The Study of FlipIt using Linear Algebra over $\mathbb{F}_2$

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Abstract

We layout and explain how the game FlipIt is generalized to "arbitrary" boards and tackle solvability and uniqueness of solutions for two particular types of graphs. The grid case is briefly mentioned, and the reader's attention is directed to some other related studies.

Introduction

The game FlipIt, which goes under many names, is played as follows. In its standard form, the initial setup consists of an $n \times n$ grid board (for instance, like an 8×8 chess board) with a double-sided coin on each square. Every coin has a black and a white side. In the beginning of the game, all coins are placed on their black side, and the goal is to flip them all to their white side.

The catch is that, once you flip a coin, you also flip its neighbouring coins. In this case, that would be its adjacent (non-diagonal) neighbours. Such a relation between coins and their neighbours is what makes the game tricky to solve.

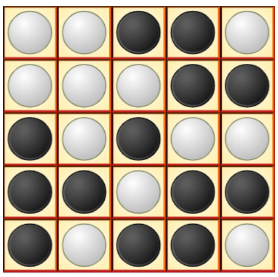


Figure 1: A 5×5 board[3]

FlipIt is an easily generalizable game using basic notions of graph theory, where coins and neighbours are defined as vertices and edges of a graph, respectively. Thus, the graph itself is seen as the board where the game is played, each vertex represents a coin, and we can read off from the graph the connections between the coins. This allows the extension of the game to arbitrary "boards" and opens up the possibility of studying its solvability on particular cases, i.e, which coins are meant to be flipped in order to have them all on their white sides. As it turns out, the standard [grid] form of the game is quite complicated to solve, or at least to give a constructive proof on how to do so. Therefore, we will start by first studying "easier" cases of the game. Towards the end, the grid case will be mentioned but not thoroughly investigated.

The following three sections describe the game and its guaranteed solvability, a remarkable result on its own. They are mostly rewordings of [5] and are included for clarity and completeness. They may be omitted by the reader familiar with the topic.

1. Definitions

We define what a finite, simple and undirected graph is. Such notion will be used throughout the rest of the paper.

Definition 1.1. *(Graph)*
A graph is a pair $\Gamma = (v(\Gamma), e(\Gamma))$, where $v(\Gamma)$ is a set of vertecies and a set of pairs $e(\Gamma) \subseteq 2^{v(\Gamma)}$ is denoted as the set of edges. Γ is finite, simple and undirected if $v(\Gamma)$ is finite and $e(\Gamma)$ is a set of distinct, unordered tuples.

Such a graph describes the interaction between each coin and its neighbours in the general setting.
Now, all there is left to define is the notion of *winning* under this abstracted definition of the game. It should be noted that the order in which we flip the coins does not matter and that flipping a coin twice is the same as not flipping it at all. Thus we define a solution as the set of vertices we choose to "flip".

Definition 1.2. *(Solution)*
Let Γ be a finite, simple, undirected graph. A set $S \subseteq v(\Gamma)$ is a solution to the game FlipIt played on Γ if the following equation is satisfied for all choices of $a \in v(\Gamma)$.

$$\#(S \cap (\{a\} \cup \{b \in v(\Gamma) | \{a, b\} \in e(\Gamma)\})) \equiv 1 \mathit{mod(2)} \tag{1}$$

The above definition embodies the fact that, in order to solve the game, each coin has to be flipped an odd amount of times. This leaves every coin with its white side facing up (eq.1 holding $\forall a \in v(\Gamma)$ guarantees such thing).

2. The game in $\mathbb{F}_2^n$

We associate the game with $\mathbb{F}_2^n$. Here there are two things to consider, the vertices we choose (a set V) and the state of the game.
First we order the vertices as

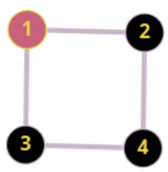
$$v(\Gamma) = \{v_1, v_2, \dots, v_n\}$$

Each v_i is associated with the i_{th} coordinate of a vector in $\mathbb{F}_2^n$. The Set V of chosen vertices is thus associated with a vector in $\mathbb{F}_2^n$ by:

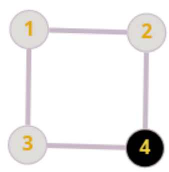
$$\begin{cases} x_i = 1 & \text{, if } v_i \in V \\ x_i = 0 & \text{, otherwise} \end{cases} \tag{2}$$

This is clearly a bijection from $2^{v(\Gamma)}$ to $\mathbb{F}_2^n$ since the set V is any collection of vertices.

