MATH 553: NOTES

GWYNETH MORELAND

Abstract. Notes for Math 553: algebraic geometry II, spring 2025. Heavily referenced from Vakil, Hartshorne. May contain typos. The referenced copy of Vakil is:
`https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`

Contents

1. Jan 13: Syllabus, sheaves

Recommended reading: Harthsorne II.1, Vakil 2.1-2

Up till now, you have been thinking of algebraic varieties more in the classical sense– they're zero sets of polynomials $V(f_1, \dots, f_n)$. From your perspective, in $k[x]$ you don't really care too much about x versus x^2 because their vanishing sets are the same, maybe you'd default to taking the one that generates a radical ideal. But you lose some things with this perspective. Certainly I wouldn't say it's great for multiplicities and whatnot.

So, we need to upgrade: instead of varieties in the classical sense, we'll eventually think of schemes. Some of the intuition will port over: we're thinking of things/geometric objects (or, topological spaces) that look like they're (locally) "cut out by polynomials," and a decent amount of the practical work of computing things will resemble some of the polynomial fiddling you've done before, but we're keeping track of more of the data of the **functions on these spaces**.

Roughly, a scheme has three levels of data.

- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space.

(The last one helps distinguish things like $V(x)$ versus $V(x^2)$.). Here is where we start brushing up against Grothendieck's perspective: that when studying an object, it's less important to study the object itself and more important to study functions between them, how they relate to other things.

Now, before that, we need to do **sheaves**, which are, informally, a bundling of data about functions on open sets of a topological space. **The usual example, which you should have in mind throughout, is the data of differentiable functions on a differentiable manifold**.

We begin with sheaves of sets, but the idea extends to sheaves of groups, rings, k-algebras, etc.

Definition 1.1. Let X be a topological space. A **presheaf** $\mathscr{F}$ on X is the following data:

- To each open set $U \subseteq X$, we have an assignment $\mathscr{F}(U)$ of a set (or group, or ring, etc...)
- For each inclusion $V \subseteq U$ of open sets, we have restriction maps $\mathrm{res}_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$. The restriction maps need to follow some reasonable properties:
  - $\mathrm{res}_U^U : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map.
  - For inclusions $W \subseteq V \subseteq U$ we have $\mathrm{res}_W^U = \mathrm{res}_W^V \circ \mathrm{res}_V^U$. That is, the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\mathrm{res}_V^U"] \arrow[dr, "\mathrm{res}_W^U"'] & \mathscr{F}(V) \arrow[d, "\mathrm{res}_W^V"] \\ & \mathscr{F}(W) \end{tikzcd}$$

Notational bits-and-bobs:

- Elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ over/on U.
- $\mathscr{F}(U)$ is notated a few other ways:
  - $\Gamma(U, \mathscr{F})$
  - $H^0(U, \mathscr{F})$
- Note that a presheaf is precisely the data of a contravariant functor from the category of open sets on X to the category of sets (of groups, rings, etc).

Definition 1.2. A presheaf $(X, \mathscr{F})$ is a **sheaf** if it satisfies two more additional axioms.

- **Identity/uniqueness:** If $\{U_i\}_{i \in I}$ is an open cover of U and $f_1, f_2 \in \mathscr{F}(U)$ are two sections/functions such that
$$\mathrm{res}_{U_i}^U f_1 = \mathrm{res}_{U_i}^U f_2$$
for all $i \in I$, then $f_1 = f_2$. (That is, two sections that line up on each piece of a cover have to have been the same).
- **Gluing:** Let $\{U_i\}_{i \in I}$ be an open cover of U. If you have an $f_i \in \mathscr{F}(U_i)$ for each i such that, for any i, j:
$$\mathrm{res}_{U_i \cap U_j}^{U_i} f_i = \mathrm{res}_{U_i \cap U_j}^{U_j} f_j$$
then there is an $f \in \mathscr{F}(U)$ such that $\mathrm{res}_{U_i}^U f = f_i$ for each i. (That is, you have an open cover, a choice of section on each piece of the cover, and these choices agree on overlaps, then you should be able to glue these to a section on the whole thing.)

Example 1.3. Let X be a differentiable manifold. Let $\mathscr{F}$ be the sheaf that assigns to an open set U the ring of differentiable real-valued functions $\mathscr{F}(U)$ defined on U. For $V \subseteq U$, the restriction map is the restriction of domain:

$$\mathscr{F}(U) \to \mathscr{F}(V)$$
$$f \mapsto f|_V$$

The fact that differentiable functions are "defined by their values" makes it clear that this is a presheaf. Likewise, the two additional sheaf properties are clear: if two functions agree on an open cover, they are the same function. And if you have a differentiable function on each piece and the overlaps agree, you can define the function on the whole manifold (or open set U).

Remark 1.4. In general, you may see things like $\mathrm{res}_V^U f$ written as $f|_V$ to save space.

Since I don't wait to shift gears too much on the first day, let's do an example of another important sheaf:

Example 1.5 ((Skyscraper shaves))**.** Let S be a set, $p \in X$ a point. Set:

$$i_{p,*}S(U) = \begin{cases} S & p \in U \\ \{e\} & p \notin U \end{cases}$$

here $\{e\}$ is any one element subset of S. If you roughly try to draw this, you see the skyscraper-type behavior around p.

2. Jan 15: Intro to Spec

Recommended reading: Hartshorne II.2, Vakil 3.1-3.4

We will eventually need to worry about morphisms of sheaves, pushforwards, pullbacks, and more. But that can come a bit later, when we better understand the topological spaces we want to look.

Recall that we are trying to define schemes, which consist of:

- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space.

Now on our journey towards schemes, which are our generalizations of algebraic varieties/sets, we need to think of the underlying topological space of our geometric objects. The building blocks of these will be the **spectrum of a ring**. These correspond to affine schemes, the building blocks of schemes in general.

There will resemble things from 552 somewhat: our first examples will be visualizable in some $\mathbb{C}^n$ with the many of the points corresponding to tuples $(a_1, \dots, a_n)$ satisfying some polynomials, along with some extra points that are useful to have.

Do note: ring here means a commutative ring with identity. For example: $\mathbb{C}, \mathbb{R}, \mathbb{F}_p, \overline{\mathbb{F}}_p, \mathbb{C}[t], \mathbb{C}(t)$, polynomial rings, quotient rings. We will often focus on $\mathbb{C}$-algebras or k-algebras with k algebraically closed, as this is the best place to start off. (Some of our tools will break down over k not algebraically closed). As appropriate, I may add in some examples over non-algebraically closed fields, but I will largely leave those examples to your future number theory courses.

The idea: given a ring A, we want the most natural/nontrivial space on which A becomes a "ring of functions." You've encountered this before with coordinate rings in 552.

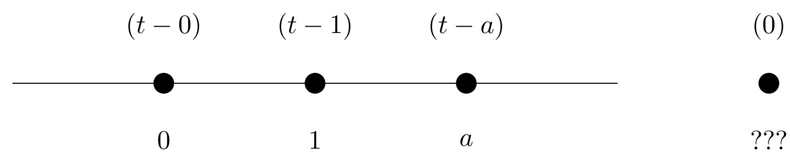
Example 2.1 ((Rough intuition))**.** The algebraic functions on the complex line $\mathbb{C}$ should be single variable polynomials: $\mathbb{C}[t]$. If you cut out the origin and consider the open set $\mathbb{C} \setminus \{0\}$, you no longer have to worry about t zeroing out, so your algebraic functions should now be $\mathbb{C}[t, t^{-1}]$.

Definition 2.2. As a set, Spec A is the set of all prime ideals of A.

Example 2.3 (The complex affine line)**.** Let us consider the case of $A = \mathbb{C}[t]$, and how we can think of $\mathbb{C}[t]$ as the ring of functions over Spec $\mathbb{C}[t]$. First, let us compute the spectrum. By the fundamental theorem of algebra, we have:

$$\mathrm{Spec}\ \mathbb{C}[t] = \{(t-a) : a \in \mathbb{C}\} \sqcup \{(0)\}$$

that is, we get a point for each element of $\mathbb{C}$, and then this extra point (0). Given that this space is "basically $\mathbb{C}$ with some extra stuff," it's not strange to think of $\mathbb{C}[t]$, i.e. complex polynomials in one variable, aka polynomials that can take in one complex input, as the ring of functions over Spec $\mathbb{C}[t]$, which is nearly $\mathbb{C}$. We visualize below:



A few things to note:

- At each point $(t-a)$ of the spectrum, we have an evaluation map
$$\begin{aligned}\mathbb{C}[t] &\to \frac{\mathbb{C}[t]}{(t-a)} \cong \mathbb{C}\\ f(t) &\mapsto f(a)\end{aligned}$$
That is, f is sent to its image in $\mathbb{C}[t]/(t-a)$, which says t can be swapped for a. That is, we send $f(t)$ to $f(a)$. This evaluates the polynomial at a. We will see a similar construction in general. Note that this means these points are keeping track of all the values of this function. If we have two different polynomials f_1, f_2 then their evaluations at some points will differ: i.e. functions are distinguished by their values. **This will not always be true!**. See Example 2.6.
- (0) is called "the generic point." It is "close" to every point, so it is "generically" on the line, but is not equal to any of the $(t-a)$. Some would choose to draw it as "fuzz" amongst the line. We will understand the generic point better when we understand the topology of Spec A.
- Spec $\mathbb{C}[t]$ will come to be known to us as the complex affine line, denote $\mathbb{A}^1_{\mathbb{C}}$.

Example 2.4 (Don't say I never gave you an example that wasn't over $\mathbb{C}$!)**.** Consider $A = \mathbb{R}[t]$. The prime ideals are of one of two forms:
$$(t-a) \quad a \in \mathbb{R} \qquad\qquad (t-a)(t-\overline{a}) \quad a \in \mathbb{C} \setminus \mathbb{R}$$
Hence we get an identification:
$$\text{Spec } \mathbb{R}[t] = \mathbb{C}/\text{Gal}(\mathbb{C}/\mathbb{R}) \sqcup \{(0)\}$$
which you can identify with the upper half plane along with a generic point.

Definition 2.5 (Evaluation map)**.** Given a ring A, $f \in A$, and $\mathfrak{p} \in \text{Spec } A$, the **value of f at $\mathfrak{p}$**, denoted by $f(\mathfrak{p})$, is the image of f under:
$$A \mapsto A/\mathfrak{p} \to \text{Frac}(A/\mathfrak{p})$$
This gives us a way to "evaluate" our sections/functions on points, but note that the field in which the values lie is thought of as varying with $\mathfrak{p}$. The field $\text{Frac}(A/\mathfrak{p}) =: k(\mathfrak{p})$ is known as the residue field at $\mathfrak{p}$.

Example 2.6 (Functions are not always separated by *values* at points)**.** Consider the set Spec $\mathbb{C}[t]/(t^2)$. As a set of points it has just one element: (t).

t is an element of the ring $\mathbb{C}[t]/(t^2)$, and we should think of it as being very small: so small that its square is zero, but it itself is not zero. If we think about the evaluations of this function note that:
$$\begin{aligned}\mathbb{C}[t]/(t^2) &\to \text{Frac}(\mathbb{C}[t]/(t,t^2)) \cong \mathbb{C}\\ t &\mapsto 0\end{aligned}$$
That is, both the function t and 0 on the LHS evaluate to 0 on the RHS.But this is the only evaluation map to consider on this spec. So, we see how functions cannot be necessarily be separated by values. Eventually, we will see that the issue is that Spec $\mathbb{C}[t]/(t^2)$ is not *reduced*. See Definition 7.13.

When drawing Spec $\mathbb{C}[t]/(t^2)$, one should visualize it as a point with a small tangent direction attached.

Now, it is time to define the topology on these spaces. The idea: closed sets should be sets of points where functions vanish (similar to 552).

$$\begin{aligned} f \text{ vanishes at } \mathfrak{p} &\iff f(\mathfrak{p}) = 0 \\ &\iff f = 0 \text{ in } A/\mathfrak{p} \\ &\iff f \in \mathfrak{p} \quad (\iff (f) \subseteq \mathfrak{p}) \end{aligned}$$

Definition 2.7 (Various vanishing loci definitions)**.** Let $f \in A$, $S \subseteq A$. Then:

$$\begin{aligned} V(f) &= \{\mathfrak{p} \in \operatorname{Spec} A : (f) \subseteq \mathfrak{p}\} \\ V(S) &= \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} A : \langle S \rangle \subseteq \mathfrak{p}\} \end{aligned}$$

Note that $V(S) = V(\langle S \rangle)$.

Definition 2.8. A (Zariski) closed subset of Spec A is any set of the form of a vanishing locus $V(\mathfrak{a})$ for $\mathfrak{a}$ an ideal.

Proposition 2.9. The collection of Zariski closed subsets forms a topology on Spec A.

Proof. Observe:

$$\begin{aligned} \varnothing &= V((1)) \\ \operatorname{Spec} A &= V((0)) \\ V(\mathfrak{a}) \cup V(\mathfrak{b}) &= V(\mathfrak{a}\mathfrak{b}) \\ \bigcap_{i \in I} V(\mathfrak{a}_i) &= V\left(\sum_{i \in I} \mathfrak{a}_i\right) \end{aligned}$$

□

Example 2.10. The closed sets in Spec $\mathbb{C}[t]$ are the whole space, the empty set, and

$$V(f(t)) = V\big(((t-a_1)\dots(t-a_n))\big) = \cup_{i=1}^n V((t-a_i)) = \{(t-a_i) : 1 \leq i \leq n\}$$

i.e. finite collections of non-generic points.

Note that $\overline{\{(0)\}} = \operatorname{Spec} \mathbb{C}[t]$. That is, the generic point is "close" to all other points, and "sits along the whole line."

Definition 2.11. Define $D(f) = \operatorname{Spec} A \setminus V((f))$. These open sets form a basis for the topology.

Proposition 2.12. Let S be a multiplicative set. By studying the map $\varphi : A \to S^{-1}A$, $a \mapsto a/1$, this induces a bijection:

$$\{\text{primes in } A \text{ with } \mathfrak{p} \cap S = \varnothing\} \longleftrightarrow \{\text{primes in } S^{-1}A\}$$

We've motivated that A should be thought of as the ring of algebraic functions over Spec A. Then, what should be the ring of functions on the open set $D(f)$? Well, since we're not working with the full spec, we should be able to invert things that don't vanish on the set. That is, things whose vanishing sets are squirreled away in $V(f)$, the set we are cutting out.

Set $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = S^{-1}A$ where S is the following multiplicative set:

$$S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

This definition only depends on $D(f)$, not on f itself. But luckily:

Proposition 2.13. The natural map

$$A_f \to \mathcal{O}_{\operatorname{Spec} A}(D(f))$$

is an isomorphism.

Lemma 2.14. *$D(f) \subseteq D(g)$ (that is, $V(g) \subseteq V(f)$) if and only if $f^n \in (g)$, if and only if g is invertible in A_f.*

Proof. $f^n \in (g) \iff \sqrt{(f)} \subseteq \sqrt{g} \iff$ the prime ideals containing (f) are a superset of those containing (g), which means $V(g) \subseteq V(f)$. Then $f^k = gm$, so g is invertible in A_f. □

That is, algebraic functions on $D(f)$ are obtained by inverting f. So, we have the makings of a *structure sheaf*, i.e. a sheaf $\mathcal{O}_{\text{Spec } A}$ where $\mathcal{O}_{\text{Spec } A}(U)$ is the ring of algebraic functions on U. But we only have it on a distinguished basis. The question becomes: is this enough to determine the sheaf overall? Will we be able to do computations in the future/check nice properties by just checking it on the basis of the $D(f)$? The answer: yes! Back to sheaf theory.

3. Jan 17: Let's understand sheaves better (stalks, morphisms)

Recommended reading: Hartshorne II.1,Vakil 2.3-2.4

We learned about the topological spaces that will be glued into schemes. These are the Spec A, and we think of A as the ring of algebraic functions on Spec A. Again, we want to assemble a structure sheaf $\mathcal{O}_{\text{Spec } A}$ on Spec A such that

$$\mathcal{O}_{\text{Spec } A}(U) = \text{ ring of algebraic functions on } U$$

From this perspective, we saw it was reasonable to set

$$\mathcal{O}_{\text{Spec } A}(D(f)) = S^{-1}A \cong A_f$$

where

$$S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

Problem: what about this would-ve sheaf on open sets in general? We would like to describe this sheaf, i.e. describe the rings of algebraic functions, on nice $D(f)$ to be enough. It is, but we need to do a bit of work to say that. (Here Vakil and Harthsorne somewhat "diverge." Vakil shows that defining a sheaf on a basis is sufficient; Hartshorne just describes the $\mathcal{O}_{\text{Spec } A}(U)$ from the get-go, with the construction being the one you'd do when defining a sheaf from a base. In the end, they are equivalent data/constructions).

We'll get to all that, but we should cover some necessary details/definitions/general knowledge first.

Definition 3.1. Let $(X, \mathscr{F})$ be a sheaf on a topological space X. Let $x \in X$ be a point. The **stalk** of $\mathscr{F}$ at x is defined as the direct limit:

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U) = \{(f, U) : f \in \mathscr{F}(U), x \in U\}$$

where $(f, U) \sim (g, V)$ if and only if there is some $W \subseteq U, V$, with W containing p, such that $f|_W = g|_W$.

You can draw a pic of this in the case of the sheaf of differentiable functions on some differentiable manifold M. Equality on the stalk means two functions, defined near at point x, agree on some smaller open set around the point. Observe that in this case, the stalk is a local ring: its unique maximal ideal is the ideal of all functions vanishing at x.

Definition 3.2. Elements of a stalk are called **germs**.

Definition 3.3. Given a section $f \in \mathscr{F}(U)$ and a point $p \in U$, we let f_p denote the image of f in the stalk:

$$\begin{aligned} \mathscr{F}(U) &\to \mathscr{F}_p \\ f &\mapsto f_p = (f, U) \end{aligned}$$

Remark 3.4. We will see later on that many properties we want to test of sheaves (or morphisms of sheaves) can be tested by checking the analogous condition on the stalks. This is reasonable, looking at the gluing axiom of a sheaf.

Definition 3.5 (Morphisms of (pre)sheaves)**.** Let $\mathscr{F}, \mathscr{G}$ be (pre)sheaves on a topological space X. A morphism $\mathscr{F} \to \mathscr{G}$ is a collection of maps $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for each U such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi_U} & \mathscr{G}(U) \\ {\scriptstyle \text{res}^U_V}\downarrow & & \downarrow{\scriptstyle \text{res}^U_V} \\ \mathscr{F}(V) & \xrightarrow{\phi_V} & \mathscr{G}(V) \end{array}$$

Consequently, we can see that φ defines a map on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ by sending $(f, U) \to (\varphi_U(f), U)$. An isomorphism is a morphism with a two-sided inverse.

Let's restrict our attention to sheaves of abelian groups at this point (we rarely fall outside this scenario).

Proposition 3.6. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on a topological space X. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then φ is an isomorphism if and only if φ_x is an isomorphism for all $x \in X$.

Proof. $\Rightarrow$ is clear. We prove the $\Leftarrow$ direction. It is enough to show that φ_U is an isomorphism for each U.

Let's start by showing injectivity: suppose $s \in \mathscr{F}(U)$ is a section such that $\varphi_U(s) = 0$ in $\mathscr{G}(U)$. Then the germ $\varphi_U(s)_x = \varphi_x(s_x)$ is zero for each $x \in U$. Then $s_x = 0$ in each $x \in U$ by injectivity on stalks. Then it follows from the definition of the stalks that we can find an open cover of U such that s restricts to zero on each piece. That is, $s = 0$.

Now we show surjectivity: suppose we are considering $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. Let $t \in \mathscr{G}(U)$. We'll piece together something that maps to it.

For each $x \in U$, we have $t_x \in \mathscr{G}_x$ and it must be the image of some $s_x \in \mathscr{F}_x$. s_x can be repped by some $s(x) \in V_x \ni x$. Then $\varphi(s(x)), t|_{V_x}$ are two elements of $\mathscr{G}(V_x)$ with the same germ, so $\varphi(s(x)), t$ agree in some neighborhood W_x of x.

