MAE 3128 : BIOMECHANICS HW 1 SOLUTIONS

SOLUTION HW1 P1

Given: $\boxed{\dfrac{d[C]}{dt} = -k[C]}$

where $[C]$ ⇝ concentration

k ⇝ specific reaction rate

⇝ There are two reactions affecting tissue maintenance

{Production of nutrients}

[k_p ⇝ specific reaction rate for production]

{Removal of waste}

[k_r ⇝ specific reaction rate for removal]

$$\therefore \frac{d[C]_p}{dt} = +k_p[C]$$

$$\frac{d[C]_r}{dt} = -k_r[C]$$

Total change in concentration ⇝ $\boxed{\dfrac{d[C]}{dt} = \dfrac{d[C]_p}{dt} + \dfrac{d[C]_r}{dt}}$

$$\Rightarrow \frac{d[C]}{dt} = (k_p - k_r)[C]$$

$$= k_o[C] \qquad \text{where } k_o = (k_p - k_r)$$

$$\therefore \frac{d[C]}{dt} = k_o [C]$$

$$\Rightarrow \int \frac{d[C]}{[C]} = \int k_o \, dt$$

$$\Rightarrow \ln[C] = k_o t + c$$

$$\Rightarrow [C] = A e^{k_o t}$$

$$\Rightarrow \boxed{[C] = A e^{(k_p - k_r)t}}$$

where A is a constant based on the initial condition at $t = 0$

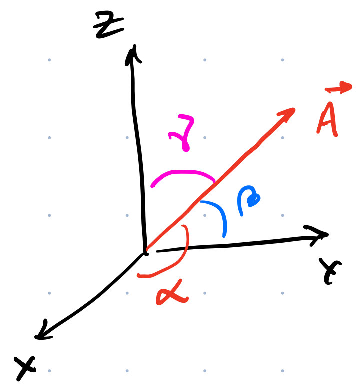
$\boxed{\text{SOLUTION HW1 P2 :}}$

Given:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\cos\alpha = \frac{A_x}{|A|} \quad \cos\beta = \frac{A_y}{|A|} \quad \cos\gamma = \frac{A_z}{|A|}$$

$$\left\{ \text{where } |A| = \sqrt{A_x^2 + A_y^2 + A_z^2} \right\}$$



$\rightarrow$ If $\hat{e} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

$$|\hat{e}| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

$\because |\hat{e}| = 1$

we have $\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \left(\frac{A_x}{|A|}\right)^2 + \left(\frac{A_y}{|A|}\right)^2 + \left(\frac{A_z}{|A|}\right)^2 = 1$$

$$\Rightarrow \frac{A_x^2 + A_y^2 + A_z^2}{|A|^2} = \frac{|A|^2}{|A|^2} = 1$$

$\because \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

we have the following inter-relationships

$$\boxed{\begin{aligned} \alpha &= \cos^{-1}\left(\sqrt{1-\cos^2\beta-\cos^2\gamma}\right) \\ \beta &= \cos^{-1}\left(\sqrt{1-\cos^2\alpha-\cos^2\gamma}\right) \\ \gamma &= \cos^{-1}\left(\sqrt{1-\cos^2\alpha-\cos^2\beta}\right) \end{aligned}}$$

These relationships work in 2D case where $\cos\gamma = 0$ or $\gamma = 90^\circ$

Consider: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \underbrace{\cancel{\cos^2(90)}}_{\to 0} = 1$$

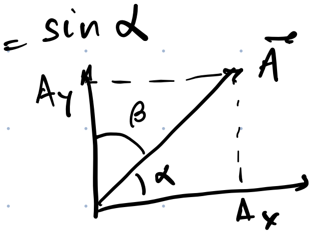
$$\Rightarrow \boxed{\cos^2\alpha + \cos^2\beta = 1}$$

