MAE 3128
Biomechanics-I

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Topics covered today:

1. Universal solutions
    1. Biological motivations and applications
    2. Pressurization of a Thin Spherical Structure
2. Non-Universal solutions involving constitutive relations and material properties
    1. Biological motivations and applications
    2. Thick-walled cylindrical tube
3. In-class problems

GW School of Engineering & Applied Science
THE GEORGE WASHINGTON UNIVERSITY

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Summary of what's been covered in this course so far

- **Lectures 1, 2:** Introduced **Biomechanics** and reviewed **Statics** under rigid body assumptions
- **Lectures 3, 4:** Introduced **general concepts of stress and strain,** and **constitutive relationship using Hooke's law** for small-strain, solid-like materials.
- **Lectures 4, 5:** Shift focus to **canonical problems** in introductory biosolid mechanics

  - **For each problem,**
    - Choose coordinate systems
    - Calculate **stress components** by applying **equilibrium equations** (force, linear and angular momentum balances without inertia).
      - **Stress** is expressed in terms of **applied load and geometry**
      - **Material properties** appearing explicitly only in a few cases.
    - **Calculate strain or deformation**, depending on **load, geometry, and material properties.**

After the midterm exam the following **canonical problems** will be addressed:

- **Extension of rods and torsion of cylindrical structures**
- **Bending of beams**
- **Buckling of columns**

Universal solutions

Reference: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Results obtained independent of the specification of particular material properties, are called **universal solutions**.

- Although not emphasized in most books on the mechanics of materials, the generality of these universal solutions allow them to be applied equally to problems involving the uniaxial extension of tendons, rubber bands, metallic wires, or concrete.

These universal solutions can be found **solely from equilibrium equations**,

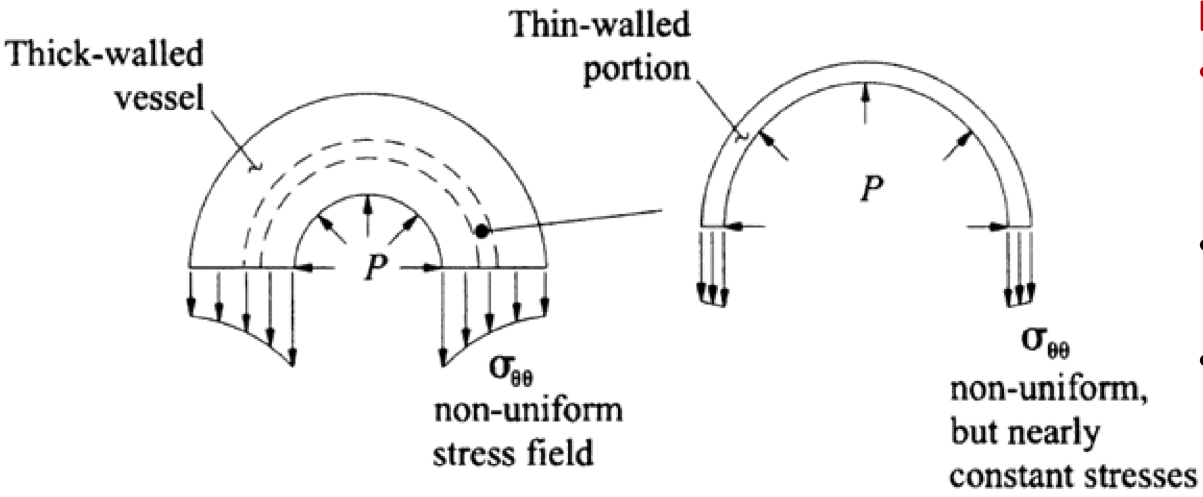
- Without using constitutive equations.
- Because they do not depend on material behavior, they apply to **any solid-like material** (metals, elastomers, biological tissues, etc.).

- Universal solutions are **experimentally valuable**,
  - as they exist independently of the material being tested and
  - assist in formulating stress–strain relations when strain is measurable.

Universal solutions

Reference: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Three special cases for **universal solutions with biological motivation**:
- Axial loading of a uniform rod.
- Inflation and extension of a thin-walled cylindrical tube.
- Inflation of a thin-walled hollow sphere.

