# MAE 3128
# Biomechanics-I

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**Topics covered today:**

1. Universal solutions
    1. Biological motivations and applications
    2. Axial loading of a uniform rod.
    3. Inflation and extension of a thin-walled cylindrical tube
2. In-class problems

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THE GEORGE WASHINGTON UNIVERSITY

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Photo: Kartik Bulusu

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# Universal solutions

**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Results obtained independent of the specification of particular material properties, are called **universal solutions**.

- Although not emphasized in most books on the mechanics of materials, the generality of these universal solutions allow them to be applied equally to problems involving the uniaxial extension of tendons, rubber bands, metallic wires, or concrete.

These universal solutions can be found **solely from equilibrium equations**,
- Without using constitutive equations.
- Because they do not depend on material behavior, they apply to **any solid-like material** (metals, elastomers, biological tissues, etc.).

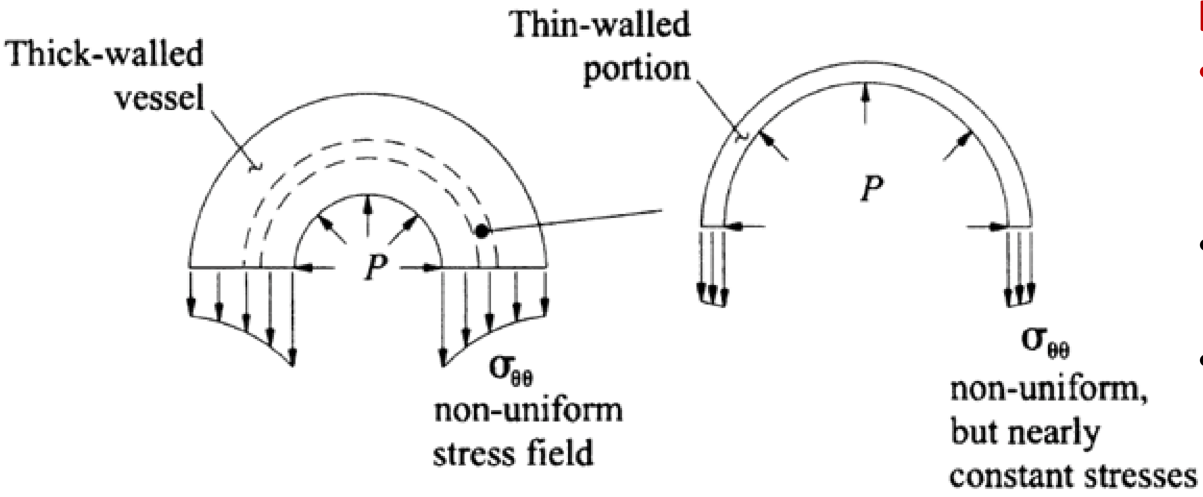
- Universal solutions are **experimentally valuable**,
  - as they exist independently of the material being tested and
  - assist in formulating stress–strain relations when strain is measurable.

# Universal solutions

Reference: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Three special cases for **universal solutions with biological motivation**:

- Axial loading of a uniform rod.
- Inflation and extension of a thin-walled cylindrical tube.
- Inflation of a thin-walled hollow sphere.

