KEY IDEA - 2

Navier-Space Equilibrium Equations :

We started with Generalized Equilibrium Equations

→ Work for any material!

→ All continua

What if we want a specialized set of equations for certain types of materials or material behavior?

For example: Hookean LEHI material (behavior)

- Linear
- Elastic
- Homogenous
- Isotropic

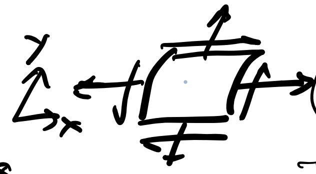
Described by Hooke's Law

Hooke's Law

→ Stress-Strain constitutive relationship

KEY IDEA - 2.1

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{yx} & 0 \\ \epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$



$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider 2D state of stress, & 2D State of Strain

Don't confuse these with plane stress & plane strain!

- Cartesian Co-ordinates (x, y, z)
- [CONSTITUTIVE RELATIONSHIP FOR HOOKEAN LEHI]
- [STRESS-STRAIN EQUATIONS]

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\,\epsilon_{xx} \quad \rightarrow (1)$$

$$\sigma_{xy} = 2\mu\,\epsilon_{xy} = \sigma_{yx} \quad \rightarrow (2)$$

$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\,\epsilon_{yy} \quad \rightarrow (3)$$

2D State of Stress & Strain

where λ, μ : material parameters (Lamé constants)

$\mu \equiv G \rightsquigarrow$ shear modulus

Let's substitute them in Generalized Eq

IN 3D Cartesian Coordinates: (x, y, z)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho g_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho g_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z = 0$$

\# We are only looking at stress variations in the x-y plane: (Neglecting body forces)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

{ From Eq ①, ② & ③ on page 16 }

$$\Rightarrow \frac{\partial}{\partial x}\left[\lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx}\right] + \frac{\partial}{\partial y}\left[2\mu\epsilon_{xy}\right] = 0$$

④ X-direction

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[2\mu \epsilon_{xy}\right] + \frac{\partial}{\partial y}\left[\lambda(\epsilon_{xx}+\epsilon_{yy}) + 2\mu\epsilon_{yy}\right] = 0 \quad (5)$$

y-direction

We know from previous lecture

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

3D Cartesian co-ordinates

normal

shear

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y}$$

We need these for this 2D state. Substitute in eq (4) on page (17)

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

Rewriting equation 4 on page 17:

$$\boxed{\Rightarrow \frac{\partial}{\partial x}\left[\lambda\left(\epsilon_{xx}+\epsilon_{yy}\right)+2\mu\,\epsilon_{xx}\right]+\frac{\partial}{\partial y}\left[2\mu\,\epsilon_{xy}\right]=0} \rightarrow 4$$

$$\epsilon_{xx} \rightarrow \frac{\partial u_x}{\partial x} \qquad \epsilon_{yy} \rightarrow \frac{\partial u_y}{\partial y} \qquad \epsilon_{xy} \rightarrow \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

$$\Rightarrow \lambda\frac{\partial^2 u_x}{\partial x^2}+2\mu\frac{\partial^2 u_x}{\partial x^2}+\lambda\frac{\partial^2 u_y}{\partial x\partial y}$$

$$+\mu\frac{\partial^2 u_x}{\partial y^2}+\mu\frac{\partial^2 u_y}{\partial x\partial y}=0$$

$$\Rightarrow \boxed{\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)=0} \rightarrow 6 \quad x\text{-direction}$$

Similarly working with equation 5 on page 18:

$$\boxed{\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)=0} \rightarrow 7 \quad y\text{-direction}$$

$\mu \equiv G$

Material properties

Summary of Navier-Space Equilibrium equations: (i.e., Eq ⑥ & Eq ⑦)

In 2D Cartesian Co-ordinates: (x, y)

↳ For a displacement vector $\vec{u} = u_x \hat{i} + u_y \hat{j}$

↳ Two coupled partial differential equations

↳ Two unknowns, u_x & u_y

$$\left.\begin{aligned} \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \\ \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \end{aligned}\right\}$$

where λ, μ are Lamé constants

& $\mu \equiv G$ or shear modulus

In 3D Vector form for Cartesian Co-ordinates: (x, y, z)

(compact form)

$$\boxed{\mu \vec{\nabla}^2 \vec{u} + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) = 0}$$

where $\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ (del operator)