MATH 553: LECTURE NOTES

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ABSTRACT. Notes for 553: algebraic geometry II, spring 2025. Heavily referenced from Vakil, Hartshorne. May contain typos.

CONTENTS

1. JAN 13: SYLLABUS, SHEAVES

Up till now, you have been thinking of algebraic varieties more in the classical sense– they're zero sets of polynomials. $V(f_1, \ldots, f_n)$. From your perspective, in $k[x]$ you don't really care too much about x versus x^2 because their vanishing sets are the same, maybe you'd default to taking the one that generates a radical idea. But you lose some things with this perspective. Certainly I wouldn't say it's great for multiplicities and whatnot.

So, we need to upgrade: instead of varieties in the classical sense, we'll eventually think of schemes. Some of the intuition will port over: we're thinking of things/geometric objects (or, topological spaces) that look like they're (locally) "cut out by polynomials," and a decent amount of the practical work of computing things will resemble some of the polynomial fiddling you've done before, but we're keeping track of more of the data of the **functions on these spaces.**

Roughly, a scheme has three levels of data.
- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space

(The last one helps distinguish things like $V(x)$ versus $V(x^2)$.)

Here is where we start brushing up on Grothendieck's perspective: that when studying on object, it's less important to study the object itself and more important to study functions between them, how they relate to other things.

Now, before that, we need to do **sheaves**, which are, informally, a bundling of data about functions on open sets of a topological space. **The usual example, which you should have in mind throughout, is the data of differentiable functions on a differentiable manifold.**

We begin with sheaves of sets, but the idea extends to sheaves of groups, rings, k-algebras, etc.

Definition 1.1. Let X be a topological space. A presheaf $\mathscr{F}$ is the following data:
- To each open set $U \subseteq X$, we have an assignment $\mathscr{F}(U)$ of a set (or group or ring, etc...)
- For each inclusion $V \hookrightarrow U$ of open sets, we have restriction maps $\operatorname{res}^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$. The restriction maps need to follow some reasonable properties:

- $\operatorname{res}_U^U : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map
- For inclusions $W \hookrightarrow V \hookrightarrow U$, the restriction maps should commute (that is, restricting from U to V to W or restricting directly U to W shouldn't change things).

$$\mathscr{F}(U) \xrightarrow{\operatorname{res}_V^U} \mathscr{F}(V)$$

with res_W^U and res_W^V to $\mathscr{F}(W)$

Notational bits-and-bobs:

- Elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ on U.
  –
- $\mathscr{F}(U)$ is notated a few other ways:
  - $\Gamma(U, \mathscr{F})$
  - $H^0(U, \mathscr{F})$
- Note that a presheaf is precisely the data of a contravariant functor from the category of open sets on X to the category of sets (or groups, rings, etc).

Definition 1.2. A presheaf $(X, \mathscr{F})$ is a sheaf if it satisfies two more additional axioms.

- **Identity/uniqueness**: If $\{U_i\}_{i \in I}$ is an open cover of U and $f_1, f_2 \in \mathscr{F}(U)$ are two sections/functions such that

$$\operatorname{res}_{U_i}^U f_1 = \operatorname{res}_{U_i}^U f_2$$

for all $i \in I$, then $f_1 = f_2$. (That is, two sections that line up on each piece of a cover have to have been the same).

- **Gluing**: Let $\{U_i\}_{i \in I}$ be an open cover of U. If you have an $f_i \in \mathscr{F}(U_i)$ for each i such that, for any i, j:

$$\operatorname{res}_{U_i \cap U_j}^{U_i} f_i = \operatorname{res}_{U_i \cap U_j}^{U_j} f_j$$

then there is an $f \in \mathscr{F}(U)$ such that $\operatorname{res}_{U_i}^U f = f_i$ for each i. (That is, if you have an open cover, a choice of section on each piece of the cover, and these choices agree on the overlaps, then you should be able to glue these to a section on the whole thing).

Example 1.3. Let X be a differentiable manifold. Let $\mathscr{F}$ be the sheaf that assigns to an open set U the ring of differentiable functions $\mathscr{F}(U)$ defined on U. For $V \subseteq U$, the restriction map is restriction of domain:

$$\mathscr{F}(U) \to \mathscr{F}(V)$$
$$f \mapsto f|_V$$

The fact that differentiable functions are "defined by their values" makes it clear that this is a pre-sheaf. Likewise, the two additional sheaf properties are clear: if two functions agree on an open cover, they are the same function. And if you have a differentiable function on each piece and the overlaps are fine, you can defined the function on the whole manifold.

Note: in general, you may see things like $\operatorname{res}_V^U f$ written as $f|_V$ to save space.

Since I don't want to shift gears too much on the first day, let's do a few bits-and-bobs about sheaves.

Example 1.4 (Skyscraper sheaves)**.** Let S be a set, $p \in X$ a point. Get:

$$i_{p,*}S(U) = \begin{cases} S & p \in U \\ \{e\} & p \notin U \end{cases}$$

here $\{e\}$ is any one element set. Skyscraper-type behavior at p. Draw pictures.

2. Jan 15: Intro to Spec

We will eventually need to worry about morphisms of sheaves, pushforwards, pullbacks, and more. But that can come a bit later, when we better understand the topological spaces we want to look at in the first place.