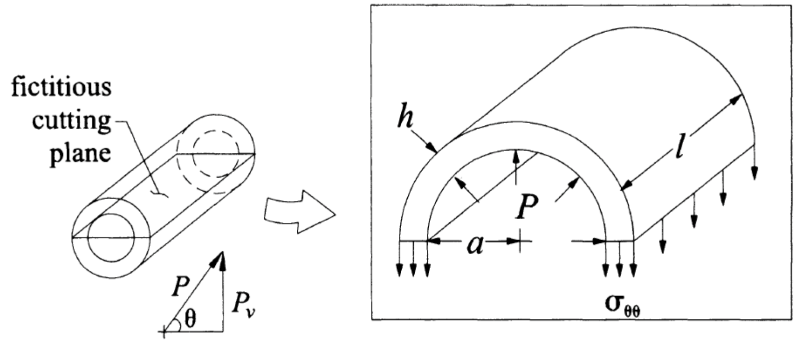


Limits of applicability
- **Increasing wall thickness** (in tubes or spheres) changes the solution approach and can make the stress distribution **non-uniform**, violating the conditions for universal solutions.
- For **thick-walled structures**, solutions require constitutive equations; the equilibrium-only approach no longer applies.
- The **average wall stress** can still be determined for both thin- and thick-walled structures, but it **better approximates reality** for thin-walled cases.

Reference: Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Circumferential (Cauchy) stress in a thin-walled pressurized cylinder

$$\sigma_{\theta\theta} = \frac{Pa}{h},$$

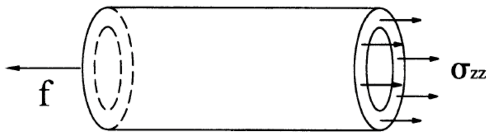


Radial stress in a thin-walled pressurized cylinder

$$\sigma_{rr} \cong \frac{-P}{2},$$

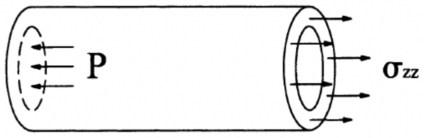
Axial stress in an inflated thin-walled pressurized cylinder

$$\sigma_{zz} \cong \frac{f}{2\pi a h}$$



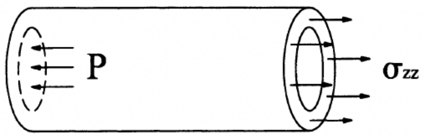
Axial stress in thin-walled pressurized cylinder: Ends of the cylinder are closed

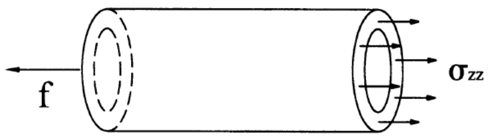
$$\sigma_{zz} = \frac{Pa}{2h},$$



Ends of the cylinder are closed + Axial load

$$\sigma_{zz} = \frac{Pa}{2h} + \frac{f}{2\pi a h}.$$





where

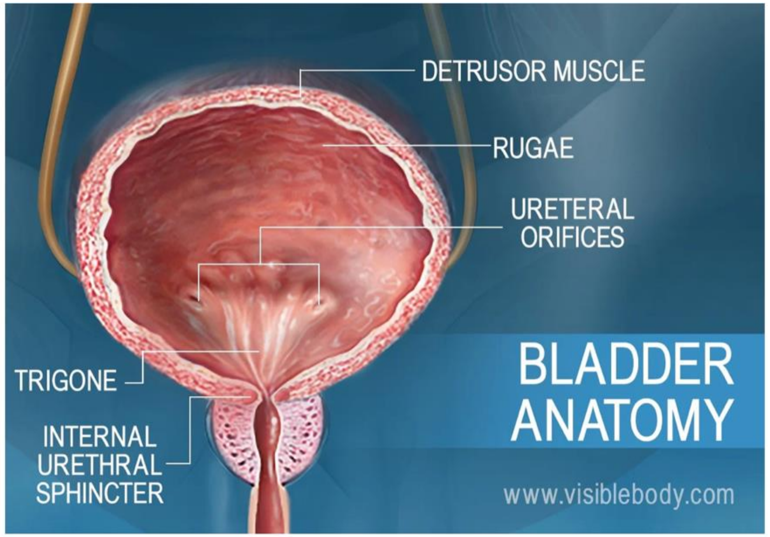
- **P** is the uniform internal pressure
- **a** is the inner radius of the cylinder in the pressurized configuration, and
- **h** is the thickness of the wall of the pressurized (i.e., deformed) cylinder.
- **a** and **h** are values in the pressurized configuration