Cover U with these W_x, and consider the $s(x)$ (well, technically $s(x)|_{W_x}$) that we get for each one. On the overlaps, these must agree due to injectivity (their overlaps go to $t|_{W_x}$). So we can piece them together to get an $s \in \mathscr{F}(U)$ that maps to t. □

Remark 3.7. Note that the proof of surjectivity needed injectivity!

Definition 3.8 ((Tentative definition of ker, image, coker))**.** Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves of abelian groups, we can define the presheaves $\ker(\varphi), \operatorname{coker}(\varphi), \operatorname{im}(\varphi)$, as follows:

$$\begin{aligned}
\ker(\varphi)(U) &= \ker(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) \quad [\subseteq \mathscr{F}(U)] \\
\operatorname{coker}(\varphi)(U) &= \operatorname{coker}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) \\
\operatorname{im}\varphi)(U) &= \operatorname{im}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) \quad [\subseteq \mathscr{G}(U)]
\end{aligned}$$

Proposition 3.9. Given $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves on X, the kernel is a sheaf

Proof. Identity is inherited from the parent sheaf. Gluing is too, though you need to check that the glued function f is still in the kernel. This works because $\varphi(f)$ restricts to zero on a cover, thus is zero globally from gluing in $\mathscr{G}$. □

Remark 3.10. Since we can view each $\ker(\mathscr{F})(U)$ as a subgroup of $\mathscr{F}(U)$, we can think of $\ker(\mathscr{F})(U)$ as a *subsheaf* of $\mathscr{F}(U)$.

Proposition 3.11. The image and cokernel of a sheaf morphism need not be a sheaf

For the cokernel: Let $X = \mathbb{C}$, and $\mathcal{O}$ be the sheaf of holomorphic function and $\mathcal{O}$ be the sheaf of nonzero holomorphic functions. Consider the map φ with

$$\begin{gathered}
\varphi_U : \mathcal{O}(U) \to \mathcal{O}^*(U) \\
f \mapsto e^f
\end{gathered}$$

we claim that the cokernel isn't a sheaf. First, note that there is no holomorphic f such that $e^f = z$ on $\mathbb{C} \setminus \{0\}$. Otherwise, differentiating both sides yields:

$$e^f \cdot f' = 1 \Rightarrow z \cdot f' = 1 \Rightarrow f' = 1/z$$

Integrating the LHS over a loop around zero yields 0, but integrating the RHS over said loop produces $2\pi i$. Contradiction.

Therefore, $[z] \neq 0$ in $\operatorname{coker}(\varphi)$. That is, $\Gamma(\mathbb{C} \setminus \{0\}, \operatorname{coker}(\varphi)) \neq 0$. But, take $U_1 = \mathbb{C} \setminus (-\infty, 0]$ and $U_2 = \mathbb{C} \setminus [0, \infty)$. These are simply connected, so every nonzero function on them can be writting as some e^f (we can define the log: we made a branch cut!). Thus $\operatorname{coker}(\varphi)(U_1), \operatorname{coker}(\varphi)(U_2)$ are both zero. So the cokernel fails the identity axiom.

Similarly, this shows why the image isn't necessarily a sheaf: we can't glue the logs of z into a log of z on all of $\mathbb{C} \setminus \{0\}$.

So, we have all these presheaves running around (including ones we'd really like to consider: the image and cokernel are imporant!). We would like some way to modify them into a sheaf, and it should have some nice universal property that relates it back to the original presheaf.

Construction 3.12 (Sheafification)**.** Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that: for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism φ^+ such that $\varphi = \varphi^+ \circ \theta$. The pair $(\mathscr{F}^+, \theta$ is unique up to unique isomorphism.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & \mathscr{F}^+ \arrow[d, dashed, "\varphi^+"] \\ & \mathscr{G} \end{tikzcd}$$

4. Jan 22: Sheafifcation, sheaves on a base

Recommended reading: Harthsorne II.1 ($\sim$ p. 64), Vakil 2.4-5

Recall: last time we saw how some ker, coker of morphism of sheaves was not necessarily a sheaf. This motivates sheafification, which will also set us up well for doing *sheaves on a base*, which will help us define the structure sheaf on Spec A.

Now, for the construction of the sheafification of a presheaf. The construction is: bundle the stalk data in a nice way. That is, make a big product of the stalks, but only allow combinations of germs that looked like they could glue together.

$$\begin{aligned} \mathscr{F}^+(U) := &\left\{ (f_p)_{p \in U} : \begin{array}{c} \text{for all } p \in U \text{, there is an open } V \text{ with } p \in V \subseteq U \\ \text{and an } s \in \mathscr{F}(V) \text{ such that } s_q = f_q \text{ for all } q \in V \end{array} \right\} \\ \subseteq &\prod_{p \in U} \mathscr{F}_p \end{aligned}$$

the morphism θ is clear: θ_U is $f \mapsto (f_p)_{p \in U}$. To describe φ^+: look at the sections that glue to your (f_p), look at their images, glue them in the target, and call that the image. This is unique: in order for the diagram to commute any other map would have to do the same thing.

Remark 4.1. Sheafification is a functor from presheaves on X to sheaves on X.

Remark 4.2. Specifically, given $i : \mathrm{Shf}_X \to \mathrm{Pre}_X$ the inclusion map from sheaves on X to presheaves on X, note that sheafification is $+ : \mathrm{Pre}_X \to \mathrm{Shf}_X$. Then $+$ is the left adjoint of i, i.e. given $\mathscr{F}$ a sheaf on X and $\mathscr{G}$ a presheaf on X, we have the natural bijection:

$$\mathrm{Hom}_{\mathrm{Pre}_X}(\mathscr{G}, i(\mathscr{F})) \cong \mathrm{Hom}_{\mathrm{Shf}_X}(\mathscr{G}^+, \mathscr{F})$$

Example 4.3 (Constant sheaves)**.** Let X be a topological space, S a set. You get the constant presheaf by assigning the same set to all open sets:

$$\mathscr{F}(U) = S$$

(On nonempty sets, you can interpret this as constant functions from U to S). Gluing is a mess because of disjoint sets, and the empty set presents problems too: all sections in $\mathscr{F}(\varnothing)$ restrict to the same thing on the empty cover. So the identity axiom says all sections on $\varnothing$ ought to be the same.

The sheafification $\mathscr{F}(U)$ will instead assign to U: locally constant maps from U to S. Denote this sheaf as $\underline{S}$.

Remark 4.4. Thinking of sheaves of abelian groups: sheafification adds the gluings that should exist but don't, and kills off the nonzero sections that are locally zero.

Proposition 4.5. $\mathscr{F} \to \mathscr{F}^+$ yields an isomorphism of stalks.

Proof. Work from the explicit description. $\square$

Definition 4.6. We say that a map of sheaves is injective if and only if the kernel sheaf is zero.

Lemma 4.7. A map of sheaves if injective $\iff$ $\varphi : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Likewise, it is injective $\iff$ it is injective on stalks.

Proof. This was done in the bijectivity proof before. □

Definition 4.8. We define the image and cokernel *sheaves* by taking the sheafification of the presheaves defined above. We generally just call them $\mathrm{im}(\varphi)$ and $\mathrm{coker}(\varphi)$ and drop any + notation, and usually refer to the presheaf versions as $\mathrm{im}(\varphi)^{\mathrm{pre}}, \mathrm{coker}(\varphi)^{\mathrm{pre}}$.

Remark 4.9. Consider a map of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ on X. Since we have a map $\mathrm{im}^{\mathrm{pre}}(\varphi) \to \mathscr{G}$, we necessarily have a map $\mathrm{im}(\varphi) \to G$. This map is injective: it is injective on the level of stalks (note that the presheaf and sheafified image have the same stalks!). Thus, we can identify $\mathrm{im}(\varphi)$ with a subsheaf of $\mathscr{G}$.

Definition 4.10. A morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if $\mathrm{im}(\varphi) = \mathscr{G}$.

Lemma 4.11. $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if $\mathscr{F}_x \to \mathscr{G}_x$ is surjective for all points x.

Proof. $\Leftarrow$: $\mathrm{im}(\mathscr{F}) = \mathscr{G}$ means the stalks are isomorphic, hence $\mathscr{F}_x \to \mathscr{G}_x$ must be surjective.

$\Rightarrow$: we want to show that $\mathrm{im}(\mathscr{F}) = \mathscr{G}$. Well, the map on stalks is an isomorphism (injective and surjective on stalks), so they are equal. □

Example 4.12. In our example with $X = \mathbb{C}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ the sheaf of non-vansihing holomorphic functions and

$$\mathcal{O}_X \to \mathcal{O}_X^*$$
$$f \mapsto e^f$$

we have that $\mathrm{im}(\varphi) = \mathcal{O}_X^*$ and $\mathrm{coker}(\varphi) = 0$. This can be seen via φ being surjective on the level of stalks (and correspondingly the cokernel is zero on the level of stalks).

Definition 4.13. A sequence of maps

$$\mathscr{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathscr{F}^i \overset{\varphi^i}{\to} \mathscr{F}^{i+1}$$

is exact if at each stage, $\ker \varphi^i = \mathrm{im}\varphi^{i-1}$.

5. Jan 24: Sheaves on a base, affine schemes

Recommended reading: Vakil 2.5, 4.1

Time to handle an issue: sometimes we understand a sheaf really well on a nice basis. But what about the rest? The details are sometimes unpleasant/obfuscating: it is mainly important to know that the data of the sheaf on a suitably nice basis is enough to determine the sheaf. The construction will be reminiscent of sheafification.

Definition 5.1. A base of a topology is a collection of open sets $\{B_j\}_{j\in J}$ such that any open set of X can be written as a union of B_j.

Remark 5.2. $(f) \subseteq \mathfrak{a} \iff V(f) \supseteq V(\mathfrak{a}) \iff D(f) \subseteq D(\mathfrak{a})$, so the $D(f)$ genuinely are a basis of the Zariski topology on Spec A.

Definition 5.3. Suppose $\{B_i\}$ is a basis on X. A presheaf of sets on the base if an assignment $F(B_i)$ for each B_i. If $B_j \subseteq B_i$, we have restriction maps $\mathrm{res}^{B_i}_{B_j}$ satisfying $\mathrm{res}^{B_i}_{B_i} = \mathrm{id}$ and $\mathrm{res}^{B_i}_{B_k} = \mathrm{res}^{B_j}_{B_k} \circ \mathrm{res}^{B_i}_{B_j}$.

For sheaves on a base: there are base identity and base gluing axioms:

- If $B \in \{B_i\}$ can be written as $B = \cup_{i\in J} B_i$ and $f, g \in F(B)$ with $\mathrm{res}^B_{B_i} f = \mathrm{res}^B_{B_i} g$ for all $i \in J$, then $f = g$.
- If we have $f_i \in B_i$ for $i \in J$ such that for any i, j we have $\mathrm{res}^{B_i}_{B_k} f_i = \mathrm{res}^{B_i}_{B_k} f_j$ for any $B_k \subseteq B_i \cap B_j$, then there is an $f \in F(B)$ such that $f|_{B_i} = f_i$ for all $i \in J$.

Theorem 5.4. Suppose $\{B_i\}$ a base on X, and F a sheaf of sets on this case. There is a sheaf $\mathscr{F}$ extending F ($\mathscr{F}(B_i) \cong F(B_i)$ with isomorphisms agreeing with restriction maps). $\mathscr{F}$ is unique up to unique isomorphism.

Proof. As before, $\mathscr{F}$ is a sheaf of compatible germs. Define the stalk of a presheaf F on a base as:

$$F_p = \varinjlim_{B_i \ni p} F(B_i)$$

Define

$$\mathscr{F}(U) := \left\{ (f_p \in F_p)_{p \in U} : \begin{array}{c} \text{for all } p \in U \text{, there is a } B \text{ with } p \in B \subseteq U \\ \text{and an } s \in \mathscr{F}(B) \text{ such that } s_q = f_q \text{ for all } q \in B \end{array} \right\} \subseteq \prod_{p \in U} F_p$$

We get a map $F(B) \to \mathscr{F}(B)$ for each B, which is an isomorphism. Checking the details is similar to the work for sheafificiation.

Note that clearly $\mathscr{F}_p \cong F_p$. □

We can finally really talk about the structure sheaf on Spec A. Consider Spec A with the Zariski topology, and for open sets $D(f)$ set

$$\mathcal{O}_{\text{Spec } A}(D(f)) = S^{-1}A \cong A_f$$

where $S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$. The restriction maps are clear enough: if $D(g) \subseteq D(f)$ then the restriction map

$$\text{res}^{D(f)}_{D(g)} : \mathcal{O}_{\text{Spec } A}(D(f)) \to \mathcal{O}_{\text{Spec } A}(D(g))$$

is further localization. This is clearly a presheaf on a distinguished base.

Lemma 5.5. Spec A is quasi-compact (every open cover has a finite subcover).

Proof. It's enough to show this for covers for the form $\{D(f_i)\}$. Note that $\cup D(f_i) = D(\sum(f_i))$. This will be all of Spec A only when $1 \in sum(f_i)$, in which case we get $1 = a_{i_1} f_{i_1} + \cdots + a_{i_k} f_{i_k}$ and you can just take the corresponding cover pieces $D(f_{i_1}), \ldots, D(f_{i_k})$. □

Theorem 5.6. This assignment of $\mathcal{O}_{\text{Spec } A}(D(f))$ gives a sheaf on a distinguished base, and thus determines a sheaf on Spec A. This sheaf is the **structure sheaf** on Spec A, and is referred to as $\mathcal{O}_{\text{Spec } A}$ or just $\mathcal{O}$ if it is clear what A is.

Proof. It's enough to show identity and gluing on just A (if you want to show it on $D(f)$, that's the same as swapping the ring out for A_f, modulo some detail-checking).

- **Identity axiom**: Write Spec $A = \cup_i D(f_i)$. Then after potentially relabeling, we can pick a finite subcover. Write Spec $A = \cup_{i=1}^n D(f_i)$. That is, $V((f_1, \ldots, f_n)) = \varnothing$, i.e. $(f_1, \ldots, f_n) = A$.

  Suppose we have a section $s \in \mathcal{O}_{\text{Spec } A}(\text{Spec } A) = A$ such that $\text{res}^{\text{Spec } A}_{D(f_i)} s = 0 \in A_{f_i}$ for each f_i. That means there is some m such that $f_i^m s = 0$ (in A) for $1 \leq i \leq n$.

  But note that $D(f_i) = D(f_i^m)$ (as f_i vanishes at $\mathfrak{p}$ if and only if f_i^m vanishes at $\mathfrak{p}$). So there are g_i such that

  $$1 = \sum_{i=1}^n g_i f_i^m$$

  But then:

  $$s = \sum_{i=1}^n g_i f_i^m s = \sum 0 = 0$$

- **Gluing**: Again, being able to write 1 as a sum of these f_i will let us piece things together in a nice way.

  Again, say we have some gluing data on an open cover. Pick a finite subcover $\{D(f_i)\}_{i=1}^n$. Let $s_i \in \mathcal{O}_{\text{Spec } A}(D(f_i))$ so that

  $$\text{res}^{D(f_i)}_{D(f_i) \cap D(f_j)} s_i = \text{res}^{D(f_j)}_{D(f_i) \cap D(f_j)} s_j$$

  noting that $D(f_i) \cap D(f_j) = D(f_i f_j)$. Identifying $\mathcal{O}_{\text{Spec } A}(D(g))$ with A_g, we get that

  $$s_i = \frac{a_i}{f_i^{\ell_i}}, \quad s_j = \frac{a_j}{f_j^{\ell_j}}$$

  and because their restrictions are the same in $A_{f_i f_j}$, it must be that there is an $m_{i,j}$ such that

  $$(a_i f_j^{\ell_j} - a_j f_i^{\ell_i})(f_i f_j)^{m_{i,j}} = 0.$$

Let $m = \max m_{i,j}$. Then the above tells us that:

$$(a_i f_i^m) f_j^{m+\ell_j} = (a_j f_j^m) f_i^{m+\ell}.$$

Now again, Spec $A = \cup D(f_i^{m+\ell_i})$, so there exists $g_i \in A$ such that

$$1 = g_1 f_1^{m+\ell_1} + \dots g_n f_n^{m+\ell_n}$$

and consider the element of A given by:

$$s = g_1 a_1 f_1^m + \dots g_n a_n f_n^m$$

Then observe that:

$$\begin{aligned} f_i^{m+\ell_i} s &= g_1(a_1 f_1^m) f_i^{m+\ell_i} + \dots + g_n(a_n f_n^m) f_i^{m+\ell_i} \\ &= g_1 f_1^{m+\ell_1}(a_i f_i^m) + \dots + g_n f_n^{m+\ell_n}(a_i f_i^m) \\ &= (g_1 f_1^{m+\ell_1} + \dots g_n f_n^{m+\ell_n}) a_i f_i^m \\ &= a_i f_i^m \end{aligned}$$

That is, $f_i^m(f_i^{\ell_i} s - a_i) = 0$. That is, $s = \frac{a_i}{f_i^{\ell_i}} = s_i$ on A_{f_i}, which is what we wanted.

You can use the identity axiom proved prior to show that the resulting glued object restricts to what you want on the other elements of the a priori infinite cover. So the identity proof does need to come first!

□

Thus, we can finally start talking about affine schemes.

6. Jan 27: Affines schemes, schemes

Recommended reading: Hartshorne II.2, Vakil 4.1-4.4

Proposition 6.1. Let A be a ring and $\mathcal{O}$ the structure sheaf on Spec A. For any $\mathfrak{p} \in$ Spec A, the stalk $\mathcal{O}_\mathfrak{p} \cong A_\mathfrak{p}$.

Proof. Fairly evident from the description of $A_\mathfrak{p}$ as a direct limit identifying a bunch of subsequent localizations. For $f \notin \mathfrak{p}, D(f)$ will appear in the direct limit. Lemma 2.14 can help. To be more concrete, we can write down the map.

And $(s, D(f)) \in \mathcal{O}_\mathfrak{p}$ can be sent to its image in $A_\mathfrak{p}$. It is surjective: any element in $A_\mathfrak{p}$ is of the form a/g for $g \notin \mathfrak{p}$, and so $\mathfrak{p} \in D(g)$. That is, $D(g)$ will be a neighborhood of $\mathfrak{p}$ and a/g will be hit by the map.

It is injective: write $s = a/f, t = b/g$, with $f, g \notin \mathfrak{p}$. These are sections on $D(f), D(g)$ respectively. If their image is the same in $A_\mathfrak{p}$, then there is some $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$ in A. But then $D(fgh) = D(f) \cap D(g) \cap D(h)$ is a neighborhood of $\mathfrak{p}$ and so s, t (after restriction to $D(fgh)$) would have been identified in the stalk $\mathcal{O}_p$. □

Now, we need some way to compare or relate sheaves on different spaces. This necessitates the direct image and inverse image functors.

Definition 6.2. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf on X. The **direct image** sheaf $f_*\mathscr{F}$ on Y is defined via

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$$

for open sets $V \subseteq Y$. This is a functor from sheaves on X to sheaves on Y.

Definition 6.3. Let $f : X \to Y$ a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. The **inverse image** sheaf $f^{-1}\mathscr{G}$ on X is the sheafification of the presheaf:

$$U \mapsto \varinjlim_{V^{\text{open}} \supseteq f(U)} \mathscr{G}(V).$$

This is a functor from sheaves on Y to sheaves on X.

Definition 6.4. If $i : Z \to X$ is a subset of X with the subspace topology, then $i^{-1}\mathscr{F}$ is the restriction of $\mathscr{F}$ to Z, denoted by $\mathscr{F}|_Z$. For open sets Z this will just turn into $\mathscr{F}|_Z(V) = \mathscr{F}(V)$.

Definition 6.5. A **ringed space** is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f : X \to Y$ and a morphism of sheaves (of rings) $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if all the stalks $\mathcal{O}_{X.p}$ are local rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced map on stalks $f_p^\sharp : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings.

Here, a local homomorphism of local rings $\varphi : A \to B$ is a ring morphism such that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. An isomorphism is a morphism (of ringed or locally ringed spaces respectively) with a two-sided inverse. Equivalently, in $(f, f^\sharp)$ the f is a homeomorphism and the $f^\sharp$ is an isomorphism of sheaves.

Remark 6.6. The induced map on stalks comes from:

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{V \ni p} \mathcal{O}_Y(V) \to \varinjlim_{f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \to \varinjlim_{U \ni p} \mathcal{O}_X(U) = \mathcal{O}_{X,P}$$

Proposition 6.7.

