Trigonometric Gaussian quadrature on subintervals of the period *

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Abstract

We construct a quadrature formula with $n+1$ angles and positive weights, exact in the $(2n+1)$-dimensional space of trigonometric polynomials of degree $\leq n$ on intervals with length smaller than 2π. We apply the formula to the construction of product Gaussian quadrature rules on circular sectors, zones, segments and lenses.

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1 Introduction

In the recent paper [2], *trigonometric interpolation* of degree $\leq n$ at the $2n+1$ angles

$$\theta_j := \theta_j(n, \omega) = 2 \arcsin(\sin(\omega/2)\tau_j) \in (-\omega, \omega) \ , \quad j = 1, 2, \ldots, 2n+1 \ , \quad (1)$$

has been studied, where $0 < \omega \leq \pi$, and

$$\tau_j := \tau_{j,2n+1} = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right) \in (-1, 1) \ , \quad j = 1, 2, \ldots, 2n+1$$

are the zeros of the $2n+1$-th Chebyshev polynomial $T_{2n+1}(x)$. Moreover, it has been proved that the Lebesgue constant of such angles is $\mathcal{O}(\log n)$, and that the associated interpolatory trigonometric quadrature formula has

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positive weights. This topic has been termed "subperiodic" trigonometric interpolation and quadrature, since it concerns subintervals of the period of trigonometric polynomials.

Denoting by

$$\ell_j(x) = T_{2n+1}(x)/(T'_{2n+1}(\tau_j)(x-\tau_j)) \tag{2}$$

the j-th algebraic Lagrange polynomial (of degree $2n$) for the nodes $\{\tau_j\}$, $\ell_j(\xi_k) = \delta_{jk}$, the *cardinal functions* for trigonometric interpolation at the angles (1) can be written explicitly as

$$L_{n+1}(\theta) = \ell_{n+1}(x) \tag{3}$$

and for $j \neq n+1$

$$L_j(\theta) = \frac{1}{2}\left(\ell_j(x) + \ell_{2n+2-j}(x)\right)\left(1 + \frac{\tau_j^2}{\sin(\theta_j)}\,\frac{\sin(\theta)}{x^2}\right) \tag{4}$$

where

$$x = x(\theta) = \frac{\sin(\theta/2)}{\sin(\omega/2)} \in [-1,1] \tag{5}$$

with inverse

$$\theta = \theta(x) = 2\arcsin(\sin(\omega/2)x) \in [-\omega,\omega]\ . \tag{6}$$

There is an explicit formula for the weights of the associated trigonometric interpolatory quadrature rule, namely

$$w_j = \frac{1}{2n+1}\left(m_0 + 2\sum_{k=1}^{n} m_{2k}\, T_{2k}(\tau_j)\right)\ ,\quad j = 1,\dots,2n+1\ , \tag{7}$$

where

$$m_s = \int_{-1}^{1} \frac{T_s(x)\, 2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\, x^2}}\, dx\ ,\quad s = 0,1,2,\dots \tag{8}$$

are the Chebyshev moments (observe that the odd moments vanish) with respect to the weight function

$$w(x) = \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\, x^2}}\ ,\quad x \in (-1,1)\ . \tag{9}$$

The availability of a "subperiodic" trigonometric quadrature formula with positive weights, opens the way to construct stable *product quadrature formulas* [17, Ch.2], which are exact for total-degree algebraic polynomials, on domains related to circular arcs. Indeed, a suitable change of variables, such as polar, spherical, or cylindrical coordinates, can transform an algebraic polynomial into a product trigonometric or mixed algebraic/trigonometric polynomial, on arc-related sections of disks, and of surfaces/solids of rotation. To this purpose, it is convenient to find subperiodic trigonometric quadrature formulas with a small number of nodes (in this framework, it is also worth quoting the work by Kim and Reichel on anti-Szegő quadrature rules [12], in the case of measures supported on a subinterval of the period).

In this paper, using the algebraic Gaussian quadrature rule for the weight function (9), we provide a trigonometric "Gaussian" quadrature formula with positive weights, that is exact in the $(2n+1)$-dimensional space of trigonometric polynomials

$$\mathbb{T}_n([-\omega,\omega]) = \text{span}\{1, \cos(k\theta), \sin(k\theta),\ 1 \le k \le n\ ,\ \theta \in [-\omega,\omega]\}\ . \tag{10}$$

We also provide a Matlab function, `trigauss`, for the computation of angles and weights of such a formula (cf. [4, 5]). Then we make an application to the construction of product Gaussian quadrature formulas exact for algebraic polynomials, on some examples of arc-related domains in a disk. Indeed, while several formulas with polynomial exactness are known for the whole disk [3], these seem to be missing, apart from special cases (e.g., [15]), for relevant disk subregions, such as *circular sectors*, *circular zones* (the portion of a disk included between any two parallel chords and their intercepted arcs), with *circular segments* as a special case, and *circular lenses* (intersection of two disks).

