MAE 3128
Biomechanics-I

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Topics covered today:

1. Constitutive relationships
2. In-class problems

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& Applied Science
THE GEORGE WASHINGTON UNIVERSITY

Spring 2026

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Biomechanics from a Continuum Mechanics perspective

Reference: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

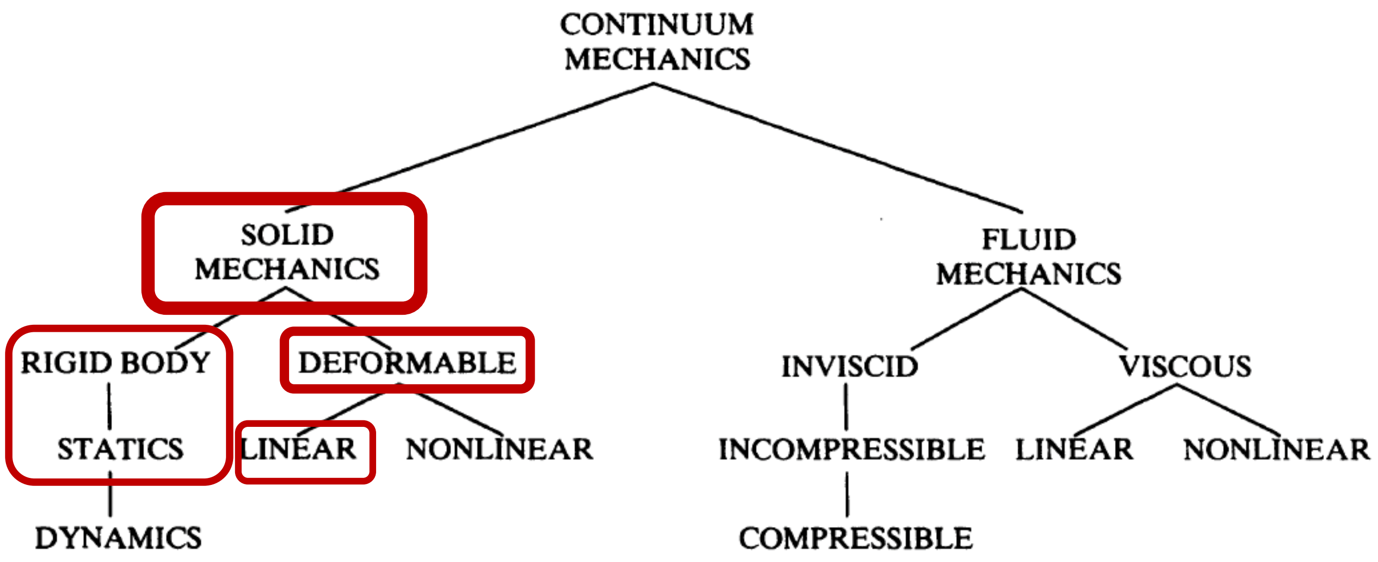

FIGURE 1.4 Flowchart of traditional divisions of study within continuum mechanics. Note that solid mechanics and fluid mechanics focus primarily on solidlike and fluidlike behaviors, not materials in their solid versus fluid/gaseous phases. Note, too, that linear and nonlinear refer to material behaviors, not the governing differential equations of motion. As we shall see in Chap. 11, many materials simultaneously exhibit solidlike (e.g., elastic) and fluidlike (e.g., viscous) behaviors, which gives rise to the study of viscoelasticity and the theory of mixtures, both of which are important areas within continuum biomechanics.

Constitutive Behavior of Materials

References: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Mathematical relations that describe the response of a material to applied loads under conditions of interest are called **constitutive relations**

Delineating whether the behavior is **solid-like** or **fluid-like**:

- **Solid-like materials** can sustain shear stress in equilibrium.
- **Fluid-like materials** cannot sustain shear stress and will flow when shear is applied.
- **Material behavior** under the condition of interest
    - Timescales and Length scales
    - Temperature
    - pH (acidity or alkalinity)
    - Electric and magnetics field etc