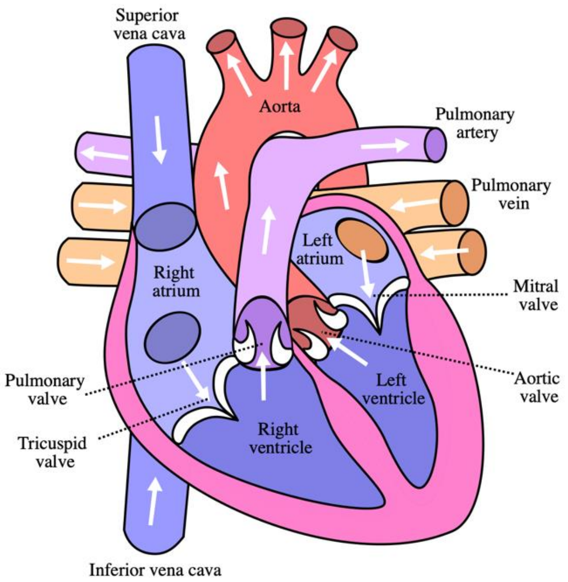


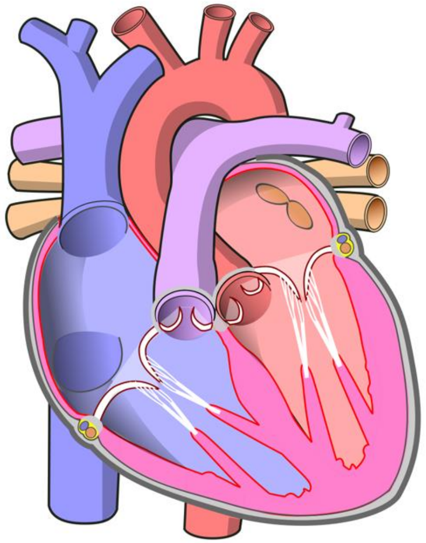

**Limits of applicability**

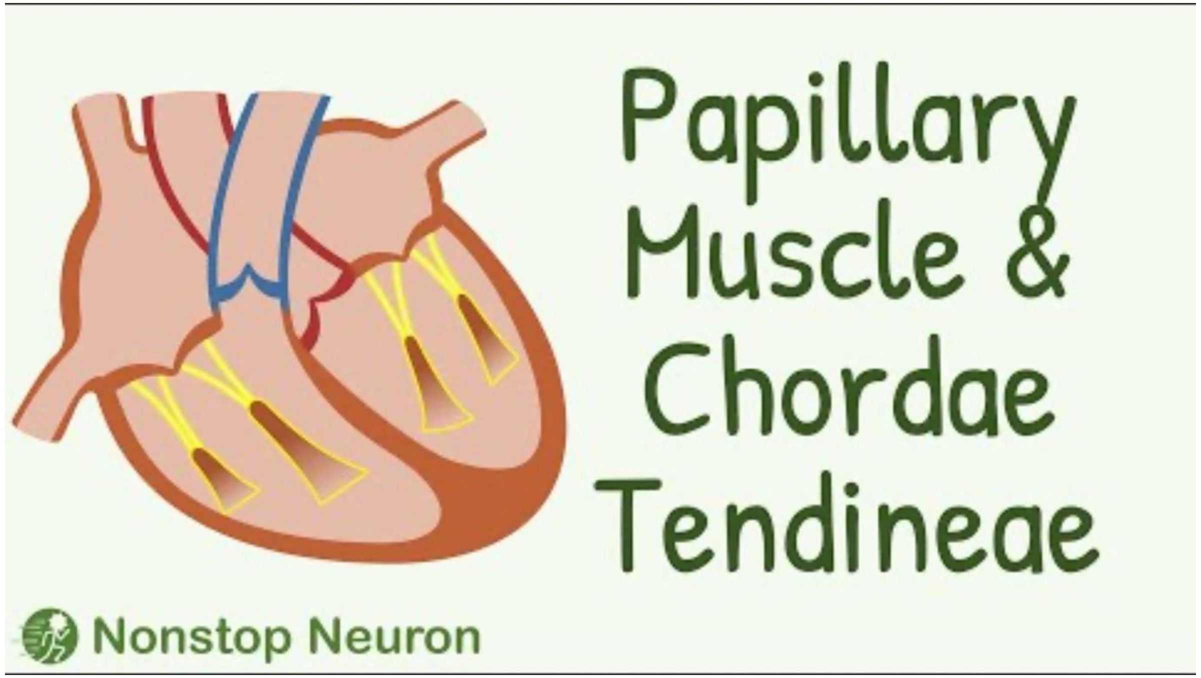
- **Increasing wall thickness** (in tubes or spheres) changes the solution approach and can make the stress distribution **non-uniform**, violating the conditions for universal solutions.
- For **thick-walled structures**, solutions require constitutive equations; the equilibrium-only approach no longer applies.
- The **average wall stress** can still be determined for both thin- and thick-walled structures, but it **better approximates reality** for thin-walled cases.

