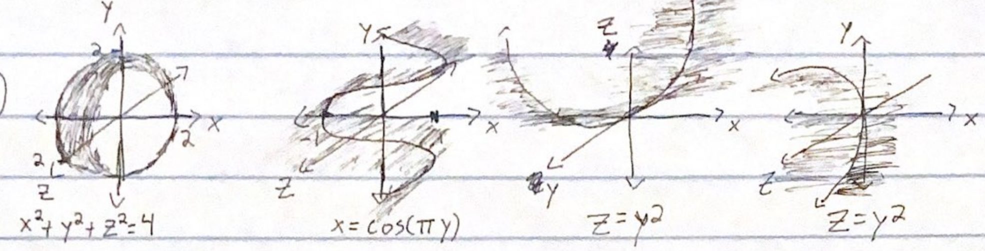
P5. (a)



$X=\cos(\pi y)$ is satisfied for any z-value, while $z=y^2$ is satisfied for any x-value. Thus, we combine them into $x=\cos(\pi\sqrt{z})$ (note: even though $\pm\sqrt{z}=y$, we can ignore $y=-\sqrt{z}$ for now because cosine is an even function and must simply remember that whenever we find a solution (x_1,y_1,z_1) we also have a solution at $(x_1,-y_1,z_1)$). Since $x^2+y^2+z^2=4$ is a sphere in $\mathbb{R}^3$ centered at the origin and since $x=\cos(\pi\sqrt{z})$ starts on the z-axis, moving only in the positive x-direction they must intersect once at some (x_1,y_1,z_1). [illegible] Recalling the note above, this means all three equations will be satisfied only at (x_1,y_1,z_1) and at $(x_1,-y_1,z_1)$. Both solutions will be inside the closed box $-1\le x\le 1$, $-2\le y\le 2$, $0\le z\le 2$ because $-2\le x,y,z\le 2$ since the radius of the sphere is 2 and centered at the origin. Since $z=y^2$, z is positive, so $0\le z\le 2$. Since $-2\le y\le 2$ and $x=\cos(\pi y)$, $\cos(-\pi)\le x\le\cos(\pi)$, $-1\le x\le 1$, as desired. ■

(b)
$$\underbrace{\begin{bmatrix}2(-1) & 2(1) & 2(1)\\ 1 & \pi\sin(\pi) & 0\\ 0 & 2(1) & -1\end{bmatrix}}_{J(X_0)}\underbrace{\begin{bmatrix}S_1\\ S_2\\ S_3\end{bmatrix}}_{S}=-1\underbrace{\begin{bmatrix}-1+1+1-4\\ -1-\cos(-\pi)\\ 1-1\end{bmatrix}}_{-1f(X_0)}$$

$X_0=(-1,1,1)$

```
function sol=myeq(x)
sol(1)=x(1)^2+x(2)^2+(x(3)^2)-4;
sol(2)=x(1)-cos(pi*x(2));
sol(3)=x(2)^2-x(3);
sol=sol';
```

```
function j=myj(x)
j=[2*x(1) 2*x(2) 2*x(3);1 pi*sin(pi*x(2)) 0;0 2*x(2) -1];
```

```
function mynewton(x)
format long
for ii=(1:5)

     f=myeq(x);
     jac=myj(x);


    s=jac\f;
    x=x-s';
end
f
x
```

```
>> x=[-1 1 1]

x =

    -1     1     1

>> mynewton(x)

f =

   1.0e-08 *

   0.284940071537676
  -0.062497951258678
   0.014789236502111


x =

  -0.856360744261663   1.172720052019146   1.375272320407788

>> x=[-1,-1,1]

x =

    -1    -1     1

>> mynewton(x)

f =

   1.0e-08 *

   0.284940071537676
  -0.062497951258678
   0.014789236502111


x =

  -0.856360744261663  -1.172720052019146   1.375272320407788

>>
```

P5 (d) Notice, $J(x_n) = \frac{df}{dx} = f'(x_n)$. Therefore, (1) turns into:

$$f'(x_n)s = -f(x_n)$$

$$s = \frac{-f(x_n)}{f'(x_n)}$$

Plugging into (2) yields: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

P6(a) $\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$

$= 1(45-48) - 2(36-42) + 3(32-35)$

$= 0$

(b) 3×3 is $(2\cdot3+3)$ multiplications
4×4 is $4(2\cdot3+3)+4$ multiplications
5×5 is $5(4(2\cdot3+3)+4)+5$ multiplications
$\vdots$