The corresponding product formulas could be useful in the field of meshless methods for PDEs, when Galerkin-type methods are applied with compactly supported basis functions, which require integration on circular segments, sectors and lenses; cf., e.g., [1, 6].

2 Subperiodic trigonometric Gaussian quadrature

The main result is the following

Proposition 1 *Let $\{(\xi_j, \lambda_j)\}_{1\le j\le n+1}$, be the nodes and positive weights of the algebraic Gaussian quadrature formula for the weight function (9). Then*

$$\int_{-\omega}^{\omega} f(\theta)\,d\theta = \sum_{j=1}^{n+1} \lambda_j f(\phi_j)\ ,\ \ \forall f \in \mathbb{T}_n([-\omega,\omega])\ ,\ \ 0 < \omega \le \pi \tag{11}$$

where

$$\phi_j = 2\arcsin(\sin(\omega/2)\xi_j) \in (-\omega,\omega)\ ,\ \ j = 1, 2, \ldots, n+1\ .$$

Proof. Assume that the Gaussian nodes be in increasing order, $-1 < \xi_1 < \xi_2 \cdots < \xi_{n+1} < 1$. It is well known that, the weight function (9) being even, such nodes are symmetric, namely $\xi_j = -\xi_{n+2-j}$ (cf. [9, Ch.1]), and that $\lambda_j = \lambda_{n+2-j}$ since the corresponding Lagrange polynomials satisfy $l_j(x) = l_{n+2-j}(-x)$.

Let us rename for convenience the nodes, $\eta_k = \xi_j$, $k = j - \lfloor n/2 \rfloor - 1$, so that $\eta_k = -\eta_{-k}$, and the corresponding weights, say u_k, satisfy $u_k = u_{-k}$, $-\lfloor n/2 \rfloor \leq k \leq \lfloor n/2 \rfloor$.

Now, if we prove that the quadrature formula in (11) is exact on the cardinal functions (3)-(4), then it will be exact for every $f \in \mathbb{T}_n([-\omega, \omega])$. Observe that by the change of variables (5)-(6) and the definition of the cardinal functions

$$\int_{-\omega}^{\omega} L_i(\theta)\, d\theta = \int_{-1}^{1} \frac{1}{2} \left(\ell_i(x) + \ell_{2n+2-i}(x)\right)\, w(x)\, dx \ , \quad i = 1, 2, \ldots, 2n+1$$

since the function $\pi_i(x) = \frac{1}{2}(\ell_i(x) + \ell_{2n+2-i}(x))$ is even for every i, and the function $\sin(\theta(x))/x^2$ is odd. But $\pi_i(x)$ is a polynomial of degree $2n$ in x, and thus

$$\int_{-1}^{1} \pi_i(x)\, w(x)\, dx = \sum_{j=1}^{n+1} \lambda_j \pi_i(\xi_j) = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} u_k \pi_i(\eta_k) \ .$$

On the other hand, since by symmetry of the nodes $\pi_i(\eta_k)$ is even and $\pi_i(\eta_k)\, \sin(\theta(\eta_k))/\eta_k^2$ is odd with respect to the index k, by (3)-(4) we get

$$\sum_{j=1}^{n+1} \lambda_j L_i(\phi_j) = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} u_k L_i(\theta(\eta_k)) = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} u_k \pi_i(\eta_k) = \int_{-\omega}^{\omega} L_i(\theta)\, d\theta \ . \quad \square$$

Remark 1 The trigonometric quadrature formula (11) is a sort of "Gaussian" formula, in view of the degree of exactness and the positivity of the weights. On the other hand, the quadrature angles are the zeros of $p_{n+1}(x(\theta))$, where $\{p_k\}_{k \geq 0}$ are the algebraic orthogonal polynomials with respect to the weight function (9); the functions $\{p_k(x(\theta))\}$ are indeed orthogonal in $d\theta$, but they are not all trigonometric polynomials, since p_k is an odd function for k odd (cf. [9, §1.2]).

Remark 2 It is worth observing that for $\omega = \pi$ the underlying algebraic Gaussian quadrature is just the Gauss-Chebyshev formula, and that the trigonometric quadrature angles $\{\phi_j\}$ become $n+1$ equally spaced angles in $(-\pi, \pi)$ with the Gauss-Chebyshev weights, that correspond to a trapezoidal composite rule.

