MATH 553: LECTURE NOTES

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Abstract. Notes for 553: algebraic geometry II, spring 2025. Heavily referenced from Vakil, Hartshorne. May contain typos.

Contents

1. Jan 13: Syllabus, sheaves

Up till now, you have been thinking of algebraic varieties more in the classical sense– they're zero sets of polynomials. $V(f_1, \dots, f_n)$. From your perspective, in $k[x]$ you don't really care too much about x versus x^2 because their vanishing sets are the same, maybe you'd default to taking the one that generates a radical idea. But you lose some things with this perspective. Certainly I wouldn't say it's great for multiplicities and whatnot.

So, we need to upgrade: instead of varieties in the classical sense, we'll eventually think of schemes. Some of the intuition will port over: we're thinking of things/geometric objects (or, topological spaces) that look like they're (locally) "cut out by polynomials," and a decent amount of the practical work of computing things will resemble some of the polynomial fiddling you've done before, but we're keeping track of more of the data of the **functions on these spaces.**

Roughly, a scheme has three levels of data.

- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space

(The last one helps distinguish things like $V(x)$ versus $V(x^2)$.)

Here is where we start brushing up on Grothendieck's perspective: that when studying on object, it's less important to study the object itself and more important to study functions between them, how they relate to other things.

Now, before that, we need to do **sheaves**, which are, informally, a bundling of data about functions on open sets of a topological space. **The usual example, which you should have in mind throughout, is the data of differentiable functions on a differentiable manifold.**

We begin with sheaves of sets, but the idea extends to sheaves of groups, rings, k-algebras, etc.

Definition 1.1. Let X be a topological space. A presheaf $\mathscr{F}$ is the following data:

- To each open set $U \subseteq X$, we have an assignment $\mathscr{F}(U)$ of a set (or group or ring, etc...)
- For each inclusion $V \hookrightarrow U$ of open sets, we have restriction maps $\text{res}^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$. The restriction maps need to follow some reasonable properties:
  – $\text{res}^U_U : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map
  – For inclusions $W \hookrightarrow V \hookrightarrow U$, the restriction maps should commute (that is, restricting from U to V to W or restricting directly U to W shouldn't change things).

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\text{res}^U_V"] \arrow[dr, "\text{res}^U_W"'] & \mathscr{F}(V) \arrow[d, "\text{res}^V_W"] \\ & \mathscr{F}(W) \end{tikzcd}$$

Notational bits-and-bobs:

- Elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ on U.
  –
- $\mathscr{F}(U)$ is notated a few other ways:
  – $\Gamma(U, \mathscr{F})$
  – $H^0(U, \mathscr{F})$
- Note that a presheaf is precisely the data of a contravariant functor from the category of open sets on X to the category of sets (or groups, rings, etc).

Definition 1.2. A presheaf $(X, \mathscr{F})$ is a sheaf if it satisfies two more additional axioms.

- **Identity/uniqueness**: If $\{U_i\}_{i\in I}$ is an open cover of U and $f_1, f_2 \in \mathscr{F}(U)$ are two sections/functions such that
$$\text{res}^U_{U_i} f_1 = \text{res}^U_{U_i} f_2$$
for all $i \in I$, then $f_1 = f_2$. (That is, two sections that line up on each piece of a cover have to have been the same).
- **Gluing**: Let $\{U_i\}_{i\in I}$ be an open cover of U. If you have an $f_i \in \mathscr{F}(U_i)$ for each i such that, for any i, j:
$$\text{res}^{U_i}_{U_i\cap U_j} f_i = \text{res}^{U_j}_{U_i\cap U_j} f_j$$
then there is an $f \in \mathscr{F}(U)$ such that $\text{res}^U_{U_i} f = f_i$ for each i. (That is, if you have an open cover, a choice of section on each piece of the cover, and these choices agree on the overlaps, then you should be able to glue these to a section on the whole thing).

Example 1.3. Let X be a differentiable manifold. Let $\mathscr{F}$ be the sheaf that assigns to an open set U the ring of differentiable functions $\mathscr{F}(U)$ defined on U. For $V \subseteq U$, the restriction map is restriction of domain:
$$\mathscr{F}(U) \to \mathscr{F}(V)$$
$$f \mapsto f|_V$$
The fact that differentiable functions are "defined by their values" makes it clear that this is a pre-sheaf. Likewise, the two additional sheaf properties are clear: if two functions agree on an open cover, they are the same function. And if you have a differentiable function on each piece and the overlaps are fine, you can defined the function on the whole manifold.

Note: in general, you may see things like $\text{res}^U_V f$ written as $f|_V$ to save space.

Since I don't want to shift gears too much on the first day, let's do a few bits-and-bobs about sheaves.

Example 1.4 (Skyscraper sheaves)**.** Let S be a set, $p \in X$ a point. Get:
$$i_{p,*}S(U) = \begin{cases} S & p \in U \\ \{e\} & p \notin U \end{cases}$$
here $\{e\}$ is any one element set. Skyscraper-type behavior at p. Draw pictures.

2. Jan 15: Intro to Spec

We will eventually need to worry about morphisms of sheaves, pushforwards, pullbacks, and more. But that can come a bit later, when we better understand the topological spaces we want to look at in the first place.

Recall that we are trying to define schemes, which consist of:

- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space

Now on our journey towards schemes, which are our generalization of algebraic varieties/sets, we need to think of the underlying topological space of our geometric objects. These will come about in the form of the **spectrum of a ring**, AKA affine schemes. These are the building blocks of schemes in general.

These will resemble things from 552 somewhat: our first examples will be visualizable in some $\mathbb{C}^n$, with the majority of the points corresponding to tuples $(a_1, \dots, a_n)$ satisfying some polynomials, along with some extra points that are useful to have in there.

Do note: ring here means a commutative ring with identity, 1. Example: $\mathbb{C}, \mathbb{R}, \mathbb{F}_p, \overline{\mathbb{F}_p}, \mathbb{C}[t], \mathbb{C}(t)$, polynomial rings, quotient rings. We will often focus on $\mathbb{C}$-algebras or k-algebras with k algebraically closed, as this is the best place to start off. As appropriate I may add in some examples over non-algebraically closed fields, but I will largely leave those examples to your future number theory courses.

The idea: given a ring A, we want the most natural/nontrivial space on which A becomes a "ring of functions." You've encountered this before with coordinate rings in 552.

Example 2.1 (Rough intuition)**.** The algebraic functions on the complex line $\mathbb{C}$ should be single variable polynomials: $\mathbb{C}[t]$. If you cut out the origin and consider the open set $\mathbb{C} \setminus \{0\}$, you no longer have to worry about t zeroing out, so your algebraic functions should now be $\mathbb{C}[t, t^{-1}]$.

Definition 2.2. As a set, Spec A is the set of all prime ideals of A.

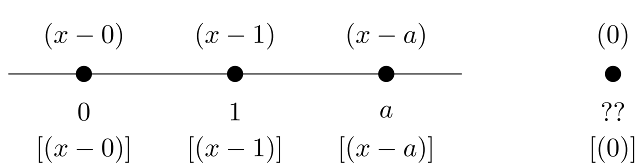
Recall that $I \subseteq A$ is an ideal $\iff$ $I + I \subseteq I, aI \subseteq I$ for all $a \in A$. An ideal I is *prime* provided that $pq \in I \Rightarrow p \in A$ or $q \in A$. Recall that for $a_1, \dots, a_n \in A$, that $(a_1, \dots, a_n)$ denotes the ideal generated by $a_1, \dots, a_n$, i.e.

$$(a_1, \dots, a_n) = \{a_1 b_1 + \dots a_n b_n \mid b_i \in A\}$$

Example 2.3 (The complex affine line)**.** Let us consider the case of $A = \mathbb{C}[t]$, and how we can think of the $\mathbb{C}[t]$ as the ring of functions over Spec $\mathbb{C}[t]$. First, let us compute the spectrum. By the fundamental theorem of algebra, we have:

$$\text{Spec } \mathbb{C}[t] = \{(x - a) \mid a \in \mathbb{C}\} \sqcup \{(0)\}$$

that is, we get a point for each element of $\mathbb{C}$, and then this extra point (0). Given that this space is "basically $\mathbb{C}$ with some extra stuff," it's not strange to think of $\mathbb{C}[t]$, i.e. complex polynomials in one variable, aka polynomials that can take in one complex input, as the ring of functions over $\mathbb{C}$.



