1. 1.2.3: Let G be the graph with vertex set $[1, ..., 15]$ in which i and j are adjacent if and only if their greatest common factor (G.C.F) exceeds 1. Count the components of G and determine the maximum length of a path in G.

Solution:
Notice that each number $u$in the subset $A = [1, 11, 13]$ of $V(G)$ has a G.C.F. of 1 when compared with any $v \epsilon V(g)$ whenever $u \neq v$. Therefore, each of those three vertices forms a component. The rest of the vertices can all be connected by a common path. An example is the ordered set $P = [7, 14, 8, 4, 10, 5, 15, 9, 3, 12, 6, 2]$. Thus, there are 4 components of G. The maximum length path in G will be length 11, since the number of vertices in the largest component is 12 and since P is a path. □

2. 1.2.5: Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Conclude that no graph has a cut-vertex of degree 1.

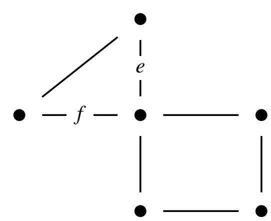
Solution:
Since G is connected, there is a u, v-path whenever $u, v \epsilon G$. Consider $G - v$. If $G - v$ has the same number of components as G, then we are done. $G - v$ will not have less components than G because there are no isolated vertices in G. If $G - v$ has more components than G, v is a cut vertex. Thus, there must be a vertex in each component of $G - v$ that is adjacent to v. Otherwise, G would not be connected. This implies that every cut-vertex v has degree at least 2 if the graph is connected. For a nontrivial (i.e. nonempty) connected graph F, with cut-vertex u, $F - u$ must have at least 2 components. Thus, since u must be adjacent to a vertex in each of the components created when it is deleted, u has a minimum degree of two. If some graph is not connected, we look at the components that are connected, and the result follows. Thus, a vertex of degree 1 cannot be a cut-vertex in any graph. □

3. 1.2.8: Determine the values m and n such that $K_{m,n}$ is Eulerian.

Solution:
From Thm.1.2.6, a graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree. Thus, $m = n \geq 2x$ for $x \epsilon \mathbb{N}$. □

4. 1.2.11: Prove or disprove: If G is an Eulerian graph with edges e, f that share a vertex, then G has an Eulerian circuit in which e, f appear consecutively.

Solution:
False. See graph below.



□

5. 1.2.22: Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Solution, part a:
→ Suppose G is connected. Being connected means that there must be a path between any two vertices. Therefore, for any nonempty bipartition of the vertices of a connected graph, there will be a path from one partition to the other. Thus, there must be an edge with endpoints in both sets to make the path possible.

Solution, part b:
← Suppose for every partition of $V(G)$ into two nonempty sets, there is an edge with endpoints in both sets. Consider a bipartition of $V(G) = b_1 \cup b_1'$, such that one set, b_1 has only one vertex, u, in it while the other has the rest, $b_1' = V(G) \backslash u$. Thus, there must be an edge from u_1 to some vertex in b_1'. Call this vertex u_2, then create a new partition with call it $b_2 = [u_1, u_2]$ and $b_2' = V(G) \backslash u_1, u_2$. There must also be an edge with endpoints in b_2 and b_2'. Suppose $V(G)$ has $n \in \mathbb{N}$ vertices. Then, continue this process until $b_{n-1} = [u_1, u_2, ...u_{n-1}]$ and $b_{n-1}' = [u_n]$. Thus, by the construction of the $n-1$ different bipartitions, and since there must be an edge from b_{n-1} to b_n, there must be u_1, u_n-walk. Therefore, there is a u_1, u_n-path. Since u_1 is arbitrary, this construction follows for any u_i, u_j, $1 \leq i, j \leq n$. Thus, for any $u, v \in V(G)$, there is a u, v-path, as desired. □

6. 1.2.25: Use ordinary induction on the number of edges to prove that absence of odd cycles is a sufficient condition for a graph to be bipartite.

Solution:
First, the base case: Consider a graph with 0 edges. It certainly has no odd cycle. Any vertices it has are all independent. Therefore, G is bipartite.

Now the inductive step. Let $k \geq 0$. Assume for all G with k edges and no odd cycles, G is bipartite. Suppose G' is a graph with length $k+1$ and no odd cycles. Let $xy \in E(G')$. Consider the graph $G = G' - xy$. By the inductive hypothesis, G is bipartite. Suppose xy is a cut-edge for G'. Then x and y are in different components of G. Thus, if we need to, we can recolor the components x and y are in, respectively, so that they have different colors without changing any of the other bipartite components of G. Therefore, $G + xy = G'$ is bipartite. Now, suppose xy is not a cut edge for G'. Then xy lies on a cycle. If x and y are in disjoint independent sets, then we are done, G' is bipartite. Suppose, on the other hand, that xy lie in one independent set of G'. In order to make a cycle going through xy, there would need to be an even number of edges to go to another independent set and back, then plus one to go from x to y. So the cycle xy lies on in this case is odd. But this is a contradiction because we assumed G' contains no odd cycle. Thus, G' is bipartite. By induction, the result follows. □

7. 1.2.26: Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

Solution, part a:
$\rightarrow$ Suppose G is bipartite. Let the partite sets of $V(G)$ be B_1 and B_2. Then $V(H)$ has vertices in B_1 or B_2. Then the result follows because B_1 and B_2 are disjoint independent sets.

Solution, part b:
$\rightarrow$ We prove the contrapositive: If G is not bipartite, then there exists a subgraph H of G such that there is no independent set consisting of at least half of $V(H)$.

Suppose G is not bipartite. Then G contains an odd cycle. Let H be an odd cycle in G. In order to make the maximal independent set of H, we alternate the color of the vertices going around the cycle. But since H is an odd cycle, it has $2k + 1$ vertices for $k \in \mathbb{N}$, the maximum vertices in one independent set is half of this. Since it must be an integer, the maximum vertices in one independent set of H is k. Since $k < \frac{2k+1}{2}$ and since k is the maximum vertices in one independent set of H, we have shown the contrapositive true, so the result follows.

□

8. 1.2.29: Let G be a connected simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique (complete bipartite graph).

Solution:
Notice that, since P_4 cannot be an induced subgraph of G, G must not have any C_k or P_k $k > 4$. Thus, since G also cannot contain C_3, G contains no odd cycle and is therefore bipartite. Let B_1 and B_2 be the partite sets of G. Suppose for some $x \in B_1$ and some $y \in B_2$, x is not adjacent to y. Since G is connected, there must be some x y-path but it would have to go from B_1 to B_2 more than once because we assumed x is not adjacent to y. There are no paths P_4 or longer, so this is impossible. Thus, x and y are neighbors, and G is a biclique, as desired. □