2.1 Computational issues

From the practical point of view, the problem is now to compute the nodes and weights of the algebraic Gaussian formula for the weight function (9). This can be done efficiently, for example in Matlab, by resorting to the *modified Chebyshev algorithm* by Gautschi, cf. [8, 9, 10].

This algorithm, implemented by the Matlab function `chebyshev` in the OPQ suite [8], computes the recurrence coefficients for the monic orthogonal polynomials with respect the weight function (9), say $\{p_k\}_{0 \le k \le n+1}$, using the so-called *modified Chebyshev moments*, that are those defined in (8) for $s = 0, 1, \ldots, 2n + 1$ (normalized in the monic case). As Gautschi says (cf. [9, §2.1.7]), "the success of the algorithm depends on the ability to compute all required modified moments accurately and reliably".

One possibility, is to compute all the moments by numerical quadrature, for example by the Matlab `quadvgk` function, which is an adaptive Gauss-Kronrod quadrature method (like the basic `quadgk`, [13]), tailored to manage vector-valued functions (cf. [18]). This approach works, but in view of efficiency we found more effective to compute the moments using the fact that the even ones (the odd ones vanish) satisfy a three-term linear non homogeneous recurrence relation, $a_n m_{2n+2} + b_n m_{2n} + c_n m_{2n-2} = d_n$, $n = 1, 2, \ldots$, as shown in [16], where the recurrence coefficients a_n, b_n, c_n, d_n are provided.

Such a recurrence is however unstable when used forward, and in order to stabilize it various algorithms are known, cf. [7]. A simple and effective method is to compute the even moments $(m_2, \ldots, m_{2n-2})$ by solving a linear system, with tridiagonal matrix (of the recurrence coefficients) and the vector $(d_1 - c_1 m_0, d_2, \ldots, d_{n-2}, d_{n-1} - a_{n-1} m_{2n})$ as right-hand side. We get immediately that $m_0 = 2\omega$, whereas the last moment m_{2n} can be computed accurately by the `quadgk` Matlab function (adaptive Gauss-Kronrod quadrature). In such a way we use numerical integration for only one moment. Since the matrix turns out to be diagonally dominant, the system can be solved quite efficiently with an $\mathcal{O}(n)$ cost, by the well-known Thomas algorithm (Gaussian elimination without pivoting), implemented by the Matlab function `tridisolve` (cf. [14]). The reduction of computing time with respect to the use of `quadvgk` for the vector of moments, is experimentally around a factor of 10.

Once the recurrence coefficients for the orthogonal polynomials $\{p_k\}$ are at hand, one can easily compute the Gaussian quadrature nodes and weights by another function of the OPQ suite, namely `gauss`, which as known performs a spectral decomposition of the Jacobi matrix of the relevant weight function. It is worth quoting that an alternative method for the computation of the recurrence coefficients could resort to the more general sub-range Jacobi polynomials approach, studied in [11].

Clearly, the trigonometric quadrature formula (11) can be extended to any angular interval, say $[\alpha, \beta]$, with $\beta - \alpha \leq 2\pi$, by using $\omega = (\beta - \alpha)/2$ and the shifted angles $\{\phi_j + (\beta + \alpha)/2\}$. In [5] we provide a Matlab function, named **trigauss**, whose call is of the form: `tw=trigauss(n,alpha,beta)`, that accepts the trigonometric degree and the endpoints of the angular interval, and returns the $(n+1) \times 2$ array of the quadrature angles and weights.

In order to test numerically the exactness of our quadrature formula (11), we have computed the integrals of the nonnegative (to avoid any cancellation problem) trigonometric basis

$$\{1, 1 + \cos(k\theta), 1 + \sin(k\theta),\ 1 \leq k \leq n\,,\ \theta \in [-\omega, \omega]\}\ , \tag{12}$$

for $n = 5, 10, 15, \ldots, 95, 100$ and several values of ω. The reference values of the integrals are known analytically. In Figure 1 we report the maximum of the componentwise relative errors for each n. We see that the error tend to increase with n and with ω, ranging from about 10^{-15} to about 10^{-14}.

The computing time of the quadrature angles and weights at a given degree turns out to be essentially invariant in ω, ranging from 10^{-2} seconds for n in the tens (it is still $3 \cdot 10^{-2}$ seconds for $n = 100$), to less than half second for n in the hundreds. The numerical experiments also show that for $n > 60$ the computing cost of the Chebyshev moments becomes rapidly a small fraction of the overall cost. These and the following tests have been made in Matlab 7.7.0 with an Athlon 64 +3800 2.40GHz processor.