This recovers 2D unit vector

$|\hat{e}| = 1 = \cos^2\alpha + \cos^2\beta$ ✓

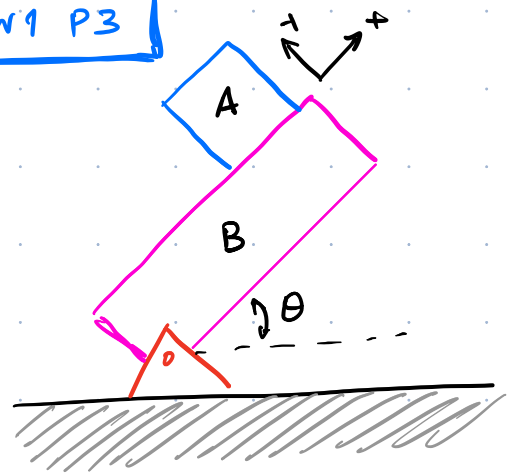
Furthermore: In 2D space $\vec{A} = A_x\hat{i} + A_y\hat{j}$

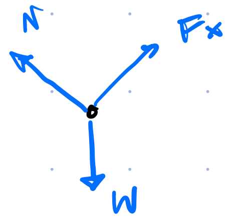
$$\cos\alpha = \frac{A_x}{|A|} \; ; \quad \cos\beta = \frac{A_y}{|A|} \; ; \quad \text{But, } \frac{A_y}{|A|} = \sin\alpha$$



$\therefore \quad \cos^2\alpha + \cos^2\beta = 1 \Rightarrow \cos^2\alpha + \sin^2\alpha = 1$ ✓

SOLUTION HW1 P3





Ans Just before the block A slips, we have equilibrium

$\therefore \sum F_x = 0 \Rightarrow \mu N - W \sin\theta = 0$

$\Rightarrow \mu \, W\cos\theta - W \sin\theta = 0$

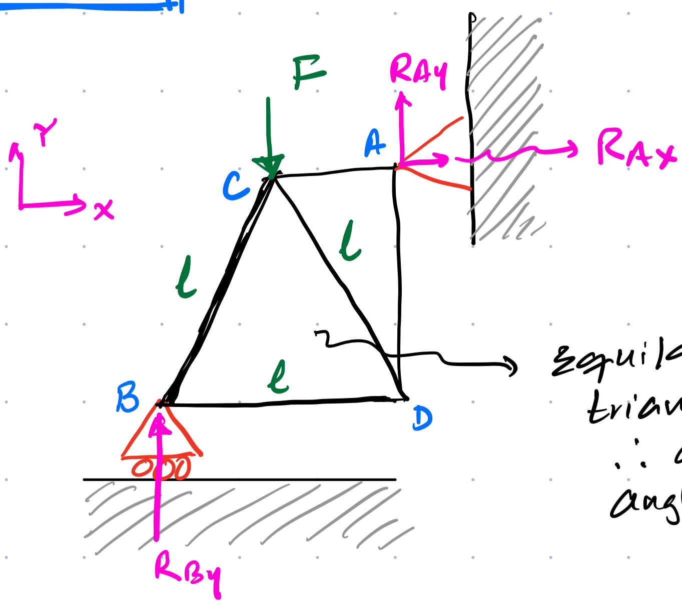
$\therefore \sum F_y = 0$

$\Rightarrow N - W\cos\theta = 0$

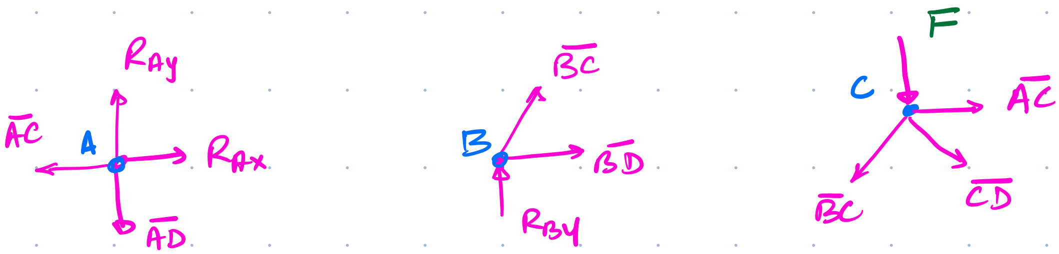
$\Rightarrow \boxed{N = W\cos\theta}$

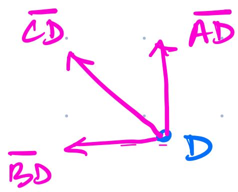
$\Rightarrow \mu = \dfrac{\cancel{W}\sin\theta}{\cancel{W}\cos\theta} \Rightarrow \boxed{\mu = \tan\theta}$