(a) $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space.

(b) $\varphi : A \to B$ a morphism of rings induces

$$(f, f^\sharp) : (\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)$$

(c) In fact, any morphism of locally ringed spaces $(\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A$ is induced by a homomorphism of rings.

Proof.

(a) Immediate from previous results.

(b) The map on topological spaces is $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$ so the map in continuous. Two ways to see the induced morphism on structure sheaves:
   - Certainly we yield a morphism on a base
   $$\mathcal{O}_A(D(f)) \to \mathcal{O}_B(D(\varphi(f)) = \mathcal{O}_B(f^{-1}(D(f)) = f_*(\mathcal{O}_B)(D(f))$$
   via $A_f \to B_{\varphi(f)}$ in the obvious way, and it respects restriction maps.
   - Localize at each prime to get a local homomorphism of loca rings $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} to B_\mathfrak{p}$. Since sheaves are isomorphic to their sheafification, you can interpret sections on U as collections of compatible germs, and so you can just map germs (and compatibility is preserved).

(c) Take global sections: we must have a map:
$$\varphi : \mathcal{O}_A(\text{Spec } A) \cong A \to \mathcal{O}_{\text{Spec } B}(\text{Spec } B) \cong B$$
One can show that φ induces all of the data of the morphism. Notably, we must have an induced morphism on stalks: $A_{f(\mathfrak{p})} \to B_\mathfrak{p}$. Due to compatibility, we must have:
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow{f_\mathfrak{p}^\sharp} & B_\mathfrak{p} \end{array}$$
Since $f_\mathfrak{p}^\sharp$ is a local homomorphism, it must be that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so that map f on points coincides with the one induced by φ. Then compatibility with restriction maps will force the $f^\sharp$ to be induced by φ as well.

□

Definition 6.8. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic, as a locally ringed space, to some $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $p \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Example 6.9 (Schemes can be glued)**.** Let X_1, X_2 be schemes. Let $U_i \subseteq X_i$ be open sets. Let $\varphi : (U_1, \mathcal{O}_X|_{U_1}) \to (U_2, \mathcal{O}_X|_{U_2})$ be an isomorphism of locally ringed spaces.

Then we can defined a scheme X obtained by gluing X_1, X_2 by identifying U_1, U_2 via the morphism φ. The topological space is the quotient of $X_1 \sqcup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$. The space is endowed with the quotient topology (a set is open $\iff$ its preimage is open).

We get maps $i_j : X_j \to X$ and the structure sheaf is defined as:

$$\mathcal{O}_X(V) = \{(s_1, s_2) : s_j \in \mathcal{O}_{X_j}(i_j^{-1}(V)), \quad \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2}\}$$

that is, sections on sets that "see" the overlap are gotten from piecing together compatible sections on (subsets of) each X_1, X_2.

Example 6.10 (More concrete: the projective line)**.** Recall that the morphisms of affine schemes are induced by ring morphisms on the globals ections. Glue Spec $\mathbb{C}[t]$ and Spec $\mathbb{C}[s]$ along $D(t) = \text{Spec } \mathbb{C}[t, t^{-1}] \cong \text{Spec } \mathbb{C}[s, s^{-1}] = D(s)$ via the following:

$$\begin{tikzcd} \text{Spec } \mathbb{C}[s] & \text{Spec } \mathbb{C}[t] & \mathbb{C}[s] \arrow[d] & C[t] \arrow[d] \\ \text{Spec } \mathbb{C}[s, s^{-1}] \arrow[u, "i_1"] \arrow[r] & \text{Spec } \mathbb{C}[t, t^{-1}] \arrow[u, "i_2"] & \mathbb{C}[s, s^{-1}] & \mathbb{C}[t, t^{-1}] \arrow[l, "t \mapsto s^{-1}"'] \end{tikzcd}$$

This yields the projective line. We will learn about the proj construction in general next lecture.

Example 6.11 (What if you take the other transition function?)**.** If we instead take the transition function as $t \mapsto s$:

$$\begin{tikzcd} \text{Spec } \mathbb{C}[s] & \text{Spec } \mathbb{C}[t] & \mathbb{C}[s] \arrow[d] & C[t] \arrow[d] \\ \text{Spec } \mathbb{C}[s, s^{-1}] \arrow[u, "i_1"] \arrow[r] & \text{Spec } \mathbb{C}[t, t^{-1}] \arrow[u, "i_2"] & \mathbb{C}[s, s^{-1}] & \mathbb{C}[t, t^{-1}] \arrow[l, "t \mapsto s"'] \end{tikzcd}$$

then this glues everything away from the origin in a "straightforward" way and we get the affine line with a doubled origin.

7. Jan 29: Proj, properties of schemes

Recommended reading: Hartshorne II.2 ($\sim$ 76-77), Vakil 4.5

Now for the proj construction: we want a big class of examples from projective varieties, and we want a big class of interesting schemes in one fell swoop. An important tool is the notion of gluing schemes from more than two charts: the details are handled in Harthsorne exercise II.2.12. Note the cocycle condition on triple overlaps.

Intuition from 552 remains: if $S_\bullet = k[x_0, \ldots, x_n]$, the proj construction yields $\mathbb{P}^n_k$ and if $S_\bullet = k[x_0, \ldots, x_n]/(f)$ where f is homogeneous, we get something "cut out" of $\mathbb{P}^n_k$ by the equation $f = 0$.

Definition 7.1 ($\mathbb{Z}$-graded rings)**.** A $\mathbb{Z}$-graded ring is a ring $S_\bullet = \oplus_{n\in\mathbb{Z}} S_n$ where multiplication respects grading: $S_m \times S_n \to S_{m+n}$. S_0 is a subring and each S_n is an S_0 module, and $S_\bullet$ is an S_0 module. A $\mathbb{Z}^{\geq 0}$-graded ring is a $\mathbb{Z}$-graded ring with no elements of negative degree. We will, in the future, use graded ring to refer to a $\mathbb{Z}^{\geq 0}$ graded ring.

Definition 7.2. An element of some S_n is a homogeneous element. If it is nonzero, nonzero, the subscript yields the degree.

Definition 7.3. An ideal I of $S_\bullet$ is homogeneous if it is generated by homogeneous elements.

Proposition 7.4. An ideal is homogeneous if and only if it contains the degree n piece of each of its elements.

Proof. An induction proof by successively lopping off the top-degree pieces. $\square$

Definition 7.5. In a graded ring $S_\bullet$, the irrelevant ideal refers to $S_+ := \oplus_{i>0} S_i$.

Definition 7.6. As a set, Proj S is the set of all homogeneous prime ideals $\mathfrak{p}$ that do not contain all of S_+. For $\mathfrak{a}$ a homogeneous ideal of S, we define the subset

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj } S : \mathfrak{a} \subseteq \mathfrak{p}\}$$

For a set T, $V(T) = V((T))$. We have distinguished open sets (well, we will eventually see they're open) $D(f) := \text{Proj } S \setminus V((f))$ for f homogeneous. Note that $D(fg) = D(f) \cap D(g)$.

Lemma 7.7.
(a) For $\mathfrak{a}, \mathfrak{b}$ homogeneous ideals in S, we have $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
(b) For any collection of homogeneous ideals $\{a_i\}$ of S, we have

$$V\left(\sum \mathfrak{a}_i\right) = \bigcap V(\mathfrak{a}_i)$$

Proof. Same as before, accounting for the following: a homogeneous prime ideal $\mathfrak{p}$ is prime $\iff$ for two homogeneous $a, b \in S$, the product $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. □

Hence we can define a Zariski topology on Proj S. Now we must define a structure sheaf on this space. The idea: on the $D(f)$ we'd like the scheme to look like Spec $((S_\bullet)_f)_0$. Think about the standard affine opens on $\mathbb{P}^n_k$ from 552, where the coordinate rings look like $k[x_0/x_i, \dots, x_n/x_i]$.

Definition 7.8. For $f \in S_+$, set

$$\mathcal{O}_{\text{Proj } S_\bullet}(D(f)) = \mathcal{O}(D(f)) = ((S_\bullet)_f)_0 = \text{"}S_{(f)}\text{"}$$

See Hartshorne p. 76 or Vakil Section 4.5 if you want to see more on the details on issues relating to, e.g., whether restriction maps will make sense.

Proposition 7.9. Let S be a graded ring.
(a) The stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local ring $SS_\mathfrak{p}$, the degree zero elements of S localized at all **homogeneous** elements not in $\mathfrak{p}$.
(b) We have that

$$D(f), \mathcal{O}|_{D(f)} \cong \text{Spec } ((S_f)_0)$$

(c) Proj S is a scheme.

It would do you well to read Exercise II.2.12 to get a sense of the work needed to glue together schemes.

Example 7.10. For A a ring, we get $\mathbb{P}^n_A = \text{Proj } A[x_0, \dots, x_n]$, the projective n-space over A. For $A = k$ algebraically closed, you get something whose set of closed points is homeomorphic to the usual variety we know as projective n-space from 552.

Definition 7.11. Let S be a fixed scheme. A **scheme over** S is a scheme X with a morphism $X \to S$. A morphism $X \to Y$ as schemes over S is a morphism of schemes $f : X \to Y$ that is compatible with the morphisms to S. Then $\mathfrak{Sch}(S)$ is the category of schemes over S. If A is a ring, $\mathfrak{Sch}(A)$ is the category of schemes over Spec A.

Proposition 7.12. Let k be algebraically closed. There is a natural, fully faithful (that is, bijective on hom sets) functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$. For any variety, the topological space is homeomorphic to the set of closed points $\text{sp}(t(V))$ and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via the homeomorphism.

Proof. See II.2, Proposition 2.6, of Harthshorne. □

Now it's time to think of all the interesting properties of schemes we could want:

Definition 7.13 (Big list of scheme adjectives)**.** Let X be a scheme.
(a) X is connected if its topological space is connected
(b) X is irreducible if the topological space is irreducible (all nonempty open sets dense).
(c) X is integral if all the $\mathcal{O}_X(U)$ are integral domains
(d) X is reduced if all the $\mathcal{O}_X(U)$ have no nilpotent elements (equivalently, by II.2.3, all the stalks have no nonzero nilpotents).

Remark 7.14. At this point, it is useful to remark that the residue field of a point $\mathfrak{p}$ in a scheme X is

$$k(\mathfrak{p}) := \mathcal{O}_{X,\mathfrak{p}}/m_\mathfrak{p}$$

This lines up with our old definition: if $\mathfrak{p}$ lies in an affine open $U \cong (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$, then $k(\mathfrak{p}) = \text{Frac}(A/\mathfrak{p})$. The presentation above has the advantage of not needing an affine chart to state it. Likewise, we have a notion of evaluation: for $f \in \mathcal{O}_X(U)$, we have $f(\mathfrak{p})$ is the image of f under $\mathcal{O}_X(U) \to \mathcal{O}_{X,\mathfrak{p}}/m_\mathfrak{p} = k(\mathfrak{p})$.

Note that for a section $f \in \mathcal{O}_X(U)$, we have that $f_\mathfrak{p} \in \mathfrak{m}_p \subseteq \mathcal{O}_{X,\mathfrak{p}}$ is the same as $f(\mathfrak{p}) = 0$.

Proposition 7.15. A scheme is integral iff it is both reduced and irreducible.

Proof. Integral certainly implies reduced. And if it's not irreducible, then it has two nonempty disjoint sets, yielding

$$\mathcal{O}_X(U_1 \sqcup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$

which is not integral.

Conversely: suppose X is reduced and irreducible. Suppose there are $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. Then look at

$$Y = \{x \in U : f_x \in m_x\} = \{x \in U : f(x) = 0\} Z \qquad = \{x \in U : g_x \in m_x\} = \{x \in U : g(x) = 0\}$$

These are closed subsets (exercise II.2.16 - on HW! Note that these are defined by vanishing conditions) of U, and $Y \cup Z = U$. But X is irreducible, so U is irreducible. So, then, say $Y = U$. But then f is nilpotent on any affine open in U (II.2.18a) meaning f is zero. □

Proposition 7.16. Suppose X is a reduced scheme. Let $f, g \in \Gamma(X, \mathcal{O}_X)$. Then:

$$f = g \iff f(x) = g(x) \text{ (in } k(x)) \text{ for all } x \in X$$

That is, evaluating the same everywhere means the two sections are the same.

Remark 7.17. What this says is that, on reduced schemes, functions are determined by their values. Recall that the example of a setting where this is not true was Spec $k[x]/(x^2)$, which is certainly not reduced.

Proof. $\Rightarrow$: this direction is obvious.

$\Leftarrow$: We may assume X is affine (you'll get equality on each open affine, and then glue to finish). In that case, $X = \text{Spec } A$ for A with nilradical equal to (0). We have:

$$A \to \prod_{\mathfrak{p} \in \text{Spec } A} \text{Frac}(A/\mathfrak{p}) = \prod_{\mathfrak{p} \in \text{Spec } A} k(\mathfrak{p}) \quad \left(\text{equivalently, } A \hookrightarrow \prod_{\mathfrak{p}} \mathcal{O}_{\text{Spec } A, \mathfrak{p}} \to \prod_{\mathfrak{p}} k(\mathfrak{p}) \right)$$

The kernel is the intersection of all prime ideals, which is (0). That is, the map is injective. So since $f - g$ maps to zero, it must be that $f - g = 0$, and we are done. □

Definition 7.18. A scheme is **locally noetherian** if it can be covered by open affine subsets Spec A_i where each A_i is notherian. X is **noetherian** if it is locally noetherian and quasi-compact. Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets Spec A_i, each A_i noetherian.

Remark 7.19. X being noetherian (so basically a.c.c. on ideals) means that the topological space is noetherian (d.c.c. on closed subsets).

8. Jan 31: More properties of schemes

Recommended reading: Harthshorne II.3 (especially Prop 3.2), Vakil 5.1-5.3

The following is an important type of proof. In our definitions of various adjectives, we often want to say that there's just one (affine) cover with a certain property (as that's easy to prove). When we use this adjective in proofs, we would like to be able to say *every* (affine) open cover has a certain property (as that's more useful to us).

These proofs tend to have a "going down, going up" sort of process: you want that if a ring B has a property then localizations B_f have the property, and then if a bunch of B_{f_i} have a property and $\cup_i \text{Spec } B_{f_i} = \text{Spec } B$ (i.e. $\sum(f_i) = 1$) implies that B has that property. More formally, you'll see this referred to as affine communication.

Proposition 8.1. A scheme X is locally noetherian iff for *every* open affine $U = \text{Spec } A$, A is noetherian.

Proof. $\Leftarrow$: this direction is clear. $\Rightarrow$: Note: if B is noetherian, so is any localization B_f. Note, then, that we have a base for the topology consisting of specs of noetherian rings, and thus our $U = \text{Spec } A$ can be covered by specs of noetherian rings.

So we may restrict to the following: if $X = \text{Spec } A$ is an affine scheme covered by spectra of noetherian rings, then A is noetherian. Let $U = \text{Spec } B$ be an open subset of X, with B noetherian. Then for some $f \in A, D(f) \subseteq U$ we have:

$$\begin{tikzcd} \text{Spec } A & \\ D(f) \arrow[u] \arrow[r] & \text{Spec } B \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d] \arrow[dr] & \\ A_f & B \arrow[l] \end{tikzcd}$$

Let $\overline{f}$ be the image of f in B. Then $A_f \cong B_{\overline{f}}$ (as both should be the coordinate ring of $D(f)$). Thus, A_f is noetherian. So we successfully shift to the "$\cup$Spec A_f = Spec A and the A_f have a property $\Rightarrow A$ has a property" part of the proof.

Cover X = Spec A with a finite number of these Spec A_f with A_f noetherian. We can do this because affine schemes are quasicompact. Now: we want to show that if $(f_1, \dots, f_n) = (1)$ and each A_f is noetherian, then A is noetherian.

Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\varphi_i : A \to A_{f_i}$. Then we claim that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \varphi^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$$

i.e. the commonality between pulling back all the extended versions of $\mathfrak{a}$ yields $\mathfrak{a}$ again. The $\subseteq$ containment is obvious. As for $\supseteq$: let b be an element of the intersection. Then:

$$\varphi_i(b) = a_i/f_i^N \in A_{f_i}$$

with $a_i \in \mathfrak{a}$ and the N the same across all A_{f_i} (take the max). Then there is an M such that for any i:

$$f_i^M(f_i^N b - a_i) = 0$$

That is, $f_i^{M+N} b \in \mathfrak{a}$ for each i. Since Spec $A = \cup D(f_i) = \cup D(f_i^{m+n})$ we get that there are c_i such that

$$1 = \sum_{i=1}^{n} c_i f_i^{M+N}$$

for $c_i \in A$. Then:

$$b = \sum c_i f_i^{M+N} b \in \mathfrak{a}$$

So, we have shown $\mathfrak{a} = \cap \varphi^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$. Now suppose that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$ is an ascending chain of ideals in A. Then for each $1 \leq i \leq n$ we get a chain of extensions in A_{f_i}

$$\varphi_i(\mathfrak{a}_1) \cdot A_{f_i} \subseteq \varphi(\mathfrak{a}_2) \cdot A_{f_i} \subseteq \dots$$

which must stabilize (and so their preimages stabilize). Then there is some step L at which all the preimages on the different A_{f_i} stabilize, since there are finitely many. Hence we get that the original chain eventually stabilizes too. □

Definition 8.2. A morphism $f : X \to Y$ of schemes is **locally of finite type** if there is a covering $\{V_i = \text{Spec } B_i\}$ of Y such that for each i, we have that $f^{-1}(V_i)$ can be covered by $U_{i,j}$ = Spec A_{ij} where each A_{ij} is a finitely generated B_i-algebra. (Note that we have Spec $A_{ij} \to$ Spec B_i induced by some $B_i \to A_{ij}$).

The morphism is of **finite type** if each $f^{-1}(V_i)$ can be covered by finitely many U_{ij}.

Remark 8.3. Note: if the morphism is $f : X \to$ Spec k, being finite type means that X looks like the finite patching of closed subsets of affine space.

Definition 8.4. A morphism $f : X \to Y$ is a **finite** morphism if there is a covering of Y by V_i = Spec B_i such that $f^{-1}(V_i) \cong$ Spec A_i with A_i a finitely generated B_i-*module*.

You will prove on your homework that having these properties on one open affine cover is the same as having them on all open affines.

Remark 8.5. Finite morphisms have finite fibers (and are closed) and preserve the dimension of the scheme (a notion we will eventually define, but lines up with the notion for varieties).

Finite fibers, however, does not imply a finite morphism. Spec $k[t, t^{-1}] \to$ Spec $k[t]$ induced by $k[t] \to k[t, t^{-1}]$ has finite fibers, but $k[t, t^{-1}]$ is not a finite $k[t]$-module.

Remark 8.6. If the morphism is flat, then the length of the fiber is constant. This can fail for non-flat morphisms. A morphism is flat if the induced stalk maps $f_P : \mathcal{O}_{Y,f(P)} \to O_{X,P}$ is flat. $\varphi : A \to B$ is flat if for every injective module morphism $M \to N$ you get that $M \otimes_A B \to N \otimes_A B$ is injective.

Example 8.7. Finite type morphisms need not have finite fibers: Spec $k[x, y] \to$ Spec $k[x]$ given by $k[x] \hookrightarrow k[x, y]$ should be thought of as projection $\mathbb{A}^2 \to \mathbb{A}^1$. This is a finite type morphism but it does not have finite fibers.

Definition 8.8. An open subscheme of a scheme X is a scheme U, with topological space an open subset of X and $\mathcal{O}_U = \mathcal{O}_X|_U$. An open immersion is a morphism $f : X \to Y$ that induces an isomorphism of X with an open subscheme of Y.

Definition 8.9. A closed immersion if a morphism $f : Y \to X$ such that

- $f(Y)$ is a closet subset of X and
- $f : Y \to f(Y) \subseteq Z$ is a homeomorphism of topological spaces
- the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective

A closed subscheme of X is an equivalence class of closed immersions, where $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$.

Remark 8.10. Closed subschemes in general look like maps induced by $A \to A/I$. This is Harthsorne exercise II.3.11.