# Biological application problem: Chordae tendinaea

Chordae tendineae, or "heart strings," thin, strong fibrous cords of connective tissue connecting the papillary muscles to the tricuspid and mitral valves within the heart ventricles.






**References:**

1. **Video**: https://youtu.be/6f0n9zJWp_8?si=aVuVdl1lu4tZAPDh
2. **Image:** https://commons.wikimedia.org/w/index.php?title=File:Diagram_of_the_human_heart_(no_labels).svg&oldid=702762353
3. **Image:** https://commons.wikimedia.org/w/index.php?title=File:Diagram_of_the_human_heart.svg&oldid=1047699286#filelinks

## Function and Structure

- **Prevent Regurgitation:** By anchoring the valve leaflets, they prevent blood from flowing backward into the atria.
- **Composition:** Composed of collagen and elastic fibers, they are durable, tough, and sometimes elastic.

**A chordae tendineae specimen initially 10 mm long is to resist an axial tensile load f of 100 g. The specimen initially has a 1.0-mm diameter.**

What is the maximum axial stress that the chordae will experience?

**Hint:** Can be considered as an axial loading of a uniform rod

$$\frac{f}{A_o} = \frac{(100\,\mathrm{g})(9.807 \times 10^{-3}\mathrm{N/g})}{\pi(0.5\,\mathrm{mm})^2}\left(\frac{1000\,\mathrm{mm}}{\mathrm{m}}\right)^2 = 1.25\,\mathrm{MPa}.$$

**Key question**

Can chordae tendineae can sustain a 1.25 MPa stress *in vivo*?

## Common Conditions and Diagnosis

- **Chordae Tendineae Rupture:** A common cause of acute, severe mitral regurgitation (leaking valve), often resulting in symptoms like sudden shortness of breath, fatigue, and chest pain.
- **Diagnosis:** Typically diagnosed using echocardiography (ultrasound), which visualizes the valve motion and chordal attachment.
- **Causes:** Often caused by degeneration (mitral valve prolapse), infection (endocarditis), or trauma.