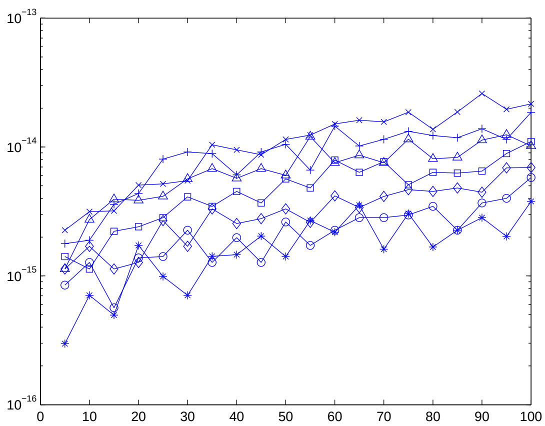

Figure 1: Maximal quadrature errors on the trigonometric basis (12) for $n = 5, 10, 15, \ldots, 95, 100$, with $\omega = \pi/16$ $(*)$, $\omega = \pi/8$ $(\circ)$, $\omega = \pi/4$ $(\Diamond)$, $\omega = \pi/2$ $(\Box)$, $\omega = 3\pi/4$ $(\triangle)$, $\omega = 7\pi/8$ $(+)$, $\omega = 15\pi/16$ $(\times)$.

2.2 Circular sectors

As a first application, we consider the integration of a bivariate function on a circular sector; up to translation and rotation, we can always take a sector centered at the origin and symmetric with respect to the x-axis, say

$$\Omega = \{(x,y) = (r\cos(\theta), r\sin(\theta))\,,\ 0 \le r \le R\,,\ \theta \in [-\omega,\omega]\}. \tag{13}$$

The key observation is that a polynomial $f \in \mathbb{P}_n^2(\Omega)$ (the space of bivariate polynomials of total degree $\le n$) becomes, in polar coordinates (r,θ), a mixed algebraic/trigonometric polynomial in the tensor-product space $\mathbb{P}_n([0,R]) \bigotimes \mathbb{T}_n([-\omega,\omega])$. Then, we get the product Gaussian formula on circular sectors

$$\iint_\Omega f(x,y)\,dx\,dy = \int_{-\omega}^{\omega}\int_0^R f(r\cos(\theta), r\sin(\theta))\,r\,drd\theta$$

$$= \sum_{j=1}^{n+1}\sum_{i=1}^{\lceil\frac{n+2}{2}\rceil} W_{ij}\, f(x_{ij}, y_{ij})\ ,\ \ \forall f \in \mathbb{P}_n^2(\Omega) \tag{14}$$

$$W_{ij} = r_i^{GL}\, w_i^{GL}\, \lambda_j\ ,\ \ (x_{ij}, y_{ij}) = (r_i^{GL}\, \cos(\phi_j), r_i^{GL}\, \sin(\phi_j))\ ,$$

cf. (11), where $\{(r_i^{GL}, w_i^{GL})\}$ are the nodes and weights of the Gauss-Legendre formula of degree of exactness $n+1$ on $[0,R]$ (cf. [9]). Observe that (14) has approximately $(n+1)(n+2)/2 \sim n^2/2$ nodes. It is also worth stressing that (14) is indeed exact not only on polynomials, but also on every function $f(x,y)$ such that $f(r\cos(\theta), r\sin(\theta)) \in \mathbb{P}_n([0,R]) \bigotimes \mathbb{T}_n([-\omega,\omega])$.

Extension to annular sectors $(0 < R_1 \le r \le R_2)$ is immediate, simply by using the corresponding Gauss-Legendre formula on $[R_1,R_2]$. We notice that there is a classical product quadrature formula for complete annuli (cf. [15]), but formulas with polynomial exactness on general annular sectors seems to be missing in the literature. In [5] we provide a Matlab function, `gqcircsect`, that computes the product Gaussian nodes and weigths for (annular) sectors. In Figure 2, we show an example of product quadrature nodes for two circular sectors. The computing time of nodes and weights at a given degree, turns out to be of some 10^{-2} seconds up to $n = 100$, independently of ω.