A few things to note:

- At each point $(x - a)$ of the spectrum, we have an evaluation map:

$$\begin{aligned} \mathbb{C}[x] &\to \frac{\mathbb{C}[x]}{(x-a)} \cong \mathbb{C} \\ f(x) &\mapsto f(a) \end{aligned}$$

That is, f is sent to its image in $\mathbb{C}[x]/(x-a)$, which says x can be swapped for a. That is, we send $f(x)$ to $f(a)$. This evaluates the polynomial at a. We will see a similar construction in general. Note that this means these points are keeping track of all the values of this function. If we have two different polynomials f_1, f_2, then their evaluations at some point will differ: i.e. functions are distinguished by their values. **This will not always be true!**

- (0) is called "the generic point." It is "close" to every point, so it is "generically" on the line, but it is not equal to any of the $(x-a)$. Some would choose to draw it as "fuzz" amongst the line. We will understand the generic point better when we understand the topology of Spec A.
- Spec $\mathbb{C}[t]$ will come to be known to us as the complex affine line, denoted $\mathbb{A}^1_{\mathbb{C}}$.

Example 2.4 (Don't say I never gave you an example that wasn't over $\mathbb{C}$!)**.** Consider $A = \mathbb{R}[t]$. The prime ideals are of one of two forms:

$$(t-a)\ a \in \mathbb{R}, \quad (t-a)(t-\overline{a})\ a \in \mathbb{C} \setminus \mathbb{R}$$

Hence we get an identification:

$$\text{Spec } \mathbb{R}[t] \cong \mathbb{C}/\text{Gal}(\mathbb{C}/\mathbb{R}) \sqcup \{(0)\}$$

which you can identify with the upper half plane along with a generic point.

Definition 2.5 (Evaluation map)**.** Given a ring A, $f \in A$, $\mathfrak{p} \in \text{Spec } A$, the **value of f at $\mathfrak{p}$** is the image of f under:

$$A \to A/\mathfrak{p} \to \text{Frac } A/\mathfrak{p}$$

Example 2.6 (Functions are not always separated by *values* at points)**.** Consider the set Spec $\mathbb{C}[t]/(t^2)$. As a set of points, it has just one element: $[(t)]$.

t is an element of the ring $\mathbb{C}[t]/(t^2)$, and we should think of it as being very small: so small that its square is zero, but it itself is not zero. If we think about the evaluations of this function, note that

$$\begin{aligned} \mathbb{C}[t]/(t^2) &\to \text{Frac } (\mathbb{C}[t]/(t, t^2)) \cong \mathbb{C} \\ t &\mapsto 0 \end{aligned}$$

That is, both the function t and 0 on the LHS evaluate to 0 on the RHS. But this is the only evaluation map to consider! So we see how functions cannot necessarily be separated by values.

Now it is time to define the topology on these spaces. The idea: closed sets should be sets of points where functions vanish.

$$\begin{aligned} f \text{ vanishes at } \mathfrak{p} &\iff f(\mathfrak{p}) = 0 \\ &\iff f = 0 \text{ in } A/\mathfrak{p} \\ &\iff f \in \mathfrak{p} \quad (\iff (f) \subseteq \mathfrak{p}) \end{aligned}$$

Definition 2.7. Various vanishing loci definitions. Let $f \in A, S \subseteq A$.

$$\begin{aligned} V(f) &= \{\mathfrak{p} \in \text{Spec } A : (f) \subseteq \mathfrak{p}\} \\ V(S) &= \{\mathfrak{p} \in \text{Spec } A : S \subseteq \mathfrak{p}\} \end{aligned}$$

Definition 2.8. A (Zariski) closed subset of Spec A is any set of the form of a vanishing locus $V(\mathfrak{a})$ for $\mathfrak{a}$ an ideal.

Proposition 2.9. The collection of Zariski closed subsets forms a topology on Spec A.

Proof. Observe:

$$\begin{aligned} \varnothing &= V((1)) \\ \text{Spec } A &= V((0)) \\ V(\mathfrak{a}) \cup V(\mathfrak{b}) &= V(\mathfrak{a}\mathfrak{b}) \\ \bigcap_{i \in I} V(\mathfrak{a}_i) &= V\left(\sum_{i \in I} \mathfrak{a}_i\right) \end{aligned}$$

∎

Example 2.10. The closed sets in Spec $\mathbb{C}[t]$ are the whole space and

$$V(f(t)) = V((t-a_1)\dots(t-a_n)) = \cup_i V((t-a_i)) = \{(t-a_i) : 1 \le i \le n\}$$

i.e. finite collections of non-generic points.

Note that $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[t]$. That is, the generic point is "close" to all other points, and "sits along the whole line".

Definition 2.11. Define $D(f) = \text{Spec } A \setminus V((f))$. These open sets form a basis for the topology.

Proposition 2.12. Let S be a multiplicative set. By studying the map $\varphi : A \to S^{-1}A, a \mapsto a/1$, this induces a bijection

$$\{\text{primes in } A \text{ with } \mathfrak{p} \cap S = \varnothing\} \longleftrightarrow \{\text{primes in } S^{-1}A\}$$

What should be the algebraic functions on $D(f)$? well, since we're not working with the full spec, we should be able to invert things that don't vanish on the set. That is, things whose vanishing sets are squirreled away in $V(f)$, a set we are cutting out.

Set $\mathcal{O}_{\text{Spec } A} = S^{-1}A$ where

$$S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

This definition depends only on $D(f)$, not on f itself. But luckily:

Proposition 2.13. The natural map

$$A_f \mapsto \mathcal{O}_{\text{Spec } A}(D(f))$$

is an isomorphism.

Lemma 2.14. $D(f) \subseteq D(g)$ (that is, $V(g) \subseteq V(f)$) if and only if $f^n \in (g)$, if any only if g is invertible in A_f.

Proof. $f^n \in (g) \iff \sqrt{(f)} \subseteq \sqrt{(g)} \iff$ the prime ideals containing (f) are a superset of those containing (g), which means $V(g) \subseteq V(f)$.

Then $f^k = gm$, so g is invertible in A_f. □

That is, the algebraic functions on $D(f)$ are basically obtained by inverting f. So, we have the makings of the structure sheaf, i.e. a sheaf $\mathcal{O}_{\text{Spec } A}$ where $\mathcal{O}_{\text{Spec } A}(U)$ is algebraic functions on U. But we only have it on a distinguished basis. The question becomes: is this enough to determine the sheaf overall? Will we be able to do computations in the future/check nice properties on just this basis? The answer: yes! Back to sheaf theory...

3. Jan 17: Let's understand sheaves better (stalks, morphisms)

We learned about the topological spaces that will be pieced into schemes. These are the Spec A, and we think of A as the ring of algebraic functions on Spec A. We had special open sets $D(f) = \text{Spec } A \setminus V(f)$, where $V(f) = \{\mathfrak{p} : f \in \mathfrak{p}\}$, i.e. the primes/points where f vanishes.

We would like to assemble a **structure sheaf** on the topological space Spec A:

$$\mathcal{O}_{\text{Spec } A}(U) = \text{ring of algebraic functions on } U$$

From this perspective, $\mathcal{O}_{\text{Spec } A}(\text{Spec } A)$ should be A. And we saw last time that it was reasonable to set:

$$\mathcal{O}_{\text{Spec } A}(D(f)) = S^{-1}A \cong A_f$$

with

$$S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

Problem: what about this would-be sheaf on open sets in general? We would like describing the sheaf, i.e. describing the rings of algebraic functions, on nice $D(f)$ to be enough. It is, but we need to do a bit of work to say that. (Here: Vakil and Hartshorne somewhat "diverge". Vakil shows that defining a sheaf on a basis is sufficient; Hartshorne just describes the $\mathcal{O}_{\text{Spec } A}(U)$ from the get-go, with the construction being the one you'd do when defining a sheaf from a base).