# Biological Motivation:
# Cardiovascular mechanics, anatomy and physiology

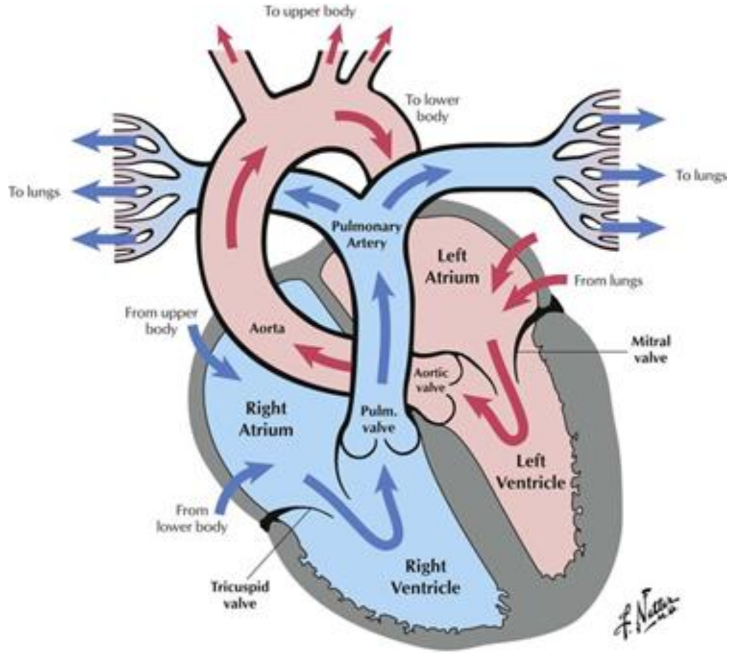
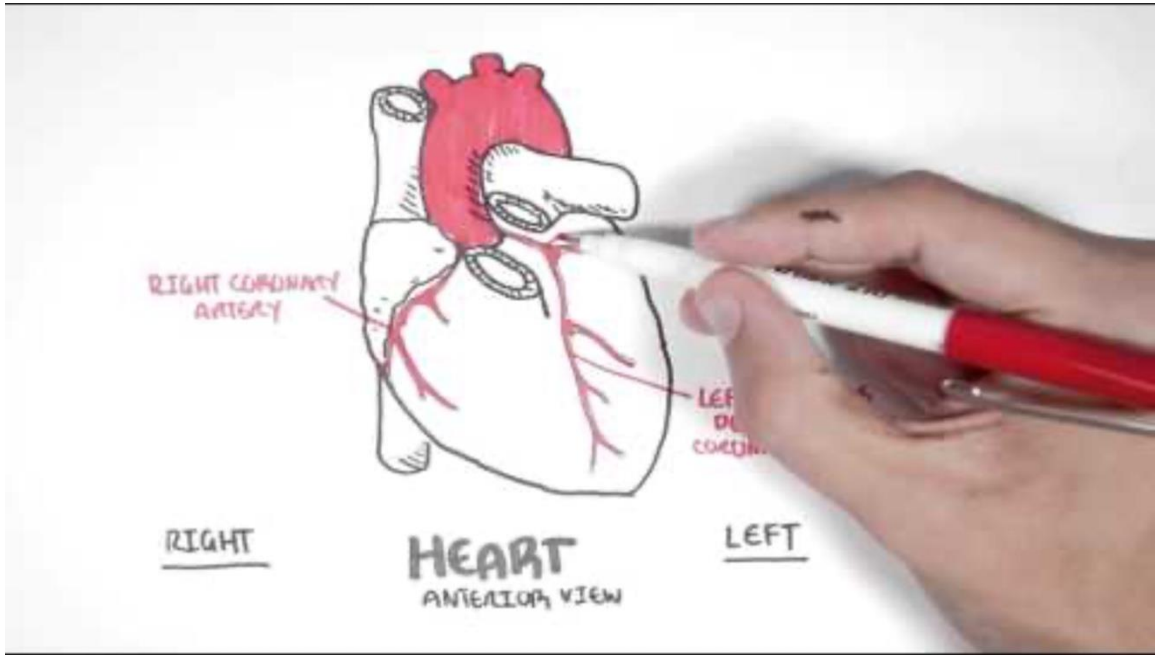
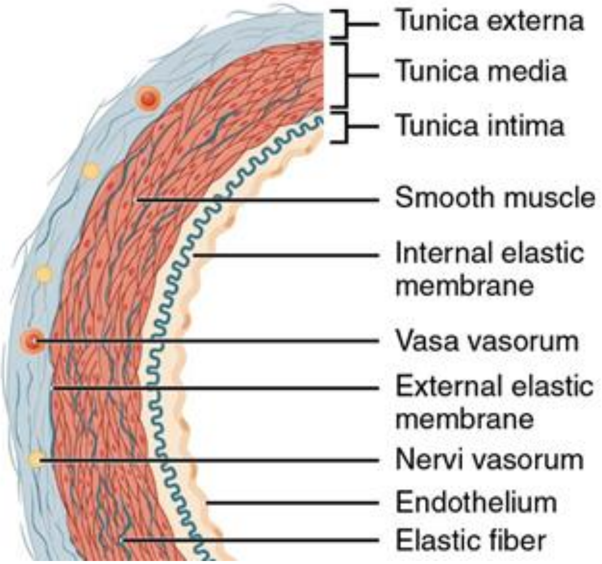


