MATH 215: CONTINUOUSLY UPDATED NOTES

Contents

Below are continuously updated notes for Math 215.

1. Jan 8: Syllabus day

- Go through syllabus
- Finish with mathematical warmup: what is an even number ($x = 2k$ with $k \in \mathbb{Z}$), what is an odd number ($y = 2k + 1$ with $k \in \mathbb{Z}$). Must every number be even or odd? (Yes, but to prove this will require later material). Idea is to get students thinking about precise, useful definitions and how to prove facts they are used to assuming.

Here $\in$ is read "in" and $\mathbb{Z}$ denotes the set of integers (i.e. the set $\{\dots, -3, -2, -, 1, 0, 1, 2, 3, \dots\}$ of positive whole numbers, zero, negative whole numbers). "$k \in \mathbb{Z}$" therefore means that k is an integer.

2. Jan 10: Statements

- Introduction to *statements* in the mathematical sense (declarative statement that can be assigned a truth value: it is either True or False, not both).
- Building statements from new statements. Negation, and, or. Truth tables below.

P	$\neg P$
T	F
F	T

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- Use examples to motivate logical equivalences. Two things are logically equivalent if they always have the same truth value. This means they can be swapped for one another in statements, expressions, etc. DeMorgan's laws:

$$\neg(A \wedge B) \text{ logically equiv to } \neg A \vee \neg B$$
$$\neg(A \vee B) \text{ logically equiv to } \neg A \wedge \neg B$$

  as well as distributive properties:

$$A \wedge (B \vee C) \text{ logically equiv to } (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \text{ logically equiv to } (A \vee B) \wedge (A \vee C)$$

3. Jan 17: Conditionals

Another example of building new statements from old ones. Given P, Q statements we can form $P \Rightarrow Q$. It has the following truth assignments, depending on those of P, Q.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We discuss what it means to "prove" a statement of the form $P \Rightarrow Q$: this means, show it is always true. Looking at the table above, we only need to worry about landing in the case of $T \Rightarrow F$. So proving a conditional means assuming P is true, and trying to use logical arguments to deduce that Q is true. We see that, for x an integer,

$$x \text{ is a multiple of } 4 \Rightarrow x \text{ is a multiple of } 2$$

is true (information on the left always implies information on the right) , but

$$x^2 > 0 \Rightarrow x > 0$$

is not true (there are situations where $x^2 > 0$ and $x < 0$).

From the conditional $P \Rightarrow Q$, we can define the inverse ($\neg P \Rightarrow \neg Q$) as well as the converse ($Q \Rightarrow P$) and the contrapositive ($\neg Q \Rightarrow \neg P$). We then go through writing their truth values:

P	Q	$P \Rightarrow Q$	$\neg P \Rightarrow \neg Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

From there we see the original $P \Rightarrow Q$ is logically equivalent to the contrapositive. We also see the inverse, converse are logically equivalent to eachother (note that the inverse is the contrapositive of the converse). We discuss how this will be useful: sometimes the contrapositive is way easier to prove than the original statement.

Lastly, if $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then we say $P \iff Q$ (read "P if and only if Q). This means P is true exactly when Q is true and vice versa: i.e. this means P, Q are logically equivalent. So from now on we write P, Q logically equivalent as $P \iff Q$. If trying to prove $P \iff Q$ on a HW/in class: there should be two parts: showing $P \Rightarrow Q, Q \Rightarrow P$.

4. Jan 19: Proof methods part 1 (direct proofs)

Three major proof methods when trying to show $P \Rightarrow Q$

- Direct proof: assume P true, use logical deductions, algebra, lemmas (small results) and theorems from class to try to show Q is true.
- Contradiction: assume P is true but Q is false. Show that this yields a logical contradiction (say, contradict a part of the assumption, or run into a logical fallacy like $0 = 1$). Then the original assumption must have been wrong, and Q is true.
  - Good to start these proofs with "Assume, for the sake of contradiction" or "Suppose P is true but Q were false." Something to indicate to the reader that you are doing a contradiction proof.
  - Colloquially, this also gets used for: if you're trying to show a statement A is true, you assume $\neg A$ is true instead and run into a contradiction.
- Contrapositive: show $\neg Q \Rightarrow \neg P$.

We focus on direct proof today.

