Introduction

1.1 Graph Isomorphism

Definition 1.1: An undirected graph G is a set of vertices, $V(G)$, and a function $g : V(G) \to V(G)$, defined by $g(v) = N(v)$, where $N(v)$ is the set of vertices that are adjacent to v.

Remark: We call $N(v)$ the *neighborhood* of v.

Remark: Since the graphs we are concerned with are undirected, for any graph G and any vertices $u, v \in V(G)$, $u \in N(V)$ if and only if $v \in N(u)$.

Definition 1.2: Two graphs, G and G', are isomorphic, denoted $G \cong G'$, if there exists a bijection $f : V(G) \to V(G')$ such that for every $u, v \in V(G)$, $u \in N(v)$ if and only if $f(u) \in N(f(v))$.

2.1 A graph as a string

Following Luks [1982], a graph can be represented as a string using the following procedure. Let G be a graph on n vertices. Let δ be the indicator function, $\delta : \binom{V(G)}{2} \to \{0, 1\}$, defined by

$$\delta(x, y) = \begin{cases} 1, & x \in N(y) \\ 0, & \text{o/w} \end{cases},$$

where $\binom{V(G)}{2}$ denotes the set of all possible edges in G (Babai 2018). Notice that the image of δ, denoted S, is a binary string that contains n number of n-tuples. Hence, S has a total length of $\binom{n}{2}$. Consider $\mathrm{Sym}(V(G))$. Let $\mathrm{Sym}(V(G))^{(2)}$ denote the group representing the action on δ by $\mathrm{Sym}(V(G))$.

Definition 2.3: Let ξ and η be string representations of a graph G. Then ξ and η are *$Sym(V(G))^{(2)}$-isomorphic*, denoted $\xi \cong_{\mathrm{Sym}\,(V(G))^{(2)}} \eta$, if there exists $\sigma \in \mathrm{Sym}(V(G))^{(2)}$ such that $\xi^\sigma = \eta$.

Example 2.1: Consider graphs G and G' in Fig. 2.1. Note that the spaces in the strings, which are added for ease of reading, create blocks that describe the neighborhood of each vertex. To show that $S \cong_{\mathrm{Sym}\,(V(G))^{(2)}} S'$, let σ be the permutation of S' that is induced by (14)(23) applied to $V(G')$. Then $\sigma \in \mathrm{Sym}(V(G))^{(2)}$. Notice that σ reverses the order of S'. QUESTION: What is σ in this case? We have

$$S'^\sigma = 0110\ 1011\ 1100\ 0100 = S.$$

Hence, by Definition 2.3, $S \cong_{\mathrm{Sym}\,(V(G))^{(2)}} S'$.

(a) G,
S =0110 1011 1100 0100

(b) G',
S' =0010 0011 1101 0110

Figure 2.1: Undirected, isomorphic graphs G and G'

Lemma 2.1: Two graphs are isomorphic if and only if their string representations are $\text{Sym}(V(G))^{(2)}$-isomorphic.

Proof. $\rightarrow$

Let S and S' be string representations of undirected graphs G and G', respectively. Suppose $G \cong G'$. Without loss of generality, suppose G and G' each have n vertices. By Definition 1.2, there exists a bijection $f : V(G) \rightarrow V(G')$ such that for every $u, v \in V(G)$, $u \in N(V)$ if and only if $f(u) \in N(f(v))$. This means that for every vertex $u \in G$, there is a vertex $f(u) \in G'$ such that $f(N(u)) = N(f(u))$. Recall that S and S' both contain n binary n-tuples that describe the neighborhood of each vertex in graphs G and G', respectively. We want to show that there exists $\sigma \in$Sym $(V(G))^{(2)}$ such that $S^\sigma = S'$. I'm stuck here. I'm having trouble figuring out how to deal with elements of Sym $(V(G))^{(2)}$.

$\square$