9. Feb 03: Closed subschemes, fiber product

Recommended reading: Hartshorne II.3, Vakil 8.1, 8.3, 9.1

Example 9.1 (The go-to example of a closed subscheme)**.** Let A be a ring, $\mathfrak{a}$ an ideal of A. Set $Y =$ Spec $A/\mathfrak{a}$ and $X =$ Spec A. Then $A \to A/\mathfrak{a}$ induces a closed immersion $f : Y \to X$ as schemes: f is a homeomorphism onto $V(\mathfrak{a})$ and the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective since it's surjective on stalks.

Any choice of $\mathfrak{b}$ with $V(\mathfrak{a}) = V(\mathfrak{b})$ yields a scheme structure on the set $V(\mathfrak{a})$ and these can very much be different. So there are lots of subscheme structures on this set. Every subscheme structure on a closed subscheme of an affine scheme arises this way.

As a fun example, consider $k[x]$ and $V((x)) = V((x^2))$ and the different subscheme structures these two ideals give you.

Example 9.2. From that example, it seems like there should be a unique "smallest" structure, something that eliminates the sort of "fuzz" that $V((x^2))$ would give. This is indeed true: it is the reduced induced closed subscheme structure.

In the above, with $V((x)) = V((x^2)) = V((x^3)) = \dots$ you want to do some sort of "taking the radical" type process.

Let Y be a closed subset of X. For X affine, set $\mathfrak{a} = \cap_{\mathfrak{p} \in Y} \mathfrak{p}$. This is the largest ideal for which $V(\mathfrak{a}) = Y$. Then the reduced induced structure on Y is the one defined by $\mathfrak{a}$. (Note that $V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$).

For X a scheme in general, take an affine open cover $\{U_i\}$, consider the closed (in U_i) subset $U_i \cap Y$, and give that the reduced induced structure. You can show this glues (Example II.3.2.6 in Hartshorne).

Now! It is time for the ever-wonderful fiber product. Let us discuss its universal property. In a given category, the fiber product of $f : X \to Z$ and $g : Y \to Z$ is the object P with morphisms $p_1 : P \to X$, $p_2 : P \to Y$ such that for any Q with maps $q_1 : Q \to X$ and $q_2 : Q \to Y$ with $f \circ q_1 = g \circ q_2$, there exists a unique morphism $u : Q \to P$ making the following diagram commute.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

The object P is usually denoted by $X \times_Z Y$. The p_1, p_2 should be thought of as projection maps, as we see below.

First, some examples from topology. Let $X \to Z$ be a map and $\{p\} \to Z$ be the inclusion of a point. Then $P = X \times_Z \{p\}$ is just the fiber (any Q with the proposed maps must land in the fiber over p and so we get the factoring).

In general, for topological spaces:

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

Let's think about affine schemes. Translating between scheme info and ring info flips all the arrows and we observe that flipping the arrows on this diagram... just yields the diagram and universal property of the tensor product of rings.

$$\begin{tikzcd}
Q' & & \\
& A \otimes_C B \arrow[ul, dashed, "u"] & B \arrow[l, "1\otimes \mathrm{id}"] \arrow[llu, bend right, "q_2"'] \\
& A \arrow[u, "\mathrm{id}\otimes 1"] \arrow[luu, bend left, "q_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

So it seems like fiber products should exist for affine schemes. Now we simply need to patch these together.

Theorem 9.3. For any two schemes $X \to S, Y \to S$ over a scheme S, the fiber product $X \times_S Y$ exists and is unique up to unique isomorphism.

Proof.

- **Step 1:** *(Handling affines)*
  For affine schemes, spec of the tensor product yields the fiber product. For $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$, consider Spec $(A \otimes_R B)$. This does not immediately have the property we want in the category of schemes, because Q may not be affine. We'll work through this subtlety using a problem from your HW.

  A morphism $Q \to \text{Spec } (A \otimes_R B)$ is the same as a homomorphism $A \times_R B \to \Gamma(Q, \mathcal{O}_Q)$ by Exercise II.2.4. Applying the universal property of the tensor product and the HW problem again, we get that QtoSpec $(A \otimes_R B)$ is exactly the same as a morphism to Spec B, Spec A with the desired composition properties.

- **Step 2:** *(Uniqueness)*
  The fiber product, if it exists, must be unique. For two candidate fiber products F_1, F_2, you'll get maps $i : F_1 \to F_2$ and $j : F_2 \to F_1$, and $i \circ j, j \circ i$ being the identity will be forced by the uniqueness part of maps to the fiber product.

- **Step 3:** *(Gluing morphisms)*
  Let X, Y be arbitrary schemes. Morphisms can be described from gluing: if $\{U_i\}$ is an open cover of X, then to describe a morphism $f : X \to Y$ it's enough to describe $f_i : U_i \to Y$ and verify that the f_i, f_j agree on $U_i \cap U_j$.

- **Step 4:** *(Fiber products are nice with open sets of one component)*
  If X, Y are schemes over S and $U \subseteq X$ open, then $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U and Y.

  (Maps $f : Z \to U$ and $g : Z \to Y$ yield $f' : Z \to U \to X$ and hence you can get $\theta : Z \to X \times_S Y$. Since $f(Z) \subseteq U$, we can regard $\theta : Z \to p_1^{-1}(U)$. It inherits uniqueness).

- **Step 5:** *(If you can get a fiber using a cover on one piece, you can get it on the whole thing)*
  Suppose X, Y are schemes over S, and $\{X_i\}$ is an open cover of X, and that $X_i \times_S Y$. exists. Then,

$X \times_S Y$ exists.

Let $p_{1,i} : X_i \times_S Y \to X_i$. Let $X_{ij} = X_i \cap X_j$, and $U_{ij} \subseteq X_i \times_X Y$ denote $p_{1,i}^{-1}(X_{i,j})$. From Step 4, U_{ij}, U_{ji} are both a fiber product for X_{ij} and Y over S. Uniqueness properties of the fiber product give unique isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$. These isomorphisms satisfy the gluing/compatibility conditions of Exercise II.2.12. (Namely, $\varphi_{ij} = \varphi_{ji}^{-1}$, and the cocycle/image condition on triple intersections.).

Thus, we can glue the $X_i \times_S Y$ to a scheme that we prematurely call $X \times_S Y$. The projection morphisms are glued from the $X_i \times_S Y$. One can check that this is indeed the fiber product.

(For a bit more detail: given $Z \to X, Z \to Y$ that yield the same map to S: we get maps $Z_i = f^{-1}(X_i) \to X_i$ yielding maps $\theta : Z_i \to X_i \times_S Y \to X \times_S Y$. These maps glue on the $Z_i \cap Z_j$ and yield $Z \to X \times_S Y$. Uniqueness can be checked locally, on the pieces $X_i \times_S Y$).

- **Step 6:** *(Gluing on the two factors, over an affine base)*
  We know that fiber products exist for X, Y, S all affine. By gluing on the first factor with step 5, we have fiber products exist for X arbitrary, Y affine, S affine. By gluing with Step 5 on the second factor, fiber products exist for X, Y arbitrary and S affine.

- **Step 7:** *(Lastly, get arbitrary bases)*
  Let X, Y, S be arbitrary schemes, with $f : X \to S, g : Y \to S$. Let $\{S_i\}$ be an open affine cover of S. Let $X_i = f^{-1}(S_i), Y_i = g^{-1}(S_i)$. We have, by step 6, that $X_i \times_{S_i} Y_i$ exists. Observe that $X_i \times_{S_i} Y_i$ functions as the fiber product $X_i \times_S Y$. If $f : Z \to X_i$ and $g : Z \to Y$ yield the same map to S, then the image of g must land in S_i. So, $X_i \times_S Y$ exists for each i, and we glue to $X \times_S Y$.

$\square$

10. Feb 05: Fiber product examples, base change

Recommended reading: Hartshorne II.3, Vakil 9.1-4

It's about time we do some examples!

Example 10.1. Given a map $f : X \to Y$ and a point $y \in Y$ we can take $i : \{y\} = \text{Spec } k(y) \hookrightarrow Y$ via $\mathcal{O}_X \to i_*(k(y))$, which will just be a skyscraper sheaf of $k(y)$ over the point y. Then $X \times_Y \text{Spec } k(y)$ is topologically the fiber $f^{-1}(y)$. **The structure on it is not necessarily reduced!!**

For example: let k be algebraically closed. Consider

$$\mathbb{A}^1_k = \text{Spec } k[t] \to \text{Spec } k[s] \mathbb{A}^1_k$$

induced by

$$k[t] \leftarrow k[s]$$
$$t^2 \leftarrow\!\shortmid s$$

Then the fiber over a point a is:

$$\text{Spec } \left(k[t] \otimes_{k[s]} \frac{k[s]}{(s-a)} \right) \cong \text{Spec } \frac{k[t]}{(t^2 - a)} \cong \begin{cases} \text{Spec } \frac{k[t]}{(t-\sqrt{a})(t+\sqrt{a})} \cong \text{Spec } (k \times k) & a \neq 0 \\ \text{Spec } \frac{k[t]}{(t^2)} & a = 0 \end{cases}$$

Both of these rings *are* 2-dimensional vector spaces over k, but one of them does not give a reduced scheme.

Example 10.2 (Reduction modulo p)**.** We can always form the following diagram:

$$\begin{array}{ccc} X_{(p)} = X \times_{\text{Spec } \mathbb{Z}} \text{Spec } (\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & (p) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \text{Spec } Z \end{array}$$

This is the reduction modulo p of the scheme X. You can also take $X_{(0)} = X \times_{\text{Spec } Z} (\mathbb{Q})$. In the case of, say, Spec $(\mathbb{Z}[x]/(x^4 + x^3 + 1))$ doing this process with $p = 5$ would yield Spec $(\mathbb{F}_5[x]/(x^4 + x^3 + 1))$.

Example 10.3 (Base extension in general)**.** . Recall that a scheme over S is a scheme X with a map $f : X \to S$. Perhaps you'd like to consider it over some other base. Well, if you have $S' \to S$, then you have the base extension:

$$\begin{array}{ccc} X_{S'} = X \times_S S' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

One of the many usages is the following: if you consider an elliptic curve C over $\mathbb{Q}$ (so $f : C \to \text{Spec } \mathbb{Q}$), then you could consider it over an extension L of $\mathbb{Q}$ by considering $C \times_{\text{Spec } \mathbb{Q}} \text{Spec } L \to \text{Spec } L$ and see if your elliptic curve acquires any more closed points.

It is interesting to see what properties are preserved by base change (and in doing these examples, we will also study properties in families).

Example 10.4 (Investigating irreducibility)**.** Consider Spec $k[x, y, t](xy - t) \to \text{Spec } k[t]$ induced by

$$\begin{aligned} k[t] &\to k[x, y, t]/(xy - t) \\ t &\mapsto t \end{aligned}$$

You should think of Spec $k[x, y, t]/(xy-t)$ as a surface in $\mathbb{A}^3$ and the morpism to $\mathbb{A}^1 = \text{Spec } k[t]$ corresponding to projection onto the third factor. Over each closed point $(t - a)$ of the affine line, we get a fiber, which looks like

$$\frac{k[x, y, t]}{(xy - t)} \otimes_{k[t]} \frac{k[t]}{(t - a)} = \frac{k[x, y]}{(xy - a)}.$$

That as, ranging over the fibers yields a family of hyperbola. For $a \neq 0$, the fiber is nice and irreducible. For $a = 0$, we get a union of two axes, and it is very much reducible. Note that the total space Spec $k[x, y, t]/(xy-t)$ is irreducible. So irreducibility does not need to be preserve by base change.

Example 10.5 (Investigating reducedness)**.** This was already done in the $k[s] \to k[t], s \mapsto t^2$ example. You can also look at Spec $k[x, y, t]/(ty - x^2) \to \text{Spec } k[t]$. The fiber over $(t - a)$ for $a \neq 0$ is a parabola, and then degenerates to the doubled line $x^2 = 0$ in Spec $k[x, y]$ for $a = 0$.

11. Feb 07: Dimension, separatedness, valuative criterion

Recommended reading: Hartshorne II.4, Vakil 10.1-10.3, 12.7

Definition 11.1. The **dimension** of a scheme X, denote $\dim X$, is its dimension as a topological space: the supremum of all n such that there is a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

with Z_i distinct, irreducible closed subsets.

Definition 11.2. Given $Z \subseteq X$ irreducible, we have $\text{codim}(Z, X)$ is the supremum of integers n such that we have a chain

$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

with Z_i irreducible and closed. For Y closed subsets in general, $\text{codim}(Y, X) = \inf_{Z^{\text{irred}} \subseteq Y} \text{codim}(Z, X)$.

Remark 11.3. For affine schemes, Krull dimension aligns with the above notion of dimension.

Remark 11.4. No, it's not true in general that for $Y \subseteq X$, that $\dim Y + \text{codim}(Y, X) = \dim X$. For most "nice" scheme we encounter this will be true, but localizations can lead to quite the messes.

Proposition 11.5. Finite morphisms are preserved under base change. That is, if $f : X \to Y$ is finite, then $f' : X \times_Y Z \to Z$ is finite for any $Z \to Y$.

Proof. We can check this on affines: if $B \to A$ makes A a finite B-module, and we have $B \to C$, then we need $A \times_B C$ is a finitely generated C-module. This is true: using the finite list of generators (generators of A over B)$\otimes$ 1 will work, by shuffling coefficients to the left as needed. $\square$

Proposition 11.6. Finite morphisms have finite fibers.

Proof. Let $f : X \to Y$ be a finite morphism. Let Spec $k(\mathfrak{p}) \to Y$ be the inclusion of a point. Form the fiber product $X \times_Y \text{Spec } k(\mathfrak{p}) \to \text{Spec } k(\mathfrak{p})$ to get the fiber over $\mathfrak{p}$.

Finiteness is respected by base change. So, we have a finite morphism to a point. This makes $X \times_Y \text{Spec } k(\mathfrak{p})$ the spec of a ring that is a finite k-module, hence Artinian. Artinian rings have Krull dimension zero. $\square$

Now for two more scheme properties (well, specifically, properties of a morphism between schemes) that correspond to two well-liked topological properties. Separatedness corresponds to the Hausdorff property: a notion of being able to separate points. Properness is meant to be analogous to the topological sense: preimage of a compact set is compact.

But we need new notions: the Zariski topology is basically never Hausdorff, and topological properties only capture so much of a scheme. Our definitions will reflect some of the functorial properties.

Definition 11.7. Let $f : X \to Y$ be a morphism of schemes. We have a diagonal morphism $\Delta : X \to X \times_Y X$ determined by the diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_2"] \arrow[ddr, bend right, "\mathrm{id}_1"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The morphism is **separated** if Δ is a closed immersion. We say that X is separated over Y. A scheme is separated if it's separated over Spec $\mathbb{Z}$.

Remark 11.8. We can now give this tidbit: when people talk about a "variety" in the context of scheme theory, they generally mean an integral (so irreducible and reduced) scheme that is separated and finite type of k.

Example 11.9 (The standard example of a scheme not separated over k)**.** Consider X, the affine line (over k) with the origin doubled. This is Spec $k[t]$ and Spec $k[s]$ glued along the opens Spec $k[t, t^{-1}]$ and Spec $k[s, s^{-1}]$ via $s \mapsto t$.

Note that $X \times_k X$ is the affine plane with doubled axes and four origins (you can think of this via intuition on closed points, and verify with chart computations). Then the image of the diagonal map is the usual diagonal in the affine plane part, with two of those origins. This is not closed, because all four origins are in the closure of $\Delta(X)$ (think about limit points).

Proposition 11.10. if $f : X = \text{Spec } A \to Y = \text{Spec } B$ is a morphism of affine schemes, then f is separated.

Proof. The fiber product $X \times_Y X$ is given by Spec $A \otimes BA$ with diagonal morphism induced by $A \otimes_B A \to A$ induced by $a \otimes a' = aa'$. This is surjective, hence the diagonal map is a closed immersion (see Exercise II.2.18(c)). $\square$

Corollary 11.11. A morphism of schemes $f : X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.

Proof. $\Rightarrow$ is obvious. We do $\Leftarrow$. Let $p_1 : X \times_Y X$ be the first projection. Since $p_1 \circ \Delta = \text{id}_X$, Δ must be a homeomorphism onto its image.

Now, we need to check that $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective. For $P \in X$, let U be an affine open containing P, such tha $f(U)$ is contained in some open affine $V \subseteq Y$. Then $U \times_V U$ is a neighborhood of $\Delta(P)$ and we know $U \to U \times_V U$ is a closed immersion. That is our map of sheaves is surjective in a neighborhood of P (we can think of as: map is surjective on stalks). $\square$

Next is the oft-cited valuative criterion for separatedness. The idea is: separated schemes shouldn't have this odd sort of "doubled point" behavior, a way to limit to two different things. Alternatively, if X is separated, then given a morphism of a punctured curve $C \setminus \{p\} \to X$, there should be at most one morphism $C \to X$ extending it. Note that the line with the doubled origin very much fails this criterion.

This criterion is local, so we swap out a curve with a punctured small neighborhood (thinking in terms of $\mathbb{C}$)/germ of a curve. This corresponds roughly to a DVR. But our schemes may be fairly general, so we just use valuation rings, and then we make the criterion relative to a morphism.

Definition 11.12. Let K be a field, and G a totally ordered abelian group. A valuation of K with values in G is a map
$$v : K \setminus \{0\} \to G$$
such that for all $x, y \in K \setminus \{0\}$ we have
(1) $v(xy) = v(x) + v(y)$
(2) $v(x + y) \geq \min(v(x), v(y))$.
The set
$$R = \{x \in K : v(x) \geq 0\} \cup \{0\}$$
is a subring of K, called the valuation ring of v. A **valuation ring** is an integral domain that is the valuation ring of some valuation of its quotient field.

Definition 11.13. A valuation is discrete if G is the integers. The valuation ring is called a discrete valuation ring.

Example 11.14. Examples of DVRs include:
- $\mathbb{Z}_{(p)}$, the integers localized at a prime
- $\mathbb{Z}_p$, the ring of p-adic integers
- Rings of formal power series $k[[T]]$
- $k[x]_{(x)}$.

Theorem 11.15 (Valuative criterion of separatedness)**.** Let $f : X \to Y$ be a morphism of schemes, and X Noetherian. Then f is separated if and only if the following condition holds (for all K, R and relevant maps). Let K be a field and R a valuation ring with quotient field K. Let i : Spec $K \to$ Spec R be the morphism induced by inclusion $R \hookrightarrow K$. Given a morphism Spec $R \to Y$ and a morphism Spec $K \to X$ yielding the following diagram
$$\begin{tikzcd} \text{Spec } K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \text{Spec } R \arrow[r] \arrow[ur, dashed, "\theta"] & Y \end{tikzcd}$$
there is at most one morphism θ : Spec $R \to X$ making the diagram commute.

Proof. See Theorem II.4.3. in Hartshorne. □

We will get more into the intuitive idea behind this criterion and its corollaries next lecture.

12. Feb 10: Valuative criterion of separatedness, properness

Recommended reading: Hartshorne II.4, Vakil 10.1-10.3, 12.7

Remark 12.1. The condition of X Noetherian is used here for niceness, namely in guaranteeing that $f : X \to Y$ is quasi-separated. (Meaning, the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact (preimage of a quasi-compact is quasi-compact).

Remark 12.2. What's the intuition here? At first, it may not seem like this setup corresponds to the intuition about nice ways to fill in a curve and such. Let's elucidate:

Let's think about one of our favorite DVRs: $k[x]_{(x)}$. This is the stalk of $\mathcal{O}_{\text{Spec } k[x]}$ at the closed point (x). So we should think of this as the ring of germs near the origin. Since $k[x]_{(x)}$ should be thought of as the ring of functions over its spec, Spec $k[x]_{(x)}$ should then be thought of as an "arbitrarily small neighborhood of the origin" or a "germ of the curve $\mathbb{A}^1$." From this perspective, if we think in a relative sense, the fractional field Frac$(k[x]_{(x)}) = k(x)$ should be thought of as functions you get after puncturing the origin. That is, Spec $k(x)$ should be thought of as a small, punctured neighborhood.

The diagram now follows the initial goal: we have a neighborhood of a curve mapping downstairs to Y, and if we have a lift of the punctured neighborhood to X, there should be at most one way to fill it in (so that it's a lift of the non-punctured neighborhood). In general, for X an irreducible Noetherian separated curve, and p a regular closed point on it $\mathcal{O}_{X,p}$ is a DVR, so this idea extends to things that don't just look like pieces of $\mathbb{A}^1$.

