BIOMECHANICS - I

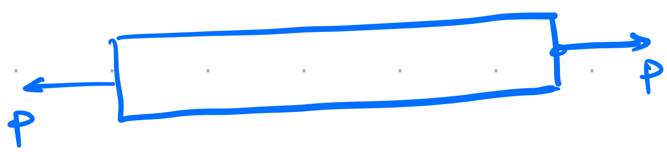
NORMAL & SHEAR STRESS

STRESS
- Normal stress
- Shear stress

KEY IDEA - I

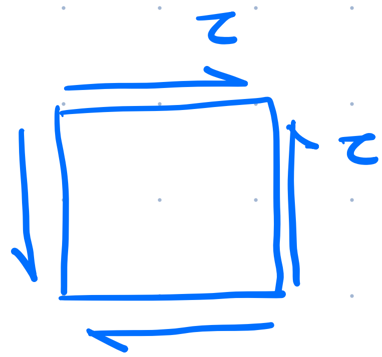
\# What is normal (σ_n) and shear (τ_n) stress
- on a plane
- at an arbitrary angle (α)
- to an applied uniaxial load?

Pure tension:



$$\sigma = \frac{P}{A}$$

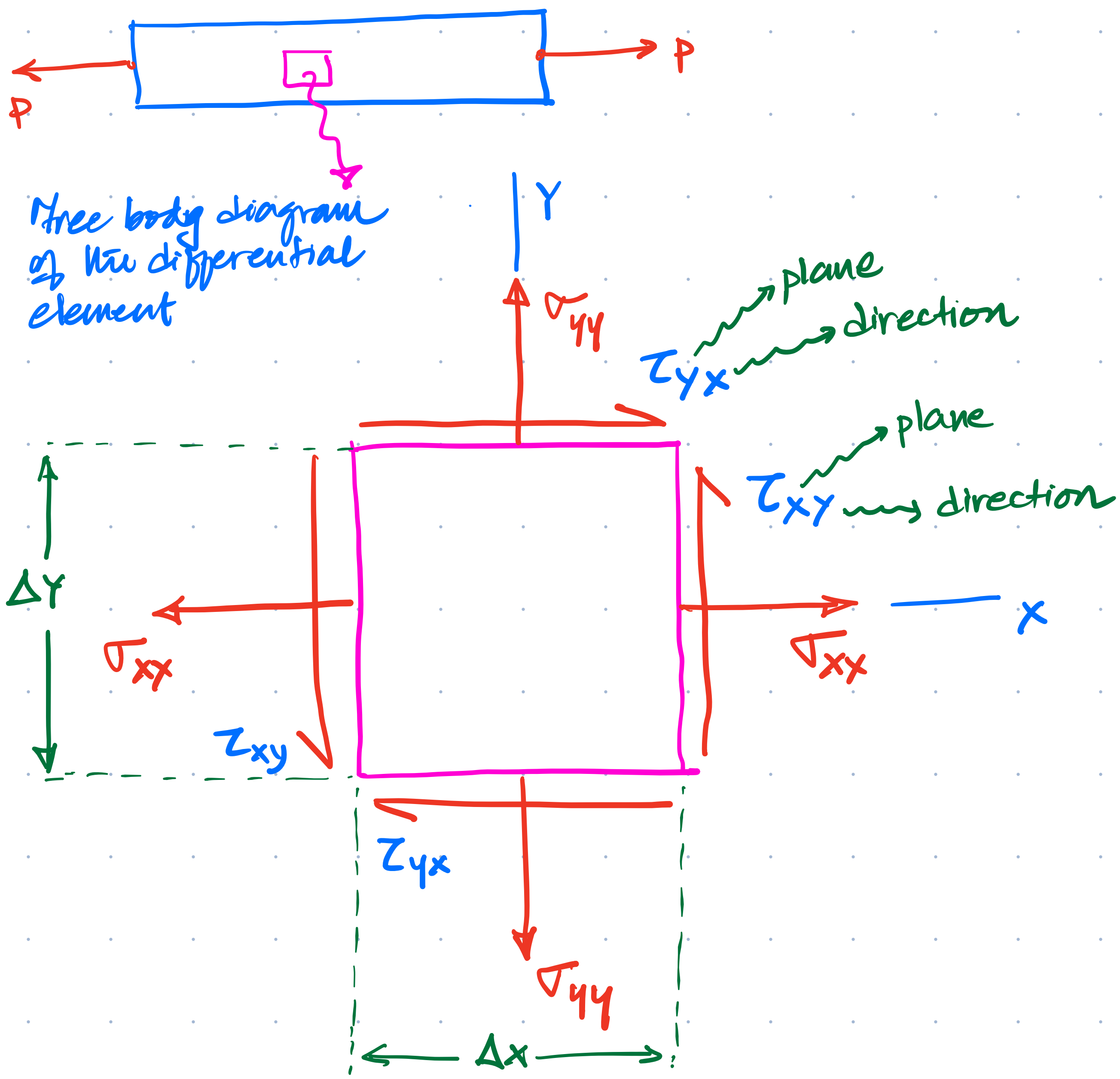
Pure shear:



