ON THE EFFECTIVE & NEF CONES OF THE CHOW RING OF THE HILBERT SCHEME OF THREE POINTS ON $\mathbb{P}^3$

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ABSTRACT. We compute the nef and effective cones in codimension and dimension 2, respectively, for the Hilbert scheme of three points in $\mathbb{P}^3$.

CONTENTS

1. INTRODUCTION

Let $\mathrm{Hilb}_m\mathbb{P}^n$ denote the Hilbert scheme of zero-dimensional, length m schemes in $\mathbb{P}^n$. Much work has been done studying Hilbert schemes of points in the surface case (ex: [Hui16], [BS17], [G$\ddot{9}$0], [RS21]) where these schemes are well behaved: $\mathrm{Hilb}_m\mathbb{P}^2$ is smooth, irreducible, reduced, and we have a birational map:

$$\mathrm{Sym}^m\mathbb{P}^2 \dashrightarrow \mathrm{Hilb}_m\mathbb{P}^2.$$

Hence a generic $\Gamma \in \mathrm{Hilb}_m\mathbb{P}^2$ corresponds to the union of m distinct points. For higher dimensions, these schemes can quickly become poorly-behaved: for $n \geq 3$, $\mathrm{Hilb}_m\mathbb{P}^3$ need not be irreducible [Iar72] and $\mathrm{Hilb}_m\mathbb{P}^4$ need not be reduced [CEVV09]. $\mathrm{Hilb}_3\mathbb{P}^3$, however, is relatively well-behaved: it is irreducible and smooth, and work has been done on writing bases for the Chow/cohomology ring [RL90], [FG93].

In this paper, we look at higher (co)dimension nef and effective cones of $\mathrm{Hilb}_3\mathbb{P}^3$. Nef and effective cones in (co)dimension 1 have been a subject of study due to their connections to the minimal model program, as the effective cone is the closure of the ample cone. Higher (co)dimension analogs have been studied recently in [DELV11], [CLO16], [CC15], as well [RS21]. In particular, Ryan and Stathis [RS21] study the (co)dimension 2, 3 nef and effective cones in $\mathrm{Hilb}_3\mathbb{P}^2$, utilizing the group action of $\mathrm{PGL}_3(\mathbb{C}^3)$. We will also use the PGL action in our work.

For these higher (co)dimension cones, they often live in vector spaces of dimension greater than 2, and *a priori* may have infinitely many extremal rays.

Our main theorem is stated below and proven in Section 5.

Theorem 1.1. *We have the following expression for the nef and effective cones in (co)dimension two for* $\mathrm{Hilb}_3\mathbb{P}^3$.

$$\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^3) = \langle \alpha,\ \epsilon,\ -\alpha+\delta,\ -\alpha+\beta-\delta,\ \alpha-\beta+\gamma-\delta-2\mu,\ -2\alpha+\beta-\gamma+2\delta+\epsilon,\ \mu\rangle$$
$$\mathrm{Nef}_2(\mathrm{Hilb}_3\mathbb{P}^3) = \langle \widetilde{B},\ \widetilde{C},\ \widetilde{D},\ \widetilde{E},\ \widetilde{A}+\widetilde{B},\ \widetilde{A}+\widetilde{E},\ 2\widetilde{A}+2\widetilde{E}+M,\ 2\widetilde{A}+2\widetilde{B}+M\rangle.$$

The theorem is stated in terms of the basis of Section 3.2. One can compare this with the result on the (co)dimension 2 nef and effective cones of $\mathrm{Hilb}_3\mathbb{P}^2$ proven in [RS21], which are recapped for the reader in Theorem 2.2.

The paper is structured as follows: in Section 2 we give some background on Hilbert schemes of points in $\mathbb{P}^2$, their Chow groups, and their nef and effective cones. In Section 3, we give some background on $\mathrm{Hilb}_3\mathbb{P}^3$ and the structure of its Chow ring. In particular, we recount a basis for the Chow groups given by [RL90], and extend a basis of [MS90] to the cases of $A_2(\mathrm{Hilb}_3\mathbb{P}^3), A_7(\mathrm{Hilb}_3\mathbb{P}^3)$. We also compute some helpful auxiliary classes in these bases. In Section 4, we study the orbits of the PGL action. Lastly, in Section 5, we prove the main theorem.

Remark. In regards to [RL90] we would like to give a word of caution to the reader: the intersection tables at the end of [RL90] contain a few errors. Namely, the values of $PP_2^4, PP_2^3\ell, PH^2P_2^2p, P^3\ell^3p$ are incorrect. They should be the following:

$$PP_2^4 = -2$$
$$PP_2^3\ell = 2$$
$$PH^2P_2^2p = -3$$
$$P^3\ell^3p = 1.$$

2. Preliminaries on Hilbert schemes of points

2.1. **Two bases for the Hilbert scheme of three points in the plane.** We consider two bases for the Chow groups $A_k(\mathrm{Hilb}_3\mathbb{P}^2)$: one from Elencwajg & Le Barz ([ELB83]), and one from Mallavibarrena & Sols ([MS90]). The Elencwajg-Le Barz basis has the advantage of being stated in terms of multiplication of a few classes, and the perk of having a $\mathrm{Hilb}^3\mathbb{P}^3$ analog with much of the multiplication data worked out. The Mallavibarrena-Sols basis is a little easier to compute intersection products with by hand, and additionally Ryan and Stathis [RS21] compute the nef cones of $A^\bullet(\mathrm{Hilb}_3\mathbb{P}^2)$ in codimensions 2 and 3 and effective cones in dimension 2 and 3 using this basis. We utilize some of their computations in the $\mathrm{Hilb}_3\mathbb{P}^3$ case. Both bases will be utilized in this paper, and we compute some changes of coordinates between the two bases in Propositions 2.1 and 3.3.

We start with the Elencwajg-Le Barz (EL) basis. Let $Q \in \mathbb{P}^2$ be a fixed point, $L, L' \subseteq \mathbb{P}^2$ be two fixed lines, and define the following classes.

$$F = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : \exists d \subset \Gamma \text{ of length 2 such that } d \text{ is collinear with } Q\}]$$
$$H = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : \Gamma \cap L \neq \emptyset\}]$$
$$\ell = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : Q \in \Gamma\}]$$
$$g = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : \Gamma \text{ is collinear with } Q\}]$$
$$p = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : \exists d \subset \Gamma \text{ of length 2 such that } d \subseteq L\}]$$
$$g_e = [\{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2 : \Gamma \subseteq L\}]$$

and lastly define β to be the locus whose generic member Γ is a nonreduced scheme given by the union of a length two subscheme supported on L and a length one subscheme supported on L'.

Then bases for the Chow groups of $\mathrm{Hilb}_3\mathbb{P}^2$ are given by:

k	$A_k(\mathrm{Hilb}^3\mathbb{P}^2)$ basis
6	$[\mathrm{Hilb}_3\mathbb{P}^2]$
5	H, F
4	H^2, HF, ℓ, p, g
3	$H^3, H^2F, H\ell, Hg, g_e, \beta$
2	$H^2\ell, H^2g, \ell^2, \ell g, \ell p$
1	$H\ell p, H\ell^2$
0	ℓ^3

and for reference we include the intersection tables for (co)dimension 1 and (co)dimension 2.

	$H\ell p$	$H\ell^2$
H	1	1
F	1	2

	$H^2\ell$	H^2g	ℓ^2	ℓg	ℓp
H^2	3	6	1	1	1
HC	6	3	2	0	1
ℓ	1	1	1	0	0
g	1	-1	0	0	0
p	1	0	0	0	0

Next we introduce the Mallavibarrena & Sols (MS) basis. In general, elements of this basis are defined by three sorts of conditons: being contained in a fixed line with a fixed point included, being contained a fixed line, and being collinear with a fixed point. We will not go into detail about the MS basis in general, and will instead describe the classes that arise in the $\mathrm{Hilb}_3\mathbb{P}^2$ case and provide figures showing a general point $\Gamma \in \mathrm{Hilb}_3\mathbb{P}^2$ in each class. A more in-depth treatment can be found in [MS90].

For the remainder of this section, L, L', L'' are fixed distinct lines in $\mathbb{P}^2$. P is a fixed point on L, and Q is a fixed point not contained in any of the three lines.

In dimension one and codimension one, the EL and MS bases are the same. In dimension five the basis elements are H, the locus of schemes incident to a line, and F, the locus of schemes with a two dimensional subscheme that is collinear with a point p. In dimension one the generators are $\phi = H\ell^2$ and $\psi = H\ell p$. ϕ is the locus whose general member is a scheme containing two fixed points P, Q, and a third point on L. ψ is the locus whose general member is contains two fixed points P, Q, and a third point on L' (which notably contains neither P nor Q). See Figure 1 for pictures of a general member of each scheme.

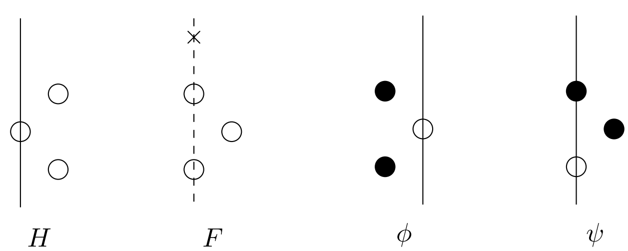

FIGURE 1. Pictures showing a general member of each of H, F, ϕ, ψ. A solid point denotes a fixed point that must be included in the scheme; an $\times$ denotes a fixed point that is not necessarily in the support. A solid line denotes a fixed line; a dashed line indicates a line that can vary, and is used for our purposes to show a collinearity requirement.

We have the following table of intersection products.

	ϕ	ψ
H	1	1
F	2	1

TABLE 1. Intersection table for $A_5(\mathrm{Hilb}_3\mathbb{P}^2) \times A_1(\mathrm{Hilb}_3\mathbb{P}^2)$.

We now turn to dimension and codimension 2. The five basis elements of $A_4(\mathrm{Hilb}_3\mathbb{P}^2)$ are as follows. A is the locus of schemes collinear with Q. B is the locus whose general member is reduced with a length two subsceheme collinear with Q, and the remaining point lying on L. C is the locus whose general member is a reduced scheme with one point on L and one point on L'. D is the locus whose general member is a scheme with a length two subscheme contained in L. E is the locus of schemes with P in the support.

The five basis elements of $A_2(\mathrm{Hilb}_3\mathbb{P}^2)$ are as follows. α is the locus of schemes contained in L and containing P. β is the locus whose general member is a reduced scheme containing two points in L, with one of the points being P, and with a third point on L'. γ is the locus whose general member is a reduced scheme containing P, a point on L', and a point on L''. δ is the locus whose general member is a reduced scheme containing P and containing two points on L'. ϵ is the locus of schemes containing P, Q in the support. See Figure 2 for pictures of a general member of each scheme.