In order to test the polynomial exactness of the product Gaussian formula (14), we have computed the integral of the positive polynomial $(x+y+2)^n$ on the sector (13) with $R = 1$, for several values of n and ω, that is by Green's formula

$$I(\omega,n) = \iint_\Omega (x+y+2)^n\,dx\,dy = \oint_{\partial\Omega} \frac{(x+y+2)^{n+1}}{n+1}\,dy$$

$$= \int_{-\omega}^{\omega} \frac{(\cos(\theta)+\sin(\theta)+2)^{n+1}}{n+1}\cos(\theta)\,d\theta - \frac{\sin(\omega)}{(n+1)(n+2)} \times$$

$$\times \left(\frac{(\cos(\omega)-\sin(\omega)+2)^{n+2} - 2^{n+2}}{\cos(\omega)-\sin(\omega)} + \frac{(\cos(\omega)+\sin(\omega)+2)^{n+2} - 2^{n+2}}{\cos(\omega)+\sin(\omega)}\right)\,.$$

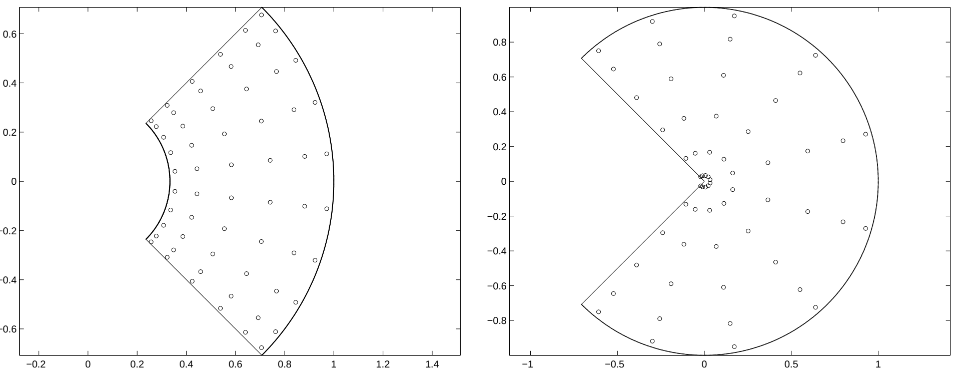

Figure 2: $60 = 6\times 10$ product quadrature nodes of algebraic exactness degree 9 on circular sectors with $R_1 = 1/3$, $R_2 = 1$, $\omega = \pi/4$ (left) and $R_1 = 0$, $R_2 = 1$, $\omega = 3\pi/4$ (right).

In Table 1 we report the maximal and average relative errors of the product Gaussian formula with respect to the reference values of $I(\omega, n)$, obtained by applying to the trigonometric polynomial $(\cos(\theta) + \sin(\theta) + 2)^n$ the quadrature formula with angles (1) and weights (7), implemented by the Matlab function `trigquad` in [5] (where the Chebyshev moments are computed by recurrence as in `trigauss`).

Table 1: Maximal and average relative errors in the integration of the polynomial $(x + y + 2)^n$ on the sector (13) with $R = 1$, for $n = 5, 10, 15, \ldots, 95, 100$.

ω	$\pi/16$	$\pi/8$	$\pi/4$	$\pi/2$	$3\pi/4$	$7\pi/8$	$15\pi/16$
E_{max}	1.9e-14	1.3e-14	1.3e-14	2.7e-14	1.3e-14	1.4e-14	1.8e-14
E_{av}	4.1e-15	4.8e-15	5.5e-15	5.6e-15	3.8e-15	4.0e-15	4.5e-15

As a second numerical test, we have integrated two parametric functions, one of infinite and the other of finite regularity (it exhibits a discontinuity of the fifth partial derivatives at a point of the sector). Observe that we obtain better results when such a singularity is located where the quadrature nodes cluster more rapidly (the origin and a curve corner of the sector). In particular, for $(a, b) = (0, 0)$ we have that the second test function is the purely radial function $f(x, y) = \exp(r^5)$.

2.3 Circular zones and segments

A *circular zone* is the portion of a disk included between any two parallel chords and their intercepted arcs. Up to translation and rotation, it can be

