$$\sigma_{rr} = \frac{c}{2} + \frac{c_1}{r^2} \rightarrow (19)$$

Apply boundary conditions

⇝ $\sigma_{rr}(r=a)$:

{ substituting equation (20) into equation (19) }

$$\frac{c}{2} + \frac{c_1}{a^2} = -P_i$$

⇝ $\sigma_{rr}(r=b)$:

{ substituting equation (21) into equation (19) }

$$\frac{c}{2} + \frac{c_1}{b^2} = -P_o$$

$-$ $+$

$$\left(\frac{c_1}{a^2} - \frac{c_1}{b^2}\right) = (P_o - P_i)$$

$$\Rightarrow c_1\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = (P_o - P_i)$$

$$\Rightarrow c_1\left(\frac{b^2 - a^2}{a^2 b^2}\right) = (P_o - P_i)$$

$$\Rightarrow \boxed{c_1 = \frac{(P_o - P_i)(a^2 b^2)}{b^2 - a^2}} \rightarrow (22)$$

Substitute C_1 into one of the boundary condition equations

$$\frac{C}{2} = -P_i + \frac{(P_o - P_i)(\cancel{a^2} b^2)}{(b^2 - a^2)\cancel{a^2}}$$

$$\Rightarrow \boxed{\frac{C}{2} = -P_i + \frac{(P_o - P_i)(b^2)}{b^2 - a^2}} \rightarrow (23)$$

Recall:

$$\sigma_{rr} = \left(\frac{C}{2}\right) + \left(\frac{C_1}{r^2}\right) \longrightarrow (19)$$

Substitute equations (22) & (23) into equation (19)

$$\sigma_{rr} = \underbrace{\left[-P_i + \frac{(P_o - P_i)(b^2)}{b^2 - a^2}\right]}_{\frac{C}{2}} + \underbrace{\left[\frac{(P_o - P_i)(a^2 b^2)}{(b^2 - a^2)(r^2)}\right]}_{\frac{C_1}{r^2}}$$

$$= \frac{(-P_i)(b^2-a^2)r^2 + (P_i - P_o)b^2 r^2 + (P_o - P_i)a^2 b^2}{(b^2-a^2)(r^2)}$$

$$= \frac{\cancel{(-P_i b^2 r^2)} + P_i a^2 r^2 + \cancel{(P_i b^2 r^2)} - P_o b^2 r^2 - (P_o - P_i)a^2 b^2}{(b^2-a^2)(r^2)}$$

$$= \frac{(P_i a^2 - P_o b^2)\cancel{(r^2)}}{(b^2-a^2)\cancel{(r^2)}} + \frac{(P_o - P_i)a^2 b^2}{(b^2-a^2)(r^2)}$$

$$\boxed{\sigma_{rr} = \frac{P_i a^2 - P_o b^2}{b^2-a^2} - \frac{(P_i - P_o)a^2 b^2}{(b^2-a^2)r^2}} \qquad (24)$$

lly

$$\boxed{\sigma_{\theta\theta} = \frac{P_i a^2 - P_o b^2}{b^2 - a^2} + \frac{(P_i - P_o) a^2 b^2}{(b^2 - a^2) r^2}} \quad \checkmark \quad \rightarrow (25)$$

Equations (24) & (25) known as "Lamé solutions"

Extended discussion: **Case 1**

Recall:

$$\boxed{\sigma_{\theta\theta} + \sigma_{rr} = C} \longrightarrow (15)$$

Sum equations (24) & (25)

$$(\sigma_{\theta\theta} + \sigma_{rr}) = 2\left(\frac{P_i a^2 - P_o b^2}{b^2 - a^2}\right) = \cancel{2}\left(\frac{C}{\cancel{2}}\right)$$

$$\boxed{\sigma_{\theta\theta} + \sigma_{rr} = C}$$

when given uniform static pressures P_i & P_o

$\boxed{\text{Case 2}}$ When $P_o = 0$

$$\boxed{\sigma_{rr} = \frac{P_i a^2 - \cancel{P_o b^2}}{b^2 - a^2} - \frac{(P_i - \cancel{P_o}) a^2 b^2}{(b^2 - a^2) r^2}} \rightarrow (24) \quad ✓$$

$$\boxed{\sigma_{\theta\theta} = \frac{P_i a^2 - \cancel{P_o b^2}}{b^2 - a^2} + \frac{(P_i - \cancel{P_o}) a^2 b^2}{(b^2 - a^2) r^2}} \rightarrow (25) \quad ✓$$

$$\sigma_{rr} = \frac{P_i a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \rightarrow (26)$$

$$\sigma_{\theta\theta} = \frac{P_i a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \rightarrow (27)$$

Case 3

~~Circumferential~~ Average or hoop stress (16)

$$\langle \sigma_{\theta\theta} \rangle = \frac{1}{b-a}\int_a^b \frac{Pa^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)dr$$

$$\boxed{\langle \sigma_{\theta\theta} \rangle = \frac{P_i(a)}{b-a}} \rightarrow (28)$$

where the thickness $h = b-a$

Compare this expression to the thin walled cylinder

$$\boxed{\sigma_{\theta\theta} = \frac{Pa}{h}}$$

$\boxed{\text{Case 4}}$

$\Rightarrow$ We want to maintain axial length of the thick-walled cylinder at a fixed value.

$$2\pi \int_a^b \sigma_{zz}\, r\,dr = P_i \pi a^2 - P_o \pi b^2 + f$$

Assumption (1c): LEHI

$\epsilon_{zz} = 0$

$$\Rightarrow \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] = 0$$

$$\Rightarrow \boxed{\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})} \rightarrow 29$$

Recall:

$$\boxed{\begin{aligned} \sigma_{rr} &= \frac{P_i a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{P_i a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \end{aligned}}$$

$\rightarrow$ 26

$\rightarrow$ 27

Substitute eq 26 & 27 into eq 29

$$\sigma_{zz} = \nu\left[\frac{P_i a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) + \frac{P_i a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)\right]$$

$$\boxed{\sigma_{zz} = \frac{2\nu P_i a^2}{b^2 - a^2} = \frac{2\nu P_i a^2}{2ah + h^2}} \rightarrow 28$$

{where $h = b - a$}

Thin-walled cylinder assumptions

① $h \ll a$ {$\therefore h^2$ is neglected}

② $\nu = \frac{1}{2}$ {$\because$ wall is incompressible}

$$\boxed{\sigma_{zz} = \frac{\cancel{2}\left(\frac{1}{\cancel{2}}\right) P_i a^2}{2ah + \cancel{h^2}} = \frac{P_i a}{2h}} \rightarrow 25$$