BIOMECHANICS - I

KVB ©

02/04/2026

①

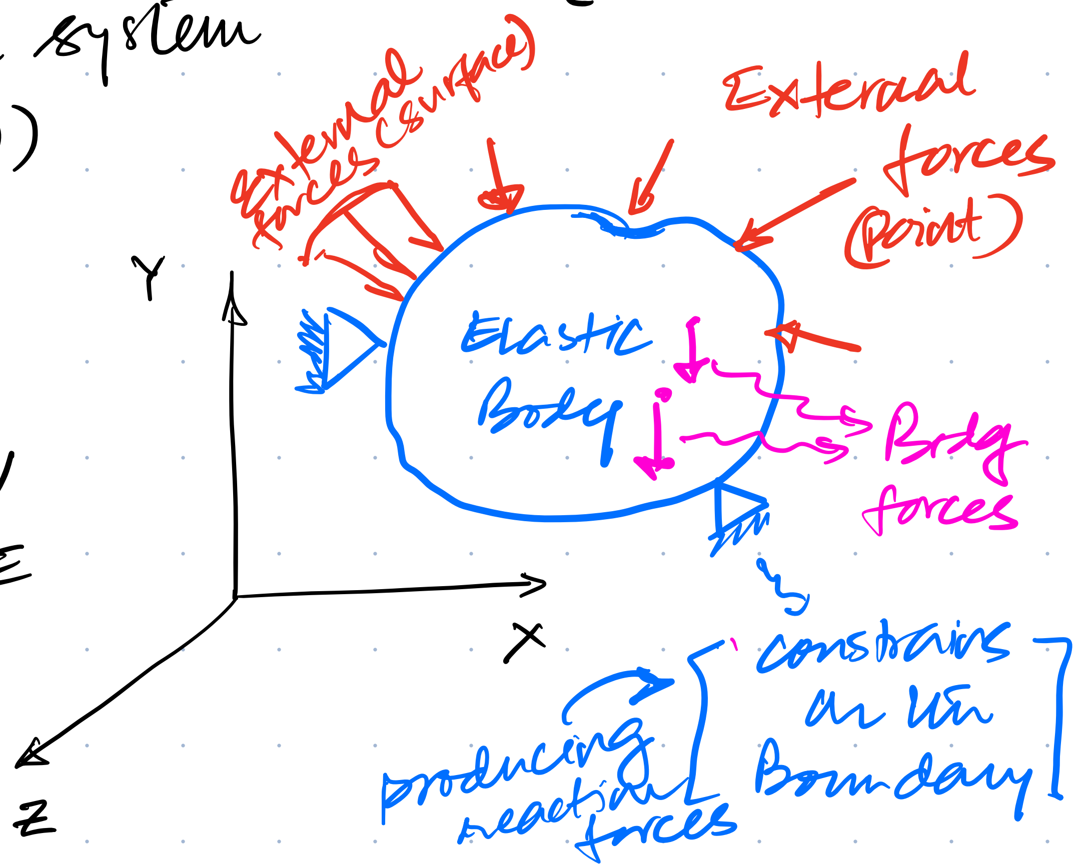
KEY IDEA-1

Generalized concept of stress:

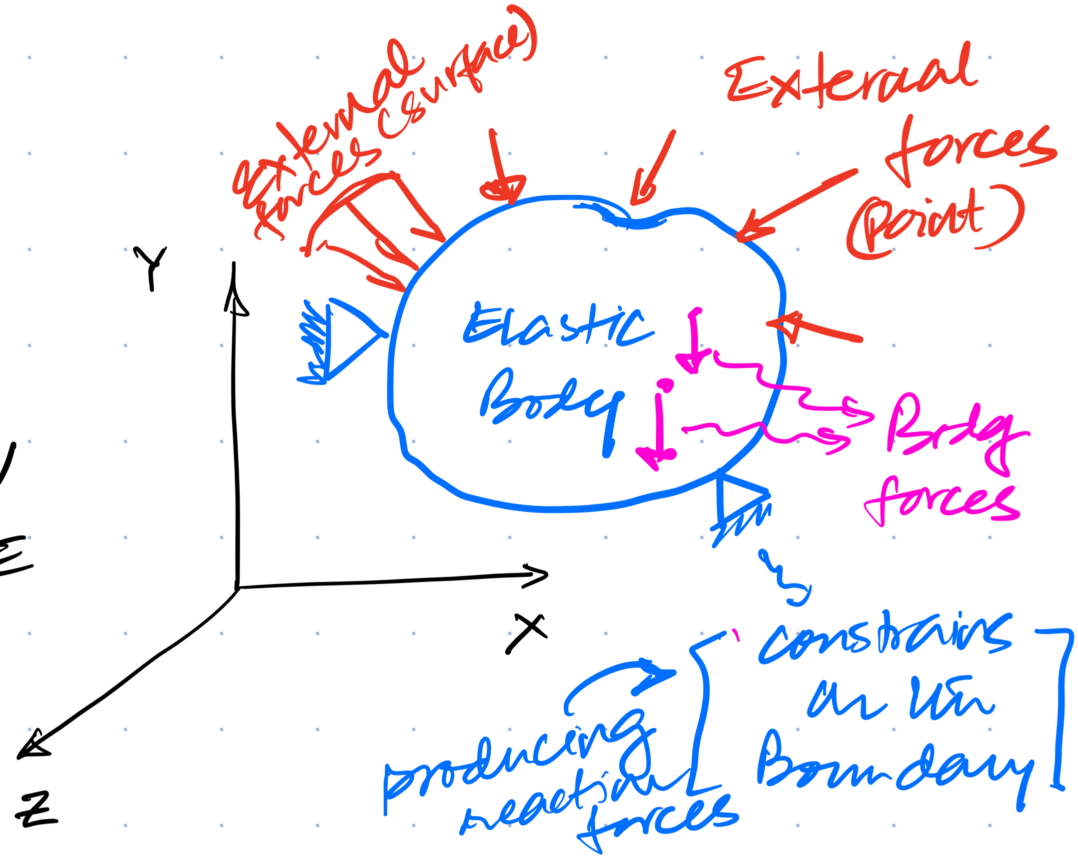
Consider a Reference axis:

- Cartesian coordinate system (X, Y, Z)
- Cylindrical cordinate system (r, θ, z)
- Spherical coordinate system (r, θ, ϕ)

CARTESIAN COORDINATE SYSTEM



CARTESIAN COORDINATE SYSTEM



- Stress → generated inside the body
- Strain → Is caused due to the stress

All this leads to deformation.