Note that the fact that we're working with schemes and bringing along all this data of the functions on our spaces is key: set-wise, Spec $k[x]_{(x)}$ is just two points, and Spec $k(x)$ is just one point, and the set/topological data is unable to tell the full story.

Corollary 12.3. Assume all schemes are noetherian.

(a) Open and closed immersions are separated
(b) A composition of two separated morphisms is separated
(c) Separated morphisms are stable under base change: $f : X \to Y$ separated implies $f' : X \times_Y Z \to Z$ is separated.
(d) If $f : X \to Y$ and $f' : X' \to Y'$ are separated with all schemes over S, then $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated.
(e) $f : X \to Y$, $g : Y \to Z$ morphisms and $g \circ f : X \to Z$ separated implies f is separated.
(f) A morphism $f : X \to Y$ is separated if and only if Y can be covered by open V_i such that $f^{-1}(V_i) \to V_i$ is separated for each i. (We say that being separated is *local on the base.*

Proof. These can all be proven using the valuative criterion (some also can be proven from the definition without much tedium). To demonstrate the style of proof, we show (c). Let $f : X \to Y$ be a separated morphism, and $f' : X' = X \times_Y Y' \to Y'$ be a base change. We wish to show that f' is separated. Consider the following diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X' = X \times_Y Y' \arrow[r] \arrow[d, "f"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\theta_1", "\theta_2"'] & Y' \arrow[r] & Y \end{tikzcd}$$

Suppose there are two distinct lifts $\theta_1, \theta_2 : \operatorname{Spec} R \to X'$. By composing with $X' \to X$, we get two maps $\tau_1, \tau_2 : \operatorname{Spec} R \to X$ which must be the same because X is separated. But then the θ_i look the same after composing with each of the two projections out of X'. By the universal property of the fiber product, $\theta_1 = \theta_2$. ☐

Corollary 12.4 (Corollary to part (f))**.** Affine morphisms (that is, morphisms where the preimage of an affine is affine) are separated.

Proposition 12.5 (Valuative criterion for separatedness: DVR version)**.** Suppose $f : X \to Y$ is a morphism of finite type of locally Noetherian schemes. Then f is separated if and only if for any <u>DVR</u> R with quotient field K with a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\theta"] & Y \end{tikzcd}$$

there is at most one morphism Spec $R \to X$ filling in this diagram.

Proof. Vakil 12.7.1 will have some exposition on this. ☐

And now for properness. In topology, a proper morphism $f : X \to Y$ is one where the preimage of a compact set is compact. For nice spaces, this is the same thing as being locally closed: $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed for any topological space. Again, with some suitable niceness conditions (e.g. X, Y Hausdorff, Y locally compact) this is the same $X \times_Y Z \to Z$ being closed for all base changes. This is the property on which the notion of a proper morphism of schemes is based on.

Definition 12.6. A morphism $f : X \to Y$ is **proper** if it separated of finite type, and universally closed (see below).

Definition 12.7. A morphism $f : X \to Y$ is **universally closed** if it is closed and for any $Z \to Y$ the base change $f' : X \times_Y Z \to Z$ is closed.

Example 12.8. Let k be a field, and $X = \text{Spec } k[t]$ the affine line over k. X is separated and finite type over k, but not proper. The fiber product $X \times_k X \to X$ is the affine plane with a projection onto one axis. If we consider the closed set $V((xy-1))$, this is closed but it projects to the punctured affine line.

We begin to see the issue: because we lack the point at infinity, nothing is getting sent to the origin. This suggests that the projective line has a good shot at being proper over k (and indeed it is: one can roughly see this through the valuative criterion). In fact, any projective variety over a field is proper. (Given that properness is meant to be an analogue of the topological notion of properness, schemes proper over k really out to be compact).

Theorem 12.9 (Valuative criterion of properness)**.** Let $f : X \to Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if, for every valuation ring R with quotient field K and $i : \text{Spec } K \to \text{Spec } R$ induced by $R \hookrightarrow K$ and diagram

$$\begin{tikzcd} \text{Spec } K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \text{Spec } R \arrow[r] \arrow[ur, dashed, "\theta"] & Y \end{tikzcd}$$

there is **exactly** one morphism $\text{Spec } R \to X$ that fills in the diagram.

Corollary 12.10. Assume all schemes are noetherian.

(a) Closed immersions are proper
(b) A composition of proper morphisms is proper
(c) Proper morphisms are stable under base change
(d) If $f : X \to Y$ and $f' : X' \to Y'$ are proper with all schemes over S, then $f \times f' : X \times_S X' \to Y \times_S Y'$ is proper.
(e) $f : X \to Y$, $g : Y \to Z$ morphisms and $g \circ f : X \to Z$ proper and g separated implies f is separated.
(f) A morphism $f : X \to Y$ is proper if and only if Y can be covered by open V_i such that $f^{-1}(V_i) \to V_i$ is proper for each i. (That is, being proper is local on the base).

Proof. See Corollary II.4.8 in Harthsorne. ☐

13. Feb 12: Projective morphisms, $\mathcal{O}_X$-modules

Recommended reading: Hartshorne II.4, Hartshorne II.5, Vakil 2.2

Lastly for Hartshorne chapter 4, we quickly define projective morphisms. The idea: emulating the form of projective k-schemes, i.e. things that look like closed subschemes in $\mathbb{P}^n_k = \text{Proj } k[x_0, \ldots, x_{n+1}]$ along with their maps to Spec k.

Recall that one can define projective n-space over any ring A. You give elements of A degree 0 the variables x_i degree 1, and take Proj $A[x_0, \ldots, x_n] := \mathbb{P}^n_A$. Note that you have $\mathbb{P}^n_A \to \text{Spec } A$.

Note that if $A \to B$ is a homomorphism of rings yielding a map $\text{Spec } B \to \text{Spec } A$, then you can form the fiber project $\mathbb{P}^n \times_{\text{Spec } A} \text{Spec } B$ and in fact:

$$\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\text{Spec } A} \text{Spec } B$$

(This can be seen by taking charts on $\mathbb{P}^n_A$ and realizing that the tensor product will yield the fiber product on the scheme side).

All this discussion motivates the following.

Definition 13.1. If Y is a scheme, we define projective n-space over Y, denoted $\mathbb{P}^n_Y$, to be $\mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} Y$.

Definition 13.2. A morphism $f : X \to Y$ of schemes is **projective** provided that it factors as $i : X \to \mathbb{P}^n_Y$ followed by a projection $\mathbb{P}^n_Y \to Y$.

A morphism is **quasi-projective** if it factors into an open immersion $j : X \to X'$ followed by a projective morphism $g : X' \to Y$.

Recall that the projective line was the answer to the affine line not being proper over k. That is, projective k-varieties have nice compactness properties over k. The most important property is the following:

Theorem 13.3. A projective morphism of noetherian schemes is proper. A quasiprojective morphism of noetherian schemes is of finite type and separated.

Proof. See Hartshorne II.4.9. □

Chow's lemma (X a scheme over S noetherian, then there is a scheme X' projective over S and surjective S-morphism $f : X' \to X$ and open dense $U \subseteq X$ such that $f^{-1}(U) \cong U$) says, roughly, that proper morphisms can be well approximated by projective ones. You can see more on Chow's lemma through Hartshorne exercise II.4.10.

Definition 13.4. An abstract variety, or just variety, is an integral (so, irreducible and reduced) separated scheme of finite type over an algebraically closed field k. If it is proper over k, we also say it is complete.

Now: we investigate sheaves of modules. We've been investigating structure sheaves for a while, but we'll get a lot more mileage out of the sheaf framework if we start considering sheaves of modules over schemes (i.e. sheaves of abelian groups with an appropriate scalar multiplication structure on each open set).

Definition 13.5. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules (or just "an $\mathcal{O}_X$-module") is a sheaf of abelian groups $\mathscr{F}$ on X such that for each U, the group $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module, and the restriction morphisms $\mathscr{F}(U) \to \mathscr{F}(V)$ are compatible with module structures via $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. That is: for $a \in \mathcal{O}_X(U)$ and $m \in \mathscr{F}(U)$ we have:

$$(a \cdot m)|_V = (a)|_V \cdot (m)|_V$$

A morphism of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open U, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

Remark 13.6. We especially like quasi-coherent and coherent sheaves, which are $\mathcal{O}_X$-modules which play the role of analogue of modules and finitely generated modules over a ring, respectively.

Example 13.7. Prototypical example: Consider Spec $k[x, y]$, the affine plane. We can think of the x-axis in here, i.e. where $y = 0$. That is, we have the closed subscheme Spec $k[x] \to$ Spec $k[x, y]$, induced by viewing the vanishing set as $V((y))$ (if we viewed it as $V((y^2))$ that would yield a different scheme structure), the vanishing of the ideal (y). We can think of a sheaf (of abelian groups) that, on each open U, keeps track of the ideal defining the portion of the x-axis in that set. That is:

$$\mathscr{F}(\text{Spec } k[x, y]) = (y)k[x, y], \quad \mathscr{F}(D(f)) = (y)k[x, y]_f$$

Note that on each open, the ideal $\mathscr{F}(U)$ has the structure of being an $\mathcal{O}_X(U) = A_f$ module. We will see it is an example of a sheaf of ideals.

Some properties:

- The kernel, cokernel, and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module (and we do mean the sheafified versions here!)
- If you have a subsheaf of an $\mathcal{O}_X$-module $\mathscr{F}$, the quotient $\mathscr{F}/\mathscr{F}'$ is again an $\mathcal{O}_X$-module
- Direct sums, direct products, direct limits, inverse limits of $\mathcal{O}_X$-modules are $\mathcal{O}_X$-modules
- If $\mathscr{F}, \mathscr{G}$ are two $\mathcal{O}_X$-modules we may define the group of morphisms $\text{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ as the group of $\mathcal{O}_X$ morphisms $\mathscr{F} \to \mathscr{G}$.
- In fact, we can think of this sheaf-wise: we can form a sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ via:
$$U \mapsto \text{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$
This is also an $\mathcal{O}_X$-module.
- The tensor product of two modules $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is the <u>sheafification</u> of the presheaf
$$U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$$
If the $\mathcal{O}_X$ is understood you may just see $\mathscr{F} \otimes \mathscr{G}$. Note: the sheafifcation step is important!
  - (When we get to Serre twists, we will have a nice example of why you need this sheafification!)

- (If you just want an example of the general phenomena of how taking tensor products on opens doesn't necessarily yield a sheaf: consider the constant sheaf $\underline{\mathbb{Z}}$ on a topological space X with multiple components. Form a new presheaf by taking $U \mapsto \Gamma(U, \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \Gamma(U, \underline{\mathbb{Z}})$. You can show that this is not a sheaf.)

Definition 13.8. A **sheaf of ideals** on X is a sheaf of modules $\mathscr{I}$ that is a subsheaf of $\mathcal{O}_X$. That is, $\mathscr{I}(U)$ is an ideal of $\mathcal{O}_X(U)$ on each U.

14. Feb 14: Locally free sheaves, vector bundle motivation

Recommended reading: Hartshorne II.5, Vakil 13.1

Definition 14.1. An $\mathcal{O}_X$-module is free if it is isomorphic to the direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open U such that $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module. You can then define the **rank** of a sheaf $\mathscr{F}$ on such a set.

Remark 14.2. If X is connected, the rank must be constant.

Definition 14.3. An **invertible sheaf** is a sheaf of rank one everywhere.

Remark 14.4. We will see why this should be thought of as "invertible" later.

The motivation for studying locally free sheaves comes from vector bundles. For the purposes of discussing motivation: it is a little easier to consider things in the topological scenario (on the scheme side: thinking about closed points yields the same sort of picture, though the full details are explored in Hartshorne Exercise II.5.18)

A rank n vector bundle on a manifold M is a map $\pi : B \to M$ such that each fiber $\pi^{-1}(x)$ has the structure of an n-dimensional real vector space, and for each point $p \in X$ we have some $U \ni p$ such that we can trivialize the bundle:

$$\phi_U : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

so that the following diagram commutes (and is an isomorphism of vector spaces over each $x \in U$).

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\text{proj onto 1st factor}"] \\ & U & \end{tikzcd}$$

A **section** over U is a map $s : U \to B$ such that $\pi \circ s = \text{id}$. On a trivialization, we see that this is the data of $U \to U \times \mathbb{R}^n$ that looks like the identity on the first part and an n-tuple of functions to $\mathbb{R}$ on the second part. That is, it looks like an element of $\mathcal{O}_X(U)^{\oplus n}$. So the sheaf of sections $\mathscr{F}$ of this vector bundle satisfies:

$$\mathscr{F}|_U \cong (\mathcal{O}_X|_U)^{\oplus n}$$

On overlaps of U_i, U_j open in X, we have:

$$\phi_{U_j}^{-1} \circ \phi_{U_i} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$$

and the map is given by some element $T_{i,j}$ of $\mathrm{GL}_n(\mathcal{O}(U_i \cap U_j))$ on the second factor, i.e. they look like $T_{i,j} : U \cap V \to \mathrm{GL}_n(\mathbb{R})$. These are the transition functions, and they determine the vector bundle.

Now, suppose we have the setup of a vector bundle and trivializations on an open cover $\{U_i\}$. Consider a section $s \in \mathscr{F}(U_i \cap U_j)$, which can be interpreted as s_i an n-tuple of functions by viewing $U_i \cap U_j$ as a subset of U_i. The various expressions s_i and s_j are related by those same transition functions:

$$T_{ij} s_i = s_j$$

Conversely, if you have a locally free sheaf $\mathscr{F}$ on M of rank n, and trivializations on neighborhoods U_i so that $\mathscr{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$, we have transition functions $T_{i,j} \in GL_n(\mathcal{O}(U_i \cap U_j))$ on the overlaps.

That is to say: the data of a locally free sheaf of rank n is equivalent to the data of a vector bundle of rank n. In algebraic geometry, we often like to study the sheaf of sections over the vector bundle. This framework has some nice features: for one: locally free sheaves slot into the category of coherent sheaves, which are nice to study. Two: this tends to be quicker to define than geometric vector bundles. Three: it suits the modern perspective of studying functions (sections) instead of just spaces (the bundle).

15. Feb 17: Direct image, inverse image, quasicoherence, coherence

Recommended reading: Hartshorne II.5, Vakil 13.1-13.5, 16.1-16.3

Construction 15.1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathscr{F}$ is a $\mathcal{O}_X$-module, then $f_*\mathscr{F}$ is an $f_*\mathcal{O}_X$ module. Because f yields a map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we obtain an $\mathcal{O}_Y$-module structure on $f_*\mathscr{F}$. This is the **direct image** of $\mathscr{F}$ by f.

If we have direct images, we want some way to turn $\mathcal{O}_Y$ modules into $\mathcal{O}_X$-modules via the data of a morphism $X \to Y$. To do this, we need to dive into the inverse image functor. Namely, its important adjoint property.

Recall that for $f : X \to Y$ a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y, the inverse image sheaf $f^{-1}\mathscr{G}$ is defined as the sheafification of the presheaf

$$U \subseteq X \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) := f^{-1}\mathscr{G}^{\text{pre}}$$

Proposition 15.2 (Hartshorne Exercise II.1.18)**.** The inverse image and direct image functors are adjoint. Namely: given a continuous map $f : X \to Y$ of topological spaces and $\mathscr{F}$ a sheaf on X and $\mathscr{G}$ a sheaf on Y, we have:

$$\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

Proof. We'll give two maps between these hom sets that are inverses. In dealing with $f^{-1}\mathscr{G}$, we will use that inclusion of sheaves into presheaves and sheafification are adjoint functors. That is, $\operatorname{Hom}_{\mathrm{Pre}_X}(\mathscr{G}, i(\mathscr{F})) \cong \operatorname{Hom}_{\mathrm{Shf}_X}(\mathscr{G}^+, \mathscr{F})$.

- Suppose we have a $\sigma : \operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$. That is, we have

  $$\sigma_U : f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \to \mathscr{F}(U)$$

  Then for $V^{\text{open}} \subseteq Y$ we can define

  $$\mathscr{G}(V) \to \varinjlim_{V' \supseteq V} \mathscr{G}(V) \overset{\sigma_{f^{-1}(V)}}{\to} \mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V)$$

  This yields a map $\mathscr{G} \to f_*\mathscr{F}$
- Suppose we have $\tau \in \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$. That is, on $V^{\text{open}} \subseteq Y$ we have

  $$\tau_V : \mathscr{G}(V) \to f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$$

  Let U in X be open. For any $V^{\text{open}} \supseteq f(U)$, we have:

  $$\mathscr{G}(V) \overset{\tau}{\to} \mathscr{F}(f^{-1}(V)) \overset{\operatorname{res}_U^{f^{-1}(V)}}{\to} \mathscr{F}(U)$$

  And compatibility of the τ means that we get a map $\varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \to \mathscr{F}(U)$, i.e. a map $f^{-1}\mathscr{G}^{\text{pre}}(U) \to \mathscr{F}(U)$ and thus a map $f^{-1}\mathscr{G} \to \mathscr{F}$.

One can check that these two assignments are inverses. □

Construction 15.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spacs and $\mathscr{G}$ an $\mathcal{O}_Y$-module. Then $f^{-1}\mathscr{G}$ is an $f^{-1}\mathcal{O}_Y$ module. Now, because we have a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we also have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. So we add in the $\mathcal{O}_X$-module structure in the usual way: tensor product! We define:

$$f^*\mathscr{G} := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

(by this we mean take the presheaf given by tensoring on opens, and then sheafify). This indeed has the structure of an $\mathcal{O}_X$-module.

Proposition 15.4. f_*, f^* are adjoint:

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathscr{G}, \mathscr{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

The next part is about a class of $\mathcal{O}_X$-modules we particularly like. There are two ways to do it: in practice, we just use the following. Let A be a ring and M an A-module. We have the sheaf associated to M on Spec A, denoted by $\widetilde{M}$. Its space of sections can be defined on distinguished opens $D(f)$ as:

$$\Gamma(D(f), \widetilde{M}) = M_f$$

As usual, one has to worry if the restriction maps make sense, if you have $D(g) \subseteq D(f)$, but the details of checking that are similar to checking them for the structure sheaf of Spec A or Proj S.

Hartshorne does it as he does the construction of the structure sheaf $\mathcal{O}_X$: the stalks of the would-be $\widetilde{M}$ should be $M_\mathfrak{p}$, and then the sheaf $\widetilde{M}$, on open sets U, look like elements $(m_\mathfrak{p})_{\mathfrak{p}\in U} \in \prod_{\mathfrak{p}\in U} M_\mathfrak{p}$ such that the $m_\mathfrak{p}$ have some compatibility conditions. See the definition on page 110 of Hartshorne for more details.

Proposition 15.5. Let A be a ring, and M an A-module. Then:

(a) $\widetilde{M}$ is an $\mathcal{O}_X$ module.
(b) The stalk $(\widetilde{M})_\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$.

Proof. Proposition II.5.1 in Hartshorne. □

Proposition 15.6. Let A be a ring, and $X = \text{Spec } A$. Let $A \to B$ a ring homomorphism, and $f : \text{Spec } B \to \text{Spec } A$ be the induced morphism on ringed spaces. Then:

(a) The map $M \mapsto \widetilde{M}$ is an exact and fully faithful functor from the category of A-modules to the category of $\mathcal{O}_X$-modules.
(b) If M, N are two A-modules, then $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$
(c) Given a family $\{M_i\}$ of A-modules, we have $\widetilde{\oplus M_i} \cong \oplus \widetilde{M_i}$.
(d) For any B-module N, we have that $f_*(\widetilde{N}) = \widetilde{({}_A N)}$, where ${}_A N$ means N considered as an A-module.
(e) For an A-module M, we have $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$.

Proof. Many of these are straightforward formality exercises. For (a): note that localization is exact and exactness of sequences sheaves can be tested on stalks, so the functor is exact. To show $\widetilde{\bullet} : \text{Hom}_A(M, N) \to \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is an isomorphism, note that by taking global sections you get a map $\text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \to \text{Hom}_A(M, N)$, which will be the inverse. □

Definition 15.7. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is **quasi-coherent** if X can be covered by open affine subsets $U_i = \text{Spec } A_i$ such that for each piece of the cover, there is an A_i-module M_i such that $\mathscr{F}|_{U_i} \cong \widetilde{M}_i$. We say that $\mathscr{F}$ is **coherent** if each M_i can be taken to be a finitely generated A_i-module.