Two exercises:

- **Show that x even, y odd $\Rightarrow x + y$ odd.**
- **Show that x odd $\iff x + 2$ odd.**

Proof of the first result: Since x even, y odd: by definition we have

$$x = 2k$$
$$y = 2\ell + 1$$

with k, ℓ integers. Then $x + y = 2k + 2\ell + 1 = 2(k + \ell) + 1$. $k + \ell$ is an integer since k, ℓ are integers. Thus, by definition, $x + y$ is odd. $\square$

Similar definition unwinding yields the second result.

Things to note: using separate variables for writing $x = 2k$, $y = 2\ell + 1$. At each point we are clear about what results/definitions/information/etc we are using. Note that for the second result: make sure proof has two parts.

5. Jan 22: Contradiction, contrapos

This lesson we focus more on contradiction, contrapositive: i.e. the methods that involve some sort of negation.

Warmup: you may assume every integer is exactly one of even or odd. Show that x^2 even $\Rightarrow x$ even.

Proof. (Note that this is hard to do directly! Contrapositive helps flip this into turning info about x into info about x^2 in a pretty straightforward manner.) We use proof by contradiction: we will show x not even $\Rightarrow x^2$ not even. Equivalently, this means showing x odd $\Rightarrow x^2$ odd. If x is odd, then $x = 2k + 1$ for some k an integer. Then:

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

$2k^2 + 2k$ is an integer because k is an integer. So by definition, x^2 is odd and we are done. □

Next, some definitions. A number x is **rational** provided that it can be written as $x = \frac{p}{q}$ where p, q are integers and $q \neq 0$. A number is **irrational** if it is not rational.

Show: if x is rational, y irrational, then $x + y$ is irrational.

One thing that jumps out: hard to do this directly. Contrapositive seems difficult because negative the left side seems tedious. So let's try contradiction. That will turn $x + y$ irrational into $x + y$ rational, which will be nice to work with.

Proof. Suppose, for the sake of contradiction, that x is rational, y irrational, $x+y$ irrational. Then $x = \frac{p}{q}$ and $x + y = \frac{a}{b}$ with p, q, a, b integers and q, b nonzero. Then:

$$y = (x + y) - x = \tfrac{a}{b} - \tfrac{p}{q} = \frac{aq - pb}{bq}.$$

The numerator and denominator are integers since a, b, p, q are. bq is nonzero because b, q are nonzero. But that means y is rational, which contradicts y being irrational. Hence our assumption is false, and $x + y$ must be irrational. □

We end by trying to prove the following: **Show $\sqrt{2}$ irrational.** Start by supposing, for the sake of contradiction, that $\sqrt{2}$ is rational. Then:

$$\sqrt{2} = \frac{p}{q}.$$

Square both sides, get $2 = p^2/q^2$, equivalently $2q^2 = p^2$. Try messing with even-ness, odd-ness to see if can get a contradiction.

(Another fun result one can do with contradiction: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ is never an integer).

6. Jan 24: Finish contradiction. Casework

We finish the proof of $\sqrt{2}$ irrational. Intuitively: look at $2q^2 = p^2$. Look at the prime factorization of each. The number of 2's on the left is odd, the number of 2's in the right is even (because the primes in the factorization of a square all have even power).

Proof that $\sqrt{2}$ irrational. Suppose, for the sake of contradiction, that $\sqrt{2}$ is irrational. Then

$$\sqrt{2} = \frac{p}{q}$$

We shall assume that p, q are in lowest terms. Squaring both sides and rearranging, we get

$$2q^2 = p^2$$

Looking at this equation, p^2 is even. By a result from last class, this means p is even. Write $p = 2k$, k an integer. Then:

$$2q^2 = (2k)^2 = 4k^2.$$

Cancelling a factor of 2, we see

$$q^2 = 2k^2,$$

hence q is even. But if p, q are both even: the fraction couldn't have been in lowest terms! We could cancel a factor of 2 from top and bottom! So we've arrived at a contradiction. Our assumption must be false, and so $\sqrt{2}$ is irrational. □

The heart of what's going on is factorization issues. We'll see more about factorization in the number theory section of the course.

A nice bookend to the proof methods chunk is to cover proof by cases. We show that $n^2 - n$ is always even by looking at even, odd cases. This and proof of $\sqrt{2}$ help motivate induction. Time permitting, show multiple proofs of $n^2 - n$.

7. Jan 26: Induction

Results like:

- all fractions can be put in least terms
- all integers are even or odd

rely on induction (or one of its equivalent formulations: strong induction, well ordering principle). It is an axiom, and a very useful and major method. Likened to "mathematical dominos."