STRESS

It is a mathematical construct

Force acting normal to the an area divided by the value of the area (F/A)

Euler (1757)

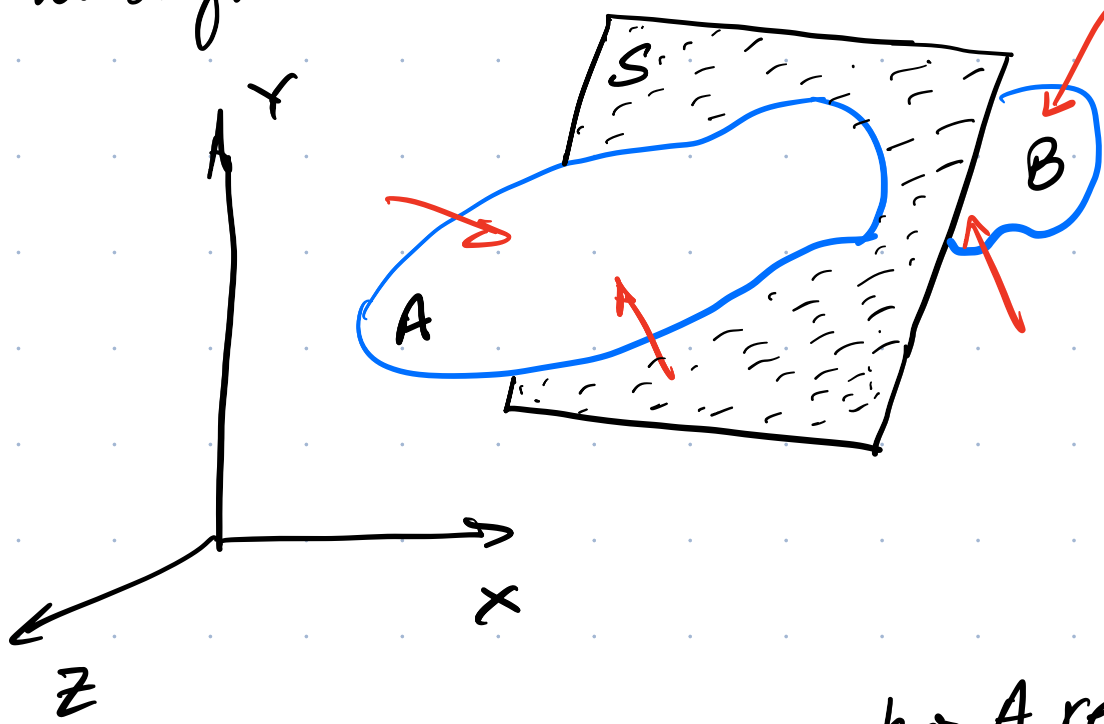
Force acting over an oriented area at any point in a body.

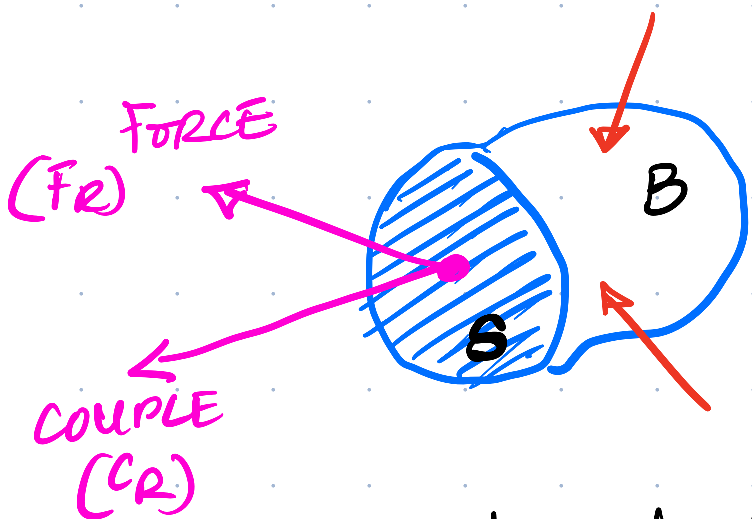
Cauchy (1823–1827)

Depends on the choice of the coordinate system.

There can be many forms of stresses acting at a point

↳ Consider a body in equilibrium with some hypothetical plane S passing through



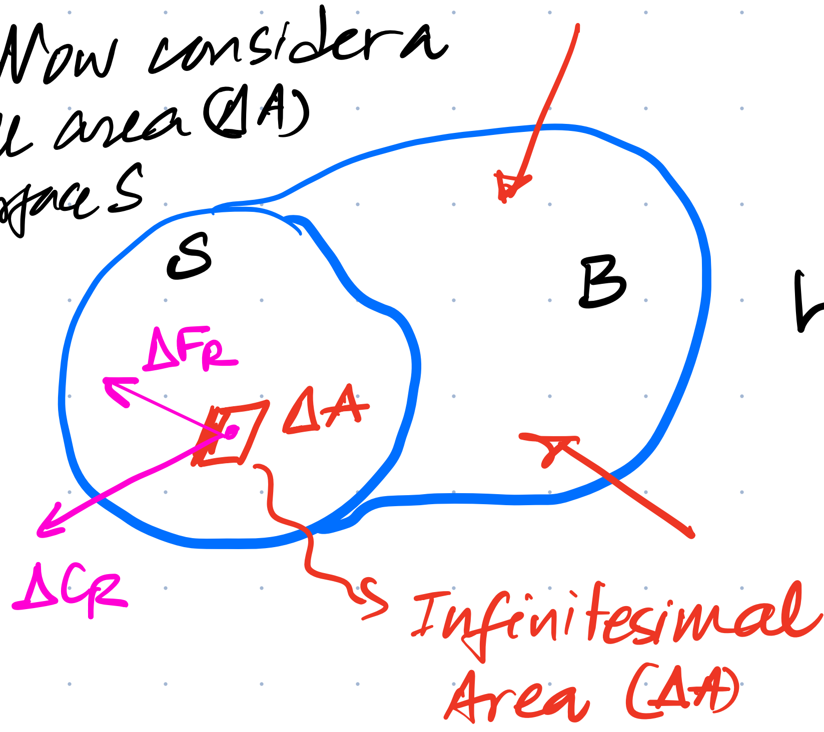


↳ A resultant force (F_R) and moment (C_R) are transmitted across the surface (S)

↳ We know this from rigid body mechanics of force and moment balance.