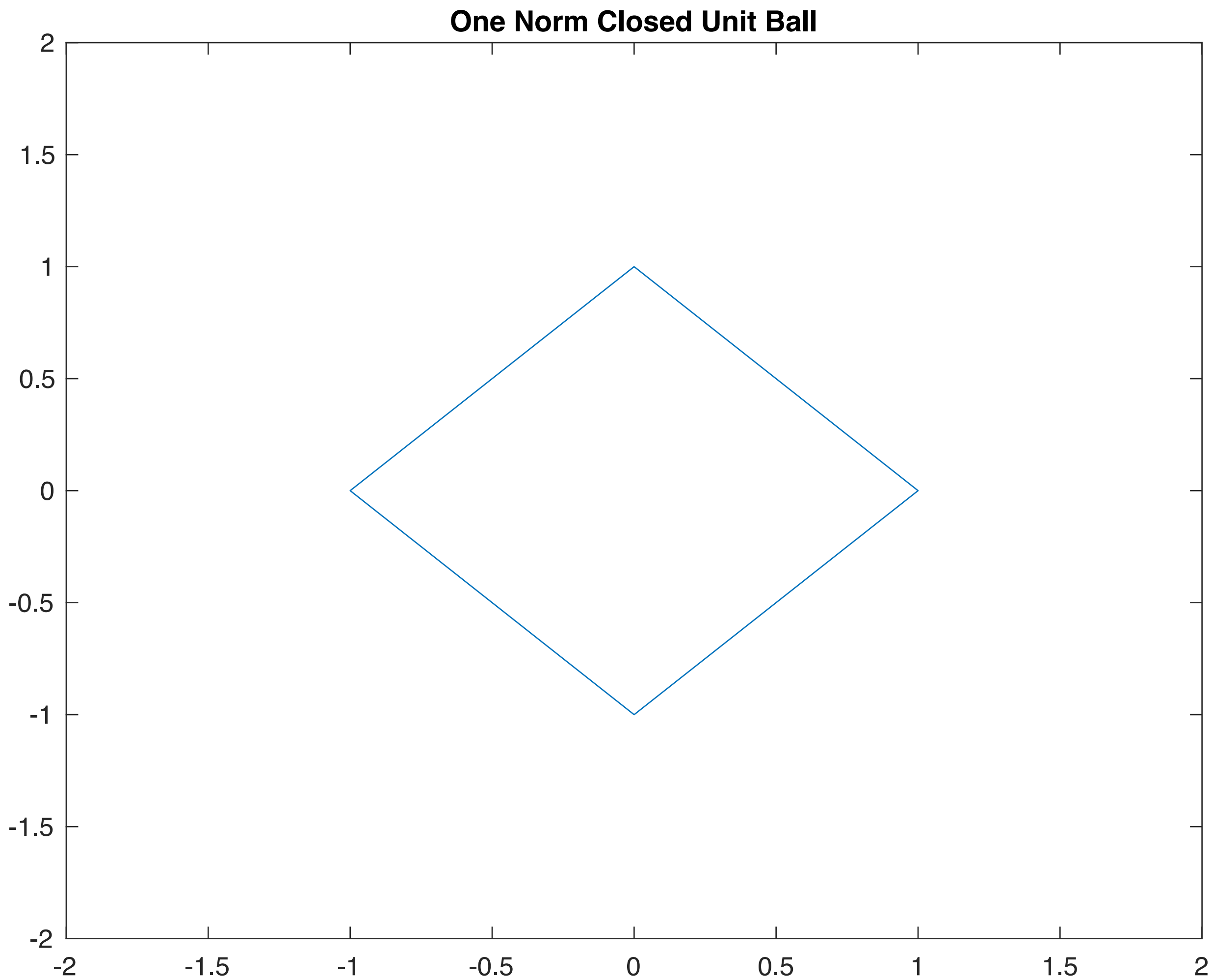
Pattern is $m + m(m-1) + m(m-1)(m-2) + \ldots + m!$
So,
$\displaystyle\sum_{i=1}^{m-1} \frac{m!}{i!}$ is # of multiplications to compute $\det(A)$ by expansion in minors ($A \in \mathbb{C}^{m\times m}$).