Consider a prismatic bar

- P is acting uniformly over the cross-section

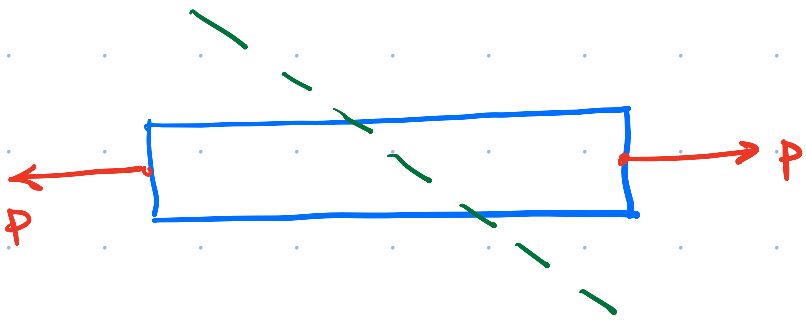
Consider a differential element

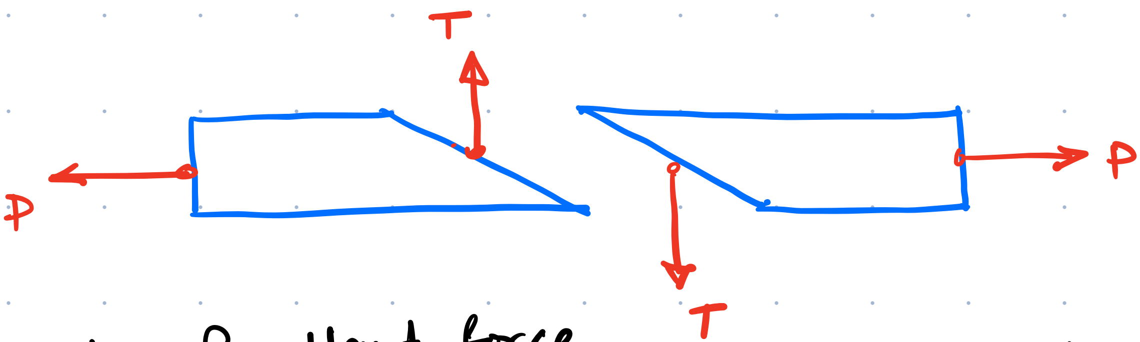


<u>Normal Stress</u>: $\sigma_{xx} = \sigma_{yy}$

<u>Shear stress</u>: τ_{xy}, τ_{yx} & $\tau_{xy} = \tau_{yx}$

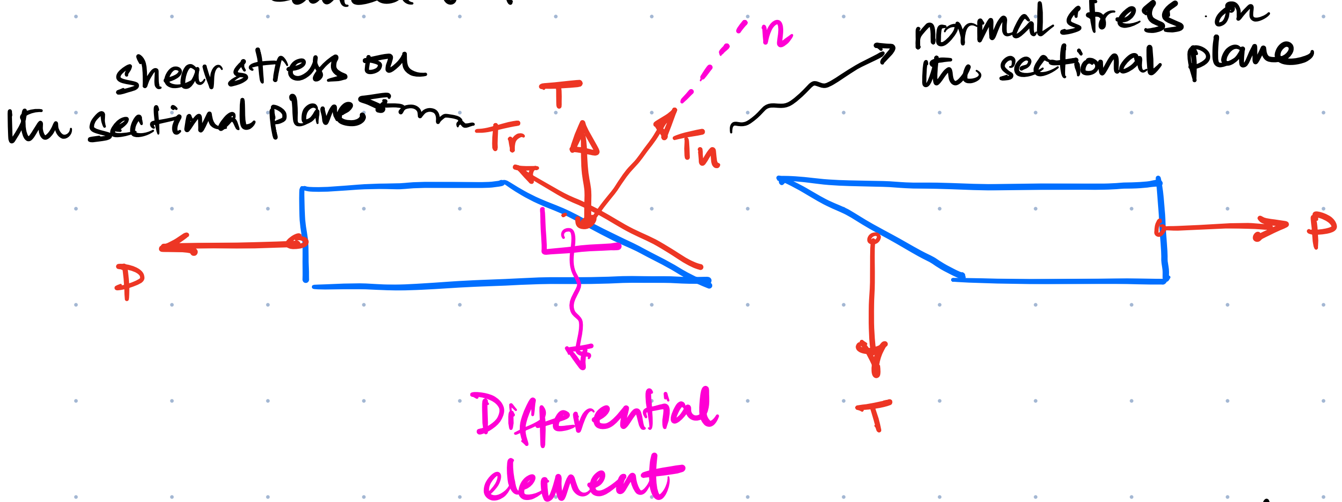
(#) Now consider a sectional plane at some arbitrary angle cutting across the prismatic bar





↳ Resultant force (T) is generated at the sectional plane

↳ We don't know the directions. But they should cancel out.