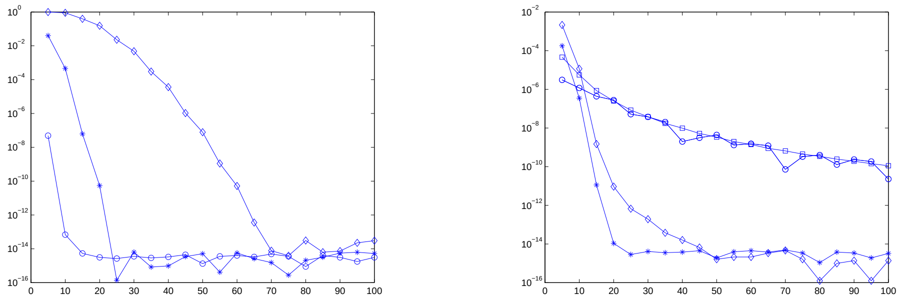

Figure 3: Relative errors in the integration of two parametric functions on the sector (13) with $R = 1$ and $\omega = \pi/3$, for $n = 5, 10, 15, \ldots, 95, 100$; left: $f(x, y) = \exp(k(x + y))$, $k = 1$ ($\circ$), $k = 10$ ($*$), $k = 100$ ($\Diamond$); right: $f(x, y) = \exp([(x - a)^2 + (y - b)^2]^{5/2})$, $(a, b) = (0.3, 0)$ ($\circ$), $(a, b) = (1, 0)$ ($\Box$), $(a, b) = (1/2, \sqrt{3}/2)$ ($\Diamond$), $(a, b) = (0, 0)$ ($*$).

represented as

$$\Omega = \{(x, y) = (R\cos(\theta), Rt\sin(\theta)),\, t \in [-1, 1]\,,\, \theta \in [\alpha, \beta]\}\ , \qquad (15)$$

where $0 \leq \alpha < \beta \leq \pi$. Now, a bivariate polynomial $f \in \mathbb{P}_n^2(\Omega)$ becomes, in the coordinates (t, θ), a mixed algebraic/trigonometric polynomial in $\mathbb{P}_n([-1, 1]) \bigotimes \mathbb{T}_n([\alpha, \beta])$. Since the transformation $(t, \theta) \mapsto (x, y)$ is injective with Jacobian $|J| = R^2 \sin^2(\theta)$, we get the product Gaussian formula on circular zones

$$\iint_\Omega f(x, y)\, dx\, dy = \sum_{j=1}^{n+3} \sum_{i=1}^{\lceil \frac{n+1}{2} \rceil} W_{ij}\, f(x_{ij}, y_{ij})\ ,\ \ \forall f \in \mathbb{P}_n^2(\Omega)$$

$$W_{ij} = R^2\, \sin^2(\varphi_j)\, w_i^{GL}\, \lambda_j\ ,\ \ (x_{ij}, y_{ij}) = (R\cos(\varphi_j), Rt_i^{GL} \sin(\varphi_j))\ , \qquad (16)$$

where $\varphi_j = \phi_j + (\beta + \alpha)/2$, $\{(\phi_j, \lambda_j)\}$ being the angles and weights of the trigonometric Gaussian formula (11) of degree of exactness $n + 2$ on $[-(\beta - \alpha)/2, (\beta - \alpha)/2]$, and $\{(t_i^{GL}, w_i^{GL})\}$ the nodes and weights of the Gauss-Legendre formula of degree of exactness n on $[-1, 1]$. Observe that formula (16) has approximately $(n + 1)(n + 3)/2 \sim n^2/2$ nodes, and is exact not only on polynomials, but also on every function $f(x, y)$ such that $f(R\cos(\theta), Rt\sin(\theta)) \in \mathbb{P}_n([-1, 1]) \bigotimes \mathbb{T}_n([\alpha, \beta])$.

A *circular segment*, one of the two portions of a disk cut by a chord, can be seen as a special degenerate case of circular zone, and up to translation and rotation corresponds to (15) with $\alpha = 0$.

In [5] we provide a Matlab function, `gqcirczone`, that computes the product Gaussian nodes and weigths for circular zones (and circular segments as a special case). Also here, the computing time of nodes and weights at a given degree is of some 10^{-2} seconds up to $n = 100$, independently of α, β. In Figure 4 and 6 (left), we show an example of product quadrature nodes for two circular segments and a circular zone.

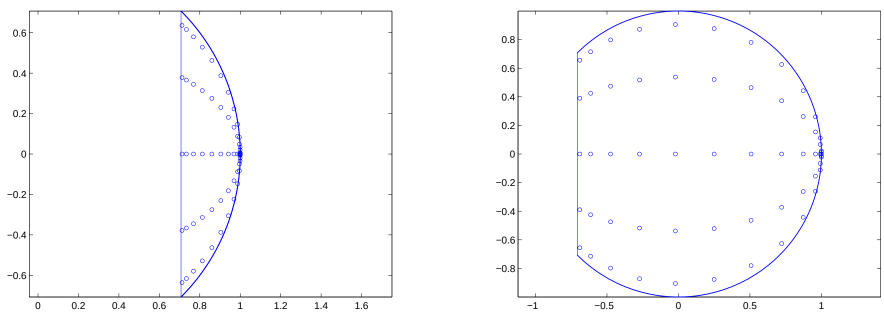

Figure 4: $60 = 5 \times 12$ product quadrature nodes of algebraic exactness degree 9 on circular segments (15) ($\alpha = 0$), with $R = 1$ and $\beta = \pi/4$ (left), $\beta = 3\pi/4$ (right).