Idea of induction: if a property holds for $k = 1$, and a property holding for k implies it holds for $k + 1$, then we can start at 1 and "domino effect" down to get a property holds for all natural numbers ($\{1, 2, 3, 4, \dots\}$). Good for proving a fact holds for all natural numbers.

Proofs by induction always have two parts: base case (the $k = 1$ part) and inductive step (showing the property holds for k implies the property holds for $k + 1$).

Examples of proofs by induction:

- Show that every integer is even or odd. (casework + induction)
- Show that
$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
for all natural numbers n Note: provides a fun proof that $n^2 - n$ is even as a corollary.
- Show that $k^3 + 2k$ is always divisible by 3.
- Show that
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
for all natural numbers n
- Show that
$$\sum_{k=1}^{n} k(k+1) = (1 \times 2) + (2 \times 3) + (3 \times 4) + \cdots + (n \times (n+1)) = \frac{n(n+1)(n+2)}{3}$$
for all natural numbers n.

Proof of the second statement. We use proof by induction. Base case: note that $1 = \frac{1(2)}{2}$ so the formula holds for $k = 1$.

Inductive step: suppose the formula is true for k. Then:

$$\begin{aligned} 1+2+\cdots+k+k+1 = \frac{k(k+1)}{2}+(k+1) &= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

which means the formula is true for $k+1$. So, by induction, the formula is true for all natural numbers n. $\square$

8. Jan 29: Induction part 2

Start with note about for all, there exists. $P(x)$ being some property of x, etc. Recall $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.

Go over

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

more slowly.

Induction

- Useful for proving things about the natural numbers (and this sometimes lets you yield statements about $\mathbb{Z}, \mathbb{Q}$)
- Useful in situations with *recursive structure* or properties that can "build up"

Let's get some motivation for why this might be true:

- $n = 1$: Well, $1 = 1(1+1)/2$.
- $n = 2$ Well, $1+2 = 3 = 2(2+1)/2$.
- $n = 3$ Well, $1+2+3 = 6 = 3(3+1)/2$.

Grouping trick: first and last add to $n+1$. Second and penultimate add to $n+1$. And so on, and there will be $n/2$ such pairs (if n even get $n/2$ pairs and if n odd get $(n-1)/2$ pairs and a loner with the value of $(n+1)/2$).

Then: circle back to induction and formal proof Say you want a property P to hold for all natural numbers, so write $P(k)$ to denote the property for the natural number k. (ex: $P(k)$ is the property that $1+2+\cdots+k = \frac{k(k+1)}{2}$).

Induction says that if you have the following:

- $P(1)$ is true
- $P(k)$ is true implies $P(k+1)$ is true

then $P(k)$ is true for all natural numbers k. That is, your desired property is true for every natural number. (With our example choice of P, this would mean the formula for $1+\cdots+n$ always holds.

Proof. We use proof by induction. Base case: note that $1 = \frac{1(2)}{2}$ so the formula holds for $k = 1$.

Inductive step: suppose the formula is true for k. Then:

$$\begin{aligned} 1+2+\cdots+k+k+1 = \frac{k(k+1)}{2}+(k+1) &= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

which means the formula is true for $k+1$. So, by induction, the formula is true for all natural numbers n. $\square$

9. Jan 31: Induction, some more examples, strong induction (if time)

Use induction to show all integers are even or odd. The main thing is that we need to start by doing this for all *natural numbers* n, and then handle 0 and negatives separately. The last part is either done by some slightly tedious algebra or just checking that (-1) is odd and citing that odd $\times$ odd, even is odd, even respectively.

This is a special case of the *division algorithm*, which we will see in the next unit (number theory). This is saying we can divide a number by 2 with remainder, and the remainder has the usual size constraints $0 \leq r < b - 1$ with $b = 2$ here.

10. Feb 2: Strong induction

Suppose we want some property P to hold for all natural numbers n. Let $P(n)$ to denote the property for the natural number n. Strong induction says: if you have the following

- $P(1)$ is true
- ($P(k)$ true for $1 \leq k < n \Rightarrow P(n)$ true) is true

then $P(n)$ is true for all natural numbers n.

One of the first applications of this is showing every fraction can be put in least terms. Another classical application is showing the fundamental theorem of arithmetic (every integer decomposes as a unique product of primes. We'll see this later).

Let $P(n)$ be the property that

$$\text{For every } m \in \mathbb{Z},\ m/n \text{ can be put in least terms.}$$

(This is saying that any fraction that can be written with a denominator of n can be written in least terms) We first show that $P(1)$ holds. Certainly $m/1 = 1$ is in least terms; the only factors 1 has is $1, -1$, so we can't do any cancellation from the top and bottom.