↳ Force (F_R) is distributed across the surface (S)

↳ Now consider a small area (ΔA) on surface S



↳ (ΔF_R) is the Resultant force and

(ΔC_R) is the Resultant moment transmitted across (ΔA).

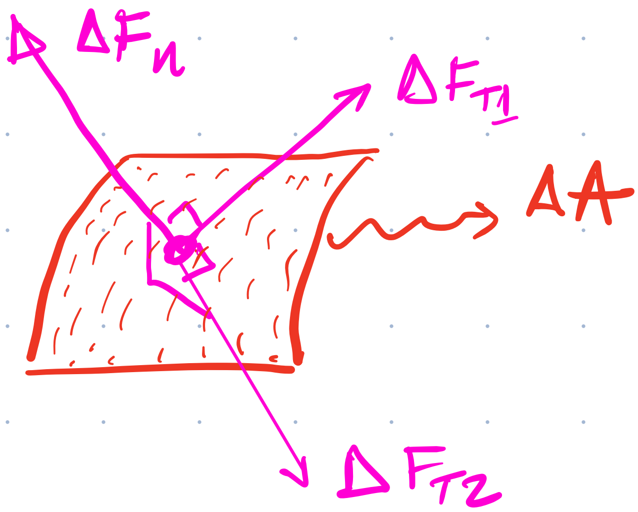
↳ Using concept of Continuum:

↳ We will let $(\Delta A) \longrightarrow 0$ and

↳ We will decompose ΔF_R into three orthogonal components

↳ To generate Force intensity.

↳ On a point over surface S.



ΔF_R

ΔF_n — Normal to the surface of ΔA.

ΔF_T, ΔF_z — Tangential to the surface of ΔA.

We can now formally define stress:

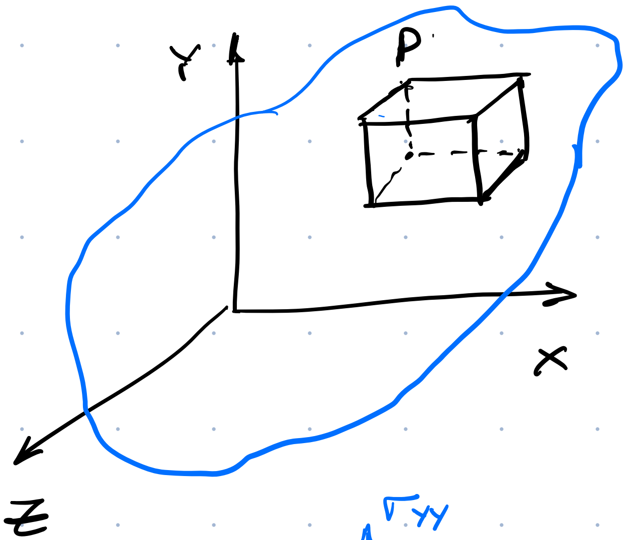
$$\tau_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} = \frac{dF_n}{dA}$$ Normal Stress

$$\tau_{T1} = \lim_{\Delta A \to 0} \frac{\Delta F_{T1}}{\Delta A} = \frac{dF_{T1}}{dA}$$ Shear Stress

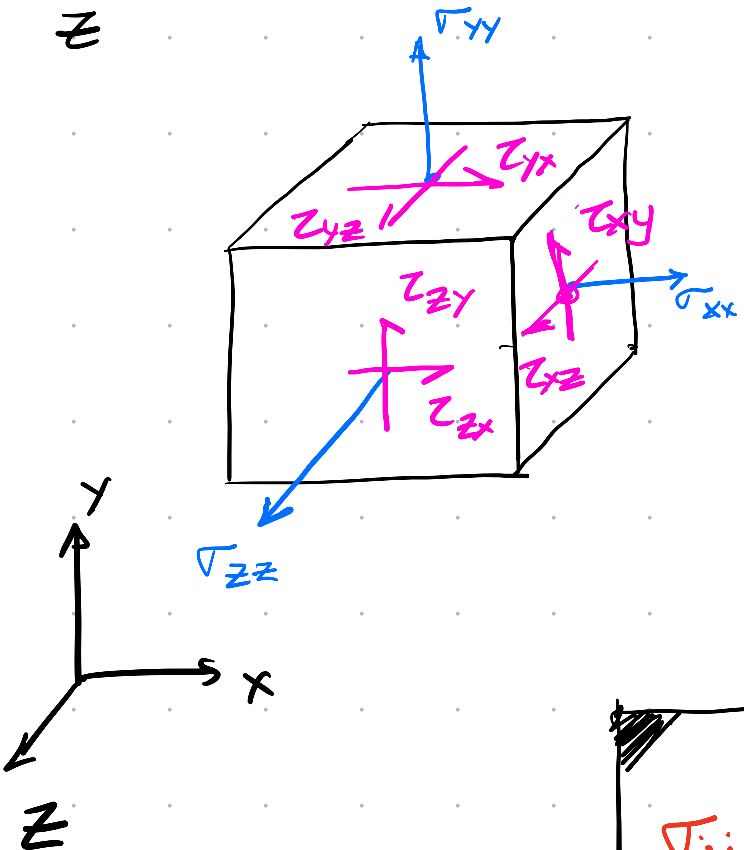
$$\tau_{T2} = \lim_{\Delta A \to 0} \frac{\Delta F_{T2}}{\Delta A} = \frac{dF_{T2}}{dA}$$ Shear Stress