Remark 15.8. One generally only studies coherent sheaves on noetherian schemes, as their behavior can be quite bad on non-noetherian schemes. Namely, we want the following property: finitely generated modules M over noetherian rings are noetherian modules, meaning any submodule of M is finitely generated. This is a property that you will want all the time when you are trying to study coherent sheaves.

Remark 15.9. Note that locally free $\mathcal{O}_X$-modules of finite rank are coherent. This is nice, because while the category of locally free sheaves on X is not abelian, we will see that the slight enlargement to the category of coherent sheaves is abelian.

Example 15.10. Here is an example of how finitely generated modules over a ring R can have non-finitely generated submodules if R is not noetherian. Note that $R = k[x_1, x_2, \ldots]$ is a finitely generated R-module, but $(x_1, x_2, \ldots,)$ is not a finitely generated R-module.

Example 15.11 (An $\mathcal{O}_X$-module that is not qcoh)**.** Consider Spec $k[t]$, and let $\mathscr{F}$ be the skyscraper sheaf supported at the origin (t) with group $k(t)$. This has an $\mathcal{O}_{\text{Spec } k[t]}$-module structure, but it is not quasicoherent.

Proposition 15.12. Let X be a scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for *every* open affine $U = \text{Spec } A$ of X, there is an A-module M such that $\mathscr{F}|_U \cong \widetilde{M}$. If X is noetherian, the analogous statement holds for $\mathscr{F}$ quasi-coherent, with the extra condition that the M are finitely generated over their respective A.

Proof. See Proposition II.5.4 in Hartshorne. □

Corollary 15.13. Let A be a ring and $X = \mathrm{Spec}\ A$. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between A modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$.

If A is noetherian, $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules.

Proof. The prior proposition makes sure that all quasicoherent $\mathscr{F}$ look like $\widetilde{\Gamma(X, \mathscr{F})}$ on X and so $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ is an inverse. □

16. Feb 19: More on quasicoherent, coherent sheaves

Recommended reading: Hartshorne II.5, Vakil 13.4-13.5, 14.1, 15.1

Proposition 16.1. Let X be an affine scheme, and $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$. an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathscr{F}'$ is quasicoherent. Then

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

Proof. We showed Γ is left-exact, so we just need to show the surjectivity of $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$. This uses that quasicoherent sheaves have nice lifting properties: emulating the relationship between M, M_f, a section s of a quasicoherent sheaf on $D(f)$ has the property that you can find some n such that $f^n s$ is a global section. Full details are in Proposition II.5.6 of Hartshorne. □

Proposition 16.2. Let X be a scheme. The kernel, cokernel, and image of any morphism of quasicoherent sheaves are quasicoherent. Any extension of quasicoherent sheaves is quasicoherent. If X is noetherian, the same is true for coherent sheaves.

Proof. All these criteria are local, so we may assume X is affine. The fact that kernels, cokernels, images are quasicoherent follows from $M \mapsto \widetilde{M}$ being fully faithful to qcoh sheaves (or coh for X Noetherian).

The nontrivial part is showing extensions play nicely. Take global sections to get:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{M'} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{M''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}'' & \longrightarrow & 0
\end{array}$$

the two outer left and two outer right columns are isomorphisms since $\mathscr{F}', \mathscr{F}''$ are quasicoherent. So the 5-lemma says the middle one is quasicoherent. Similarly for coherent (M', M'' finitely generated implies M finitely generated). □

Proposition 16.3. Let $f : X \to Y$ be a morphism of schemes.

(a) If $\mathscr{G}$ is a quasicoherent $\mathcal{O}_Y$-module then $f^*\mathscr{G}$ is a quasicoherent $\mathcal{O}_X$ module
(b) If X, Y noetherian and $\mathscr{G}$ coherent, then $f^*\mathscr{G}$ is coherent
(c) If either X noetherian **or** f quasi-compact and separated, then $\mathscr{F}$ quasicoherent on X implies that $f_*\mathscr{F}$ is quasicoherent on Y.

Proof. Note that (a), (b) are local (on both X, Y) and so we can reduce to the case of $\mathrm{Spec}\ A \to \mathrm{Spec}\ B$. Then it follows from Proposition 15.6. For (c), the property is only local on Y, so you can only assume Y is affine. See Hartshorne Proposition II.5.8 for the full proof. □

Remark 16.4. If X, Y are noetherian, it is not necessarily true that f_* of a coherent sheaf is coherent. It is true if f is finite or projective. Or, most generally, proper.

Definition 16.5. Let Y be a closed subscheme of a scheme X. Let $i : Y \to X$ be the inclusion morphism. The **ideal sheaf of** Y, denoted $\mathscr{I}$, is the kernel of the morphism $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Remark 16.6. Consider the map Spec $A/I \to$ Spec A induced by $A \to I$. By Proposition 15.6, we get that $i_*\mathcal{O}_{\text{Spec } A/I}$ is $\widetilde{A/I}$ considered as an A-module. Then our map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$ is the map $\widetilde{A} \to \widetilde{A/I}$. Then the map $i^\sharp$ has kernel $\widetilde{I}$. So we see in this simple case that the construction does line up with what it should be.

Proposition 16.7. Let X be a scheme. There is a one-to-one correspondence between closed subschemes Y of X and <u>quasicoherent</u> ideal sheaves.

Corollary 16.8. If $X = \text{Spec } A$ is an affine scheme, there is a one-to-one correspondence between ideals $\mathfrak{a}$ in A and closed subschemes Y of X, given by $\mathfrak{a} \mapsto$ image of Spec $A/\mathfrak{a}$ in X. Notably, every closed subscheme of an affine scheme is affine.

Proof. Follows from the equivalence of categories of A-modules and quasicoherent sheaves on Spec A. $\square$

Now that we've gotten a good sense of quasicoherent and coherent sheaves on affine things, let's study them on something a little more complicated: projective space! And projective varieties. We will define an important class of modules now.

Construction 16.9. For S a graded ring, we have a notion of a graded module. A graded S-module M is an S-module M with a decomposition $M = \oplus_{d\in\mathbb{Z}} M_d$ such that $S_d \cdot M_e \subseteq M_{d+e}$. For any graded S-module M and $n \in \mathbb{Z}$ we have the *twisted* modules $M(n)$ where

$$M(n)_d = M_{d+n}$$

That is, $M(n)$ is M but with the degree assignments shifted. $M(n)$ is also a graded S-module.

Construction 16.10. Let S be a graded ring and M a graded S-module. Then we can construct an $\mathcal{O}_{\text{Proj } S}$ module from it, which we will denote $\widetilde{M}$. It is defined on distinguished opens $D(f)$ with $f \in S_+$ as follows:

$$\widetilde{M}(D(f)) = (M_f)_0$$

That is, it assigns to $D(f))$ the degree zero elements of (M_f).

Proposition 16.11. Let S be a graded ring and M a graded S-module.

(a) For any $\mathfrak{p} \in \text{Proj } S$, $(\widetilde{M})_\mathfrak{p} = M_{(\mathfrak{p})}$, the degree zero elements of M localized at $\mathfrak{p}$.
(b) For any homogeneous f_+, recall that $D(f)$ is isomorphic to Spec $(S_f)_0$ as schemes. With this in mind, we have that as $\mathcal{O}_{D(f)} = \mathcal{O}_{\text{Spec } (S_f)_0}$ modules that

$$\widetilde{M}|_{D(f)} \cong (\widetilde{(M_f)_0}) \quad = \left[\widetilde{M_{(f)}}\right]$$

(c) $\widetilde{M}$ is a quasicoherent sheaf. If S is noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.

Definition 16.12. Let S be a graded ring and $X = \text{Proj } S$. For any $n \in \mathbb{Z}$ we have

$$\mathcal{O}_X(n) := \widetilde{S(n)}$$

$\mathcal{O}_X(1)$ is called *Serre's twisting sheaf.* You may see the $\mathcal{O}_X(n)$ referred to as the Serre twists.

Definition 16.13. For $X = \text{Proj } S$ and $\mathscr{F}$ an $\mathcal{O}_X$-module, we set

$$\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

Proposition 16.14. Let S be a graded ring and $X = \text{Proj } S$. Assume that S is generated by S_1 as an S_0-algebra.

(a) $\mathcal{O}_X(n)$ is locally free
(b) For any graded S-module M, we have that

$$\widetilde{M}(n) \cong \widetilde{M(n)}$$

that is, twisting and applying the $\sim$ construction can be done in either order. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

Proof. (a) It is most instructive to do this for $\mathbb{P}^n_k = \text{Proj } k[x_0, \ldots, x_n]$. $\mathbb{P}^n$ is covered by affine opens $D(x_i) \cong \text{Spec } k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \cong \mathbb{A}^n$. By the previous proposition, we have that

$$\mathcal{O}_{\mathbb{P}^n}(m)|_{D(x_i)} \cong ((\widetilde{k[x_0, \ldots, x_n](m)})_{x_i})_0 = (\widetilde{k[x_0, \ldots, x_n]_{x_i}})_m$$

Because quasicoherent modules over affine schemes Spec A are equivalent to modules over A, we just need to give a module isomorphism

$$(k[x_0, \ldots, x_n]_{x_i})_0 \xrightarrow{\cong} (k[x_0, \ldots, x_n]_{x_i})_m$$

Multiplication by x_i^m yields such an isomorphism. Note that the map and its inverse both make sense because x_i is invertible. Since $\mathbb{P}^n_k$ is covered by the $D(x_i)$, this shows that $\mathcal{O}_{\mathbb{P}^n}(m)$ is locally free.

In the general scenario, one picks $f \in S_1$ and needs so show that $(S_f)_0 \cong (S_f)_m$ are isomorphic as S_f-modules. Again, multiplication by f^m works. Since S is generated by S_1 as an S_0-algebra, the $D(f)$ cover S_0.

(b) Follows from $\widetilde{(M \otimes_S N)} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Namely we can check this on affines by $(M \otimes_S N)_{(f)} = M_{(f)} \otimes S_{(f)} N_{(f)}$. Note that $\deg f = 1$ is crucial: things can get messed up if $\deg f > 1$ (try working out an example!)

□

17. Feb 21: Modules on Proj

Recommended reading: Hartshorne II.5, Vakil 14.1, 15.1-15.3

As discussed prior, a locally free sheaf on X can be determined by its transition functions. That is, given the knowledge that $\mathscr{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ then the bundle is determined by the functions $GL_n(U_i \cap U_j)$ on the $U_i \cap U_j$.

Let us determine the transition functions on $\mathcal{O}_{\mathbb{P}^n}(d)$. Write $\mathbb{P}^n = \text{Proj } k[x_0, \ldots, x_n]$. We have trivializable neighborhoods $D(x_i), D(x_j)$ with their intersection $D(x_i), D(x_j)$

$$\begin{tikzcd}
D(x_i) & & D(x_j) \\
& D(x_i x_j) = D(x_i) \cap D(x_j) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and we need to see how section of $\mathcal{O}_{\mathbb{P}^n}(d)$ changes between the two trivializations. Recall that $\mathcal{O}_{\mathbb{P}^n}(d)(D(f)) = (k[x_0, \ldots, x_n]_f)_d$.

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n}(D(x_i)) = (k[x_0, \ldots, x_n]_{x_i})_0 \arrow[r, "\times x_i^d"] & \mathcal{O}_{\mathbb{P}^n}(d)(D(x_i)) \arrow[d] & \mathcal{O}_{\mathbb{P}^n}(d)(D(x_j)) \arrow[d] & (k[x_0, \ldots, x_n]_{x_j})_0 \arrow[l, "\times x_j^d"'] \\
\mathcal{O}_{\mathbb{P}^n}(D(x_i x_j)) = (k[x_0, \ldots, x_n]_{x_i})_0 \arrow[r, "\times x_i^d"] \arrow[rrr, bend right, "\times\left(\frac{x_i}{x_j}\right)^d"'] & \mathcal{O}_{\mathbb{P}^n}(d)(D(x_i x_j)) & \mathcal{O}_{\mathbb{P}^n}(d)(D(x_i x_j)) & (k[x_0, \ldots, x_n]_{x_i x_j})_0 \arrow[l, "\times x_j^d"']
\end{tikzcd}$$

Proposition 17.1. Let $k = \mathbb{C}$. The global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ is isomorphic to the space of degree d homogeneous polynomials in $x_0, \ldots, x_n$.

Proof. Giving a global section is the same as giving as an element of $\Gamma(D(x_i), \mathcal{O}_{\mathbb{P}^n}(d))$ for each i such that they agree on the overlaps. That is, this is the same as giving a choice of $f_i \in k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ for each i such that f_i satisfies $f_j = (x_i/x_j)^d f_i$.

We see that a necessary condition is that $f_i(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$ must be degree $\leq d$ in order for $(x_i/x_j)^d f_i$ to even be a valid element of $\Gamma(D(x_j), \mathcal{O}_{\mathbb{P}^n}(d))$. This is in fact sufficient. Further, a global section is determined by any one of the f_i

By homogenizing, we can get $\widetilde{f_i}$ a homogeneous degree d polynomial corresponding to this section. Dividing through by x_i^d gives the representative for it in any given $D(x_i)$.

One can also see this by just viewing all the $\Gamma(\widetilde{S(d)}, D(x_i)) = (k[x_0, \ldots, x_n]_{x_i})_d$ in $k(x_0, \ldots, x_n)$ and intersecting. $\square$

Remark 17.2. Note that this gives us an example of why you need sheafification in the tensor product of $\mathcal{O}_X$ modules. After all:

$$\begin{aligned}\dim_k \left(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_{\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)\right) &= \dim_k \left(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)\right) \\ &= (n+1) \cdot 0 \\ &= 0\end{aligned}$$

Meanwhile:

$$\begin{aligned}\dim_k(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(-1))) &= \dim_k(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(0))) \\ &= \dim_k(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})) \\ &= \dim_k k \\ &= 1\end{aligned}$$

If the sheafification step were not necessary, these numbers would be the same.

Example 17.3 (A more concrete example: the tautological bundle)**.** Let's think about algebraic sets/closed points for a bit. Interpret $\mathbb{P}^n$ as lines through the origin in $\mathbb{C}^{n+1}$. We have the *tautological bundle* in the manifolds sense:

$$T = \{(p, v) : v \in < p >\} \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

This is a line bundle: it continuously assigns a line to each point of $\mathbb{P}^n$. On $U_i = x_i \neq 0$, the points look like:

$$\left(\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right], c\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right)$$

So we have this coordinate c that allows us to trivialize:

$$\begin{gathered}U_i \times \mathbb{C} \to T|_{U_i} \\ \left(\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right], c\right) \mapsto \left(\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right], c\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right)\end{gathered}$$

What if we have a point where both x_i, x_j are nonzero? Then let's write a point in the standardized form of both coordinate sets to see the transition functions.

$$\begin{aligned}\left(\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right], c\right) &\mapsto \left(\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right], c\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right) \\ &= \left(\left[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right], c\left(\frac{x_j}{x_i}\right)\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right)\right) \\ &\mapsfrom \left(\left[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right], c\left(\frac{x_j}{x_i}\right)\right)\end{aligned}$$

Since the transition function is $(\frac{x_i}{x_j})^{-1}$, we see that the tautological bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Definition 17.4. Let S be a graded ring, and $X = \operatorname{Proj} S$. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We can define the graded S-module associated to $\mathscr{F}$ as:

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

It has the structure of a graded S-module: if $s \in S_d$, then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$. For $t \in \Gamma(X, \mathscr{F}(n))$ we have the product $s \cdot t \in \Gamma(X, \mathscr{F}(n+d))$ via the tensor product $s \otimes t$ and using that

$$\mathscr{F}(n) \otimes \mathcal{O}_X(d) \cong \mathscr{F}(n+d) = \mathscr{F} \otimes \mathcal{O}_X(n+d)$$

Example 17.5. If $S = k[x_0, \ldots, x_n]$ and $X = \operatorname{Proj} S$ and $\mathscr{F} = \mathcal{O}_X(d) = \widetilde{S(d)}$ then note that the $\Gamma_*(\mathscr{F})$ returns back $S(d)$.

Proposition 17.6. Let $X = \operatorname{Proj}{}^n_A$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.

18. Feb 24: Modules on Proj, very ample sheaves, starting divisors

Recommended reading: Hartshorne II.5, II.6. Vakil 15.1-15.4, 16.1-16.4

Proposition 18.1. Let S be a graded ring, which is finitely generated by S_1 as an S_0 algebra. Set $X = \text{Proj } S$. Let $\mathscr{F}$ be a quasi-coherent sheaf on X (not necessarily graded!). Then there is a natural isomorphism $\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$.

Proof. First we define β. Let $f \in S_1$. Since S is generated by a finite number of the S_1 elements, just need to give the map over $D(f)$. Note that $D(f)$ is affine, so defined by the map on $D(f)$. We consider sections of $\Gamma_*(\mathscr{F})$ of the form m/f^d where $m \in \Gamma(X, \mathscr{F}(d))$ for some $d \geq 0$. We can think of f^-d as a section of $\mathcal{O}_X(-d)$ defined over $D(f)$. Then we can think of $m \otimes f^{-d}$ as a section of $\mathscr{F} \cong \mathscr{F}(d) \otimes \mathcal{O}_X(-d)$ over $D(f)$. This defines β.

One can show this is an isomorphism (see Hartshorne Proposition II.5.15). □

Corollary 18.2. Let A be a ring.

(a) If Y is a closed subscheme of $\mathbb{P}^n_A$, there exists a homogeneous ideal $I \subseteq S = A[x_0, \ldots, x_n]$ such that Y is the closed subscheme determined by the ideal I. That is, it looks like

$$\text{Proj } S/I \to \text{Proj } S$$

Proof. Y defines an ideal sheaf $\mathscr{I}_Y$, a subsheaf of $\mathcal{O}_X$. Twisting is exact (invertible process) and global sections is left-exact, so we get that $\Gamma_*(\mathscr{I}_Y)$ is a submodule of $\Gamma_*(\mathcal{O}_X) \cong S$. Hence $\Gamma_*(\mathscr{I}_Y)$ corresponds to a homogeneous ideal of S, call it I. Since $\mathscr{I}_Y$ is quasicoherent, we have that $\mathscr{I}_Y \cong \widetilde{\Gamma_*(\mathscr{I}_Y)} = \widetilde{I}$. Hence Y is the subscheme determined by I.

□

Definition 18.3. For Y a scheme, the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ is $g^*(\mathcal{O}(1))$ where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map (note $\mathbb{P}^r_Y = \mathbb{P}^r_Z \otimes_Z Y$). If $Y = \text{Spec } A$ this returns the old definition of $\mathcal{O}(1)$.

Definition 18.4. If X is a scheme over Y, an invertible sheaf $\mathscr{L}$ on X is very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r such that $i^*(\mathcal{O}(1)) \cong \mathscr{L}$. A morphism $i : X \to Z$ if it gives an iso of X with an open subscheme of a closed subscheme of Z.

Remark 18.5. Roughly speaking, very ample line bundles are line bundles with a lot of sections: enough that they can be used to define an embedding of a scheme into some projective space. Consider $\mathbb{P}^1_k = \text{Proj } k[s,t]$. Then consider $\mathcal{O}_{\mathbb{P}^1}(3)$, which has global sections $\langle s^3, s^2t, st^2, t^3\rangle$. We can use these to write a map:

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [s,t] &\mapsto [s^3, s^2t, st^2, t^3] \end{aligned}$$

which embeds $\mathbb{P}^1$ as a twisted cubic in $\mathbb{P}^3$. Then it turns out that $f^*(\mathcal{O}_{\mathbb{P}^3}(1)) = \mathcal{O}_{\mathbb{P}^1}(3)$, showing that $\mathcal{O}_{\mathbb{P}^1}(3)$ is very ample. This is the embedding of $\mathbb{P}^1$ using the *complete linear series* $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, also denoted by $|\mathcal{O}_{\mathbb{P}^1}(3)|$.