Constitutive Behavior of Materials

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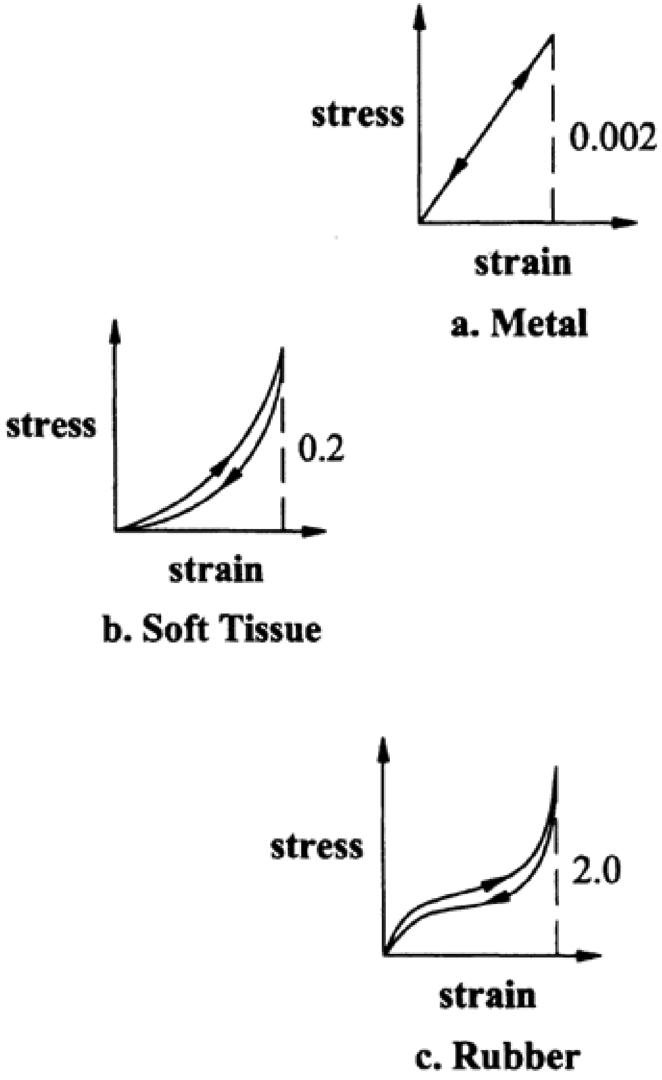
Understanding the material's response i.e., linear or nonlinear under applied load:

- **Linear response:** Stress and strain are proportional; typical of metals and bone under small, reversible strains.
- **Nonlinear response:** Stress and strain are not proportional; common in elastomers and soft tissues under large, reversible strains.
    - Nonlinear behavior is more complex to quantify than linear behavior.
- **Examples for discussion**
    - **Elastomers and soft tissues** differ from traditional engineering materials because of their long-chain polymeric structures.
    - Their mechanical behavior depends on changes in molecular conformations (degrees of order/disorder).
        - Such behavior is governed by **entropic mechanisms**, unlike the **energetic mechanisms** in metals that depend on atomic lattice structure.
    - **In biological tissues, elastin and collagen** are key biopolymers controlling mechanical response through entropic effects, making quantification challenging.

