Introduction

1.1 Graph Isomorphism

Definition 1.1: An undirected graph G is a set of vertices, $V(G)$, and a function $g : V(G) \to P(V(G))$, defined by $g(v) = N(v)$, where $N(v)$ is the set of vertices that are adjacent to v and $P(V(G))$ is the power-set of $V(G)$. For this paper, every graph G shall be labelled using the integers.

Remark: We call $N(v)$ the *neighborhood* of v.

Remark: Since the graphs we are concerned with are undirected, for any graph G and any vertices $u, v \in V(G)$, $u \in N(V)$ if and only if $v \in N(u)$.

Definition 1.2: Two graphs, G and G', are isomorphic, denoted $G \cong G'$, if there exists a bijection $f : V(G) \to V(G')$ such that for every $u, v \in V(G)$, $u \in N(v)$ if and only if $f(u) \in N(f(v))$.

The String Isomorphism Problem

2.1 A graph as a string

Following Luks [1982], a graph can be represented as a string using the following procedure. Let $\Omega = [1, \ldots, n]$. Let G be an undirected graphs on n vertices, labelled using Ω. Let δ be the indicator function, $\delta : \binom{\Omega}{2} \to \{0, 1\}$, defined by

$$\delta(x, y) = \begin{cases} 1, & x \in N(y) \\ 0, & \text{o/w} \end{cases},$$

where $\binom{\Omega}{2}$ denotes the set of all unordered pairs in Ω (Babai 2018). For ease of indexing, we choose an order for all strings created in this way. We choose a straightforward ordering.

Definition 2.3: The binary string representation of a graph G is

$$B_G = \delta(1,2)\ldots\delta(1,n)\delta(2,3)\ldots\delta(2,n)\ldots\delta(k,k+1)\ldots\delta(k,n)\ldots\delta(n-1,n).$$

Notice that B_G has a total length of $\binom{n}{2}$.

Definition 2.4: We define $\mathrm{Sym}(\Omega)$ as the group of possible permutations of a finite set Ω. For our purposes, $\Omega = [1, \ldots, n]$. Therefore, we let $\mathrm{Sym}(\Omega) = S_n$.

(a) G_1 (b) G_2

Figure 2.1: Two non-isomorphic graphs on 4 vertices.

Example 2.1: Consider graphs G_1 and G_2 in Figure 2.1. In this case, we consider $\Omega =$

$[1, 2, 3, 4]$. Since G_1 has a vertex with degree 2 while G_2 does not, we know immediately that the two are non-isomorphic. The string representations of G_1 and G_2 are both of length $\binom{4}{2} = 6$. Applying the ordering used above, we have

$$B_{G_1} = 001100,$$

and

$$B_{G_2} = 101000.$$

We also have

$$\text{Sym}\ (\Omega) = S_4.$$

Definition 2.5: Let S_G be the string representation of a graph G with n vertices labelled using $\Omega = [1, \ldots, n]$.

Then

$$S_n^{(2)} = \left\{ f_\sigma : \binom{\Omega}{2} \to \binom{\Omega}{2} \right\}$$

where

$$f_\sigma(\{a, b\}) = \{\sigma a, \sigma b\}$$

for $\sigma \in S_n$.

Definition 2.6: Let $\sigma \in S_n$ and let G be a graph. Then $\sigma(G)$ is the graph obtained by permuting the vertex labels of G according to σ.

(a) G (b) $\sigma(G)$

Figure 2.2: Graphs G and $\sigma(G)$ where $\sigma = (123)$.

Example 2.2:

Definition 2.7: Let $\sigma \in S_n$, $f_\sigma \in S_n^{(2)}$ and let B be the binary string representation of a graph G. Then

$$B^{f_\sigma} = B_{\sigma(G)}$$

where $B_{\sigma(G)}$ is as defined in Definition 2.3.

Definition 2.8: Let B_1 and B_2 be string representations of undirected graphs G_1 and G_2, respectively. Then B_1 and B_2 are $S^{(2)}$*-isomorphic*, denoted $B_1 \cong B_2$, if there exists $f_\sigma \in S_n^{(2)}$ such that $B_1^{f_\sigma} = B_2$.

Example 2.3: Consider graphs G and G' in Fig. 2.2. Let B and B' be the string representations of G and G', respectively. Notice that if we swap the labels 1 and 4 as well as 2 and 3 in G (i.e. $\sigma = (14)(23)$), we have G'. Hence, we easily see that $G \cong G'$. To show that $B \cong B'$, we find a function $f_\sigma \in S_n^{(2)}$ that results from applying σ to G. We have

$$f_\sigma = \begin{cases} \{1,2\} & \to \{4,3\} \\ \{1,3\} & \to \{4,2\} \\ \{1,4\} & \to \{1,4\} \\ \{2,3\} & \to \{2,3\} \\ \{2,4\} & \to \{3,1\} \\ \{3,4\} & \to \{2,1\} \end{cases}.$$

We now compute B^{f_σ} using Definition 2.3. We have

$$\begin{aligned} B^{f_\sigma} &= 010111 \\ &= B' \end{aligned}$$

Hence, by Definition 2.8, $B \cong B'$.

(a) G, $B = 110110$

(b) G', $B' = 010111$

Figure 2.3: Undirected, isomorphic graphs G and G'

Lemma 2.1: Two graphs are isomorphic if and only if their string representations are $S_n^{(2)}$-isomorphic.

Proof. SKETCH

Let B and B' be binary string representations of undirected graphs G and G', respectively.

$\rightarrow$

Suppose $G \cong G'$. By Definition 1.2, there exists bijection $g : V(G) \rightarrow V(G')$ that preserves neighborhoods. Let $\sigma = g$. Since we know g preserves neighborhoods, $B^{f_\sigma} = B'$.

$\leftarrow$

Suppose $B \cong B'$. By Definition 2.8, there exists f_σ such that $B^{f_\sigma} = B'$. Hence, there exists $\sigma : V(G) \rightarrow V(G')$ that is onto. Since the $V(G)$ and $V(G')$ have the same cardinality, σ is also one-to-one. Hence, σ is bijective. Lastly, since the strings describe all adjacency relations and $B \cong B'$, we know that σ preserves neighborhoods. Hence, $G \cong G'$.

$\square$