Recall that we are trying to define schemes, which consist of:

- underlying set of points
- topology on the set (so the first two are the data of the topological space)
- **and** the "structure sheaf:" the data of algebraic functions on your space

Now on our journey towards schemes, which are our generalization of algebraic varieties/sets, we need to think of the underlying topological space of our geometric objects. These will come about in the form of the **spectrum of a ring**, AKA affine schemes. These are the building blocks of schemes in general.

These will resemble things from 552 somewhat: our first examples will be visualizable in some $\mathbb{C}^n$, with the majority of the points corresponding to tuples $(a_1, \ldots, a_n)$ satisfying some polynomials, along with some extra points that are useful to have in there.

Do note: ring here means a commutative ring with identity, 1. Example: $\mathbb{C}, \mathbb{R}, \mathbb{F}_p, \overline{\mathbb{F}_p}, \mathbb{C}[t], \mathbb{C}(t)$, polynomial rings, quotient rings. We will often focus on $\mathbb{C}$-algebras or k-algebras with k algebraically closed, as this is the best place to start off. As appropriate I may add in some examples over non-algebraically closed fields, but I will largely leave those examples to your future number theory courses.

The idea: given a ring A, we want the most natural/nontrivial space on which A becomes a "ring of functions." You've encountered this before with coordinate rings in 552.

Example 2.1 (Rough intuition)**.** The algebraic functions on the complex line $\mathbb{C}$ should be single variable polynomials: $\mathbb{C}[t]$. If you cut out the origin and consider the open set $\mathbb{C} \setminus \{0\}$, you no longer have to worry about t zeroing out, so your algebraic functions should now be $\mathbb{C}[t, t^{-1}]$.

Definition 2.2. As a set, Spec A is the set of all prime ideals of A.

Recall that $I \subseteq A$ is an ideal $\iff$ $I + I \subseteq I, aI \subseteq I$ for all $a \in A$. An ideal I is *prime* provided that $pq \in I \Rightarrow p \in A$ or $q \in A$. Recall that for $a_1, \ldots, a_n \in A$, that $(a_1, \ldots, a_n)$ denotes the ideal generated by $a_1, \ldots, a_n$, i.e.

$$(a_1, \ldots, a_n) = \{a_1 b_1 + \ldots a_n b_n \mid b_i \in A\}$$

Example 2.3 (The complex affine line)**.** Let us consider the case of $A = \mathbb{C}[t]$, and how we can think of the $\mathbb{C}[t]$ as the ring of functions over Spec $\mathbb{C}[t]$. First, let us compute the spectrum. By the fundamental theorem of algebra, we have:

$$\text{Spec } \mathbb{C}[t] = \{(x - a) \mid a \in \mathbb{C}\} \sqcup \{(0)\}$$

that is, we get a point for each element of $\mathbb{C}$, and then this extra point (0). Given that this space is "basically $\mathbb{C}$ with some extra stuff," it's not strange to think of $\mathbb{C}[t]$, i.e. complex polynomials in one variable, aka polynomials that can take in one complex input, as the ring of functions over $\mathbb{C}$.

A few things to note:

- At each point $(x - a)$ of the spectrum, we have an evaluation map:

$$\mathbb{C}[x] \to \frac{\mathbb{C}[x]}{(x - a)} \cong \mathbb{C}$$
$$f(x) \mapsto f(a)$$

That is, f is sent to its image in $\mathbb{C}[x]/(x - a)$, which says x can be swapped for a. That is, we send $f(x)$ to $f(a)$. This evaluates the polynomial at a. We will see a similar construction in general. Note that this means these points are keeping track of all the values of this function. If we have two different polynomials f_1, f_2, then their evaluations at some point will differ: i.e. functions are distinguished by their values. **This will not always be true!**

- (0) is called "the generic point." It is "close" to every point, so it is "generically" on the line, but it is not equal to any of the $(x - a)$. Some would choose to draw it as "fuzz" amongst the line. We will understand the generic point better when we understand the topology of Spec A.
- Spec $\mathbb{C}[t]$ will come to be known to us as the complex affine line, denoted $\mathbb{A}^1_{\mathbb{C}}$.

Example 2.4 (Don't say I never gave you an example that wasn't over $\mathbb{C}$!)**.** Consider $A = \mathbb{R}[t]$. The prime ideals are of one of two forms:

$$(t - a)\ a \in \mathbb{R}, \quad (t - a)(t - \bar{a})\ a \in \mathbb{C} \setminus \mathbb{R}$$

Hence we get an identification:

$$\text{Spec } \mathbb{R}[t] \cong \mathbb{C}/\text{Gal}(\mathbb{C}/\mathbb{R}) \sqcup \{(0)\}$$

which you can identify with the upper half plane along with a generic point.

Definition 2.5 (Evaluation map)**.** Given a ring A, $f \in A$, $\mathfrak{p} \in \text{Spec } A$, the **value of f at $\mathfrak{p}$** is the image of f under:

$$A \to A/\mathfrak{p} \to \text{Frac } A/\mathfrak{p}$$

Example 2.6 (Functions are not always separated by *values* at points)**.** Consider the set Spec $\mathbb{C}[t]/(t^2)$. As a set of points, it has just one element: $[(t)]$.

t is an element of the ring $\mathbb{C}[t]/(t^2)$, and we should think of it as being very small: so small that its square is zero, but it itself is not zero. If we think about the evaluations of this function, note that

$$\mathbb{C}[t]/(t^2) \to \text{Frac } (\mathbb{C}[t]/(t, t^2)) \cong \mathbb{C}$$
$$t \mapsto 0$$

That is, both the function t and 0 on the LHS evaluate to 0 on the RHS. But this is the only evaluation map to consider! So we see how functions cannot necessarily be separated by values.

