Studying FlipIt using Linear Algebra over $\mathbb{F}_2$

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Abstract

This is the abstract of the paper. It should provide a brief summary of the main findings or arguments presented in the paper. The abstract should be no more than 250 words and should not contain any references or citations.

Introduction

The game FlipIt, which goes under many names, is played as follows. In its standard form, the initial setup consists of a grid board of $n \times n$ squares (like an 8×8 chess board for instance) with a double-sided coin on each square. All coins have a black and a white side. In the beginning, every coin is on its black side, and the goal is to flip them all to their white side.

The catch is that, once you flip a coin, you also flip its neighbouring coins. In this case, that would be its adjacent (non-diagonal) neighbours. Such a relation between coins and their neighbours is what makes the game tricky to solve.

FlipIt is an easily generalizable game using basic notions of graph theory, where coins and neighbours are defined as vertices and edges of a graph, respectively. Thus, the graph itself is seen as the board where the game is played, each vertex represents a coin, and we can read off from the graph the connections of each coin. This allows the extension of the game to arbitrary "boards" and opens up the possibility of studying its solvability.

The following three sections describe the game and its guaranteed solvability, a remarkable result on its own. They are mostly rewordings of [:)] and are included for clarity and completion. They may be omitted by the reader familiar with the topic.

1. Definitions

We define what a finite, simple and undirected graph is. Such notion will be used throughout the rest of the paper.

Definition 1.1. *$\Gamma = (e(\Gamma), v(\Gamma))$, where $v(\Gamma)$ is a set of vertecies and $e(\Gamma) \subseteq 2^{v(\Gamma)}$ a set of edges. Γ is finite, simple and undirected if $v(\Gamma)$ is finite and $e(\Gamma)$ is a set of unordered tuples.*

Such a graph describes the interaction between each coin and it's neighbours in the general setting. Now, all there is left to define is a notion of winning under this abstracted definition of the game.

Definition 1.2. *Let Γ be a finite, simple, undirected graph. A set $S \subseteq v(\Gamma)$ is a solution to the game FlipIt played in Γ if the following equation is satisfied for all choices of $a \in v(\Gamma)$.*

$$\#(S \cap (\{a\} \cup \{b \in v(\Gamma) | \{a, b\} \in e(\Gamma)\})) \equiv 1 mod(2) \tag{1}$$

The above definition translates to the fact that, in order to solve the game, each coin has had to be flipped an odd amount of times, and thus its white side is facing up at the end (1 holding $\forall a \in v(\Gamma)$ guarantees such thing). It is important to note that a coin flipped an even amount of times yields the same result as not flipping the coin at all.

2. The game in $\mathbb{F}_2^n$

We associate the game with $\mathbb{F}_2^n$. Here there are two things to consider, the vertices we choose, V, and the state of the game.
First we order the vertices as

$$v(\Gamma) = \{v_1, v_2, \dots, v_n\}$$

Each v_i is associated with the i_{th} coordinate of a vector in $\mathbb{F}_2^n$. The Set V of chosen vertices is thus associated with a vector in $\mathbb{F}_2^n$ by:

$$\begin{cases} x_i = 1 & \text{, if } v_i \in V \\ x_i = 0 & \text{, otherwise} \end{cases}$$

This is clearly a bijection from $2^{v(\Gamma)}$ to $\mathbb{F}_2^n$ since the set V.

We can also associate the state of the game with a vector in $\mathbb{F}_2^n$ by the same rule but instead of the set V we choose the set of all vertices that are already flipped. This forms a subspace of $\mathbb{F}_2^n$.

Solvability

A remarkable property of this game is that, for any connected simple graph, it is solvable. in this section we will prove this:

Theorem 2.1. *let $\Gamma = (v(\Gamma), e(\Gamma))$ be a graph, then:*
Γ is simple and connected $\implies \exists S \subseteq v(\Gamma)$ such that

$$\#(S \cap (\{a\} \cup \{b \in v(\Gamma) | \{a, b\} \in e(\Gamma)\})) \equiv 1 mod(2) \tag{2}$$

for all vertecies $a \in v(\Gamma)$.