We can also associate the state of the game with a vector in $\mathbb{F}_2^n$ by the same rule but instead of the set V we choose the set of all vertices that are already flipped. This forms a subspace of $\mathbb{F}_2^n$.
It should be noted that hereafter we work over the binary field $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$. We also refer to "solving" or "winning" the game, these are understood to be equivalent to satisfying equation 1. Moreover, a notion of "flipping vertices" is also used throughout the paper. The reader can think of it as having coins placed on top of each vertex of the graph and playing in such a way. The authors of the paper found it convenient to use such terminology.



(a) Selecting vertex 1, [1]



(b) Action of flipping vertex 1

Figure 2: FlipIt on a 2×2 grid

Consider the example in Figure 2. By the rule above we associate the vertices chosen (Figure 2a) with the vector $(1, 0, 0, 0)^T$; and the state of the board after flipping the first vertex (Figure 2b) with the vector $(1, 1, 1, 0)^T$.

3. Solvability

A remarkable property of this game is that it is solvable for any connected simple graph. In other words, FlipIt always has a solution!
This is stated in Theorem 3.7. We first introduce results required to prove it. Unfortunately, the proof of this is non-constructive and does not give insight on how to solve the game.

3.1 The map φ

We define a map sending an ordered n-tuple of vertices, the ones we choose to flip, to the resulting state of the game.

Definition 3.1. *The map φ*
Let V be the set of vectors representing the vertices we can choose to flip (this is $\mathbb{F}_2^n$) and let $x \in V$, we define the map $\varphi : V = \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $x \mapsto r \in \mathbb{F}_2^n$ where r is the state of the game after flipping the vertices x.

It should be emphasized that the elements in the domain of this map represent the vertices selected to flip and its image represents the possible outcomes after the flipping. Next, we prove a key property of the map.

Lemma 3.2. *(Linearity)*
The map φ is linear.

Proof. Let $x, y \in \mathbb{F}_2^n$ be the vertices selected. Note that this is a linear space over $\mathbb{F}_2$. Therefore, $(x + y) \in \mathbb{F}_2^n$ and is such that $(x + y)_i = 1$ only if $x_i \neq y_i$ and exactly one of them is one. Hence $\varphi(x + y)$ is the same as first flipping the vertices in x and then the vertices in y. If the vertices selected are both in x and y then they flip the same neighbouring vertices (and themselves). Therefore $\varphi(x + y) = \varphi(x) + \varphi(y)$ and hence φ is linear. □

This leads us to consider the matrix representation of φ.

Definition 3.3. *(The matrix representation of φ)*
We define the matrix representation of φ as:

$$A = [\varphi] = \begin{bmatrix} \varphi(e_1) & \varphi(e_2) & \dots & \varphi(e_n) \end{bmatrix} \tag{3}$$

where e_i are the standard basis vectors in $\mathbb{F}_2^n$.

Each e_i corresponds to only the i_{th} vertex being chosen and the columns of A represent the state of the game after flipping only the i_{th} vertex. Note that since we take e_i to be the standard basis vectors, **the diagonal elements of A are all ones**. An important observation is that A is symmetric.

Lemma 3.4. *(Symmetric property)*
A is symmetric

Proof.

$$a_{ij} = 1 \iff (i = j) \vee \{v_i, v_j\} \in e(\Gamma) \iff a_{ji} = 1 \tag{4}$$

□

Exploiting such a strong property of A allows us to show that a solution must exist. In addition, constructive solutions to certain "simple" graphs can be worked out.