Now it is time to define the topology on these spaces. The idea: closed sets should be sets of points where functions vanish.

$$f \text{ vanishes at } \mathfrak{p} \iff f(\mathfrak{p}) = 0$$
$$\iff f = 0 \text{ in } A/\mathfrak{p}$$
$$\iff f \in \mathfrak{p} \ (\iff (f) \subseteq \mathfrak{p})$$

Definition 2.7. Various vanishing loci definitions. Let $f \in A, S \subseteq A$.

$$V(f) = \{\mathfrak{p} \in \text{Spec } A : (f) \subseteq \mathfrak{p}\}$$
$$V(S) = \{\mathfrak{p} \in \text{Spec } A : S \subseteq \mathfrak{p}\}$$

Definition 2.8. A (Zariski) closed subset of Spec A is any set of the form of a vanishing locus $V(\mathfrak{a})$ for $\mathfrak{a}$ an ideal.

Proposition 2.9. The collection of Zariski closed subsets forms a topology on Spec A.

Proof. Observe:

$$\varnothing = V((1))$$
$$\text{Spec } A = V((0))$$
$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$$
$$\bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$$

$\square$

Example 2.10. The closed sets in Spec $\mathbb{C}[t]$ are the whole space and
$$V(f(t)) = V((t-a_1)\dots(t-a_n)) = \cup_i V((t-a_i)) = \{(t-a_i) : 1 \le i \le n\}$$
i.e. finite collections of non-generic points.

Note that $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[t]$. That is, the generic point is "close" to all other points, and "sits along the whole line".

Definition 2.11. Define $D(f) = \text{Spec } A \setminus V((f))$. These open sets form a basis for the topology.

Proposition 2.12. Let S be a multiplicative set. By studying the map $\varphi : A \to S^{-1}A, a \mapsto a/1$, this induces a bijection
$$\{\text{primes in } A \text{ with } \mathfrak{p} \cap S = \varnothing\} \longleftrightarrow \{\text{primes in } S^{-1}A\}$$

What should be the algebraic functions on $D(f)$? well, since we're not working with the full spec, we should be able to invert things that don't vanish on the set. That is, things whose vanishing sets are squirreled away in $V(f)$, a set we are cutting out.

Set $\mathcal{O}_{\text{Spec } A} = S^{-1}A$ where
$$S = \{g \in A : g(\mathfrak{p}) \ne 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

This definition depends only on $D(f)$, not on f itself. But luckily:

Proposition 2.13. The natural map
$$A_f \mapsto \mathcal{O}_{\text{Spec } A}(D(f))$$
is an isomorphism.

Lemma 2.14. $D(f) \subseteq D(g)$ (that is, $V(g) \subseteq V(f)$) if and only if $f^n \in (g)$, if any only if g is invertible in A_f.

Proof. $f^n \in (g) \iff \sqrt{(f)} \subseteq \sqrt{(g)} \iff$ the prime ideals containing (f) are a superset of those containing (g), which means $V(g) \subseteq V(f)$.

Then $f^k = gm$, so g is invertible in A_f. $\qquad\square$

That is, the algebraic functions on $D(f)$ are basically obtained by inverting f. So, we have the makings of the structure sheaf, i.e. a sheaf $\mathcal{O}_{\text{Spec } A}$ where $\mathcal{O}_{\text{Spec } A}(U)$ is algebraic functions on U. But we only have it on a distinguished basis. The question becomes: is this enough to determine the sheaf overall? Will we be able to do computations in the future/check nice properties on just this basis? The answer: yes! Back to sheaf theory...

3. Jan 17: Let's understand sheaves better (stalks, morphisms)

We learned about the topological spaces that will be pieced into schemes. These are the Spec A, and we think of A as the ring of algebraic functions on Spec A. We had special open sets $D(f) = \text{Spec } A \setminus V(f)$, where $V(f) = \{\mathfrak{p} : f \in \mathfrak{p}\}$, i.e. the primes/points where f vanishes.

We would like to assemble a **structure sheaf** on the topological space Spec A:
$$\mathcal{O}_{\text{Spec } A}(U) = \text{ring of algebraic functions on } U$$
From this perspective, $\mathcal{O}_{\text{Spec } A}(\text{Spec } A)$ should be A. And we saw last time that it was reasonable to set:
$$\mathcal{O}_{\text{Spec } A}(D(f)) = S^{-1}A \cong A_f$$
with
$$S = \{g \in A : g(\mathfrak{p}) \ne 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$$

Problem: what about this would-be sheaf on open sets in general? We would like describing the sheaf, i.e. describing the rings of algebraic functions, on nice $D(f)$ to be enough. It is, but we need to do a bit of work to say that. (Here: Vakil and Hartshorne somewhat "diverge". Vakil shows that defining a sheaf on a basis is sufficient; Hartshorne just describes the $\mathcal{O}_{\text{Spec } A}(U)$ from the get-go, with the construction being the one you'd do when defining a sheaf from a base).

We'll get to all that, but we should do some necessary details and in general shore up our knowledge.

