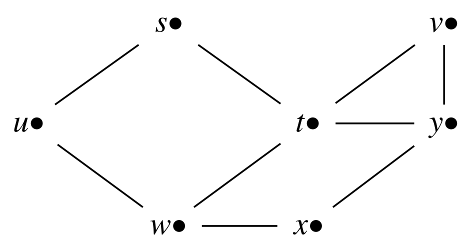
1. 4.2.1 Determine $k(u,v)$ and $k'(u,v)$ in the graph. (Hint: Use the dual problems to give short proofs of optimality.)

Solution:
There are three internally disjoint paths from u to v. The three vertices with distance 2 from u form a vertex cut. Hence, $k(u,v) = 3$. On the other hand, we can find 5 internally edge disjoint u,v-paths. By Theorem 4.2.19, $k'(u,v) = 5$.

2. 4.2.4. Prove or disprove: If P is a u,v-path in a 2-connected graph G, then there is a u,v-path Q that is internally disjoint from P.

Solution:
False. The graph below is 2-connected. Let $P = u,s,t,y,v$. There does not exists a u,v-path that is internally disjoint from P.



3. 4.2.5. Let G be a simple graph, and let $H(G)$ be the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if u,v appear on a common cycle in G. Characterize the graphs G such that H is a clique.

Solution:
The graphs G will produce H that is a clique if and only if G is 2-connected. Observe:

$$\begin{aligned} G \text{ is 2-connected} &\iff \delta(G) \geq 1, \text{ and every pair of edges in } G \text{ lies on a common cycle.} \\ &\iff \forall u,v \in V(G), uv \in E(H) \\ &\iff \text{H is a clique.} \end{aligned}$$

□

4. 4.2.8 Prove that a simple graph G is 2-connected if and only if for every triple (x,y,z) of distinct vertices, G has an x,z-path through y.

Solution:
$\rightarrow$
Assume G is 2-connected and x,y, and z are three distinct vertices. We construct G' by adding vertex y' adjacent to x and z. Then G' is 2-connected, so we can find two internally disjoint y,y'-paths in G'. Deleting y' yields a x,z-path through y in G.
$\leftarrow$
Suppose for every triple of distinct vertices, G has an x,z-path through y. Hence G is connected. Suppose, for a contradiction, that G has a cut-vertex, x. Let C_1 and C_2 be

any two components in $G - v$. Then let $y \in C_1$ and $z \in C_2$. Then G has no x, z-path through y, a contradiction. □

5. 4.2.12. Use Menger's Theorem to prove that $k(G) = k'(G)$ when G is 3-regular.

Solution:
First, notice that it is sufficient to show that $k'(G) \leq k(G)$ because we know $k(G) \leq k'(G)$. Without loss of generality, assume there exists $x, y \in V(G)$ such that x and y are not neighbors. Let $k = k'(G)$. By Theorem 4.2.19., for all x and y not adjacent in G, there exists k edge disjoint paths. Suppose for a contradiction that any two of these paths share a single vertex v. Then v has four edges incident to it because the paths are edge disjoint. This is a contradiction because we assumed G is 3-regular. □