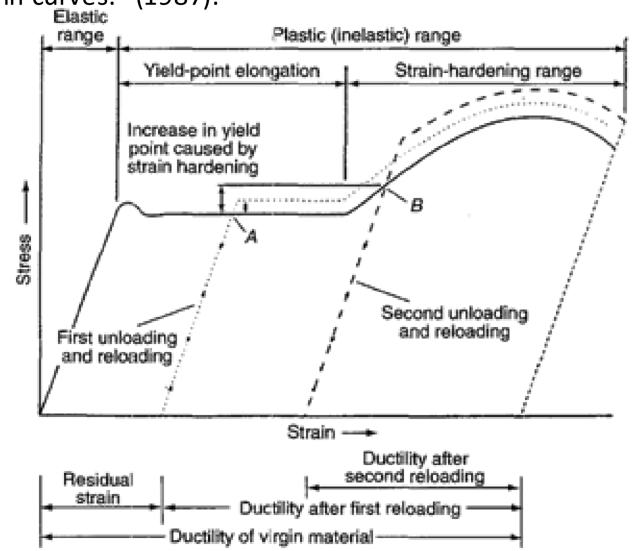


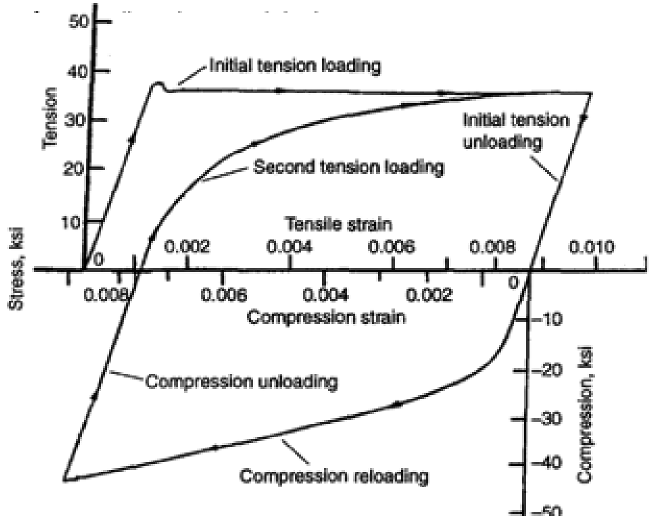
Constitutive Behavior of Material

References:
Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)
Boyer, Howard E.. “Atlas of stress-strain curves.” (1987).

Understanding material behavior and composition:

- **Elastic materials:** Materials that return instantly to its original size and shape when the load is removed, implying an instantaneous response to applied stress.
  - Metals typically show **elastic behavior** under small strains.
- **Pseudoelastic materials:** Materials with slight hysteresis (energy loss) due to internal friction within structural proteins and ground substance
  - **Soft tissues and rubber** exhibit **nearly elastic (pseudoelastic)** behavior under normal conditions.
- **Homogenous materials:** A material is called **homogeneous** if its behavior is uniform throughout, independent of the sample’s location within the body or structure.
  - **Soft tissues**, though composite in nature (**elastin, collagen, proteoglycans,** etc.), can sometimes be treated as approximately homogeneous for modeling purposes. **E.g., skin, lung, myocardium, bone, or brain tissue**
  - **Non-homogeneous materials** include fiber-reinforced composites (e.g., steel-reinforced concrete) where components have distinct properties.
  - Layered structures (**blood vessel walls or cortical vs. cancellous bone**)—**heterogeneity must be explicitly considered**.





Constitutive Behavior of Materials

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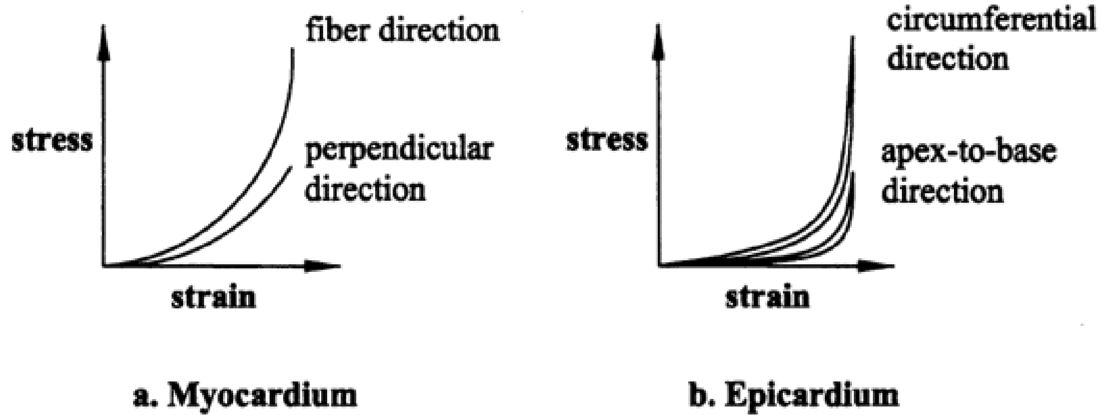
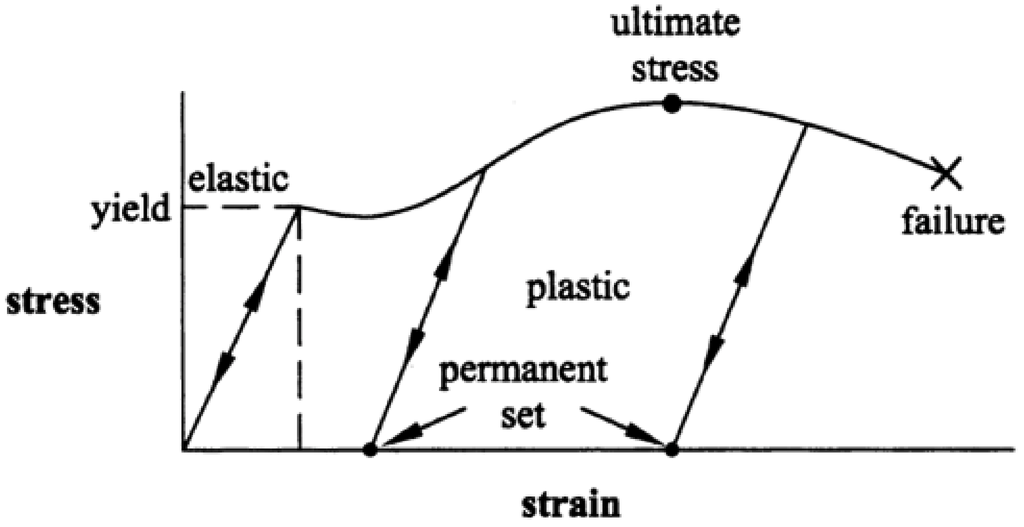
Understanding material behavior and microstructure:

Isotropy: A material's behavior is **isotropic** if its response is the same regardless of orientation within the structure.

- **Metals** typically show isotropy under small strains, while **rubber** exhibits isotropy under large strains.
- **Tendons** (due to aligned collagen fibers) and **plant stalks** do not exhibit isotropic behavior; they are **anisotropic**.
- Most biological tissues display **anisotropy**, meaning their response varies with direction, making quantification more complex.

Terms like **linearity, elasticity, homogeneity**, and **isotropy** are merely **descriptors of behavior**, not absolute properties.
No real material is perfectly linear, elastic, homogeneous, or isotropic.
Material response always depends on specific **conditions of interest**.





Hookean LEHI Behavior

Nonlinearity, inelasticity, anisotropy, and heterogeneity are common characteristics of soft tissues.

However, we simply note that we will focus a class of material behaviors that we refer to as LEHI:

- **Linear:** linear stress-strain behavior and linearized kinematics
- **Elastic:** no dissipation and the loading/unloading curve coincide
- **Homogeneous:** same material behavior everywhere in the material/body
- **Isotropic:** same material response in all directions at a point

References: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - v\left(\sigma_{yy} + \sigma_{zz}\right)\right] + \beta\Delta T, \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy},$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - v\left(\sigma_{xx} + \sigma_{zz}\right)\right] + \beta\Delta T, \qquad \varepsilon_{xz} = \frac{1}{2G}\sigma_{xz},$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - v\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \beta\Delta T, \qquad \varepsilon_{yz} = \frac{1}{2G}\sigma_{yz},$$

where

- **T** is the temperature
- **E** is called Young's modulus (after T. Young, a physician interested in biomechanics in 1808)
  - **E** is a measure of the extensional stiffness (i.e., change of stress with respect to strain)
- **v** is called Poisson's ratio; it describes a coupling between strains in orthogonal directions
- **G** is called the shear modulus; it provides a measure of the resistance to shear. **G = E/2(1 + v)** for LEHI
- **β** is a coefficient of thermal expansion; material expansion/contraction due to changes in temperature from some reference temperature T_o; that is, $\mathbf{\Delta T = T - T_o}$

Hooke's Law for Transverse Isotropy

"Hooke's law" as stated in LEHI holds if the material behavior is isotropic (i.e., the behavior is independent of the direction in which the force is applied at a point within the material).

- When a material has a different behavior in one direction compared to all directions in an orthogonal plane
  - the behavior is said to be **transversely isotropic (i.e., isotropic in a plane transverse to a preferred or different direction).**
- If the transversely isotropic behavior is otherwise linear, elastic, and homogeneous under small strains, it is describable via a transversely isotropic Hooke's law.
  - Wood, fiberglass, and other man-made composites as well as tendons, ligaments, skin, bone and most other biological tissues exhibit an anisotropy