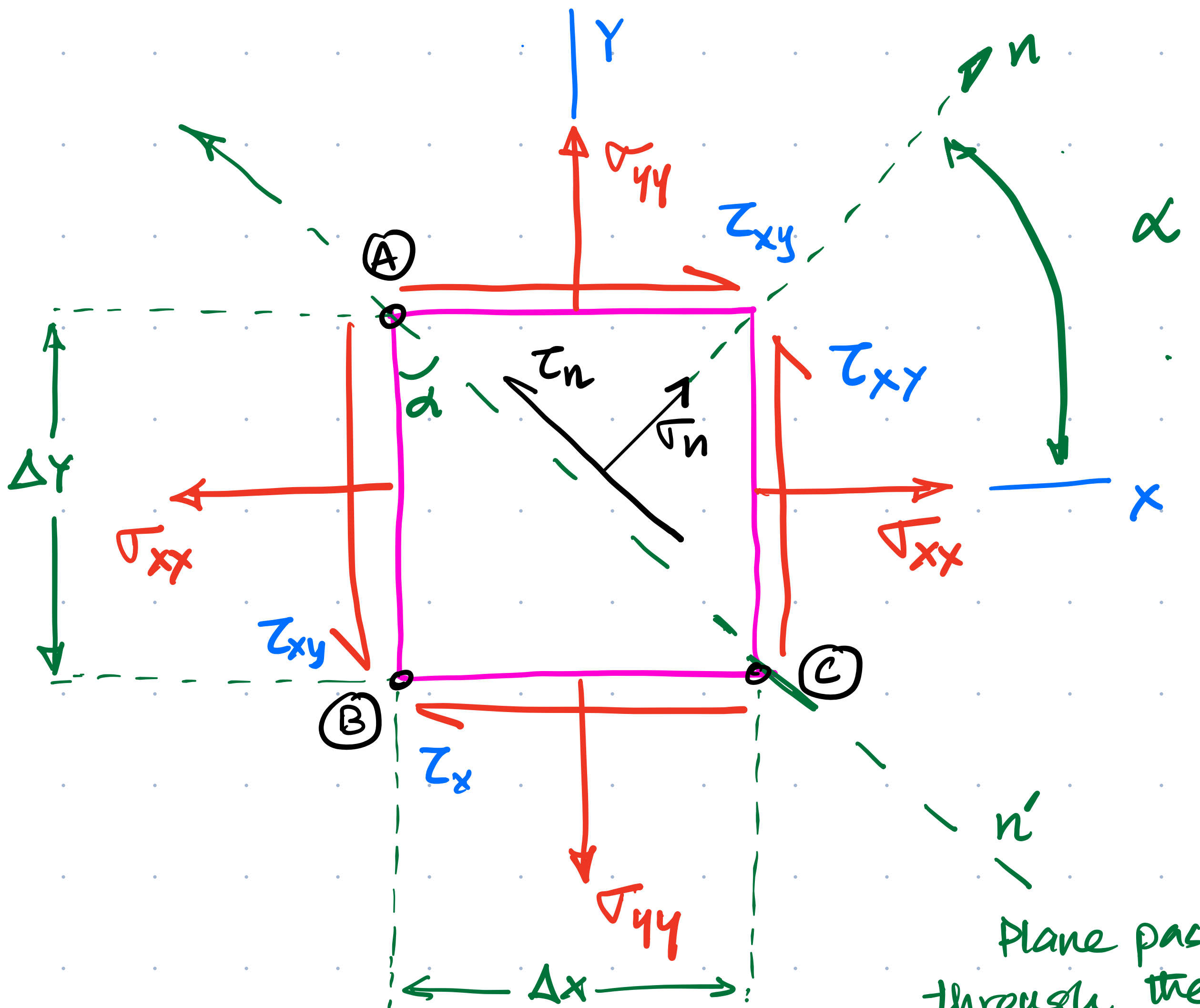
$$\boxed{T^2 = T_n^2 + T_r^2}$$

T_r → Shear stress on the sectional plane

T → Resultant force (T)

T_n → Normal stress on the sectional plane

Free body diagram of the differential element



Plane passing through the element

$n \rightarrow$ normal vector

$\sigma_n \rightarrow$ normal stress

$\tau_n \rightarrow$ shear stress

$\alpha \rightarrow$ plane angle

Given a plane with angle α, ESTIMATE σ_n & τ_n

Consider the wedge ABC :

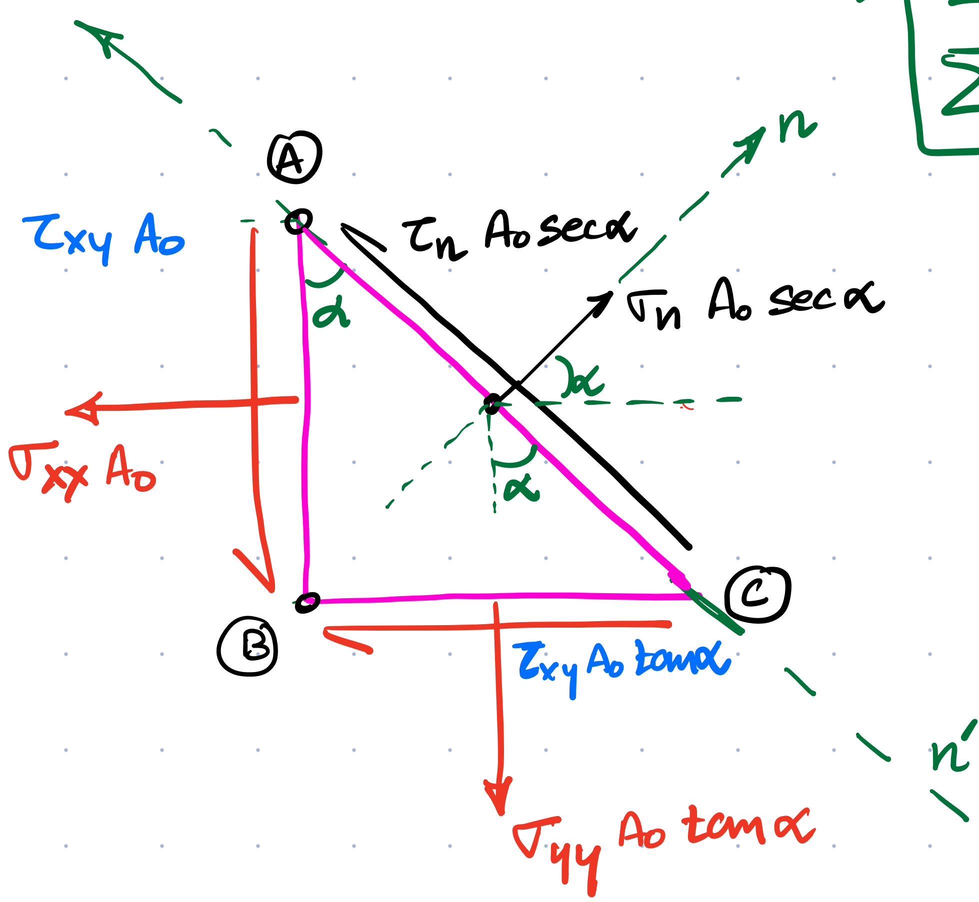
↳ Let the cross-sectional area of side $AB = A_o$

↳ Identify forces acting on wedge

↳ Apply equilibrium conditions

↳ Express each force in terms of A_o

↳ $\boxed{\begin{array}{l}\sum F_n = 0 \\ \sum F_{n'} = 0\end{array}}$



$\boxed{\sum F_n = 0}$

$$\Rightarrow \sigma_n \cancel{A_o} \sec\alpha - \sigma_{yy} \cancel{A_o} \tan\alpha \sin\alpha - \tau_{xy} \cancel{A_o} \tan\alpha \cos\alpha$$
$$- \sigma_{xx} \cancel{A_o} \cos\alpha - \tau_{xy} \cancel{A_o} \sin\alpha = 0$$

$$\Rightarrow \sigma_n = \sigma_{xx} \cos^2\alpha + \sigma_{yy} \sin^2\alpha + \tau_{xy} \sin\alpha \cos\alpha + \tau_{xy} \sin\alpha \cos\alpha$$

$$\Rightarrow \sigma_n = \sigma_{xx} \cos^2\alpha + \sigma_{yy} \sin^2\alpha + \frac{\tau_{xy}}{2} \sin 2\alpha + \frac{\tau_{xy}}{2} \sin 2\alpha$$