Definition 3.1. Let $(X, \mathscr{F})$. Let $x \in X$ be a point. The **stalk** of $\mathscr{F}$ at x is defined as the direct limit:

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U) = \{(f, U) : f \in \mathscr{F}(U), p \in U\}$$

where $(f, U) \sim (g, V)$ if and only if there is some $W \subseteq U, V$, with W containing p, such that $f|_W = g|_W$. **Draw pic in differentiable functions case! This means they need to be same on some smaller open set around the point.** Observe that in the case of differentiable functions on a differentiable manifold, the stalk is a local ring: its unique maximal ideal is the ideal of all functions vanishing at p.

Definition 3.2. Elements of a stalk are called **germs.**

Remark 3.3. We will see later on that many properties we want to test of sheaves (or structure sheaves of schemes) can be tested by checking the analogous condition on the stalks. This is reasonable, looking at the gluing axiom.

Definition 3.4 (Morphisms of (pre)sheaves)**.** Let $\mathscr{F}, \mathscr{G}$ be (pre)sheaves on a topological space X. A morphism $\mathscr{F} \to \mathscr{G}$ is a collection of maps $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for each U such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\
{\scriptstyle \mathrm{res}^U_V} \downarrow & & \downarrow {\scriptstyle \mathrm{res}^U_V} \\
\mathscr{F}(V) & \xrightarrow{\varphi_V} & \mathscr{G}(V)
\end{array}
$$

Consequently, we can see that φ defines a map on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ by sending (f, U) to $(\varphi(f), U)$. An isomorphism is a morphism with a two-sided inverse.

Let's restrict our attention to sheaves of abelian groups at this point (we rarely fall outside this scenario).

Proposition 3.5. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on a topological space X. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then φ is iso if and only if ϕ_x is iso.

Proof. Iso $\iff$ iso will cover the whole proof. ϕ an isomorphism implies ϕ_x an isomorphism: this is clear.

Each ϕ_x an isomorphism implies ϕ an iso. Let's start with injectivity: consider a fixed $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. For injectivity: if $\phi(s) = 0$, then $\phi(s)_x = 0$ for each $x \in U$. Then s_x is zero for each $x \in U$ by injectivity on stalks. That is s restricted to zero on an open cover of U. That is, $s = 0$.

For surjectivity: suppose we are considering $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$. Let $t \in \mathscr{G}(U)$. We'll piece together something that maps to it.

For each $x \in U$, we have $t_x \in \mathscr{G}_x$, and it must be the image of some $s_x \in \mathscr{F}_x$. s_x can be repped by some $s(x) \in V_x \ni x$. Then $\varphi(s(x)), t|_{V_x}$ are two elements of $\mathscr{G}(V_x)$ with the same germ, so $\varphi(s(x)), t$ agree in some neighborhood W_x of x.

Cover U with these W_x, and consider the $s(x)$ (well, $s(x)|_{W_x}$ we get for each one. On the overlaps, these must agree due to injectivity (their overlaps go to $t|_{set}$). So we can piece them together to get an $s \in \mathscr{F}(U)$ that maps to t.

Note that the proof of surjectivity needed injectivity! $\square$

Definition 3.6 (**Tentative** definition of kernel, image, cokernel)**.** Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves of abelian groups, we can define the <u>presheaves</u> $\ker(\varphi), \operatorname{coker}(\varphi), \operatorname{im}(\varphi)$, as follows:

$$\ker(\varphi)(U) = \ker(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) \, [\subseteq \mathscr{F}(U)]$$
$$\operatorname{coker}(\varphi)(U) = \operatorname{coker}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U))$$
$$\operatorname{im}(\varphi)(U) = \operatorname{im}(\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)) [\subseteq \mathscr{G}(U)]$$

Proposition 3.7. Given $\varphi : \mathscr{F} \to \mathscr{G}$, the kernel is a sheaf.

Proof. Identity is inherited from the parent sheaf. Gluing is too, though you need to check that the glued function f is still in the kernel. This works because $\varphi(f)$ restricts to zero on a cover, thus is zero globally from gluing in $\mathscr{G}$. $\square$

Proposition 3.8. The image and cokernel of a sheaf need not be a sheaf.

For cokernel. Let X be $\mathbb{C}$, and let $\mathcal{O}$ be the sheaf of holomorphic functions and $\mathcal{O}$ be the sheaf of nonzero holomorphic functions. Consider the map φ with

$$\varphi_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$$
$$f \mapsto e^f$$

we claim that the cokernel isn't a sheaf. First, note that there is no holomorphic f such that $e^f = z$ on $\mathbb{C} \setminus \{0\}$. Otherwise, differentiating both sides yields:

$$e^f \cdot f' = 1 \Rightarrow z \cdot f' = 1 \Rightarrow f' = 1/z$$

Integrating the LHS over a loop around zero yields 0, but integrating the RHS over said loop produces $2\pi i$. Contradiction.

Therefore, $[z] \neq 0$ in $\operatorname{coker}(\varphi)$. That is, the global sections of the cokernel has nonzero elements.

But, take $U_1 = \mathbb{C} \setminus (-\infty, 0]$ and $U_2 = \mathbb{C} \setminus [0, \infty)$. These are simply connected, so every nonzero function on them can be written as some e^f (we can define the log: we made a branch cut!). Thus $\operatorname{coker}(U_1)$, $\operatorname{coker}(U_2)$ both equal zero. So the cokernel fails the identity axiom.

