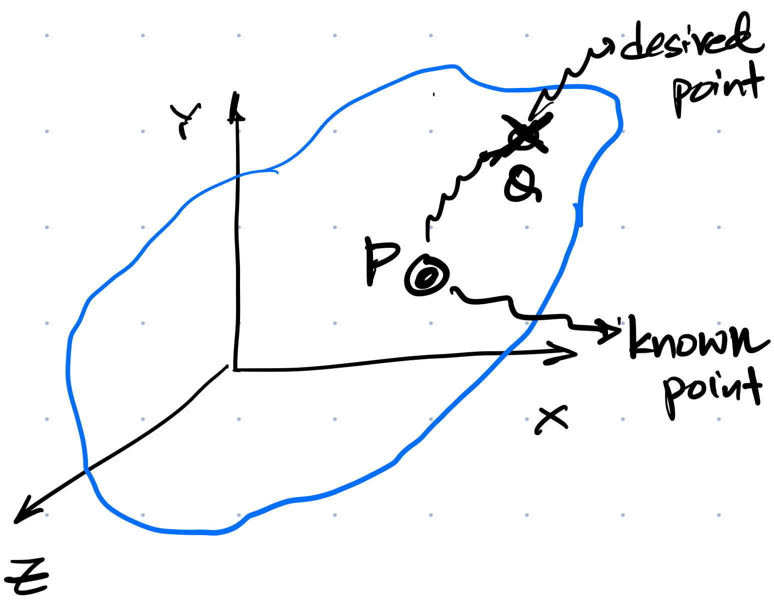
## (ii) Apply Taylor series

→ We want to write the generalized equilibrium equations for



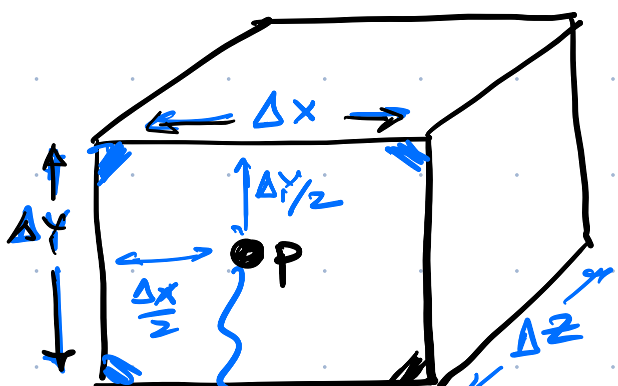

(i) small stress variations between P & Q, and

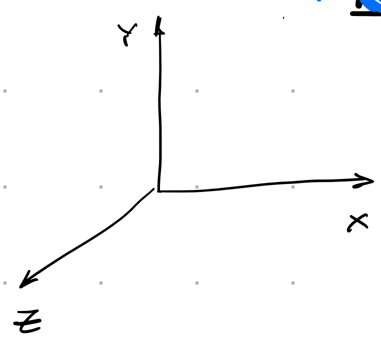
(ii) within the continuum of the body

→ consider a body in 3D

(i) Cartesian coordinates,

(ii) with point, P located at the center



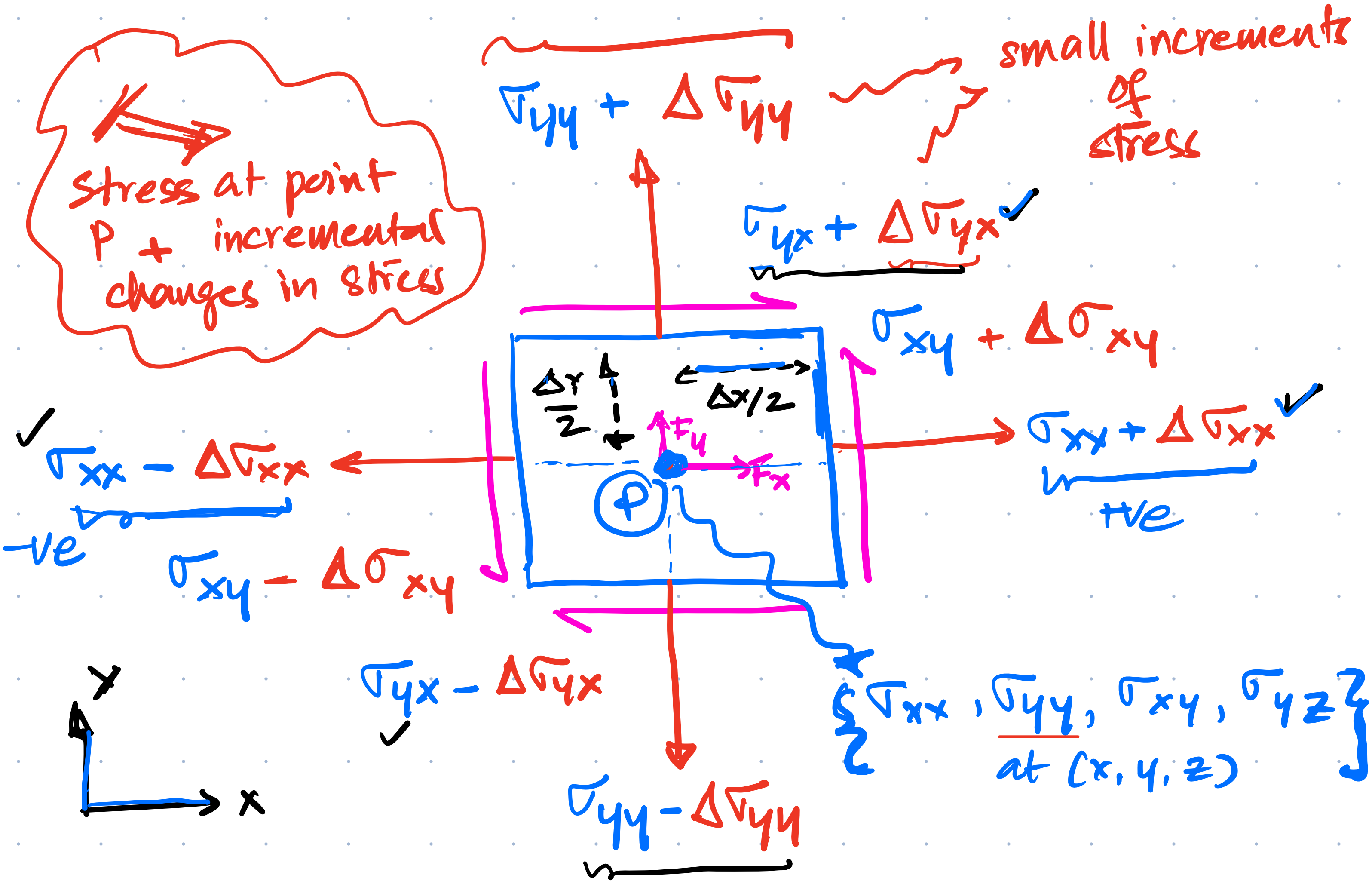




$\underbrace{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz} \ldots\ldots}$

→ How these stress vary at some distance away from that point (P)

↳ Let's now consider the 2D state of stress

↳ This is only for simplicity of mathematical analysis.




## → CONDITIONS FOR EQUILIBRIUM ←

① $\sum F = 0$

② $\sum M = 0$

Note: P is located at (x, y, z), center!

Note: $F_x$, $F_y$ are body force intensities acting at P.