SOLUTION HW1 P4



Equilateral triangle
∴ all internal angles are 60°





Reaction forces and Applied forces across the entire truss:

$\boxed{\sum F_x = 0}$ $\quad \boxed{R_{Ax} = 0} \rightarrow (1)$

$\boxed{\sum F_y = 0}$ $\quad F - R_{By} - R_{Ay} = 0$

$\Rightarrow F = R_{By} + R_{Ay} \Rightarrow \boxed{R_{By} = F/2} \rightarrow (3)$

$\boxed{\sum M_B = 0}$

$R_{Ay}(\ell) - F(\ell/2) = 0$

$\Rightarrow \boxed{R_{Ay} = \frac{F}{2}} \rightarrow (2)$

At point A :

$\boxed{\sum F_x = 0} \Rightarrow R_{Ax} - \overline{AC} = 0 \Rightarrow \boxed{\overline{AC} = 0} \rightarrow (4)$ From eq (1)

$\boxed{\sum F_y = 0} \Rightarrow R_{Ay} - \overline{AD} = 0 \Rightarrow \boxed{\overline{AD} = \frac{F}{2}} \rightarrow (5)$ From eq (2)

At point B :

$\boxed{\sum F_x = 0} \Rightarrow \overline{BD} + \overline{BC} \cos 60^\circ = 0 \Rightarrow \overline{BD} = \frac{F}{\sqrt{3}} \cos 60^\circ \Rightarrow \boxed{\overline{BD} = \frac{F}{2\sqrt{3}}} \rightarrow (7)$

$\boxed{\sum F_y = 0} \Rightarrow \overline{BC} \sin 60^\circ + R_{By} = 0 \Rightarrow \boxed{\overline{BC} = -\frac{F}{\left(\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right)} = -F/\sqrt{3}} \rightarrow (6)$ From eq (3)

At point C:

$\boxed{\sum F_x = 0} \Rightarrow \overline{AC} + \overline{CD} \cos 60^\circ - \overline{BC} \cos 60^\circ = 0$

$\Rightarrow \boxed{\overline{CD} = \overline{BC}} \rightarrow (8)$ $\because \overline{AC} = 0$ from Eq (4)

$\boxed{\sum F_y = 0} \Rightarrow F + \overline{BC} \sin 60^\circ + \overline{CD} \sin 60^\circ = 0$

$\Rightarrow \boxed{F = -2\overline{BC} \sin 60^\circ}$ $\because \overline{CD} = \overline{BC}$ from Eq (7)

At point D :

From eqs (7) & (8)

$\boxed{\sum F_x = 0}$ $\overline{BD} + \overline{CD} \cos 60^\circ = 0 \checkmark \Rightarrow \cancel{\frac{F}{2\sqrt{3}}} + \cancel{\left(\frac{-F}{\sqrt{3}}\right)\frac{1}{2}} = 0$

$\boxed{\sum F_y = 0}$ $\overline{AD} + \overline{CD} \sin 60^\circ = 0 \Rightarrow \overline{AD} = -\frac{F}{\sqrt{3}} \frac{\sqrt{3}}{2}$

$\Rightarrow \boxed{\overline{AD} = -\frac{F}{2}}$

SOLUTION HW1 P5

Given: $\sigma_{xy} = \sigma_{yx} = 5\,MPa$

$\sigma_{xx} = \sigma_{yy} = 0$

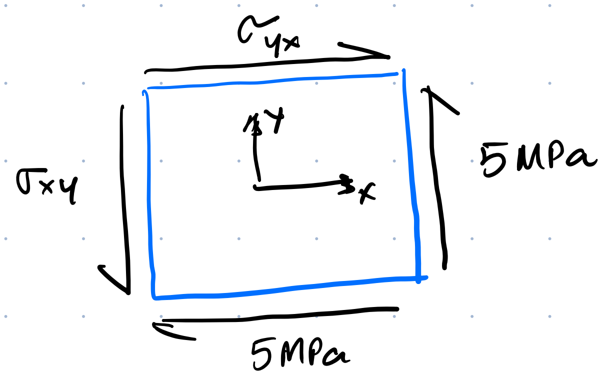
$$\sigma_{1,2} = \frac{\cancel{\sigma_{xx}}^{0} + \cancel{\sigma_{yy}}^{0}}{2} \pm \sqrt{\frac{\cancel{\sigma_{xx}}^{2}_{0} - \cancel{\sigma_{yy}}^{2}_{0}}{2} + \sigma_{xy}^{2}}$$