(c) Suppose A is a diagonal matrix.
When A is 3×3, $\det(A) = a_{11}a_{22}a_{33}$.
When A is $\mathbb{C}^{m\times m}$, $\det(A) = a_{11}a_{22}a_{33}\cdots a_{mm}$.
Thus, $\det(A) = \prod_{i=1}^{m} a_{ii}$ and $A^{-1} = \frac{1}{\prod_{i=1}^{m} a_{ii}} A^*$ $(\det(A) \neq 0)$.

```
function detchecker(~)
format long
n=10;
detA=[];
detB=[];
for i=[1:n]
    A = rand(10,10);
    d = diag(A);

    if (max(d, [], 'all')/min(d, [], 'all'))<10    %well conditioned
        detA(i)=prod(d);
    else              %not well conditioned
        detB(i)=prod(d);
    end

end
detA
detB

%both are small
```

```
>> detchecker

detA =

  Columns 1 through 6

   0.000050322789348                   0                   0   0.005524035757532↙
0   0.001394211470358

  Columns 7 through 10

   0.007142256188470                   0                   0   0.002008324794775


detB =

   1.0e-03 *

  Columns 1 through 6

                   0   0.196077060646668   0.014910904085730                   0↙
0.093568754356011                   0

  Columns 7 through 9

                   0   0.000652621027165   0.015980434733288

>>
```

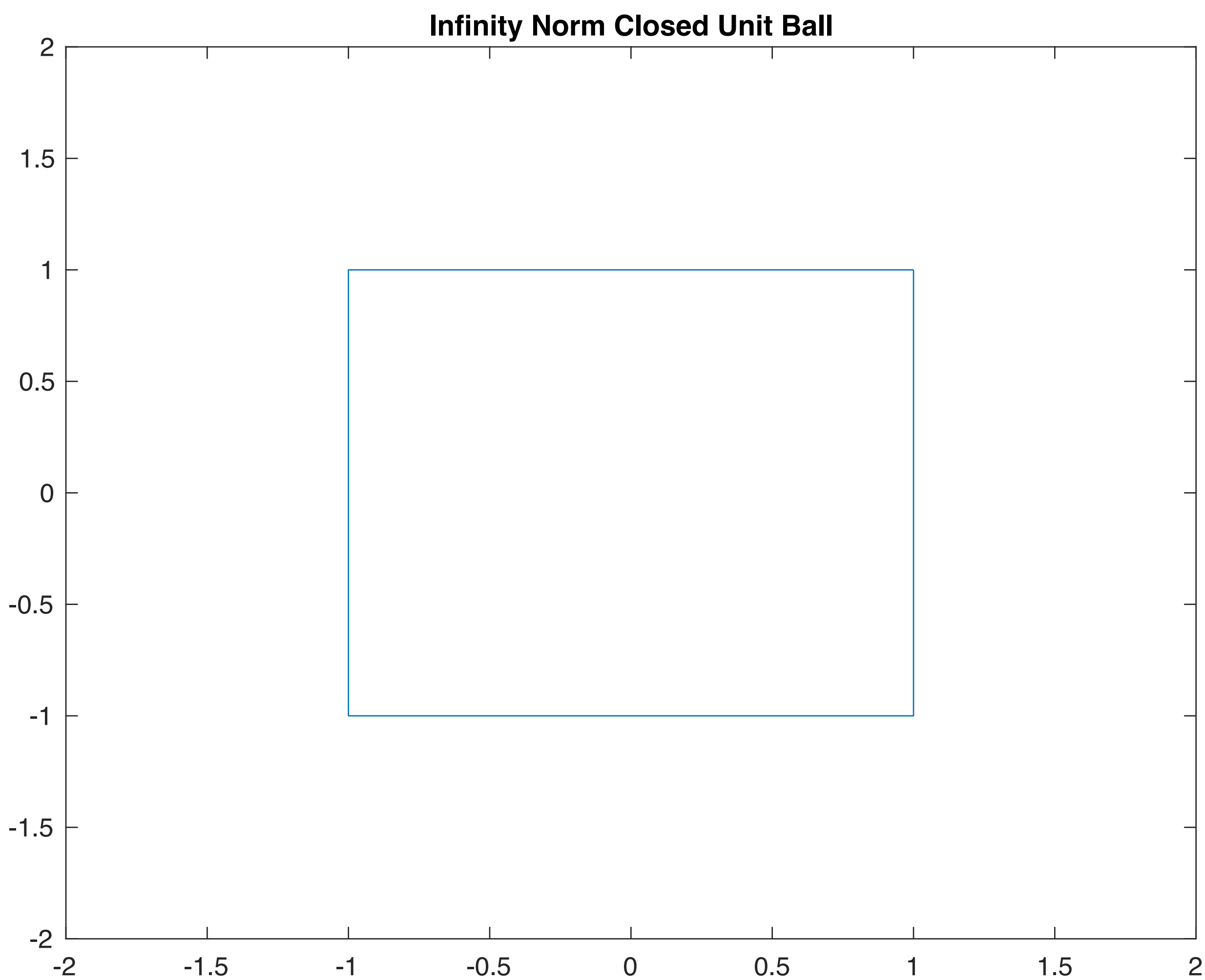
```
x=pi/4;
y=x+pi/2;
z=y+pi/2;
m=z+pi/2;
t=[x y z m x];
a=cos(t+pi/4);
b=sin(t+pi/4);
plot(a,b)
xlim([-2 2])
ylim([-2 2])
title('One Norm Closed Unit Ball')
```