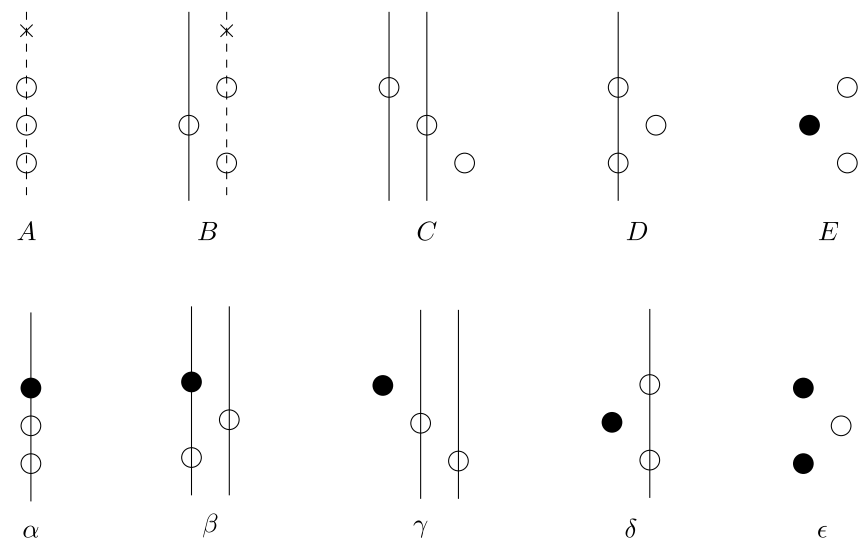

FIGURE 2. General members of each of the basis elements of $A_4(\mathrm{Hilb}_3\mathbb{P}^2), A_2(\mathrm{Hilb}_3\mathbb{P}^2)$.

The intersection pairings between these bases are given below. We now describe the six basis elements of

	α	β	γ	δ	ϵ
A	0	0	1	0	0
B	0	1	2	1	0
C	1	2	2	1	0
D	0	1	1	0	0
E	0	0	0	0	1

TABLE 2. Intersection matrix for $A_4(\mathrm{Hilb}_3\mathbb{P}^2) \times A_2(\mathrm{Hilb}_3\mathbb{P}^2)$.

$A_3(\mathrm{Hilb}_3\mathbb{P}^2)$. U is the locus of schemes containing P and incident to L'. V is the locus of schemes whose general member is the union of a fixed point P and a length two subscheme collinear with Q. W is the locus of schemes whose general member is a reduced subscheme given by the union of one point each on L, L', L''. X is the locus of schemes whose general member is the union of a length two subscheme contained in L and a point contained in L'. Y is locus of schemes contained in L. Z is the locus of schemes with a length two subscheme that contains P and is contained in L. See below for a picture describing a general member of each locus, as well as the table of intersection products.

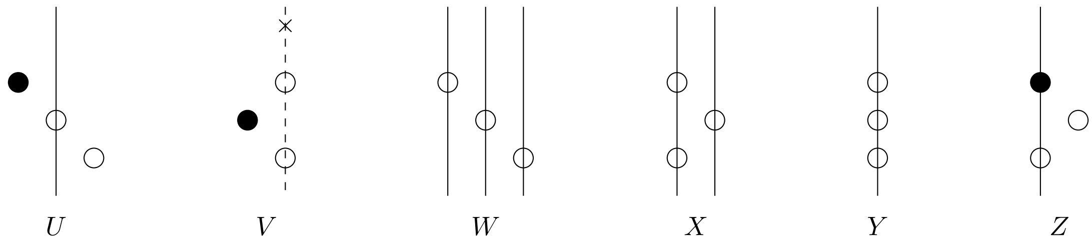

FIGURE 3. General members of the basis elements of $A_3(\mathrm{Hilb}_3\mathbb{P}^2)$

	U	V	W	X	Y	Z
U	1	1	0	0	0	1
V	1	1	0	0	0	0
W	0	0	6	3	1	0
X	0	0	3	1	0	0
Y	0	0	1	0	0	0
Z	1	0	0	0	0	1

TABLE 3. Intersection table for $A_3(\mathrm{Hilb}_3\mathbb{P}^2) \times A_3(\mathrm{Hilb}_3\mathbb{P}^2)$

Proposition 2.1. We have the following change of basis between the EL and MS bases in $A_2(\mathrm{Hilb}_3\mathbb{P}^2)$.

$$\begin{aligned} H^2\ell &= \gamma + \varepsilon \\ H^2 g &= 3\alpha + \beta - \gamma + 2\delta + \epsilon \\ \ell^2 &= \epsilon \\ \ell g &= \alpha \\ \ell p &= \delta \end{aligned}$$

Correspondingly, we have the following conversion between the EL and MS bases in $A_7(\mathrm{Hilb}_3\mathbb{P}^2)$.

$$\begin{aligned} H^2 &= C + E \\ HC &= B + 2D + 2E \\ \ell &= E \\ g &= A \\ p &= D \end{aligned}$$

Proof. The dimension 7 change of basis can be obtained by the dimension 2 change of basis. In dimension 2, the only nontrivial computation is $H^2 g$. We compute its expression in the MS basis by first computing the class of Hg. We see that Hg has intersection numbers:

	U	V	W	X	Y	Z
Hg	1	0	3	0	0	0

Hence $Hg = (Z - U + V) + 3Y$. Then $Z \cdot H = \beta + \epsilon$, $U \cdot H = \gamma + \epsilon$, and $Y \cdot H = \alpha$. $V \cdot H$ can be checked by computing the intersection products with the complementary dimension:

	A	B	C	D	E
$V \cdot H$	0	2	2	0	1

yielding that $V \cdot H = 2\delta + \epsilon$. This yields $H^2 g = 3\alpha + \beta - \gamma + 2\delta + \epsilon$. $\square$

2.2. **Techniques for computing the effective and nef cones.** We begin by defining the nef and effective cones, and then give an overview of the methods in Ryan and Stathis's work [RS21] computing the nef and effective cones of $A^\bullet(\mathrm{Hilb}_3\mathbb{P}^2)$.

Let $N_k(X)$ denote the $\mathbb{Q}$-vector space of algebraic cycles, modulo numerical equivalence. The effective cone $\mathrm{Eff}_k(X)$ is the cone spanned by classes of k-dimensional varieties, and the pseudoeffective cone is its closure $\overline{\mathrm{Eff}}_k(X)$.

Furthermore, let $N^k(X)$ denote the dual space of $N_k(X)$. We define $\mathrm{Nef}_k(X)$ as the dual cone of $\overline{\mathrm{Eff}}_k(X)$ in $N^k(X)$. In the case of X smooth, we may identify $N^k(X)$ with $N_{\dim X-k}(X)$ via the intersection pairing, and thus may think of $\mathrm{Nef}_k(X)$ as a cone in $N_{\dim X-k}(X)$.

In Theorem 5 and 6 of [RS21], the authors prove the following result.

Theorem 2.2 (Ryan, Stathis '21). *We have:*

$$\begin{aligned}\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^2) &= \langle \alpha, \epsilon, -\alpha+\delta, -\alpha+\beta-\delta, \alpha-\beta+\gamma-\delta, -2\alpha+\beta-\gamma+2\delta+\epsilon\rangle \\ \mathrm{Nef}^2(\mathrm{Hilb}_3\mathbb{P}^2) &= \langle B, C, D, E, A+B, A+E\rangle\end{aligned}$$

And furthermore:

$$\begin{aligned}\mathrm{Eff}_3(\mathrm{Hilb}_3\mathbb{P}^2) = \langle &Y, -U+V+Z, 3U-2V-W+4X-6Y-Z, \\ &W-3X+3Y, X-3Y, U-V, U-Z\rangle \\ \mathrm{Nef}^3(\mathrm{Hilb}_3\mathbb{P}^2) = \langle &U, V, W, X, Z, V+Y, X+Y, 2Y+Z\rangle.\end{aligned}$$

In a variety with a transitive action by an algebraic group G, determining the nef and effective cones can be greatly aided by usage of Kleiman's transversality theorem (for example, Kleiman's quickly allows one to determine that the nef and effective cones of a Grassmannian are spanned by the Schubert classes). While the action of $\mathrm{PGL}_4(k)$ on $\mathrm{Hilb}_3\mathbb{P}^3$ is not transitive, it does have finitely many orbits, which is enough to yield the following useful tool.

Lemma 2.3. Let G be an infinite connected algebraic group acting on a smooth variety X with finitely many orbits $\mathcal{O}_1,\dots,\mathcal{O}_m$. Suppose Y is a subvariety of X of codimension k. If Y intersects $\mathcal{O}_1,\dots,\mathcal{O}_\ell$ in the expected dimension, intersects $\mathcal{O}_{\ell+1},\dots,\mathcal{O}_m$ in dimension one higher than expected, and intersects every effective k-cycle in $\overline{\mathcal{O}_{\ell+1}},\dots,\overline{\mathcal{O}_m}$ non-negatively, then Y is nef.

Proof. See Lemma 3 in [RS21]. □

In Ryan and Stathis's work, understanding the nef and effective cones of the closures of the orbits of the $\mathrm{PGL}_k(3)$ action on $\mathrm{Hilb}_3\mathbb{P}^2$ is key to working out $\mathrm{Nef}_4, \mathrm{Nef}_3, \mathrm{Eff}_2, \mathrm{Eff}_3$ of $\mathrm{Hilb}_3\mathbb{P}^2$. Thus we too work out many effective and nef cones of the orbit closures in Section 4. We also wish to identify these "positive " cones computed in $\mathrm{Hilb}_3\mathbb{P}^2$ as part of the nef and effective cones in $\mathrm{Hilb}_3\mathbb{P}^3$. To do this we will need a notion of "lifting" cycles in $\mathrm{Hilb}_3\mathbb{P}^2$, which we will define in Section 3.