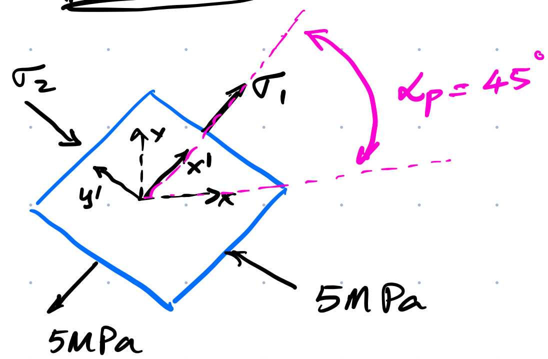
$$= \pm\sqrt{\sigma_{xy}^2} \Rightarrow \boxed{\sigma_{1,2} = \pm 5\,MPa}$$

Orientation:

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{xy}}{\cancel{\sigma_{xx}}_{0} - \cancel{\sigma_{yy}}_{0}}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty) \Rightarrow \boxed{\alpha_p = \frac{\pi}{4} = 45^\circ}$$





SOLUTION HW1 P6

Given: $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ pressure

$$\sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$= \frac{-p-p}{2} \pm \sqrt{\underbrace{\cancel{\left(\frac{-p-(-p)}{2}\right)^2}}_{0} + 0}$$

$$\Rightarrow \boxed{\sigma_{1,2} = \frac{-2p}{2} = -p}$$

$$\sigma_{x,y,max} = \pm\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}} = \tau_m$$

$$= \pm\sqrt{\underbrace{\cancel{\left\{\frac{-p-(-p)}{2}\right\}^2}}_{0} + 0}$$

$$\Rightarrow \boxed{\tau_m = 0}$$

SOLUTION HW1 P7

Given: $\sigma_{xx} = 3\ \text{MPa}$

$\sigma_{yy} = -3\ \text{MPa}$

$$\tau_m = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}}$$

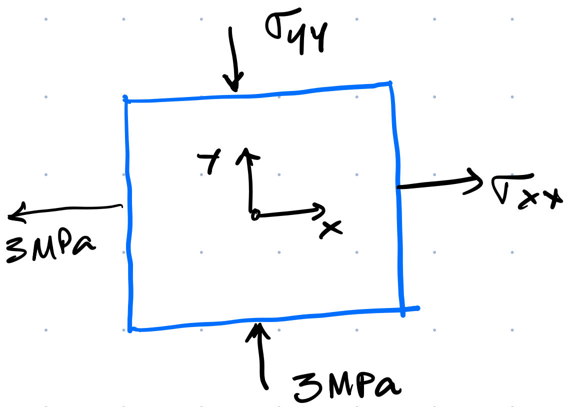
$$= \sqrt{\left\{\frac{3-(-3)}{2}\right\}^2 + 0^2}$$

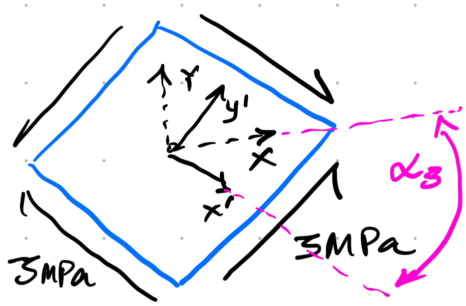
$$\boxed{\tau_m = 3\ \text{MPa}}$$

$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{\sigma_{yy} - \sigma_{xx}}{2\sigma_{xy}}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty)$$