Theorem 3.5. *(linear equation)*
Suppose $A = (a_{i,j})$ is a symmetric $n \times n$ matrix with coefficients in the finite field $\mathbb{F}_2$. Let $\boldsymbol{d} = (d_{1,1}, d_{2,2}, \dots, d_{n,n})^T$ be the diagonal of A, which we regard as a column vector. Then the equation $A\boldsymbol{x} = \boldsymbol{d}$ admits a solution $\boldsymbol{x} \in \mathbb{F}_2$ (here $\boldsymbol{x}$ is a column vector).

The proof may be found in the lecture notes (Theorem 3.5 of [4]). Now we have sufficient results to prove that a solution always exists.

Remark 3.6. *By the bijective association of subsets of $v(\Gamma)$ with vectors in $\mathbb{F}_2^n$ defined in 2, we can equivalently define a solution to the game as a vector $\boldsymbol{x} \in \mathbb{F}_2^n$ such that $A\boldsymbol{x} = \boldsymbol{1}$.*

Theorem 3.7. *(**Solvability of FlipIt**)*
Let $\Gamma = (v(\Gamma), e(\Gamma))$ be a finite, simple, undirected graph, $\implies \exists x \in \mathbb{F}_2^n$ such that

$$A\boldsymbol{x} = \boldsymbol{1} \tag{5}$$

Satisfying equation 5 is equivalent to solving the game flip it on Γ. Thus the conclusion of the theorem can be restated as *The game FlipIt always admits a solution on any finite and simple* Γ.

Proof. Note that, by lemma 3.4 A is symmetric. By definition 3, A has ones as its diagonal entries. This gives $diag(A) = \mathbf{1}$. By Theorem 3.5, equation 5 has a solution in $\mathbb{F}_2^n$. □

The linear map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ defined above solves the linear equation $\varphi(\mathbf{x}) = \mathbf{y}$ for the case $\mathbf{y} = \mathbf{1}$. If we require φ to be surjective, it becomes clear that a solution must exist for any final state y. Moreover, if φ is bijective, it is invertable and "one-to-one" which immediately yields that a solution (for any final state) exists, is unique and can be computed by inverting the matrix representation $[\varphi]$.
As may be expected, the behaviour of the map φ depends on the layout of the graph.

3.2 Elementary example

After having laid out all the definitions and theory on how the game is played, the reader may find it useful to go through an example. Consider a 3×3 board:

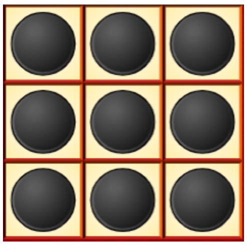


Figure 3: Initial board set up

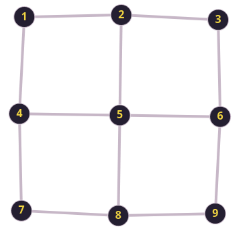


Figure 4: Underlying graph

At first, as explained above, we find the underlying graph (figure 4) that describes the neighbouring connections between coins. The case for an $n \times n$ grids isn't too troublesome. From this, one can easily read the neighbouring connections of each vertex and construct the matrix $[\varphi]$.

$$[\varphi] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{6}$$

As previously mentioned, each column j represents the action of flipping said vertex j. For example, turning vertex 1 flips the vertices 1, 2 and 4. Hence the first column of $[\varphi]$. Moreover, note how $[\varphi]$ is the sum of the Adjacent Matrix of the Graph plus the Identity Matrix. Now, playing around for a bit one will eventually find that pressing the vertices $\{1, 3, 5, 7, 9\} = S$ flips all coins white. Thus, $\mathbf{x} = (1, 0, 1, 0, 1, 0, 1, 0, 1)$ solves the game, i.e, $\varphi(\mathbf{x}) = \mathbf{1}$.