Next we show $P(k)$ true for $1 \leq k < n$ implies that $P(n)$ is true (n is an arbitrary natural number). Boils down to: if you can put fractions with denominator $< n$ in least terms, can you put fractions with denominator $= n$ in least terms?

Note that our goal is to show every m/n can be put in least terms. Assume that indeed, $P(k)$ is true for $1 \leq k < n$. Well, either m/n is in least terms (and we are fine), or it is not. In that case,

$$\frac{m}{n} = \frac{m'}{n'}$$

with $1 \leq n < n'$. Note that the fraction on the right has a smaller denominator, so $P(n')$ is true. In particular, m'/n' can be put in least terms. So $m/n = m'/n'$ can be put in least terms. No matter what, we can always put m/n in least terms. So $P(n)$ holds. By strong induction, $P(n)$ holds for all n, and so all fractions can be put in least terms.

(Rephrase to students: we're saying: need to show: if we can put m/k in least terms for every $k < n$, then m/n can always be put in least terms).

End on an example of finding an error in a proof:

False theorem: if the sum of two integers is even, then both integers are even ($m+n$ even $\Rightarrow m, n$ even).

Proof. Assume, for the sake of contradiction, that the result is false, i.e. either m or n is odd. Then $m = 2k + 1$ and $n = 2j$ with $j, k \in \mathbb{Z}$ (swap the label of m and n as needed). Then:

$$m + n = 2(k + \ell) + 1$$

is odd. Contradicts our assumption that $m + n$ is even. So our assumption is false and the theorem is true. $\square$

Where is the error? (Where: in the contradiction setup, assumed *either* m or n is odd. Why: it's assumed precisely one odd one even, when the failure could come from both odd. In failing to account for this, they miss the phenomena that m odd and n odd will yield $m+n$ even, which is where this theorem fails).

Can also write a proof of $\sqrt{m}\sqrt{n}$ an integer, then $\sqrt{m}/\sqrt{n}$ is rational.

Can also do a strong induction example or take HW 1/ HW 2 questions. Strong induction example: We will define a sequence of numbers. Let $a_1 = 1, a_2 = 2$, and then for $n \geq 3$ we set

$$a_n := a_{n-1} + a_{n-2}.$$

Use strong induction to show that $a_n < 2^n$ for all $n \in \mathbb{N}$. (Need two base cases!)

11. Feb 5: Division alg, reducing fractions

Do reducing fractions proof. Then also do division algorithm.

For $a \in \mathbb{N}, b \in \mathbb{N}$ with b nonzero, can find *unique* q, r such that $a = bq + r$ with $0 \leq r < b$. This is the division algorithm. It is division with remainder. q is the quotient, and r is the remainder.

So, with $a = 12, b = 5$ performing the division algorithm is $12 = 5 \cdot 2 + 2$, i.e. we put as many copies of 5 as we can into 12, and then we have a remainder that is non-neg and strictly less than the thing we're dividing with.

With $a = 27, b = 8$, performing the division algorithm yields $27 = 8 \cdot 3 + 3$. When $a = 16, b = 8$ we get $16 = 8 \cdot 2 + 0$.

In fact, can let a just be an integer, no necessarily positive. Still get that $a = bq + r$ with $0 \leq r < b$.

Restrict to $b = 2$ case: Note that diving by 2 always gets a remainder of 0 or 1. i.e. division algorithm being true $\Rightarrow$ every natural number even or odd.

So splitting into even and odd cases in proofs was like splitting into cases based on remainder when dividing by 2. Leads us to another example of useful cases: we can split into cases based on remainder when dividing by, say, 3 or 5.

Example: Can split into cases by remainder ($3k$, $3k+1$, $3k+2$) to show that $n(n+1)(n+2)$ is always divisible by 3. (You could do this with induction too, but it's a little painful and less intuitive).

12. Feb 7: (Number theory 1) Division algorithm

Hamkins 3

Talk about division with remainder: put as many copies of b into a, get a leftover bit. The remainder should be $< b$, otherwise I could shrink it. Well-ordering might be useful here.

If time, talk about Euclid's algorithm. Have students do examples, look for patterns. Suggest GCD stuff.

13. Feb 9 (Number theory 2): Euclid's algorithm

Show that we can always write GCD(a, b) as a linear combo of a, b by induction. Perhaps do some examples first of the nasty substitution one has to do. Remind them: induction good for things with recursive structure.