Recall that satisfying equation 2 is equivalent to solving the game flip it on Γ. Thus the conclusion of the theorem can be restated as *The game flipit admits a solution on* Γ.

2.1 The φ map

We define a map sending and ordered tuple of vertecies, the ones we choose to flip, to states of the game.

Definition 2.2. *let $x \in \mathcal{V}$ be the set of vectors representing the vertices we can choose to flip(this is $\mathbb{F}_2^n$), we define the map $\varphi : \mathcal{V} = \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $x \mapsto r \in \mathbb{F}_2^n$ where r is the state of the game after flipping the vertices x.*

It should be emphasized that the elements in the domain of this map represent the vertecies select to flip and its image represents the possible outcomes after the fliping.

Lemma 2.3. *The map φ is linear.*

Proof. let $x, y \in \mathbb{F}_2^n$ be the vertices selected. note that this is a linear space over $\mathbb{F}_2^n$ and $(x + y) \in \mathbb{F}_2^n$ is such that $(x + y)_i = 1$ only if $x_i \neq y_i$ and zero otherwise. Hence $\varphi(x + y)$ is the same as selecting first flipping the vertices in x and then the vertices in y, if the vertices selected are both in x and y then they flip the same neighbouring vertices. therefore $\varphi(x + y) = \varphi(x) + \varphi(y)$ and hence φ is linear. $\square$

Definition 2.4. *we define the matrix representation of φ as:*

$$A = [\varphi] = \begin{bmatrix} \varphi(e_1) & \varphi(e_2) & \dots & \varphi(e_n) \end{bmatrix} \tag{3}$$

where e_i are the standard basis vectors in $\mathbb{F}_2^n$.

Note that since we take e_i to be the standard basis vectors, the diagonal elements of A are all one.

Lemma 2.5. *A is symmetric*

Proof.

$$a_{ij} = 1 \iff (i = j) \vee \{v_i, v_j\} \in e(\Gamma) \iff a_{ji} = 1 \tag{4}$$

□

The linear map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ defined above solves the linear equation $\varphi(x) = y$ for the case $y = 1$. If we require φ to be surjective(bijective), it becomes clear that a solution must exist for any final state y. Moreover if φ is bijective, it is invertable and "one-to-one" which immediately yields that a solution(for any final state) exists, is unique and can be computed by inverting the matrix representation $[\varphi]$.
As may be expected the behaviour of the map φ depends on the rules of the game(the edges of the graph). In the following we define a structure which...

Proposition 2.6. *notes theorem*

Proof. (**Of theorem 1**)
the Game on Γ is solvable if $\exists x \in \mathbb{F}_2^n$ such that

$$Ax = \vec{1} \tag{5}$$

By the lemmas below()change order), $\vec{1} = diag(A)$. By theorem from Notes(0state) this equation alwayes has a solution. □

2.2 Examples

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What does uniqness mean?

Given a solution S, defined as a set of vertecies. S is said to be uniqe if no other collection of vertecies satysfies equation 2 (wins the game). Note that the order in which we select the vertices does not matter thus a solution can be defined as a set.

2.3 uniqness in special settings

2.3.1 Straight graph

We define straight graphs as a chain of vertices In order to construct solutions for such graphs, we need to distinguish 2 cases: n divisible by 3 and n not divisible by 3. We will later show how the solution is not unique when $n = 2mod3$.

Proposition 2.7. *If $3|n$ the solution vector has the following form: $x = (x_1, x_2, ..., x_n)$ where $x_{3k-1} = 1$ for $k \in \{1, 2, .., n/3\}$ and all other coordinates are zero.*
If $3 \nmid n$, then $x = (x_1, x_2, ..., x_n)$ where $x_{3k-2} = 1$ for $k \in \{1, 2, .., \lceil n/3 \rceil\}$ and all other coordinates are zero, is a solution.