References: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - v\sigma_{yy}) - \frac{v'}{E'}\sigma_{zz}, \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy},$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - v\sigma_{xx}) - \frac{v'}{E'}\sigma_{zz}, \qquad \varepsilon_{xz} = \frac{1}{2G'}\sigma_{xz},$$

$$\varepsilon_{zz} = \frac{1}{E'}\sigma_{zz} - \frac{v'}{E}(\sigma_{xx} + \sigma_{yy}), \qquad \varepsilon_{yz} = \frac{1}{2G'}\sigma_{yz},$$

where

- **G = E/2(1 + v)**
- **Z-direction** (arbitrarily) taken to be the **preferred direction**
- **Transversely isotropic behavior is described by five independent parameters**
  - Two Young's moduli **E** and **E'**,
  - Two Poisson's ratios **v** and **v'**, and
  - a shear modulus **G'**, where **G** is, again, related to E and v and thus is not independent).

Hooke's Law for Orthotropy

References: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Orthotropic response is one that differs in three orthogonal directions.

- **Example-1: Artery**
  - Its behavior differs in the axial (due to axially oriented adventitial collagen),
  - circumferential (due to the nearly circumferentially oriented smooth muscle in the media), and radial directions.
- **Example-2: Bone**
  - Tends to exhibit an orthotropic response
  - Nearly transversely isotropic in some cases.

When the response is otherwise linear, elastic, and homogeneous under small strains, Hooke's law can be generalized to account for the orthotropy.

$$\varepsilon_{xx} = \frac{1}{E_1}\sigma_{xx} - \frac{\nu_{21}}{E_2}\sigma_{yy} - \frac{\nu_{31}}{E_3}\sigma_{zz}, \qquad \varepsilon_{xy} = \frac{1}{2G_{12}}\sigma_{xy},$$

$$\varepsilon_{yy} = \frac{1}{E_2}\sigma_{yy} - \frac{\nu_{12}}{E_1}\sigma_{xx} - \frac{\nu_{32}}{E_3}\sigma_{zz}, \qquad \varepsilon_{xz} = \frac{1}{2G_{13}}\sigma_{xz},$$

$$\varepsilon_{zz} = \frac{1}{E_3}\sigma_{zz} - \frac{\nu_{13}}{E_1}\sigma_{xx} - \frac{\nu_{23}}{E_2}\sigma_{yy}, \qquad \varepsilon_{yz} = \frac{1}{2G_{23}}\sigma_{yz},$$

Nine independent material parameters:

- Three Young's moduli $\mathbf{E_1}$, $\mathbf{E_2}$, and $\mathbf{E_3}$,
- Three shear moduli $\mathbf{G_{12}}$, $\mathbf{G_{13}}$, and $\mathbf{G_{23}}$, and
- Six Poisson's ratios ν_{12}, ν_{21}, ν_{13}, ν_{31}, ν_{23}, and ν_{32},
  - Only three of which are independent as shown by the following relations:

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \qquad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \qquad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}.$$

Hookean LEHI in Cylindrical and Transformed Coordinate Systems

References: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

$$\varepsilon_{rr} = \frac{1}{E}[\sigma_{rr} - v(\sigma_{\theta\theta} + \sigma_{zz})] + \beta\Delta T, \qquad \varepsilon_{r\theta} = \frac{1}{2G}\sigma_{r\theta},$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}[\sigma_{\theta\theta} - v(\sigma_{rr} + \sigma_{zz})] + \beta\Delta T, \qquad \varepsilon_{rz} = \frac{1}{2G}\sigma_{rz},$$

$$\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - v(\sigma_{rr} + \sigma_{\theta\theta})] + \beta\Delta T, \qquad \varepsilon_{\theta z} = \frac{1}{2G}\sigma_{\theta z}.$$

$$\varepsilon'_{xx} = \frac{1}{E}\left[\sigma'_{xx} - v\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \qquad \varepsilon'_{xy} = \frac{1}{2G}\sigma'_{xy},$$

$$\varepsilon'_{yy} = \frac{1}{E}\left[\sigma'_{yy} - v\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \qquad \varepsilon'_{xz} = \frac{1}{2G}\sigma'_{xz},$$

$$\varepsilon'_{zz} = \frac{1}{E}\left[\sigma'_{zz} - v\left(\sigma'_{xx} + \sigma'_{yy}\right)\right], \qquad \varepsilon'_{yz} = \frac{1}{2G}\sigma'_{yz}.$$

Plane Stress and Plane Strain in matrix form

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Plane Stress

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

Plane Strain

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$