3. The Chow ring of $\mathrm{Hilb}_3\mathbb{P}^3$

3.1. **Basic structure.** $\mathrm{Hilb}_3\mathbb{P}^3$ is an irreducible, projective, smooth scheme and its Chow ring is isomorphic to the cohomology ring [ELB83]. Any $\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3$ is contained in a plane in $\mathbb{P}^3$, and if Γ is not contained in a line that plane is unique. This motivates viewing $\mathrm{Hilb}_3\mathbb{P}^3$ as a blowup of the incidence variety $\widetilde{\mathrm{Hilb}}_3\mathbb{P}^3$.

$$\widetilde{\mathrm{Hilb}}_3\mathbb{P}^n = \{(\Gamma,\Pi) \in \mathrm{Hilb}_3\mathbb{P}^3 \times \mathrm{Gr}(2,n) : \Gamma \subseteq \Pi\}.$$

Note that $\widetilde{\mathrm{Hilb}}_3\mathbb{P}^n$ is a $\mathrm{Hilb}_3\mathbb{P}^2$-bundle over $\mathrm{Gr}(2,n)$. Rosselló-Llompart uses this presentation to give an analogue of the EL basis for $\mathrm{Hilb}_3\mathbb{P}^3$ in [RL90]. We give an abbreviated primer on this basis for the reader. Let $\mathrm{Al}_3\mathbb{P}^n$ denote the locus of collinear schemes in $\mathrm{Hilb}_3\mathbb{P}^n$.

Proposition 3.1. Let $q : \widetilde{\mathrm{Hilb}}_3\mathbb{P}^n \to \mathrm{Hilb}_3\mathbb{P}^n$ be the projection map onto the first coordinate. Then q is the blowup of $\mathrm{Hilb}_3\mathbb{P}^n$ along $\mathrm{Al}_3\mathbb{P}^n$.

Proof. See [ELB83]. □

Consider the following commutative diagram, using $\widetilde{\mathrm{Al}}_3\mathbb{P}^n$ to denote the proper transform of $\mathrm{Al}_3\mathbb{P}^n$:

$$\begin{tikzcd} \widetilde{\mathrm{Al}}_3\mathbb{P}^n \arrow[r, "j"] \arrow[d, "q'"] & \widetilde{\mathrm{Hilb}}_3\mathbb{P}^n \arrow[d, "q"] \\ \mathrm{Al}_3\mathbb{P}^n \arrow[r, "h"] & \mathrm{Hilb}_3\mathbb{P}^n \end{tikzcd}$$

Roberts [Rob88], Rosselló-Llompart and Xambó-Descamps [RLXD91] use this to generate the split exact sequences

$$0 \to (\mathrm{Ker}{q'}_*)_k \xrightarrow{j_*} A_k(\widetilde{\mathrm{Hilb}}_3\mathbb{P}^n) \xrightarrow{q_*} A_k(\mathrm{Hilb}_3\mathbb{P}^n) \to 0$$

for $k \geq 0$. Then one can use bases of $(\mathrm{Ker}{q'}_*)_k$, $A_k(\widetilde{\mathrm{Hilb}}_3\mathbb{P}^n)$ to write a basis of $A_k(\mathrm{Hilb}_3\mathbb{P}^n)$, see Section 3 in [RL90] for more details.

To define the basis of $\mathrm{Hilb}_3\mathbb{P}^3$ given in [RL90], we first need to define some loci. Let Π, Π' denote fixed planes, L denote a fixed line, and p denote a fixed point.

$$\begin{aligned} H &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \Gamma \cap \Pi \neq \emptyset\} \\ P &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \Gamma \text{ is coplanar with } p\} \\ P_3 &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \Gamma \subseteq \Pi\} \\ P_2 &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \Gamma \text{ coplanar with } L\} \\ \ell &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \Gamma \cap L \neq \emptyset\} \\ p &= \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \exists d \subseteq \Gamma \text{ of length 2 such that } d \subseteq \Pi\} \end{aligned}$$

In addition, we define β to the locus of schemes in $\mathrm{Hilb}_3\mathbb{P}^3$ whose general member is the union of a length two scheme contained in Π and a length one scheme contained in Π'. It will also be useful to define

$$F = \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \exists d \subseteq \Gamma \text{ of length 2 such that } d \text{ coplanar with } L\}$$

Remark. By abuse of terminology, H, ℓ, p, β, F are being used in two ways (for both a subvariety of $\mathrm{Hilb}_3\mathbb{P}^2$ and a subvariety of $\mathrm{Hilb}_3\mathbb{P}^3$). We do this for convenience, as it is clear that H, ℓ, p, β, F restrict to their $\mathrm{Hilb}_3\mathbb{P}^2$ analogues when intersected with a plane Π' in $\mathbb{P}^3$, and because in context it will be clear which usage we are referring to in any given situation.

Lemma 3.2. The Chow groups of $\mathrm{Hilb}_3\mathbb{P}^3$ have the following bases.

k	Basis for $A_k(\mathrm{Hilb}_3\mathbb{P}^3)$
9	$\mathrm{Hilb}_3\mathbb{P}^3$
8	P, H
7	$P_2, P^2, PH, H^2, \ell, p$
6	$P_3, P_2P, P_2H, PH^2, P^2H, P\ell, Pp, H^3, H\ell, \beta$
5	$P_3P, P_3H, P_2H^2, P_2HP, P_2\ell, P_2^2, P_2p,\ (P^2H^2 + PH^3),\ PH\ell, P\beta, H^2\ell, \ell^2, \ell p$
4	$P_3HP, P_3H^2, P_3\ell, P_3P_2, P_3p, P_2H\ell, P_2^2H, P_2\beta, P_2H^3, P_2PH^2, PH^2\ell, P\ell^2, H\ell p$
3	$P_3H\ell, P_3P_2H, P_3^2, P_3\beta, P_3H^3, P_3PH^2,\ (P_2^2H^2 + P_2H^2\ell),\ P_2\ell p, P_2\ell^2, PH\ell p$
2	$P_3H^2\ell, P_3P_2H^2, P_3\ell^2, P_3\ell p, P_3P_2\ell, P_2H\ell p$
1	$P_3H\ell p, P_3H\ell^2$
0	$P_3\ell^3$

Proof. See Lemma 4.5 in [RL90]. □

Remark. Rosselló-Llompart provides the integer values of all weight 9 monomials in $P, H, P_3, \beta, P_2\ell, p$ in the tables of Proposition 4.7 of [RL90]. Hence if you can write a cycle in $\mathrm{Hilb}_3\mathbb{P}^3$ in terms of $P, H, P_3, \beta, P_2\ell, p$, then you can use this table and the intersection tables below to write the cycle in terms of the bases. Hence in this paper we will aim to write all pertinent cycles in terms of these seven generators, but may not always give their expression in the bases of $A_k(\mathrm{Hilb}_3\mathbb{P}^3)$.

For reference, we include some of the intersection tables for this basis in Tables 4, 5, 6.

	P_2	P^2	PH	H^2	ℓ	p
$P_3H^2\ell$	1	0	3	3	1	1
$P_3P_2H^2$	-1	0	-3	6	1	0
$P_3\ell^2$	0	-1	1	1	1	0
$P_3\ell p$	0	0	0	1	0	0
$P_3P_2\ell$	0	1	-1	1	0	0
$P_2H\ell p$	0	-3	4	9	2	2

TABLE 4. The intersection matrix for $A_2(\mathrm{Hilb}_3\mathbb{P}^3) \times A_7(\mathrm{Hilb}_3\mathbb{P}^3)$

	P_3	P_2P	P_2H	PH^2	P^2H	$P\ell$	Pp	H^3	$H\ell$	β
$P_3H\ell$	0	-1	1	3	0	1	0	3	1	0
P_3P_2H	0	1	-1	-3	0	-1	0	6	1	0
P_3^2	0	0	0	-1	1	0	0	1	0	0
$P_3\beta$	0	0	0	1	-1	0	0	3	0	1
P_3H^3	1	-3	6	15	3	3	3	15	3	3
P_3PH^2	-1	0	-3	3	-12	0	-3	15	3	1
$(P_2^2H^2 + P_2H^2\ell)$	0	-2	2	21	-6	4	1	65	16	4
$P_2\ell p$	0	0	0	4	-3	0	0	9	2	1
$P_2\ell^2$	0	-2	2	7	1	3	0	7	3	0
$PH\ell p$	0	-3	4	18	4	5	2	25	7	2

TABLE 5. The intersection matrix for $A_3(\mathrm{Hilb}_3\mathbb{P}^3) \times A_6(\mathrm{Hilb}_3\mathbb{P}^3)$

	P_3P	P_3H	P_2H^2	P_2HP	$P_2\ell$	P_2^2	P_2p	$(P^2H^2 + PH^3)$	$PH\ell$	$P\beta$	$H^2\ell$	ℓ^2	ℓp
P_3HP	1	-1	-3	0	-1	1	0	-9	0	-1	3	1	0
P_3H^2	-1	1	6	-3	1	-1	0	18	3	1	3	1	1
$P_3\ell$	0	0	1	-1	0	0	0	3	1	0	1	1	0
P_3P_2	0	0	-1	1	0	0	0	-3	-1	0	1	0	0
P_3p	0	0	0	0	0	0	0	0	0	0	1	0	0
$P_2H\ell$	-1	1	9	-3	2	-2	0	33	7	1	7	3	2
P_2^2H	1	-1	-7	1	-2	2	0	-18	-3	-1	9	2	0
$P_2\beta$	0	0	1	-1	0	0	0	7	1	1	3	0	1
P_2H^3	-3	6	40	-4	9	-7	3	150	25	9	25	7	9
P_2PH^2	0	-3	-4	-14	-3	1	-3	6	8	-2	25	7	4
$PH^2\ell$	0	3	25	8	7	-3	4	116	20	6	20	6	7
$P\ell^2$	-1	1	7	1	3	-2	0	32	6	1	6	0	3
$H\ell p$	0	1	9	4	2	0	2	43	7	2	7	3	2

TABLE 6. The intersection matrix for $A_4(\mathrm{Hilb}_3\mathbb{P}^3) \times A_5(\mathrm{Hilb}_3\mathbb{P}^3)$

3.2. **Lifts and pushforwards of the MS basis.** Occassionally, it is useful to identify certain subspaces of the Chow groups of $\mathrm{Hilb}_3\mathbb{P}^3$ that "act" like a copy of the MS basis in $\mathrm{Hilb}_3\mathbb{P}^2$. That is, we want to identify the intersection matrix of $A_k(\mathrm{Hilb}_3\mathbb{P}^2), A^k(\mathrm{Hilb}_3\mathbb{P}^2)$ as a submatrix of $A_k(\mathrm{Hilb}_3\mathbb{P}^3), A^k(\mathrm{Hilb}_3\mathbb{P}^3)$, and we would also like the elements of $A(\mathrm{Hilb}_3\mathbb{P}^3)$ yielding this submatrix to be defined based on MS basis elements. In this way, we can better utilize and compare against $\mathrm{Hilb}_3\mathbb{P}^2$ computations. Furthermore, this leads to cleaner expressions for the final cones $\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^3), \mathrm{Nef}^2(\mathrm{Hilb}_3\mathbb{P}^3)$ than in the Rosselló-Llompart basis.

