1. Prove that a matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.

Solution:
$\rightarrow$
We prove the contrapositive. Suppose G has an M-augmenting path. By definition, there exists a larger matching than M. Hence, M is not a maximum matching, as desired.
$\leftarrow$
We prove the contrapositive. Suppose M is not a maximum matching in G. First, if M is also not a maximal matching, then there must be an edge in G with vertices unsaturated by M. Hence, there exists an M-augmenting path. Otherwise, suppose M is a maximal but not a maximum matching. Then there exists matching M' in G such that $|M| < |M'|$. Hence, there exists a uv-path on which M and M' both have edges, but where M' has more edges along the path than M. Since M is maximal, there is an alternating path on the uv-path that is in M. The only way for M' to have more edges than M on this path is for M' to saturate the endpoints while M doesn't. Hence, we have found an M-augmenting path in G. $\square$

2. Let M be a matching in G and let Q be a vertex cover of G.

a) Prove that $|M| \leq |Q|$

b) Give an example of a graph G such that the cardinality of a maximum matching M in G is strictly smaller than the cardinality of a minimum vertex cover of G or explain why no such example exists.

Solution, part a:
Let M be a matching in G and let Q be a vertex cover of G. Then G has at least $|M|$ edges that do not share vertices. Hence, $|Q|$ must be at least $|M|$ just to cover every edge in M. If M doesn't saturate every vertex, then $|M| < |Q|$. Otherwise, if M is a perfect matching, $|Q| = |M|$. Therefore, $|M| \leq |Q|$. $\square$

Solution, part b:
In the graph below, a minimum vertex cover is $Q = \{a, b\}$, so $|Q| = 2$, but a maximum matching, $M = ab$, has cardinality 1.