We'll get to all that, but we should do some necessary details and in general shore up our knowledge.

Definition 3.1. Let $(X, \mathscr{F})$. Let $x \in X$ be a point. The **stalk** of $\mathscr{F}$ at x is defined as the direct limit:

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U) = \{(f, U) : f \in \mathscr{F}(U), p \in U\}$$

where $(f, U) \sim (g, V)$ if and only if there is some $W \subseteq U, V$, with W containing p, such that $f|_W = g|_W$. **Draw pic in differentiable functions case! This means they need to be same on some smaller open set around the point.** Observe that in the case of differentiable functions on a differentiable manifold, the stalk is a local ring: its unique maximal ideal is the ideal of all functions vanishing at p.

Definition 3.2. Elements of a stalk are called **germs.**

Remark 3.3. We will see later on that many properties we want to test of sheaves (or structure sheaves of schemes) can be tested by checking the analogous condition on the stalks. This is reasonable, looking at the gluing axiom.

Definition 3.4 (Morphisms of (pre)sheaves)**.** Let $\mathscr{F}, \mathscr{G}$ be (pre)sheaves on a topological space X. A morphism $\mathscr{F} \to \mathscr{G}$ is a collection of maps $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for each U such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}^U_V"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\
\mathscr{F}(V) \arrow[r, "\varphi_V"] & \mathscr{G}(V)
\end{tikzcd}$$

Consequently, we can see that φ defines a map on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ by sending (f, U) to $(\varphi(f), U)$. An isomorphism is a morphism with a two-sided inverse.

Let's restrict our attention to sheaves of abelian groups at this point (we rarely fall outside this scenario).

Proposition 3.5. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on a topological space X. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then φ is iso if and only if ϕ_x is iso.

Proof. Iso $\iff$ iso will cover the whole proof. ϕ an isomorphism implies ϕ_x an isomorphism: this is clear.

Each ϕ_x an isomorphism implies ϕ an iso. Let's start with injectivity: consider a fixed $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. For injectivity: if $\phi(s) = 0$, then $\phi(s)_x = 0$ for each $x \in U$. Then s_x is zero for each $x \in U$ by injectivity on stalks. That is s restricted to zero on an open cover of U. That is, $s = 0$.

For surjectivity: suppose we are considering $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. Let $t \in \mathscr{G}(U)$. We'll piece together something that maps to it.

For each $x \in U$, we have $t_x \in \mathscr{G}_x$, and it must be the image of some $s_x \in \mathscr{F}_x$. s_x can be repped by some $s(x) \in V_x \ni x$. Then $\varphi(s(x)), t|_{V_x}$ are two elements of $\mathscr{G}(V_x)$ with the same germ, so $\varphi(s(x)), t$ agree in some neighborhood W_x of x.

Cover U with these W_x, and consider the $s(x)$ (well, $s(x)|_{W_x}$ we get for each one. On the overlaps, these must agree due to injectivity (their overlaps go to $t|_{set}$). So we can piece them together to get an $s \in \mathscr{F}(U)$ that maps to t.

Note that the proof of surjectivity needed injectivity! ∎

Definition 3.6 (**Tentative** definition of kernel, image, cokernel)**.** Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves of abelian groups, we can define the <u>presheaves</u> $\ker(\varphi), \mathrm{coker}(\varphi), \mathrm{im}(\varphi)$, as follows:

$$\begin{aligned}
\ker(\varphi)(U) &= \ker(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U))\,[\subseteq \mathscr{F}(U)] \\
\mathrm{coker}(\varphi)(U) &= \mathrm{coker}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) \\
\mathrm{im}(\varphi)(U) &= \mathrm{im}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U))[\subseteq \mathscr{G}(U)]
\end{aligned}$$

Proposition 3.7. Given $\varphi : \mathscr{F} \to \mathscr{G}$, the kernel is a sheaf.

Proof. Identity is inherited from the parent sheaf. Gluing is too, though you need to check that the glued function f is still in the kernel. This works because $\varphi(f)$ restricts to zero on a cover, thus is zero globally from gluing in $\mathscr{G}$. ∎

Proposition 3.8. The image and cokernel of a sheaf need not be a sheaf.

For cokernel. Let X be $\mathbb{C}$, and let $\mathcal{O}$ be the sheaf of holomorphic functions and $\mathcal{O}$ be the sheaf of nonzero holomorphic functions. Consider the map φ with

$$\begin{aligned} \varphi_U : \mathcal{O}(U) &\to \mathcal{O}^*(U) \\ f &\mapsto e^f \end{aligned}$$

we claim that the cokernel isn't a sheaf. First, note that there is no holomorphic f such that $e^f = z$ on $\mathbb{C} \setminus \{0\}$. Otherwise, differentiating both sides yields:

$$e^f \cdot f' = 1 \Rightarrow z \cdot f' = 1 \Rightarrow f' = 1/z$$

Integrating the LHS over a loop around zero yields 0, but integrating the RHS over said loop produces $2\pi i$. Contradiction.

Therefore, $[z] \neq 0$ in $\operatorname{coker}(\varphi)$. That is, the global sections of the cokernel has nonzero elements.

But, take $U_1 = \mathbb{C} \setminus (-\infty, 0]$ and $U_2 = \mathbb{C} \setminus [0, \infty)$. These are simply connected, so every nonzero function on them can be written as some e^f (we can define the log: we made a branch cut!). Thus $\operatorname{coker}(U_1)$, $\operatorname{coker}(U_2)$ both equal zero. So the cokernel fails the identity axiom.

Similarly, this shows why the image isn't necessarily a sheaf: we can't glue the logs of z into a log for z on all of $\mathbb{C} \setminus \{0\}$ □

So, we have all these presheaves running around (including ones we'd really like to consider: the image and cokernel are important!). We would like some way to modify them into a sheaf, and it should have some nice universal property that relates it back to the original presheaf.

Construction 3.9 (Sheafification)**.** Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that: for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism φ^+ such that $\varphi = \varphi^+ \circ \theta$. The pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism. $\mathscr{F}^+$

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & \mathscr{F}^+ \arrow[d, dashed, "\varphi^+"] \\ & \mathscr{G} \end{tikzcd}$$

To be continued next lecture...

4. Jan 22: Sheafifcation, sheaves on a base

Recall: last time: nice properties of stalks. Recall that we have $\mathscr{F}(U) \to \mathscr{F}_p$ for each $p \in U$, given by $f \mapsto f_p = (f, U)$. Also $\varphi : \mathscr{F} \to \mathscr{G}$ induces $\mathscr{F}_p \to \mathscr{G}_p$ for each $p \in X$. Also, covered morphisms of sheaves. Saw that naive definition of im, ker of sheaves not necessarily sheaves. This necessitated sheafification.

Now for sheafification. The construction is: bundle the stalks together in a nice way. That is, make a big product but only allow combinations of germs that looked like they could glue together.

$$\begin{aligned} \mathscr{F}^+(U) := \{(f_p)_{p \in U} : &\text{ for all } p \in U, \text{ there is an open } V \text{ with } p \in V \subseteq U \\ &\text{ with } s \in \mathscr{F}(V), s_q = f_q \text{ for all } q \in V\} \\ \subseteq \prod_{p \in U} \mathscr{F}_p \end{aligned}$$

the morphism θ is clear ($f \mapsto (f_p)_{p \in U}$). To describe φ^+: look at the sections that glue to your (f_p), look at their images, glue them in the target, and call that the image. This is unique: in order for the diagram to commute any other map would have to do the same thing.

Remark 4.1. Sheafification is a functor from presheaves on X to sheaves on X.