$\{ \because 2 \sin\alpha \cos\alpha = \sin 2\alpha \}$

$$\therefore \quad \boxed{\sigma_n = \sigma_{xx}\cos^2\alpha + \sigma_{yy}\sin^2\alpha + \tau_{xy}\sin 2\alpha} \quad \to (1)$$

$$\left\{ \because \begin{aligned} \cos^2\alpha &= \tfrac{1}{2}(1+\cos 2\alpha) \\ \sin^2\alpha &= \tfrac{1}{2}(1-\cos 2\alpha) \end{aligned} \right\}$$

$$\boxed{\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha} \quad \to (2)$$

$$\boxed{\sum F_{n'} = 0}$$

$$\Rightarrow \tau_n \cancel{A_0}\sec\alpha - \sigma_{yy}\cancel{A_0}\tan\alpha\cos\alpha + \sigma_{xx}\cancel{A_0}\sin\alpha + \tau_{xy}\cancel{A_0}\tan\alpha\sin\alpha - \tau_{xy}\cancel{A_0}\cos\alpha = 0$$

$$\Rightarrow \tau_n = \tau_{xy}\cos^2\alpha - \tau_{xy}\sin^2\alpha - \sigma_{xx}\sin\alpha\cos\alpha + \sigma_{yy}\sin\alpha\cos\alpha$$

$$\left\{ \because \begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \end{aligned} \right\}$$

$$\therefore \quad \boxed{\tau_n = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha} \quad \to (3)$$

If we have a stress field given by σ_{xx}, σ_{yy} & τ_{xy} and we have an angle (α) of a sectional plane w.r.t to the X-axis, we can calculate σ_n & τ_n, the normal & shear stress on that plane using Eqns (1) (2) & (3)

Special Cases

$\boxed{\alpha = 0^\circ}$ $\Rightarrow \begin{cases} \cos\alpha = 1 \\ \sin\alpha = 0 \\ \cos 2\alpha = 1 \\ \sin 2\alpha = 0 \end{cases}$

$$\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cancelto{1}{\cos 2\alpha} + \tau_{xy}\cancelto{0}{\sin 2\alpha}$$

$$\Rightarrow \sigma_n = \frac{\sigma_{xx}}{2} + \cancel{\frac{\sigma_{yy}}{2}} + \frac{\sigma_{xx}}{2} - \cancel{\frac{\sigma_{yy}}{2}}$$

$$\Rightarrow \sigma_n = \sigma_{xx}$$

$$\boxed{\sigma_n = \sigma_{xx}}$$

$$\tau_n = \left(\frac{\tau_{yy}-\tau_{xx}}{2}\right)\cancelto{0}{\sin 2\alpha} + \tau_{xy}\cancelto{1}{\cos 2\alpha}$$

$$\Rightarrow \tau_n = \tau_{xy}$$

$$\boxed{\tau_n = \tau_{xy}}$$

$\boxed{\alpha = 90^\circ}$ $\Rightarrow \begin{cases} \cos\alpha = 0 \\ \sin\alpha = 1 \\ \cos 2\alpha = -1 \\ \sin 2\alpha = 0 \end{cases}$

$$\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cancelto{-1}{\cos 2\alpha} + \tau_{xy}\cancelto{0}{\sin 2\alpha}$$

$$\Rightarrow \sigma_n = \cancel{\frac{\sigma_{xx}}{2}} + \frac{\sigma_{yy}}{2} - \cancel{\frac{\sigma_{xx}}{2}} + \frac{\sigma_{yy}}{2}$$