$$\Rightarrow \boxed{\alpha_s = \frac{\pi}{4} \text{ OR } 45^\circ}$$





SOLUTION HW1 P8

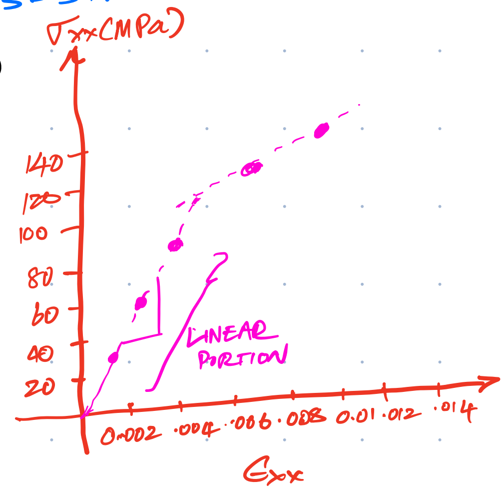
Axial force (N)	94	190	284	376	440
Change in length (mm)	0.009	0.018	0.027	0.050	0.094

ⓐ Plot the associated stress-strain relationship

$$\sigma_{xx} = \frac{F}{A}$$

$$\epsilon_{xx} = \frac{\Delta u_x}{\Delta x}$$

ϵ_{xx}	σ_{xx} (MPa)
0.0015	29.921
0.003	60.479
0.0045	90.4
0.0083	119.68
0.0154	140.06



ⓑ Calculate the Young's Modulus

↳ E is the slope of the line on the linear portion (first 3 points) of the stress-strain curve

↳ the trend line for the linear portion is fit by $y = 20176\,x$ (i.e., $\sigma_{xx} = E\,\epsilon_{xx}$)

So, $\boxed{E = 20.186 \text{ GPa}}$

ⓒ Show that the yield stress is 118 MPa (recall that the yield stress reveals the transition from elastic to plastic deformation)

↳ The bone will yield before the 4th point but after the 3rd point on the stress-strain data shown in the figure.

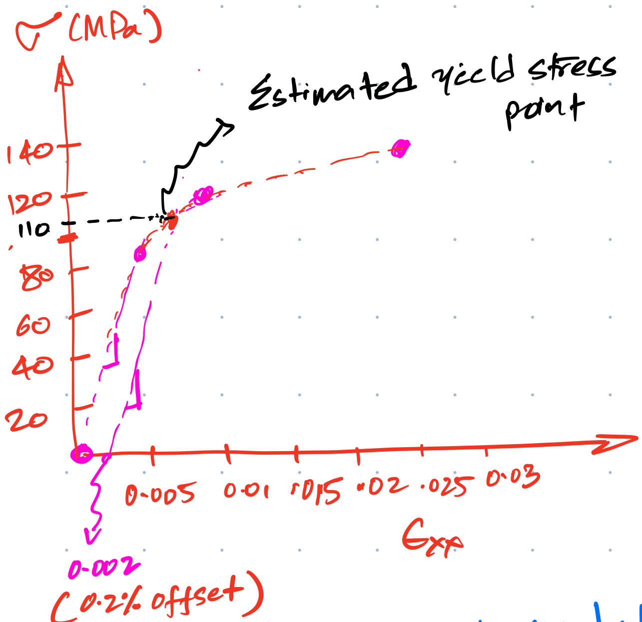
↳ $90 < \sigma_{yield} < 119$

↳ We can estimate that to be approximately $\boxed{118 \text{ MPa}}$.

↳ OR we can draw a 0.2% strain offset line parallel to the linear portion & look for the point of intersection.

SOLUTION HW1 P9

(a) Plot the data given. Interpret the plot



→ Linear region is upto 85 MPa

→ Material may be brittle - based on the data shown

(b) Estimate the Young's modulus, yield stress and Ultimate stress

→ Young's modulus is the slope of the linear portion of the data

$y_1 = Ex \Rightarrow 8.5\times10^6\ Pa = E\ (0.005)$

$\Rightarrow \boxed{E = 17\ GPa}$

→ Yield stress: Point of intersection of 0.2% strain offset line, drawn parallel to the linear portion of the data, i.e., $\approx \boxed{110\ MPa}$

→ Ultimate stress: Stress value where the material fails.

→ From the problem statement that is $\boxed{128\ MPa}$