Proof. Let $[\varphi]$ be the n*n symmetric matrix representation of the linear map φ where $n = 0mod3$ and φ acts on a n-long straight graph. Then,

$$[\varphi]x = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_3 + x_4 + x_5 \\ x_4 + x_5 + x_6 \\ x_5 + x_6 + x_7 \\ \vdots \\ x_{n-1} + x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{6}$$

Note how for a fixed i, $(\varphi)_{ij} x_s = 1 \iff s = j$, otherwise either $(\varphi)_{ij} = 0$ or $x_s = 0$. The proof for $3 \nmid n$ is analogous, again for a fixed i

$\square$

In simpler words, what the aforementioned theorem says is that the solution vector for a straight graph of n vertices, where n=3k for $k \in \mathbb{N}$, you start by having the first coordinate equal to 1 and then leave two spaces of 0's before writing the next 1, and keep filling up your vector until you have a 1 in the last coordinate. For the case which n is not a multiple of 3, you start with a 0, proceed with a 1, and then start like before leaving two spaces of 0's before you write the next one, and your last coordinate will be a zero.

Theorem 2.8. ***(Uniqueness in Straight Graphs)***
Let Γ be a straight graph of length $n \geq 3$, Then:
$n \not\equiv 2mod(3) \implies$ the Matrix representation of φ is invertable.

Proof. Note that the matrix can be easily computed from the defining equation 3. We take the linear combination of the columns.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ a_1 + a_2 + a_3 \\ a_2 + a_3 + a_4 \\ a_3 + a_4 + a_5 \\ \vdots \\ a_{n-2} + a_{n-1} + a_n \\ a_{n-1} + a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Assume that $[\varphi]$ is singular. This implies that there exists a non-trivial solution to equation 7. Note that if the first two or last two values are zero then all subsequent values must also be zero and the solution is trivial. Thus assume that they are non-zero. This implies that $a_1 = a_2 = 1$, to satisfy the first coordinate. Then necessarily $a_3 = 0$ and so on.
We claim that $a_k = 0$ whenever $k \equiv 0 mod(3)$. This implies that $a_{k+1} = a_{k+2} = 1$ to satisfy the $(k+1)_{th}$ coordinate but also $a_{k+1} + a_{k+2} + a_{k+3} = 0$ thus $a_{k+3} = 0$ which shows the claim.
Consider the last column, by our assumption it is necessary that $a_{n-1} = a_n = 1$, however, to satisfy the $(n-1)_{th}$ coordinate we have that that $a_{n-2} = 0$. By the claim above, this means that $n - 2 \equiv 0\text{mod}(3)$. which implies that $n \equiv 2\text{mod}(3)$. By contraposition, this proves the proposition. □

Corollary 2.9. *For a straight graph of length $n \geq 3$ such that $n \not\equiv 2 mod(3)$, the solution to FlipIt is unique.*

Proof. Apply Theorem 2.8. This implies that $[\varphi]$ is invertible and thus a bijection. □

2.3.2 Looopie loopie graph

A loopie loopie graph $(v(\Gamma), e(\Gamma))$ is defined as follows; given a finite set of n vertecies $v(\Gamma) = \{v_0, v_2, v_3, .., v_{n-1}\}$ the set of edges is then defined to be $e(\Gamma) = \bigcup_{k \in \mathbb{F}_n} \{\{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}\}$. All that this cumbersome definition says is that a loopie loopie graph is a "circular" arrangement of vertices where each one is connected only to its two closest neighbours(like hours on a clock).
The game flipit played on this graph is relatively simple and a sure way to obtain a solution is known. First, we make the observation that for any loop of length divisible by 3, pressing every third vertex wins the game. It also becomes apparent that for any loop we must press at least every third element since each press turns around exactly three elements. Hence $\#v(\Gamma)/3 \leq \#S$ for any solution S. Note that for graphs of length divisible by 3 the solution is not unique since we can always start with another vertex and press every third vertex thereafter. In fact, for this case, there are at least 4 distinct solutions; three of which arise by pressing every third vertex.
For this case we can also see that the matrix A is singular by applying the row operation of adding every third row to the first one we arive at a zero row.

