1. Five jobs, five applicants, etc... Complete two iterations of the Gale-Shapley Proposal Algorithm. For each iteration, you should stat the proposals and the rejections.

Solution:
An X indicates a rejection.
First iteration:

$$\begin{array}{l} \text{Proposals}||\text{Rejections} \\ j_1, a_1|| \\ j_2, a_1||X \\ j_3, a_1||X \\ j_4, a_2|| \\ j_5, a_3|| \end{array}$$

Second iteration:

$$\begin{array}{l} \text{Proposals}||\text{Rejections} \\ j_1, a_1|| \\ j_2, a_2|| \\ j_3, a_3||X \\ j_4, a_2||X \\ j_5, a_3|| \end{array}$$

□

2. Answer questions about the graph G.

a) Find $k(G)$ and $k(G')$.

b) Give and example of a pair of vertices x, y such that $k(x, y) > k(G)$.

c) Give an example of a disconnecting set of edges F of G that is minimal but not minimum.

Solution, part a:
We have: $k(G) = k'(G) = 2$.

Solution, part b:
Consider $(x, y) = (b_2, b_1)$. For this pair, $k(b_2, b_1) = 4 > 2$.

Solution, part c:
Consider the disconnecting set $F = \{b_2a_z, b_2, a_3, b_2a_4, b_2a_5\}$. □

3. Assume G is a connected graph on at least 3 vertices. Prove that G is 2-edge-connected if and only if every edge of G lies on a cycle.

Solution:
Let G be a connected graph on at least 3 vertices.
We proceed via contradiction.

$$\begin{aligned} G \text{ is not 2-edge-connected} &\iff \exists\, S \subset E(G) : G - S \text{ is disconnected and } |S| = 1 \\ &\iff G \text{ has a cut-edge } e \\ &\iff e \text{ is not on a cycle.} \end{aligned}$$

□

4. Prove that if G is k-connected and G' is constructed from G by adding a vertex v with at least k neighbors in $V(G)$, then G' must also be k-connected.

Solution:
Let $S \subset V(G')$ be any separating set of G'. We consider 3 cases. If $v \in S$, then $G - (S + \{v\})$ is disconnected. Therefore, $|S| \geq k + 1$. On the other hand, if $v \notin S$ but $N(v) \subseteq S$, then $G - S$ is disconnected. Hence, $|S| \geq k$. Lastly, if $v \notin N(S)$ and $v \notin S$, then $G - S$ is disconnected, so $|S| \geq k$. Since in every possible case, $|S| \geq k$, G' is k-connected. □

5. Let G be a k-connected graph. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ be disjoint subsets of $V(G)$ such that $|A| = |B| = k$. Prove that G contains k pairwise disjoint A, B-paths. (That is, G contains k paths each of which start with a vertex from the set A, end with a vertex from the set B and share no vertices.)

Solution:
Create G' by adding to G two vertices, v_A and v_B and add k edges to incident to both of them such that $v_A \leftrightarrow a_i$ and $v_B \leftrightarrow b_i$ for every $1 \leq i \leq k$. From the previous result, since G is connected, G' is k-connected as well. Hence, there exists k pairwise disjoint $v_A v_B$-paths in G' through A and B. Deleting v_A and v_B leaves us with G, but there remains k disjoint A, B-paths. □