Embed $\mathbb{P}^2 \overset{i}{\hookrightarrow} \mathbb{P}^3$ by identifying it with a distinguished plane Π in $\mathbb{P}^3$. Then certainly we have

$$\begin{aligned} i_* : A_\bullet(\mathbb{P}^2) &\to A_\bullet(\mathbb{P}^3) \\ [X] &\mapsto [i(X)] \end{aligned}$$

For a class in the MS-basis, we will sometimes omit the i_* notation if it is clear enough whether we are working in $\mathbb{P}^2$ or $\mathbb{P}^3$. For example, if it is clear that we are working in $\mathbb{P}^3$, we will use E (instead of $i_*(E)$) to denote the locus in $\mathrm{Hilb}_3\mathbb{P}^3$ of schemes contained in Π and containing a fixed point P in Π.

For a map $A^\bullet(\mathrm{Hilb}_3\mathbb{P}^2) \to A^\bullet(\mathrm{Hilb}_3\mathbb{P}^3)$, we define the notion of a lift of an MS basis element. For each basis element b of $A(\mathrm{Hilb}_3\mathbb{P}^2)$, we want an element $\widetilde{b}$ such that $\widetilde{b} \cdot P_3 = i^*(\widetilde{b}) = b$. We accomplish this changing all the conditions to their higher-dimensional analogues: the condition of being incident to a line in $\mathbb{P}^2$ becomes the condition of being incident to a plane in $\mathbb{P}^3$; the condition of containing a fixed point becomes being incident to a fixed line in $\mathbb{P}^3$; the condition of being collinear with a point in $\mathbb{P}^2$ becomes being coplanar with a line in $\mathbb{P}^3$. See the figures below for illustrations of a generic member of each lift.

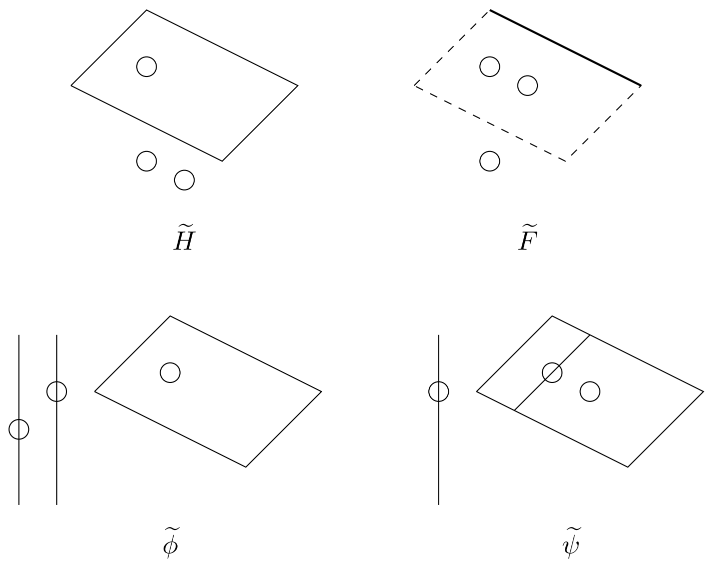

FIGURE 4. The lifts of the MS basis elements of codimension 1 and 5.

For ease of some later computators, we define an extended analogue of the MS basis in (co)dimension 2 in $\mathrm{Hilb}_3\mathbb{P}^3$. Consider the basis for $A^2(\mathrm{Hilb}_3\mathbb{P}^3)$ given by $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, M$, where M is the locus of schemes containing a length two subscheme collinear with a fixed point Q. Additionally, consider the basis for $A_2(\mathrm{Hilb}_3\mathbb{P}^3)$ given by $\alpha, \beta, \gamma, \delta, \epsilon, \mu$, where μ is the locus parametrizing a given by the union of a fixed point q and a length two scheme supported at a second fixed point q'. Then the intersection matrix is as follows.

	α	β	γ	δ	ϵ	μ
$\widetilde{A}$	0	0	1	0	0	0
$\widetilde{B}$	0	1	2	1	0	0
$\widetilde{C}$	1	2	2	1	0	0
$\widetilde{D}$	0	1	1	0	0	0
$\widetilde{E}$	0	0	0	0	1	0
M	0	0	0	0	0	1

The upper-left 5×5 comes from the push-pull formula reducing the intersections to a computation in $\mathrm{Hilb}_3\mathbb{P}^2$. For the intersections with M, μ: the zeros are clear. To calculate that $Z \cdot \zeta = 1$, we check transversality in coordinates. Note that this is equivalent to checking the intersection number between M', the locus of 2 points collinear with a fixed point q, and μ', the locus of nonreduced points supported at a point q', in $\mathrm{Hilb}_2\mathbb{P}^3$.

Pick a copy of $\mathbb{A}^3$ in $\mathbb{P}^3$ and give it coordinates x, y, z. Then we have coordinates $\beta, \gamma, b_1, b_2, c_1, c_2$ on an open subset of $\mathrm{Hilb}_2(\mathbb{A}^3)$ by writing the ideal of a general length 2 subscheme as

$$(x^2 + \beta x + \gamma) \cap (y - b_1 x - b_2) \cap (z - c_1 x - c_2)$$

Set $p = (0, 0, 0)$ and $q = (1, 0, 0)$. Then the equations cutting out Z' are

$$b_2 = 0, c_2 = 0,$$

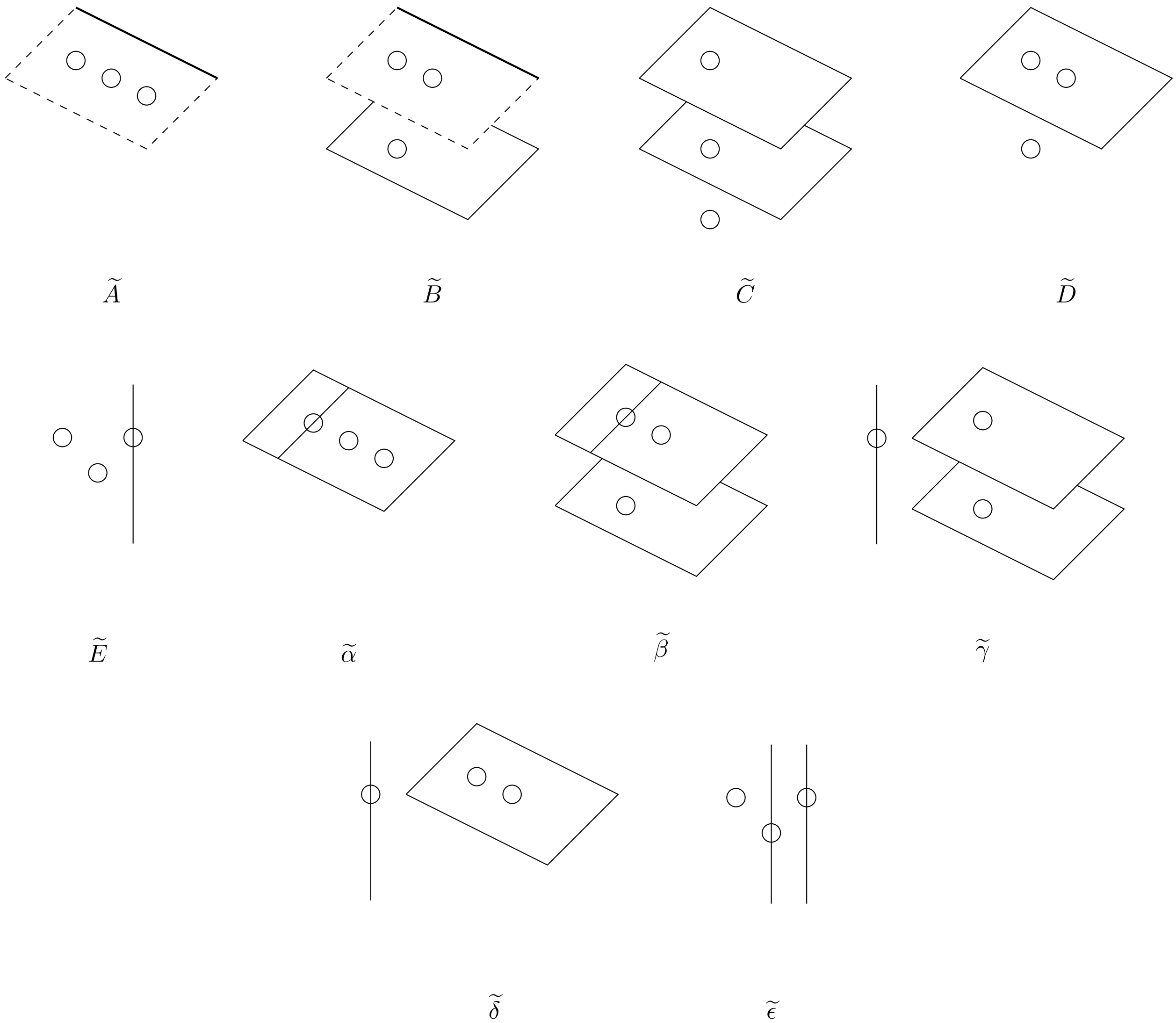

FIGURE 5. The lifts of the MS basis elements of codimension 2 and 4.

and the equations defining ζ' are

$$b_1 = -b_2,\ c_1 = -c_2,\ \beta = -2, \gamma = 1.$$

Since $\dim_k k[\beta, \gamma, b_1, b_2, c_1, c_2]/(b_2, c_2, b_1 + b_2, c_1 + c_2, \beta + 2, \gamma - 1) = 1$, we see that $M \cdot \mu = M' \cdot \mu' = 1$. Note that we are taking the reduced structure here.