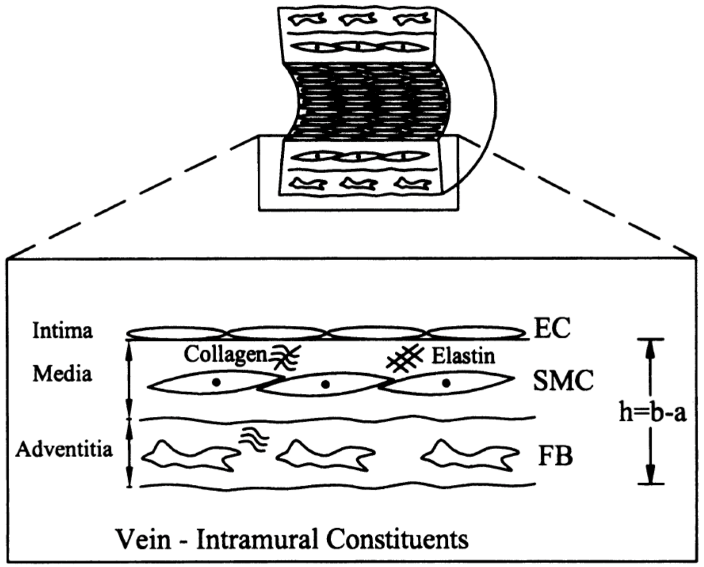

Image reference: :
https://basicmedicalkey.com/drugs-used-in-disorders-of-the-cardiovascular-system/