# Expanding each stress-term in the x-direction using Taylor series:

(x-direction)

$[\sigma_{xx} + \Delta\sigma_{xx}]$

$$\sigma_{xx}\left(x + \frac{\Delta x}{2}\right) = \sigma_{xx}(x) + \frac{1}{1!}\frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2!}\frac{\partial^2 \sigma_{xx}}{\partial x^2}\left(\frac{\Delta x}{2}\right)^2 + \underbrace{O(\Delta x)^3}_{HOT}$$

NORMAL

$[\sigma_{xx} - \Delta\sigma_{xx}]$

$$\sigma_{xx}\left(x - \frac{\Delta x}{2}\right) = \sigma_{xx}(x) - \frac{1}{1!}\frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2!}\frac{\partial^2 \sigma_{xx}}{\partial x^2}\left(\frac{\Delta x}{2}\right)^2 + O(\Delta x)^3$$

$[\sigma_{yx} + \Delta\sigma_{yx}]$

$$\sigma_{yx}\left(y + \frac{\Delta y}{2}\right) = \sigma_{yx}(y) + \frac{1}{1!}\frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right) + \frac{1}{2!}\frac{\partial^2 \sigma_{yx}}{\partial y^2}\left(\frac{\Delta y}{2}\right)^2 + O(\Delta y)^3$$

SHEAR

$[\sigma_{yx} - \Delta\sigma_{yx}]$

$$\sigma_{yx}\left(y - \frac{\Delta y}{2}\right) = \sigma_{yx}(y) - \frac{1}{1!}\frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right) + \frac{1}{2!}\frac{\partial^2 \sigma_{yx}}{\partial y^2}\left(\frac{\Delta y}{2}\right)^2 + O(\Delta y)^3$$

(11)

Multiplying each stress-term by their respective areas & volume to get the force terms

$\boxed{\sum F_x = 0}$ ⟶ Equilibrium condition in X-direction

$$\underbrace{\left[\sigma_{xx}\left(x+\frac{\Delta x}{2}\right)\right]}_{\hookrightarrow \#}\Delta y\,\Delta z - \underbrace{\left[\sigma_{xx}\left(x-\frac{\Delta x}{2}\right)\right]}_{\hookrightarrow \#}\Delta y\,\Delta z$$

$$+$$

$$\underbrace{\left[\sigma_{yx}\left(y+\frac{\Delta y}{2}\right)\right]}_{\hookrightarrow \#}\Delta x\,\Delta z - \underbrace{\left[\sigma_{yx}\left(y-\frac{\Delta y}{2}\right)\right]}_{\hookrightarrow \#}\Delta x\,\Delta z$$

$$+$$

$$F_x\,(\Delta x\,\Delta y\,\Delta z) = 0$$

Taylor series expansion

$$\left[\sigma_{xx}(x) + \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\frac{\partial^2 \sigma_{xx}}{\partial x^2}\frac{(\Delta x)^2}{4} + \cancel{O(\Delta x)^3}_{\,0}\right]\Delta y\,\Delta z$$

$$- \left[\sigma_{xx}(x) - \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\frac{\partial^2 \sigma_{xx}}{\partial x^2}\frac{(\Delta x)^2}{4} + \cancel{O(\Delta x)^3}_{\,0}\right]\Delta y\,\Delta z$$

$$+ \left[\sigma_{yx}(y) + \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right) + \frac{1}{2}\frac{\partial^2 \sigma_{yx}}{\partial y^2}\frac{(\Delta y)^2}{4} + \cancel{O(\Delta y)^3}_{\,0}\right]\Delta x\,\Delta z$$

$$- \left[\sigma_{yx}(y) - \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right) + \frac{1}{2}\frac{\partial^2 \sigma_{yx}}{\partial y^2}\frac{(\Delta y)^2}{4} + \cancel{O(\Delta y)^3}_{\,0}\right]\Delta x\,\Delta z$$

$$+ F_x\,(\Delta x\,\Delta y\,\Delta y) = 0$$

$\boxed{\sum F_x = 0}$

Balancing all the force-terms including the body force in the x-direction

$$\cancel{\tau_{xx}(x)(\Delta y \Delta z)} + \underbrace{\frac{\partial \tau_{xx}}{\partial x}\frac{\Delta x}{2}(\Delta y \Delta z)} + \cancel{\frac{1}{2}\frac{\partial^2 \tau_{xx}}{\partial x^2}\frac{(\Delta x)^2}{4}}(\Delta y)(\Delta z)$$