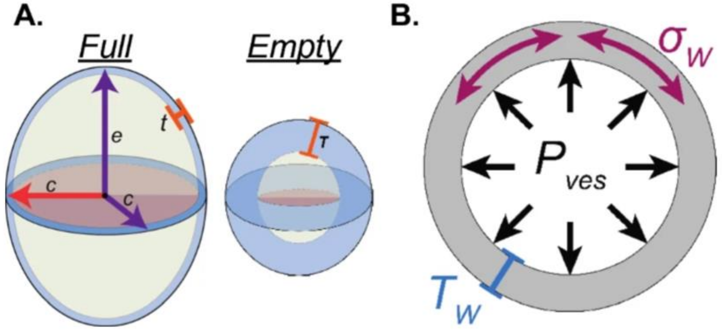
Assumption of thinness
(a/h >> 1)

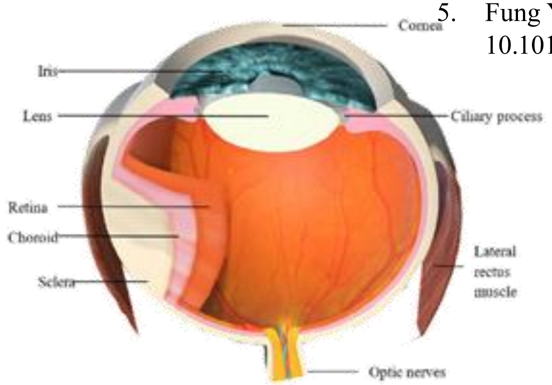
Biological motivation: Urinary bladder, Eye and the RBC

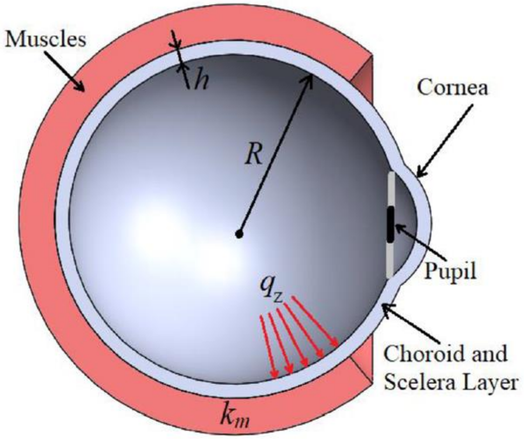
References:

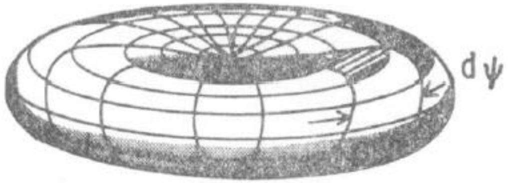
1. Hennig, G., Saxena, P., Broemer, E. *et al.* Quantifying whole bladder biomechanics using the novel pentaplanar reflected image macroscopy system. *Biomech Model Mechanobiol* **22**, 1685–1695 (2023). https://doi.org/10.1007/s10237-023-01727-0
2. https://www.bafound.org/understanding-brain-aneurysms/about-brain-aneurysms/
3. https://www.visiblebody.com/learn/urinary/urine-storage-and-elimination
4. Shahriar Dastjerdi, Bekir Akgöz, Ömer Civalek, On the shell model for human eye in Glaucoma disease, International Journal of Engineering. Science, Volume 158, 2021, 103414, ISSN 0020-7225, https://doi.org/10.1016/j.ijengsci.2020.103414
5. Fung YC, Tong P. Theory of the sphering of red blood cells. Biophys J. 1968 Feb;8(2):175-98. doi: 10.1016/S0006-3495(68)86484-7. PMID: 5639934; PMCID: PMC1367371.