Proposition 3.3. We have the following change of bases between the EL basis and the extended MS basis in $A_2(\mathrm{Hilb}_3\mathbb{P}^3)$.

$$\begin{aligned}
P_3H^2\ell &= \gamma + \epsilon \\
P_3H^2P_2 &= 3\alpha + \beta - \gamma + 2\delta + \epsilon \\
P_3\ell^2 &= \epsilon \\
P_3\ell p &= \delta \\
P_3\ell P_2 &= \alpha \\
P_2H\ell p &= 2\beta + 3\delta + 2\epsilon + \mu
\end{aligned}$$

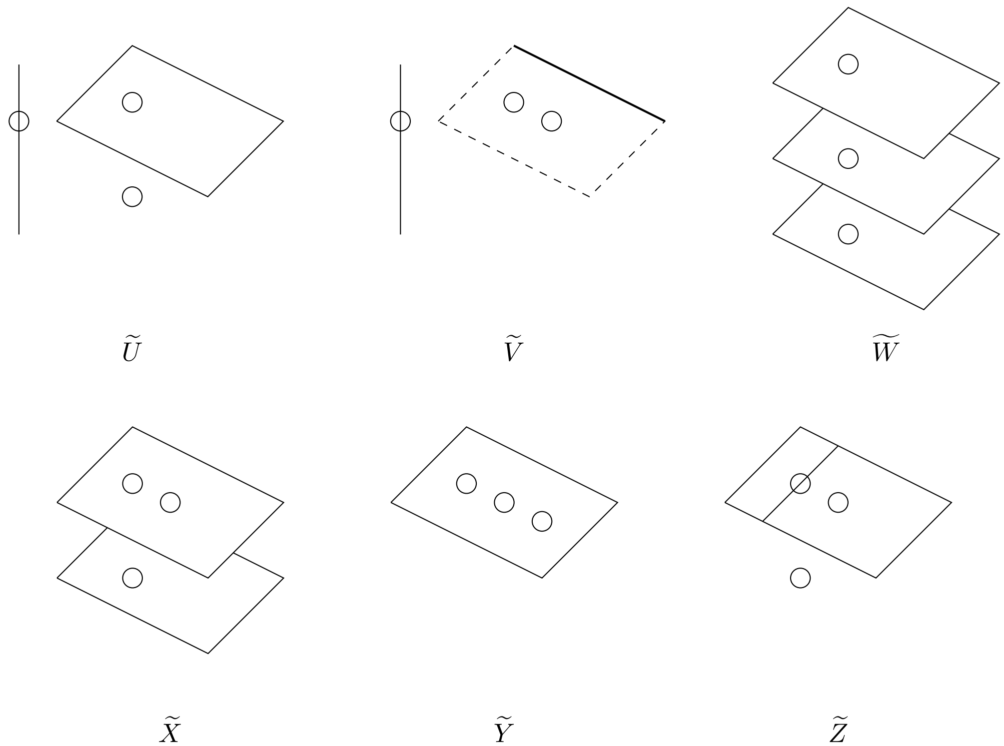

FIGURE 6. The lifts of the MS basis elements of (co)dimension 3

Correspondingly, we have the following conversion between the EL and extended MS bases in $A_7(\mathrm{Hilb}_3\mathbb{P}^3)$.

$$\begin{aligned}
P_2 &= \widetilde{A} \\
P^2 &= 3\widetilde{A} - \widetilde{B} + \widetilde{C} - 2\widetilde{D} - \widetilde{E} + M \\
PH &= \widetilde{B} - \widetilde{C} + 2\widetilde{D} + \widetilde{E} \\
H^2 &= \widetilde{C} + \widetilde{E} \\
\ell &= \widetilde{E} \\
p &= \widetilde{D}
\end{aligned}$$

Proof. For $A_2(\mathrm{Hilb}_3\mathbb{P}^3)$, the first five equations follow from pushing forward the formulas in Proposition 2.1. For the last equation, note that:

	P_2	P^2	PH	H^2	ℓ	p
μ	0	1	0	0	0	0

The zeroes in this table can be seen from the cycles being disjoint, and $\mu \cdot P^2 = 1$ follows from $P^2 = P_2 + \mathrm{Al}_3\mathbb{P}^3$. See Lemma 4.1 in [RL90]. Then, using Table 4, that tells us

$$\mu = -2P_3H^2\ell - 2P_3H^2P_2 + 2P_3\ell^2 + P_3\ell p + 6P_3\ell P_2 + P_2H\ell p,$$

and therefore

$$P_2H\ell p = 2\beta + 3\delta + 2\epsilon + \mu.$$

The $A_7(\mathrm{Hilb}_3\mathbb{P}^3)$ conversions follow. □

We now compute the class of a few useful cycles that feature in the computations of Section 4.

Lemma 3.4. Let Y_1 be the locus of $\gamma \in \mathrm{Hilb}_3\mathbb{P}^3$ containing a fixed point p. Let Y_2 denote the locus of $\Gamma \in \mathrm{Hilb}_3$ with a length two subscheme contained in a fixed line L. Furthermore, let Y_3 be the locus of schemes that are incident ot a line L and coplanar with L. Then:

$$Y_1 = 3P_3 - P_2H + P\ell$$
$$Y_2 = -P_3P - P_3H + P_2p$$
$$Y_3 = P\ell - Y_1 = -3P_3 + P_2H$$

Proof. We compute the products of Y_1, Y_2 with the classes of the EL basis of complimentary codimension.

	$P_3H\ell$	P_3P_2H	P_3^2	$P_3\beta$	P_3H^3	P_3PH^2	$(P_2^2H^2 + P_2H^2\ell)$	$P_2\ell p$	$P_2\ell^2$	$PH\ell p$
Y_1	0	0	0	0	0	0	2	0	1	1

	P_3HP	P_3H^2	$P_3\ell$	P_3P_2	P_3p	$P_2H\ell$	P_2^2H	$P_2\beta$	P_2H^3	P_2PH^2	$PH^2\ell$	$P\ell^2$	$H\ell p$
Y_2	0	0	0	0	0	0	0	0	0	0	1	0	1

Then using Tables 5, 6, this yields $Y_1 = 3P_3 - P_2H + P\ell$ and $Y_2 = -P_3P - P_3H + P_2p$. For Y_3, consider $P \cdot \ell$, the locus of schemes incident to a line and coplanar with a fixed point. By specializing the fixed point onto the line, we see that

$$P\ell = c_1Y_3 + c_2Y_1$$

Intersecting both sides of the equation with $P_3\ell H$ we see that $c_1 = 1$, and intersecting with Y_1^2, we see that $c_2 = 1$. Hence we are done. □

4. Analyzing the seven orbits

There are seven orbits under $\mathrm{PGL}_n(\mathbb{C})$ action on $\mathbb{P}^3$; We describe them now. The subscripts indicate the codimension of the orbit.

1) $\mathcal{O}_0$, the locus consisting of three distinct, non collinear points.
2) $\mathcal{O}_1$, the locus of non-collinear schemes whose support consists of two points.
3) $\mathcal{O}_{2,\mathrm{col}}$, the locus consisting of three distinct collinear points.
4) $\mathcal{O}_{2,\mathrm{nonred}}$, the locus of curvilinear schemes supported at a single point that are not contained in a line.
5) $\mathcal{O}_3$, the locus of collinear schemes supported at two distinct points.
6) $\mathcal{O}_{4,\mathrm{col}}$, the locus of schemes supported at a single point and contained in a line.
7) $\mathcal{O}_{4,\max}$, the locus of schemes isomorphic to a fat point in a plane, i.e. isomorphic to the variety defined by $\mathcal{I}_p^2$ where $\mathcal{I}_p$ is the ideal of a point viewed inside of a plane.

In order to even begin to utilize Lemma 2.3, we need to analyze the classes of the orbit closures in $A^\bullet(\mathrm{Hilb}_3(\mathbb{P}^3))$, as well as the Chow groups of each orbit closure. Many of our computations take after those in [RS21].

4.1. **Planar fat points.** Note that $\overline{\mathcal{O}_{4,\max}} = \mathcal{O}_{4,\max}$. To compute its class, note that $\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot F^2$ is supported on two components: $\mathcal{O}_{4,\max}$ and $\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot D'$, where D' denotes the locus of schemes $\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3$ containing a length two subscheme Γ' such that the line containing Γ' is incident to two fixed lines L, L'. Thus,

$$\ell\mathcal{O}_4 = \overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot (F^2 - kD').$$

Before determining ℓ, k, we first work on rewriting D'. We have a linear map defined by

$$A^2(\mathrm{Gr(1,3)}) \to A^2(\mathrm{Hilb}_3\mathbb{P}^3)$$
$$\Sigma \mapsto Z_\Sigma,$$

where, if Σ is a subvariety of $Gr(1,3)$, Z_Σ is defined as follows

$$Z_\Sigma = \{\Gamma \in \mathrm{Hilb}_3\mathbb{P}^3 : \exists \Gamma' \subseteq \Gamma \text{ of length two such that } \Gamma' \subseteq z \text{ for some line } z \in \Sigma\}.$$

We get the full map by extending linearly. Observe that D' is the image of Σ_1^2, meaning D' is the sum of $Z_{\Sigma_{1,1}} + Z_{\Sigma_2}$. Thus, $D' = \widetilde{D} + M$.

$$\ell\mathcal{O}_4 = \overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot (F^2 - k(\widetilde{D} + M))$$

Now, observe that $\mathcal{O}_{4,\max} \cdot P_3 \cdot P \cdot H$ is equal to $\mathcal{O}_{4,\max,\mathbb{P}^2} \cdot \overline{\mathcal{O}_{1,\mathrm{col},\mathbb{P}^2}} H = 0$. On the other hand,

$$\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot F^2 \cdot P_3PH = \overline{\mathcal{O}_{2,\mathrm{nonred},\mathbb{P}^2}} \cdot \overline{\mathcal{O}_{1,\mathrm{col},\mathbb{P}^2}} \cdot HF^2 = 27$$
$$\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot (\widetilde{D} + M) \cdot P_3PH = \overline{\mathcal{O}_{2,\mathrm{nonred},\mathbb{P}^2}} \cdot \overline{\mathcal{O}_{1,\mathrm{col},\mathbb{P}^2}} \cdot D \cdot H = 9,$$

see [RS21]. Here, $\overline{\mathcal{O}_{2,\mathrm{nonred}}}$ denotes the locus of totally nonreduced schemes in $\mathrm{Hilb}_3\mathbb{P}^2$, and $\overline{\mathcal{O}_{1,\mathrm{col},\mathbb{P}^2}}$ denotes the locus of collinear schemes in $\mathrm{Hilb}_3\mathbb{P}^2$, and the intersection products on the right side are taken in $A_\bullet(\mathrm{Hilb}_3\mathbb{P}^2)$. So, $k = 3$. Furthermore, since

$$\overline{\mathcal{O}_{4,\max}} \cdot P_3H^2 = \overline{\mathcal{O}_{4,\max,\mathbb{P}^2}} \cdot H^2 = 9,$$
$$\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot (F^2 - 3(\widetilde{D} + M)) \cdot P_3H^2 = \overline{\mathcal{O}_{2,\mathrm{nonred},\mathbb{P}^2}} \cdot (F^2 - 3D)H^2 = 27,$$

(see Section 5.1 in [RS21] and [ELB88]) we have that $\ell = 3$. Later on we compute that $\overline{\mathcal{O}_{2,\mathrm{nonred}}} = 3(P_2 - PH + \ell + p)$. This yields the following formula.