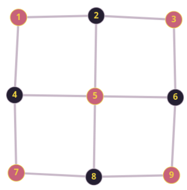


Figure 5: Selected vertices

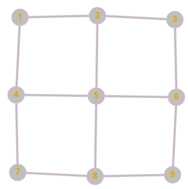


Figure 6: Solved graph

4. Solutions and Uniqueness in Special Graph Settings

Let S be a solution defined as a set of vertices. S is said to be unique if no other collection of vertices satisfies equation 1 or equivalently its associated vector which satisfies equation 5, is unique. These two notions of winning are used interchangeably in the following section.

4.1 Straight graphs

We define straight graphs as a chain of $n \geq 3$ vertices (see figure 7).

Figure 7: Straight graph, $n = 4$

In order to construct solutions for such graphs we distinguish 2 cases: n divisible by 3 and n not divisible by 3. We will later deal with uniqueness of solutions.

Proposition 4.1. *If* $3|n$, *the solution vector has the following form:* $\boldsymbol{x} = (x_1, x_2, ..., x_n)^T$ *where* $x_{3k-1} = 1$ *for* $k \in \{1, 2, .., n/3\}$ *and all other coordinates are zero.*

In simpler words, what the aforementioned proposition says is that the solution vector $\mathbf{x}$ for a straight graph, where $n = 3k$ for $k \in \mathbb{Z}_{\geq 1}$, is constructed according to the following procedure: You start by having the first coordinate $x_1 = 0$ followed by $x_2 = 1$ and then leave two spaces of coordinates being zeroes before writing the next one, and so on until you fill up the whole vector. Naturally, since n is a multiple of 3, following this procedure will result in $x_n = 0$. For example, for $n = 6$ the vector would look as follows: $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T = (0, 1, 0, 0, 1, 0)^T$.

Proof. Let $[\varphi]$ be the $n \times n$ matrix representation of the linear map φ where $n \equiv 0 \bmod 3$ and φ acts on an n-long straight graph. Then,

$$[\varphi]\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_2 + x_3 + x_4 \\ x_3 + x_4 + x_5 \\ x_4 + x_5 + x_6 \\ \vdots \\ x_{n-1} + x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{7}$$

Where the matrix is computed in accordance with definition 3. Without loss of generality, the vertices are taken to be ordered in a path from one end of the straight graph to the other.

Claim: By the way we defined $\mathbf{x}$, equation (6) holds for any $k \geq 1$, where $n = 3k$.
Proof: By induction, we see that for $n = 3$ ($n = 3k$, where $k = 1$), $x_1 = 0$, $x_2 = 1$ and the next two coordinates of $\mathbf{x}$ are meant to be 0. The vector has length three and thus $x_3 = 0$ and we are done.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0+1+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \end{bmatrix}$$

Thus, we have shown that for the base case $k = 1$ our construction of $\mathbf{x}$ solves the game. We assume that for $n = 3k$ the statement is true, and we show it holds for the case $n = 3(k+1)$.

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3k} \\ x_{3k+1} \\ x_{3k+2} \\ x_{3k+3} \end{bmatrix} = \begin{bmatrix} \\ \mathbf{1} \\ \\ x_{3k} + x_{3k+1} + x_{3k+2} \\ x_{3k+1} + x_{3k+2} x_{3k+3} \\ x_{3k+2} + x_{3k+3} \end{bmatrix}$$

By our assumption $x_{3k-1} = 1$ which makes both $x_{3k} = x_{3k+1} = 0$ and $x_{3k+2} = 1$. Finally, $x_{3k+3} = 0$ and therefore:

$$\begin{bmatrix} \\ \mathbf{1} \\ \\ x_{3k} + x_{3k+1} + x_{3k+2} \\ x_{3k+1} + x_{3k+2} x_{3k+3} \\ x_{3k+2} + x_{3k+3} \end{bmatrix} = \begin{bmatrix} \\ \mathbf{1} \\ \\ 0+0+1 \\ 0+1+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \end{bmatrix}$$

By the principle of mathematical induction, we have shown the claim holds true $\forall k \in \mathbb{Z}_{\geq 1}$.
■
Having proved the claim, we see that that $[\varphi]\mathbf{x} = \mathbf{1}$ $\forall n \in 3\mathbb{Z}_{\geq 3}$ and we are done □

The way of construction of a solution for $3 \nmid n$ is almost identical and is discussed hereunder.

