MATH 215: CONTINUOUSLY UPDATED NOTES

Below are continuously updated notes for Math 215.

1. Jan 8: Syllabus day

- Go through syllabus
- Finish with mathematical warmup: what is an even number ($x = 2k$ with $k \in \mathbb{Z}$), what is an odd number ($y = 2k + 1$ with $k \in \mathbb{Z}$). Must every number be even or odd? (Yes, but to prove this will require later material). Idea is to get students thinking about precise, useful definitions and how to prove facts they are used to assuming.

Here $\in$ is read "in" and $\mathbb{Z}$ denotes the set of integers (i.e. the set $\{\dots, -3, -2, -, 1, 0, 1, 2, 3, \dots\}$ of positive whole numbers, zero, negative whole numbers). "$k \in \mathbb{Z}$" therefore means that k is an integer.

2. Jan 10: Statements

- Introduction to *statements* in the mathematical sense (declarative statement that can be assigned a truth value: it is either True or False, not both).
- Building statements from new statements. Negation, and, or. Truth tables below.

P	$\neg P$
T	F
F	T

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- Use examples to motivate logical equivalences. Two things are logically equivalent if they always have the same truth value. This means they can be swapped for one another in statements, expressions, etc. DeMorgan's laws:

$$\neg(A \wedge B) \text{ logically equiv to } \neg A \vee \neg B$$
$$\neg(A \vee B) \text{ logically equiv to } \neg A \wedge \neg B$$

as well as distributive properties:

$$A \wedge (B \vee C) \text{ logically equiv to } (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \text{ logically equiv to } (A \vee B) \wedge (A \vee C)$$

3. Jan 17: Conditionals

Another example of building new statements from old ones. Given P, Q statements we can form $P \Rightarrow Q$. It has the following truth assignments, depending on those of P, Q.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We discuss what it means to "prove" a statement of the form $P \Rightarrow Q$: this means, show it is always true. Looking at the table above, we only need to worry about landing in the case of $T \Rightarrow F$. So proving a conditional means assuming P is true, and trying to use logical arguments to deduce that Q is true. We see that, for x an integer,

$$x \text{ is a multiple of } 4 \Rightarrow x \text{ is a multiple of } 2$$

is true (information on the left always implies information on the right) , but

$$x^2 > 0 \Rightarrow x > 0$$

is not true (there are situations where $x^2 > 0$ and $x < 0$).

From the conditional $P \Rightarrow Q$, we can define the inverse ($\neg P \Rightarrow \neg Q$) as well as the converse ($Q \Rightarrow P$) and the contrapositive ($\neg Q \Rightarrow \neg P$). We then go through writing their truth values:

P	Q	$P \Rightarrow Q$	$\neg P \Rightarrow \neg Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

From there we see the original $P \Rightarrow Q$ is logically equivalent to the contrapositive. We also see the inverse, converse are logically equivalent to eachother (note that the inverse is the contrapositive of the converse). We discuss how this will be useful: sometimes the contrapositive is way easier to prove than the original statement.

Lastly, if $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then we say $P \iff Q$ (read "P if and only if Q). This means P is true exactly when Q is true and vice versa: i.e. this means P, Q are logically equivalent. So from now on we write P, Q logically equivalent as $P \iff Q$. If trying to prove $P \iff Q$ on a HW/in class: there should be two parts: showing $P \Rightarrow Q, Q \Rightarrow P$.

4. Jan 19: Proof methods part 1 (direct proofs)

Three major proof methods when trying to show $P \Rightarrow Q$

- Direct proof: assume P true, use logical deductions, algebra, lemmas (small results) and theorems from class to try to show Q is true.
- Contradiction: assume P is true but Q is false. Show that this yields a logical contradiction (say, contradict a part of the assumption, or run into a logical fallacy like $0 = 1$). Then the original assumption must have been wrong, and Q is true.
  - Good to start these proofs with "Assume, for the sake of contradiction" or "Suppose P is true but Q were false." Something to indicate to the reader that you are doing a contradiction proof.
  - Colloquially, this also gets used for: if you're trying to show a statement A is true, you assume $\neg A$ is true instead and run into a contradiction.
- Contrapositive: show $\neg Q \Rightarrow \neg P$.

We focus on direct proof today.

Two exercises:

- **Show that x even, y odd $\Rightarrow x + y$ odd.**
- **Show that x odd $\iff x + 2$ odd.**

Proof of the first result: Since x even, y odd: by definition we have

$$x = 2k$$
$$y = 2\ell + 1$$

with k, ℓ integers. Then $x + y = 2k + 2\ell + 1 = 2(k + \ell) + 1$. $k + \ell$ is an integer since k, ℓ are integers. Thus, by definition, $x + y$ is odd. □

Similar definition unwinding yields the second result.

Things to note: using separate variables for writing $x = 2k$, $y = 2\ell + 1$. At each point we are clear about what results/definitions/information/etc we are using. Note that for the second result: make sure proof has two parts.