$$\begin{aligned}\overline{\mathcal{O}_{4,\max}} &= (P_2 - PH + \ell + p)((P + H)^2 - 3(\widetilde{D} + M)) \\ &= (P_2 - PH + \ell + p)((P + H)^2 - 3(p + (P^2 + PH - 3P_2))) && \text{(Prop 3.3)} \\ &= (P_2 - PH + \ell + p)(-2P^2 - PH + H^2 - 3p + 9P_2).\end{aligned}$$

To compute the nef and effective cones, we first observe that $\mathcal{O}_{4,\max}$ is isomorphic to the flag variety of pairs (p, Λ) of a point and a plane in $\mathbb{P}^3$, and as such its Chow ring is generated by two elements of degree 1 each. Hence we have:

k	$\dim A_k(\mathcal{O}_{4,\max})$
5	1
4	2
3	3
2	3
1	2
0	1

We want to determine generators for the nef and eff cones. Let P be a fixed point and L be a fixed line on a fixed plane Π. We begin with the curve and fourfold classes. Let $X_{1,1}$ be the locus of planar fat points where the support is a fixed point and the plane Λ must contain P. Let $X_{1,2}$ be the locus of planar fat points supported on L and contained in Π. Then:

$$\begin{aligned}X_{1,1} &= \mathcal{O}_{4,\max} \cdot Y_1P && \text{(see Lemma 3.4)} \\ X_{1,2} &= \mathcal{O}_{4,\max} \cdot \widetilde{\alpha} = \mathcal{O}_{4,\max} \cdot P_3H.\end{aligned}$$

Let $X_{4,1}$ be the locus of planar fat points Γ such that Λ, the plane containing Γ, also contains P. Let $X_{4,2}$ be the locus of planar fat points whose support (that is, the underlying point p) is in Π.

$$\begin{aligned}X_{4,1} &= \mathcal{O}_{4,\max} \cdot P \\ X_{4,2} &= \mathcal{O}_{4,\max} \cdot H\end{aligned}$$

For surfaces, let $X_{2,1}$ be the locus of planar fat points contained in the plane Π, and let $X_{2,2}$ be the locus of planar fat points Γ whose support is contained in L and satisfying the condition that the plane containing Γ contains L. Lastly, let $X_{2,3}$ be the locus of planar fat points supported at P.

$$\begin{aligned}X_{2,1} &= \mathcal{O}_{4,\max} \cdot P_3 = 3(\alpha + \delta + \epsilon) = 3(P_3\ell^2 + P_3\ell p + P_3\ell P_2) \\ X_{2,2} &= \mathcal{O}_{4,\max} \cdot Y_3 \\ &= \mathcal{O}_{4,\max} \cdot (P\ell - Y_1) && \text{(Lemma 3.4)} \\ &= 3(-5P_3H^2\ell - 5P_3P_2H^2 + 4P_3\ell^2 + P_3\ell p + 13P_3P_2\ell + 3P_2H\ell p) \\ &= 3(-2\alpha + \beta + 3\mu) \\ X_{2,3} &= \mathcal{O}_{4,\max} \cdot Y_1 \\ &= 3(4P_3H^2\ell + 3P_3P_2H^2 - 3P_3\ell^2 - P_3\ell p - 8P_3P_2\ell - 2P_2H\ell p) \\ &= 3(\alpha - \beta + \gamma - \delta - 2\mu)\end{aligned}$$

For threefolds, let $X_{3,1}$ be the locus of planar fat points supported on L, let $X_{3,2}$ be the locus of planar fat points supported on Π and coplanar with P, and let $X_{3,3}$ be the locus of planar fat points coplanar with L.

$$\begin{aligned} X_{3,1} &= \mathcal{O}_{4,\max} \cdot \ell \\ X_{3,2} &= \mathcal{O}_{4,\max} \cdot HP \\ X_{3,3} &= \mathcal{O}_{4,\max} \cdot P_2 \end{aligned}$$

Proposition 4.1. We have the following expressions for the nef and effective cones of $\mathcal{O}_{4,\max}$.

$$\begin{aligned} \mathrm{Nef}^1(\mathcal{O}_{4,\max}) &= \mathrm{Eff}_4(\mathcal{O}_{4,\max}) = \langle X_{4,1}, X_{4,2} \rangle \\ \mathrm{Nef}^2(\mathcal{O}_{4,\max}) &= \mathrm{Eff}_3(\mathcal{O}_{4,\max}) = \langle X_{3,1}, X_{3,2}, X_{3,3} \rangle \\ \mathrm{Nef}^3(\mathcal{O}_{4,\max}) &= \mathrm{Eff}_2(\mathcal{O}_{4,\max}) = \langle X_{2,1}, X_{2,2}, X_{2,3} \rangle \\ \mathrm{Nef}^4(\mathcal{O}_{4,\max}) &= \mathrm{Eff}_1(\mathcal{O}_{4,\max}) = \langle X_{1,1}, X_{1,2} \rangle. \end{aligned}$$

Proof. This follows the emptiness of the apppropriate intersections and from the action of $\mathrm{PGL}_4(\mathbb{C})$ being transitive on $\mathcal{O}_{4,\max}$. □

4.2. **Collinear schemes supported at a single point.** Note that $\overline{\mathcal{O}_{4,\mathrm{col}}} = \mathcal{O}_{4,\mathrm{col}}$. It has the class

$$\begin{aligned} \overline{\mathcal{O}_{4,\mathrm{col}}} &= \overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \overline{O_{2,\mathrm{nonred}}} \\ &= 3(P_2 - PH + \ell + p)(P^2 - P_2) \end{aligned}$$

Further, it is isomorphic to the flag variety of pairs (p, L) of a point and a line in $\mathbb{P}^3$. Hence its Chow groups have the following dimensions.

k	$\dim A_k(\mathcal{O}_{4,\mathrm{col}})$
5	1
4	2
3	3
2	3
1	2
0	1

Let $P \subseteq L \subseteq \Pi$ be a fixed point, line, and plane respectively. For curve classes, let $X_{1,3}$, be the locus of collinear, totally nonreduced schemes contained in L, and let $X_{1,4}$ be the locus of collinear, totally nonreduced schemes Γ supported at P and contained in Π.

$$\begin{aligned} X_{1,3} &= \frac{1}{9}\mathcal{O}_{4,\mathrm{col}} \cdot p^2 = 3(-\phi + 2\psi) = 3(-H\ell^2 + 2H\ell p) \\ X_{1,4} &= \frac{1}{3}\mathcal{O}_{4,\mathrm{col}} \cdot p\ell = 9(\phi - \psi) = 9(H\ell^2 - H\ell p) \end{aligned}$$

For fourfolds, let $X_{4,3}$ be the locus of collinear, totally nonreduced schemes supported on Π, and $X_{4,4}$ be the locus of collinear, totally nonreduced schemes Γ that are coplanar with L.

$$\begin{aligned} X_{4,3} &= \mathcal{O}_{4,\mathrm{col}} \cdot H \\ X_{4,4} &= \frac{1}{3}\mathcal{O}_{4,\mathrm{col}} \cdot F \end{aligned}$$

For surfaces, let $X_{2,4}$ be the locus of collinear, totally nonreduced schemes supported at a fixed point. Let $X_{2,5}$ be the locus of collinear, totally nonreduced scehemes supported at L and contained in Π. Let $X_{2,6}$ be the locus of collinear, totally nonreduced schemes Γ that are collinear with P and contained in Π.

$$\begin{aligned}
X_{2,4} &= \mathcal{O}_{4,\text{col}} \cdot Y_1 \\
&= 3(-6P_3H^2\ell - 8P_3P_2H^2 + 8P_3\ell^2 + 5P_3\ell p + 26P_3P_2\ell + 3P_2H\ell p) \\
&= 3(2\alpha - 2\beta + 2\gamma - 2\delta + 3\mu) \\
X_{2,5} &= \frac{1}{3}\mathcal{O}_{4,\text{col}} \cdot Hp = 9(-2\alpha + \beta) \\
&= 9(P_3H^2\ell + P_3P_2H^2 - 2P_3\ell^2 - 2P_3\ell p - 5P_3P_2\ell) \\
X_{2,6} &= \frac{1}{9}\mathcal{O}_{4,\text{col}} \cdot Fp = 3(\beta - \gamma + 2\delta + \epsilon) \\
&= P_3H^2P_2 - 3P_3\ell P_2
\end{aligned}$$

Lastly, for threefolds, let $X_{3,4}$ be the locus of collinear, totally nonreduced schemes that are collinear with P. Then let $X_{3,5}$ be the locus of collinear, totally nonreduced schemes contained in a plane, and let $X_{3,6}$ be the locus of collinear, totally nonreduced schemes such that the support p is contained in L.

$$\begin{aligned}
X_{3,4} &= \frac{1}{3}\mathcal{O}_{4,\text{col}} \cdot M \\
X_{3,5} &= \mathcal{O}_{4,\text{col}} \cdot p \\
X_{3,6} &= \mathcal{O}_{4,\text{col}} \cdot \ell
\end{aligned}$$

Proposition 4.2. We have the following expressions for the nef and effective cones of $\mathcal{O}_{4,\text{col}}$.

$$\begin{aligned}
\text{Nef}^1(\mathcal{O}_{4,\text{col}}) &= \text{Eff}_4(\mathcal{O}_{4,\text{col}}) = \langle X_{4,3}, X_{4,4}\rangle \\
\text{Nef}^2(\mathcal{O}_{4,\text{col}}) &= \text{Eff}_3(\mathcal{O}_{4,\text{col}}) = \langle X_{3,4}, X_{3,5}, X_{3,6}\rangle \\
\text{Nef}^3(\mathcal{O}_{4,\text{col}}) &= \text{Eff}_2(\mathcal{O}_{4,\text{col}}) = \langle X_{2,4}, X_{2,5}, X_{2,6}\rangle \\
\text{Nef}^4(\mathcal{O}_{4,\text{col}}) &= \text{Eff}_1(\mathcal{O}_{4,\text{col}}) = \langle X_{1,3}, X_{1,4}\rangle.
\end{aligned}$$

Proof. This follows the emptiness of the apppropriate intersections and from the action of $\text{PGL}_4(\mathbb{C})$ being transitive on $\mathcal{O}_{4,\text{col}}$. ☐

4.3. **Collinear and nonreduced schemes.** Note that

$$\begin{aligned}
\overline{\mathcal{O}_3} = \overline{\mathcal{O}_{2,\text{col}}} \cdot \overline{\mathcal{O}_1} &= (P^2 - P_2) \cdot 2(H - P) \\
&= 2(P^2H - P^3 - P_2H + P_2P) \\
&= 6P_3 - 6P_2P - 4P_2H + 4P^2H
\end{aligned}$$

For the Chow groups, note that $\overline{\mathcal{O}_3}$ is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $\text{Gr}(1,3)$ and as such its Chow groups have the following dimensions:

k	$\dim A_k(\overline{\mathcal{O}_3})$
6	1
5	3
4	5
3	6
2	5
1	3
0	1

Let $P \subseteq L \subseteq \Pi$ be a point, line, and plane in $\mathbb{P}^3$ respectively. Let Π' denote an additional plane in $\mathbb{P}^3$, and let L' denote an additional line in $\mathbb{P}^3$.