Proposition 4.2. *If* $3 \nmid n$, *then* $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ *where* $x_{3k-2} = 1$ *for* $k \in \{1, 2, .., \lceil n/3 \rceil\}$ *and all other coordinates are zero, is a solution.*

Note how the main difference in the construction of $\mathbf{x}$ when $3 \nmid n$ is that instead of starting with a zero, we start with a 1 and then proceed as above leaving two coordinates being zero before having the next 1 up until termination. Finally, $3 \nmid n$ implies either $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}_{\geq 1}$. For the first case, $x_n = 1$ and for the latter, $x_n = 0$. This will later on have some connection with the uniqueness of the solutions.

Proof. The proof is directly analogous to the case where $3|n$ and it boils down to the same principle, showing the constructed vector $\mathbf{x}$ is a solution $\forall n \in \mathbb{Z}_{\geq 3}$ inductively. □

4.1.1 Uniqueness in Straight Graphs

Having constructed solutions for all cases of straight graphs, a natural question to ask is whether they are unique. It turns out that, for the case $n \not\equiv 2\text{mod}(3)$, this is true. We show this by considering the matrix $[\varphi]$ (see the proof of proposition 4.1).

Theorem 4.3. ***(Uniqueness in Straight Graphs)***
Let Γ *be a straight graph of length* $n \geq 3$, *Then:*
$n \not\equiv 2mod(3) \implies$ *the Matrix representation of* φ *is invertable.*

Proof. Note that the matrix can be easily computed from the defining equation 3. We take the linear combination of the columns.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ a_1 + a_2 + a_3 \\ a_2 + a_3 + a_4 \\ a_3 + a_4 + a_5 \\ \vdots \\ a_{n-2} + a_{n-1} + a_n \\ a_{n-1} + a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Assume that $[\varphi]$ is singular. This implies that there exists a non-trivial solution to equation 8. Note that if the first two or last two values are zero then all subsequent values must also be zero and the solution is trivial. Thus, assume that they are non-zero. This implies that $a_1 = a_2 = 1$, to satisfy the first coordinate. Then, necessarily $a_3 = 0$ and so on.
We claim that $a_k = 0$ whenever $k \equiv 0mod(3)$. This implies that $a_{k+1} = a_{k+2} = 1$ to satisfy the $(k+1)_{th}$ coordinate but also $a_{k+1} + a_{k+2} + a_{k+3} = 0$ thus $a_{k+3} = 0$ which shows the claim.

Consider the last coordinate, by our assumption, it is necessary that $a_{n-1} = a_n = 1$, however, to satisfy the $(n-1)_{th}$ coordinate we have that $a_{n-2} = 0$. By the claim above, this means that $n - 2 \equiv 0 \text{mod}(3)$, which implies that $n \equiv 2\text{mod}(3)$. By contraposition, this proves the proposition. □

Corollary 4.4. *For a straight graph of length $n \geq 3$ such that $n \not\equiv 2$mod(3), the solution to FlipIt is unique.*

Proof. Apply Theorem 4.3. This implies that $[\varphi]$ is invertible and thus a bijection. □

(Explain why for the case n=2mod3 has two solutions, the second one mainly being shitfing all ones 1 position)

4.2 Loopie graphs

A loopie graph $(v(\Gamma), e(\Gamma))$ is defined as follows; given a finite set of n vertecies $v(\Gamma) = \{v_0, v_2, v_3, .., v_{n-1}\}$ the set of edges is then defined to be
$e(\Gamma) = \{v_n, v_0\} \cup (\bigcup_{k=1}^{n-1} \{\{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}\})$. All that this cumbersome definition says is that a loopie graph is a "circular" arrangement of vertices where each one is connected only to its two closest neighbours (like hours on a clock, see figure 8).

