1. Let G be a simple graph such that $\delta(G) = k \geq 2$. Prove that G contains a cycle on at least $k + 1$ vertices. (3 pts extra credit: Give an example that demonstrates this result is best possible.)

Solution:
Let G be a simple graph such that $\delta(G) = k \geq 2$. Let u be the endpoint of a maximal length path in G. Since P is maximal, every neighbor of u is in $V(P)$. Hence, since u has at least k neighbors, P has at least k edges. Let v be the farthest neighbor of u's along P. Then there is a uv-path that has length k, but there is also edge uv. Hence, there is a cycle of length $k + 1$. Consider the graph below, which has minimum degree 2 with C_3 as the maximum length cycle, as a demonstration that this result is best possible:

$\square$

2. Let G be a connected graph on n vertices. Prove that G has exactly one cycle if and only if G has exactly n edges.

Solution:
$\rightarrow$
Suppose G has exactly one cycle and G is connected on n vertices. Let C_k be the cycle, $k \leq n$. If $k = n$, then G has n edges, as desired. On the other hand, suppose $k < n$. Consider a graph containing only C_k. To yield G, we must connect $n - k$ vertices to C_k. Yet, each edge added must be a cut-edge in order to avoid creating a cycle. Thus, exactly one edge must be added for each of the $n - k$ missing vertices. Hence, the total number of edges in G is:

$$\begin{aligned} e(G) &= e(C_k) + n - k \\ &= k + n - k \\ &= n \end{aligned}$$

$\leftarrow$
Suppose G is a connected graph on n vertices and n edges. G is not a tree because G has one too many edges to be a tree. Since G is connected but not a tree, G contains a cycle. Remove an edge from cycle C_k in G to produce G' with n vertices and $n-1$ edges. Since the edge was in a cycle, it is not a cut-edge. Thus, G' is connected on $n-1$ edges, which means G' is a tree. Therefore, G' is acyclic. Hence, C_k is the only cycle in G. $\square$

3. Let G be a simple graph having no isolated vertex and no *induced* subgraph with exactly two edges. Prove that G must be the complete graph.

Solution:
Assume G is a simple graph with no isolated vertices, and no induced subgraph with exactly two edges. Assume, for a contradiction, that G is not complete. That is, there exists $u, v \in V(G)$ such that $uv \notin E(G)$. Since there are no isolated vertices in G, u and v have neighbors. Delete every vertex in G except for u, v, a neighbor of u, and a neighbor

of v. If u and v share one or more neighbors, delete every vertex except u, v, and one vertex from their neighborhood. In either case, the induced subgraph has exactly two edges, a contradiction. Hence, G must be the complete graph. $\square$

4. Let d be a sequence $d_1, d_2, \cdots, d_n$ be positive integers and assume $n \geq 2$. Prove that there exists a tree T with degree sequence d if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

Solution:
$\rightarrow$
Suppose there exists tree T with degree sequence $d_1, d_2, ..., d_n$, $n \geq 2$. Then T has $n-1$ edges. Hence, the degree sum of T is $\sum_{i=1}^{n} d_i = 2n-2$, as desired.
$\leftarrow$
Let d be a sequence $d_1, d_2, \cdots, d_n$ of positive integers and, without loss of generality, suppose they are arranged in non-increasing order. Assume $n \geq 2$. Suppose $\sum_{i=1}^{n} d_i = 2n-2$. I claim $d_n = d_{n-1} = 1$. Otherwise, if either $d_{n-1} \geq 2$ and $d_n = 1$, or if $d_{n-1}, d_n \geq 2$, then $\sum_{i=1}^{n} d_i \geq 2n-1$, which is a contradiction. Let $\Delta = max_{1 \leq i \leq n} d_i$. The maximum Δ could be is when all the other terms of d are equal to 1. In this extremal case, we have:

$$\sum_{i=1}^{n} d_i = 2n - 2 = \Delta + n - 1.$$

Which means,

$$n - 1 = \Delta$$

in this extremal case. Hence, in general, $\Delta \leq n-1$. Furthermore, for $n > 2$, $\Delta \geq 2$, for otherwise, $\sum_{i=1}^{n} d_i = n-2$, a contradiction. Proceeding inductively on n, when $n = 2$, $d = 1, 1$, which is certainly the degree sequence of a tree. Let $2 < k \in \mathbb{N}$. For our inductive hypothesis, suppose that for a non-increasing sequence $d = d_1, d_2, \cdots, d_k$ of positive integers where $\sum_{i=1}^{k} d_i = 2k-2$ there exists a tree with degree sequence d. Consider the sequence of positive integers arranged in non-increasing order $d' = d'_1, d'_2, \cdots, d'_{k+1}$ where $\sum_{i=1}^{k+1} d'_i = 2(k+1) - 2$. Now, remove d'_{k+1} from the sequence and subtract 1 from $d'_s \geq 2$, $1 \leq s < k$, (d'_s must exists since $\Delta \geq 2$) to create d''. Evaluating the sum of the terms in d'', we have:

$$\begin{aligned}
\sum_{i=1}^{k} d''_i &= 2(k+1) - 2 - 2 \\
&= 2k + 2 - 4 \\
&= 2k - 2.
\end{aligned}$$

Hence, by the inductive hypothesis, there exists a tree T with degree sequence d''. In order to get a tree with degree sequence d', we simply add a vertex v to T and an edge from v to a vertex with degree $d'_s - 1$ in T. Since this is merely adding a leaf to a tree, the resulting graph must also be a tree. $\square$