Similarly, this shows why the image isn't necessarily a sheaf: we can't glue the logs of z into a log for z on all of $\mathbb{C} \setminus \{0\}$ $\qquad\qquad\square$

So, we have all these presheaves running around (including ones we'd really like to consider: the image and cokernel are important!). We would like some way to modify them into a sheaf, and it should have some nice universal property that relates it back to the original presheaf.

Construction 3.9 (Sheafification). Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that: for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism φ^+ such that $\varphi = \varphi^+ \circ \theta$. The pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism. $\mathscr{F}^+$

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\ \theta\ } & \mathscr{F}^+ \\
& \varphi \searrow & \downarrow \varphi^+ \\
& & \mathscr{G}
\end{array}$$

To be continued next lecture...

4. Jan 22: Sheafifcation, sheaves on a base

Recall: last time: nice properties of stalks. Recall that we have $\mathscr{F}(U) \to \mathscr{F}_p$ for each $p \in U$, given by $f \mapsto f_p = (f, U)$. Also $\varphi : \mathscr{F} \to \mathscr{G}$ induces $\mathscr{F}_p \to \mathscr{G}_p$ for each $p \in X$. Also, covered morphisms of sheaves. Saw that naive definition of im, ker of sheaves not necessarily sheaves. This necessitated sheafification.

Now for sheafification. The construction is: bundle the stalks together in a nice way. That is, make a big product but only allow combinations of germs that looked like they could glue together.

$$\mathscr{F}^+(U) := \{(f_p)_{p \in U} : \text{ for all } p \in U, \text{ there is an open } V \text{ with } p \in V \subseteq U$$
$$\text{with } s \in \mathscr{F}(V), s_q = f_q \text{ for all } q \in V\}$$
$$\subseteq \prod_{p \in U} \mathscr{F}_p$$

the morphism θ is clear ($f \mapsto (f_p)_{p \in U}$). To describe φ^+: look at the sections that glue to your (f_p), look at their images, glue them in the target, and call that the image. This is unique: in order for the diagram to commute any other map would have to do the same thing.

Remark 4.1. Sheafification is a functor from presheaves on X to sheaves on X.

Remark 4.2. Specifically, given $i : \text{Shf} \to \text{Pre}$ note that sheafification $+ : \text{Pre} \to \text{Shf}$, we have that $+$ is the left adjoint of i, i.e. given $\mathscr{F}$ a sheaf and $\mathscr{G}$ a presheaf, we have natural bijection

$$\operatorname{Hom}_{\text{Pre}}(\mathscr{G}, i(\mathscr{F})) \cong \operatorname{Hom}_{\text{Shf}}(\mathscr{G}^+, \mathscr{F})$$

Example 4.3 (Constant sheaves)**.** Let X be a topological space, S a set. Get constant presheaf by assigning same set to all open sets:

$$\mathscr{F}(U) = S$$

(On nonempty sets, you can interpret this as constant functions from U to S). Gluing is a mess because of disconnects. The empty set presents problems too: the empty cover $\{\}$ is an open cover of the empty set, and each section vacuously restricts to the same thing on this open cover. So the identity axiom says all sections on $\varnothing$ should be the same!

The sheafification $\mathscr{F}(U)$ will instead assign to U: locally constant maps from U to S. Denote this sheaf as $\underline{S}$.

Remark 4.4. Thinking of sheaves of abelian groups: sheafification adds in the gluings that should exist but don't, and kills off the nonzero sections that are locally zero.

Proposition 4.5. $\mathscr{F} \to \mathscr{F}^+$ yields an isomorphism of stalks.

Proof. Work from the explicit description. $\qquad\square$

Definition 4.6. We say that a map of sheaves is injective if and only if the kernel sheaf is zero.

Lemma 4.7. A map of sheaves is injective $\iff$ $\varphi : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Likewise it is injective $\iff$ is injective on stalks.

Proof. This was done in the bijectivity proof before. $\qquad\square$

Definition 4.8. We define the image and cokernel *sheaves* by taking the sheafification of the presheaves defined above.

Given a map $\varphi : \mathscr{F} \to \mathscr{G}$. Since we have a map $\mathrm{im}_{\mathrm{pre}}(\varphi) \to \mathscr{G}$, we necessarily have a map $\mathrm{im}(\varphi) \to \mathscr{G}$. This map is injective: it is injective on the level of stalks (note that the unsheafified and sheafified im have the same stalks!!). Thus we can identify $\mathrm{im}(\varphi)$ with a subsheaf of $\mathscr{G}$.

Definition 4.9. A morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is **surjective** if $\mathrm{im}(\varphi) = \mathscr{G}$.

Lemma 4.10. $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if $\mathscr{F}_x \to \mathscr{G}_x$ is surjective for all points x.

Proof. $\Rightarrow$: $\mathrm{im}(\mathscr{F}) = \mathscr{G}$ means their stalks are isomorphic, hence $\mathscr{F}_x \to \mathscr{G}_x$ must be surjective.