Video reference: :
https://youtu.be/3wpT-4bSmoU?si=wG114i_ym4ekaDmU

## The Structure of An Artery Wall



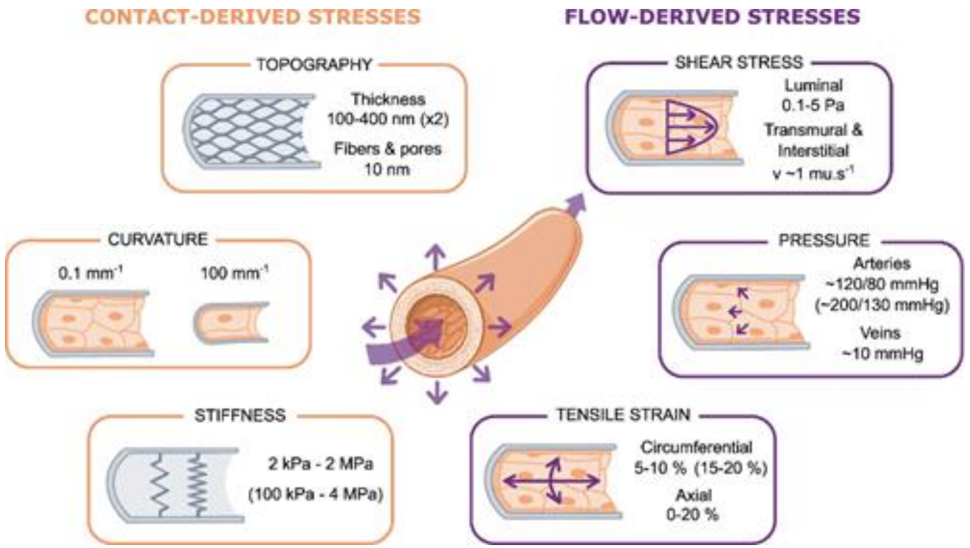



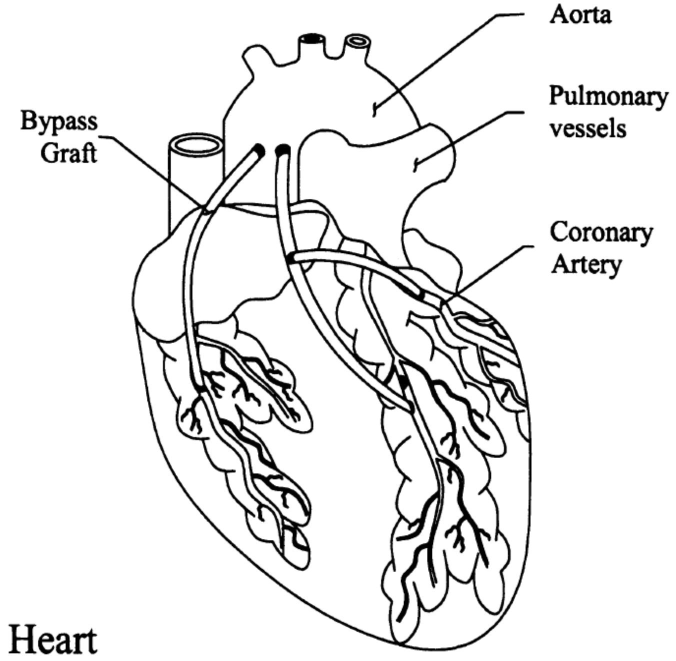

**Image reference:**
https://commons.wikimedia.org/wiki/File:Art%C3%A9re_membrane.png

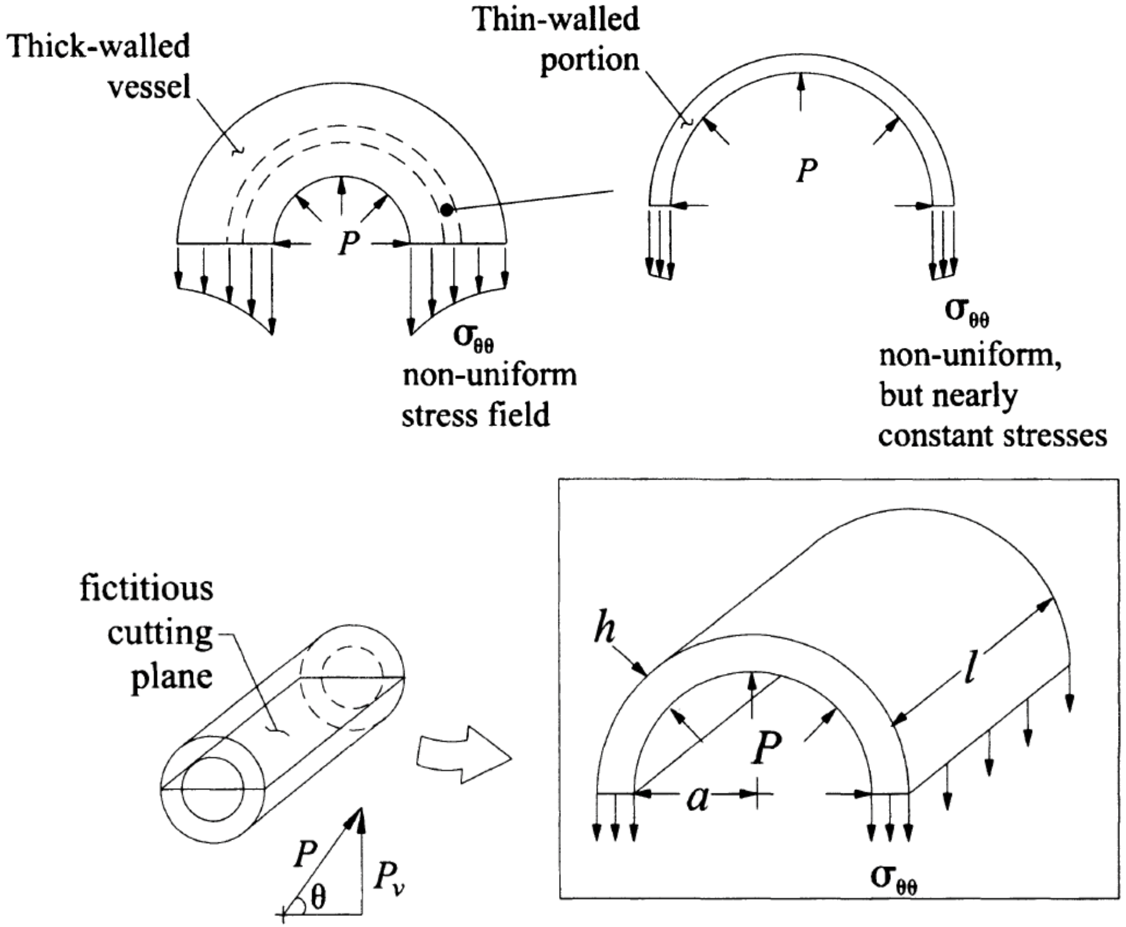
**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

**Image reference:**
Dessalles, C.A., Leclech, C., Castagnino, A. *et al.* Integration of substrate- and flow-derived stresses in endothelial cell mechanobiology. *Commun Biol* **4,** 764 (2021). https://doi.org/10.1038/s42003-021-02285-w

