From equation 7 & 8

$$\underline{\epsilon_{rr}} = \frac{\partial u_r}{\partial r} \qquad \underline{\epsilon_{\theta\theta}} = \frac{u_r}{r}$$

$$\boxed{\frac{d}{dr}(r\epsilon_{\theta\theta}) = \epsilon_{rr}} \rightarrow 11$$

compatibility equation

Aside:

$$\because \frac{d}{dr}\Big(\underbrace{\cancel{r}\frac{u_r}{\cancel{r}}}_{(r\epsilon_{\theta\theta})}\Big) = \frac{d}{dr}(u_r) = \epsilon_{rr}$$

From equations 5 & 11

{LEHI Assumptions}

$$\epsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right] \rightarrow 5$$

substitute eq 5 into eq 11

$$\Rightarrow \frac{d}{dr}\left[r\left\{\frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz})\right]\right] = \epsilon_{rr}$$

$$\Rightarrow \boxed{\frac{1}{\cancel{E}}(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz}) + \frac{r}{\cancel{E}}\left(\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\frac{d\sigma_{zz}}{dz}\right) = \frac{1}{\cancel{E}}(\sigma_{rr} - \nu\sigma_{\theta\theta} - \nu\sigma_{zz})}$$

④

$$r\left(\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\frac{d\sigma_{zz}}{dz}\right)$$

$$= \sigma_{rr} - \nu\sigma_{\theta\theta} - \cancel{\nu\sigma_{rr}} - \sigma_{\theta\theta} + \nu\sigma_{rr} + \cancel{\nu\sigma_{zz}}$$

$$= (1+\nu)\sigma_{rr} - (1+\nu)\sigma_{\theta\theta}$$

$$\boxed{r\left(\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\frac{d\sigma_{zz}}{dz}\right) = (\sigma_{rr} - \sigma_{\theta\theta})(1+\nu)}$$

{Compatibility equation} → ⑫

From assumption 4C we know that Axial strain (ϵ_{zz}) is zero or constant

$$\epsilon_{zz} = 0 \Rightarrow \frac{d}{dr}(\epsilon_{zz}) = 0$$

$$\left\{ \text{LEHI Assumption: Equation 6} \quad \epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] \right\}$$

$$\Rightarrow \frac{d}{dr}(\epsilon_{zz}) = 0$$

$$\Rightarrow \frac{d}{dr}\left[\frac{1}{E}\left\{\sigma_{zz} - \nu\sigma_{rr} - \nu\sigma_{\theta\theta}\right\}\right] = 0$$

$$\Rightarrow \frac{1}{E}\left[\frac{d\sigma_{zz}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\frac{d\sigma_{\theta\theta}}{dr}\right] = 0 \quad (5)$$

$$\Rightarrow \boxed{\frac{d\sigma_{zz}}{dr} = \nu\left(\frac{d\sigma_{rr}}{dr} + \frac{d\sigma_{\theta\theta}}{dr}\right)} \rightarrow (13)$$

Recall

$$\boxed{\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\underbrace{(\sigma_{rr} - \sigma_{\theta\theta})} = 0} \longrightarrow (10)$$

Substitute Equation (10) & (13) onto equation (12)

$$\boxed{r\left(\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\frac{d\sigma_{zz}}{dz}\right) = \frac{\overbrace{(\sigma_{rr} - \sigma_{\theta\theta})}}{(1+\nu)}}$$

{compatibility equation} $\rightarrow$ (12)

$$r\left[\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \nu\left\{\overbrace{\nu\left(\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{\theta\theta}}{dr}\right)}^{\text{from } (10)}\right\}\right]$$

$$= (1+\nu)\underbrace{\left(-r\frac{d\sigma_{rr}}{dr}\right)}_{\text{from } (13)}$$

||

$\Downarrow$

6

$$\Rightarrow r\frac{d\sigma_{\theta\theta}}{dr} - \cancel{r\nu\frac{d\sigma_{rr}}{dr}} - r\nu^2\frac{d\sigma_{rr}}{dr} - r\nu^2\frac{d\sigma_{\theta\theta}}{dr} + r\frac{d\sigma_{rr}}{dr} + \cancel{r\nu\frac{d\sigma_{rr}}{dr}} = 0$$

$$\Rightarrow \cancel{r}\left(\frac{d\sigma_{\theta\theta}}{dr} - \nu^2\frac{d\sigma_{\theta\theta}}{dr}\right) + \cancel{r}\left(\frac{d\sigma_{rr}}{dr} - \nu^2\frac{d\sigma_{rr}}{dr}\right) = 0$$

$$\Rightarrow \underbrace{\left(\frac{d\sigma_{\theta\theta}}{dr} + \frac{d\sigma_{rr}}{dr}\right)} - \nu^2\underbrace{\left(\frac{d\sigma_{\theta\theta}}{dr} + \frac{d\sigma_{rr}}{dr}\right)} = 0$$

$$\Rightarrow (1-\nu^2)\left(\frac{d\sigma_{\theta\theta}}{dr} + \frac{d\sigma_{rr}}{dr}\right) = 0$$

$$\Rightarrow \boxed{(1-\nu^2)\,\frac{d}{dr}\left(\sigma_{\theta\theta} + \sigma_{rr}\right) = 0} \rightarrow (14)$$