Remark 4.2. Specifically, given $i : \mathrm{Shf} \to \mathrm{Pre}$ note that sheafification $+ : \mathrm{Pre} \to \mathrm{Shf}$, we have that $+$ is the left adjoint of i, i.e. given $\mathscr{F}$ a sheaf and $\mathscr{G}$ a presheaf, we have natural bijection

$$\mathrm{Hom}_{\mathrm{Pre}}(\mathscr{G}, i(\mathscr{F})) \cong \mathrm{Hom}_{\mathrm{Shf}}(\mathscr{G}^+, \mathscr{F})$$

Example 4.3 (Constant sheaves)**.** Let X be a topological space, S a set. Get constant presheaf by assigning same set to all open sets:

$$\mathscr{F}(U) = S$$

(On nonempty sets, you can interpret this as constant functions from U to S). Gluing is a mess because of disconnects. The empty set presents problems too: the empty cover $\{\}$ is an open cover of the empty set, and each section vacuously restricts to the same thing on this open cover. So the identity axiom says all sections on $\varnothing$ should be the same!

The sheafification $\mathscr{F}(U)$ will instead assign to U: locally constant maps from U to S. Denote this sheaf as $\underline{S}$.

Remark 4.4. Thinking of sheaves of abelian groups: sheafification adds in the gluings that should exist but don't, and kills off the nonzero sections that are locally zero.

Proposition 4.5. $\mathscr{F} \to \mathscr{F}^+$ yields an isomorphism of stalks.

Proof. Work from the explicit description. □

Definition 4.6. We say that a map of sheaves is injective if and only if the kernel sheaf is zero.

Lemma 4.7. A map of sheaves is injective $\iff$ $\varphi : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Likewise it is injective $\iff$ is injective on stalks.

Proof. This was done in the bijectivity proof before. □

Definition 4.8. We define the image and cokernel *sheaves* by taking the sheafification of the presheaves defined above.

Given a map $\varphi : \mathscr{F} \to \mathscr{G}$. Since we have a map $\mathrm{im}_{\mathrm{pre}}(\varphi) \to \mathscr{G}$, we necessarily have a map $\mathrm{im}(\varphi) \to \mathscr{G}$. This map is injective: it is injective on the level of stalks (note that the unsheafified and sheafified im have the same stalks!!). Thus we can identify $\mathrm{im}(\varphi)$ with a subsheaf of $\mathscr{G}$.

Definition 4.9. A morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is **surjective** if $\mathrm{im}(\varphi) = \mathscr{G}$.

Lemma 4.10. $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if $\mathscr{F}_x \to \mathscr{G}_x$ is surjective for all points x.

Proof. $\Rightarrow$: $\mathrm{im}(\mathscr{F}) = \mathscr{G}$ means their stalks are isomorphic, hence $\mathscr{F}_x \to \mathscr{G}_x$ must be surjective.

$\Leftarrow$: we want to show that $\mathrm{im}(\mathscr{F}) = \mathscr{G}$. Well, the map on stalks is an isomorphism (injective and surjective on stalks), so they are equal. □

Example 4.11. In our example with $X = \mathbb{C}$, $\mathcal{O}_X$ the sheaf of holomorphic functions and $\mathcal{O}_X^*$ the sheaf of non-vanishing holomorphic functions and

$$\mathcal{O}_X \to \mathcal{O}_X^*$$
$$f \mapsto e^f$$

we have that $\mathrm{im}(\varphi) = \mathcal{O}_X^*$ and $\mathrm{coker}(\varphi) = 0$. This can be seen via φ being surjective on the level of stalks (and correspondingly the cokernel is 0 on the level of stalks).

Definition 4.12. A sequence of maps:

$$\mathscr{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathscr{F}^i \overset{\varphi^i}{\to} \mathscr{F}^{i+1}$$

is *exact* if at each stage, $\ker \varphi^i = \mathrm{im}\ \varphi^{i-1}$.

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5. Jan 24: Sheaves on a base, affine schemes

Time to handle an issue: sometimes we understand a sheaf really well on a nice collection of open sets. But what about the rest? The details are sometimes unpleasant/obfuscating: it is mainly important to know that the data of the sheaf on a suitably nice basis is enough to determine the sheaf. The construction will be reminiscent of sheafification.

Definition 5.1. A base of a topology is a collection of open sets $\{B_j\}_{j\in J}$ such that any open set of X can be written as a union of B_j.

Remark 5.2. $(f) \subseteq \mathfrak{a} \iff V(f) \supseteq V(\mathfrak{a}) \iff D(f) \subseteq D(\mathfrak{a})$ so the $D(f)$ genuinely are a basis of the Zariski toplogy on Spec A.

Definition 5.3. Suppose $\{B_i\}$ is a basis on X. A presheaf of sets on the base is an assignment $F(B_i)$ for each B_i. If $B_j \subseteq B_i$, we have restriction maps $\mathrm{res}^{B_i}_{B_j}$ satisfying $\mathrm{res}^{B_i}_{B_i} = \mathrm{id}$ and $\mathrm{res}^{B_i}_{B_k} = \mathrm{res}^{B_j}_{B_k} \circ \mathrm{res}^{B_i}_{B_j}$.

For sheaves on a base: base identity and base gluing axioms:

- If $B \in \{B_i\}_{i\in I}$ can be written $B = \cup_{i\in J} B_i$ and $f, g \in F(B)$ with $\mathrm{res}^B_{B_i} f = \mathrm{res}^B_{B_i} = g$ for all $i \in J$, then $f = g$.
- If we have $f_i \in B_i$ for $i \in J$ such that f_i agrees with f_j on any B_k contained in $B_i \cap B_j$, then there is a $f \in F(B)$ such that $f|_{B_i} = f_i$ for all $i \in J$.

Theorem 5.4. *Suppose $\{B_i\}$ a base on X, and F a sheaf of sets on this base. There is a sheaf $\mathscr{F}$ extending F ($\mathscr{F}(B_i) \cong F(B_i)$ with isomorphisms agreeing with restriction maps). $\mathscr{F}$ is unique up to unique isomorphism.*

Proof. As before, $\mathscr{F}$ is a sheaf of compatible germs. Define the stalk of a presheaf F on a base as:

$$F_p = \varinjlim_{B_i \ni p} F(B_i)$$

Define

$$\mathscr{F}(U) := \{(f_p \in F_p)_{p\in U} : \text{for all } p \in U, \text{ there is a } B \text{ with } p \in B \subseteq U \\ \text{and } s \in F(B) \text{ such that } s_q = f_q \text{ for } q \in B\}$$

We get a map $F(B) \to \mathscr{F}(B)$ for each B, which is an isomorphism. Checking the details is similar to the hassle of sheafification.

Note that clearly $\mathscr{F}_p \cong F_p$. □

We can finally really talk about affine schemes. Consider Spec A with the Zariski topology, and for open sets $D(f)$ set

$$\mathcal{O}_{\mathrm{Spec}\ A}(D(f)) = S^{-1}A$$

where $S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$. The restriction maps are clear enough: if $D(g) \subseteq D(f)$ then the restriction map

$$\mathrm{res}^{D(f)}_{D(g)} : \mathcal{O}_{\mathrm{Spec}\ A}(D(f)) \to \mathcal{O}_{\mathrm{Spec}\ A}(D(g))$$

is further localization. This is clearly a presheaf on a distinguished base.

Lemma 5.5. Spec A is quasi-compact (every open cover has a finite subcover).

Proof. It's enough to show this for covers for the form $\{D(f_i)\}$. Note that $\cup D(f_i) = D(\sum(f_i))$. This will be all of Spec A only when $1 \in \sum(f_i)$, in which case we get $1 = a_{i_1}f_{i_1} + \dots a_{i_k}f_{i_k}$ and you can just take the corresponding cover pieces $D(f_{i_1}), \dots, D(f_{i_k})$. □

Proposition 5.6. This description in fact gives a sheaf on a distinguished base, and thus determine a sheaf on Spec A. This sheaf is the **structure sheaf** on Spec A, and is referred to as $\mathcal{O}_{\mathrm{Spec}\ A}$ or just $\mathcal{O}$ if it is clear what A is.

Proof. It's enough to show identity and gluability on just A (if you want to show it on $D(f)$, that's the same as swapping the ring out for A_f, modulo some detail-checking).