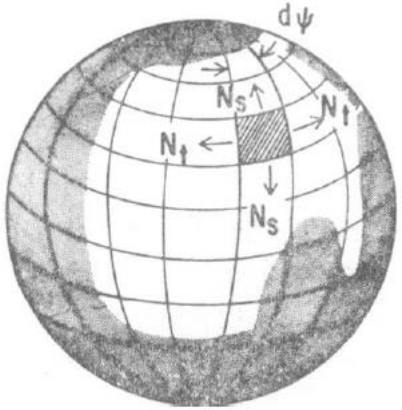


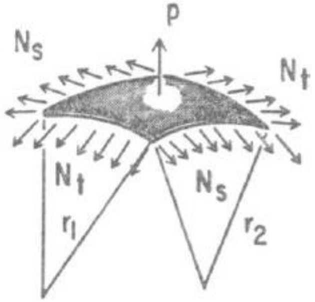








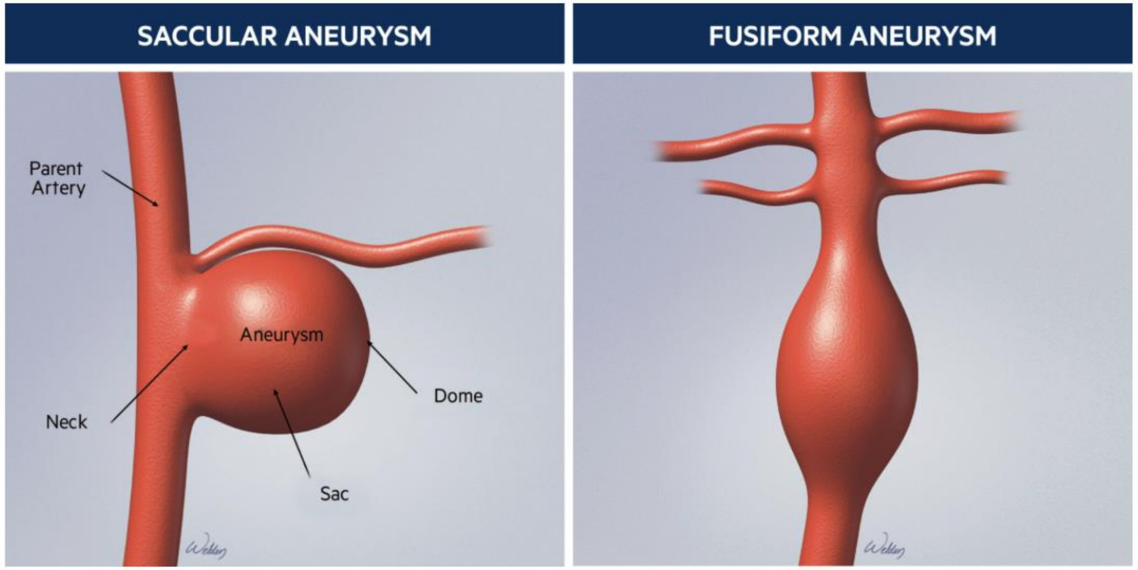


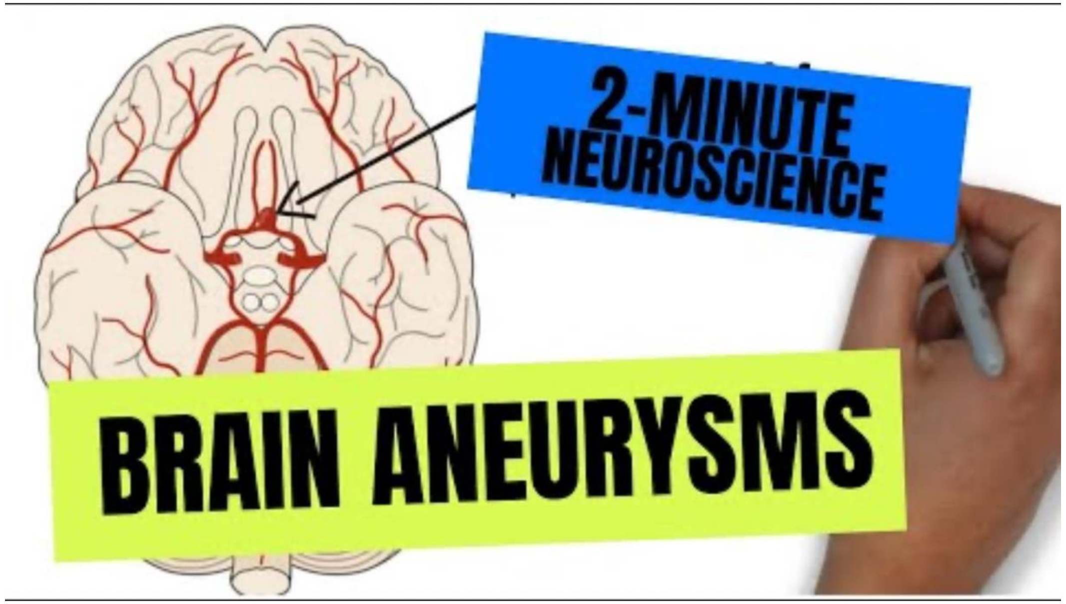


Biological motivation: Brain aneurysms