14. Feb 12 (Number theory 3):

Fundamental theorem of algebra!!! Unique factorization!!

15. Feb 14 (NT 4/MA 1):

Introduction to modular arithmetic. Clockwork algebra. Motivation from CS. Add/subtract copies of m to get equivalence classes.

16. Feb 16 (NT 5/MA 2):

More modular arithmetic. Squares or powers mod p, taking equations mod p. Give another proof that $n^2 - n$ is always even.

17. Feb 19 (NT 6/MA 3):

Definitely some mod p stuff. Look at *units*. Show that every nonzero element is a unit mod p.

18. Feb 21 (NT 7)/MA 4 :

Perhaps $a^{p-1} \equiv 1 \bmod p$ for all $a \not\equiv 1$.

19. Feb 23 (NT 8)/MA 5:

Maybe fun modular arithmetic stuff not on exam OR preview set theory.
May even want to have a day about how to use LaTeX

20. Feb 26: Exam review

Do practice problems from sheet, discuss proof strategies.

21. Feb 28: N/A due to exam

Administer exam.

22. March 1

Set theory. Taylor 4.

23. March 4

Set theory. Taylor 4.

24. March 6

Set theory. Taylor 4.

25. March 8

Combinatorics. Hamkins 5.6. **Note: as going through this: always be doing concrete examples. Start with way to order 3 objects. Then ways to order 2 things chosen from 3. Then what about choosing 2 from 3 without order? Then what about choosing 3 from 4 without order?** Start with working out counting *permutations*. Ways to order n objects? n options for first, $n-1$ options for second... results in $n(n-1)(n-2)\ldots 2\cdot 1 = n!$. Now suppose you have n objects and want to pick k with order. Then, get

$$n(n-1)(n-2)\ldots(n-(k-1)) = n(n-1)(n-2)\ldots(n-k+1)$$

(have to stop because only choosing k objects). Can express this as:

$$\frac{n!}{(n-k)!}.$$

What about just picking k objects without caring about the order? Maybe you're just picking groceries and tossing them in a cart. Don't really care what order they came in. Well, think about a collection of k objects. The above quantity counts it multiple times, because it cares about all the orders they came in. $k!$ ways to reorder it, so the above counts the same grocery cart $k!$ times. Need to divide out by this redundancy.

$$\binom{n}{k} = \text{number of ways to choose } k \text{ objects from } n = \frac{n!}{(n-k)!k!}$$

These are binomial coefficients One really fun consequence: this number above, which *a priori* looks like a fraction, in fact has to be a integer! It's counting an integer quantity! Super fun!

26. March 11

Combinatorics. Hamkins 5.6

Today: the fun of combinatorics: can bash things out with algebra, or do clever proofs with words!!

Do Pascal's triangle $\binom{n}{k}$ with $n \geq k \geq 0$. Row number is index of n on top. Observe an additive pattern. Then: two proofs!

One way: note that

$$\binom{n+1}{k} = \frac{(n+1)!}{(n+1-k)!k!}$$

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-(k-1))!(k-1)!} + \frac{n!}{(n-k)!k!} + \\
&= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!} \\
&= \frac{n!(n+1)}{(n-k+1)!k!} \\
&= \frac{(n+1)!}{((n+1)-k)!k!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

But perhaps an easier way to see it is: $\binom{n+1}{k}$ counts the number of ways to choose k objects from $n+1$. Consider this new added $n+1$-th object. When you pick the k objects, you can include this new object or not. If you do, you are picking the remaining $k-1$ objects from the remaining n objects. If you do not pick it, you are choosing k objects from the n original objects. These two scenarios are disjoint and cover everything. Hence you get Pascal's formula.

(Draw this with circles and dot-dot-dots)

27. March 13

Combinatorics

28. March 15

Combinatorics

29. March 25

Functions and equivalence relations

30. March 27

Functions and equivalence relations

31. March 29

Functions and equivalence relations

32. April 1

Functions and equivalence relations

33. April 3

Functions and equivalence relations: now in combination with modular arithmetic!

34. April 5

Graph theory

35. April 8

Graph theory

36. April 10

Graph theory

37. April 12

Graph theory

38. April 15

Graph theory

39. April 17

Finite games

40. April 19

Finite games

41. April 22

Finite games

42. April 24

Finite games

43. April 26: Review for final exam

Go over practice problems and strategies