We have tested polynomial exactness of the product Gaussian formula (16), by computing the integral of the positive polynomial $(x + y + 2)^n$ on circular segments (15) ($\alpha = 0$) with $R = 1$, for several values of n and β, that is

$$I(\beta, n) = \int_\Omega (x + y + 2)^n \, dx \, dy$$

$$= \int_0^\beta \frac{(\cos(\theta) + \sin(\theta) + 2)^{n+1} - (\cos(\theta) - \sin(\theta) + 2)^{n+1}}{n + 1} \sin(\theta) \, d\theta \ . \quad (17)$$

In Table 2 we report the maximal and average relative errors of the product Gaussian formula with respect to the reference values of $I(\beta, n)$, obtained by integrating the trigonometric polynomial in (17) by the Matlab function `trigquad` in [5].

Table 2: Maximal and average relative errors in the integration of the polynomial $(x + y + 2)^n$ on circular segments (15) ($\alpha = 0$) with $R = 1$, for $n = 5, 10, 15, \ldots, 95, 100$.

β	$\pi/16$	$\pi/8$	$\pi/4$	$\pi/2$	$3\pi/4$	$7\pi/8$	$15\pi/16$
E_{max}	4.8e-15	8.4e-15	1.3e-14	1.6e-14	1.3e-14	1.5e-14	1.5e-14
E_{av}	1.4e-15	2.7e-15	3.9e-15	4.2e-15	3.9e-15	3.8e-15	4.2e-15

In Figure 5, we show the results of a numerical integration test, on the same parametric functions of Figure 3. Again, we obtain better results for the finite regularity one when the singularity is located where the quadrature nodes cluster more rapidly (the point $(1,0)$ and a corner of the circular segment).

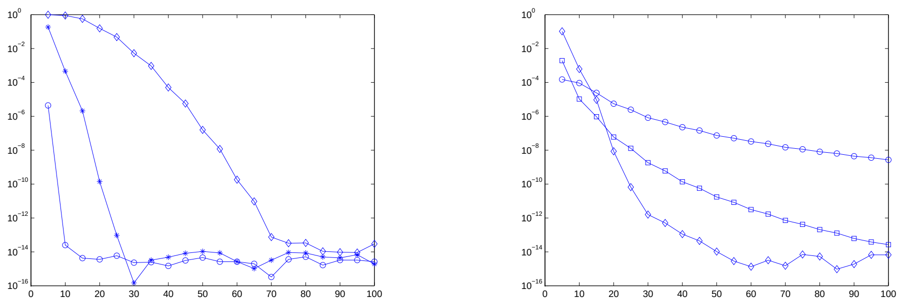

Figure 5: Relative errors in the integration of two parametric functions on the circular segment (15) ($\alpha = 0$), with $R = 1$ and $\beta = \pi/3$, for $n = 5, 10, 15, \ldots, 95, 100$; left: $f(x,y) = \exp(k(x+y))$, $k = 1$ ($\circ$), $k = 10$ ($*$), $k = 100$ ($\Diamond$); right: $f(x,y) = \exp([(x-a)^2 + (y-b)^2]^{5/2})$, $(a,b) = (0.7, 0)$ ($\circ$), $(a,b) = (1,0)$ ($\Box$), $(a,b) = (1/2, \sqrt{3}/2)$ ($\Diamond$).

2.3.1 Circular lenses

It is worth observing that (16) specialized to circular segments ($\alpha = 0$) can be used straightforwardly to construct Gaussian quadrature formulas on *lenses* (intersection of two disks), which are the union of two circular segments of possibly different angle, with the chord in common. The number of quadrature nodes doubles, becoming approximately $(n+1)(n+3) \sim n^2$.

These formulas could be useful whenever one has to integrate the product of two functions supported on disks, for example in the field of meshless methods for PDEs, when Galerkin-type methods are applied with compactly supported basis functions, centered at scattered points. Indeed, integration of the product of such functions or of their derivatives (with possibly different radii of the support disks), reduces to integration on (generally asymmetric) lenses; cf., e.g., [1, 6].