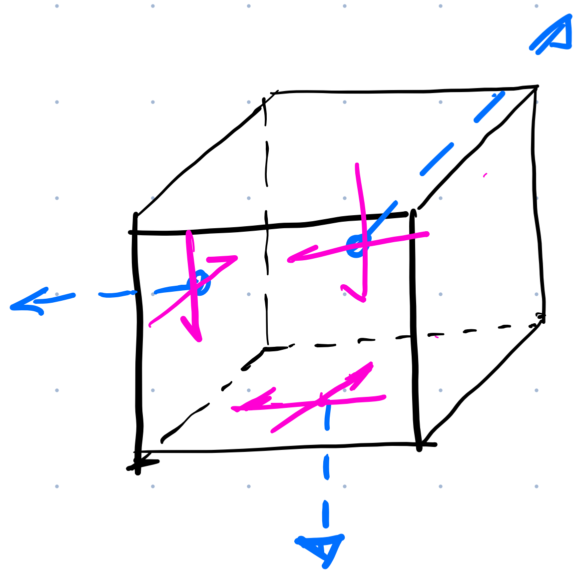
KEY IDEA - 2

Stress components in Cartesian co-ordinate system (X, Y, Z):



→ consider a point (P) with interface parallel to reference planes





$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

plane normal to the axis

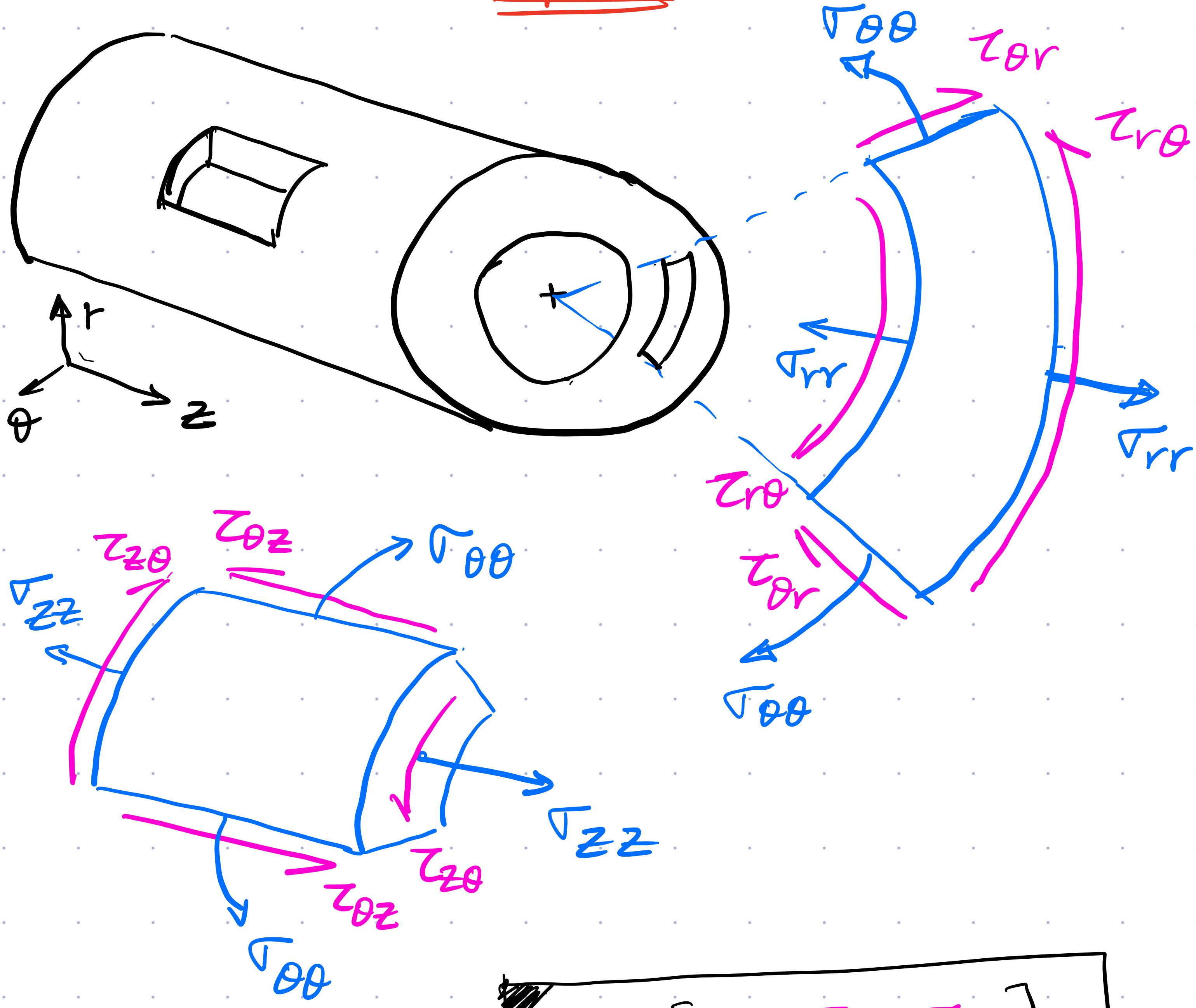
Direction of reference axis

STRESS TENSOR

2^{nd} order tensor

KEY IDEA - 3

Stress components in cylindrical coordinate system (r, θ, z):



$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix}$$

plane normal to the axis (i)

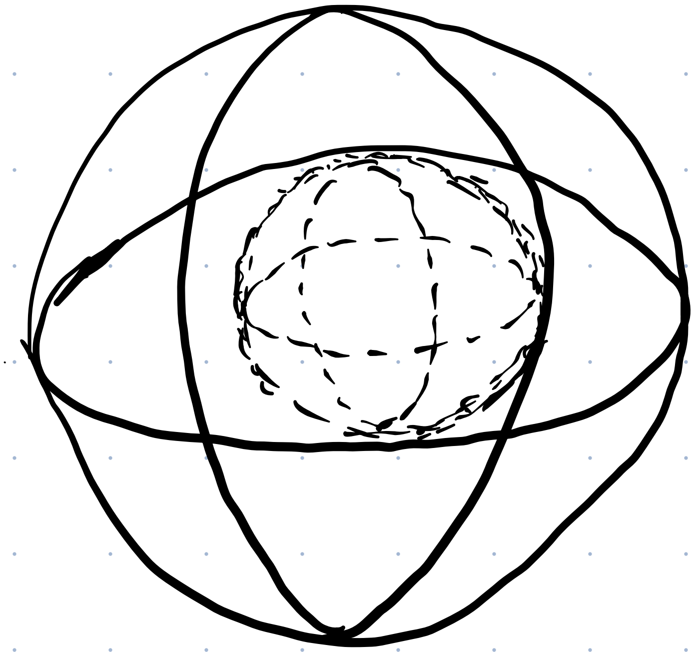
Direction of reference axis (j)