- **Identity axiom**: Write Spec $A = \cup_i D(f_i)$. Then after potentially relabeling, we can pick a finite subcover. Spec $A = \cup_{i=1}^n D(f_i)$. That is, $V((f_1, \dots, f_n\rangle)) = \varnothing$, i.e. $(f_1, \dots, f_n) = A$.

  Suppose we have a section $s \in \mathcal{O}_{\text{Spec } A}(\text{Spec } A) = A$ such that $\text{res}_{D(f_i)} s = 0 \in A_{f_i}$ for each f_i. That means that there is some m such that $f_i^m s = 0$ (in A).

  But note that $D(f_i) = D(f_i^m)$ (as f_i vanishes at $\mathfrak{p}$ if and only if f_i^m vanishes at $\mathfrak{p}$). So there are g_i such that
$$1 = \sum g_i f_i^m$$
  But then:
$$s = \sum g_i f_i^m s = \sum 0 = 0$$
- **Gluing:** Again, being able to write 1 as a sum of these f_i will let us piece things together in a nice way.

  Again, say we have gluing data on some open cover. Pick a finite subcover $\{D(f_1), \dots, D(f_n)\}$. Let s_i be the section on each $D(f_i)$ so that
$$\text{res}^{D(f_i)}_{D(f_i)\cap D(f_j)} s_i = \text{res}^{D(f_j)}_{D(f_i)\cap D(f_j)} s_j$$
  Identifying $D(f_i) = A_{f_i}, D(f_j) = A_{f_j}, D(f_i f_j) = A_{f_i \cdot f_j}$ we get that
$$s_i = \frac{a_i}{f_i^{\ell_i}}, \quad s_j = \frac{a_j}{f_j^{\ell_j}}$$
  and because their restrictions are the same in $A_{f_i f_j}$ it must be that there is an mi, j such that
$$(a_i f_j^{\ell_j} - a_j f_i^{\ell_i})(f_i f_j)^{m_{i,j}} = 0.$$
  Let $m = \max_{i,j} m_{i,j}$. Then the above tells us that
$$(a_i f_i^m) f_j^{m+\ell_j} = (a_j f_j^m) f_i^{m+\ell_i}$$
  Now again, Spec $A = \cup D(f_i) = \cup D(f_i^{m+\ell_i})$, so there exists $g_i \in A$ such that
$$1 = g_1 f_1^{m+\ell_1} + \cdots + g_n f_n^{m+\ell_n}$$
  and consider the element of A given by:
$$s = g_1 a_1 f_1^m + \cdots + g_n a_n f_n^m$$
  Then observe that
$$\begin{aligned} f_i^{m+\ell_i} s &= g_1(a_1 f_1^m) f_i^{m+\ell_i} + \cdots + g_n(a_n f_n^m) f_i^{m+\ell_i} \\ &= g_1 f_1^{m+\ell_1}(a_i f_i^m) + \cdots + g_n f_n^{m+\ell_n}(a_i f_i^m) \\ &= (g_1 f_1^{m+\ell_1} + \cdots + g_n f_n^{m+\ell_n}) a_i f_i^m \\ &= a_i f_i^m \end{aligned}$$
  That is, $f_i^m(f_i^{\ell_i} s - a_i)$. That is, $s = \frac{a_i}{f_i^{\ell_i}} = s_i$ on A_{f_i}, which is what we wanted.

  (You can use identity to show that the resulting glued object restricts to what you want on the other elements of the a priori infinite cover. So the identity proof does need to come first!).

□

Thus, we can finally start talking about affine schemes!!

6. Jan 27: Affine schemes, schemes

Proposition 6.1. Let A be a ring and $\mathcal{O}$ the structure sheaf on Spec A. For any $\mathfrak{p} \in \text{Spec}A$, the stalk $\mathcal{O}_p \cong A_{\mathfrak{p}}$.

Proof. Fairly evident from the description of $A_\mathfrak{p}$ as a direct limit identifying a bunch of subsequent localizations. For $f \notin \mathfrak{p}$, $D(f)$ will appear in the direct limit. Lemma 2.14 can help. To be more concrete, we can write down the map.

Any $(s, D(f)) \in \mathcal{O}_p$ can be sent to its image in $A_\mathfrak{p}$. It is surjective: any element in $A_\mathfrak{p}$ is of the form a/g with $g \notin \mathfrak{p}$, and so $\mathfrak{p} \in D(g)$. That is, $D(g)$ will be a neighborhood of $\mathfrak{p}$ and a/f will be hit by this map.

It is injective: write $s = a/f, t = b/g$, with $f, g \notin \mathfrak{p}$. If their image is the same in $A_\mathfrak{p}$, then there is some $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$. But then $D(fgh) = D(f) \cap D(g) \cap D(h)$ is a neighborhood of $\mathfrak{p}$, and so s, t would have been identified in the stalk $\mathcal{O}_p$ □

Definition 6.2. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf on X. The **direct image** sheaf $f_*\mathscr{F}$ on Y is defined via

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$$

for open sets $V \subseteq Y$. This is a functor from sheaves on X to sheaves on Y.

Definition 6.3. The **inverse image** sheaf $f^{-1}\mathscr{G}$ on X is the sheafification of the presheaf

$$U \mapsto \varinjlim_{V^{\text{open}} \supseteq f(U)} \mathscr{G}(V)$$

This is a functor from sheaves on Y to sheaves on X.

Definition 6.4. If $i : Z \hookrightarrow X$ a subset of X with the subspace topology, then $i^{-1}\mathscr{F}$ is the restriction of $\mathscr{F}$ to Z, denoted $\mathscr{F}|_Z$. For open sets Z this will just turn into $\mathscr{F}|_Z(V) = \mathscr{F}(V)$.

Definition 6.5. A **ringed space** is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a morphism of sheaves (of rings) $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if all the stalks $\mathcal{O}_{X,p}$ are local rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced map on stalks $f^\sharp_P : \mathcal{O}_{Y,f(P)} \to (f_*\mathcal{O}_X)_{f(P)} = \mathcal{O}_{X,P}$ is a local homomorphism of local rings.

Here, a local homomorphism of local rings $\varphi : A \to B$ is a ring morphism such that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

An isomorphism is a morphism (of ringed or locally ringed spaces) with a two-sided inverse. Equivalently, in $(f, f^\sharp)$, the f is a homeomorphism and $f^\sharp$ is a isomorphism of sheaves.

Proposition 6.6. (a) $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space.
(b) $\varphi : A \to B$ a morphism of rings induces

$$(f, f^\sharp) : (\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)$$

(c) In fact, any morphism of locally ringed spaces $(\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)$ is induced by a homomorphism of rings.

Proof.

(a) Immediate from previous results.
(b) The map on topological spaces is $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$ so the map is continuous.Two ways to see this:
   - Certainly this gives a morphism on a base $\mathcal{O}_A(D(f)) \to \mathcal{O}_B(D(\phi(f)) = \mathcal{O}_B(f^{-1}(D(f)) = f_*(\mathcal{O}_B)(D(f))$ via $A_f \to B_{\phi(f)}$ in the obvious way, and it respects restriction maps.
   - Localize at each prime to get a local homomorphism of local rings $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. Since sheaves are isomorphic to their sheafifcation, you can interpret sections on U as a compatible collection of germs, and so you can just map germs (and compatibility is preserved).
(c) Take global sections: we must have a map

$$\phi : \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong A \to \mathcal{O}_{\text{Spec } B}(\text{Spec } B) \cong B$$

One can show that φ induces all the data of the morphism.

Notably, we must have an induced morphism on stalks: $A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$. Due to compatibility, we must have

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^\sharp_{\mathfrak{p}}"] & B_{\mathfrak{p}} \end{tikzcd}$$

Since $f^\sharp$ a local homomorphism, it must be that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so the map f on points coincides with the one induced by φ. Then compatibility with restriction maps will force the $f^\sharp$ to be induced by φ as well.