References:

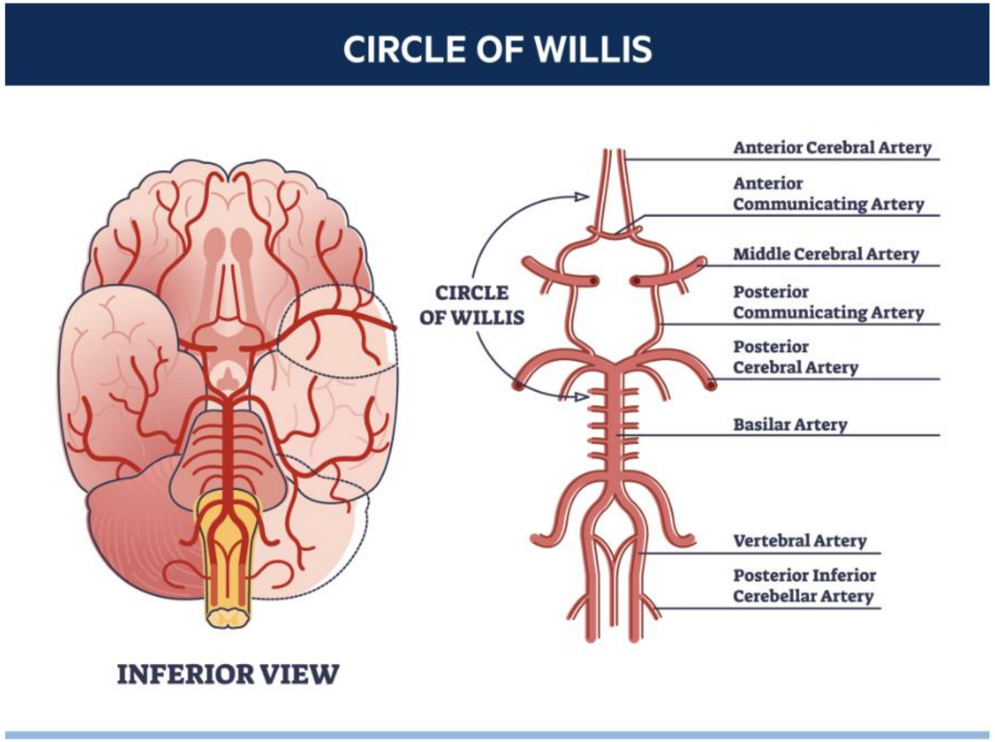
1. https://youtu.be/ZRjJp4Jl1QM?si=07syHDpIaEFNbC0_
2. https://www.bafound.org/understanding-brain-aneurysms/about-brain-aneurysms/