$\Leftarrow$: we want to show that $\mathrm{im}(\mathscr{F}) = \mathscr{G}$. Well, the map on stalks is an isomorphism (injective and surjective on stalks), so they are equal. $\qquad\square$

Example 4.11. In our example with $X = \mathbb{C}$, $\mathcal{O}_X$ the sheaf of holomorphic functions and $\mathcal{O}_X^*$ the sheaf of non-vanishing holomorphic functions and

$$\mathcal{O}_X \to \mathcal{O}_X^*$$
$$f \mapsto e^f$$

we have that $\mathrm{im}(\varphi) = \mathcal{O}_X^*$ and $\mathrm{coker}(\varphi) = 0$. This can be seen via φ being surjective on the level of stalks (and correspondingly the cokernel is 0 on the level of stalks).

Definition 4.12. A sequence of maps:

$$\mathscr{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathscr{F}^i \overset{\varphi^i}{\to} \mathscr{F}^{i+1}$$

is *exact* if at each stage, $\ker \varphi^i = \mathrm{im}\, \varphi^{i-1}$.

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5. Jan 24: Sheaves on a base, affine schemes

Time to handle an issue: sometimes we understand a sheaf really well on a nice collection of open sets. But what about the rest? The details are sometimes unpleasant/obfuscating: it is mainly important to know that the data of the sheaf on a suitably nice basis is enough to determine the sheaf. The construction will be reminiscent of sheafification.

Definition 5.1. A base of a topology is a collection of open sets $\{B_j\}_{j \in J}$ such that any open set of X can be written as a union of B_j.

Remark 5.2. $(f) \subseteq \mathfrak{a} \iff V(f) \supseteq V(\mathfrak{a}) \iff D(f) \subseteq D(\mathfrak{a})$ so the $D(f)$ genuinely are a basis of the Zariski toplogy on Spec A.

Definition 5.3. Suppose $\{B_i\}$ is a basis on X. A presheaf of sets on the base is an assignment $F(B_i)$ for each B_i. If $B_j \subseteq B_i$, we have restriction maps $\text{res}^{B_i}_{B_j}$ satisfying $\text{res}^{B_i}_{B_i} = \text{id}$ and $\text{res}^{B_i}_{B_k} = \text{res}^{B_j}_{B_k} \circ \text{res}^{B_i}_{B_j}$.

For sheaves on a base: base identity and base gluing axioms:

- If $B \in \{B_i\}_{i \in I}$ can be written $B = \cup_{i \in J} B_i$ and $f, g \in F(B)$ with $\text{res}^B_{B_i} f = \text{res}^B_{B_i} = g$ for all $i \in J$, then $f = g$.
- If we have $f_i \in B_i$ for $i \in J$ such that f_i agrees with f_j on any B_k contained in $B_i \cap B_j$, then there is a $f \in F(B)$ such that $f|_{B_i} = f_i$ for all $i \in J$.

Theorem 5.4. *Suppose $\{B_i\}$ a base on X, and F a sheaf of sets on this base. There is a sheaf $\mathscr{F}$ extending F ($\mathscr{F}(B_i) \cong F(B_i)$ with isomorphisms agreeing with restriction maps). $\mathscr{F}$ is unique up to unique isomorphism.*

Proof. As before, $\mathscr{F}$ is a sheaf of compatible germs. Define the stalk of a presheaf F on a base as:
$$F_p = \varinjlim_{B_i \ni p} F(B_i)$$

Define
$$\mathscr{F}(U) := \{(f_p \in F_p)_{p \in U} : \text{for all } p \in U, \text{ there is a } B \text{ with } p \in B \subseteq U$$
$$\text{and } s \in F(B) \text{ such that } s_q = f_q \text{ for } q \in B\}$$

We get a map $F(B) \to \mathscr{F}(B)$ for each B, which is an isomorphism. Checking the details is similar to the hassle of sheafification.

Note that clearly $\mathscr{F}_p \cong F_p$. $\qquad\square$

We can finally really talk about affine schemes. Consider Spec A with the Zariski topology, and for open sets $D(f)$ set
$$\mathcal{O}_{\text{Spec } A}(D(f)) = S^{-1}A$$
where $S = \{g \in A : g(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in D(f)\} = \{g \in A : V(g) \subseteq V(f)\}$. The restriction maps are clear enough: if $D(g) \subseteq D(f)$ then the restriction map
$$\text{res}^{D(f)}_{D(g)} : \mathcal{O}_{\text{Spec } A}(D(f)) \to \mathcal{O}_{\text{Spec } A}(D(g))$$
is further localization. This is clearly a presheaf on a distinguished base.

Lemma 5.5. Spec A is quasi-compact (every open cover has a finite subcover).

Proof. It's enough to show this for covers for the form $\{D(f_i)\}$. Note that $\cup D(f_i) = D(\sum(f_i))$. This will be all of Spec A only when $1 \in \sum(f_i)$, in which case we get $1 = a_{i_1} f_{i_1} + \ldots a_{i_k} f_{i_k}$ and you can just take the corresponding cover pieces $D(f_{i_1}), \ldots, D(f_{i_k})$. $\qquad\square$

Proposition 5.6. This description in fact gives a sheaf on a distinguished base, and thus determine a sheaf on Spec A. This sheaf is the **structure sheaf** on Spec A, and is referred to as $\mathcal{O}_{\text{Spec } A}$ or just $\mathcal{O}$ if it is clear what A is.

Proof. It's enough to show identity and gluability on just A (if you want to show it on $D(f)$, that's the same as swapping the ring out for A_f, modulo some detail-checking).