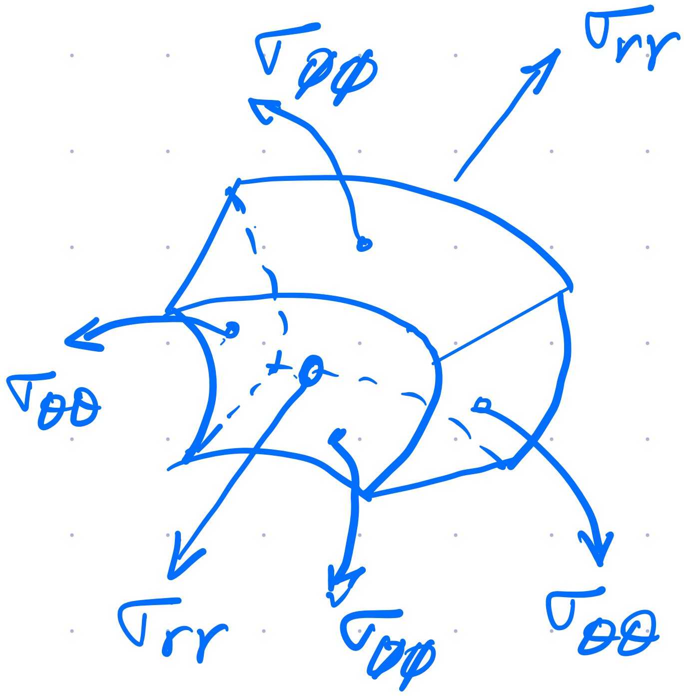
STRESS TENSOR | 2nd order tensor

KEY IDEA - 3

Stress components in spherical coordinate system (r, θ, ϕ):

Sphere !





$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

plane normal to the axis

Direction of reference axis

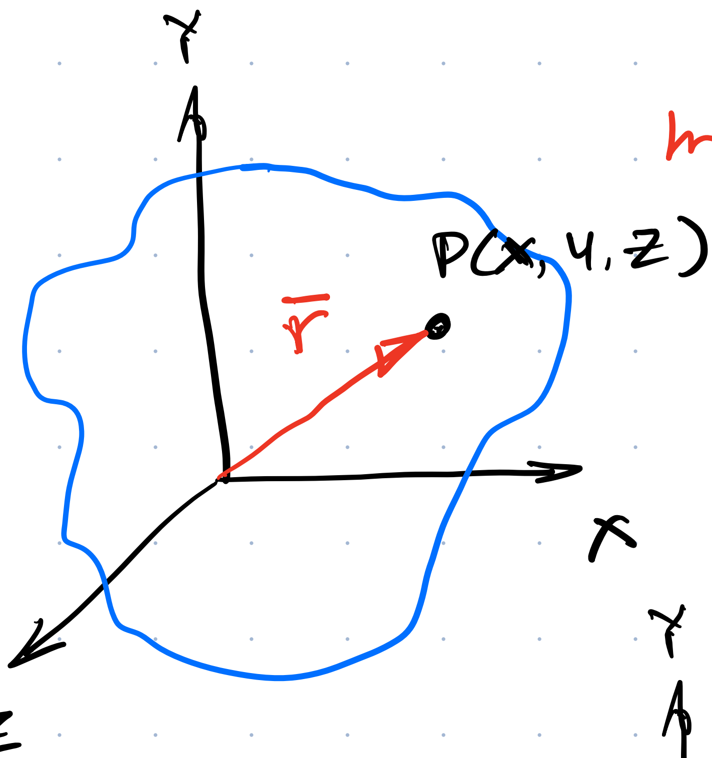
STRESS TENSOR

2nd order tensor

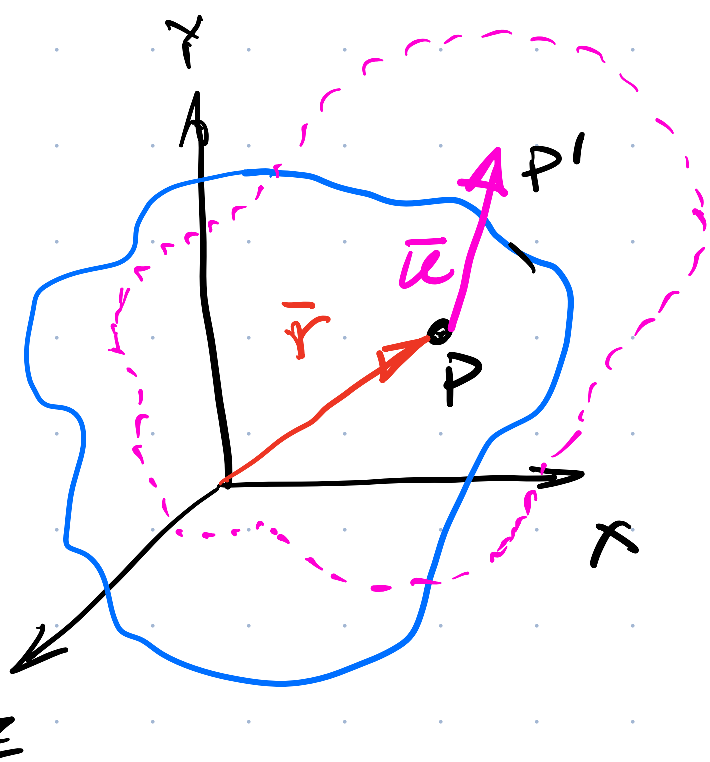
KEY IDEA - 4

Generalized concept of strain:

STRAIN: Quantitative measure of deformation of a body.



