Introduction

1.1 Graph Isomorphism

Definition 1.1: An undirected graph G is a set of vertices, $V(G)$, and a function $g : V(G) \to P(V(G))$, defined by $g(v) = N(v)$, where $N(v)$ is the set of vertices that are adjacent to v and $P(V(G))$ is the power-set of $V(G)$.

Remark: We call $N(v)$ the *neighborhood* of v.

Remark: Since the graphs we are concerned with are undirected, for any graph G and any vertices $u, v \in V(G)$, $u \in N(V)$ if and only if $v \in N(u)$.

Definition 1.2: Two graphs, G and G', are isomorphic, denoted $G \cong G'$, if there exists a bijection $f : V(G) \to V(G')$ such that for every $u, v \in V(G)$, $u \in N(v)$ if and only if $f(u) \in N(f(v))$.

2.1 A graph as a string

Following Luks [1982], a graph can be represented as a string using the following procedure. Let $\Omega = [1, \ldots, n]$. Let G_1 and G_2 be undirected graphs on n vertices, labelled using Ω. Let δ be the indicator function, $\delta : \binom{V(G)}{2} \to \{0, 1\}$, defined by

$$\delta(x, y) = \begin{cases} 1, & x \in N(y) \\ 0, & \text{o/w} \end{cases},$$

where $\binom{V(G)}{2}$ denotes the set of all unordered pairs in $V(G)$ for some undirected graph G (Babai 2018). To create a string from delta that represents G, we must order the pre-image of δ. We choose a straightforward ordering. We have

$$S = \delta(1, 2) \ldots \delta(1, n)\delta(2, 3) \ldots \delta(2, n) \ldots \delta(k, k + 1) \ldots \delta(k, n) \ldots \delta(n - 1, n).$$

Notice that S has a total length of $\binom{n}{2}$.

(a) G_1 (b) G_2

Figure 2.1: Two non-isomorphic graphs on 4 vertices.

Example 2.1: Consider graphs G_1 and G_2 in Figure 2.1. Since G_1 has a vertex with degree 2 while G_2 does not, we know immediately that the two are non-isomorphic. The string representations of G and G' are both of length $\binom{4}{2} = 6$. Applying the ordering used above, the string representation of G_1 is

$$001100,$$

while that of G_2 is

$$101000.$$

Notice that the string representation of graph is only unique after an order has been established on $\binom{V(G)}{2}$. We also have

$$\text{Sym }(V(G)) = \{(), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)\}.$$

Definition 2.3: Let G be an undirected graph on n vertices. We define $\text{Sym}(V(G))$ as the group of possible permutations of $V(G)$.

Definition 2.4: Let G be an undirected graph on n ordered vertices. Let δ be the indicator function for G while S is the string representation of G produced by δ and the order on $\binom{V(G)}{2}$.

Then

$$\text{Sym }(V(G)))^{(2)} = \{\{\sigma a, \sigma b\} : a, b \in V(G), \sigma \in \text{Sym }(V(G))\}.$$

When does $\delta(\{\sigma a, \sigma b\}) = \delta(\{a, b\})$ come into play (i.e. valid vs invalid?)?

Definition 2.5: Let S_1 and S_2 be string representations of undirected graphs G_1 and G_2, respectively. Then S_1 and S_2 are *Sym*$(V(G_1))^{(2)}$*-isomorphic*, denoted $S_1 \cong_{\text{Sym }(V(G_1))^{(2)}} S_2$, if there exists $\overline{\sigma} \in \text{Sym}(V(G_1))^{(2)}$ such that $S_1^{\overline{\sigma}} = S_2$.

Example 2.2: Consider graphs G and G' in Fig. 2.1. Let the order of the string be 12, 13, 14, 23, 24, 34. To show that $S \cong_{\text{Sym }(V(G))^{(2)}} S'$, let σ be the permutation of S' that is induced by (14)(23) applied to $V(G')$. Then $\sigma \in \text{Sym}(V(G))^{(2)}$. QUESTION: What is σ in this case? Hence, by Definition 2.3, $S \cong_{\text{Sym }(V(G))^{(2)}} S'$.

(a) G,
$S = 110110$

(b) G',
$S' = 010111$

Figure 2.2: Undirected, isomorphic graphs G and G'

Lemma 2.1: Two graphs are isomorphic if and only if their string representations are $\mathrm{Sym}(V(G))^{(2)}$-isomorphic.

Proof. $\rightarrow$

Let S and S' be string representations of undirected graphs G and G', respectively. Suppose $G \cong G'$. Without loss of generality, suppose G and G' each have n vertices. By Definition 1.2, there exists a bijection $f : V(G) \rightarrow V(G')$ such that for every $u, v \in V(G)$, $u \in N(V)$ if and only if $f(u) \in N(f(v))$. This means that for every vertex $u \in G$, there is a vertex $f(u) \in G'$ such that $f(N(u)) = N(f(u))$. Recall that S and S' both contain n binary n-tuples that describe the neighborhood of each vertex in graphs G and G', respectively. We want to show that there exists $\sigma \in$Sym $(V(G))^{(2)}$ such that $S^\sigma = S'$. I'm stuck here. I'm having trouble figuring out how to deal with elements of Sym $(V(G))^{(2)}$.

$\square$