Definition 18.6. Let X be a scheme, and $\mathscr{F}$ an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is generated by global sections if there is a collection of global sections $\{s_i\}_{i \in I}$ with $s_i \in \Gamma(X, \mathscr{F})$ such that for each x, the images of s_i in the stalk $\mathscr{F}_x$ generated that stalk as an $\mathcal{O}_x$-module.

Equivalently, this means you can write a surjective $\mathcal{O}_x$-module map

$$\mathcal{O}_X^{\oplus I} \to \mathscr{F}$$

and realize $\mathscr{F}$ as a quotient of a free module.

Example 18.7. A quasicoherent sheaf on an affine scheme is generated by global sections.

Example 18.8. The $\mathcal{O}_{\mathbb{P}^n}(d)$ are globally generated (work on affines to show the morphsms of sheaves is surjective).

Theorem 18.9 (Serre)**.** Let X be projective scheme over noetherian ring A. Let $\mathscr{L}$ be a very ample invertible sheaf on X and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ can be generated by a finite number of global sections.

Proof. See Hartshorne Theorem II.5.17. □

We now turn our attention to divisors, which are a great tool for studying the geometry of a scheme. The divisor class group is a useful invariant of a variety as well. We start with Weil divisors, which are nice/intuitive geometrically, but there are conditions on when you can define them. Cartier divisors will be definable more broadly. And then we will see how this info relates to invertible sheaves.

Definition 18.10. A generic point η of a scheme X is one such that $\overline{\{\eta\}} = X$, topologically. Note that any nonempty open set must contain η.

Remark 18.11. If X is integral, then there is a unique generic point, and is it obtained by taking any affine open Spec A in X and taking the zero ideal in Spec A. We will restrict our attention to integral schemes in this lecture.

Definition 18.12. The **function field** of an integral scheme X, denoted $K(X)$, is the field of rational functions on X. Note that since every open set contains the generic point η, we get that $\mathcal{O}_{X,\eta} = K(X)$. Note that $K(X) = K(U)$ for any affine open U in X.

Remark 18.13. In the non-integral case we need to be a little more careful about the construction– we will see this when we deal with Cartier divisors.

Example 18.14. Note that for $\mathbb{A}^n = \text{Spec } k[x_1, \ldots, x_n]$, we have $K(\mathbb{A}^n) = k(x_1, \ldots, x_n)$. For $\mathbb{P}^n = \text{Proj } k[x_0, \ldots, x_n]$, we have that

$$K(\mathbb{P}^n) = k(x_0, \ldots, x_n)_0 \cong k(x_1, \ldots, x_n)$$

Definition 18.15. A regular local ring is a Noetherian local ring where $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$. Alternatively, $\mathfrak{m}$ has a minimal generating set of $\dim A$ elements.

If A has Krull dimension one, this is precisely the same as being a DVR. Which means we have all sorts of equivalent characterizations (local ring, PID, not a field)

Definition 18.16. We say a scheme X is **regular in codimension one** (or *nonsingular in codimension one*) if every local ring $\mathcal{O}_{X,x}$ of dimension one is regular. You should think of this as the singular locus having codimension at least 2.

Note: If Y is a closed irreducible subspace of X, with $y \in Y$ the generic point of Y, then

$$\operatorname{codim}(Y, X) = \dim(\mathcal{O}_{X,y})$$

coming from the equivalence between prime ideals of the latter and closed irreducible subspaces of X containing Y.

Example 18.17. As a quick example, observe that in $\mathbb{A}^3 = \text{Spec } k[x, y, z]$, we can take the hypersurface $Y = \{y = 0\}$. This is given by the data:

$$Y \to \text{Spec } k[x, y, z]$$
$$\frac{k[x, y, z]}{(y)} \hookleftarrow k[x, y, z]$$

the generic point $(0) \subseteq k[x.y, z]/(y)$ is mapped to the ideal $(y) \in \text{Spec } k[x, y, z]$ and we see that

$$\operatorname{codim}(Y, \text{Spec } k[x, y, z]) = 1 = \dim k[x, y, z]_{(y)}$$

19. Feb 26: Divisors

Recommended reading: Hartshorne II.6, Vakil 14.2

Definition 19.1. Hartshorne refers throughout the chapter to the following condition, which is denoted as just $(*)$. $(*)$ is the property that X is noetherian, integral, and separated scheme which is regular in codimension one.

Definition 19.2. Let X satisfy $(*)$. A **prime divisor** on X is a closed integral subscheme Y of codimension one.

Definition 19.3. A **Weil divisor** is an element of the free abelian group Div X, which is generated by prime divisors. That is, elements of Div X look like $D = \sum n_i Y_i$ where the Y_i are prime divisors, and only finitely many n_i are nonzero. If all the $n_i \geq 0$, then we say D is **effective**. (That is, effective divisors are things that look like actual subschemes).

Definition 19.4 (Valuation associated to a prime divisor)**.** Let Y be a prime divisor, and η_Y the generic point of Y. The local ring $\mathcal{O}_{\eta_Y,X}$ is a DVR with quotient field K, the function field of X.

The corresponding discrete valuation is denoted v_Y. X is separated so Y is uniquely determined by its valuation (this is the content of Hartshorne exercise II.4.5). Let $f \in K^*$ be a nonzero rational function. Then $v_Y(f)$ is an integer. If it is zero, f is said to have a zero along Y. If it is negative, f is said to have a pole along Y (of order $-v_Y(f)$).

Definition 19.5. Suppose X satisfies $(*)$, and let $f \in K(X)^*$ be a nonzero rational function. Then $v_Y(f) = 0$ for all but finitely many prime divisors.

Proof. Let $U = \text{Spec } A$ be an open affine on which f is regular. $Z = X \setminus U$. Since X is Noetherian, Z contains at most finitely many prime divisors (use d.c.c. on closed subsets and quotient ideals to remove pieces). All other prime divisors must meet U.

So now we need to show there are only finitely many prime divisors Y *of* U such that $v_Y(f) \neq 0$. We necessarily have that $v_Y(f) \geq 0$ since f is regular on U. And $v_Y(f) > 0$ only when Y is contained in fA, the ideal generated by f. There are only finitely many such closed irreducible subsets within Spec A/fA . (Descending chain condition on Noetherian topological spaces). ☐

Example 19.6. Consider $f = \frac{x_0}{x_i}$ which is a rational function on $\mathbb{P}^n_k$. From the proof above, we see that we only need to consider prime divisors in $V(x_1)$ for poles. For zeroes, consider closed subsets in $\mathbb{A}^n \cong U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n$. Any prime divisors where the valuation is positive must be contained in $(x_0/x_i)k[\frac{x_0}{x_i}, \ldots, \frac{x_0}{x_i}]$. That is, we see that we only need to compute $v_Y(f)$ for $Y = V(x_0), V(x_1)$.

Consider $Y_1 = V(x_0)$ first. So we view $f = \frac{x_0}{x_1}$ as sitting in $\mathcal{O}_{\eta_{Y_1},X} = (k[x_0, \ldots, x_n]_{\langle x_0 \rangle})_0$. This is a DVR with uniformizer x_0, and we see that the valuation $v_{Y_1}(f) = 1$.

Likewise, for $Y_2 = V(x_1)$, we get that $\mathcal{O}_{\eta_{Y_2},X} = (k[x_0, \ldots, x_n]_{\langle x_1 \rangle})_0$ and in the fraction field of this, we see that $v_{Y_2}(f) = -1$.

Definition 19.7. Suppose X is a scheme satisfying $(*)$ and $f \in K^*$. We define the divisor of f, denoted (f), by:

$$(f) = \sum v_Y(f) \cdot Y.$$

By the lemma, this sum is finite and therefore an actual member of Div X. Any divisor of this form is a called a principal divisor.

Remark 19.8. Observe that for $f, g \in K(X)^*$, we have $(f/g) = (f) - (g)$ and $(fg) = (f) + (g)$. So we get a group homomorphism $K(X)^* \to \text{Div } X$. The image is a subgroup.

Definition 19.9. Suppose X satisfies $(*)$. The **divisor class group** of X, denoted $\text{Cl}(X)$, is obtained by taking Div X and quotienting by the subgroup of principal divisors.

Two divisors D, D' are linearly equivalent (written $D \sim D'$) if their difference $D - D'$ is a principal divisor.

Example 19.10. Let's do some more concrete examples about how being smooth in codimension 1 is important.

Let's observe what happens with the cuspidal cubic Spec $k[x, y]/(y^2 - x^3)$ and try to figure out some notion of order of vanishing at the origin. We get that we are considering the ring

$$\left(\frac{k[x, y]}{(y^2 - x^3)}\right)_{(y,x)}$$

But this is not a DVR: it is not principal, and we have no uniformizer. Even if we try to take some sort of degree (in x, y) of a polynomial representative, we run into issues like: what should be the order of $x^2 = y^3$? Should it be 2? 3? 6?

The divisor class group is an invariant, but often tricky to calculate. Part of Hartshorne II.6 will be dedicated towards some techniques and examples.

Proposition 19.11. Consider the scheme $\mathbb{P}^n_k$. For any divisor $D = \sum n_i Y_i$, define the degree to be $\sum n_i \deg Y_i$ where $\deg Y_i$ is taken as a the degree of the hypersurface. Let H be the hyperplane $x_0 = 0$. Then:

(a) If D is any divisor of degree d, then $d \sim dH$.
(b) For any $f \in K(\mathbb{P}^n_k)$, we have $\deg(f) = 0$.
(c) The degree function gives can isomorphism $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$.

Proof. For (b): we did this computation last time. For (a): collect positive and negative terms, so that $D = D_1 - D_2$ with each D_i effective and $\deg(D_1) - \deg(D_2) = e$. Write $D_1 = (g_1), D_2 = (g_2)$, then $D - dH = (f)$ where $f = g_1/(x_0^e g_2)$. This proves part (a). Part (c) follows. □

20. Feb 28: Computational tools for divisors, divisors on curves

Recommended reading: Harthsorne II.6, Vakil 14.1-14.2 We continue with some examples of computing class groups, some tools for computation, and some bits on the class group on curves.

Proposition 20.1. Let A be a noetherian domain. Then A is a UFD if and only if $X = \operatorname{Spec} A$ is normal and $\mathrm{Cl}(X) = 0$

Proof. UFDs are integrally closed, so X will be normal. Then the Spec A covering X will be UFDs if and only if every prime ideal of height 1 is principal. So need: if A integrally closed, then every prime ideal of height 1 if principal if and only if $\mathrm{Cl}(\operatorname{Spec} A) = 0$.

If every prime ideal of ht 1 is principal consider a prime divisor $Y = (f = 0)$, then Div $(f) = Y$, and Y zeroes out in the class group.

For converse: suppose $\mathrm{Cl}(X) = 0$. Then $Y = (f)$ (in the class group) corresponding to prime with ht 1 $\mathfrak{p}$. From $V_Y(f) = 1$ we have $f \in A_\mathfrak{p}$ and that (f) generates $\mathfrak{p}A_\mathfrak{p}$.If $\mathfrak{q}$ is any other ht 1 prime, then $\mathfrak{q}$ corresponds to some Y' and $v_{Y'}(f) = 0$, so $f \in A_\mathfrak{q}$. Now Matsumura (intersection of $A_\mathfrak{p}$ ht one primes is A) gives us that $f \in \mathbb{A} \cap \mathfrak{p}A_\mathfrak{p} = \mathfrak{p}$.

To show it generates: let g be any other element of $\mathfrak{p}$. Then $v(g/f) \geq 0$ for all prime divisors, so regular, so $g/f \in A$, that is $g \in fA$. Thus $\mathfrak{p} = (f)$ as ideals. □

Remark 20.2. In general, for a Dedekind domain, $\mathrm{Cl}(\operatorname{Spec} A)$ is just the ideal class group of A.

Proposition 20.3 (Excision exact sequence)**.** Suppose X satisfies $(*)$, and let Z be a proper closed subset of X. Let $U = X \setminus Z$. Then:

(a) There is a surjective homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by
$$D = \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$$
where we drop the terms where the $Y_i \cap U$ is empty.
(b) If $\operatorname{codim}(Z, X) \geq 2$ then $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism
(c) if Z is an **irreducible** subset of codimension 1, there is an exact sequence
$$\mathbb{Z} \xrightarrow{1 \mapsto 1 \cdot Z} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \to 0$$

Proof.
(a) This is well defined since $(f) = \sum n_i Y_i$ on X and $(f)_U = \sum n_i (Y_i \cap U)$ on U. It is surjective because every prime divisor of U is the restriction of its closure.
(b) Div and Cl depend on codimension 1 data, so excising a codimension 2 thing shouldn't change anything.
(c) The kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is divisors whose support is contained in Z. If Z is irreducible, then the kernel is just multiples of Z.

□

Example 20.4. It immediately follows that $\mathrm{Cl}(\mathbb{P}^2 \setminus D) = \mathbb{Z}/d\mathbb{Z}$ for irreducible degree d hypersurfaces.

Example 20.5. Let k be a field, and let $A = k[x, y, z]/(xy - z^2)$, the cone over a quadric. Then $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$ and it is generated by the ruling of a cone, say $Y = \{y = z = 0\}$. See details in Hartshorne Example II.6.5.2.

Proposition 20.6. Suppose X satisfies $(*)$. Then $X \times \mathbb{A}^1 = X \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}[t]$ also satisfies $(*)$ and $\text{Cl}(X) \cong \text{Cl}(X \times A^1)$.

Proof. $X \times \mathbb{A}^1$ is noetherian, integral, and separated. To see that it's regular in codim 1: there are two kinds of points of codimension one. We have

- First type is points $x \in X \times \mathbb{A}^1$ whose image in X are points of codimension 1, i.e. some $y \in X$ of codimension 1 with $\overline{\{y\}} = Y$ codimension 1 in X. Then x is the generic point of $\pi_1^{-1}(Y)$ and the local ring at that point is
$$\mathcal{O}_{X \times \mathbb{A}^1, x} \cong \mathcal{O}_Y[t]_{m_y}$$
- Second time is a point $x \in X \times \mathbb{A}^1$ whose image under projection to X is codim 0, i.e. the generic point. Then $\mathcal{O}_{X,x}$ looks like the localization of $K[t]$ at some maximal ideal, where K is the function field of X. It is a DVR because $K[t]$ is a PID.

We define a map $\text{Cl}(X) \to \text{Cl}(X \times \mathbb{A}^1)$ by $D = \sum n_i Y_i \to \pi^* D = \sum n_i \pi^{-1}(Y_i)$. If $f \in K^*$, then $\pi^*((f))$ is the divisor of f considered as an element of $K(t)$, the function field of $X \times \mathbb{A}^1$.

- Injectivity: If $\pi^*(D) = (f)$ for some $f \in K(t)$, then note that (f) must in fact lie in K, otherwise we would get components of the second kind ($X \times \text{pt}$).
- Surjectivity: We show that any prime divisor of type 2 is linearly equivalent to a combo of the type 1 sort. Let Z be a type 2 prime divisor. Localizing at the generic point of X, we get a prime divisor in Spec $K[t] \subseteq X \times \mathbb{A}^1$, yielding a prime divisor in Spec $K[t]$ corresponding to some $\mathfrak{p} \in K[t]$. This is principal, so let f be a generator. Then $f \in K(t)$, and Div f consists of Z and some pieces of type 1 that could be accrued when passing to $K[t]$. Thus Z is linearly equivalent to something of type 1.

□

Proposition 20.7 (Exercise 6.1 from your HW)**.** On HW, you will show that $X \times \mathbb{P}^n$ satisfies
$$\text{Cl}(X \times \mathbb{P}^n) \cong \text{Cl}(X) \times \mathbb{Z}$$
for schemes X satisfying $(*)$.

Time for divisors on curves. By curves, we mean a integral separated scheme X of finite type over some field k.

If all the local rings of X are regular local rings, we say X is nonsingular. For X nonsingular, then a curve over an algebraically closed field k is projective $\iff$ it is proper.

Proposition 20.8. Let X be a nonsingular curve, proper over k (AKA complete). Let $f : X \to Y$ be a morphism to another curve over k. Then either:

- $f(X)$ is a point
- $f(X) = Y$ and in this case, $K(X)$ is a finite field extension of $K(Y)$, f is a finite morphism, and Y is complete (proper over k)

Proof. Full proof in Hartshorne Prop II.6.8.

Basic split in cases is from the map needing to be closed. In the latter cae, $K(Y) \subseteq K(X)$ and they are both transcendence degree 1 over k, so $K(X)$ a finite extension of $K(Y)$. It follows that the morphism is finite ($V = \text{Spec } B$ affine open of Y, then let A be integral closure of B in $K(X)$. Then A is finite module/B and Spec A is an affine open of X). □

Definition 20.9. For $f : X \to Y$ a finite morphism of curves, define the degree of f to be the degree of the field extension $[K(X) : K(Y)]$.

Note that for nonsingular curves on X, prime divisors just look like closed points. So may write divisors as $D = \sum n_i P_i$. The degree of D is then $\sum n_i$.

Definition 20.10. Let $f : X \to Y$ be a finite morphism of nonsingular curves. We define a homomorphism $f^* : \text{Div }(Y) \to \text{Div }(X)$ as follows.

For $Q \in Y$, let $t \in \mathcal{O}_{Y,Q}$ be a local parameter at Q, i.e. $t \in K(Y)$ with $v_Q(t) = 1$. Then we define
$$f^* Q = \sum_{f(P)=Q} v_P(t) \cdot P$$

f is a finite morphism, so this sum is finite. Extend linearly to get a morphism on Weil divisors. f^* preserves linear equivalence, so we get a map $f^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ on class groups.

Proposition 20.11. Let $f : X \to Y$ be a finite morphism of nonsingular curves. For any divisor D on Y, we have $\deg f^*D = \deg f \cdot \deg D$.

Corollary 20.12. A principal divisor on a X nonsingular curve proper over k has degree zero. Thus the degree map is a surjective function $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$.

Remark 20.13. It is strongly recommended that you read some of the specifics on divisor theory for elliptic curves in Hartshorne (Example 6.10.2)

21. Mar 03: Cartier divisors

Recommended reading: Hartshorne II.6, Vakil 14.2-14.3

We would like a notion of divisor for arbitrary schemes. The codimension 1 subvariety idea does not generalize so well, so we instead try to preserve the notion of taking things that locally looks like the divisor of a rational function. First, we need something that replaces the notion of a function field on an integral scheme.

Definition 21.1. Let X be a scheme. For each open U in general, can pick $S(U)$ = elements of $\Gamma(U, \mathcal{O}_X)$ that are not zero divisors. We get a presheaf

$$U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$$

and the sheaf associated to this presheaf is $\mathscr{K}$, the sheaf of total quotient rings of $\mathcal{O}_X$. $\mathscr{K}^*$, $\mathcal{O}_X^*$ denotes the invertible elements in each.

Definition 21.2. A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathscr{K}^*/\mathcal{O}_X^*$. One can describe a Cartier divisor as being described via a collection of (f_i, U_i) where the $\{U_i\}$ form an open cover of X, and each $f_i \in \Gamma(U_i, \mathscr{K}^*)$ such that for each i, j, the $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ (that is, you can transition with a regular function on the overlaps).

A Cartier divisor is principal if it is in the image

$$\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$$

Two Cartier divisors are linearly equivalent if their "difference" (that is, quotient) is principal.

Proposition 21.3. Let X be integral, separated, noetherian scheme, whose local rings are all UFDs (that is, X is *locally factorial*). Let Div X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$ and the isomorphism descends to an isomorphism

$$\mathrm{Cl}(X) \xrightarrow{\cong} \mathrm{CaCl}(X) \quad \big(= \text{Cartier div. mod lin equiv}\big)$$

Proof. In this case, $\mathscr{K}$ is the constant sheaf $\underline{K(X)}$. Then a Cartier divisor C is given on a cover by $\{(f_i, U_i)\}$ and $f_i \in K(X)^*$. The associated Weil divisor is the following:

- For each prime divisor Y, take the coefficient of Y to be $v_Y(f_i)$, where f_i is any i such that $U_i \cap Y \neq \varnothing$. This is well-defined because on overlaps $U_i \cap U_j$ we have $f_i/f_j \in \mathcal{O}_X^*$ invertible, so $v_Y(f_i/f_j) = 0$ where appropriate to yield that $v_Y(f_i) = v_Y(f_j)$.