□

Definition 6.7. An **affine scheme** if a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic, as a locally ringed space, to some $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $p \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Example 6.8 (Schemes can be glued)**.** Let X_1, X_2 be schemes. Let $U_i \subseteq X_i$ be open sets. Let $\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_1}|_{U_2})$ be an isomorphism of locally ringed spaces.

Then we can define a scheme X, obtained by gluing X_1, X_2 via identifying U_1, U_2 with the morphism φ. The topological space is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology (set open iff preimage is open).

We get maps $i_j : X_j \to X$ and the structure sheaf is defined as:

$$\mathcal{O}_X(V) = \{(s_1, s_2) : s_j \in \mathcal{O}_{X_j}(i_j^{-1}(V)), \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2})\}$$

that is, sections on sets that "see" the overlap are gotten from piecing together compatible sections on each X_1, X_2.

Example 6.9 (More concrete)**.** Recall that morphisms of affine schemes are induced by ring morphisms on the global sections. Glue Spec $(\mathbb{C}[t]$ and Spec $(\mathbb{C}[s]$ along $D(t) = \text{Spec }(\mathbb{C}[t, t^{-1}]) \cong \text{Spec }(\mathbb{C}[s, s^{-1}]) = D(s)$ via the following.

$$\begin{tikzcd} \text{Spec } \mathbb{C}[s] & \text{Spec } \mathbb{C}[t] \\ \text{Spec } \mathbb{C}[s, s^{-1}] \arrow[u, "i_1"] \arrow[r, "t \mapsto s^{-1}"'] & \text{Spec } \mathbb{C}[t, t^{-1}] \arrow[u, "i_1"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[s] \arrow[d] & C[t] \arrow[d] \\ \mathbb{C}[s, s^{-1}] & \mathbb{C}[t, t^{-1}] \arrow[l, "t \mapsto s^{-1}"'] \end{tikzcd}$$

This yields the projective line. We will learn about the *proj construction* in general next lecture.

Example 6.10 (What if you take the other transition function)**.** We could instead glue two copies of the line without any sort of inversion. Take Spec $(\mathbb{C}[t])$ and Spec $(\mathbb{C}[s])$. Take the open sets $D(t) = \text{Spec } \mathbb{C}[t, t^{-1}], D(s) = \text{Spec } \mathbb{C}[s, s^{-1}]$, and glue them via $t \to s$. This glues everything away from the origin in a "straightforward" way and we get the affine line with a doubled origin.

7. Jan 29: Proj, properties of schemes

Now for the proj construction: we want a big class of examples from the projective varieties/we want to handle them in one fell swoop.

Intuition from 552 remainds: if $S_\bullet = k[x_0, \ldots, x_n]$, the proj construction yields $\mathbb{P}^n_k$, and if $S_\bullet = A[x_0, \ldots, x_n]/(f)$ where f is homogeneous, we get something "cut out" of $\mathbb{P}^n$ by the equation $f = 0$.

Definition 7.1 ($\mathbb{Z}$-graded rings)**.** A $\mathbb{Z}$-graded ring is a ring $S_\bullet = \oplus_{n\in\mathbb{Z}} S_n$ where multiplication respects grading: $S_m \times S_n \to S_{m+n}$. S_0 is a subring and each S_n is an S_0 module, and $S_\bullet$ an S_0 module. A $\mathbb{Z}^{\geq 0}$-graded ring is a $\mathbb{Z}$-graded ring with no elements of negative degree. We will, in the future, use graded ring to refer to a $\mathbb{Z}^{\geq 0}$-graded ring.

Definition 7.2. An element of some S_n is a homogeneous element. If it is nonzero, the subscript yields the degree.

Definition 7.3. An ideal I of $S_\bullet$ is homogeneous if it is generated by homogeneous elements.

Proposition 7.4. An ideal is homogeneous if and only if it contains the degree n piece of each its elements.

Proof. An induction proof by successively lopping off the top-degree pieces. □

Definition 7.5. In a graded ring $S_\bullet$, the irrelevant ideal refers to $S_+ := \oplus_{i>0} S_i$

Definition 7.6. As a set, Proj S is the set of all homogeneous prime ideals $\mathfrak{p}$ that do not contain all of S_+. For $\mathfrak{a}$ a homogeneous ideal of S, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj } S : \mathfrak{p} \supseteq \mathfrak{a}\}$. For a set T, $V(T) = V((T))$. We have distinguished open sets (well, we'll eventually see they're open) $D(f) := \text{Proj } S \setminus V((f)) = \{\mathfrak{p} \in \text{Proj } S : f \notin \mathfrak{p}\}$ for $f \in S_+$. Note that $D(fg) = D(f) \cap D(g)$.

Lemma 7.7.

(a) For $\mathfrak{a}$ and $\mathfrak{b}$ homogeneous ideals in S, we have $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$
(b) For any collection of homogeneous ideals $\{\mathfrak{a}_i\}$ any family of homogeneous ideals of S, we have

$$V\left(\sum \mathfrak{a}_i\right) = \cap V(\mathfrak{a}_i)$$

Proof. Same as before, accounting for the following: a homogeneous ideal $\mathfrak{p}$ is prime iff for two homogeneous $a, b \in S$, the product $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. □

Hence we can define a Zariski topology on Proj S. Now we must define a structure sheaf on this space. Idea: on the $D(f)$ we'd like the scheme to look like Spec $((S_\bullet)_f)_0$.

Definition 7.8. For $f \in F_+$, set $\mathcal{O}_{\text{Proj } S_\bullet} = \mathcal{O}(D(f)) = ((S_f)_0 = \text{"}S_{(f)}\text{"}$. See Hartshorne p. 76 or Vakil Section 4.5 if you want to see more on details on issues relating to, e.g., whether localization maps will make sense.

Proposition 7.9. Let S to be a graded ring.

(a) The stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local ring $S_\mathfrak{p}$, the degree zero elements of S localized at all **homogeneous** elements not contained in $\mathfrak{p}$.
(b) We have that

$$(D(f), \mathcal{O}|_{D(f)}) \cong \text{Spec } ((S_f)_0)$$

(c) Proj S is a scheme, as (check triple intersections in the vein of II.2.12)

It would do you well to read Exercise II.2.12, to get a sense of the work needed to glue together schemes. The proj construction thankfully allows us to construct a bunch of interesting non-affine schemes in one fell swoop, so that we do not have to keep doing this gluing.

Example 7.10. For A a ring, we set $\mathbb{P}^n_A = \text{Proj } A[x_0, \ldots, x_n]$, the projective n-space over A. For $A = k$ algebraically closed, you get something whose set of closed points is homeomorphic to the usual *variety* we know as projective n-space.

Definition 7.11. Let S be a fixed scheme. A scheme over S is a scheme X with a morphism $X \to S$. A morphism X to Y as schemes over S is a morphism $f : X \to Y$ of schemes that is compatible with the morphisms to S. Then $\mathfrak{Sch}(S)$ is the category of schemes over S. If A is a ring, $\mathfrak{Sch}(A)$ is the category of schemes over Spec A.

Proposition 7.12. Let k be algebraically closed. There is a natural, fully faithful (that is, bijective on hom sets) functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$. For any varierty, the top space is homeomorphic to the set of closed points $\text{sp}(t(V))$ and its sheaf of regular functons is obtained by restricting the structure sheaf of $t(V)$ via this homeo.

Proof. See II.2, Proposition 2.6, of Hartshorne. □

Now it's about time to think of all the interesting properties of schemes we could want:

Definition 7.13 (Big list of scheme adjectives)**.** Let X be a scheme.

(a) X is **connected** if its topological space is connected.
(b) X is **irreducible** if the topological space is irreducible (all nonempty open sets dense).
(c) X is **integral** if all the $\mathcal{O}_X(U)$ are integral domains

(d) X is **reduced** if all the $\mathcal{O}_X(U)$ have no nilpotent elements (equivalently, by II.2.3, all the stalks have no nonzero nilpotents).

Proposition 7.14. A scheme is integral iff it is both reduced and irreducible.

Proof. Integral certainly implies reduced. And if it's not irreducible, then it has two nonempty disjoint sets, yielding:

$$\mathcal{O}_X(U_1 \sqcup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$

which is not integral.