# Biological application problem:
# Pressurization and Extension of a Thin-Walled Tube

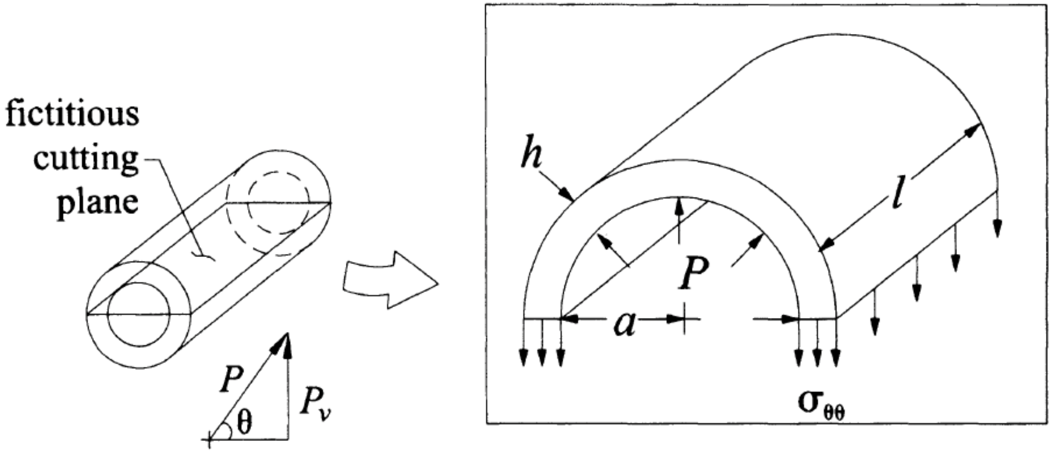
**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)



# Circumferential (Cauchy) stress in a thin-walled pressurized cylinder

**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)




$$\sum F_v = 0 \rightarrow \int P_v dA - 2\int \sigma_{\theta\theta} dA = 0.$$

$$\iint P \sin\theta a d\theta dz - 2\iint \sigma_{\theta\theta} dr dz = 0.$$

$$Pa\int_0^l \int_0^\pi \sin\theta d\theta dz = 2\sigma_{\theta\theta}\int_0^l \int_0^{a+h} dr dz,$$

$$Pa\int_0^l 2dz = 2\sigma_{\theta\theta}\int_0^l hdz \rightarrow P(2a)(l) = 2\sigma_{\theta\theta}(h)(l).$$

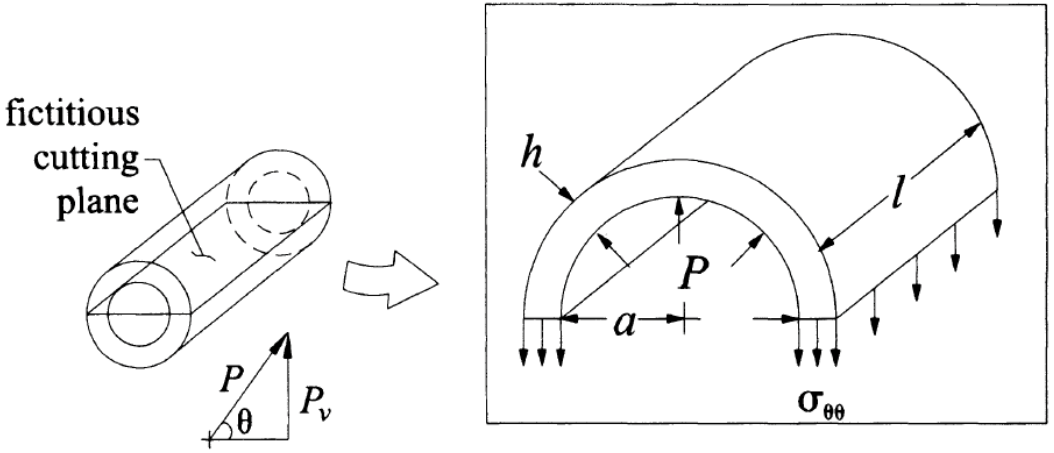
$$\sigma_{\theta\theta} = \frac{Pa}{h},$$

where

- **P** is the uniform internal pressure
- **a** is the inner radius of the cylinder in the pressurized configuration, and
- **h** is the thickness of the wall of the pressurized (i.e., deformed) cylinder.
- **a** and **h** are values in the pressurized configuration