$$\boxed{\sigma_n = \sigma_{yy}}$$

$$\Rightarrow \sigma_n = \sigma_{yy}$$

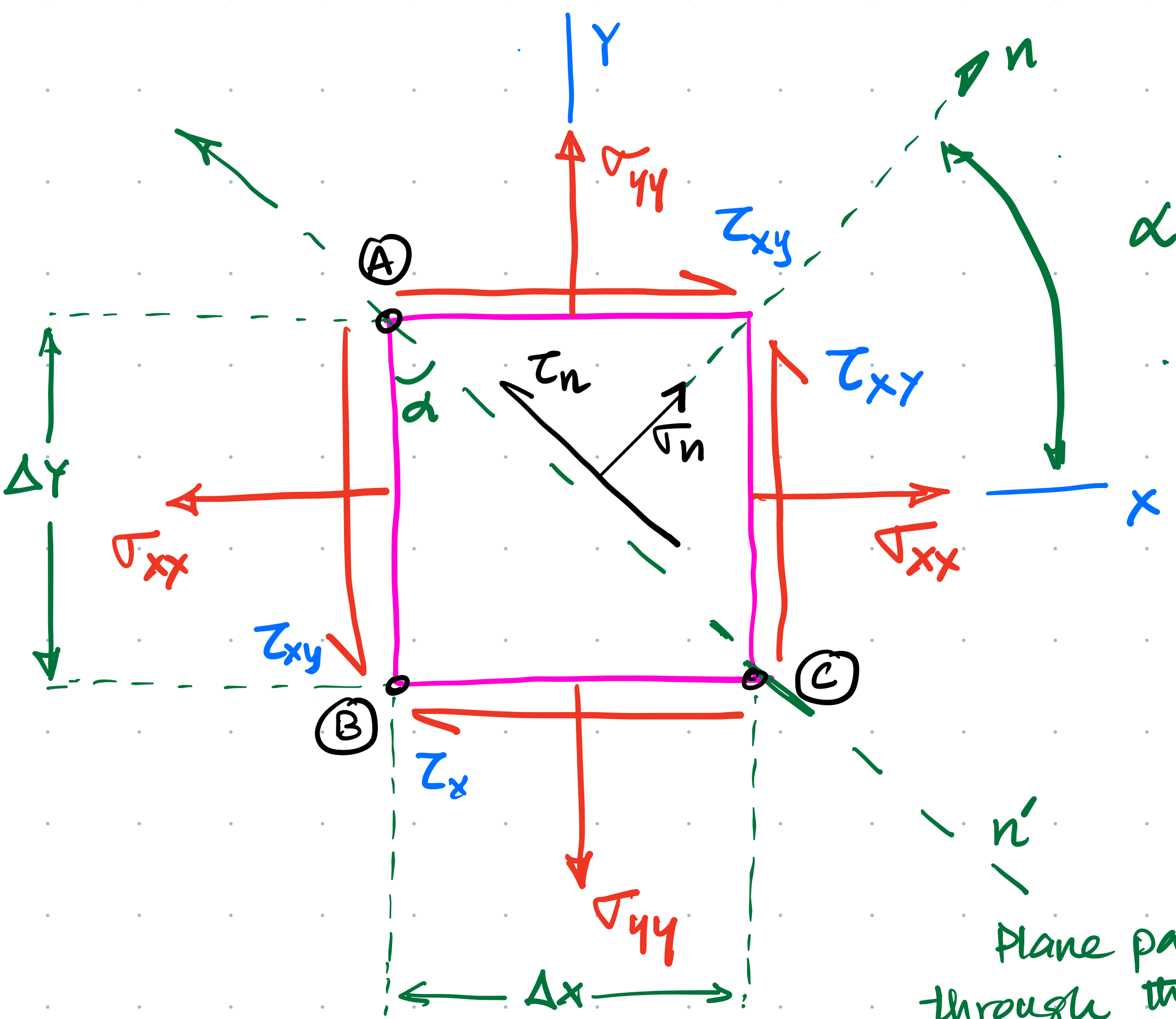
$$\tau_n = \left(\frac{\tau_{yy}-\tau_{xx}}{2}\right)\cancelto{0}{\sin 2\alpha} + \tau_{xy}\cancelto{-1}{\cos 2\alpha}$$

$$\Rightarrow \tau_n = -\tau_{xy}$$

$$\boxed{\tau_n = -\tau_{xy}}$$

KEY IDEA - 2

2D Stress field:



Plane passing through the element

$n \rightarrow$ normal vector

$\sigma_n \rightarrow$ normal stress

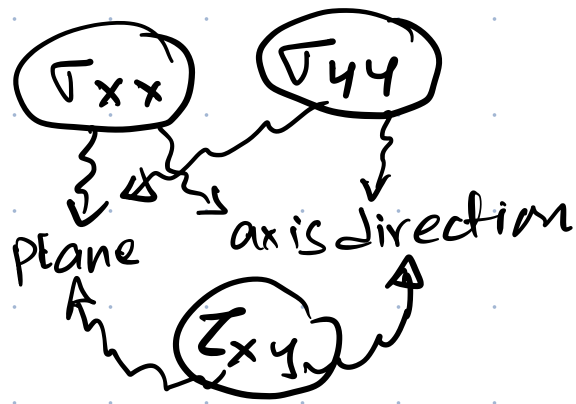
$\tau_n \rightarrow$ shear stress

$\alpha \rightarrow$ plane angle

Can we vary $\alpha = [-180°, 180°]$ and then ESTIMATE (σ_n, τ_n)?

Normal stress : σ_{xx}, σ_{yy}

Shear stress : $\tau_{xy} = \tau_{yx}$



→ Stress components acting on a plane defined by the angle α are given by the following equations:

$$\boxed{\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha}$$

→ (2)

$$\boxed{\tau_n = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha}$$

→ (3)

Example:

Given: $\sigma_{xx} = 20\,MPa$

$\sigma_{yy} = 96\,MPa$

$\tau_{xy} = 8\,MPa$

α	$\sin 2\alpha$	$\cos 2\alpha$	σ_n	τ_n
0°				
30°				
45°			26	-2
60°	-0.5	0.866	15.93	-5.73
90°				
120°				
⋮				
180°				

KEY IDEA - 3 Why do we need to vary α?

As a designer of structures such as prosthetics, stents, bone reconstruction structures etc.,

↪ You are either selecting or configuring a material that

① Meets the DEMAND → Applied load

② Offers the CAPACITY → offers sufficient strength against applied load

↪ $\boxed{\text{CAPACITY}} > \boxed{\text{DEMAND}}$

⇒ Structure is safe!

↪ $\boxed{\text{SAFETY}}$ → Non-dimensional safety factor.