One Norm Closed Unit Ball
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

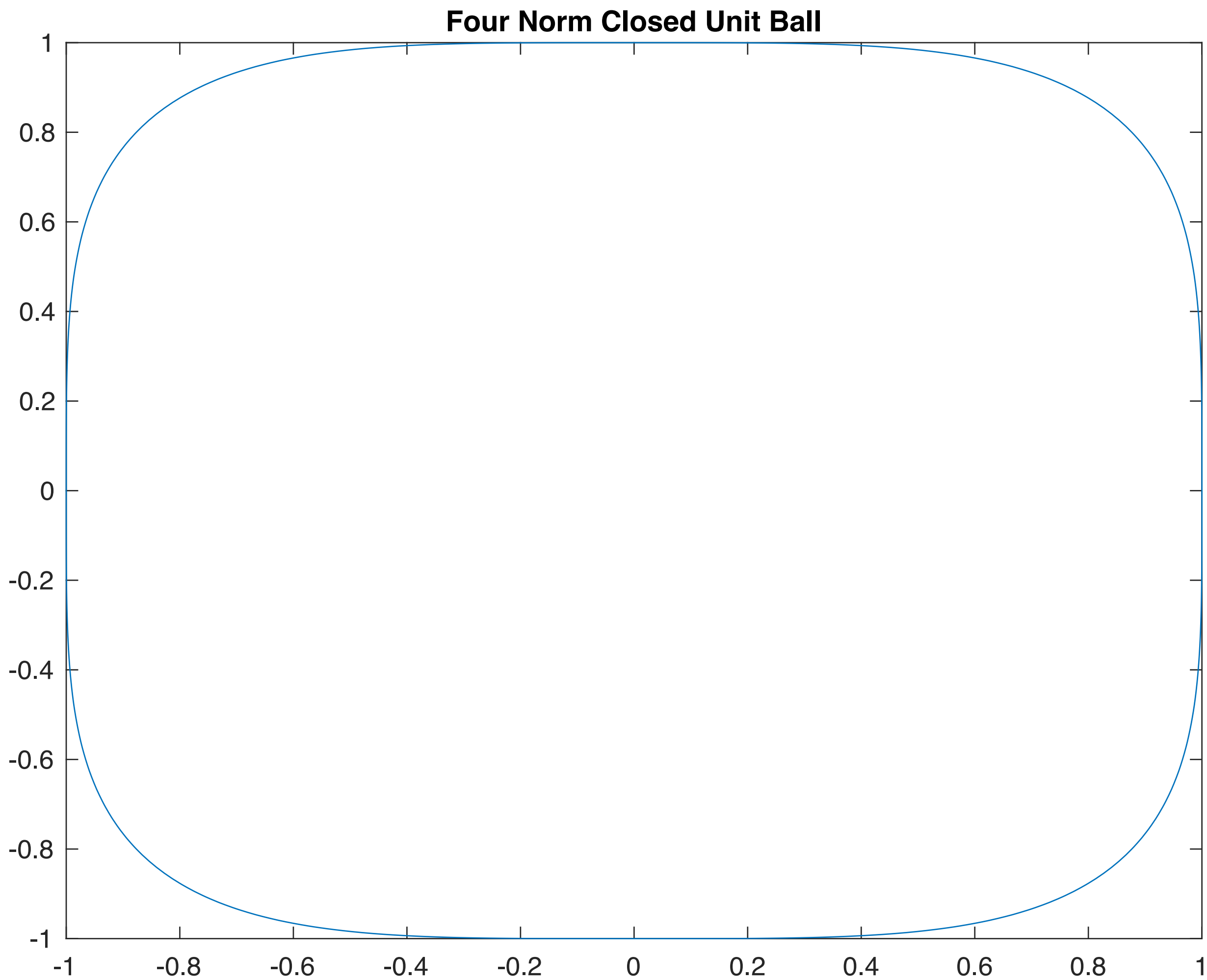
```
fimplicit(@(x,y)x.^2+y.^2-1)
title('Two Norm Closed Unit Ball')
```