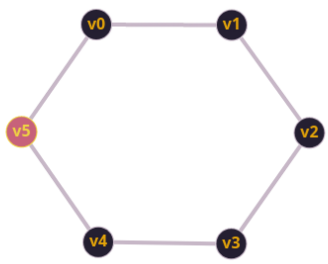


Figure 8: A Loopie graph of length 6

The game FlipIt played on this graph is relatively simple and a sure way to obtain a solution is known. First, we make the observation that for any loop of length divisible by 3, pressing every third vertex wins the game. It also becomes apparent that for any loop we must press at least every third element since each press turns around exactly three elements. Hence $\#v(\Gamma)/3 \leq \#S$ for any solution S. Note that for graphs of length divisible by 3 the solution is not unique since we can always start with another vertex and press every third vertex thereafter. In fact, for this case, there are at least 4 distinct solutions; three of which arise by pressing every third vertex.
For this case we can also see that the matrix A is singular by applying the row operation of adding every third row to the first one we arive at a zero row.
After some investigation one may realize that the loopie graph has the property that the

state of each vertex v_k can be affected only by three vertices, v_k, v_{k-1} and v_{k+1}. This leads us to the following observation.

Proposition 4.5. *(Solution for loopie graph)*
For a loopie graph Γ of length $n \geq 3$, the set $S = v(\Gamma)$ wins the game.

Proof. Recall that winning the game is equivalent to satisfying equation 1. Let $v_k \in v(\Gamma) = S$ be arbitrary, then;

$$\begin{aligned}
\#(\mathcal{S} \cap (\{v_k\} \cup \{b \in v(\Gamma) | \{v_k, b\} \in e(\Gamma)\})) \\
= \#(v(\Gamma) \cap (\{v_k, v_{k-1}, v_{k+1}\})) \\
= \#\{v_{k-1}, v_k, v_{k+1}\} \\
= 3 \equiv 1 \text{mod}(2)
\end{aligned}$$

Since we took v_k to be arbitrary this holds for all $v_k \in v(\Gamma)$. Hence, $S = v(\Gamma)$ always satisfies 1 and solves the game. □

This means that for any loopie graph pressing all vertices solves the game. This is a strong result since it is always true. It follows that if a solution is unique, it must be this one. The reader may have realized that the proof requires very few assumptions (concerning loopie graphs) and thus can be generalized.

Theorem 4.6. ***(Solution of even neighboured graphs)***
Given any finite graph ζ with the property that each vertex has an even amount of neighbours, then $S = v(\zeta)$ wins the game FlipIt on ζ.

Proof. let $v(\zeta) = \{v_0, v_1, .., v_{n-1}\}$. fix $v_k \in v(\zeta)$ to be arbitrary, and let $B = \{b \in v(\zeta)\ s.t\ \{v_k, b\} \in e(\zeta)\} \subseteq v(\zeta) = S$ be the neighboring vertices of v_k. Note that $v_k \notin B$ and by assumption $\#B \equiv 0 \text{mod}(2)$, thus:

$$\#(S \cap (\{v_k\} \cup B)) = \#(\{v_k\} \cup B) \equiv 1 \text{mod}(2)$$

□

This last result applies to the game FlipIt on any graph with the property that each vertex has an even amount of neighbours. It provides an easy way to find a solution and similarly to the loopie graph, if the solution is unique [for such a graph] it is this one.

4.2.1 Uniqueness in Loopie Graphs

By computing the matrix representation of φ according to equation 3, we find that the columns have a cyclic nature. This is as should be expected from a loop. The matrix $[\varphi]$

is given below.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

Theorem 4.7. ***(Uniqueness For Loops)***
Given a loopie graph Γ *such that* $\#v(\Gamma) = n \geq 3$, *then:*
$n \not\equiv 0 mod(3) \implies A = [\varphi]$ *is invertable.*

The proof of this is analogous to the proof of Theorem 4.3, however, there are multiple cases to consider and so it is more tedious.