In the case of *symmetric lenses*, that are the intersection of two disks with equal radius R and distance between the centers in $[0, 2R)$, we can even reduce the number of product quadrature nodes. Indeed, a symmetric lens can be represented, up to translation and rotation, as

$$\Omega = \{(x,y) = (Rt(\cos(\theta) - \cos(\omega)), R\sin(\theta)),\, t \in [-1,1]\,,\, \theta \in [-\omega, \omega]\}\,, \tag{18}$$

where $0 < \omega \leq \pi/2$. A bivariate polynomial $f \in \mathbb{P}_n^2(\Omega)$ becomes, in the coordinates (t, θ), a mixed algebraic/trigonometric polynomial in the tensor-product space $\mathbb{P}_n([-1,1]) \bigotimes \mathbb{T}_n([-\omega, \omega])$. Since the transformation $(t, \theta) \mapsto$

(x, y) is injective with Jacobian $|J| = R^2 \cos(\theta)(\cos(\theta) - \cos(\omega))$, we get the product Gaussian formula for symmetric lenses

$$\iint_{\Omega} f(x,y)\,dx\,dy = \sum_{j=1}^{n+3} \sum_{i=1}^{\lceil \frac{n+1}{2} \rceil} W_{ij}\, f(x_{ij}, y_{ij})\ , \quad \forall f \in \mathbb{P}_n^2(\Omega)$$

$$W_{ij} = R^2\, \cos(\phi_j)(\cos(\phi_j) - \cos(\omega))\, w_i^{GL}\, \lambda_j\ ,$$
$$(x_{ij}, y_{ij}) = (Rt_i^{GL}(\cos(\phi_j) - \cos(\omega)), R\sin(\phi_j))\ , \qquad (19)$$

$\{(\phi_j, \lambda_j)\}$ being the angles and weights in (11), and $\{(t_i^{GL}, w_i^{GL})\}$ the nodes and weights of the Gauss-Legendre formula of degree of exactness n on $[-1, 1]$. This formula has approximately $(n + 1)(n + 3)/2 \sim n^2/2$ nodes.

In [5] we provide a Matlab functions, named `gqsymmlens`, which computes the product Gaussian nodes and weights for symmetric lenses. The computational costs and the numerical results on test functions are similar to those obtained for circular segments, and are not reported for brevity. In Figure 6 (right) we show an example of product quadrature nodes for a symmetric lens.

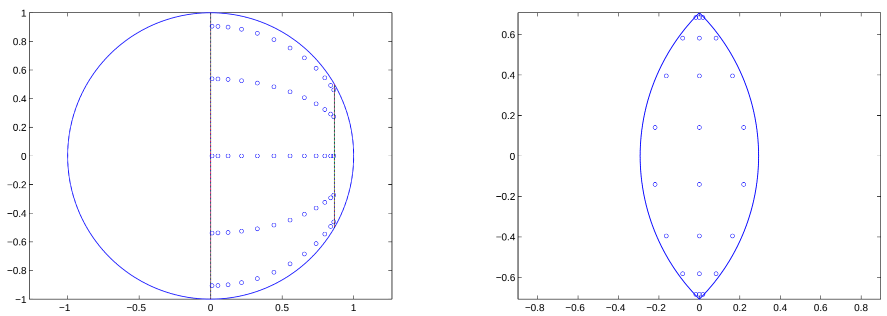

Figure 6: $60 = 5 \times 12$ product quadrature nodes of algebraic exactness degree 9 on a circular zone (15) with $R = 1$, $\alpha = \pi/6$ and $\beta = \pi/2$ (left), and $24 = 3\times 8$ product cubature nodes of exactness degree 5 on a symmetric lens (18) with $R = 1$ and $\omega = \pi/4$ (right).

3 Conclusions

We have constructed a quadrature formula with $n + 1$ nodes (angles) and positive weights, exact on the $(2n + 1)$-dimensional space of trigonometric polynomials of degree not greater than n, restricted to an angular interval $[-\omega, \omega]$ of length $2\omega < 2\pi$. Such a formula is related by a simple nonlinear transformation to an algebraic Gaussian quadrature formula on $[-1, 1]$, and can be implemented efficiently by Gautschi's OPQ Matlab suite [8]. This gives an improvement with respect to a recent result, concerning a quadrature formula with positive weights and $2n+1$ angles, based on subperiodic trigonometric interpolation [2].

We have applied the new subperiodic trigonometric quadrature formula to product Gaussian quadrature on relevant sections of the disk, such as circular sectors, zones, segments, lenses. All the corresponding Matlab codes are available online [5].

Subperiodic trigonometric Gaussian quadrature and related formulas could be useful in several applications, for example within Galerkin-type meshless methods for PDEs, or in the construction of product Gaussian rules on arc-related sections of surfaces/solids of rotation.

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