5. Jan 22: Dontradiction, contrapos

This lesson we focus more on contradiction, contrapositive: i.e. the methods that involve some sort of negation.

Warmup: you may assume every integer is exactly one of even or odd. Show that x^2 even $\Rightarrow x$ even.

Proof. (Note that this is hard to do directly! Contrapositive helps flip this into turning info about x into info about x^2 in a pretty straightforward manner.) We use proof by contradiction: we will show x not even $\Rightarrow x^2$ not even. Equivalently, this means showing x odd $\Rightarrow x^2$ odd. If x is odd, then $x = 2k + 1$ for some k an integer. Then:

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

$2k^2 + 2k$ is an integer because k is an integer. So by definition, x^2 is odd and we are done. □

Next, some definitions. A number x is **rational** provided that it can be written as $x = \frac{p}{q}$ where p, q are integers and $q \neq 0$. A number is **irrational** if it is not rational.

Show: if x is rational, y irrational, then $x + y$ is irrational.

One thing that jumps out: hard to do this directly. Contrapositive seems difficult because negative the left side seems tedious. So let's try contradiction. That will turn $x + y$ irrational into $x + y$ rational, which will be nice to work with.

Proof. Suppose, for the sake of contradiction, that x is rational, y irrational, $x + y$ irrational. Then $x = \frac{p}{q}$ and $x + y = \frac{a}{b}$ with p, q, a, b integers and q, b nonzero. Then:

$$y = (x + y) - x = \tfrac{a}{b} - \tfrac{p}{q} = \frac{aq - pb}{bq}.$$

The numerator and denominator are integers since a, b, p, q are. bq is nonzero because b, q are nonzero. But that means y is rational, which contradicts y being irrational. Hence our assumption is false, and $x + y$ must be irrational. □

We end by trying to prove the following: **Show $\sqrt{2}$ irrational.** Start by supposing, for the sake of contradiction, that $\sqrt{2}$ is rational. Then:

$$\sqrt{2} = \frac{p}{q}.$$

Square both sides, get $2 = p^2/q^2$, equivalently $2q^2 = p^2$. Try messing with even-ness, odd-ness to see if can get a contradiction.

(Another fun result one can do with contradiction: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ is never an integer).

6. Jan 24: Finish contradiction. Casework

We finish the proof of $\sqrt{2}$ irrational. Intuitively: look at $2q^2 = p^2$. Look at the prime factorization of each. The number of 2's on the left is odd, the number of 2's in the right is even (because the primes in the factorization of a square all have even power).

Proof that $\sqrt{2}$ irrational. Suppose, for the sake of contradiction, that $\sqrt{2}$ is irrational. Then

$$\sqrt{2} = \frac{p}{q}$$

We shall assume that p, q are in lowest terms. Squaring both sides and rearranging, we get

$$2q^2 = p^2$$

Looking at this equation, p^2 is even. By a result from last class, this means p is even. Write $p = 2k$, k an integer. Then:

$$2q^2 = (2k)^2 = 4k^2.$$

Cancelling a factor of 2, we see

$$q^2 = 2k^2,$$

hence q is even. But if p, q are both even: the fraction couldn't have been in lowest terms! We could cancel a factor of 2 from top and bottom! So we've arrived at a contradiction. Our assumption must be false, and so $\sqrt{2}$ is irrational. □

The heart of what's going on is factorization issues. We'll see more about factorization in the number theory section of the course.

A nice bookend to the proof methods chunk is to cover proof by cases. We show that $n^2 - n$ is always even by looking at even, odd cases. This and proof of $\sqrt{2}$ help motivate induction. Time permitting, show multiple proofs of $n^2 - n$.

7. Jan 26: Induction

Results like:

- all fractions can be put in least terms
- all integers are even or odd

rely on induction (or one of its equivalent formulations: strong induction, well ordering principle). It is an axiom, and a very useful and major method. Likened to "mathematical dominos."

Idea of induction: if a property holds for $k = 1$, and a property holding for k implies it holds for $k + 1$, then we can start at 1 and "domino effect" down to get a property holds for all natural numbers ($\{1, 2, 3, 4, \dots\}$). Good for proving a fact holds for all natural numbers.

Proofs by induction always have two parts: base case (the $k = 1$ part) and inductive step (showing the property holds for k implies the property holds for $k + 1$).

Examples of proofs by induction:

- Show that every integer is even or odd. (casework + induction)
- Show that
$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
for all natural numbers n Note: provides a fun proof that $n^2 - n$ is even as a corollary.
- Show that $k^3 + 2k$ is always divisible by 3.
- Show that
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
for all natural numbers n
- Show that
$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$
for all natural numbers n.

Proof of the second statement. We use proof by induction. Base case: note that $1 = \frac{1(2)}{2}$ so the formula holds for $k = 1$.

Inductive step: suppose the formula is true for k. Then:

$$\begin{aligned} 1+2+\cdots+k+k+1 = \frac{k(k+1)}{2}+(k+1) &= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

which means the formula is true for $k+1$. So, by induction, the formula is true for all natural numbers n. $\square$