$ is $n - 1$. Thus, to fined $\max(e(G))$, we find the total edges possible for a graph with n vertices and subtract $n - 1$ from it: $\max(e(G)) \leq \binom{n}{2} - (n - 1)$. □

5. Exercise 1.3.57: Let n be a positive integer. Let d be a list of n nonnegative integers with even sum whose largest entry is less than n and differs from the smallest entry by at most 1. Prove that d is graphic.

Solution:
Let Δ =max(d) and let δ =min(d). We proceed using induction on the n. First, the base case. If n=1, then $d = [0]$, which satisfies the conditions. Now the inductive step. Suppose the result is true for some $k > 0$. We must show it true for $k + 1$. Since $\Delta < n$, applying Havel-Hakimi yields: $d = [\Delta - 1]$

6. 2.1.2. Let G be a graph.

a) Prove that G is a tree if and only if G is connected and every edge is a cut edge.

b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle

Solution, part a:
$\rightarrow$ Suppose G is a tree. By definition, G is connected. By Corollary 2.1.5., every edge of a tree is a cut-edge, as desired. $\leftarrow$ Suppose G is connected and every edge is a cut-edge. Then for $u, v \in G$, there exists a uv-path. Since every edge is a cut edge, this path must be unique. If it were not unique, there would be a cut-edge separating partite sets.

Solution, part b:
$\rightarrow$ Suppose G is a tree. If we add any edge with endpoints in $V(G)$, we will create exactly one cycle because G is connected and every uv-path $\in G$ is unique. $\leftarrow$ Suppose adding any edge, uv, with endpoints in $V(G)$ creates exactly one cycle in $G + uv$. Then $G - uv$ has no cycles, so G is a tree. $\square$

7. 2.1.5 Let G be a graph. Prove that a maximal acyclic subgraph of G consists of a spanning tree from each component of G.

Solution:
Suppose $G = G_1 \cup G_2 \cup ...G_k$, where k is a positive integer. Suppose some component of G, G_i, $1 \leq i \leq k$, has n vertices. Then the maximal acyclic subgraph of G_i happens whenever G_i is connected and has $n-1$ edges, which spans all n vertices of G_i and is guaranteed to exist by Corollary 2.1.5. The same follows for every component of G because G_i was picked arbitrarily. Since G is maximized when each component is maximized, the result follows. $\square$

8. 2.1.6. Let T be a tree with average degree a. In terms of a, determine $n(T)$.

Solution:
Let $n = n(T)$ and $k = e(T)$. Then, by Theorem 2.1.4., $n - 1 = k$. Thus, we have:

$$\begin{aligned}
\frac{2k}{n} &= a \\
\frac{2(n-1)}{n} &= a \\
2n - 2 &= an \\
-2 &= n(a-2) \\
\frac{-2}{a-2} &= n \\
\frac{2}{2-a} &= n
\end{aligned}$$

$\square$