- **Identity axiom**: Write Spec $A = \cup_i D(f_i)$. Then after potentially relabeling, we can pick a finite subcover. Spec $A = \cup_{i=1}^n D(f_i)$. That is, $V((f_1, \ldots, f_n)) = \varnothing$, i.e. $(f_1, \ldots, f_n) = A$.

  Suppose we have a section $s \in \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$ such that $\operatorname{res}_{D(f_i)} s = 0 \in A_{f_i}$ for each f_i. That means that there is some m such that $f_i^m s = 0$ (in A).

  But note that $D(f_i) = D(f_i^m)$ (as f_i vanishes at $\mathfrak{p}$ if and only if f_i^m vanishes at $\mathfrak{p}$). So there are g_i such that
  $$1 = \sum g_i f_i^m$$

  But then:
  $$s = \sum g_i f_i^m s = \sum 0 = 0$$

- **Gluing:** Again, being able to write 1 as a sum of these f_i will let us piece things together in a nice way.

  Again, say we have gluing data on some open cover. Pick a finite subcover $\{D(f_1), \ldots, D(f_n)\}$. Let s_i be the section on each $D(f_i)$ so that
  $$\operatorname{res}_{D(f_i) \cap D(f_j)}^{D(f_i)} s_i = \operatorname{res}_{D(f_i) \cap D(f_j)}^{D(f_j)} s_j$$

  Identifying $D(f_i) = A_{f_i}, D(f_j) = A_{f_j}, D(f_i f_j) = A_{f_i \cdot f_j}$ we get that
  $$s_i = \frac{a_i}{f_i^{\ell_i}}, \quad s_j = \frac{a_j}{f_j^{\ell_j}}$$

  and because their restrictions are the same in $A_{f_i f_j}$ it must be that there is an $m_{i,j}$ such that
  $$(a_i f_j^{\ell_j} - a_j f_i^{\ell_i})(f_i f_j)^{m_{i,j}} = 0.$$

  Let $m = \max_{i,j} m_{i,j}$. Then the above tells us that
  $$(a_i f_i^m) f_j^{m+\ell_j} = (a_j f_j^m) f_i^{m+\ell_i}$$

  Now again, Spec $A = \cup D(f_i) = \cup D(f_i^m)$, so there exists g_i such that
  $$1 = g_1 f_1^{m+\ell_1} + \cdots + g_n f_n^{m+\ell_n}$$

  and consider the element of A given by:
  $$s = g_1 a_1 f_1^{m+\ell_1} + \cdots + g_n a_n f_n^{m+\ell_n}$$

  Then observe that
  $$\begin{aligned}
  f_i^{m+\ell_i} s &= g_1 (a_1 f_1^m) f_i^{m+\ell_i} + \cdots + g_n (a_n f_n^m) f_i^{m+\ell_i} \\
  &= g_1 f_1^{m+\ell_1} (a_i f_i^m) + \cdots + g_n f_n^{m+\ell_n} (a_i f_i^m) \\
  &= (g_1 f_1^{m+\ell_1} + \cdots + g_n f_n^{m+\ell_n}) a_i f_i^m \\
  &= a_i f_i^m
  \end{aligned}$$

  That is, $f_i^m(f_i^{\ell_i} s - a_i)$. That is, $s = \frac{a_i}{f_i^{\ell_i}} = s_i$ on A_{f_i}, which is what we wanted.

  (You can use identity to show that the resulting glued object restricts to what you want on the other elements of the a priori infinite cover. So the identity proof does need to come first!).

  $\square$

Thus, we can finally start talking about affine schemes!!

6. Jan 27: Affine schemes, schemes

Proposition 6.1. Let A be a ring and $\mathcal{O}$ the structure sheaf on Spec A. For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_p \cong A_{\mathfrak{p}}$.

Proof. Fairly evident from the description of $A_{\mathfrak{p}}$ as a direct limit identifying a bunch of subsequent localizations. Lemma 2.14 can help. To be more concrete, we can write down the map.

Any $(s, D(f)) \in \mathcal{O}_p$ can be sent to its image in $A_{\mathfrak{p}}$. It is surjective: any element in $A_{\mathfrak{p}}$ is of the form a/g with $g \notin \mathfrak{p}$, and so $\mathfrak{p} \in D(g)$. That is, $D(g)$ will be a neighborhood of $\mathfrak{p}$ and a/f will be hit by this map.

It is injective: write $s = a/f, t = b/g$, with $f, g \notin \mathfrak{p}$. If their image is the same in $A_{\mathfrak{p}}$, then there is some $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$. But then $D(fgh) = D(f) \cap D(g) \cap D(h)$ is a neighborhood of $\mathfrak{p}$, and so s, t would have been identified in the stalk $\mathcal{O}_p$ $\qquad\square$

Definition 6.2. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf on X. The **direct image** sheaf $f_*\mathscr{F}$ on Y is defined via

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$$

for open sets $V \subseteq Y$. This is a functor from sheaves on X to sheaves on Y.

Definition 6.3. The **inverse image** sheaf $f^{-1}\mathscr{G}$ on X is the sheafification of the presheaf

$$U \mapsto \varinjlim_{V^{\text{open}} \supseteq f(U)} \mathscr{G}(V)$$

This is a functor from sheaves on Y to sheaves on X.