- Position vector $(\bar{r})$ in a body
- Undeformed elastic body



- Point P(x, y, z) moves to a new point P'.
- P' is in the deformed geometry
- Displacement field is defined by $\bar{u}(x, y, z)$

DISPLACEMENT FIELD

- $\bar{u}(x, y, z)$
- Mathematical description of how a body moves and deforms, point-by-point.
  - change in geometry

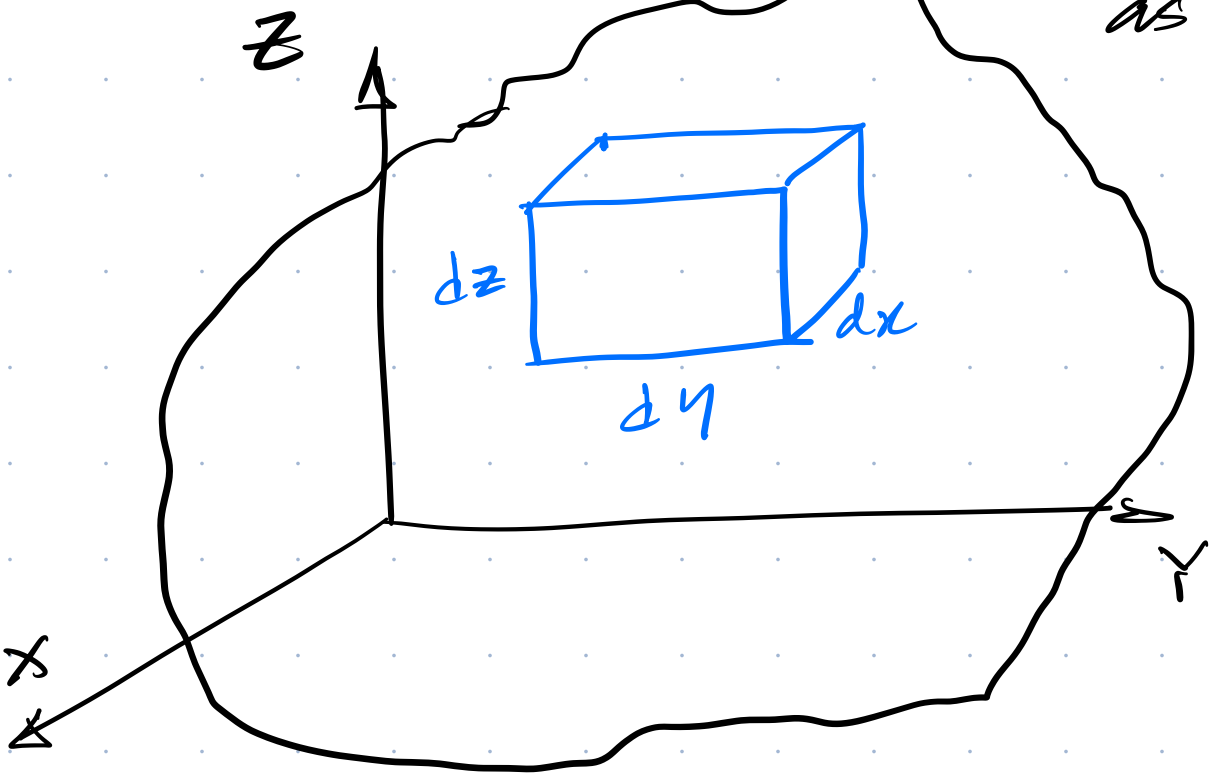
- We need to somehow relate the change in geometry to stress distributions.
  - Recall: Stress is expressed as a stress tensor.

- [Relative movement of adjacent points lying along 3 orthogonal axes at a point]

Strain components

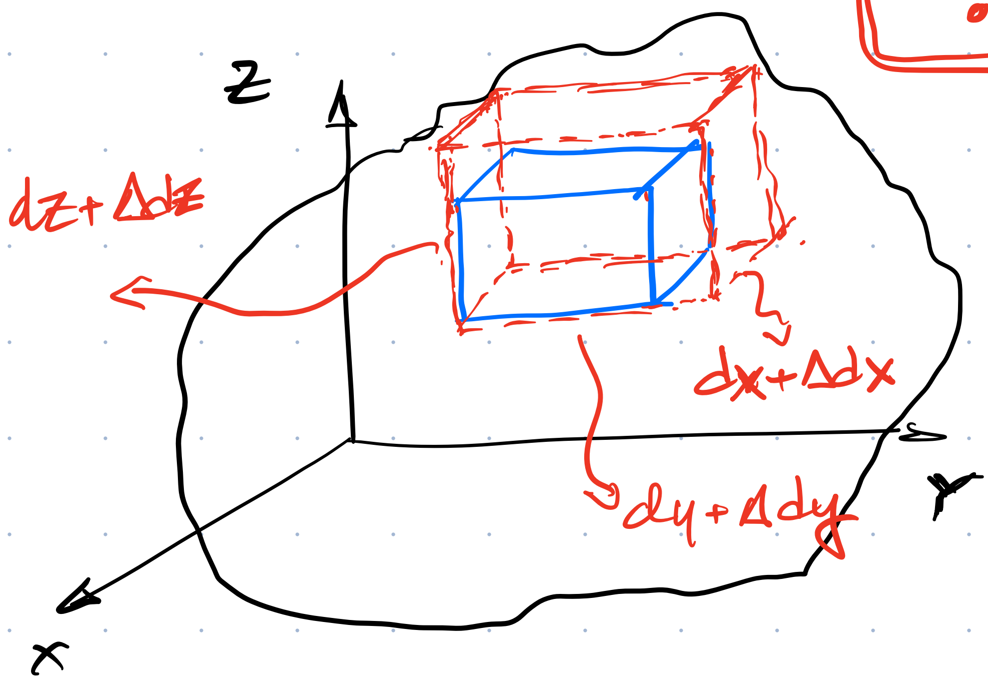
consider a parallelopied as shown:

↳ It is an infinitesimal entity & an undeformed body.