Let $X_{5,1}$ denote the locus whose generic member is the union of a nonreduced point supported on Π and a reduced point, and let $X_{5,2}$ be the locus whose generic member is the union of a reduced point supported on Π and a nonreduced point. Furthermore, let $X_{5,3}$ be the locus of collinear, nonreduced schemes that are coplanar with L. Also, let $X_{5,4}$ be the locus of totally nonreduced collinear schemes. Then:

$$\begin{aligned}
X_{5,1} &= \overline{\mathcal{O}_{2,\text{col}}} \cdot Y_2 \\
X_{5,2} &= \overline{\mathcal{O}_{2,\text{col}}} \cdot Y_3 \\
X_{5,3} &= \frac{1}{3}\overline{\mathcal{O}_3} \cdot F \\
X_{5,4} &= \overline{\mathcal{O}_{2,\text{nonred}}} \cdot \overline{\mathcal{O}_{2,\text{col}}}
\end{aligned}$$

And now let $X_{1,5}$ be the locus of collinear nonreduced schemes contained in L with a length two subscheme supported at P, and let $X_{1,6}$ be the locus whose generic member is the union of P and a nonreduced scheme of length two contained in L. Let $X_{1,7}$ be the locus of collinear, totally nonreduced schemes contained in Π and supported at P.

Also, let $X_{1,8}$ be the locus of nonreduced collinear schemes containing P and incident to L', and $X_{1,9}$ be the locus of nonreduced collinear schemes with a length two subscheme supported at P and are incident to L'. Using 5.4 in [RS21], we see:

$$\begin{aligned}
X_{1,5} &= 2(-P_3H\ell^2 + 2P_3H\ell p) \\
X_{1,6} &= 4(-P_3H\ell^2 + 2P_3H\ell p) \\
X_{1,7} &= X_{1,4} \\
X_{1,8} &= 2(P_3H\ell^2 + P_3H\ell p) \\
X_{1,9} &= 2(2P_3H\ell^2 - P_3H\ell p)
\end{aligned}$$

For fourfolds, let $X_{4,5}$ be the locus of collinear nonreduced schemes with a length two scheme supported on L, and let $X_{4,6}$ be the locus of collinear nonreduced schemes whose generic member has a reduced point on L. Let $X_{4,7}$ be the locus of schemes whose general member Γ is the union of a reduced point on Π and a nonreduced point whose support is in Π'. Let $X_{4,8}$ denote the locus of collinear nonreduced schemes contained in Π. Lastly, let $X_{4,9}$ denote the locus of collinear nonreduced schemes that are collinear with P.

$$\begin{aligned}
X_{4,7} &= \overline{\mathcal{O}_3} \cdot \widetilde{C} = \mathcal{O}_3 \cdot (H^2 - \ell) \\
X_{4,8} &= \frac{1}{3}\overline{\mathcal{O}_3} \cdot p \\
X_{4,9} &= \frac{1}{3}\overline{\mathcal{O}_3} \cdot M
\end{aligned}$$

Then for surfaces, let $X_{2,7}$ be the locus of collinear nonreduced schemes contained in Π and containing P, and let $X_{2,8}$ be the locus of collinear nonreduced schemes contained in Π with containing a length two subscheme supported at P. Then let $X_{2,9}$ denote the locus of collinear nonreduced schemes contained in L. We also consider $X_{2,5}$, the locus of totally nonreduced schemes contained in Π and supported at L, and $X_{2,4}$, the locus of totally nonreduced schemes supported at a fixed point. Note that

$$\begin{aligned}
X_{2,7} &= 2(-\alpha + \beta - \gamma + 2\delta + \epsilon) && \text{(See } S_7 \text{ in [RS21])} \\
&= 2(P_3H^2P_2 - 4P_3\ell P_2) \\
X_{2,8} &= 2(-2\alpha + \beta - \gamma + 2\delta + \epsilon) && \text{(See } S_9 \text{ in [RS21])} \\
&= 2(P_3H^2P_2 - 5P_3\ell P_2) \\
X_{2,9} &= \frac{1}{3}\overline{\mathcal{O}_3} \cdot p^2 = 12\alpha = 12P_3P_2\ell
\end{aligned}$$

For threefolds: let $X_{3,7}$ be the locus of collinear nonreduced schemes contained in Π and containing a point on L, and let $X_{3,8}$ be the locus of collinear nonreduced schemes contained in Π and with a nonreduced length two subscheme supported on L. Let $X_{3,9}$ be the locus of collinear nonreduced schemes whose generic member is the union of a reduced point on L and a length two subscheme supported on Π', and let $X_{3,10}$ be the locus of collinear nonreduced schemes whose generic member is the union of a length two subscheme supported on L and a reduced point on Π'. Let $X_{3,11}$ be the locus of collinear nonreduced schemes with a length two subscheme supported at P, and let $X_{3,12}$ be the locus of collinear nonreduced schemes containing P. Finally, let $X_{3,13}$ denote the locus of collinear nonreduced schemes contained in Π and collinear with P. Also, recall that $X_{3,5}$ is the locus of totally nonreduced collinear schemes contained in Π and $X_{3,6}$ is the locus of totally nonreduced collinear schemes supported on L.

Proposition 4.3. We have the following expressions for the nef and effective cones of $\overline{\mathcal{O}_3}$.

$$\mathrm{Nef}^1(\overline{\mathcal{O}_3}) = \langle X_{5,1}, X_{5,2}, X_{5,3}\rangle, \quad \mathrm{Eff}_1(\overline{\mathcal{O}_3}) = \langle X_{1,5}, X_{1,6}, X_{1,7}\rangle$$
$$\mathrm{Nef}^2(\overline{\mathcal{O}_3}) = \langle X_{4,5}, X_{4,6}, X_{4,7}, X_{4,8}, X_{4,9}\rangle, \quad \mathrm{Eff}_2(\overline{\mathcal{O}_3}) = \langle X_{2,7}, X_{2,8}, X_{2,9}, X_{2,5}, X_{2,4}\rangle$$
$$\mathrm{Nef}^3(\overline{\mathcal{O}_3}) = \langle X_{3,7}, X_{3,8}, X_{3,9}, X_{3,10}, X_{3,11}, X_{3,12}, X_{3,13}\rangle, \quad \mathrm{Eff}_3(\overline{\mathcal{O}_3}) = \langle X_{3,7}, X_{3,8}, X_{3,13}, X_{3,5}, X_{3,11}, X_{3,12}, X_{3,6}\rangle$$
$$\mathrm{Nef}^5(\overline{\mathcal{O}_3}) = \langle X_{1,5}, X_{1,6}, X_{1,8}, X_{1,9}\rangle, \quad \mathrm{Eff}_5(\overline{\mathcal{O}_3}) = \langle X_{5,1}, X_{5,2}, X_{5,3}, X_{5,4}\rangle$$

Proof. For the first two lines, duality follows from the intersection matrix between the generators being diagonal. For $\mathrm{Nef}^5, \mathrm{Eff}_5$, the intersections between the proposed generators are

	$X_{1,5}$	$X_{1,6}$	$X_{1,8}$	$X_{1,9}$
$X_{5,1}$	0	$*$	$*$	0
$X_{5,2}$	$*$	0	0	$*$
$X_{5,3}$	0	0	$*$	$*$
$X_{5,4}$	$*$	$*$	0	0

All the $*$ are positive numbers. After scaling the columns and rows in a certain order, we may assume all the nonzero entries are 1. Then relabeling the rows as $e_1, e_2, e_3, e_1 + e_2 - e_3$, the duality of these cones is equivalent to the duality of $\langle e_1, e_2, e_3, e_1 + e_2 - e_3\rangle$ and $\langle e_1^*, e_2^*, e_1^* + e_3^*, e_2^* + e_3^*\rangle$, which is evident.

For the fourth line of equations, we have the following intersection matrix.

	$X_{3,7}$	$X_{3,8}$	$X_{3,9}$	$X_{3,10}$	$X_{3,11}$	$X_{3,12}$	$X_{3,13}$
$X_{3,7}$	0	$*$	$*$	0	0	0	0
$X_{3,8}$	$*$	0	0	$*$	0	0	0
$X_{3,13}$	0	0	$*$	$*$	0	0	0
$X_{3,5}$	$*$	$*$	0	0	0	0	0
$X_{3,11}$	0	0	0	0	0	$*$	0
$X_{3,12}$	0	0	0	0	$*$	0	0
$X_{3,6}$	0	0	0	0	0	0	$*$

Once again, after scaling the rows in the columns in the correct order, we may assume all the nonzero intersections are 1. Then a similar argument to the $\mathrm{Nef}^5, \mathrm{Eff}_5$ case yields that the two cones are dual.

Nef-ness of the proposed nef classes follows from all generators intersecting the suborbits in the correct dimension. □

4.4. **Schemes supported at a single point.** We now consider $\overline{\mathcal{O}_{2,\mathrm{nonred}}}$, parametrizing schemes supported at a single point (both curvilinear and planar fat points).

Note that $\overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot \mu = 0$. Hence $\overline{\mathcal{O}_{2,\mathrm{nonred}}}$ is determined by its restriction to $\mathrm{Hilb}_3\mathbb{P}^2$. Then since the locus of totally nonreduced schemes in $\mathrm{Hilb}_3\mathbb{P}^2$ has class

$$\overline{\mathcal{O}_{2,\mathrm{nonred},\mathbb{P}^2}} = 3(A - B + C - D)$$

by Section 5.3 in [RS21], we have

$$\begin{aligned}\overline{\mathcal{O}_{2,\mathrm{nonred}}} &= 3(\widetilde{A} - \widetilde{B} + \widetilde{C} - \widetilde{D}) \\ &= 3(P_2 - PH + \ell + p). \qquad \text{(Prop 3.3)}\end{aligned}$$

Since all the classes we consider in this section intersect this orbit in the correct dimension, we do not need to calculate the nef and effective cones of $\overline{\mathcal{O}_{2,\mathrm{nonred}}}$.