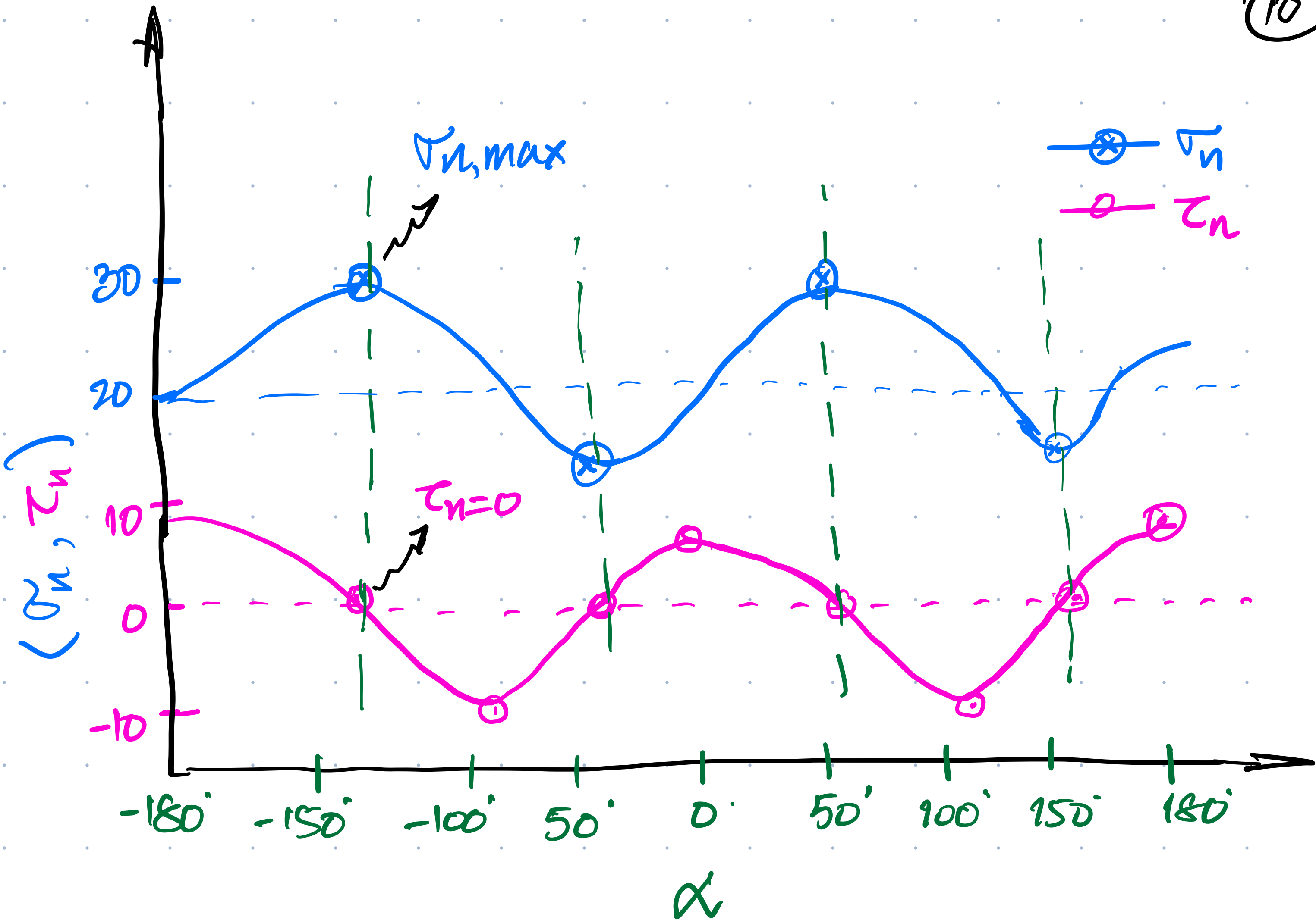
↪ We can then

① Sweep through α

② Calculate σ_n & τ_n

③ Identify a plane where $\boxed{\sigma_n \text{ is WORST or } \tau_n \text{ is WORST}}$

④ Select a material so that it can withstand that level of stress or a combination of σ_n & τ_n



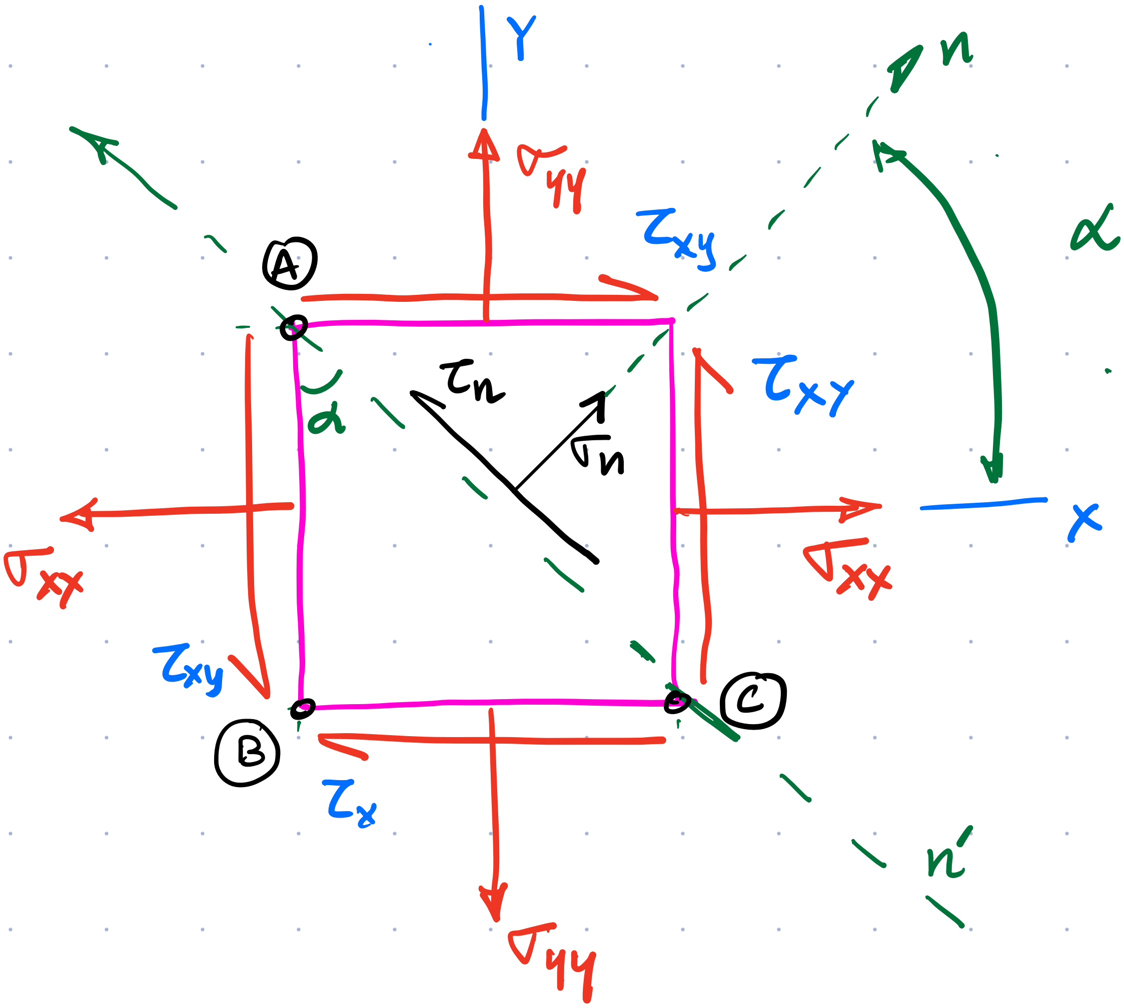
KEY TAKEAWAYS

① At every α, there is a combination of σ_n & τ_n.