# Radial stress in a thin-walled pressurized cylinder

**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)




- $\sigma_{rr}$ = -P at the inner surface
- $\sigma_{rr}$ = 0 at the outer surface

Assumption of thinness (a/h >> 1):

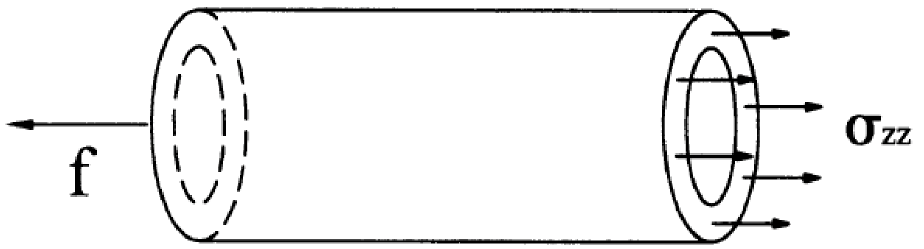
- $\sigma_{rr}$ can vary from -P to 0
  - linearly or nonlinearly with radial location $r \in [a, a + h]$
  - Therefore, mean value can be estimated
  - $\sigma_{rr}$ can be neglected because $\sigma_{\vartheta\vartheta} \gg \sigma_{rr}$

$$\sigma_{rr}\Big)_{mean} = \frac{-P + 0}{2} \rightarrow \sigma_{rr} \cong \frac{-P}{2},$$

# Axial stress in an inflated thin-walled pressurized cylinder

**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Blood vessels retract considerably when cut (e.g., the murine carotid artery will shorten 80 % when cut), which reveals that significant axial loads are present *in vivo.*




$$\sigma_{zz}\left[2\pi\left((a+h)^2-a^2\right)\left(\frac{1}{2}\right)\right]=f \rightarrow \sigma_{zz}\cong\frac{f}{2\pi ah}$$

$$-f+\int_0^{2\pi}\int_a^{a+h}\sigma_{zz}r\,dr\,d\theta=0.$$

Assumption of thinness (a/h >> 1):

- The deformed cross-sectional area over which **f** acts is $A \approx 2\pi ah$

# Axial stress in thin-walled pressurized cylinder: Ends of the cylinder are closed

**Reference:** Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

- An internal pressure acting on a closed-ended tube.
- The cut exposes the stress of interest but does not depressurize the tube because it is fictitious.
- Internal pressure and the axial stress are assumed to be uniformly distributed


$$\iint \sigma_{zz} r d\theta dr - \iint P r d\theta dr = 0.$$

$$\sigma_{zz} \int_0^{2\pi} \int_a^{a+h} r dr d\theta = P \int_0^{2\pi} \int_0^a r dr d\theta,$$

$$\sigma_{zz} \int_0^{2\pi} \frac{1}{2}\left[(a+h)^2 - a^2\right] d\theta = P \int_0^{2\pi} \frac{1}{2} a^2 d\theta,$$

$$\sigma_{zz} \left\{\frac{1}{2}\left[(a+h)^2 - a^2\right]\right\}(2\pi) = P\left(\frac{1}{2}a^2\right)(2\pi),$$

$$\sigma_{zz}\left(2ah + h^2\right) = Pa^2.$$

## Assumption of thinness (a/h >> 1):

- The term **$h^2$** is small compared to **$2ah$** **and therefore, can be neglected**

$$\sigma_{zz} = \frac{Pa}{2h},$$

Ends of the cylinder are closed + Axial load

$$\sigma_{zz} = \frac{Pa}{2h} + \frac{f}{2\pi a h}.$$