Case(i): Let the parallelopied only develop normal stresses on its faces

DILATION
- Volume changes
- Shape does not change



Strains:

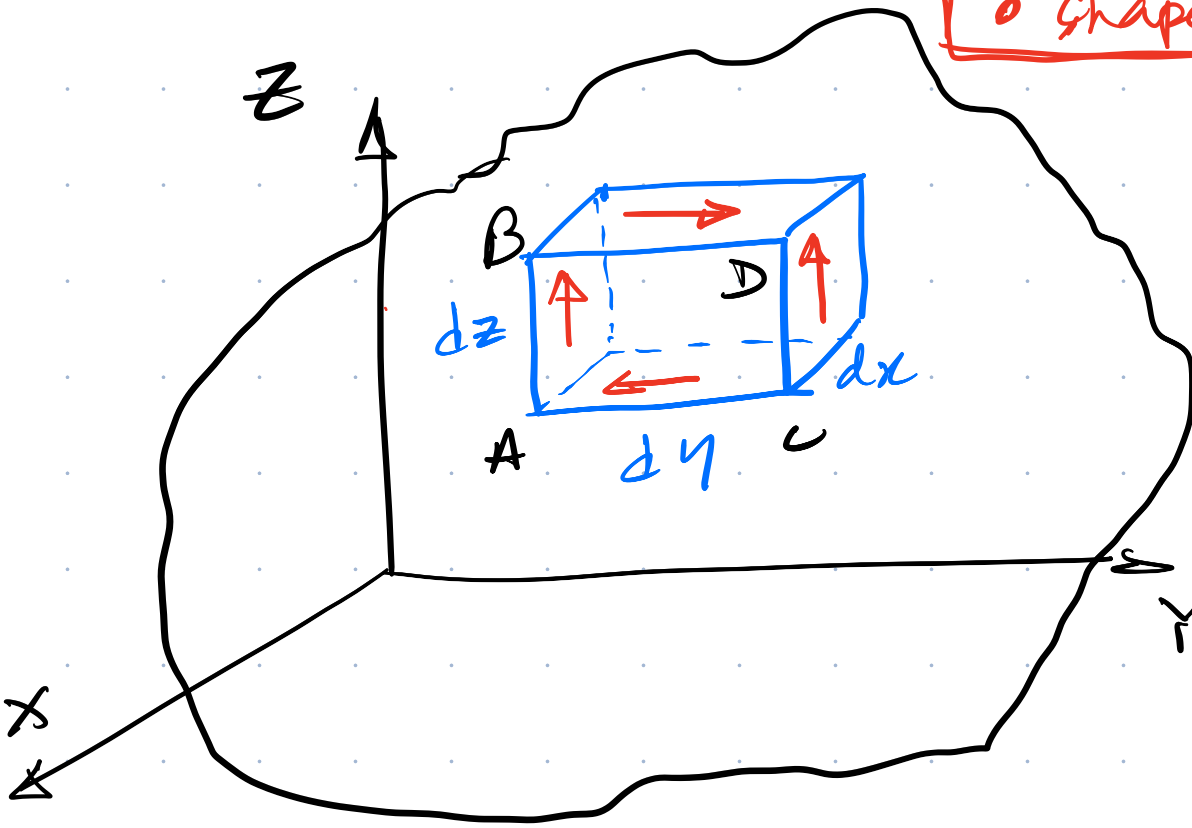
$$\epsilon_{xx} = \frac{\Delta dx}{dx}$$

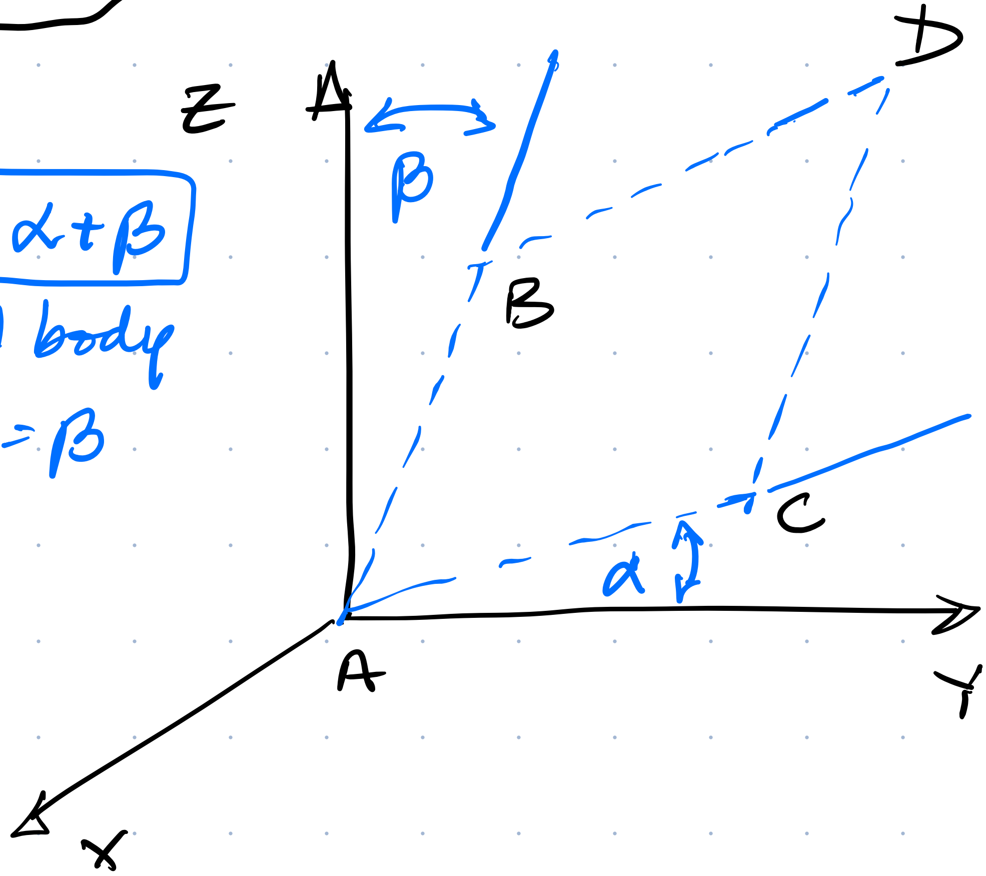
$$\epsilon_{yy} = \frac{\Delta dy}{dy}$$

$$\epsilon_{zz} = \frac{\Delta dz}{dz}$$

Case (ii) Let the parallelopiped only develop shear stress on its faces.

- Volume does not change
- Shape changes





$$\boxed{\gamma_{yz} = \gamma_{zy} = \alpha + \beta}$$

Assume: Rigid body rotation $\Rightarrow \alpha = \beta$

Strains:

$$\epsilon_{zy} = \epsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\gamma_{zy}$$

COMBINATION OF CASE(i) & CASE(ii) can lead to a tensor, similar to a stress tensor!