Conversely: supposed X is reduced and irreducible. Suppose there are $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. Then look at $Y = \{x \in U : f_x \in m_x\}, Z = \{x \in U : g_x \in m_x\}$. These are closed subsets (exercise II.2.16– on HW!) of U, and $Y \cup Z = U$. But X is irreducible, so U is irreducible. So, then, say, $Y = U$. But then f is nilpotent on any affine open in U (II.2.18a), meaning f is zero. □

Proposition 7.15. Suppose X is a reduced scheme. Let $f, g \in \Gamma(X, \mathcal{O}_X)$. Then

$$f = g \iff f(x) = g(x) \ \text{ (in } k(x)) \text{ for all } x \in X$$

That is, evaluating the same everywhere means the two sections are the same.

Remark 7.16. Here $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field. Taking an affine open Spec A containing x , this aligns with Frac(A/x), so it's the same evaluation map as before. Viewing $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ has the benefit of not needing to select a chart to write it down. Also, recall that our example of a setting where evaluation didn't determine the section was $k[x]/(x^2)$, which has nonzero nilpotents.

Proof. $\Leftarrow$: This direction is obvious.

$\Rightarrow$: We may assume X is affine (you'll get equality on each open affine, and then glue to finish). We have:

$$A \to \prod_{\mathfrak{p} \in \text{ Spec } A} \text{Frac}(A/\mathfrak{p}) = \prod_{\mathfrak{p} \in \text{Spec } A} k(\mathfrak{p}) \qquad \left(\text{equivalently, } A \hookrightarrow \prod_{\mathfrak{p}} \mathcal{O}_{\text{Spec } A, \mathfrak{p}} \to \prod_{\mathfrak{p}} k(\mathfrak{p}) \right)$$

The kernel is the intersection of all prime ideals, which is (0). That is, the map is injective. So since $f - g$ maps to zero, it must be that $f - g = 0$, and we are done. □

Definition 7.17. A scheme is locally noetherian if it can be covered by open affine subsets Spec A_i where each A_i is Noetherian. X is Noetherian if it is locally noetherian and quasi-compact. Equivalently, X is Noetherian if it can be covered by a finite number of open affine subsets Spec A_i, each A_i noetherian.

Remark 7.18. X being Noetherian (so basically a.c.c on ideals) means that the topological space is Noetherian (d.c.c. on closed subsets).

8. Jan 31: More properties of schemes

The following is an important type of proof. In our definitions of various adjectives we often want to say that there's just one cover with a certain property (as that's easy to prove!). When we use this adjective in proofs, we would like to be able to say *every* open cover has a certain property (as that's more useful to us).

Proposition 8.1. A scheme X is locally noetherian iff for every open affine $U =$ Spec A, A is a noetherian ring.

Proof. One direction is obvious. We must show the forward direction, then.

These proofs tend to have a "go down, then up" sort of process: you want B has a property implies B_f has some property, and then from there you want a bunch of B_{f_i} having a property and $\cup_i$Spec $B_{f_i} =$ Spec B (i.e. $\sum (f_i) = 1$) implies that Spec B has that property.

Note: if B is noetherian, so is any localization B_f. Note, then, that we have a base for the topology consisting of specs noetherian rings Spec B_f, and thus our $U =$ Spec A can be covered by specs of noetherian rings.

So we may restrict to showing the following: if $X =$ Spec A is an affine scheme covered by spectra of noetherian rings, then A is noetherian. Let $U =$ Spec B be an open subset of X, with B noetherian. Then for some $f \in A$, $D(f) \subseteq U$.

$$\begin{tikzcd} \text{Spec } A & \\ D(f) \arrow[u] \arrow[r] & \text{Spec } B \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d] \arrow[dr] & \\ A_f & B \arrow[l] \end{tikzcd}$$

Let $\overline{f}$ be the image of f in B. Then $A_f \cong B_{\overline{f}}$ (as both should be the coordinate ring of $D(f)$). **Thus, A_f is noetherian.** So we successfully shift to the "$\cup$Spec A_f = Spec A, and the A_f have a property $\Rightarrow A$ has a property" part of the proof.

Cover X = Spec A with a finite number of these Spec A_f with the A_f noetherian (quasi-compactness).

Now: want to show: if A is a ring, $\langle f_1, \ldots, f_n \rangle = 1$, and each A_f noetherian, is A noetherian?

Let $\mathfrak{a} \subseteq A$ be an ideal, let $\varphi : A \to A_{f_i}$ be localization (this induces $D(f) \hookrightarrow specA$). Then:

$$\mathfrak{a} = \bigcap \varphi^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$$

i.e. the commonality between pulling back all the extended versions of $\mathfrak{a}$ yields $\mathfrak{a}$ again. $\subseteq a$ is obvious. As for $\supseteq$: let b be an element of the intersection. Then

$$\varphi_i(b) = a_i / f_i^n \in A_{f_i}$$

with $a_i \in \mathfrak{a}$, and the n the same across all A_{f_i} (take the max). Then:

$$f_i^m(f_i^n b - a_i) = 0$$

(again, at first we get m_i for each i, then take a max). That is, $f_i^{m+n} b \in \mathfrak{a}$ for each i. Since Spec $A = \cup D(f_i) = \cup D(f_i^{m+n})$ we get that there are c_i such that

$$1 = \sum c_i f_i^{m+n}$$

for $c_i \in A$. Then:

$$b = \sum c_i f_i^{m+n} b \in \mathfrak{a}$$

So, we have shown $\mathfrak{a} = \cap \varphi^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$. Now suppose $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \ldots$ is an ascending chain of ideas in A. Then we get a chain of the extensions

$$\varphi_i(\mathfrak{a}_1) \cdot A_{f_i} \subseteq \varphi_i(\mathfrak{a}_2) \cdot A_{f_i} \subseteq \ldots$$

which must stabilize (and so their preimages stabilize). Then there is some N at which all the preimages on the different A_{f_i} stabilize, since there are finitely many. Hence we get that the original chain eventually stabilizes too. $\square$

Definition 8.2. A morphism $f : X \to Y$ of schemes is **locally of finite type** is there is a covering $\{V_i = \text{Spec } B_i\}$ of Y such that for each i, we have $f^{-1}(V_i)$ can be covered by $U_{i,j} = \text{Spec } A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra. (note that we have Spec $A_{ij} \to$ Spec B_i, induced by some $B_i \to A_{ij}$).

The morphism is of **finite type** if each $f^{-1}(V_i)$ can be covered by finitely many U_{ij}. (Note: if mapping to k, this says X looks like finite union of closed subsets of affine space).

Definition 8.3. A morphism $f : X \to Y$ is a **finite** morphism if there is a covering of Y by $V_i = \text{Spec } B_i$ such that $f^{-1}(V_i) \cong \text{Spec } A_i$ with A_i a finitely generated B_i-*module.*

You will prove on your homework that having these properties on one open affine cover is the same as having them on all open affine covers.

Remark 8.4. Finite morphisms have finite fibers (and are closed), and preserve the dimension of the scheme (a notion we will eventually define, but lines up with the notion for varieties).

Finite fibers, however, does not imply a finite morphism. Spec $k[t, t^{-1}] \to$ Spec $k[t]$ induced by $k[t] \to k[t, t^{-1}]$ has finite fibers, but $k[t, t^{-1}]$ is not a finite $k[t]$-module.

Remark 8.5. If the morphism is flat, then the length of the fiber is constant. This can fail for non flat morphisms. A morphism is flat if the induced stalk maps $f_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is flat. $\varphi : A \to B$ is flat if for every injective module morphism $M \to N$ you get $M \otimes_A B \to N \otimes_A B$ is injective.

Example 8.6. Finite type morphisms need not have finite fibers: Spec $k[x, y] \to$ Spec $k[x]$ given by $k[x] \hookrightarrow k[x, y]$ should be thought of projection $\mathbb{A}^2 \to \mathbb{A}^1$. This is a finite type morphism but does not have finite fibers.