4.5. **Collinear schemes.** The locus of collinear schemes $\overline{\mathcal{O}_{2,\mathrm{col}}}$ is given by $P^2 - P_2$.

As for the nef and effective cones of this orbit closure, note that $\overline{\mathcal{O}_{2,\mathrm{col}}}$ has the structure of a $\mathrm{Sym}^3\mathbb{P}^1 \cong \mathbb{P}^3$ bundle over $\mathrm{Gr}(1,3)$. Hence the dimensions of the Chow groups of $\overline{\mathcal{O}_{2,\mathrm{col}}}$ are

k	$\dim A_k(\overline{\mathcal{O}_{2,\mathrm{col}}})$
7	1
6	2
5	4
4	5
3	5
2	4
1	2
0	1

Once again let $P \subseteq L \subseteq \Pi$ denote a point, line,and plane in $\mathbb{P}^3$ respectively. Let Π', Π'' denote additional planes in $\mathbb{P}^3$. Let $X_{6,1}$ denote the locus of collinear schemes incident to Π, and let $X_{6,2}$ denote the locus of collnear schemes coplanar with L. Let $Z_{1,1}$ denote the locus of collinear schemes contained in L and containing two fixed points, and let $Z_{1,2}$ denote the locus of totally nonreduced collinear schemes contained in Π and supported at P.

$$\begin{aligned}
X_{6,1} &= \overline{\mathcal{O}_{2,\mathrm{col}}} \cdot H \\
X_{6,2} &= \frac{1}{3}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot F \\
Z_{1,1} &= \frac{1}{9}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot H^2p^2 = i_*(\alpha)H = HP_3P_2\ell \\
Z_{1,2} &= \frac{1}{3}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot \ell p
\end{aligned}$$

For fivefolds, let $Z_{5,1}$ be the locus of collinear schemes contained in Π, let $Z_{5,2}$ be the locus of collinear schemes whose generic member contains a point in $\Pi \setminus \Pi'$ and a point in $\Pi' \setminus \Pi$, and let $Z_{5,3}$ be the locus of collinear schemes incident to a line. Furthermore, let $Z_{5,4}$ denote the locus of schemes collinear with a fixed point.

$$\begin{aligned}
Z_{5,1} &= \frac{1}{3}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot p \\
Z_{5,2} &= \overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \widetilde{C} = \overline{\mathcal{O}_{2,\mathrm{col}}}(H^2 - \ell) \\
Z_{5,3} &= \overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \ell \\
Z_{5,4} &= \frac{1}{3}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot M
\end{aligned}$$

And then for surfaces, let $Z_{2,1}$ be the locus of totally nonreduced collinear schemes contained in Π and supported on L. Let $Z_{2,2}$ be the locus of of collinear schemes contained in L and containing P. Additionally let $Z_{2,3}$ be the locus of collinear schemes contained in Π and containing a length 2 scheme supported at P, and let $Z_{2,4}$ be the locus of totally nonreduced collinear schemes supported at P.

$$\begin{aligned}
Z_{2,1} &= \frac{1}{3}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot Hp \\
Z_{2,2} &= \frac{1}{9}\overline{\mathcal{O}_{2,\mathrm{col}}} \cdot Hp^2 = \alpha = P_3\ell P_2 \\
Z_{2,3} &= 2(-2\alpha + \beta - \gamma + 2\delta + \epsilon) && \text{(see } S_9 \text{ in Section 5.4 of [RS21])} \\
&= 2(P_3H^2P_2 - 5P_3\ell P_2) \\
Z_{2,4} &= \overline{\mathcal{O}_{2,\mathrm{col}}} \cdot \overline{\mathcal{O}_{2,\mathrm{nonred}}} \cdot Y_1
\end{aligned}$$

For fourfolds, let $Z_{4,1}$ be the locus of collinear schemes contained in Π and incident to L. Let $Z_{4,2}$ be the locus of collinear schemes whose generic member contains a point in $\Pi' \setminus L$ and a point in $L \setminus \Pi'$. Let $Z_{4,3}$ be the locus of collinear schemes whose generic member is the union of a point on Π, Π', and Π''. Let $Z_{4,4}$ be the locus of collinear schemes contained in Π and collinear with P, and let $Z_{4,5}$ be the locus of collinear schemes that contain P.

Let $P \subseteq L \subseteq \Pi$ be a point, line, and plane respectively. Another class we'll need is the classes of the locus of schemes whose general member is a union of P and a length two scheme supported on L. This has class $k(-\alpha+\beta-\delta)$ and appears in the boundary of $\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^3)$. We also need the class of the locus whose general member is the union of a length two scheme supported at P and a point on L. This has class $-\alpha+\delta$ and is also in the boundary.

Proposition 4.4. We have the following expressions for the nef and effective cones of $\overline{\mathcal{O}_{2,\mathrm{col}}}$.

$$\begin{aligned}
&\mathrm{Nef}^1 = \langle X_{6,1}, X_{6,1}\rangle, \quad \mathrm{Eff}_1 = \langle Z_{1,1}, Z_{1,2}\rangle\\
&\mathrm{Nef}^2 =, \langle Z_{5,1}, Z_{5,2}, Z_{5,3}, Z_{5,4}\rangle, \quad \mathrm{Eff}_2 = \langle Z_{2,1}, Z_{2,2}, Z_{2,3}, Z_{2,4}\rangle\\
&\mathrm{Nef}^3 = \langle Z_{4,1}, Z_{4,2}, Z_{4,3}, Z_{4,4}, Z_{4,5}\rangle, \quad \mathrm{Eff}_3 = \langle Z_{3,1}, Z_{3,2}, Z_{3,3}, Z_{3,4}, Z_{3,5}\rangle
\end{aligned}$$

Proof. Duality follows from the generators yielding a diagonal intersection matrix, and the nefness of the proposed generators follows from the fact they intersect all the suborbits in the appriopriate dimension. □

4.6. **Non-reduced schemes.** The class of the locus of nonreduced schemes $\overline{\mathcal{O}_1}$ can be computed using the push-pull formula to reduce it to a computation in $\mathrm{Hilb}_3\mathbb{P}^2$. We have that

$$\begin{aligned}
\overline{\mathcal{O}_1}\cdot P_3H\ell p &= 1\\
\overline{\mathcal{O}_1}\cdot P_3H\ell^2 &= 0
\end{aligned}$$

hence $\overline{\mathcal{O}_1} = 2(2H-F) = 2(H-P)$. Since all relevant schemes intersect this orbit in the correct dimension, we do not calculate the effective and nef cones of this orbit closure.

5. Nef and effective cones in (co)dimension 2,3

Theorem 1.1. We have the following expression for the nef and effective cones in (co)dimension two for $\mathrm{Hilb}_3\mathbb{P}^3$.

$$\begin{aligned}
\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^3) &= \langle \alpha,\ \epsilon,\ -\alpha+\delta,\ -\alpha+\beta-\delta,\ \alpha-\beta+\gamma-\delta-2\mu,\ -2\alpha+\beta-\gamma+2\delta+\epsilon,\ \mu\rangle\\
\mathrm{Nef}_2(\mathrm{Hilb}_3\mathbb{P}^3) &= \langle \widetilde{B},\ \widetilde{C},\ \widetilde{D},\ \widetilde{E},\ \widetilde{A}+\widetilde{B},\ \widetilde{A}+\widetilde{E},\ 2\widetilde{A}+2\widetilde{E}+M,\ 2\widetilde{A}+2\widetilde{B}+M\rangle.
\end{aligned}$$

Proof. We first show that these two cones are dual. For the purposes of the proof, we scale the proposed generators by positive constants so that we are showing the following duality:

$$\begin{aligned}
&\langle \alpha, \epsilon, -\alpha+\delta, -\alpha+\beta-\delta, \alpha-\beta+\gamma-\delta-2\mu, -2\alpha+\beta-\gamma+2\delta+\epsilon, 2\mu\rangle^*\\
&= \langle \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{A}+\widetilde{B}, \widetilde{A}+\widetilde{E}, \widetilde{A}+\widetilde{E}+(1/2)M, \widetilde{A}+\widetilde{B}+(1/2)M\rangle
\end{aligned}$$

With this scaling, we can label the proposed generators of $\mathrm{Eff}_2(\mathrm{Hilb}_3\mathbb{P}^3)$ as $e_1, e_2, e_3, e_4, e_5, e_6, e_2+e_3-e_5-e_6$. Then certainly the dual of this cone is $\langle e_1^*, e_2^*, e_3^*, e_4^*, e_2^*+e_5^*, e_2^*+e_6^*, e_3^*+e_5^*, e_3^*+e_6^*\rangle$. We compute that

$$\begin{aligned}
e_1^* &= \widetilde{C}\\
e_2^* &= \widetilde{E}+\widetilde{A}+\tfrac{1}{2}M\\
e_3^* &= \widetilde{B}+\widetilde{A}+\tfrac{1}{2}M\\
e_4^* &= \widetilde{D}\\
e_5^* &= -\tfrac{1}{2}M\\
e_6^* &= (-\widetilde{A}-\tfrac{1}{2}M),
\end{aligned}$$

yielding the desired dual cone $\langle \widetilde{C}, \widetilde{E}+\widetilde{A}+(1/2)M, \widetilde{B}+\widetilde{A}+(1/2)M, \widetilde{E}+\widetilde{A}, \widetilde{E}, \widetilde{B}+\widetilde{A}, \widetilde{B}\rangle$.

All the proposed effective classes are indeed effective since we've given effective representations of them in Section 4.

For nef-ness: note that $\widetilde{A}$ intersects $\overline{\mathcal{O}_{2,\mathrm{col}}}, \overline{\mathcal{O}_3}, \overline{\mathcal{O}_{4,\mathrm{col}}}$ in one higher dimension than expected, and that $\widetilde{B}, \widetilde{D}$, and M intersects $\overline{\mathcal{O}_{4,\max}}$ in one higher dimension than expected. All other intersections between $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, M$ and the orbit closures are of expected dimension. Then note that $\widetilde{A}, \widetilde{A}+\widetilde{E}, \widetilde{A}+\widetilde{B}, 2\widetilde{A}+2\widetilde{B}+M$, $2\widetilde{A}+2\widetilde{E}+M$, intersect all the effective surface classes in $\overline{\mathcal{O}_{2,\mathrm{col}}}, \overline{\mathcal{O}_3}, \overline{\mathcal{O}_{4,\mathrm{col}}}, \overline{\mathcal{O}_{4,\max}}$ non-negatively. Furthermore, $\widetilde{B}, \widetilde{D}, M$ intersect all the effective surface classes in $\overline{\mathcal{O}_{4,\max}}$ non-negatively. Therefore, the proposed nef classes are nef. □

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