Convolutional Neural Networks

Image classification

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Introduction

Convolutions are mathematical operations that, among other things, are useful for image processing. In this context, a convolution involves sliding a small matrix, called a convolution kernel, across a matrix representation of an image to produce a new matrix. Convolutional neural networks (CNNs) are deep neural networks (artificial neural networks with more than two layers) where the hidden layers use a convolution kernel to alter their input. CNNs are powerful tools for image classification. This talk goes over the basics of convolutions, CNNs, and shows an example CNN that can be used to classify images of handwritten numbers.

Farey Sequences

Definition 1: A Farey sequence of order n, F_n, is a sorted list of all the irreducible fractions between 0 and 1 that have denominators less than or equal to n. Here is F_5:

$$F_5 = \{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\}.$$

The infinite Farey sequence is an ordering of all the rational numbers between 0 and 1.

Farey Addition

John Farey (1816), a geologist, is credited with the conjecture that the new terms in F_n could be obtained from the *mediants* of consecutive terms in F_{n-1}. We call this the Farey sum. Using $\oplus$, we define the Farey sum of two fractions as:

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}.$$

Cauchy proved Farey's conjecture, coining the term Farey sequences. However, C. Haros introduced Farey sequences in 1802. (Zhang and Comellas 2010)

Recursive Construction of F_n

We obtain F_n from F_{n-1} by computing the Farey sum of consecutive terms in F_{n-1} whose Farey sum has a denominator equal to n. Here is $F_1, \ldots, F_5$:

$$
\begin{aligned}
F_1 &= \{\frac{0}{1}, \frac{1}{1}\} \\
F_2 &= \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\} \\
F_3 &= \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\} \\
F_4 &= \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\} \\
F_5 &= \{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\}
\end{aligned}
$$

Puncturing a Farey Sequence

There are many ways to create a graph from Farey sequences. In order to characterize maximal outerplanar graphs, we intend to construct a Farey graph from a *punctured Farey sequence*.

Definition 2: A punctured Farey sequence is a sequence obtained from any number of puncturing operations: select three fractions $p'/q', p/q, p''/q''$ that are adjacent in the sequence and for which $q > q', q''$; then delete p/q from the sequence.

The possible punctured Farey sequences are the subsequences of F_n that are created by iteratively deleting mediants, e.g. $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}$ is a punctured Farey sequence but $\frac{0}{1}, \frac{1}{3}, \frac{2}{3}, \frac{1}{1}$ is not.

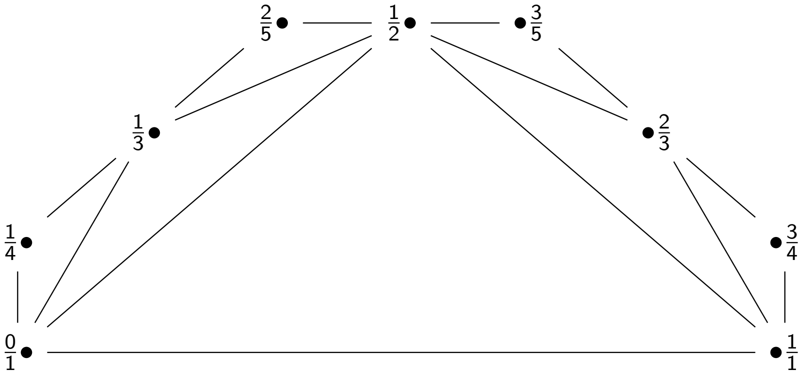
Farey Graph

We now have the tools to define the Farey Graph we are interested in.

Definition 3: A Farey graph, $\mathcal{F}_n$, is a graph whose n vertices are the fractions in a punctured Farey series; two vertices p/q and p'/q' are adjacent if and only if $|hk' - h'k| = 1$ (Colbourn 1982)

$\mathcal{F}_9$

Here is an example of $\mathcal{F}_9$ constructed using Definition 3 applied to a punctured F_5 (i.e. removing $1/5$ and $4/5$ from F_5).



Outerplanar

Definition 4: A graph is outerplanar if it can be embedded in the plane so that no two edges cross and every vertex lies on the exterior face. A maximal outerplanar graph is an outerplanar graph to which no edge can be added without destroying outerplanarity. (Colbourn 1981)

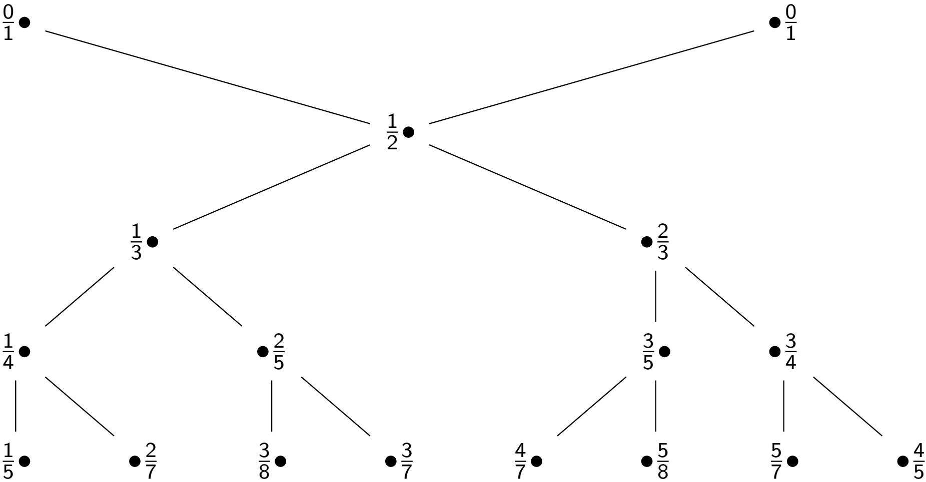
Coulborn's Theorem

Theorem: The $\mathcal{F}_n$ are maximal outerplanar and maximal outerplanar graphs are Farey graphs. (1981)

This means that the $\mathcal{F}_n$ characterize maximal outerplanar graphs. This also implies that the $\mathcal{F}_n$ is minimally 3-colorable.

Farey Trees

As mentioned earlier, Coulborn's construction is not the only way to construct a Farey graph from a Farey sequence. Another option is to create a binary tree that branches at every mediant:



Future directions

It seems possible that there could be other applications of graphic representations of Farey sequences as Farey trees. For example, if we let each vertex in a Farey tree have weight equal to its rational number label, we can approximate any number we like in a deterministic fashion: start at $1/2$, move left if the number is smaller than the weight of the vertex, move right if the number is larger than the weight of the vertex. Furthermore, at the nth level of the tree, there are n pairs of numbers that sum to 1. This could be useful in modeling probabilities. The interesting part in each of these is that since there is a way to move between constructions, namely using the Farey sequence itself, there is a way to move between a binary weighted tree (Farey tree) and any maximal outerplanar graph in a deterministic way.

References

- Colbourn, C. J. (1982). Farey series and maximal outerplanar graphs. SIAM Journal on Matrix Analysis and Applications, 3(2), 187-3. doi:http://dx.doi.org/10.1137/0603018
- Zhang Z.Z., Comellas F., Farey graphs as models for complex networks, Theoret. Comput. Sci., 412 (2011), pp. 865-875, 10.1016/j.tcs.2010.11.036