For the map in the other direction: let D be a Weil divisor on X. We want to produce a Cartier divisor from this.

- Let $x \in X$ be any point. Then we get a divisor D_x on the local scheme Spec $O_{X,x}$. $\mathcal{O}_{X,x}$ is a UFD, so $D_x = (f_x)$ for some $f_x \in K(X)$. Because they agree in Spec $\mathcal{O}_X$, they could only differ on prime divisors (subschemes) not passing through x that are in the expression of D or (f_x). There are finitely many, so they agree on some U_x of x.

  Then the principal divisor (f_x), viewed on X, agree on some open U_x of x. (TO see this: they only differ on prime divisors not passing through x, and only finitely many have a nonzero coefficient in D or (f_x)). Then the Cartier divisor associated to the D is this data of all the $\{(f_x, U_x)\}$. This is well-defined.

These two constructions are inverse. Details can be found in Harthsorne Proposition II.6.11 □

Remark 21.4. The Cartier divisor constructed from a Weil divisor has an open set for each $x \in X$, but in practice one does not need so many pieces. See the example below.

Example 21.5. Consider the projective line over k with coordinates $[x, y]$, so $\mathbb{P}^1_k = \operatorname{Proj} k[x, y]$. We could consider the Weil divisor $V(x)$, which is represented by the point $[0 : 1]$.

As a Cartier divisor: consider $U = \{x \neq 0\}$ and $V = \{y \neq 0\}$. Then on the set U, the function cutting out our divisor, x, looks like $x|_U = 1$ (capturing that our divisor doesn't really have any support over U). So $(U, 1)$ is one piece of our Cartier divisor.

Consider the piece $V = \{y \neq 0\}$. On the set V, the polynomial cutting out our divisor, the x coordinate function, looks like $x/y = v$. So we get that the associated Cartier divisor is

$$\{(U, 1), (V, \tfrac{x}{y} = v)\}$$

Note that on the overlaps, the $1/(x/y)$ is invertible on $U \cap V$.

We'll later see that there is an invertible sheaf you can associate to this divisor, and we will compute that.

Remark 21.6. For X normal, not necessarily locally factorial, we see that the Cartier divisors correspond to a subgroup of div consisting of *locally principal* Weil divisors: i.e. $D|_U$ is principal for each U.

Example 21.7. For the affine quadric cone $k[x, y, z]/(xy - z^2)$ the ruling $z = 0$ generates the class group $\mathbb{Z}/2\mathbb{Z}$, but $\operatorname{CaCl}(X) = 0$, as the generator of the divisor class group is not locally principal.

To wrap it all up, we see how it all ties together with invertible sheaves/line bundles (locally free of rank 1).

Proposition 21.8. If $\mathscr{L}, \mathscr{M}$ are invertible sheaves then so is their product $\mathscr{L} \otimes \mathscr{M}$. For any invertible sheaf $\mathscr{L}$ on X there is an invertible sheaf $\mathscr{L}^{-1}$ such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules.

Proof. For (a): locally, we can take covers such that on each piece $\mathscr{L}, \mathscr{M}$ are trivializable, and then use that $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$. For (b): let $\mathscr{L}$ be an invertible sheaf and set $\mathscr{L}^{-1} = \mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$. Then $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathscr{H}om(\mathscr{L}, \mathscr{L}) = \mathcal{O}_X$ by Exercise II.5.1 from your HW. $\square$

Definition 21.9. For any ringed space X, define the **Picard** group of X, denoted $\operatorname{Pic}(X)$, to be the group of isomorphism classes of invertible sheaves, with group operation $\otimes$.

Remark 21.10. When we learn sheaf cohomology (and Čech cohomology will help) we will see that $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$.

Definition 21.11. Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$ as above. We define a subsheaf $\mathscr{L}(D)$ of the sheaf of total quotient rings $\mathscr{K}$ by taking $\mathscr{L}(D)$ to be $f_i^{-1}\mathcal{O}_X \in \mathscr{K}(U)$. This is well-defined as f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1}, f_j^{-1} generate the same $\mathcal{O}_{U_i \cap U_j}$−module.

Remark 21.12. You may see people refer to this as $\mathcal{O}_X(D)$, which overlaps with terminology for the construction of a line bundle from a Weil divisor. In most cases, it's either clear whether you're using a Cartier or Weil divisor, *or*, perhaps more likely, you're in a scenario where it does not matter.

Proposition 21.13. Let X be a scheme.

(a) For any Cartier D, $\mathscr{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathscr{L}(D)$ gives a bijection between Cartier divisors on X and invertible subsheaves of $\mathscr{K}$
(b) $\mathscr{L}(D_1 - D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$
(c) $D_1 \sim D_2 \iff L(D_1) \cong L(D_2)$ as abstract invertible sheaves (so ignoring how they embed in $\mathscr{K}$)

Proof.

(1) Clearly locally free of rank 1 by definition. The cartier divisor can be recovered (i.e. a map in other direction) by taking U_i a cover such that on U_i it is locally generated by f_i.
(2) If D_1 locally generated by f_i, and D_2 locally generated by some g_i, then $\mathscr{L}(D_1 - D_2)$ is locally generated by $f_i^{-1} g_i$ and then certainly $\mathscr{L}(D_1 - D_2) = \mathscr{L}(D_1) \cdot \mathscr{L}(D_2)^{-1}$ as subsheaves on the right, yielding $\mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$.
(3) Use that you can globally generate the difference, so $1 \mapsto f^{-1}$ trivializes the difference f the sheaves. $\square$

Corollary 21.14. On any scheme this assignment gives an assignment CaCl to Pic that is injective

Remark 21.15. May not by surjective: there may be invertible sheaves that can not be realized as subsheaves of $\mathscr{K}$. The examples of such tend to be fairly bizarre: in most scenarios these groups are the same.

Proposition 21.16. If X is an integral scheme, then CaCl to Pic is an iso.

Proof. Need that every invertible sheaf on X is realizable as a subsheaf of $\mathscr{K}$, which is the constant sheaf $\underline{K(X)}$. On trivializable neighborhoods, $\mathscr{L} \otimes \mathscr{K} \cong \mathscr{K}$, so constant on U. X irreducible, so $\mathscr{L} \otimes \mathscr{K} \cong \mathscr{K}$ overall, and $L \to L \otimes \mathscr{K}$ expresses it as a subsheaf. $\square$

Corollary 21.17. Noetherian, integral, separated, locally factorial implies class group and pic same.

Remark 21.18. And so we see another reason to care about divisors: their ties to line bundles, which, as we will see in Hartshorne II.7, are tied to morphisms to projective space.

22. Mar 05: Misc. divisor & l.b. constructions, morphisms to projective space

Recommended reading: Hartshorne II.7, Vakil 14.1-14.2, 15.1-15.4

Some last-bits-and bobs on Hartshorne II.6 material:

Let's circle back to our last example. We saw that the Cartier divisor associated to $V(x)$ in $\mathbb{P}^1_k = \text{Proj } k[x,y]$ was $\{(D(x), 1), (D(y), \frac{x}{y}\} = C$. We get that the associated line bundle $\mathscr{L}(C) = \mathcal{O}_X(C)$ is given on $D(x)$ by $\mathcal{O}_X|_{D(x)}$ and on $D(y)$ by $(\frac{y}{x})(\mathcal{O}_X|_{D(y)}) \subseteq K(X)$. That is, on $D(x)$ the module is generated by 1 and on $D(y)$ the module is generated by $(\frac{y}{x})$

Then the transition function on $D(x) \cap D(y)$ (when viewing in $D(x)$ versus $D(y)$) is:

$$\mathcal{O}_{D(x)\cap D(y)} \xrightarrow{\times(x/y)} \mathcal{O}_{D(x)\cap D(y)}$$

which is the transition function of $\mathcal{O}_{\mathbb{P}^1}(1)$, as we'd like it to be.

Below recaps some info about constructions relating divisor data and line bundle data. We will assume that X is integral henceforth.

Definition 22.1. Let $\mathscr{L}$ be an invertible sheaf on a scheme X. A **rational section of** $\mathscr{L}$ is a section of $\mathscr{L}$ over a nonempty dense open set U.

Remark 22.2. Equivalently, this is a global section of $\mathscr{L} \otimes \mathscr{K}$. Locally, $X = \text{Spec } A$, $\mathscr{L} = \widetilde{M}$, $\mathscr{K} = \mathscr{K}(A)$. Then:

$$\Gamma(X, \mathscr{L} \otimes \mathscr{K}) = M \otimes \mathscr{K}(A) = \mathscr{M}_\eta$$

Given a line bundle $\mathscr{L}$ and a rational section s, there are a few constructions we can do.

- We always get a map div to Cartier divisors: write $X = \cup U_i$ where $\mathscr{L}$ is trivializable on the U_i, let s be a rational section, then take s_i to be the image of s under
$$(\mathscr{L} \otimes \mathscr{K})|_{U_i} \to (\mathcal{O}_X \otimes \mathscr{K})|_{U_i}$$
On the overlaps the transitions are $s_i/s_j \in \mathcal{O}^*$ so we get a well-defined element $\{(U_i, s_i)\}$.
- In scenarios were Weil divisors make sense, we can also define the associated Weil divisor
$$\text{div}(s) = \sum_Y v_Y(s)$$
where we make sense of $v_Y(s)$ as follows: take any open U containing the generic point of Y on which $\mathscr{L}$ is trivializable. Then s is a nonzero rational function on U, and has a valuation. In this case, get Div $: \{(\mathscr{L}, s\}/\cong \to$ Weil.

One can more directly define the bundle associated to a Weil divisor $\mathcal{O}_X(D)$:

$$\Gamma(U, \mathcal{O}_X(D)) := \{t \in \mathscr{K}(X)^* : \operatorname{div}|_U(t) + D|_U \geq 0\} \cup \{0\}$$

Where $\operatorname{div}|_U(t)$ means take the divisor as a rational function of U, i.e. ignore prime divisors Y outside of U and the ≥ 0 condition means that the coefficients of the Weil divisor are all non-negative. That is, your sections have certain permissible zeroes/poles that are controlled by D.

For $H = V(x)$ in $\mathbb{P}^1_{[x,y]}$, consider the line bundle $\mathcal{O}_X(H)$. We see that on $D(x)$ this looks generated as an $\mathcal{O}_X$-module by 1, and on $D(y)$ this looks generated by $\frac{y}{x}$, as we expected. That is, in this sheaf the rational functions on $D(y)$ are allowed a pole of order one at $x = 0$.

Remark 22.3. We indeed have that, for a line bundle $\mathscr{L}$ and rational section s, that $\mathcal{O}(\operatorname{div}(s)) \cong \mathscr{L}$. This is quite useful in certain pullback computations.

Remark 22.4. One can find a summary of the various maps and constructions relating to Cartier divisors, Weil divisors, and line bundles in 14.2.7 of Vakil.

Certainly in our Grothendieck-type perspective we care about morphisms from a scheme X, and of particular note are morphisms to projective space. Projective varieties are so important, and the ways to can map and embed X into projective space are controlled by line bundles and their sections.

Namely, we see how a morphisms X to a projective space is determined by giving an invertible sheaf $\mathscr{L}$ and some collection of the global sections of $\mathscr{L}$. Under certain criteria, it will be an immersion. Through this we see how *ampleness* is a useful/important property and run into the terminology of *linear systems*.

Now, onto studying morphisms to $\mathbb{P}^n$!

Let A be a fixed ring, and consider $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. This has a line bundle $\mathcal{O}(1)$ and homogeneous coordinates $x_0, \dots, x_n$ yield global sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$. Note that the images of these in any stalk will generate, so $\mathcal{O}(1)$ is globally generated.

Theorem 22.5. Let A be a ring, and X a scheme over A.

(a) If $\varphi : X \to \mathbb{P}^n_A$ is a morphism as A-schemes, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, generated by the global sections $s_i = \varphi^*(x_i)$ (where $i = 0, 1, \dots, n$).

(b) Conversely, given $\mathscr{L}$ invertible sheaf on X and $s_0, \dots, s_n$ generating $\mathscr{L}$ globally, there is a unique A-morphism $X \to \mathbb{P}^n_A$ such that $\mathscr{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under the iso.

That is, roughly: equivalence between two things:

- Morphisms $X \to \mathbb{P}^n_A$ and
- Invertible sheaves $\mathscr{L}$ on X and a choice of $n+1$ global sections $x_0, \dots, x_n$ that globally generate $\mathscr{L}$. (i.e. images in stalk generate).

$\Rightarrow$: $\mathcal{O}(1) \Rightarrow \varphi^*$ invertible, and note that $(\varphi^*\mathcal{O}(1))_p = \mathcal{O}(1)_{\varphi(p)} \otimes_{\mathcal{O}_{\mathbb{P}^n, \varphi(p)}} \mathcal{O}_{X,p}$ get that the $s_i \otimes 1 = \varphi^*(s_i)$ generate.

$\Leftarrow$: proof will be finished next time.

23. Mar 07: Closed immersion maps to projective space

Recommended reading: Hartshorne II.7, Vakil 15.2, 15.3 Our first goal is to finish the proof from last time.

$\Leftarrow$: We want to show that given $\mathscr{L}$ and global sections $s_0, \dots, s_n$ globally generating $\mathscr{L}$, that there is a unique A-morphism $X \to \mathbb{P}^n_A$ such that $\mathscr{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$.

Intuitively, this map is given by $x \mapsto [s_0(x), \dots, s_n(x)]$. For example, for $X = \mathbb{P}^1_k, \mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(3)$ and generating sections x^3, x^2y, xy^2, y^3, the map should be, on closed points,

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x,y] &\mapsto [x^3, x^2y, xy^2, y^3] \end{aligned}$$

and the pullback of x_0 would be x^3, and so on. But we need to formalize this process: we should give the map as schemes, and we should be careful about what we mean when the global sections aren't clearly things that could give you a function to projective space.

Note with this example: a hyperplane condition $x_0 = 0$ becomes something like $x^3 = 0$ when pulled back. Take div of this to get $3H$, which is a Weil divisor that can produce $\mathcal{O}_{\mathbb{P}^1}(3)$ under the correspondence $D \mapsto \mathcal{O}_{\mathbb{P}^1}(D)$. This is what we want!

First, we cover X in $D(s_i) = \{p \in X : (s_i)_p \notin m_p\mathscr{L}_p\}$, the "nonvanishing of s_i". Note that $(s_i)_p \in \mathfrak{m}_p\mathscr{L}_p$ means that $s_i|_p = 0$ in the fiber $\mathscr{L}|_p = \mathscr{L} \otimes k(p) = \mathscr{L}_p/\mathfrak{m}_p\mathscr{L}_p$. This means, in the section-of-a-bundle interpretation of s_i, that s_i takes the value zero at p.

The fact that the s_i generate $\mathscr{L}$ means that $X = \cup D(s_i)$. Set $D(s_i) := U_i$. On this, define the map

$$\varphi_i : D(s_i) \to D(x_i) \subseteq \mathbb{P}^n$$

induced by

$$\varphi : A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(D(s_i), \mathcal{O}_{D(s_i)})$$
$$\frac{x_j}{x_i} \to \frac{s_j}{s_i}$$

This is well-defined: the quotient must be an element of the structure sheaf. This is because $\mathscr{L}$ is locally free of rank 1. One way that might help to see it is

$$\mathcal{O}_X|_{U_i} \xrightarrow{\cong} \mathscr{L}|_{U_i}$$
$$1 \mapsto s_i$$

is an isomorphism. There are a couple mays to see this. The first, and perhaps the most formal, is that you can check this is an isomorphism on trivializable affines $U_{i,j}$, in which s_i restricts to things looking like units in each $U_{i,j}$. Hence s_i must generate $\mathscr{L}$ on each $U_{i,j}$. Another way to see this from the idea that a line bundle on some X or in this case U_i with a nonvanishing section is trivializable (it gives you a consistent way to pick a basis of the fibers!).

On overlaps $U_i \cap U_j$ we get $\frac{s_k}{s_i} = \frac{s_k}{s_j} \cdot (\frac{s_j}{s_i})$, so the morphisms glue to

$$\varphi : X \to \mathbb{P}^n$$

By construction, the $\mathscr{L} = \varphi^*(\mathcal{O}(1))$ (it has the right transition functions) and $\varphi^*(x_i) = s_i$. Any other morphism with these properties would need to be the one we've constructed.

Remark 23.1. In our beloved $\mathbb{P}^1 \to \mathbb{P}^3$ example, with the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and sections x^3, x^2y, y^2x, y^3, this describes the map via 4 charts:

$$D(x^3) \to D(x_0) \subseteq \mathbb{P}^3$$
$$[x, y] = [1, \tfrac{y}{x}] \mapsto \left[1, \tfrac{x^2y}{x^3}, \tfrac{xy^2}{x^3}, \tfrac{y^3}{x^3}\right] = [1, \tfrac{y}{x}, (\tfrac{y}{x})^2, (\tfrac{y}{x})^3]$$
$$D(x^2y) \to D(x_1) \subseteq \mathbb{P}^3$$
$$[x, y] \mapsto \left[\tfrac{x}{y}, 1, \tfrac{y}{x}, (\tfrac{y}{x})^2\right]$$
$$D(xy^2) \to D(x_2) \subseteq \mathbb{P}^3$$
$$[x, y] \mapsto \left[(\tfrac{x}{y})^2, \tfrac{x}{y}, 1, \tfrac{y}{x}\right]$$
$$D(y^3) \to D(x_3) \subseteq \mathbb{P}^3$$
$$[x, y] \mapsto \left[(\tfrac{x}{y})^3, (\tfrac{x}{y})^2, \tfrac{x}{y}, 1\right]$$

which all patches to $[x, y] \to [x^3, x^2y, xy^2, y^3]$.

Proposition 23.2. Likewise, rational maps $X \to \mathbb{P}^n_A$ are in 1-to-1 correspondence with line bundles $\mathscr{L}$ with $n + 1$ global sections $s_0, \dots, s_n$.

Proof. Largely the same proof, but the s_i not globally generating means that possibly we only have this morphism defined on $U = \cup D(s_i)$, which may be a proper subset of X. $\square$

Definition 23.3. In the notation above, the **base locus** of the morphism is, as a set, $X \setminus U = V(s_0, \ldots, s_n)$. It is the closed set where the morphism is not defined. We will worry about its specific scheme structure later, when we deal with blowups.

It is immediate that $s_0, \ldots, s_n$ globally generate $\mathscr{L}$ if and only if the base locus is empty.

Example 23.4. If we take $X = \mathbb{P}^2$ and line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$ with sections yz, xz, xy, we would get the morphism

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

$$[x, y, z] \mapsto [yz, xz, xy] = [\tfrac{1}{x}, \tfrac{1}{y}, \tfrac{1}{y}]$$

The base locus is $[1, 0, 0], [0, 1, 0], [0, 0, 1]$.

Example 23.5. If one takes $X = \mathbb{P}^{n+1}_k$ and $\mathscr{L} = \mathcal{O}(1)$ and sections $x_1, \ldots, x_{n+1}$, then the base locus is $[1, 0, \ldots, 0] = P$ and the map $\mathbb{P}^{n+1} \to \mathbb{P}^n$ is projection from P onto the hyperplane $x_0 = 0$.

Corollary 23.6. The automorphisms of $\mathbb{P}^n_k$ are given by $\mathrm{PGL}_n(k) = \mathrm{GL}_{n+1}(k)/k^*$.

Proof. An element of GL induces an automorphism of $k[x_0, \ldots, x_n]$ and thus of projective space. Scalar multiples induce the same automorphism, so may consider as an element of PGL.

To see that all automorphisms are of this form: consider an automorphism $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$. Then pullback induces an automorphism of $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$. So then $\varphi^*(\mathcal{O}(1))$ must be a generator of $\mathrm{Pic}(\mathbb{P}^n_k)$ and it also must have global sections, so $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$. The $s_i = \varphi^*(x_i)$ must also be a basis of $\langle x_0, \ldots, x_n \rangle$ and thus determine an element of M of GL. Scalar multiples don't change the map, so just care about PGL rep. Note that the map looks like $[x_0, \ldots, x_n] \to [Mx_0, \ldots, Mx_n]$. $\square$

E-mail address: gwynm@uic.edu

Department of Mathematics, Stat., & CS, University of Illinois Chicago, Chicago, IL 60607