Definition 6.4. If $i : Z \hookrightarrow X$ a subset of X with the subspace topology, then $i^{-1}\mathscr{F}$ is the restriction of $\mathscr{F}$ to Z, denoted $\mathscr{F}|_Z$. For open sets Z this will just turn into $\mathscr{F}|_Z(V) = \mathscr{F}(V)$.

Definition 6.5. A **ringed space** is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a morphism of sheaves (of rings) $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if all the stalks $\mathcal{O}_{X,p}$ are local rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced map on stalks $f_P^\sharp : \mathcal{O}_{Y,f(P)} \to f_*\mathcal{O}_X$ is a local homomorphism of local rings.

Here, a local homomorphism of local rings $\varphi : A \to B$ is a ring morphism such that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

An isomorphism is a morphism (of ringed or locally ringed spaces) with a two-sided inverse. Equivalently, in $(f, f^\sharp)$, the f is a homeomorphism and $f^\sharp$ is a isomorphism of sheaves.

Proposition 6.6. (a) $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space.
(b) $\varphi : A \to B$ a morphism of rings induces

$$(f, f^\sharp) : (\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)$$

(c) In fact, any morphism of locally ringed spaces $(\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)$ is induced by a homomorphism of rings.

Proof.

(a) Immediate from previous results.
(b) The map on topological spaces is $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$ so the map is continuous. Two ways to see this:
   - Certainly this gives a morphism on a base $\mathcal{O}_A(D(f)) \to \mathcal{O}_B(D(\phi(f)) = \mathcal{O}_B(f^{-1}(D(f)) = f_*(\mathcal{O}_B)(D(f))$ via $A_f \to B_{\phi(f)}$ in the obvious way, and it respects restriction maps.
   - Localize at each prime to get a local homomorphism of local rings $\varphi_{\mathfrak{p}} : A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. Since sheaves are isomorphic to their sheafifcation, you can interpret sections on U as a compatible collection of germs, and so you can just map germs (and compatibility is preserved).
(c) Take global sections: we must have a map

$$\phi : \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong A \to \mathcal{O}_{\text{Spec } B}(\text{Spec } B) \cong B$$

One can show that φ induces all the data of the morphism.

Notably, we must have an induced morphism on stalks:$A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$. Due to compatibility, we must have

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_{f(\mathfrak{p})} & \xrightarrow{f_{\mathfrak{p}}^{\sharp}} & B_{\mathfrak{p}}
\end{array}
$$

Since $f^{\sharp}$ a local homomorphism, it must be that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so the map f on points coincides with the one induced by φ. Then compatibility with restriction maps will force the $f^{\sharp}$ to be induced by φ as well.

$\square$

Definition 6.7. An **affine scheme** if a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic, as a locally ringed space, to some $(\mathrm{Spec}\ A, \mathcal{O}_{\mathrm{Spec}\ A})$.

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $p \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Example 6.8 (Schemes can be glued)**.** Let X_1, X_2 be schemes. Let $U_i \subseteq X_i$ be open sets. Let $\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_1}|_{U_2})$ be an isomorphism of locally ringed spaces.

Then we can define a scheme X, obtained by gluing X_1, X_2 via identifying U_1, U_2 with the morphism φ. The topological space is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology (set open iff preimage is open).

We get maps $i_j : X_j \to X$ and the structure sheaf is defined as:

$$
\mathcal{O}_X(V) = \{(s_1, s_2) : s_j \in \mathcal{O}_{X_j}(i_j^{-1}(V)), \varphi(s_1|_{v_1^{-1}(V) \cap U_1}) = s_2|_{i_2^{-1}(V) \cap U_2})\}
$$

that is, sections on sets that "see" the overlap are gotten from piecing together compatible sections on each X_1, X_2.

Example 6.9 (More concrete)**.** Recall that morphisms of affine schemes are induced by ring morphisms on the global sections. Glue $\mathrm{Spec}\ (\mathbb{C}[t]$ and $\mathrm{Spec}\ (\mathbb{C}[s]$ along $D(t) = \mathrm{Spec}\ (\mathbb{C}[t, t^{-1}]) \cong \mathrm{Spec}\ (\mathbb{C}[s, s^{-1}]) = D(s)$ via the following.

$$
\begin{array}{cccc}
\mathrm{Spec}\ \mathbb{C}[s] & \mathrm{Spec}\ \mathbb{C}[t] & \mathbb{C}[s] & C[t] \\
\uparrow i_1 & \uparrow i_1 & \downarrow & \downarrow \\
\mathrm{Spec}\ \mathbb{C}[s, s^{-1}] \xrightarrow[t \mapsto s^{-1}]{} \mathrm{Spec}\ \mathbb{C}[t, t^{-1}] & & \mathbb{C}[s, s^{-1}] \xleftarrow{t \mapsto s^{-1}} \mathbb{C}[t, t^{-1}]
\end{array}
$$

This yields the projective line. We will learn about the *proj construction* in general next lecture.

Example 6.10 (What if you take the other transition function)**.** We could instead glue two copies of the line without any sort of inversion. Take $\mathrm{Spec}\ (\mathbb{C}[t])$ and $\mathrm{Spec}\ (\mathbb{C}[s])$. Take the open sets $D(t) = \mathrm{Spec}\ \mathbb{C}[t, t^{-1}], D(s) = \mathrm{Spec}\ \mathbb{C}[s, s^{-1}]$, and glue them via $t \to s$. This glues everything away from the origin in a "straightforward" way and we get the affine line with a doubled origin.

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