$$\cancel{-\tau_{xx}(x)}(\Delta y \Delta z) + \underbrace{\frac{\partial \tau_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right)(\Delta y \Delta z)} - \cancel{\frac{1}{2}\frac{\partial^2 \tau_{xx}}{\partial x^2}\frac{(\Delta x)^2}{4}}(\Delta y)(\Delta z)$$

$$+\cancel{\tau_{yx}(y)(\Delta x \Delta z)} + \underbrace{\frac{\partial \tau_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right)(\Delta x \Delta z)} + \cancel{\frac{1}{2}\frac{\partial^2 \tau_{yx}}{\partial y^2}\frac{(\Delta y)^2}{4}}(\Delta x \Delta z)$$

$$\cancel{-\tau_{yx}(y)(\Delta x \Delta z)} + \underbrace{\frac{\partial \tau_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right)(\Delta x)(\Delta z)} - \cancel{\frac{1}{2}\frac{\partial^2 \tau_{yx}}{\partial y^2}\frac{(\Delta y)^2}{4}}(\Delta x \Delta z)$$

$$+ F_x(\Delta x \Delta y \Delta z) = 0$$

$$\boxed{\frac{\partial \tau_{xx}}{\partial x}(\Delta x \Delta y \Delta z) + \frac{\partial \tau_{yx}}{\partial y}(\Delta x \Delta y \Delta z) + F_x(\Delta x \Delta y \Delta z) = 0}$$

$$\boxed{\sum F_x = 0}$$

Dividing by $(\Delta x\,\Delta y\,\Delta z)$ & taking limit as $(\Delta x\,\Delta y\,\Delta z) \longrightarrow 0$

$$\lim_{\underbrace{(\Delta x\,\Delta y\,\Delta z)}_{\text{volume}} \to 0} \frac{\left\{ \frac{\partial \sigma_{xx}}{\partial x}(\Delta x\,\Delta y\,\Delta z) + \frac{\partial \sigma_{yx}}{\partial y}(\Delta x\,\Delta y\,\Delta z) + F_x(\Delta x\,\Delta y\,\Delta z) \right\}}{(\Delta x\,\Delta y\,\Delta z)}$$

$$= 0$$

$$\boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho g_x = 0} \qquad \boxed{\Sigma F_x = 0}$$

Normal stress variation in x-direction

Shear stress variation

Body force

2D states

$$\boxed{\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y = 0} \qquad \boxed{\Sigma F_y = 0}$$

# GENERALIZED EQUILIBRIUM EQUATIONS

IN 3D Cartesian Coordinates: $(x, y, z)$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho g_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z = 0$$

All continua

## IN 3D Cylindrical Co-ordinates:

$(r, \theta, z)$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \left(\frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right) + \rho g_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \left(\frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r}\right) + \rho g_\theta = 0$$

$$\left(\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z = 0$$

## IN 3D SPHERICAL CO-ORDINATES:

$(r, \theta, \phi)$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\theta r}\cot\theta\right) + \rho g_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{r}\left(2\sigma_{r\theta} + \sigma_{\theta r} + \{\sigma_{\theta\theta} - \sigma_{\phi\phi}\}\cot\theta\right) + \rho g_\theta = 0$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}\left(2\sigma_{r\phi} + \sigma_{\phi r} + \{\sigma_{\phi\theta} - \sigma_{\theta\phi}\}\cot\theta\right) + \rho g_\phi = 0$$

## Generalized Equilibrium Equations:

⊕ Valid for all continua

- Would be cool to have specialized equations for particular materials
- 3 Equations and 6 unknown stress
- We need additional equations to get to mathematically well-posed problems