Definition 8.7. An open subscheme of a scheme X is a scheme U, with topological space an open subset of X and $\mathcal{O}_U = \mathcal{O}_X|_U$. An open immersion is a morphism $f : X \to Y$ that induces an iso of X with an open subscheme of Y.

Definition 8.8. A closed immersion is a morphism $f : Y \to X$ such that

- $f(Y)$ is a closed subset of X and
- $Y \cong f(Y) \subseteq X$ is a homeomorphism of topological spaces
- the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

A closed subscheme of X is an equivalence class of closed immersions, where $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$.

Remark 8.9. Closed subschemes in general look like maps induced by $A \to A/I$. This is Hartshorne exercise II.3.11.

9. Feb 03: Closed subschemes, the fiber product

Example 9.1 (The go-to example of a closed subscheme)**.** Let A be a ring, $\mathfrak{a}$ an ideal of A. Set $Y =$ Spec $A/\mathfrak{a}$ and $X =$ Spec A. Then $A \to A/\mathfrak{a}$ induces a closed immersion $f : Y \to X$ as schemes: f is a homeomorphism onto $V(\mathfrak{a})$, and the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective since it's surjective on stalks.

Any choice of $\mathfrak{b}$ with $V(\mathfrak{a}) = V(\mathfrak{b})$ yields a scheme structure on the set $V(\mathfrak{a})$, and these can very much be different. So lots of subscheme structures on this set. Every subscheme structure on a subscheme of an affine scheme arises in this way.

As a fun example, consider $k[x]$ and $V((x)) = V((x^2))$ and the different subscheme structures these two ideals give you.

Example 9.2. From that example, it seems like there should be a unique "smallest" structure, something that eliminates the sort of "fuzz" that $V(x^2)$ would give. This is indeed true: it is the **reduced induced closed subscheme structure**.

In the above, with $V((x)) = V((x^2)) = V((x^3)) = \dots$ you want to do some more of "taking the radical" type process.

For X affine, set $\mathfrak{a} = \cap_{\mathfrak{p} \in Y} \mathfrak{p}$. This is the largest ideal for which $V(\mathfrak{a}) = Y$. Then the reduced induced structure on Y is the one defined by $\mathfrak{a}$. (Note that $V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$).

For X a scheme in general, take an affine open cover $\{U_i\}$, consider the closed (in U_i) subset $U_i \cap Y$, and give that the reduced induced structure. You can show this glues (Example 3.2.6 in Hartshorne).

Now! it is time for the ever-wonderful fiber product. Let us discuss its universal property. In a given category, the fiber product of $f : X \to Z$ and $g : Y \to Z$ is the object P with morphisms $p_1 : P \to X$, $p_2 : P \to Y$ such that for any Q with maps $q_1 : Q \to X$ and $q_2 : Q \to Y$ such that $f \circ q_1 = g \circ q_2$, there exists a unique morphism $u : Q \to P$ making the following diagram commute.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

First, some examples from topology. Let $X \to Z$ be a map and $p \to Z$ be the inclusion of a point. Then P is just the fiber (any Q with the proposed maps must land in the fiber over p and so we get the factoring).

In general, for topological spaces:

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

Let's think about affine schemes. Translating between scheme info and ring info flips all the arrows and we observe that flipping the arrows on this diagram... just yields the diagram and property of the tensor product of rings.

$$\begin{tikzcd}
\widetilde{Q} & & \\
& A \otimes_C B \arrow[ul, dashed, "u"'] & A \arrow[l] \arrow[ull, bend right] \\
& B \arrow[u] \arrow[uul, bend left] & C \arrow[l, "f^\sharp"'] \arrow[u, "g^\sharp"']
\end{tikzcd}$$

Now we simply need to patch these together.

Theorem 9.3. *For any two schemes $X \to S, Y \to S$ over a scheme S, the fiber product $X \times_S Y$ exists and is unique up to unique isomorphism.*

Proof.

- **Step 1:** *(Handling affines)*
  For affine schemes, spec of the tensor product yields the fiber product. For $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$, consider Spec $(A \otimes_R B)$. This does not immediately have the property we want in the category of schemes, because Q may not be affine. We'll work through this subtlety using a problem from your HW.

  A morphism $Q \to \text{Spec } (A \otimes_R B)$ is the same as a homomorphism $A \otimes_R B \to \Gamma(Q, \mathcal{O}_Q)$ by II.2.4 on your HW. Applying the universal property of the tensor product and the HW problem again, we get that $Q \to \text{Spec } (A \otimes_R B)$ is exactly the same as a morphism to Spec B, Spec A with the desired properties.

- **Step 2:** *(Uniqueness)*
  The fiber product, if it exists, must be unique. For two candidate fiber products F_1, F_2, you'll get maps $i : F_1 \to F_2$ and $j : F_2 \to F_1$, and $i \circ j$, $j \circ i$ being the identity will be forced by the uniqueness part of maps to the fiber product.

- **Step 3:** *(Glueing morphisms)*
  Let X, Y be arbitrary schemes. Morphisms can be described from gluing: if $\{U_i\}$ is an open cover of X, then to describe a morphism $f : X \to Y$ it's enough to describe $f_i : U_i \to Y$ and verify that the f_i, f_j agree on $U_i \cap U_j$.

- **Step 4:** *(Fiber products are nice with open subsets of one component)*
  If X, Y are schemes over S and $U \subseteq X$ open, then $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U and and Y.

  (Maps $f : Z \to U$ and $g : Z \to Y$ yield $f' : Z \to U \to X$, and hence you can get $\theta : Z \to X \times_s Y$. Since $f(Z) \subseteq U$, we can regard $\theta : Z \to p_1^{-1}(U)$. It inherits uniqueness.)

- **Step 5:** *(If you can get a fiber product using a cover of one piece, you can get it on the whole thing)*
  Suppose X, Y are schemes over S, and that $\{X_i\}$ is an open cover of X, and that $X_i \times_S Y$ exists. Then, $X \times_S Y$ exists.

  Let $p_{1,i} : X_i \times_S Y \to X_i$. Let $X_{ij} = X_i \cap X_j$, and $U_{ij} \subseteq X_i \times_S Y$ denote $p_{1,i}^{-1}(X_{i,j})$. From Step 4, U_{ij} U_{ji} are both a fiber product for X_{ij} and Y over S. Uniqueness gives unique isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$. These isomorphisms satisfy the gluing/compatibility conditions of II.2.12. (Namely, $\varphi_{ij} = \varphi_{ji}^{-1}$, and the cocycle/image condition on triple intersections).

  Thus, we can glue the $X_i \times_S Y$ to a scheme that we prematurely call $X \times_S Y$. The projection morphisms are glued from the $X_i \times_S Y$. One can check that this indeed is the fiber product.

(For a bit more detail: given $Z \to X, Z \to Y$ that yield the same map to S: we get maps $Z_i = f^{-1}(X_i) \to X_i$, yielding maps $\theta_i : Z_i \to X_i \times_S Y \to X \times_S Y$. These maps glue on the $Z_i \cap Z_j$ and yield $Z \to X \times_S Y$. Uniqueness can be checked locally, on the pieces $X_i \times_S Y$.)

- **Step 6:** *(Gluing on the two factors, over an affine base)* We know that fiber products exist for X, Y, S all affine. By gluing with step 5, we have fiber products exist for X arbitrary, Y affine, S affine. By gluing with step 5 again, fiber products exist for X, Y arbitrary and S affine.

- **Step 7:** *(Lastly, get arbitrary bases)* Let X, Y, S be arbitrary schemes with $q : X \to S$, $r : Y \to S$. Let S_i be an open affine cover of S. Let $X_i = q^{-1}(S_i)$, $Y_i r^{-1}(S_i)$. We have, by Step 6, that $X_i \times_{S_i} Y_i$ exists. Observe that $X_i \times_{S_i} Y_i$ functions as the product $X_i \times_S Y$. If $f : Z \to X_i$ and $g : Z \to Y$ yield the same map to S, then the image of g must land in S_i. So, $X_i \times_S Y$ exists for each i, and we glue to $X \times_S Y$.

$\square$

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