Proposition 2.10. *For a loopie loopie graph Γ, the set $S = v(\Gamma)$ wins the game*

Proof. Winning the game is equivalent to satisfying equation 2. Let $v_k \in v(\Gamma) = S$ be arbitrary, then;

$$\begin{aligned}
&\#(\mathcal{S} \cap (\{v_k\} \cup \{b \in v(\Gamma) | \{v_k, b\} \in e(\Gamma)\})) \\
&= \#(v(\Gamma) \cap (\{v_k\} \cup \{\{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}\})) \\
&= \#\{v_{k-1}, v_k, v_{k+1}\} \\
&= 3 \equiv 1 \text{mod}(2)
\end{aligned}$$

Since we took v_k to be arbitrary this holds for all $v_k \in v(\Gamma)$. Hence, $S = v(\Gamma)$ always satisfies 2 and wins the game. $\square$

This means that for any loopie loopie graph pressing all vertices solves the game. This is a strong result since it is always true. It follows that if a solution is unique, it must be this one.
The vivid reader may have realized that this result may be generalized.

Theorem 2.11. *Given any finite graph ζ with the property that each vertex has an even amount of neighbours, then $S = v(\zeta)$ wins the game flip it on ζ.*

Proof. let $v(\zeta) = \{v_0, v_1, .., v_{n-1}\}$. fix $v_k \in v(\zeta)$ to be arbitrary, and let $B = \{b \in v(\zeta)\ s.t\ \{v_k, b\} \in e(\zeta)\} \subseteq v(\zeta) = S$ be the neighboring vertices of v_k. Note that $v_k \notin B$. By assumption $\#B \equiv 0\text{mod}(2)$, thus:

$$\#(\mathcal{S} \cap (\{v_k\} \cup B)) = \#(\{v_k\} \cup B) \equiv 1\text{mod}(2)$$

$\square$

UNIQNESSSSSSSSSSS the for the loopie loopie graphs the matrix represetnation of φ is:

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1
\end{pmatrix}$$

we show that if n is not a multiple of 3 then the matrixs is non singular and hence φ a bijection and the solution given in Theorem llll is uniqe. Consider the linear combination of the columns:

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \ldots + a_n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence we obtain:

$$\begin{bmatrix} a_1 + a_2 + a_n \\ a_1 + a_2 + a_3 \\ a_2 + a_3 + a_4 \\ a_3 + a_4 + a_5 \\ a_4 + a_5 + a_6 \\ \vdots \\ a_{n-2} + a_{n-1} + a_n \\ a_{n-1} + a_n + a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

We can directly see that exactly three values contribute to each coordinate. Thus equation 8 is satisfied whenever either all zero all exactly two of them are non-zero. Considering the first coordinate we can derive the subsequent values.

First, consider the case that $a_1 = a_2 = a_n = 0$, this directly implies that $a_3 = 0$, to satisfy the second coordinate beeing zero. This in turn implies that $a_4 = 0$ to satysfy the third coordinate and so on we find that all $a_i = 0$. This forces the trivial solution.

Now we consider the case that exatly two are non zero:

Choose $a_1 = a_2 = 1$ and $a_n = 0$ to fix the first coordinate to be zero. This makes $a_3 = 0$ hence $a_4 = a_5 = 1$ and likewise $a_6 = 0$ and so on we find that every third coordinate must be zero but since we chose a_n to be zero this implies that $n \equiv 0 \text{mod}(3)$, otherwise we arrive at a contradiction.

Similarly if we choose $a_2 = a_n = 1$ and $a_1 = 0$ then $a_3 = 1$ and $a_4 = 0$ which implies that $a_5 = a_6 = 1$ and $a_7 = 0$ and so on we find that $a_k = 0$ only if $k \equiv 1 mod(3)$. Similarly, this implies that $n + 1 = 1 \equiv 1 \text{mod}(3)$, thus $n \equiv 0 \text{mod}(3)$.

Finally, if we choose $a_1 = a_n = 1$ and $a_2 = 0$ we find that $a_3 = a_4 = 1$ and $a_5 = 0$, ... Thus every coordinate 2modulo(3) must be zero. and hence, again, $n \equiv 0 \text{mod}(3)$.

Hence the matrix A has trivial kernel if $n \equiv 0 \text{mod}(3)$.

We conclude that the matrix A is singular if and only if n is divisible by 3. In turn, A is non-singular and the map φ is bijective if and only if n is not a multiple of 3.

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References