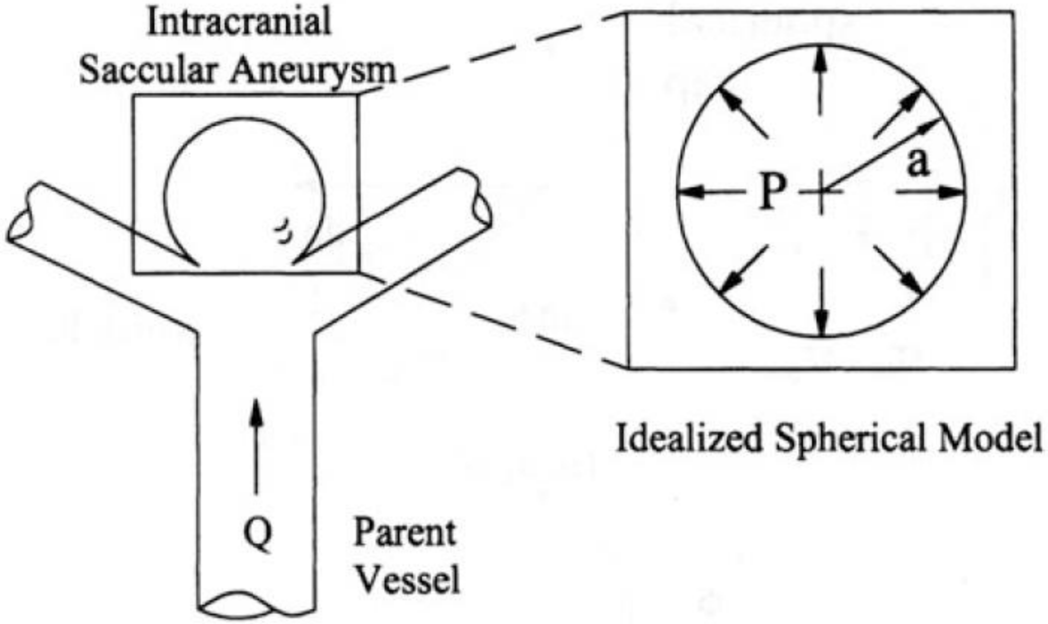




Biological application problem: Pressurization of a Thin Spherical Structure

References: H
1. Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)
2. https://www.bafound.org/understanding-brain-aneurysms/about-brain-aneurysms/





Circumferential (Cauchy) stress in a thin-walled pressurized sphere

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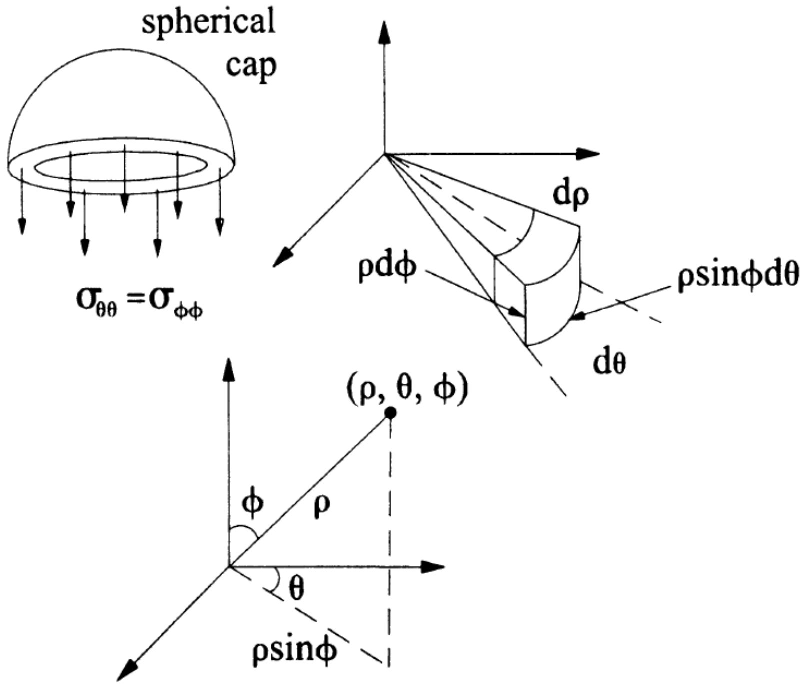

$$\sum F_v = 0 \rightarrow \int P_v dA - \int \sigma_{\theta\theta} dA = 0,$$

$$\iint P \cos\phi \rho \sin\phi d\theta \rho d\phi = \iint \sigma_{\theta\theta} \rho d\theta d\rho$$