Two Norm Closed Unit Ball
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

```
x=pi/4;
y=x+pi/2;
z=y+pi/2;
m=z+pi/2;
t=[x y z m x];
a=(2/sqrt(2))*cos(t);
b=(2/sqrt(2))*sin(t);
plot(a,b)
xlim([-2 2])
ylim([-2 2])
title('Infinity Norm Closed Unit Ball')
```


Infinity Norm Closed Unit Ball
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

```
fimplicit(@(x,y)x.^4+y.^4-1)
title('Four Norm Closed Unit Ball')
```


Four Norm Closed Unit Ball
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Hypothesis: If A is both triangular and unitary, A is diagonal.

Exercise 2.1, (Lecture 2)

pf: WLOG, assume $A \in \mathbb{C}^{m \times m}$ is upper triangular,

$$A = \begin{bmatrix} a_{11} & \cdots & & a_{1m} \\ 0 & & & \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{mm} \end{bmatrix} ; \quad A^* = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ \vdots & & \ddots & 0 \\ a_{1m} & & & a_{mm} \end{bmatrix}$$

$$A^*A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ \vdots & & \ddots & 0 \\ a_{1m} & \cdots & & a_{mm} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & & a_{1m} \\ 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a_{mm} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I_{m \times m}$$

Since A is unitary

If there existed $a_{ij} \neq 0$, $i \neq j$, then A^*A would contain a_{ij}^2 not on the diagonal. This is a contradiction because $A^*A = I_{m \times m}$. Thus, A is diagonal. ■

Exercise 2.3 (a) Hypothesis: Let $A \in \mathbb{C}^{m \times m}$ be hermitian. All eigenvalues of A are real.

Pf: For eigenvectors of A, we have a nonzero vector $v \in \mathbb{C}^m$ such that $Av = \lambda v$ for some $\lambda \in \mathbb{C}$. We have:

$$(Av)^* = (\lambda v)^*$$
$$v^* A^* = \lambda^* v^*$$
$$v^* A = \lambda^* v^* \qquad \text{Since } A \text{ is hermitian, } A^* = A.$$
$$v^* A v = \lambda^* v^* v$$
$$\lambda v^* v - \lambda^* v^* v = 0$$
$$\lambda = \lambda^*$$

Hence, λ must be real. ■

Exercise 23(b) Hypothesis: If x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

pf. We have $Ax=\lambda_x x$ and $Ay=\lambda_y y$.

Observe:

$$(Ay)^*=(\lambda_y y)^*$$

$$y^*A^*=\lambda_y y^* \qquad \lambda_y=\lambda_y^* \text{ b/c part (a)}$$

$$y^*A^*x=\lambda_y y^*x$$

$$y^*Ax=\lambda_y y^*x \qquad A=A^* \text{ b/c } A \text{ is hermitian}$$

$$y^*\lambda_x x=\lambda_y y^*x$$

$$\lambda_x y^*x-\lambda_y y^*x=0$$

$$y^*x(\lambda_x-\lambda_y)=0$$

$$y^*x=0 \qquad \text{since } \lambda_x\neq\lambda_y \quad \blacksquare$$