② When σ_n is max, $\tau_n = 0$. → Only σ_n is acting on this plane

③ When τ_n is max, $\sigma_n \neq 0$. → A combination of σ_n & τ_n is acting on this plane

KEY IDEA-4 Principal Stresses:



$$\boxed{\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha} \rightarrow (2)$$

$$\boxed{\tau_n = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha} \rightarrow (3)$$

- Equations (2) & (3) are functions of α.
- can we combine equations (2) & (3)?
- How can we estimate σ_n & τ_n w.r.t α?

(i) Maximize Normal Stress:

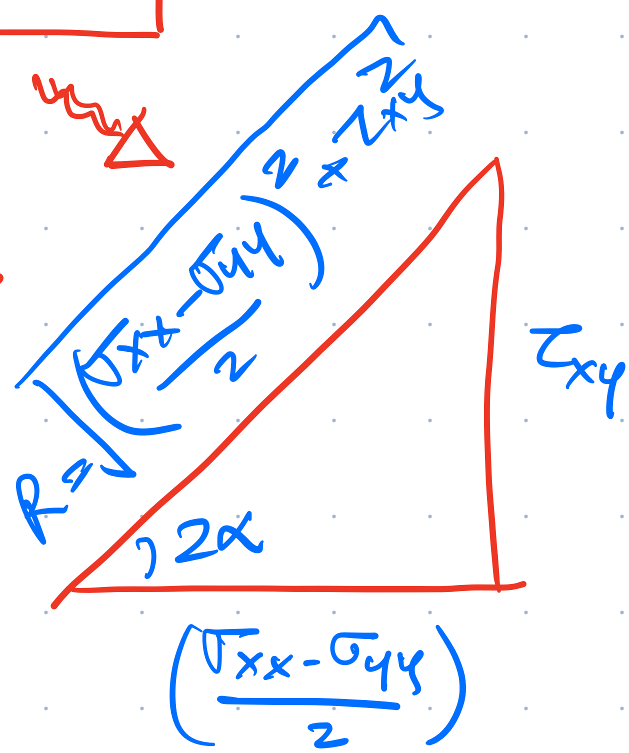
$$\boxed{\frac{d\sigma_n}{d\alpha}=0} \Rightarrow \frac{d}{d\alpha}\left[\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)+\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha\right]=0$$

$$\Rightarrow -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)2\sin 2\alpha + \tau_{xy}\,2\cos 2\alpha = 0$$

$$\Rightarrow \boxed{\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}} \rightarrow (4)$$

$$\Rightarrow \boxed{\cos 2\alpha = \frac{\sigma_{xx}-\sigma_{yy}}{2R}} \rightarrow (5)$$

$$\Rightarrow \boxed{\sin 2\alpha = \frac{\tau_{xy}}{R}} \rightarrow (6)$$



where $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$

=> Put equations (4), (5) & (6) into equations (2) & (3) to get maximum σ_n

Recall equation (2):

$$\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

Substituting $\cos 2\alpha$, $\sin 2\alpha$ & R

$$\Rightarrow \sigma_{n,max} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\left(\frac{\sigma_{xx}-\sigma_{yy}}{2R}\right) + \tau_{xy}\frac{\tau_{xy}}{R}$$

$$= \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \frac{1}{R}\Bigg[\underbrace{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}_{R^2}\Bigg]$$

$$= \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \frac{1}{\cancel{R}}R^{\cancel{2}}$$

$$\Rightarrow \sigma_{n,max} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad \rightarrow (7)$$

$$\Rightarrow \sigma_{n,min} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) - \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad \rightarrow (8)$$

PRINCIPAL PLANES

there are two planes offering two different values of σ_n where $\tau_n = 0$
(i) where $\sigma_n = \sigma_{n,max}$, and (ii) where $\sigma_n = \sigma_{n,min}$

From 7 & 8 we have

PRINCIPAL STRESSES

$$\boxed{\sigma_{n_{max,min}} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}} \quad (9)$$

→ + ve for larger value
→ − ve for smaller value
} 90° apart

→ $\tau_n = 0$ on this plane

→ $\sigma_{n_{max}} \perp \sigma_{n_{min}}$

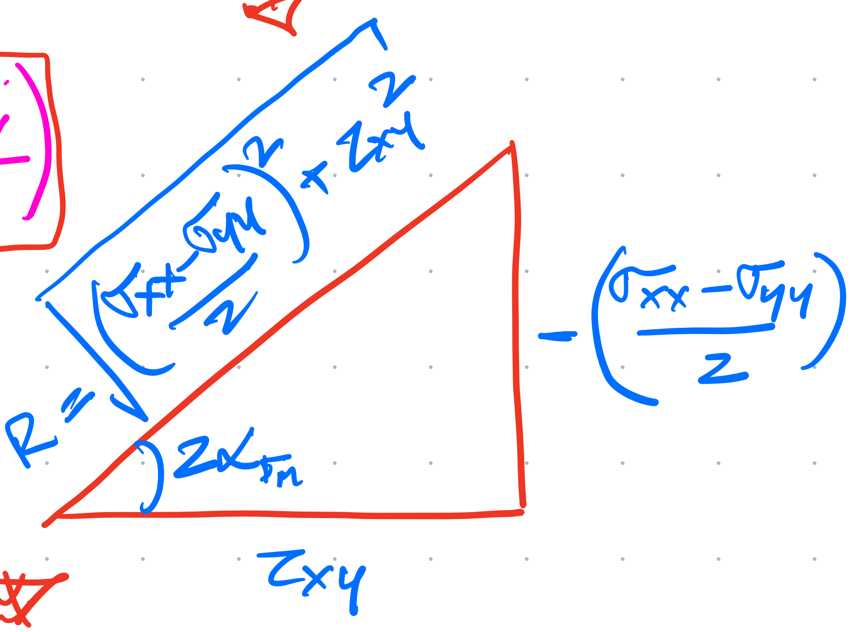
(ii) Maximize Shear Stress:

$$\boxed{\frac{d\tau_n}{d\alpha} = 0}$$

$$\Rightarrow \frac{d\tau_n}{d\alpha} = \frac{d}{d\alpha}\left[-\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha\right] = 0$$

$$\Rightarrow \frac{d\tau_n}{d\alpha} = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right) 2\cos 2\alpha - 2\,\tau_{xy}\sin 2\alpha = 0$$

$$\Rightarrow \boxed{\tan(2\alpha_{\tau n}) = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2\tau_{xy}}\right)} \quad (10)$$



$$\boxed{\sin(2\alpha_{\tau n}) = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2R}\right)} \quad (11)$$

$$\boxed{\cos(2\alpha_{\tau n}) = \frac{\tau_{xy}}{R}} \quad (12)$$

where

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Recall equation (3):

$$\tau_n = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin(2\alpha_{zn}) + \tau_{xy}\cos(2\alpha_{zn})$$

Substituting equations (10), (11) & (12) representing $\cos 2\alpha_{zn}$, $\sin 2\alpha_{zn}$, $\tan 2\alpha_{zn}$ & R.

$$\tau_{n,max} = \left(-\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\left(-\frac{\sigma_{xx}-\sigma_{yy}}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

$$= \frac{1}{R}\left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2\right]$$

$$= \frac{1}{\cancel{R}} R^{\cancel{2}} = R$$

$$\boxed{\tau_{n,max} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}}$$

Maximum shear stress

(13)

SUMMARY OF EQUATIONS FOR NORMAL & SHEAR STRESS

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$$\sigma_n = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

NORMAL STRESS

along a plane at angle α

(2)

$$\tau_n = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

SHEAR STRESS

(3)

PRINCIPAL STRESSES

(9)

$$\sigma_{n_{max,min}} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Angle of max. Normal stress

$$\tan(2\alpha_{\sigma_n}) = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}$$

(2)

$$\tau_{n,max} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

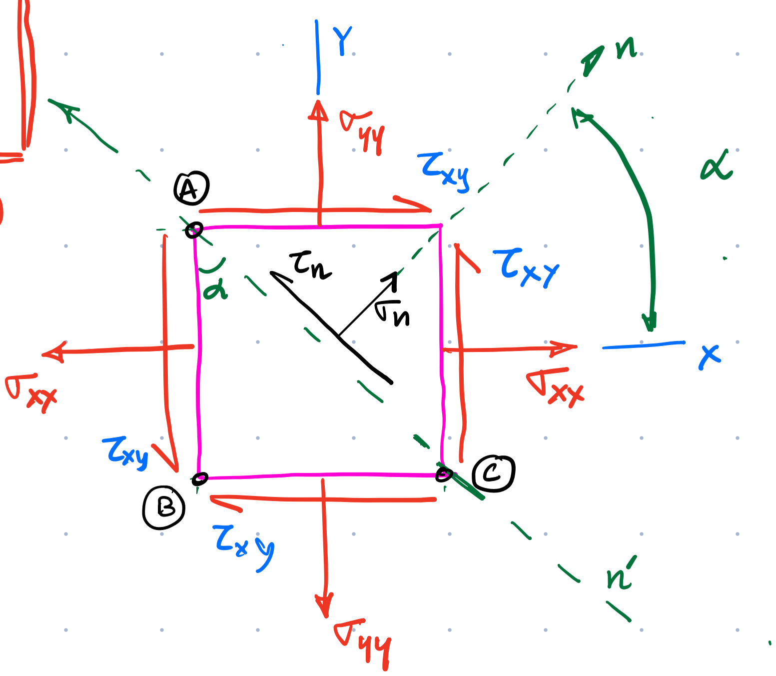
Maximum shear stress

(13)

Angle of max. shear stress

$$\tan(2\alpha_{\tau_n}) = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2\tau_{xy}}\right)$$

(10)



SUMMARY OF EQUATIONS FOR NORMAL & SHEAR STRAIN

$$\epsilon_{xx}' = \epsilon_{xx}\cos^2\alpha + 2\epsilon_{xy}\sin\alpha\cos\alpha + \epsilon_{yy}\sin^2\alpha$$

$$\epsilon_{yy}' = \epsilon_{xx}\sin^2\alpha - 2\epsilon_{xy}\sin\alpha\cos\alpha + \epsilon_{yy}\cos^2\alpha$$

$$\epsilon_{xy}' = 2\sin\alpha\cos\alpha\left(\frac{\epsilon_{yy}-\epsilon_{xx}}{2}\right) + \left(\cos^2\alpha - \sin^2\alpha\right)\epsilon_{xy}$$

where angle α, relates the change in position from x, y, z to x', y', z' in Cartesian coordinates

PRINCIPAL STRAINS

$$\left.\begin{matrix}\epsilon_{xx}'\big|_{max,min} \\ \epsilon_{yy}'\big|_{max,min}\end{matrix}\right. = \left(\frac{\epsilon_{xx}+\epsilon_{yy}}{2}\right) \pm \sqrt{\left(\frac{\epsilon_{xx}-\epsilon_{yy}}{2}\right)^2 + \epsilon_{xy}^2}$$

Angle of max. Normal strain

$$\alpha_p = \frac{1}{2}\tan^{-1}\left[\frac{\epsilon_{xy}}{2(\epsilon_{xx}-\epsilon_{yy})}\right]$$

$$\epsilon_{xy}'\Big|_{max, min} = \pm\sqrt{\left(\frac{\epsilon_{xx}-\epsilon_{yy}}{2}\right)^2 + \epsilon_{xy}^2}$$

Maximum shear strain

Angle of max. shear strain

$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{\epsilon_{yy}-\epsilon_{xx}}{2\epsilon_{xy}}\right)$$