$$Pa^2 \int_0^{\pi/2} \int_0^{2\pi} \cos\phi \sin\phi d\theta d\phi = \sigma_{\theta\theta} \int_a^{a+h} \int_0^{2\pi} \rho d\theta d\rho,$$

$$Pa^2 \int_0^{\pi/2} 2\pi \cos\phi \sin\phi d\phi = \sigma_{\theta\theta} \int_a^{a+h} 2\pi\rho d\rho,$$

$$Pa^2 \int_0^{\pi/2} \frac{1}{2} \sin 2\phi d\phi = \sigma_{\theta\theta} \int_a^{a+h} \rho d\rho,$$

$$Pa^2 \left(\frac{1}{2}\right)(1) = \sigma_{\theta\theta} \left[\frac{1}{2}(2ah + h^2)\right],$$

$$Pa^2 = \sigma_{\theta\theta}(2ah + h^2),$$

$$\sigma_{\theta\theta} = \frac{Pa^2}{2ah + h^2} \rightarrow \sigma_{\theta\theta} = \frac{Pa}{2h} = \sigma_{\phi\phi}.$$

Assumption of thinness (a/h >> 1)

where

- **P** is the uniform internal pressure
- **a** is the inner radius of the cylinder in the pressurized configuration, and
- **h** is the thickness of the wall of the pressurized (i.e., deformed) cylinder.
- **a** and **h** are values in the pressurized configuration

Modeling a saccular aneurysm as a thin-walled sphere, assume that it has an inner radius of 2.5 mm and a thickness of 15 µm at a mean blood pressure of 120 mmHg.

Calculate the stress $\sigma_{\theta\theta}$ or $\sigma_{\varphi\varphi}$ and determine if rupture is likely if the critical stress is on the order of 5 MPa.

Hint: 1 mm/ Hg ≈ 133.32 N/m²

Given:
P = 120 mm/Hg ≈16,000 N/m²
a = 2.5 mm = 2.5 x 10^{-3} m
h = 15 µm = 15 x 10^{-6} m

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{Pa}{2h} = \frac{(16000\,\mathrm{N/m^2})(2.5 \times 10^{-3}\mathrm{m})}{2(15 \times 10^{-6}\mathrm{m})} \cong 1{,}333{,}333\,\mathrm{N/m^2}$$

$$\cong 1.3\,\mathrm{MPa}$$

Key question
Given the 50 % mortality rate associated with rupture and the sparseness of data on the mechanical behavior of saccular aneurysms, what is the factor of safety ?

Biological application problem: Pressurization of a Thick-walled cylinder

References: H
1. Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

Case study:
Both Stress and strain are determined for a defined material model by directly solving the equilibrium equations

Motivation:
- Biological and physiological contexts, such as veins, aneurysms, the aorta, and the left ventricle.
- Emphasis of understanding thick-walled structures and comparing them conceptually with simpler thin-walled approximations.
  - Previous results provided general solutions for stresses under restrictive conditions (e.g., centroidal loading and thin-walled assumptions)
  - Did not include the corresponding strains or specific deformation measures.

To obtain strains:
One must **incorporate a constitutive relation**, which makes the solution **material-specific rather than universal**.

Assumptions:
Pressurization of a Thick-walled cylinder

References: H
1. Humphrey, J. D., & O'Rourke, S. L. (2015). *An introduction to biomechanics: Solids and fluids, analysis and design* (2nd ed.)

The problem is simplified to the **inflation of a thick-walled cylindrical vessel in cylindrical coordinates**:

1. **Material:**
   - **Linearly Elastic, Homogeneous, and Isotropic (LEHI)**.
   - Experiences **small strains**.
   - Allows the **axial strain (ε_{zz})** to be either zero or a constant.
2. **Stresses**
   - Material is subject to **axisymmetric conditions** with **no axial variation** in stress.
3. **Internal Pressure**
   - **Pressurized uniformly**, possibly with a single **axial load applied through the centroid**.
4. **Body forces:**
   - Has **no body forces** acting within it.

These assumptions reduce the stress state to only three components:
1. Radial (σ_{rr}),
2. Circumferential ($\sigma_{\theta\theta}$), and
3. Axial (σ_{zz})

each depending only on the radial coordinate.