Proof. Assume that A is singular, hence there exists a nontrivial solution to:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-2} \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_n \\ a_1 + a_2 + a_3 \\ a_2 + a_3 + a_4 \\ a_3 + a_4 + a_5 \\ a_4 + a_5 + a_6 \\ \vdots \\ a_{n-2} + a_{n-1} + a_n \\ a_{n-1} + a_n + a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

We can directly see that exactly three values contribute to each coordinate. Thus equation 9 is satisfied whenever either all are zero or exactly two of them are non-zero. Considering the first coordinate we can derive the subsequent values.
First, consider the case that $a_1 = a_2 = a_n = 0$, this directly implies that $a_3 = 0$, to satisfy the second coordinate being zero. This in turn implies that $a_4 = 0$ to satisfy the third coordinate and so on we find that all $a_i = 0$, $\forall i$. This forces the trivial solution. Also note that, by the cyclic nature of the columns, this is true regardless which coordinate we consider first.
Now we consider the case that exactly two are non-zero (we will have to consider all such possibilities):

Choose $a_1 = a_2 = 1$ and $a_n = 0$ to fix the first coordinate to be zero. This makes

$a_3 = 0$ hence $a_4 = a_5 = 1$ and likewise, $a_6 = 0$ and so on. This is identical to the claim in the proof of 4.3. We find that every third coordinate must be zero but since we chose a_n to be zero this implies that $n \equiv 0 \mathrm{mod}(3)$, otherwise we arrive at a contradiction.

Similarly if we choose $a_2 = a_n = 1$ and $a_1 = 0$ then $a_3 = 1$ and $a_4 = 0$ which implies that $a_5 = a_6 = 1$ and $a_7 = 0$. Now we claim that $a_k = 0$ whenever $k \equiv 1 \mathrm{mod}(3)$. Since $a_k = 1 \implies a_{k+1} = a_{k+2} = 1 \implies a_{k+3} = 0$ where $k + 3 \equiv 1 \mathrm{mod}(3)$.
Since $a_1 = 0$ we have that $a_n = a_{n-1} = 1$ thus $a_{n-2} = 0$ thus $n - 2 \equiv 1 \mathrm{mod}(3)$, hence $n \equiv 0 \mathrm{mod}(3)$.

Finally, we choose $a_1 = a_n = 1$ and $a_2 = 0$. Thus $a_3 = a_4 = 1$ and $a_5 = 0$, In perfect analogy to the claims above we have that $a_k = 0$ whenever $k \equiv 2 \mathrm{mod}(3)$. By assumption $a_1 = a_n = 1 \implies n - 1 \equiv 2 \mathrm{mod}(3)$. Hence, again, $n \equiv 0 \mathrm{mod}(3)$.

Hence the matrix A is singular only if $n \equiv 0 \mathrm{mod}(3)$. By contraposition, this proves the theorem.

□

The converse is also true simply by considering the solutions given by pressing every third vertex if $3|n$. This leads us to a complete description of the uniqueness of loopie graphs.

Corollary 4.8. *Let Γ be a loopie graph such that $\#v(\Gamma) = n \geq 3$, then:*
$3 \nmid n \iff$ *The solution to FlipIt played on Γ is unique*

Proof. The forward direction. By Theorem 4.7, A is invertible and φ is a bijection thus the solution is unique.
The reverse direction is shown by contra-position. Assume $n \equiv 0 \mathrm{mod}(3)$ then pressing every third vertex solves the game trivially but pressing all is also a solution thus it is not unique. Hence the solution is unique implies that $3 \nmid n$. □

Remark 4.9. *Recall that if the solution is unique, it has to be $S = v(\Gamma)$ since that is always a solution (proposition 4.5).*

5. Brief mention on solving Grid Graphs

Unlike the graphs defined above, the original game FlipIt played on a grid is much more difficult to solve. No constructive proof of a solution is known to the authors at the present moment, however, a numerical method may be found in [2](Dutch). The author tackles the problem by solving the game in three ways; by Gauß elimination on $[\varphi]$, a direct iterative method and by what is called the Lanczos-algorithm. Moreover, the paper also includes a comparation of their efficiencies by checking computational times on large size grids.

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