Note: We are dealing with Cartesian co-ordinates in this discussion

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Off-diagonal terms:

↳ 6 symmetric terms

↳ change in shape

↳ No change in volume

Diagonal terms:

↳ 3 Normal strains

↳ Dilation effect

↳ Change in volume

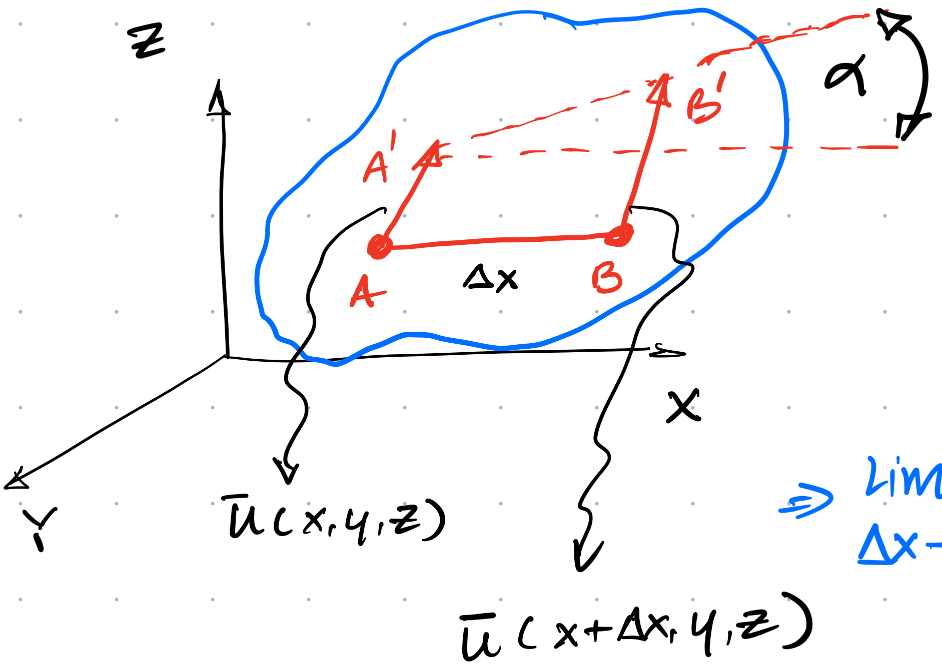
↳ No change in shape

STRAIN-DISPLACEMENT RELATIONS:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

→ Express each term in the matrix in terms of displacement field.

Assume: small deformations



$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{A'B' - \Delta x}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \left[\frac{u_x(x+\Delta x, y, z) - u_x(x, y, z)}{\Delta x} \right]$$

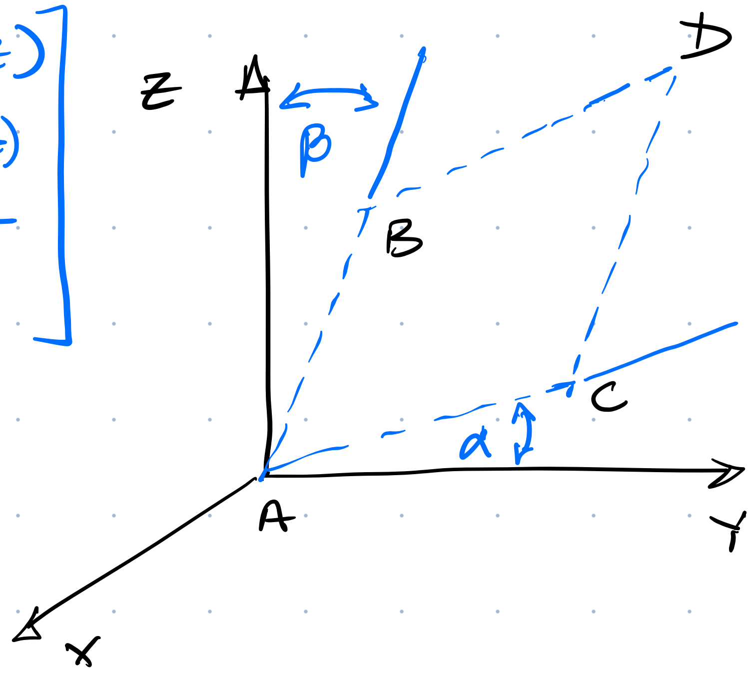
$$\Rightarrow \boxed{\varepsilon_{xx} = \frac{\partial u_x}{\partial x}}$$

Similarly,

$$\boxed{\varepsilon_{yy} = \frac{\partial u_y}{\partial y}}$$

$$\boxed{\varepsilon_{zz} = \frac{\partial u_z}{\partial z}}$$

$$\alpha = \lim_{\Delta x \to 0} \left[\frac{u_y(x+\Delta x, y, z) - u_y(x, y, z)}{\Delta x} \right]$$



$$\alpha = \frac{\partial u_y}{\partial x}$$

Similarly

$$\beta = \frac{\partial u_x}{\partial y}$$

Recall: $\epsilon_{yz} = \epsilon_{zy}$

$$= \frac{1}{2}\gamma_{yz}$$

$$= \frac{1}{2}\gamma_{zy}$$

$$\gamma_{yz} + \gamma_{zy} = \alpha + \beta$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\therefore \boxed{\begin{aligned} \epsilon_{xy} &= \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \epsilon_{xz} &= \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \epsilon_{yz} &= \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned}}$$

Let's now fill out all terms in the matrix

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Cartesian co-ordinates

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Cartesian co-ordinates

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Cartesian co-ordinates

$$\epsilon_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix}$$

Cylindrical co-ordinates

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{bmatrix}$$

Cylindrical co-ordinates

$$\epsilon_{ij} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \left(\dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\epsilon_{\theta z} = \epsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\epsilon_{rz} = \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

Spherical co-ordinates

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{r\phi} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta\phi} \\ \epsilon_{\phi r} & \epsilon_{\phi\theta} & \epsilon_{\phi\phi} \end{bmatrix}$$

Spherical co-ordinates

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\epsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\epsilon_{\theta\phi} = \epsilon_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r}\cot\phi\right)$$

$$\epsilon_{r\phi} = \epsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)$$